University of New Brunswick
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SHGEO software package

The UNB Application to Stokes-Helmert Approach for Precise Geoid Computation

Reference Manual I.

Theory of Stokes-Helmert's Method of Geoid Determination

Fredericton, New Brunswick, Canada
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Preface

The Stokes-Helmert Geoid software (SHGeo software) is a scientific software package for a precise geoid determination based on the Stokes-Helmert theory of determination of the gravimetric geoid. The software has been developed during a period of more then 10 years under the leadership of professor Petr Vaníček at the Department of Geodesy and Geomatics Engineering, University of New Brunswick in Fredericton. Authors of particular programs are: Juraj Janák, Pavel Novák, Mehdi Najafi-Alamdari, Jianliang Huang, Sander van Eck van der Sluijs, Robert Tenzer and Artu Ellmann. We also have to mention Z. Martinec, A. Kleusberg, L.E. Sjöberg, W.E. Featherstone and W. Sun whose research presented in their papers was incorporated into the SHGeo software. SHGeo software uses global geopotential models (e.g., GRIM4-S4, EGM-96, GGM-02) and a global elevation model (TUG-87 or JGP95 for instance). These global models play an important role in the geoid computation scheme. Therefore we acknowledge the contribution of all research teams that have developed these or other global models.

The present software version (SHGEO 3.1) is an upgrade of the three earlier SHGeo packages:
SHGEO vers. 1 (standard Helmert approach) was compiled by Dr. J. Janak in 2001
SHGEO vers. 2 (formulated for the NoTopography space) was compiled by Dr. R. Tenzer in 2003
SHGEO vers. 3 (capable of standard Helmert and the NoTopography) was compiled by Dr. A. Ellmann in 2005

This reference manual and the current version of the package, SHGEO 3.1 (standard Helmert and NoTopography) is compiled by D. Avalos in 2009. New programs (dwnc08.c, res_anomaly.c and cogeoid2geoid.c) as well as bug-fixes for the Stokes integral computation are included in this version.

The manual consists of three parts. Part I contains the theoretical description of the Stokes-Helmert method of the geoid determination by Dr. A. Ellmann (September 2005). Part II is the reference user's guide with the description of the particular programs for the geoid computation, and Part III has the description of auxiliary programs, which can be used for data manipulation and format transformations.

Contents:

Stokes-Helmert's method for precise geoid determination

Part I of the manual gives a brief theoretical overview of the precise geoid determination by the Stokes-Helmert method. Particular details can be found in the reference papers listed at the end of this manual.

1. Introduction

The geoid as an equipotential surface of the Earth's gravity field plays an essential role in geosciencies and in many practical applications. In geodesy, it serves as the reference surface for topographic height and depth measurements.

The solution of the boundary value problem by Stokes's method requires gravity observations that refer to the geoid as a boundary surface while gravity measurements are taken at the topographic surface. Thus, to satisfy the boundary condition the gravity anomalies need to be downward continued to the geoid level. Downward continuation requires harmonicity of the quantities to be downward continued; thus a number of different corrections related to the existence of topography and atmosphere need be accounted for very carefully. As is well known the evaluation of topographical effects is one of the most serious limits in precise geoid modeling nowadays. Therefore, the topographical effects need be rigorously formulated and evaluated for the spherical Earth.

One way of estimating the effect of topographical masses is to use Helmert's second condensation model. According to this model the Earth's topographical masses can be replaced by an infinitesimal condensation layer located on the geoid. To use this model, the real field quantities are first transformed into corresponding quantities in this model space which we call the Helmert space. So "Helmertizised" gravity field can be then decomposed into low- and high-frequency parts (see Vaníček and Kleusberg, 1987). Global geopotential models are the most accurate source for the low-frequency information, whereas the short-wavelength information is obtained from the Stokesian integration. For practical considerations, the integration is limited to a spatial domain around the computation points. Importantly, the Stokes kernel modification scheme is employed. This scheme mitigates the error in the contribution from the field in the remote zone which exists even though the contribution is rigorously evaluated from global geopotential model.

The Stokesian integration with the so-called residual gravity anomalies results with the Helmert residual co-geoid. The complete Helmert co-geoid is obtained as a sum of the Helmert reference spheroid and the residual co-geoid. In the final computational step the geoid in the real space is evaluated by subtracting the primary indirect topographical and atmospheric effects from the Helmert co-geoidal heights.

The principles of the Stokes-Helmert method of the geoid determination are summarized by the following scheme (see Fig. 1):

- **Formulation of the fundamental formula of physical geodesy at the surface of the Earth in the real space.**

- **Transformation of the real space gravity anomalies into the Helmert gravity anomalies (referred to the Earth's surface)**

- **Solution to Dirichlet's inverse boundary value problem by applying the Poisson integral equation, i.e., the downward continuation of the gravity anomalies from the Earth's surface onto the geoid.**

- **Reformulation of the geodetic boundary-value problem by decomposition of Helmert's gravity field into the low-frequency and high-frequency part of the gravity field.**

- **Solution to the Stokes boundary value problem for the high-frequency Helmert gravity field (by using the modified spheroidal Stokes' kernel) and the evaluation of the Helmert reference spheroid (by using the "satellite-only" spherical harmonics).**
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- **Transformation of the geoid from the Helmert gravity space back into the real space.**

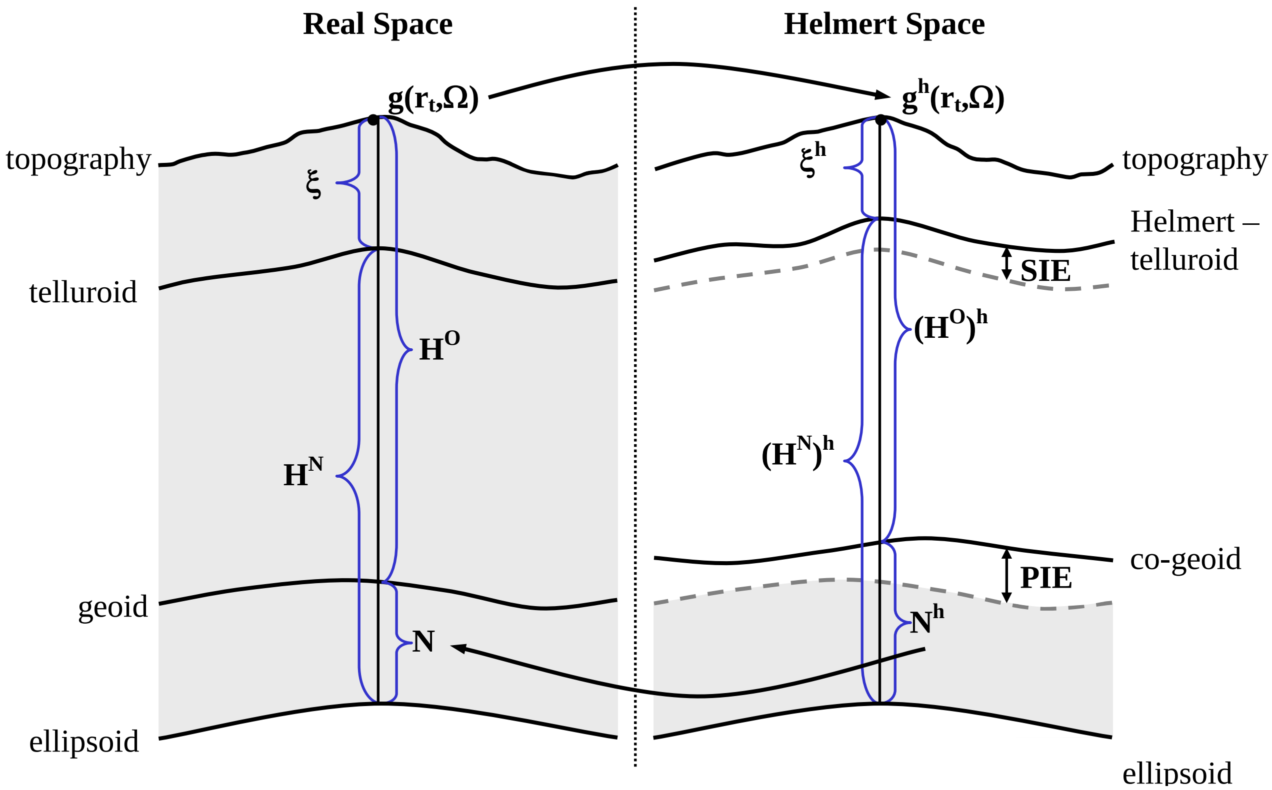

Fig. 1: The quantities involved in the real and Helmert's spaces

2. Formulation of fundamental formula of physical geodesy

2.1 Real space

In the classical sense of Gauss and Listing (1873), the geoid is defined as an equipotential surface with the gravity potential value W_0. Gauss (1828) defined this surface in the strict mathematical sense as a surface which is intersected everywhere by directions of gravity at right angle and which best approximates the mean sea level over the whole Earth.
In order to estimate the geoid shape a reference field (so called normal field) generated by the reference ellipsoid of revolution is introduced. The normal potential on this reference ellipsoid is chosen to be equal to the actual potential on the geoid.

The **disturbing gravity potential** $T(r_t,\Omega)$, which is reckoned at the Earth's surface, is defined as a difference of the **Earth's gravity potential** $W(r_t,\Omega)$ and the **normal gravity potential** $U(r_t,\Omega)$ generated by the geocentric reference ellipsoid (of which parameters: U_o, ω_o, and GM_o are equal to the fundamental physical parameters of the Earth: W_o, ω, and GM; where GM is the geocentric gravitational constant, ω is the mean angular velocity of the Earth's spin, $W_o \equiv W(r_g, \Omega)$ is the gravity potential referred to the geoid surface $\forall \Omega \in \Omega_o : r_g(\Omega)$, and $U_o \equiv U(r_o, \phi)$ is the normal gravity potential referred to the surface of the geocentric reference ellipsoid $\forall \Omega \in \Omega_o : r_o(\phi)$, so that

$$T(r,\Omega) = W(r,\Omega) - U(r,\Omega). \tag{2.1}$$

The geocentric position (r,Ω) is represented by the geocentric radius r and the geocentric direction $\Omega = (\phi, \lambda)$, where ϕ and λ are the geocentric spherical coordinates, and Ω_O stands for the geocentric total solid angle $[\phi \in \langle -\pi/2, \pi/2 \rangle, \lambda \in \langle 0, 2\pi \rangle]$. The geocentric radius of the Earth's surface $r_t(\Omega)$ is given (with an accuracy of a few millimeters) by the geocentric radius of the geoid surface $r_g(\Omega)$, plus the **orthometric height** $H^O(\Omega)$, i.e. $r_t(\Omega) \cong r_g(\Omega) + H^O(\Omega)$.

If the topographic and atmospheric attractions were completely absent, then *T* would be harmonic above the geoid surface and the Laplace equation would be satisfied.

$$\nabla^2 T(r,\Omega) = 0 \qquad r \geq r_g. \tag{2.2}$$

Once *T* is known on the geoid, the separation between the reference ellipsoid and the geoid can be obtained by the Bruns formula (Bruns, 1878)

$$N(\Omega) = \frac{T(r_g,\Omega)}{\gamma_o(\Omega)}, \tag{2.3}$$

where $T(r_g,\Omega)$ is the disturbing potential on the geoid and γ_o is the normal gravity on the reference ellipsoid. The problem is now reduced to the determination of *T* on and outside the geoid. To determine $T(r,\Omega)$, the boundary value problem of the third kind outside the geoid has to be solved. In this problem the gravity anomalies on the geoid itself serve as the boundary values.

To find a relation between the disturbing potential and the gravity anomalies, the radial derivative of the disturbing potential is introduced:

$$\frac{\partial T(r,\Omega)}{\partial r} = \frac{\partial W(r,\Omega)}{\partial r} - \frac{\partial U(r,\Omega)}{\partial r}. \tag{2.4}$$

The above expression, evaluated at the earth's surface, can be approximated by (cf. Vanicek et al 1999)

$$\left. \frac{\partial T(r,\Omega)}{\partial r} \right|_{r=r_t} \cong -g(r_t,\Omega) + \gamma(r_t,\Omega) + \varepsilon_{\delta g}(r_t,\Omega) = -\delta g(r_t,\Omega) + \varepsilon_{\delta g}(r_t,\Omega), \tag{2.5}$$

where the difference between the actual **gravity** $g(r_t,\Omega)$ and the **normal gravity** $\gamma(r_t,\Omega)$ is the gravity disturbance $\delta g(r_t,\Omega)$, and $\varepsilon_{\delta g}(r_t,\Omega)$ is the ellipsoidal correctcion to the gravity disturbance.

The gravity disturbance is usually not considered to be a measurable quantity on the surface of the Earth. Therefore the gravity disturbance has to be transformed to gravity anomaly, which is still the most widely available data type. Gravity anomaly $\Delta g(r_t,\Omega)$:

$$\Delta g(r_t,\Omega) = g(r_t,\Omega) - \gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right], \tag{2.6}$$

is related to the gravity disturbance $\delta g(r,\Omega)$ by the following formula (Vaníček et al., 1999)

$$\Delta g(r_t,\Omega) = \delta g(r_t,\Omega) + \gamma(r_t,\Omega) - \gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right], \tag{2.7}$$

where $H^{\mathrm{N}}(\Omega)$ is the **normal height** (Molodensky, 1945).

The difference of the normal gravity $\gamma(r_t,\Omega)$ referred to the Earth's surface and the normal gravity $\gamma[r_o(\phi) + H^{\mathrm{N}}(\Omega)]$ referred to the telluroid $r_o(\phi) + H^{\mathrm{N}}(\Omega)$ can be expressed using the Bruns formula

$$\gamma(r_t,\Omega) - \gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right] \cong \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \varsigma(\Omega) = \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \frac{T(r_t,\Omega)}{\gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]}, \tag{2.8}$$

where $\varsigma(\Omega)$ is the **height anomaly** (Molodensky et al., 1960), and $\partial\gamma(r,\phi)/\partial \mathrm{n}$ is the normal gravity gradient.

Applying the spherical approximation, then Eq. (2.8) becomes (Vaníček and Martinec, 1994)

$$\left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \frac{T(r_t,\Omega)}{\gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]} = -\frac{2}{r_t(\Omega)} T(r_t,\Omega) - \varepsilon_{\mathrm{n}}(r_t,\Omega), \tag{2.9}$$

where $\varepsilon_{\mathrm{n}}(r_t,\Omega)$ is the **ellipsoidal correction for the spherical approximation**

Substituting Eqns. (2.5) and (2.9) into Eq. (2.7), the **fundamental formula of physical geodesy takes** the following form (Vaníček et al., 1999)

$$\Delta g(r_t,\Omega) = -\left.\frac{\partial T(r,\Omega)}{\partial r}\right|_{r=r_t(\Omega)} + \varepsilon_{\delta g}(r_t,\Omega) - \frac{2}{r_t(\Omega)} T(r_t,\Omega) - \varepsilon_{\mathrm{n}}(r_t,\Omega). \tag{2.10}$$

The above equation, formulated for the real space, can be applied in Helmert's space for the purpose of the computation of Helmert's gravity anomaly.

2.2 Helmert space

However, T in Eq. (2.1) does not satisfy the Laplace equation inside the topographical masses where the geoid is often located. Therefore in order to establish a harmonicity of the disturbing potential, the atmospheric and topographical masses have to be (mathematically) removed or replaced. This can be done by using Helmert's second condensation method.

When the masses are condensed to a layer that is located right on the geoid, the Earth's gravity field will slightly change. The space obtained after such a condensation is called the Helmert space. The quantities given in the Helmert space will be denoted by superscript h.

Helmert's gravity potential is defined as follows

$$W^h(r,\Omega) = W(r,\Omega) - \delta V^t(r,\Omega) - \delta V^a(r,\Omega), \tag{2.11}$$

where $\delta V^t(r,\Omega)$ is the residual topographical potential, which is the difference between the potential of the topographical masses and the potential of the condensation layer

$$\delta V^t(r,\Omega) = V^t(r,\Omega) - V^c(r,\Omega). \tag{2.12}$$

Similarly, $\delta V^a(r,\Omega)$ is defined as the residual atmospheric potential, which is obtained by subtracting the potential of the atmospheric condensation layer from the potential of the atmospheric masses, i.e

$$\delta V^a(r,\Omega) = V^a(r,\Omega) - V^{ca}(r,\Omega). \tag{2.13}$$

By subtracting the normal reference field U from equation (2.4), the disturbing potential in Helmert's space becomes

$$T^h(r,\Omega) = W^h(r,\Omega) - U(r,\Omega) = T(r,\Omega) - \delta V^t(r,\Omega) - \delta V^a(r,\Omega). \tag{2.14}$$

It has been shown (Vanicek and Martinec 1994) that the Helmert's disturbing potential is harmonic in every point outside the geoid – this is rather obvious from the fact that mass density above the geoid is equal to zero everywhere, so that the Laplace equation

$$\nabla^2 T^h(r,\Omega) = 0 \qquad\qquad r \geq r_g \tag{2.15}$$

holds everywhere above the geoid level.

The Helmert gravity $g^h(r_t,\Omega)$ on the earth surface is obtained from the observed gravity g on the earth surface, by adding the direct topographical effect $\delta A^t(r_t,\Omega)$ and the direct atmospheric effect $\delta A^a(r_t,\Omega)$ (referred to the earth surface):

$$g^h(r_t,\Omega) = g(r_t,\Omega) + \delta A^t(r_t,\Omega) + \delta A^a(r_t,\Omega). \tag{2.16}$$

The direct topographical effect (DTE) on gravity is a residual quantity. It is obtained by subtracting the gravitational attraction of the condensed topographical masses from the the attraction of topographical masses. It should be evaluated on the earth surface. Analogously, the direct atmospheric effect (DAE) on gravity is the gravitational attraction of the whole atmosphere minus the gravitational attraction of the condensed atmosphere. The direct effect is obtained by taking the radial derivative of the residual topographical and atmospheric potentials (Eqs. (2.12) and (2.13)):

$$\delta A^t(r_t,\Omega) = \frac{\partial \delta V^t(r_t,\Omega)}{\partial r} = \frac{\partial V^t(r_t,\Omega)}{\partial r} - \frac{\partial V^c(r_t,\Omega)}{\partial r} = A^t(r_t,\Omega) - A^c(r_t,\Omega), \tag{2.17a}$$

$$\delta A^a(r_t,\Omega) = \frac{\partial \delta V^a(r_t,\Omega)}{\partial r} = \frac{\partial V^a(r_t,\Omega)}{\partial r} - \frac{\partial V^{ca}(r_t,\Omega)}{\partial r} = A^a(r_t,\Omega) - A^{ca}(r_t,\Omega). \tag{2.17b}$$

More details on the estimation of $\delta A^t(r_t,\Omega)$ and $\delta A^a(r_t,\Omega)$ will be given in Sections 3.1. and 3.2. By analogy with Eq. (2.5) the Helmert gravity disturbance is defined as the negative gradient of the Helmert disturbing gravity potential and the ellipsoidal correction to the gravity disturbance, i.e.

$$\delta g^h(r_t,\Omega) = -\frac{\partial T^h(r_t,\Omega)}{\partial r} + \varepsilon_{\delta g}(r_t,\Omega) = g(r_t,\Omega) - \gamma(r_t,\Omega) + \varepsilon_{\delta g}(r_t,\Omega) + \delta A^t(r_t,\Omega) + \delta A^a(r_t,\Omega) \tag{2.18}$$

The relation between the Helmert gravity disturbance and the Helmert gravity anomaly $\Delta g^h(r_t,\Omega)$ can be obtained from the boundary condition (cf. Eq. (2.10)):

$$\Delta g^h(r_t,\Omega) = -\left.\frac{\partial T^h(r,\Omega)}{\partial r}\right|_{r=r_t(\Omega)} + \varepsilon_{\delta g}(r_t,\Omega) + \left.\frac{\partial \gamma(r,\Omega)}{r_t(\Omega)}\right|_{r=r_t(\Omega)} \frac{T^h(r_t,\Omega)}{\gamma(r+H^o,\Omega)} \tag{2.19}$$

Considering Eqs. (2.8), (2.14) and (2.18) one arrives at (cf. Vanicek et al 1999, Eq. 37)

$$\begin{aligned}
\Delta g^h(r_t,\Omega) &= \delta g^h(r_t,\Omega) + \gamma(r_t,\Omega) - \gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right] - \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \frac{T(r_t,\Omega)}{\gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]} + \\
&+ \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \frac{T^h(r_t,\Omega)}{\gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]} = \\
&= \delta g^h(r_t,\Omega) + \gamma(r_t,\Omega) - \gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right] - \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \frac{\left[\delta V^t(r_t,\Omega) + \delta V^a(r_t,\Omega)\right]}{\gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]} = \\
&= g(r_t,\Omega) + \varepsilon_{\delta g}(r_t,\Omega) + \delta A^t(r_t,\Omega) + \delta A^a(r_t,\Omega) - \gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right] - \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} \frac{\left[\delta V^t(r_t,\Omega) + \delta V^a(r_t,\Omega)\right]}{\gamma\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]}
\end{aligned} \tag{2.20}$$

The Helmert anomaly can be expressed via commonly used free-air anomalies $\Delta g^{FA}\left(r_t,\Omega\right)$ (see e.g. Heiskanen and Moritz, 1967),

$$\Delta g^{FA}\left(r,\Omega\right)=g\left(r,\Omega\right)-\gamma(r_o,\phi)-\frac{\partial\gamma\left(r,\Omega\right)}{\partial n}H^{o}\left(\Omega\right)\approx g\left(r,\Omega\right)-\gamma(r_o,\phi)-0.3086H^{o}\left(\Omega\right) \tag{2.21}$$

Applying also the spherical approximation from Eq. (2.9) one arrives at

$$\Delta g^{h}\left(r_t,\Omega\right)=\Delta g^{FA}\left(r_t,\Omega\right)+\delta A^{t}\left(r_t,\Omega\right)+\delta A^{a}\left(r_t,\Omega\right)+\varepsilon_{\delta g}\left(r_t,\Omega\right)+\frac{2}{r_t}\delta V^{t}\left(r_t,\Omega\right)+\frac{2}{r_t}\delta V^{a}\left(r_t,\Omega\right)-\varepsilon_{n}\left(r_t,\Omega\right) \tag{2.22}$$

In Eq. (2.22), the second term on the right-hand side is the direct topographic effect on the gravitational attraction, the third term represents the direct atmospheric effect on the gravitational attraction, the fourth term is the ellipsoidal correction to gravity disturbance, the fifth term is the secondary indirect topographic effect (SITE) on the gravitational attraction, the sixth term is the secondary indirect atmospheric effect (negligible in practical computations), the last term is the ellipsoidal correction for spherical approximation.

2.2.1. Geoid – quasigeoid correction

If in Eq. (2.21) the Helmert orthometric heights (rather than normal heights) are used, the so-called "geoid-quasigeoid correction" has to be applied to the boundary condition formulated in the Helmert space (Vanicek et al 1999). The geoid-quasigeoid correction, i.e. due to the difference of the normal and orthometric heights, can be approximately described as a function of the simple Bouguer gravity anomaly $\Delta g^{B}\left(r_t,\Omega\right)$, see (Martinec, 1993),

$$H^{\mathrm{N}}(\Omega)-H^{\mathrm{O}}(\Omega)\cong H^{\mathrm{O}}(\Omega)\frac{\Delta g^{\mathrm{SB}}(r_t(\Omega))}{\gamma_o(\phi)}, \tag{2.23}$$

where $\gamma_o(\phi)$ is the normal gravity referred to the ellipsoid surface.
The formula for the simple Bouguer gravity anomaly $\Delta g^{\mathrm{SB}}(r_t(\Omega))$ reads (Heiskanen and Moritz, 1967)

$$\Delta g^{\mathrm{SB}}(r_t(\Omega))=g(r_t(\Omega))-\gamma\left[r_o(\phi)+H^{\mathrm{O}}(\Omega)\right]-2\pi\mathrm{G}\rho_{\mathrm{o}}H^{\mathrm{O}}(\Omega), \tag{2.24}$$

where G is Newton's gravitational constant. The third term on the right-hand side of Eq. (2.13) stands for the gravitational attraction generated by the infinite Bouguer plate (of **mean topographical density** ρ_{o} and thickness equal to the orthometric height $H^{\mathrm{O}}(\Omega)$ of the computation point). The geoid-quasigeoid correction can be expressed as

$$\chi(r_t(\Omega)) \cong -\frac{1}{\gamma_o(\Omega)} \frac{\partial \gamma(r,\Omega)}{\partial \mathrm{n}}\bigg|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega)\,\Delta g^{\mathrm{SB}}(r_t(\Omega))$$
$$\approx \frac{2}{r_t(\Omega)} H^{\mathrm{O}}(\Omega)\,\Delta g^{\mathrm{SB}}(r_t(\Omega)), \tag{2.25}$$

The Helmert gravity anomaly, Eq. (2.22), with the geoid-quasigeoid correction is computed as

$$\Delta g^h(r_t,\Omega) = \Delta g^{FA}(r_t,\Omega) + \delta A^t(r_t,\Omega) + \delta A^a(r_t,\Omega) + \varepsilon_{\delta g} + \frac{2}{r_t}\delta V^t(r_t,\Omega) + \frac{2}{r_t}\delta V^a(r_t,\Omega) - \varepsilon_n + \chi(r_t,\Omega) \tag{2.26}$$

2.2.2. A comment on the ellipsodial corrections

To solve the Stokes boundary value problem, the gravity anomalies referred to the Earth's surface have to be downward continued onto the geoid surface. For this reason, the external gravitational field above the geoid has to be harmonic. In the Helmert space the product $\Delta g^h \cdot r$ is harmonic (Vanicek and Martinec, 1994). Note the presence of the two ellipsoidal corrections in the expressions (2.22) and (2.26).

Wong (2001) showed that ellipsoidal corrections are harmonic. Moreover, if the ellipsoidal form of a function is harmonic then so is its spherical form. Thus, the downward continuation can conventionally be processed with the "spherical" Helmert anomalies (i.e. without considering the ellipsoidal corrections $\varepsilon_{\delta g}$ and ε_n). It means that the Helmert gravity anomalies do not need be corrected for the ellipsoidal correction <u>before</u> DWC. The appropriate ellipsoidal corrections are added to the Helmert anomalies only on the geoid level. Note that this is done in the Helmert space rather than in the real space. More details on evaluation of the ellipsoidal corrections will be given in Section 5.

3.1. Treatment of the atmospheric effects

To evaluate the Helmert anomalies $\Delta g^{\mathrm{h}}(r_t,\Omega)$ referred to the Earth's surface according to Eq. (2.22), the effects of the atmospheric and topographical masses on the gravitational attraction, i.e., the direct and secondary indirect atmospheric and topographical effects, have to be evaluated at the Earth surface.

According to Eq. (2.13) the residual gravitational potential of the atmospheric masses $\delta V^t(r_t,\Omega)$ is given by the difference of the gravitational potential $V^a(r_t,\Omega)$ of atmospheric masses and the gravitational potential $V^{ca}(r_t,\Omega)$ of atmospheric masses condensed onto the geoid (Vanicek et al 1999).

Considering the radially distributed **atmospheric density** $\rho^a(r)$ and under the spherical approximation of the geoid surface ($r_g(\Omega) \approx \mathrm{R}$, where $\mathrm{R} = \sqrt[3]{\mathrm{a}^2\,\mathrm{b}}$ is the mean radius of the Earth, Bomford, 1971), the gravitational potential $V^a(r_t,\Omega)$ of the atmospheric masses reads (Novák, 2000)

$$V^a(r_t,\Omega) = \mathrm{G} \iint_{\Omega'\in\Omega_\mathrm{O}} \int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{r_{lim}}} \rho^a(r')\, l^{-1}\left[r_t(\Omega),\psi(\Omega,\Omega'),r'\right] r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega' , \tag{3.1}$$

where $\mathrm{r_{lim}}$ is the upper limit of the atmosphere, $l[r,\psi(\Omega,\Omega'),r']$ denotes the spatial distance between the computation point (r,Ω) and integration points (r',Ω')

$$\forall\, \Omega,\Omega' \in \Omega_\mathrm{O},\, r,r' \in \Re^+ : \qquad l[r,\psi(\Omega,\Omega'),r'] = \sqrt{r^2 + r'^2 - 2r\, r' \cos\psi(\Omega,\Omega')} , \tag{3.2}$$

and the spherical distance $\psi(\Omega,\Omega')$ is given by the cosine theorem

$$\forall\, \psi \in \langle 0, \pi \rangle : \qquad \cos\psi(\Omega,\Omega') = \sin\phi' \sin\phi + \cos\phi' \cos\phi \cos(\lambda' - \lambda) . \tag{3.3}$$

Formally, the Earth's atmospheric masses can be split into the spherical shell between the upper limit of the topography $\forall\, \Omega \in \Omega_\mathrm{O} : \mathrm{R} + \mathrm{H_{lim}};\ \mathrm{H_{lim}} = \max H^\mathrm{O}(\Omega)$, and the upper limit of the atmosphere (~ 50 km), and the atmospheric roughness term between the Earth's surface and the upper limit of the topography (Novák, 2000).
The gravitational potential $V^a(r,\Omega)$ of the atmospheric masses is then expressed by (Novák, 2000)

$$\begin{aligned} V^a(r_t,\Omega) = {} & \mathrm{G} \iint_{\Omega'\in\Omega_\mathrm{O}} \int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{R+H_{lim}}} \rho^a(r')\, l^{-1}\left[r_t(\Omega),\psi(\Omega,\Omega'),r'\right] r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega' \\ & + \mathrm{G} \iint_{\Omega'\in\Omega_\mathrm{O}} \int_{r'=\mathrm{R+H_{lim}}}^{\mathrm{r_{lim}}} \rho^a(r')\, l^{-1}[r_t(\Omega),\psi(\Omega,\Omega'),r']\, r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega' . \end{aligned} \tag{3.4}$$

The gravitational potential of the condensed atmospheric masses $V^{ca}(r_t,\Omega)$ referred to the earth surface reads (Novák, 2000)

$$V^{ca}(r_t,\Omega) = \mathrm{G}\, \mathrm{R}^2 \iint_{\Omega'\in\Omega_\mathrm{O}} \sigma^a(\Omega')\, l^{-1}\left[r_t,\psi(\Omega,\Omega'),\mathrm{R}\right] \mathrm{d}\Omega' , \tag{3.5}$$

where $\sigma^a(\Omega)$ is the **surface density of the condensed atmospheric masses**.
According to the principle of the mass-conservation condensation of the atmospheric masses, the atmospheric surface density $\sigma^a(\Omega)$ is defined by (Novák, 2000)

$$\sigma^a(\Omega) = \frac{1}{\mathrm{R}^2} \int_{r=\mathrm{R}+H^\mathrm{O}(\Omega)}^{\mathrm{r_{lim}}} \rho^a(r)\, r^2\, \mathrm{d}r . \tag{3.6}$$

Substituting Eq. (3.6) to Eq. (3.5), the gravitational potential $V^{ca}(r_t,\Omega)$ of the condensed atmospheric masses takes the following form (Novák, 2000)

$$V^{ca}\left(r_t,\Omega\right)=\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{r_{lim}}}\rho^a\left(r'\right)r'^2\,\mathrm{d}r'\,l^{-1}\left[r_t,\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\mathrm{d}\Omega'\,. \tag{3.7}$$

Dividing the integration domain of the atmospheric surface density $\sigma^a(\Omega)$ into the atmospheric spherical shell between the upper limit of the topography and the upper limit of the atmosphere, and the atmospheric roughness term between the Earth's surface and the upper limit of the topography (Novák, 2000)

$$\sigma^a(\Omega)=\frac{1}{\mathrm{R}^2}\int_{r=\mathrm{R}+H^\mathrm{O}(\Omega)}^{\mathrm{R}+\mathrm{H_{lim}}}\rho^a(r)\,r^2\,\mathrm{d}r+\frac{1}{\mathrm{R}^2}\int_{r=\mathrm{R}+\mathrm{H_{lim}}}^{\mathrm{r_{lim}}}\rho^a(r)\,r^2\,\mathrm{d}r\,, \tag{3.8}$$

the gravitational potential $V^{ca}(\mathrm{R},\Omega)$ of the condensed atmospheric masses in Eq. (5.4) becomes (Novák, 2000)

$$\begin{aligned}V^{ca}\left(r_t,\Omega\right)=\mathrm{G}&\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{R}+\mathrm{H_{lim}}}\rho^a\left(r'\right)r'^2\,\mathrm{d}r'\,l^{-1}\left[r,\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\mathrm{d}\Omega'\\&+\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+\mathrm{H_{lim}}}^{\mathrm{r_{lim}}}\rho^a\left(r'\right)r'^2\,\mathrm{d}r'\,l^{-1}\left[r,\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\mathrm{d}\Omega'\,.\end{aligned} \tag{3.9}$$

Combining Eqs. (3.4) and (3.9) an expression for computing the secondary indirect atmospheric effect, $\dfrac{2}{r_t}\delta V^a\left(r_t,\Omega\right)$, can be derived. However, the magnitude of this effect is negligible in the practical computations.

The gravitational attraction of the atmospheric spherical shell at the inner point $r<\mathrm{R}+\mathrm{H_{lim}}$ is equal to zero (Mac Millan, 1930)

$$\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+\mathrm{H_{lim}}}^{\mathrm{r_{lim}}}\rho^a(r')\left.\frac{\partial\,l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial\,r}\right|_{r<\mathrm{R}+\mathrm{H_{lim}}}r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega'=0\,. \tag{3.10}$$

Therefore, the direct atmospheric effect on the gravitational attraction is given by the radial derivative of the gravitational potential of the atmospheric spherical roughness term (Novák, 2000)

$$\left.\frac{\partial V^a(r,\Omega)}{\partial\,r}\right|_{r=r_t(\Omega)}=\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{R}+\mathrm{H_{lim}}}\rho^a(r')\left.\frac{\partial\,l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial\,r}\right|_{r=r_t(\Omega)}r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega'\,. \tag{3.11}$$

The radial derivative of the inverse spatial distance reads (Martinec, 1998)

$$\left.\frac{\partial\,l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial\,r}\right|_{r=r_t(\Omega)}=-\frac{r_t(\Omega)-r'\cos\psi(\Omega,\Omega')}{l^3\left[r_t(\Omega),\psi(\Omega,\Omega'),r'\right]}\,. \tag{3.12}$$

The gravitational attraction of the condensed atmospheric masses reads

$$-\frac{\partial V^{ca}(r,\Omega)}{\partial r}\bigg|_{r=r_t} = -\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{R}+\mathrm{H}_\mathrm{lim}} \rho^a(r')\,r'^2\,\mathrm{d}r'\,\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]}{\partial r}\bigg|_r\,\mathrm{d}\Omega'$$

$$-\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+\mathrm{H}_\mathrm{lim}}^{\mathrm{r}_\mathrm{lim}} \rho^a(r')\,r'^2\,\mathrm{d}r'\,\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]}{\partial r}\bigg|_r\,\mathrm{d}\Omega'\,. \qquad (3.13)$$

Considering that the gravitational attraction of the spherical condensation layer with the surface atmospheric density $\sigma^a(\Omega)$ at the outer point above the condensation layer (i..e. $r>R$) is equal to a constant (Mac Millan, 1930)

$$\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+\mathrm{H}_\mathrm{lim}}^{\mathrm{r}_\mathrm{lim}} \rho^a(r')\,r'^2\,\mathrm{d}r'\,\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]}{\partial r}\bigg|_r\,\mathrm{d}\Omega' = -4\pi\frac{R^2}{r^2}\int_{r'=\mathrm{R}+\mathrm{H}_\mathrm{lim}}^{\mathrm{r}_\mathrm{lim}} \rho^a(r')\,r'^2\,\mathrm{d}r' \qquad (3.14)$$

The final expression for the direct atmospheric effect becomes thus

$$\frac{\partial\delta V^a(r,\Omega)}{\partial r}\bigg|_{r=r_t(\Omega)} = \mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{R}+\mathrm{H}_\mathrm{lim}} \rho^a(r')\,\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial r}\bigg|_{r=r_t(\Omega)} r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega' +$$

$$+4\pi\frac{R^2}{r^2}\int_{r'=\mathrm{R}+\mathrm{H}_\mathrm{lim}}^{\mathrm{r}_\mathrm{lim}} \rho^a(r')\,r'^2\,\mathrm{d}r' - \mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}+H^\mathrm{O}(\Omega')}^{\mathrm{R}+\mathrm{H}_\mathrm{lim}} \rho^a(r')\,r'^2\,\mathrm{d}r'\,\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]}{\partial r}\bigg|_{rr=r_t(\Omega)}\mathrm{d}\Omega' \qquad (3.15)$$

.
There is an alternative approach to account for the direct atmospheric effect. The International Association of Geodesy (IAG) recommends that the effect of the atmospheric masses may be directly reduced on gravity anomalies at the computation point (Moritz 1992). This method considers the direct gravity effect (maximum at the sea level, +0.87 mGal) only. However, many datasets ignore the IAG recommendation. Therefore, before applying Eq. (3.15) one has to be aware whether thr IAG atmospheric effect is (or is not) applied to the gravity data to be used in geoid computations.

3.2. Treatment of the topographical effects

Note the gravitational potential and attraction of the condensed topographical are evaluated at the earths surface. The topographic potential can be evaluated from the classical Newtonian integration, i.e,

$$V^t(r_t,\Omega) = \mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}}\int_{r'=\mathrm{R}}^{\mathrm{R}+H^\mathrm{O}(\Omega')} \rho(\Omega')\,l^{-1}\left[r_t(\Omega),\psi(\Omega,\Omega'),r'\right]r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega' \qquad (3.16)$$

The radial integral of the inverse special distance $l^{-1}\left[r_t(\Omega),\psi(\Omega,\Omega'),r'\right]$ multiplied by r'^2 can be described by the analytical form (Gradstein and Ryzhik, 1980, see also Martinec 1993)

$$\int_{r'=R}^{R+H^o(\Omega')} l^{-1}\left[r\ (\Omega),\psi\left(\Omega,\Omega'\right),r'\right]r'^2 dr' = \left|\frac{1}{2}\left[r'+3r\cos\psi\left(\Omega,\Omega'\right)\right]l\left[r\ (\Omega),\psi\left(\Omega,\Omega'\right),r'\right]+\right.$$
$$\left.\frac{r^2}{2}\left(3\cos^2\psi\left(\Omega,\Omega'\right)-1\right)\ln\left|r'-r\cos\psi\left(\Omega,\Omega'\right)+l\left[r\ (\Omega),\psi\left(\Omega,\Omega'\right),r'\right]\right|\right|_{r'=R}^{R+H^o(\Omega')} \tag{3.17}$$

Alternatively, in the computations of all topographical effects the earth's topomasses can be split into the Bouguer shell attraction, and the spherical roughness term. This approach has been introduced to eliminate the singularity which may occur in the vicinity of the computation points when integrating with a high-resolution data-set.

The gravitational potential $V^t\left(r_t,\Omega\right)$ of the topographical masses in the secondary indirect topographical effect (the fourth term on the right-hand side of Eq. 2.22) is given by (Martinec and Vaníček, 1994b; Martinec, 1998)

$$\begin{aligned} V^t\left(r_t,\Omega\right) &= 4\pi \mathrm{G}\rho_o \frac{\mathrm{R}^2}{r_t\left(\Omega\right)} H^{\mathrm{O}}\left(\Omega\right)\left[1+\frac{H^{\mathrm{O}}\left(\Omega\right)}{\mathrm{R}}+\frac{1}{3}\left(\frac{H^{\mathrm{O}}\left(\Omega\right)}{\mathrm{R}}\right)^2\right] \\ &+\mathrm{G}\rho_o \iint_{\Omega'\in\Omega_{\mathrm{O}}} \int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),r'\right] r'^2\ \mathrm{d}r'\,\mathrm{d}\Omega' \\ &+\mathrm{G}\iint_{\Omega'\in\Omega_{\mathrm{O}}} \delta\rho\left(\Omega'\right)\int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),r'\right] r'^2\ \mathrm{d}r'\,\mathrm{d}\Omega'\ . \end{aligned} \tag{3.18}$$

The first term on the right-hand side of Eq. (3.18) is the **gravitational potential of the spherical Bouguer shell** (of mean topographical density ρ_o and thickness equal to the orthometric height $H^{\mathrm{O}}(\Omega)$ of the computation point (r,Ω)), see Wichiencharoen (1982). The second term stands for the **gravitational potential of the spherical roughness term**, and the third term represents the **effect of the anomalous topographical density $\delta\rho(\Omega)$ distribution on the gravitational potential**.

The direct topographical effect on the gravitational attraction reads (Martinec and Vaníček, 1994a; Martinec, 1998)

$$\begin{aligned} \left.\frac{\partial V^t\left(r,\Omega\right)}{\partial r}\right|_{r=r_t(\Omega)} &= -4\pi \mathrm{G}\rho_o \frac{\mathrm{R}^2}{r_t^2\left(\Omega\right)} H^{\mathrm{O}}\left(\Omega\right)\left[1+\frac{H^{\mathrm{O}}\left(\Omega\right)}{\mathrm{R}}+\frac{1}{3}\left(\frac{H^{\mathrm{O}}\left(\Omega\right)}{\mathrm{R}}\right)^2\right] \\ &+\mathrm{G}\rho_o \iint_{\Omega'\in\Omega_{\mathrm{O}}} \int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \left.\frac{\partial\, l^{-1}\left[r,\psi\left(\Omega,\Omega'\right),r'\right]}{\partial r}\right|_{r=r_t(\Omega)} r'^2\ \mathrm{d}r'\,\mathrm{d}\Omega' \\ &+\mathrm{G}\iint_{\Omega'\in\Omega_{\mathrm{O}}} \delta\rho\left(\Omega'\right)\int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \left.\frac{\partial\, l^{-1}\left[r,\psi\left(\Omega,\Omega'\right),r'\right]}{\partial r}\right|_{r=r_t(\Omega)} r'^2\ \mathrm{d}r'\,\mathrm{d}\Omega'\ . \end{aligned} \tag{3.19}$$

The first term on the right-hand side of Eq. (3.19) is the negative value of the **gravitational attraction of the spherical Bouguer shell** (Wichiencharoen, 1982). The second term stands for the negative value of the **gravitational attraction of the spherical roughness term**, i.e., the **spherical**

terrain correction, and the third term represents the negative value of the **effect of the anomalous topographical density** $\delta\rho(\Omega)$ **distribution on the gravitational attraction**.

The gravitational potential $V^{ct}(r_t,\Omega)$ of the condensed topographical masses is given by (Martinec, 1998)

$$V^{ct}(r_t,\Omega) = \mathrm{G\,R}^2 \iint_{\Omega'\in\Omega_\mathrm{O}} \sigma(\Omega')\, l^{-1}\left[r_t, \psi(\Omega,\Omega'), \mathrm{R}\right] \mathrm{d}\Omega', \tag{3.20}$$

where $\sigma(\Omega)$ is the **surface density of the condensed topographical masses**

$$\sigma(\Omega) = \frac{\rho(\Omega)}{\mathrm{R}^2} \int_{r=\mathrm{R}}^{\mathrm{R}+H^\mathrm{O}(\Omega)} r^2\,\mathrm{d}r = \rho(\Omega)\frac{r_t^3(\Omega)-\mathrm{R}^3}{3\mathrm{R}^2} =$$
$$= \rho(\Omega)\,H^\mathrm{O}(\Omega)\left[1+\frac{H^\mathrm{O}(\Omega)}{\mathrm{R}}+\frac{1}{3}\left(\frac{H^\mathrm{O}(\Omega)}{\mathrm{R}}\right)^2\right]. \tag{3.21}$$

According to Martinec (1998) the gravitational potential $V^{ct}(\mathrm{R},\Omega)$ of the condensed topographical masses in Eq. (3.20) can be expressed as

$$V^{ct}(r_t,\Omega) = 4\pi\,\mathrm{G}\,\rho_\mathrm{o}\frac{\mathrm{R}^2}{\mathrm{r_t}(\Omega)} + \mathrm{G}\,\rho_\mathrm{o}\iint_{\Omega'\in\Omega_\mathrm{O}} \frac{r^3(\Omega')-r^3(\Omega)}{3}\, l^{-1}\left[r_t,\psi(\Omega,\Omega'),\mathrm{R}\right]\mathrm{d}\Omega'$$
$$+\mathrm{G}\iint_{\Omega'\in\Omega_\mathrm{O}} \delta\rho(\Omega')\,\frac{r^3(\Omega')-\mathrm{R}^3}{3}\, l^{-1}\left[r_t,\psi(\Omega,\Omega'),\mathrm{R}\right]\mathrm{d}\Omega', \tag{3.22}$$

where the first term on the right-hand side is the **gravitational potential of the condensed spherical material single layer**, the second term stands for the **gravitational potential of the spherical roughness term of the condensed topographical masses**, and the third term represents the **effect of the anomalous condensed topographical density distribution on the gravitational potential**.

The gravitational attraction of the condensed topographical masses is defined as the negative radial derivative of the gravitational potential $V^{ct}(r,\Omega)$. The radial derivative of the gravitational potential $V^{ct}(r,\Omega)$ of the condensed topographical masses reads

$$\left.\frac{\partial V^{ct}(r,\Omega)}{\partial r}\right|_{r=r_t} = \mathrm{G\,R}^2 \iint_{\Omega'\in\Omega_\mathrm{O}} \sigma(\Omega') \left.\frac{\partial\, l^{-1}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]}{\partial r}\right|_{r=r_t} \mathrm{d}\Omega'. \tag{3.23}$$

The gravitational attraction of the condensed topographical masses can further be expressed as (cf. Eq. (3.22))

$$\left.\frac{\partial V^{ct}(r,\Omega)}{\partial r}\right|_{r=r_t} = -4\pi G\sigma_o \frac{R^2}{r_t^2(\Omega)} H^O(\Omega) + G\rho_o \iint_{\Omega'\in\Omega_O} \frac{r^3(\Omega')-r^3(\Omega)}{3} \left.\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),R\right]}{\partial r+}\right|_{r=r_t} d\Omega' +$$

$$+G\iint_{\Omega'\in\Omega_O} \delta\rho(\Omega') \frac{r^3(\Omega')-R^3}{3} \left.\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),R\right]}{\partial r}\right|_{r=r_t} d\Omega' . \qquad (3.24)$$

where the first term on the right-hand side is the **gravitational attraction of the condensed spherical Bouguer shell**, the second term stands for the **gravitational attraction of the spherical roughness term of the condensed topographical masses**, and the third term represents the **effect of the anomalous condensed topographical density distribution on the gravitational attraction**.

Considering the gravitational potential $V^t(r_t,\Omega)$ of topographical masses (in Eq (3.17)) and the gravitational potential $V^{ct}(r_t,\Omega)$ of condensed topographical masses (in Eq. 20)**, the secondary indirect topographic effect** $\frac{2}{r_t}\delta V^t(r_t,\Omega)$ becomes

$$\frac{2}{r_t}\delta V^t(r_t,\Omega) = \frac{2}{r_t}\left[V^t(r_t,\Omega) - V^{ct}(r_t,\Omega)\right] = \frac{2}{r_t} G \iint_{\Omega'\in\Omega_O} \int_{r'=R}^{R+H^O(\Omega')} \rho(\Omega') l^{-1}\left[r_t(\Omega),\psi(\Omega,\Omega'),r'\right] r'^2 \, dr' \, d\Omega' -$$

$$-\frac{2}{r_t} G \iint_{\Omega'\in\Omega_O} \rho(\Omega') \frac{r_t^3(\Omega')-R^3}{3} l^{-1}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right] d\Omega'$$

(3.25)

In order to obtain the residual direct topographical effect in the Helmert space the attraction of condensation layer (at the surface!) is subtracted from attraction of topographical masses.
Note that the gravitational attraction of the spherical Bouguer shell (referred to the Earth surface, see the first term of Eq. (3.19)) and the attraction of the spherical condensation layer (that is given by the first term on th eright hand side of Eq. (3.24)) are equal. Thus they cancel each other efficiently out when computing the direct topographic effect, i.e. (cf. Martinec and Vaníček, 1994)

$$\delta A^t(r_t,\Omega) = \frac{\partial\left\{V^t(r_t,\Omega) - V^{ct}(r_t,\Omega)\right\}}{\partial r} =$$

$$= G\rho_o \iint_{\Omega'\in\Omega_O} \frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial r} r'^2 \, dr' \Bigg|_{R+H(\Omega)}^{R+H(\Omega')} d\Omega' - G\rho_o \iint_{\Omega'\in\Omega_O} \frac{r_t^3(\Omega)-r^3(\Omega')}{3} \left.\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),R\right]}{\partial r}\right|_{R+H(\Omega)} d\Omega' +$$

$$+G\iint_{\Omega'\in\Omega_O} \delta\rho(\Omega') \frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial r} r'^2 \, dr' \Bigg|_{R+H(\Omega)}^{R+H(\Omega')} d\Omega' - G\iint_{\Omega'\in\Omega_O} \delta\rho(\Omega') \frac{r_t^3(\Omega)-R^3}{3} \left.\frac{\partial l^{-1}\left[r,\psi(\Omega,\Omega'),R\right]}{\partial r}\right|_{R+H(\Omega)} d\Omega'$$

(3.26)

The first term in the right hand side of Eq. (3.26) is the so called " spherical terrain correction", and th second term stands for the "spherical condensed terrain correction" (Martinec and Vanicek

1994). The third and fourth terms represent together the contribution of the laterally varying topographical density to the direct topographic effect.

4. Dirichlet's boundary value problem and downward continuation

Once the Helmert gravity anomaly field is evaluated on the earth surface (obtained by Eq. (2.22) or Eq. (2.26), minus the ellipsoidal corrections, which will be accounted for at the geoid level) it has to be continued to the boundary, in our case geoid. This process is called downward continuation of the Helmert gravity anomaly.

The downward continuation is evaluated by the inverse operation to the Poisson integral equation. The Poisson integral is given by the following formula (Kellogg, 1929)

$$\Delta g^{\mathrm{h}}\left(r_t,\Omega\right)=\frac{\mathrm{R}}{4\pi\, r_t\left(\Omega\right)}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\mathrm{K}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\Delta g^{\mathrm{h}}\left(\mathrm{R},\Omega'\right)\mathrm{d}\Omega', \tag{4.1}$$

where $\mathrm{K}[r_t(\Omega),\psi(\Omega,\Omega'),\mathrm{R}]$ is the spherical Poisson integral kernel (e.g., Sun and Vaníček, 1998)

$$\mathrm{K}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]=\sum_{n=0}^{\infty}\left(2n+1\right)\left[\frac{\mathrm{R}}{r_t\left(\Omega\right)}\right]^{n+1}\mathrm{P_n}\left(\cos\psi\left(\Omega,\Omega'\right)\right)=\mathrm{R}\,\frac{r_t^2\left(\Omega\right)-\mathrm{R}^2}{l^3\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]} \tag{4.2}$$

and $\mathrm{P_n}(\cos\psi(\Omega,\Omega'))$ are the Legendre polynomials (Hobson, 1931).

The discrete form of the Poisson integral equation, of which the generic form is the Fredholm integral equation of the first kind, can be expressed as (Martinec, 1996; Huang, 2002)

$$\mathbf{\Delta g}^h\left(r_t\left(\Omega\right)\right)=\mathbf{K}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\ \mathbf{\Delta g}^h\left(\mathrm{R},\Omega'\right), \tag{4.3}$$

where $\mathbf{\Delta g}^h\left(r_t,\Omega\right)$ is the vector of the gravity anomalies referred to the Earth's surface, $\mathbf{\Delta g}^h\left(\mathrm{R},\Omega'\right)$ is the vector of the gravity anomalies referred to the geoid surface (approximated again by the reference sphere of radius R), and $\mathbf{K}[r_t(\Omega),\psi(\Omega,\Omega'),\mathrm{R}]$ is the matrix of the values of the Poisson integral kernel multiplied by the factor $\mathrm{R}/r_t(\Omega)$, by constant $1/4\pi$, and also by the size of the discretized surface element $\Delta\Omega=\Delta\lambda\,\Delta\phi\cos\phi$.

According to Jacobi's iteration approach (Ralston, 1965) for a solution of the system of linear algebraic equations, the matrix $\mathbf{K}[r_t(\Omega),\psi(\Omega,\Omega'),\mathrm{R}]$ is expressed in the form

$$\mathbf{K}[r_t(\Omega),\psi(\Omega,\Omega'),\mathrm{R}]=\mathbf{E}-\mathbf{B}[r_t(\Omega),\psi(\Omega,\Omega'),\mathrm{R}], \tag{4.4}$$

where $\mathbf{E}$ is the unit matrix.

Substituting Eq. (4.4) into Eq. (4.3), the following system of the linear algebraic equations is subsequently obtained (Martinec, 1996)

$$\mathbf{\Delta g}^h\left(\mathrm{R},\Omega'\right)=\mathbf{\Delta g}^h\left(r_t\left(\Omega\right)\right)+\mathbf{B}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\ \mathbf{\Delta g}^h\left(\mathrm{R},\Omega'\right). \tag{4.5}$$

The system of the linear equations in Eq. (4.5) can be solved iteratively starting with the vector $\mathbf{\Delta g}^{FA}\left(r_t\left(\Omega\right)\right)$ of the geoid-generated gravity anomalies referred to the Earth's surface

$$\mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right)\Big|_{0} = \mathbf{\Delta g}^{h}\left(r_t\left(\Omega\right)\right). \tag{4.6}$$

The k-th stage of iteration ($k>0$) $\mathbf{\Delta g}^{\mathbf{NT}}\left(\mathrm{R},\Omega'\right)\Big|_{k}$ is carried out according to the equation (Martinec, 1996)

$$\mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right)\Big|_{k} = \mathbf{B}\left[r_t\left(\Omega\right),\psi\left(\Omega,\Omega'\right),\mathrm{R}\right]\ \mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right)\Big|_{k-1} \tag{4.7}$$

When the difference of results from two successive steps $\left|\mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right)\Big|_{k} - \mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right)\Big|_{k-1}\right|$ is smaller than some tolerance ε , the iterative process stops. The result of this operation yields the solution of Eq. (4.3), see Martinec (1996),

$$\mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right) = \mathbf{\Delta g}^{h}\left(r_t\left(\Omega\right)\right) + \sum_{k=1}^{\bar{k}} \mathbf{\Delta g}^{h}\left(\mathrm{R},\Omega'\right)\Big|_{k}\ , \tag{4.8}$$

where $\bar{k}$ is the final number of iteration steps.

The Poisson downward continuation is known to be a unstable problem. Due to the instability, existing errors in $\mathbf{\Delta g}^{h}\left(r_t\left(\Omega\right)\right)$ may appear magnified in the solution. However, when mean values are used instead of point values, this problem is somewhat alleviated, as the mean values do not exhibit the highest frequencies.

5. Ellipsoidal corrections

As already mentioned, that according to Wong's (2001) investigations one may add the ellipsoidal correction at the geoid level.
In order to obtain ellipsoidal anomalies from spherical type of anomalies (both referred to the geoid level) the following expression can be used

$$\begin{aligned} &-\frac{\partial T^{h}(R,\Omega)}{\partial r} - \frac{2}{R}T^{h}(R,\Omega) = \Delta g^{h}(R,\Omega) + \varepsilon_{n}(R,\Omega) - \varepsilon_{\delta g}(R,\Omega) = \\ &= \Delta g^{h}(R,\Omega) - \frac{e^{2}}{R}(3\cos^{2}\theta - 2)T^{h}(R,\Omega) + \frac{e^{2}}{R}\cos\theta\sin\theta\frac{\partial T^{h}(R,\Omega)}{\partial\theta} \end{aligned} \tag{5.1}$$

where θ is geocentric co-latitude and e^{2} is the eccentricity of the reference ellipsoid. The second term in the left-hand side is the ellipsoidal correction to the gravity disturbance, and the last term is the ellipsoidal correction for the spherical approximation.

Conveniently, the disturbing potential $T\left[r_t(\Omega)\right]$ (outside of topographic masses) can be estimated from spherical – harmonic models of geopotential:

$$T^{H}\left[r_t(\Omega)\right] = \frac{\mathrm{GM}}{r_t(\Omega)} \sum_{n=2}^{\infty} \left[\frac{\mathrm{a}}{r_t(\Omega)}\right]^{n} \sum_{m=0}^{n} \left[\mathrm{C}^{\mathrm{T}}_{\mathrm{n,m}} \cos m\lambda + \mathrm{S}^{\mathrm{T}}_{\mathrm{n,m}} \sin m\lambda\right] \mathrm{P}_{\mathrm{n,m}}\left(\sin\phi\right) \tag{5.2}$$

Analogeously, the first derivative of the disturbing potential can be estimated via first derivative of the Legendre associated functions, as

$$\frac{\partial T^{H}\left[R,(\Omega)\right]}{\partial\phi} = \frac{\mathrm{GM}}{R(\Omega)} \sum_{n=2}^{\infty} \left[\frac{\mathrm{a}}{R(\Omega)}\right]^{n} \sum_{m=0}^{n} \left[\mathrm{C}^{\mathrm{T}}_{\mathrm{n,m}} \cos m\lambda + \mathrm{S}^{\mathrm{T}}_{\mathrm{n,m}} \sin m\lambda\right] \frac{\partial\, \mathrm{P}_{\mathrm{n,m}}\left(\sin\phi\right)}{\partial\phi} \tag{5.3}$$

Note that T^H in Eq. (5.2) denotes "Helmertized" disturbing potential (i.e., since T^H is referred to the geoid level then the contribution of topographic masses and DWC need be accounted for). For the "Helmertization" the spherical-harmonic coefficients of topoheights can be used (for more details see Section 6).

6. Reference field and spheroid in Helmert's gravity space

To solve the Stokes boundary value problem, the gravity anomalies over the entire Earth are required. To reduce the truncation errors, i.e., the far-zone contribution in the Stokes integration, the low and high-frequency parts of Helmert's gravity field are defined (Vaníček and Sjöberg, 1991).

The reference gravity field of degree $\bar{n}$ is described by the **reference gravity potential** $W_{\mathrm{ref}}(r,\Omega)$ as follows (Vaníček et al., 1995)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\geq r_t(\Omega): \qquad W_{\mathrm{ref}}(r,\Omega) = \frac{\mathrm{GM}}{r} - \sum_{n=2}^{\bar{n}} \left(\frac{\mathrm{a}_{\mathrm{o}}}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{W}_{\mathrm{n,m}}(\Omega)\, \mathrm{Y}_{\mathrm{n,m}}(\Omega), \tag{6.1}$$

where $\mathrm{W}_{\mathrm{n,m}}(\Omega)$ denote the geopotential coefficients of the harmonic expansion of the Earth's gravity field, $\mathrm{Y}_{\mathrm{n,m}}(\Omega)$ are the normalized spherical functions of degree n and order m; a_{o} is an arbitrary parameter of length (usually the major semi-axis of the geocentric reference ellipsoid), and $\bar{n}$ stands for the maximum degree of retained harmonics.

In the Helmert gravity space the reference gravity potential $W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)$ reads

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^{+}: \qquad W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega) = W_{\mathrm{ref}}(r,\Omega) - \delta V^{t}_{\mathrm{ref}}(r,\Omega) - \delta V^{a}_{\mathrm{ref}}(r,\Omega)\,, \tag{6.2}$$

where $\delta V^{t}_{\mathrm{ref}}(r,\Omega)$ is the **reference residual gravitational potential of the topographical masses**, and $\delta V^{a}_{\mathrm{ref}}(r,\Omega)$ is the **reference residual gravitational potential of the atmospheric masses**.

6.1 Reference residual gravitational potential of topographical masses

The reference residual gravitational potential of the topographical masses $\delta V^t_{\mathrm{ref}}(r,\Omega)$ is defined as the difference of the **reference gravitational potential of the topographical masses** $V^t_{\mathrm{ref}}(r,\Omega)$ (Vaníček et al., 1995)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>\mathrm{R}+\mathrm{H}_{\mathrm{lim}}: \quad V^t_{\mathrm{ref}}(r,\Omega)\approx \mathrm{G}\,\rho_{\mathrm{o}}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\int\limits_{H'=0}^{H^{\mathrm{O}}(\Omega')}\frac{1}{r}\sum_{n=0}^{\bar{n}}\left(\frac{\mathrm{R}+H'}{r}\right)^{n}\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)(\mathrm{R}+H')^2\,\mathrm{d}H'\,\mathrm{d}\Omega', \tag{6.3}$$

and the **reference gravitational potential of the condensed topographical masses** $V^{ct}_{\mathrm{ref}}(r,\Omega)$ (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}, r>\mathrm{R}: \qquad V^{ct}_{\mathrm{ref}}(r,\Omega)\approx \mathrm{G\,R}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\sigma(\Omega')\sum_{n=0}^{\bar{n}}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'. \tag{6.4}$$

For the space $\forall\,\Omega\in\Omega_{\mathrm{O}}: r>\mathrm{R}+\mathrm{H}_{\mathrm{lim}}$ outside the Brillouin sphere (minimal geocentric sphere containing all the Earth's mass) the reference gravitational potential $V^t_{\mathrm{ref}}(r,\Omega)$ of the topographical masses in Eq. (6.3) takes the following form (Vaníček et al., 1995)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>\mathrm{R}+\mathrm{H}_{\mathrm{lim}}:$$
$$V^t_{\mathrm{ref}}(r,\Omega)=\mathrm{G}\,\rho_{\mathrm{o}}\,\mathrm{R}^2\sum_{n=0}^{\bar{n}}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\frac{1}{n+3}\sum_{k=1}^{n+3}\binom{n+3}{k}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\left[\frac{H^{\mathrm{O}}(\Omega')}{\mathrm{R}}\right]^k\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'. \tag{6.5}$$

Differencing the reference gravitational potential $V^t_{\mathrm{ref}}(r,\Omega)$ of the topographical masses, see Eq. (6.5), and the reference gravitational potential $V^{ct}_{\mathrm{ref}}(r,\Omega)$ of the condensed topographical masses, see Eq. (6.4), the reference residual gravitational potential $\delta V^t_{\mathrm{ref}}(r,\Omega)$ of the topographical masses becomes (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>\mathrm{R}+\mathrm{H}_{\mathrm{lim}}:$$
$$\begin{aligned}\delta V^t_{\mathrm{ref}}(r,\Omega)\cong \mathrm{G}\,\rho_{\mathrm{o}}\mathrm{R}^2\sum_{n=0}^{\bar{n}}\left(\frac{\mathrm{R}}{r}\right)^{n+1}&\left\{\frac{1}{n+3}\sum_{k=1}^{n+3}\binom{n+3}{k}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\left[\frac{H^{\mathrm{O}}(\Omega')}{\mathrm{R}}\right]^k\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'\right.\\ &\left.-\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\frac{H^{\mathrm{O}}(\Omega')}{\mathrm{R}}\left[1+\frac{H^{\mathrm{O}}(\Omega')}{\mathrm{R}}+\frac{1}{3}\left(\frac{H^{\mathrm{O}}(\Omega')}{\mathrm{R}}\right)^2\right]\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'\right\}.\end{aligned} \tag{6.6}$$

Since for $H^{\mathrm{O}}(\Omega')<<\mathrm{R}$ the summation over k converges very quickly (Vaníček et al., 1995), Eq. (6.6) can be rewritten into the following form (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>\mathrm{R}+\mathrm{H}_{\mathrm{lim}}: \quad \delta V^t_{\mathrm{ref}}(r,\Omega)\cong \mathrm{G}\,\rho_{\mathrm{o}}\,\mathrm{R}^2\sum_{n=1}^{\bar{n}}\frac{n}{2}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\left\{\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\frac{\left[H^{\mathrm{O}}(\Omega')\right]^2}{\mathrm{R}^2}\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'\right.$$

$$+\frac{n+3}{3}\iint_{\Omega'\in\Omega_O}\frac{\left[H^{O}(\Omega')\right]^3}{R^3}\,P_n\left(\cos\psi(\Omega,\Omega')\right)d\Omega'\Bigg\}. \tag{6.7}$$

Expressing the surface harmonics of the orthometric height as

$$\sum_{n=0}^{\infty}\iint_{\Omega'\in\Omega_O}H^{O}(\Omega')\,P_n\left(\cos\psi(\Omega,\Omega')\right)d\Omega' = \sum_{n=0}^{\infty}\frac{4\pi}{2n+1}\sum_{m=-n}^{n}H_{n,m}(\Omega)\,Y_{n,m}(\Omega), \tag{6.8}$$

the reference residual gravitational potential $\delta V^{t}_{\text{ref}}(r,\Omega)$ of the topographical masses finally becomes (Novák, 2000)

$$\forall\,\Omega\in\Omega_O,\, r>R+H_{\lim}: \qquad \delta V^{t}_{\text{ref}}(\Omega)\cong 2\pi G\rho_o\sum_{n=1}^{\bar{n}}\frac{n}{2n+1}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n}H^{2}_{n,m}(\Omega)\,Y_{n,m}(\Omega)$$
$$+\frac{2\pi}{3R}G\,\rho_o\sum_{n=1}^{\bar{n}}\frac{n(n+3)}{2n+1}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n}H^{3}_{n,m}(\Omega)\,Y_{n,m}(\Omega). \tag{6.9}$$

6.2 Reference residual gravitational potential of atmospheric masses

The **reference gravitational potential of the atmospheric masses** $V^{a}_{\text{ref}}(r,\Omega)$ is expressed in the following form (Novák, 2000)

$$\forall\,\Omega\in\Omega_O,\, r>r_{\lim}: \qquad V^{a}_{\text{ref}}(r,\Omega)=G\sum_{n=0}^{\bar{n}}\frac{1}{r^{n+1}}\iint_{\Omega'\in\Omega_O}P_n\left(\cos\psi(\Omega,\Omega')\right)\int_{r'=R+H^{O}(\Omega')}^{r_{\lim}}\rho(r')\,r'^{\,n+2}dr'\,d\Omega'. \tag{6.10}$$

The atmospheric density $\rho^{a}(r)$ can be expressed by (Sjöberg, 1998; Novák, 2000)

$$\forall\,\Omega\in\Omega_O, r\in\left\langle R+H^{O}(\Omega),\, r_{\lim}\right\rangle, \nu>2\wedge\nu\in\Im^{+}: \qquad \rho^{a}(r)=\rho^{a}_{o}\left[\frac{R}{R+H^{O}(\Omega)}\right]^{\nu}, \tag{6.11}$$

where ρ^{a}_{o} is the atmospheric density at the sea level, and the positive integer constant $\nu\in\Im^{+}$ ($\Im^{+}=1,2,\ldots$) determines the radial atmospheric density distribution model.
If the integration over the geocentric radius r from the Earth's surface to the upper limit of the atmosphere is evaluated using the atmospheric model density from Eq. (6.11)

$$\forall\,\Omega\in\Omega_O, \nu>2\wedge\nu\in\Im^{+}, n=0,2,\ldots,\bar{n}: \qquad \int_{r=R+H^{O}(\Omega)}^{r_{\lim}}\rho^{a}(r)\,r^{n+2}\,dr=\rho^{a}_{o}\int_{r=R+H^{O}(\Omega)}^{r_{\lim}}\left(\frac{R}{r}\right)^{\nu}r^{n+2}\,dr, \tag{6.12}$$

the reference gravitational potential $V^{a}_{\text{ref}}(r,\Omega)$ of the atmospheric masses is rewritten as (Novák, 2000)

$$\forall\,\Omega\in\Omega_O, r>r_{\lim}, \nu>2\wedge\nu\in\Im^{+}:$$

$$V_{\mathrm{ref}}^{a}(r,\Omega) \cong \mathrm{G}\,\mathrm{R}^{\nu}\,\rho_{\mathrm{o}}^{\mathrm{a}} \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \left| \frac{r'^{\,n-\nu+3}}{n-\nu+3} \right|_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega')}^{\mathrm{r_{lim}}} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega'$$

$$\cong \mathrm{G}\,\mathrm{R}^{\nu}\,\rho_{\mathrm{o}}^{\mathrm{a}} \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{\mathrm{R}^{n-\nu+3}}{n-\nu+3} \sum_{k=1}^{n-\nu+3} \binom{n-\nu+3}{k} \frac{(r_{\mathrm{lim}}-\mathrm{R})^k - \left[H^{\mathrm{O}}(\Omega')\right]^k}{\mathrm{R}^k} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \,. \quad (6.13)$$

Applying the binomial theorem to the evaluation of the surface atmospheric density $\sigma^{a}(\Omega)$, see Novák (2000),

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, \nu > 2 \wedge \nu \in \Im^{+}: \qquad \sigma^{a}(\Omega) = \frac{\rho_{\mathrm{o}}^{\mathrm{a}}}{\mathrm{R}^2} \int_{r=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{r_{lim}}} \left(\frac{\mathrm{R}}{r}\right)^{\nu} r^2 \,\mathrm{d}r$$

$$= \rho_{\mathrm{o}}^{\mathrm{a}} \frac{\mathrm{R}^{3-\nu}}{3-\nu} \sum_{k=1}^{3-\nu} \binom{3-\nu}{k} \frac{(r_{\mathrm{lim}}-\mathrm{R})^k - \left[H^{\mathrm{O}}(\Omega)\right]^k}{\mathrm{R}^k}\,, \qquad (6.14)$$

the **reference gravitational potential of the condensed atmospheric masses** $V_{\mathrm{ref}}^{ca}(r,\Omega)$ takes the following form (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > \mathrm{r_{lim}},\, \nu > 2 \wedge \nu \in \Im^{+}:$$

$$V_{\mathrm{ref}}^{ca}(r,\Omega) \cong \frac{\mathrm{G}\,\rho_{\mathrm{o}}^{\mathrm{a}}}{\mathrm{R}^{1-\nu}} \sum_{n=0}^{\bar{n}} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \left| \frac{r'^{\,3-\nu}}{3-\nu} \right|_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega')}^{\mathrm{r_{lim}}} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega'$$

$$= \frac{\mathrm{G}\,\rho_{\mathrm{o}}^{\mathrm{a}}}{\mathrm{R}^{1-\nu}} \sum_{n=0}^{\bar{n}} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{\mathrm{R}^{3-\nu}}{3-\nu} \sum_{k=1}^{3-\nu} \binom{3-\nu}{k} \frac{(\mathrm{r_{lim}}-\mathrm{R})^k - \left[H^{\mathrm{O}}(\Omega')\right]^k}{\mathrm{R}^k} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \,. \qquad (6.15)$$

The reference residual gravitational potential of the atmospheric masses $\delta V_{\mathrm{ref}}^{a}(r,\Omega)$ is then obtained as the difference of the reference gravitational potential $V_{\mathrm{ref}}^{a}(r,\Omega)$ of the atmospheric masses from Eq. (6.13), and the reference gravitational potential $V_{\mathrm{ref}}^{ca}(r,\Omega)$ of the condensed atmospheric masses given by Eq. (6.15), so that (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > \mathrm{r_{lim}},\, \nu > 2 \wedge \nu \in \Im^{+}:$$

$$\delta V_{\mathrm{ref}}^{a}(r,\Omega) \cong -2\pi\mathrm{G}\,\rho_{\mathrm{o}}^{\mathrm{a}} \sum_{n=1}^{\bar{n}} \frac{n}{2n+1} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H}_{\mathrm{n,m}}^{2}(\Omega)\, \mathrm{Y_{n,m}}(\Omega)$$

$$-2\pi\mathrm{G} \frac{\rho_{\mathrm{o}}^{\mathrm{a}}}{3\mathrm{R}} \sum_{n=1}^{\bar{n}} \frac{n(n-2\nu+3)}{2n+1} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H}_{\mathrm{n,m}}^{3}(\Omega)\, \mathrm{Y_{n,m}}(\Omega) \qquad (6.16)$$

6.3 Reference gravity potential in Helmert's gravity space

The reference gravity potential $W_{\mathrm{ref}}^{\mathrm{H}}(r,\Omega)$ in the Helmert gravity space given by Eq. (6.2) can be expressed by the following formula (Vaníček et al., 1995)

$$\forall\,\Omega \in \Omega_{\mathrm{O}},\, r > \mathrm{R}: \qquad W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega) = \frac{\mathrm{GM}}{r} - \sum_{n=2}^{\bar{n}} \left(\frac{\mathrm{a_o}}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{W}^{\mathrm{H}}_{\mathrm{n,m}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{6.17}$$

Since the summation in the expansion of Helmert's reference gravity potential $W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)$ is finite, i.e., the validity of this expression is not limited to the outside of the Brillouin sphere (in the case of the topographical effect) and to the upper limit of the atmosphere (in the case of the atmospheric effect), the series in Eq. (6.17) can be used at the geoid surface to evaluate the reference gravity field in the Helmert space (Vaníček et al., 1995). If this surface is unknown, the appropriate approximation of the geoid surface by the surface of the geocentric reference ellipsoid ($\forall\,\Omega \in \Omega_{\mathrm{O}}$: $r_g(\Omega) \approx r_o(\phi)$) can be applied (Vaníček et al., 1995)

$$\forall\,\Omega \in \Omega_{\mathrm{O}}: \qquad r_g(\Omega) \approx r_o(\phi) \cong \mathrm{a}\left(1 - \mathrm{f}\sin^2\varphi\right). \tag{6.18}$$

Substituting (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\, n = 1, 2, \dots, \bar{n}: \qquad \left[\frac{\mathrm{a_o}}{r_g(\Omega)}\right]^{n+1} = 1 + (n+1)\mathrm{f}\sin^2\varphi - \dots, \tag{6.19}$$

into Eq. (6.17), Helmert's reference gravity potential in the ellipsoidal approximation takes the following form (Vaníček et al., 1995)

$$\forall\,\Omega \in \Omega_{\mathrm{O}},\, r > \mathrm{R}: \qquad W^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) \approx \frac{\mathrm{GM}}{r_o(\Omega)} - \sum_{n=2}^{\bar{n}} \left[1 + (n+1)\mathrm{f}\sin^2\varphi\right] \sum_{m=-n}^{n} \mathrm{W}^{\mathrm{H}}_{\mathrm{n,m}}(\Omega)\,\mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{6.20}$$

6.4 Accounting for the differences of the GGM and GRS constants

According to Vanicek and Kleusberg (1987) also the differences between the constants of the used GGM and adopted geodetic reference ellipsoid need be considered.
Below, the subscript "*G*" at symbols is related to the EGM values, subscript "*E*" at some quantities will be related to the parameters of the geodetic reference ellipsoid of interest.
The gravity potential W_G can be computed as

$$W_G(r,\theta,\lambda) = \frac{GM_G}{r}\left(1 + \sum_{n=2}^{n_{\max}} \sum_{m=0}^{n} \left(\frac{a_G}{r}\right)^n \{\bar{C}^G_{nm}\cos m\lambda + \bar{S}^G_{nm}\sin m\lambda\}\bar{P}_{nm}(\cos\theta)\right) \tag{6.21}$$

Similarly, the gravitational potential of the normal ellipsoid, i.e. normal potential U_E can be expressed as a harmonic series

$$U_E = \frac{GM_E}{r}\left(1 + \sum_{n=2}^{10} \left(\frac{a_E}{r}\right)^n \bar{C}^E_{n0}\bar{P}_{n0}(\cos\theta)\right), \tag{6.22}$$

The difference between the gravitational potentials of the GGM from Eq. (6.21) and the normal potential Eq. (6.22) at the same point on the geoid defines the disturbing potential T_G as

$$T_G = \frac{GM_G - GM_E}{r} + \frac{GM_G}{r} \sum_{n=2}^{n_{\max}} \sum_{m=0}^{n} \left(\frac{a_G}{r} \right)^n \{ \Delta \bar{C}_{nm} \cos m\lambda + \bar{S}_{nm}^G \sin m\lambda \} \bar{P}_{nm}(\cos\theta) \tag{6.23}$$

where

$$\Delta \bar{C}_{n0} = \bar{C}_{n0}^G - \frac{GM_E}{GM_G} \left(\frac{a_E}{a_G} \right)^n \bar{C}_{n0}^E \tag{6.24}$$

The zero degree geoid undulation term is thus

$$N_0 = \frac{GM_G - GM_E}{r\gamma} - \frac{W_0 - U}{\gamma} = \delta(GM) - \delta W \tag{6.25}$$

At the compilation of the AUS-SEGM (Baran et al, submitted) the NASA GSFC and NIMA Joint Geopotential Model EGM96, complete to degree and order 360 (*Lemoine et al. 1998*) was used. The parameters of the Geodetic Reference System GRS-80 (Moritz 1992) are conventionally used in the geoid computations nowadays, the main characteristics of both models are compared in Table 1.

Table 1 The parameters of the EGM96 and GRS-80

Parameter	**EGM96**	**GRS-80**
Equatorial radius	*6378136.3 m*	6378137 m
Gravity Mass constant	*398600.4415* km^3/s^2	*398600.5* km^3/s^2

From the presented values and taking into account the following values R=6371 *km* and γ = 981 *Gal*, then the zero degree term becomes

$$N_0 = \frac{GM_G - GM_E}{r\gamma} \cong -0.936\ m \tag{6.26}$$

6.5 Reference gravity anomaly and reference spheroid in Helmert's gravity space

According to the boundary condition (Heiskanen and Moritz, 1967), Helmert's reference gravity anomaly $\Delta g_{\mathrm{ref}}^{\mathrm{H}}(r_g(\Omega))$ can be expressed as follows

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \qquad \Delta g^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) \approx -\left.\frac{\partial T^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)}{\partial\, r}\right|_{r=r_g(\Omega)} + \frac{2}{\mathrm{R}}\, T^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right), \tag{6.27}$$

where $T^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) = W^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) - U\left(r_g(\Omega)\right)$ is Helmert's reference disturbing gravity potential.

The reference spheroid is given by the reference co-geoidal heights $N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$. Applying the Bruns formula (Bruns, 1878), the reference co-geoidal height $N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$ reads

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{\mathrm{ref}}(\Omega) = \frac{T^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right)}{\gamma_o(\phi)}\,. \tag{6.28}$$

7. Stokes' boundary value problem in Helmert's gravity space

The equipotential boundary surface in the Helmert gravity space, which is given by the co-geoidal heights $N^{\mathrm{H}}(\Omega)$, can be evaluated from Helmert's gravity anomalies $\Delta g^{\mathrm{H}}(\mathrm{R},\Omega)$ referred to the reference sphere of radius R by applying the Stokes integral formula (Stokes, 1849) and the Bruns formula (Bruns, 1878) in the following equation (Heiskanen and Moritz, 1967)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}(\Omega) = \frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(\mathrm{R},\Omega')\, \mathrm{S}\left(\psi(\Omega,\Omega')\right) \mathrm{d}\Omega'\,. \tag{7.1}$$

The homogenous spherical Stokes function $\mathrm{S}(\psi(\Omega,\Omega'))$, see Stokes (1849), is given by (e.g., Heiskanen and Moritz, 1967)

$$\begin{aligned}\forall\, \Omega,\Omega' \in \Omega_{\mathrm{O}}: \quad \mathrm{S}\left(\psi(\Omega,\Omega')\right) &= \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\, \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) = 1 + \operatorname{cosec}\frac{\psi(\Omega,\Omega')}{2} - 6\sin\frac{\psi(\Omega,\Omega')}{2} \\ &\quad - 5\cos\psi(\Omega,\Omega') - 3\cos\psi(\Omega,\Omega')\ln\left(\sin\frac{\psi(\Omega,\Omega')}{2} + \sin^2\frac{\psi(\Omega,\Omega')}{2}\right).\end{aligned} \tag{7.2}$$

To evaluate the co-geoidal height $N^{\mathrm{H}}(\Omega)$ by a surface integration according to the Stokes integral in Eq. (7.1), the gravity anomalies $\Delta g^{\mathrm{H}}(\mathrm{R},\Omega)$ have to be known over the entire Earth.

7.1 Spheroidal Stokes' function

In practice, the gravity anomalies over the entire Earth are not available. For this reason Vaníček and Kleusberg (1987) introduced the idea to separate the summation over n in the Stokes function given by Eq. (7.2) into the low-degree part and the high-degree part:

$$\forall\, \Omega,\Omega' \in \Omega_{\mathrm{O}}: \qquad \mathrm{S}\left(\psi(\Omega,\Omega')\right) = \sum_{n=2}^{\bar{n}} \frac{2n+1}{n-1}\, \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) + \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1}\, \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right). \tag{7.3}$$

The second term on the right-hand side of Eq. (7.3) represents the **spheroidal Stokes function** $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))$, see Vaníček and Kleusberg (1987); Vaníček and Featherstone (1998),

$$\forall\, \Omega,\Omega' \in \Omega_{\mathrm{O}}: \qquad \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega')) = \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega')) \,. \tag{7.4}$$

Considering Eq. (7.3), Eq. (7.1) becomes (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad N^{\mathrm{H}}(\Omega) = N^{\mathrm{H}}_{\mathrm{ref}}(\Omega) + N^{\mathrm{H}}_{n>\bar{n}}(\Omega) = \frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(\mathrm{R},\Omega') \sum_{n=2}^{\bar{n}} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega'))\, \mathrm{d}\Omega'$$
$$+ \frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(\mathrm{R},\Omega') \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' \,. \tag{7.5}$$

The **reference co-geoid** (spheroid) of degree $\bar{n}$ is given by the reference co-geoidal heights $N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$, and $N^{\mathrm{H}}_{n>\bar{n}}(\Omega)$ represents the high-frequency (**residual**) **co-geoid** (Novák et al., 2001). According to this approach the reference spheroid determined from the satellite data is assumed (Vaníček and Kleusberg, 1987). The surface integration by the Stokes integral formula is employed to compute the high-frequency part of the co-geoid only from the terrestrial gravity data.

7.2 Modified spheroidal Stokes' function

The integration domain Ω_{O} of the Stokes integral formula can be divided into the near-zone integration sub-domain $\Omega_{\psi_{\mathrm{o}}}$ (defined on the interval $\psi \in \langle 0,\psi_o \rangle$) and the far-zone integration sub-domain $\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}$ (on the interval $\psi \in \langle \psi_o, \pi \rangle$), see Vaníček and Kleusberg (1987).

The near-zone contribution to the high-frequency co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ reads (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}(\mathrm{R},\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' \,, \tag{7.6}$$

and the far-zone contribution to the high-frequency co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is given by

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}(\mathrm{R},\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' \,. \tag{7.7}$$

According to Molodensky et al. (1960), Vaníček and Kleusberg (1987) proposed to modify the spheroidal Stokes function $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))$ so that the far-zone contribution (truncation error) $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is minimal in the least-squares sense. The modified spheroidal Stokes's function $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o,\psi(\Omega,\Omega'))$ is expressed by (Vaníček and Kleusberg, 1987)

$$\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega')) = \begin{cases} 0, & \psi \in \langle 0, \psi_o \rangle, \\ \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega')), & \psi \in (\psi_o, \pi \rangle, \end{cases} \tag{7.8}$$

and then expanded into the series of the Legendre polynomials

$$\forall\, \psi \in \langle 0, \pi \rangle: \qquad \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega')) = \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{2} \mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega')), \tag{7.9}$$

where $\mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega'))$ are the **truncation coefficients for the modified spheroidal Stokes function** $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))$, see Molodensky et al. (1960). Multiplying Eq. (7.9) by the Legendre polynomials $\mathrm{P}_{\mathrm{m}}(\cos\psi(\Omega,\Omega'))$, i.e.,

$$\forall\, \psi \in \langle 0, \pi \rangle:$$
$$\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{m}}(\cos\psi(\Omega,\Omega')) = \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{2} \mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{m}}(\cos\psi(\Omega,\Omega')), \tag{7.10}$$

and integrating the result over the interval $\psi \in \langle 0, \pi \rangle$, the following expression is found

$$\int_{\psi=0}^{\pi} \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{m}}(\cos\psi(\Omega,\Omega')) \sin\psi(\Omega,\Omega')\, \mathrm{d}\psi$$
$$= \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{2} \mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega')) \int_{\psi=0}^{\pi} \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{m}}(\cos\psi(\Omega,\Omega')) \sin\psi(\Omega,\Omega')\mathrm{d}\psi\ . \tag{7.11}$$

Using the orthogonality property of the Legendre polynomials (Hobson, 1931)

$$\forall\, \psi \in \langle 0, \pi \rangle,\, n \neq m: \qquad \int_{\psi=0}^{\pi} \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{m}}(\cos\psi(\Omega,\Omega')) \sin\psi(\Omega,\Omega')\mathrm{d}\psi = 0\ , \tag{7.12}$$

$$\forall\, \psi \in \langle 0, \pi \rangle,\, n = m: \qquad \int_{\psi=0}^{\pi} \left[\mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega'))\right]^2 \sin\psi(\Omega,\Omega')\mathrm{d}\psi = \frac{2}{2n+1}\ , \tag{7.13}$$

and substituting for $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))$ in Eq. (7.8), the truncation coefficients $\mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega'))$ of the modified spheroidal Stokes function become (Molodensky et al., 1960)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \qquad \mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega')) = \int_{\psi=0}^{\pi} \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{P}_{\mathrm{n}}(\cos\psi(\Omega,\Omega')) \sin\psi(\Omega,\Omega')\mathrm{d}\psi\ . \tag{7.14}$$

7.3 Near-zone contribution to high-frequency co-geoid

Helmert's gravity anomaly referred to the geoid surface can be divided into the low-frequency (**reference**) **gravity anomaly** $\Delta g^{\mathrm{H}}_{n<\bar{n}}(\mathrm{R},\Omega) \equiv \Delta g^{\mathrm{H}}_{\mathrm{ref}}(\mathrm{R},\Omega)$ and the high-frequency (**residual**) **gravity anomaly** $\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega)$. The low-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{\mathrm{ref}}(\mathrm{R},\Omega)$ are evaluated according to Eq. (6.21). The high-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$ are evaluated by subtracting the reference gravity anomalies $\Delta g^{\mathrm{H}}_{\mathrm{ref}}(\mathrm{R},\Omega)$ from Helmert's gravity anomalies computed according to Eq. (5.1).

The near-zone contribution of the high-frequency Helmert's gravity anomalies to the co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_o}(\Omega)$ is expressed by (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_o}}(\Omega)=\frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)}\iint_{\Omega'\in\Omega_{\psi_o}}\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega')\,\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o,\psi(\Omega,\Omega'))\,\mathrm{d}\Omega'\,. \tag{7.15}$$

The Stokes integral is only weakly singular for the spherical distance $\psi=0$ (Martinec, 1993). A classical method for treating a removable singularity consists of adding and subtracting the value of gravity anomaly at the singular point, see Martinec (1993),

$$\begin{aligned}\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_o}}(\Omega)&=\frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)}\iint_{\Omega'\in\Omega_{\psi_o}}\left[\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega')-\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega)\right]\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o,\,\psi(\Omega,\Omega'))\,\mathrm{d}\Omega'\\&+\frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)}\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega)\iint_{\Omega'\in\Omega_{\psi_o}}\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o,\,\psi(\Omega,\Omega'))\,\mathrm{d}\Omega'\,.\end{aligned} \tag{7.16}$$

7.4 Far-zone contribution to high-frequency co-geoid

The far-zone contribution of the high-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega)$ to the co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_o}}(\Omega)$ is given by

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_o}}(\Omega)=\frac{\mathrm{R}}{4\pi\gamma_{\mathrm{o}}(\phi)}\iint_{\Omega'\in\Omega_{\mathrm{O}}-\Omega_{\psi_o}}\Delta g^{\mathrm{H}}_{n>\bar{n}}(\mathrm{R},\Omega')\,\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o,\,\psi(\Omega,\Omega'))\,\mathrm{d}\Omega'\,. \tag{7.17}$$

If the gravity anomalies are not available over the entire Earth, the numerical computation can be done by using the following equation (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_o}}(\Omega)=\frac{\mathrm{R}}{2}\sum_{n=\bar{n}+1,\dots}\mathrm{Q}_{\mathrm{n}}(\psi_o,\psi(\Omega,\Omega'))\sum_{m=-n}^{n}\mathrm{T}^{\mathrm{H}}_{\mathrm{n,m}}\;\mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{7.18}$$

8. Primary indirect effect

After the evaluation of the Stokes boundary value problem in the Helmert gravity space the Helmert co-geoid is obtained. To find the geoid in the real space, the primary indirect topographical and atmospheric effects on the geoidal heights are evaluated (Vaníček and Martinec, 1994).

Helmert's disturbing gravity potential referred to the geoid surface (in the spherical approximation) reads

$$\forall\, \Omega \in \Omega_o: \qquad T^{\mathrm{H}}(\mathrm{R},\Omega) = T(\mathrm{R},\Omega) - \delta V^{t}(\mathrm{R},\Omega) - \delta V^{a}(\mathrm{R},\Omega), \tag{8.1}$$

where $\delta V^{t}(\mathrm{R},\Omega)$ is the **residual gravitational potential of the topographical masses**, and $\delta V^{a}(\mathrm{R},\Omega)$ is the **residual gravitational potential of the atmospheric masses**.

The residual gravitational potential of the topographical masses $\delta V^{t}(\mathrm{R},\Omega)$ is defined as the difference of the gravitational potential $V^{t}(\mathrm{R},\Omega)$ of the topographical masses and the gravitational potential $V^{ct}(\mathrm{R},\Omega)$ of the condensed topographical masses (Martinec et al., 1993)

$$\forall\, \Omega \in \Omega_o: \qquad \delta V^{t}(\mathrm{R},\Omega) = V^{t}(\mathrm{R},\Omega) - V^{ct}(\mathrm{R},\Omega). \tag{8.2}$$

The residual gravitational potential of the atmospheric masses $\delta V^{a}(\mathrm{R},\Omega)$ is given by the difference of the gravitational potential $V^{a}(\mathrm{R},\Omega)$ of the atmospheric masses and the gravitational potential $V^{ca}(\mathrm{R},\Omega)$ of the condensed atmospheric masses (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_o: \qquad \delta V^{a}(\mathrm{R},\Omega) = V^{a}(\mathrm{R},\Omega) - V^{ca}(\mathrm{R},\Omega). \tag{8.3}$$

Applying the Bruns formula (1878) to the disturbing gravity potential $T(\mathrm{R},\Omega)$ and Helmert's disturbing gravity potential $T^{\mathrm{H}}(\mathrm{R},\Omega)$, i.e.,

$$\forall\, \Omega \in \Omega_o: \qquad N(\Omega) = \frac{T(\mathrm{R},\Omega)}{\gamma_o(\phi)}, \tag{8.4}$$

$$\forall\, \Omega \in \Omega_o: \qquad N^{\mathrm{H}}(\Omega) = \frac{T^{\mathrm{H}}(\mathrm{R},\Omega)}{\gamma_o(\phi)} = \frac{T(\mathrm{R},\Omega) - \delta V^{t}(\mathrm{R},\Omega) - \delta V^{a}(\mathrm{R},\Omega)}{\gamma_o(\phi)}, \tag{8.5}$$

the relation between the geoidal height $N(\Omega)$ and the co-geoidal height $N^{\mathrm{H}}(\Omega)$ is obtained (Martinec, 1993)

$$\forall\, \Omega \in \Omega_o: \qquad \delta N(\Omega) = N(\Omega) - N^{\mathrm{H}}(\Omega) = \frac{T(\mathrm{R},\Omega)}{\gamma_o(\phi)} - \frac{T^{\mathrm{H}}(\mathrm{R},\Omega)}{\gamma_o(\phi)} = \frac{\delta V^{t}(\mathrm{R},\Omega)}{\gamma_o(\phi)} + \frac{\delta V^{a}(\mathrm{R},\Omega)}{\gamma_o(\phi)}. \tag{8.6}$$

The first term on the right-hand side of Eq. (8.6) is the **primary indirect topographical effect on the geoidal heights**, and the second term represents the **primary indirect atmospheric effect on the geoidal heights**.

8.1 Primary indirect topographical effect

The gravitational potential $V^{t}(\mathrm{R},\Omega)$ of the topographical masses referred to the geoid surface is given by (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad V^{t}(\mathrm{R},\Omega) = 4\pi\,\mathrm{G}\,\rho_{\mathrm{o}}\, H^{\mathrm{O}}(\Omega)\left[\mathrm{R} + \frac{1}{2} H^{\mathrm{O}}(\Omega)\right] + \mathrm{G}\,\rho_{\mathrm{o}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), r'\right] r'^{2}\, \mathrm{d}r'\, \mathrm{d}\Omega' + \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), r'\right] r'^{2}\, \mathrm{d}r'\, \mathrm{d}\Omega' . \tag{8.7}$$

The gravitational potential $V^{ct}(\mathrm{R},\Omega)$ of the condensed topographical masses referred to the geoid surface reads (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad V^{ct}(\mathrm{R},\Omega) = 4\pi\,\mathrm{G}\,\rho_{\mathrm{o}} \frac{r_t^3(\Omega) - \mathrm{R}^3}{3\mathrm{R}} + \mathrm{G}\,\rho_{\mathrm{o}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3}\, l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), \mathrm{R}\right] \mathrm{d}\Omega' + \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - \mathrm{R}^3}{3}\, l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), \mathrm{R}\right] \mathrm{d}\Omega' . \tag{8.8}$$

Substituting the gravitational potential $V^{t}(\mathrm{R},\Omega)$ of the topographical masses from Eq. (8.7) and the gravitational potential $V^{ct}(\mathrm{R},\Omega)$ of the condensed topographical masses from Eq. (8.8) into the residual gravitational potential of the topographical masses $\delta V^{t}(\mathrm{R},\Omega)$ in Eq. (8.2), the primary indirect topographical effect on the geoidal heights takes the following form (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad \frac{\delta V^{t}(\mathrm{R},\Omega)}{\gamma_o(\phi)} = -4\pi\,\mathrm{G}\,\rho_{\mathrm{o}} \frac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{\gamma_o(\phi)} \left[\frac{1}{2} + \frac{H^{\mathrm{O}}(\Omega)}{3\mathrm{R}}\right] + \frac{\mathrm{G}}{\gamma_o(\phi)} \rho_{\mathrm{o}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), r'\right] r'^{2}\, \mathrm{d}r'\, \mathrm{d}\Omega' - \frac{\mathrm{G}}{\gamma_o(\phi)} \rho_{\mathrm{o}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3}\, l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), \mathrm{R}\right] \mathrm{d}\Omega' + \frac{\mathrm{G}}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), r'\right] r'^{2}\, \mathrm{d}r'\, \mathrm{d}\Omega' - \frac{\mathrm{G}}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - \mathrm{R}^3}{3}\, l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), \mathrm{R}\right] \mathrm{d}\Omega' . \tag{8.9}$$

8.2 Primary indirect atmospheric effect

The gravitational potential $V^{a}(\mathrm{R},\Omega)$ of the atmospheric masses referred to the geoid surface is given by (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad V^{a}(\mathrm{R},\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega')}^{\mathrm{R}+\mathrm{H}_{\mathrm{lim}}} \rho^{a}(r')\, l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), r'\right] r'^{2}\, \mathrm{d}r'\, \mathrm{d}\Omega' + \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=\mathrm{R}+\mathrm{H}_{\mathrm{lim}}}^{r_{\mathrm{lim}}} \rho^{a}(r')\, l^{-1}\left[\mathrm{R}, \psi(\Omega,\Omega'), r'\right] r'^{2}\, \mathrm{d}r'\, \mathrm{d}\Omega' . \tag{8.10}$$

The gravitational potential $V^{ca}(\mathrm{R},\Omega)$ of the condensed atmospheric masses referred to the geoid surface reads (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \quad V^{ca}(\mathrm{R},\Omega) = \mathrm{G}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega')}^{\mathrm{R}+\mathrm{H_{lim}}} \rho^{a}(r')\,r'^{2}\,\mathrm{d}r'\,l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),\mathrm{R}]\,\mathrm{d}\Omega' + \mathrm{G}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}+\mathrm{H_{lim}}}^{r_{\mathrm{lim}}} \rho^{a}(r')\,r'^{2}\,\mathrm{d}r'\,l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),\mathrm{R}]\,\mathrm{d}\Omega' \,. \tag{8.11}$$

Substituting Eqns. (8.10) and (8.11) into Eq. (8.3), the primary indirect atmospheric effect on the geoidal heights takes the following form

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \quad \frac{\delta V^{a}(\mathrm{R},\Omega)}{\gamma_{o}(\phi)} = \frac{\mathrm{G}}{\gamma_{o}(\phi)}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega')}^{\mathrm{R}+\mathrm{H_{lim}}} \rho^{a}(r')\left(l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),r'] - l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),\mathrm{R}]\right) r'^{2}\,\mathrm{d}r'\,\mathrm{d}\Omega' + \frac{\mathrm{G}}{\gamma_{o}(\phi)}\iint\limits_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}+\mathrm{H_{lim}}}^{r_{\mathrm{lim}}} \rho^{a}(r')\left(l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),r'] - l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),\mathrm{R}]\right) r'^{2}\,\mathrm{d}r'\,\mathrm{d}\Omega' \,. \tag{8.12}$$

9. Effect of Helmert's condensation to the first degree term

Conventionally, the mass center of the Earth is located at the origin of the coordinate system. Often the Helmert condensation is done under the condition that the topographical mass will be preserved. Due to this the mass-center of the Helmert body (a geoid model from the Stokes-Helmert scheme) will be at a new location, with Cartesian coordinates as follows
Xh = (-0.006 ; -0.015 ; +0.002) [m] (Martinec (1998), p. 30).
In other words, the maximum shift, 16 mm, of the Helmert body is toward geodetic Longitude ~ 248° and geodetic latitude +7°. At this location one needs to introduce a correction to the Helmert geoidal heights: -16 mm.
On the opposite site of the globe (i.e at B= -7° and L=68°) the correction to the Helmert body is +16 mm.

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9. Effect of Helmert's condensation to the first degree term

Conventionally, the mass center of the Earth is located at the origin of the coordinate system. Often the Helmert condensation is done under the condition that the topographical mass will be preserved. Due this the mass-center of the Helmert body (a geoid model from the Stokes-Helmert scheme) will be at a new location, which Cartesian coordinates are as follows
Xh = (-0.006 ; -0.015 ; +0.002) [m] (Martinec (1998), p. 30).
In other words, the maximum shift, 16 mm, of the Helmert body is toward geodetic Longitude ~ 248° and geodetic latitude +7°. At this location one needs to introduce a correction to the Helmert geoidal heights: -16 mm.
On the opposite site of the globe (i.e at B= -7° and L=68°) the correction to the Helmert body is +16 mm

However, a double check shows that the magnitude and the direction of this shift differes slightly from the one of Martinec (1998). He has used a set of spherical-harmonic coefficients of the squared topographical heights. He claims that the following coefficients are originating from TUG-87
$(H^2)_{10}$ = -0.847 e5 (Cnm)
$(H^2)_{11}$ = -0.207e6 (Cnm)
$(H^2)_{11}$ = +0.509e6 (Snm)

The above coefficients are inserted into Eq. 2.17, the results are shown by Eq. 2.19 (both Martinec (1998) .
As a matter of fact the TUG-87 coefficients (squared heights are as follows):

```
 H2 (H. squred) coefficients up to 90/90 in m2 originated from TUG87 data
  0  0   0.4464025065D+06   0.0000000000D+00
  1  0  -0.2161658303D+05   0.0000000000D+00
  1  1   0.8572045522D+05   0.2030582602D+06
  2  0   0.2837114475D+06   0.0000000000D+00
  2  1  -0.2092626193D+05   0.2064845705D+06
  2  2  -0.2356146467D+06   0.3491766549D+05
```

Well, considering the "correct" coefficients, then the triplet Xh of Cartesian coordinates becomes
Xh = (+0.0022 ; -0.0053 ; +0.0004) Unit is metre [m]
The magnitude of the above vector is equal to 6 mm (i.e. three times less than estimated in Martinec (1998)).
Now the maximum shift, 6 mm, of the Helmert body is toward geodetic Long ~ 292° and geodetic latitude +4° (South Venezuela). At this location one needs to introduce a correction to the Helmert geoidal heights: -6 mm. On the opposite site of the globe (i.e at B= -4° and L=112°, island Java) the correction to the Helmert body is +6 mm.

```
a=6378137 %suur pooltelg GRS
e2=0.006694380023 % GRS
GEOID=ones(690*690)*10;
N=a*((1-e2*sin(B*pi/180).*sin(B*pi/180)).^(-0.5));
X=(N+GEOID).*cos(B*pi/180).*cos(L*pi/180);
Y=(N+GEOID).*cos(B*pi/180).*sin(L*pi/180);
Z=(N*(1-e2)+GEOID).*sin(B*pi/180);
%clear N

% martinec (1998) page 31
X_new=X-0.0022;
Y_new=Y+0.0053;
Z_new=Z-0.00042;

P=sqrt(X_new.*X_new+Y_new.*Y_new);

THETA=atan(Z_new./(P*sqrt(1-e2)));
FI=atan((Z+a*e2*sin(THETA).*sin(THETA).*sin(THETA)./sqrt(1-e2))./(P-
a*e2*cos(THETA).*cos(THETA).*cos(THETA)));

H=P./cos(FI)-N;
mean(mean(H-GEOID))
```