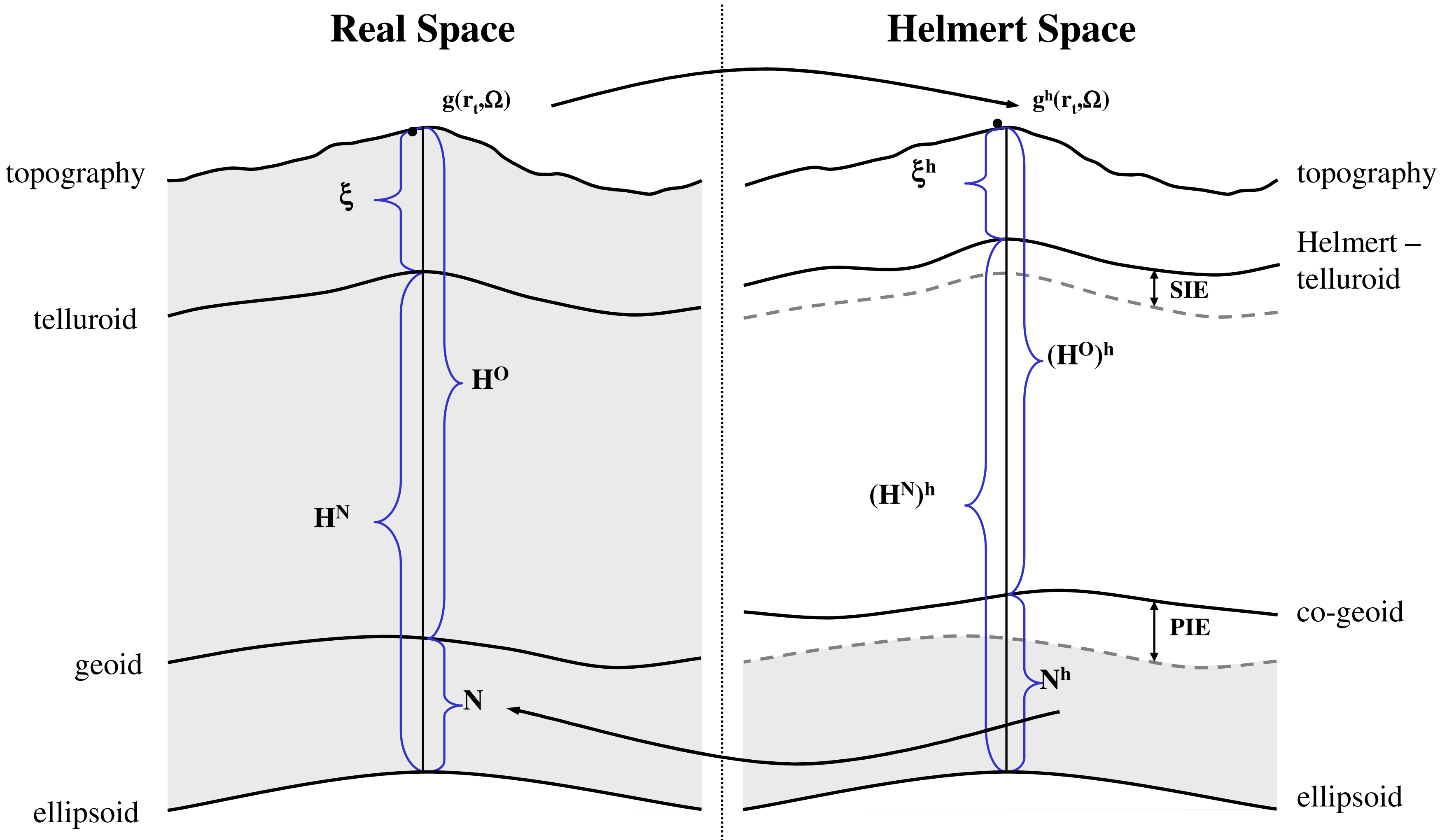
ROLE OF "NO TOPOGRAPHY SPACE" IN STOKES-HELMERT SCHEME FOR GEOID DETERMINATION

Petr Vaníček, Robert Tenzer
and
Jianliang Huang

Annual meeting of CGU

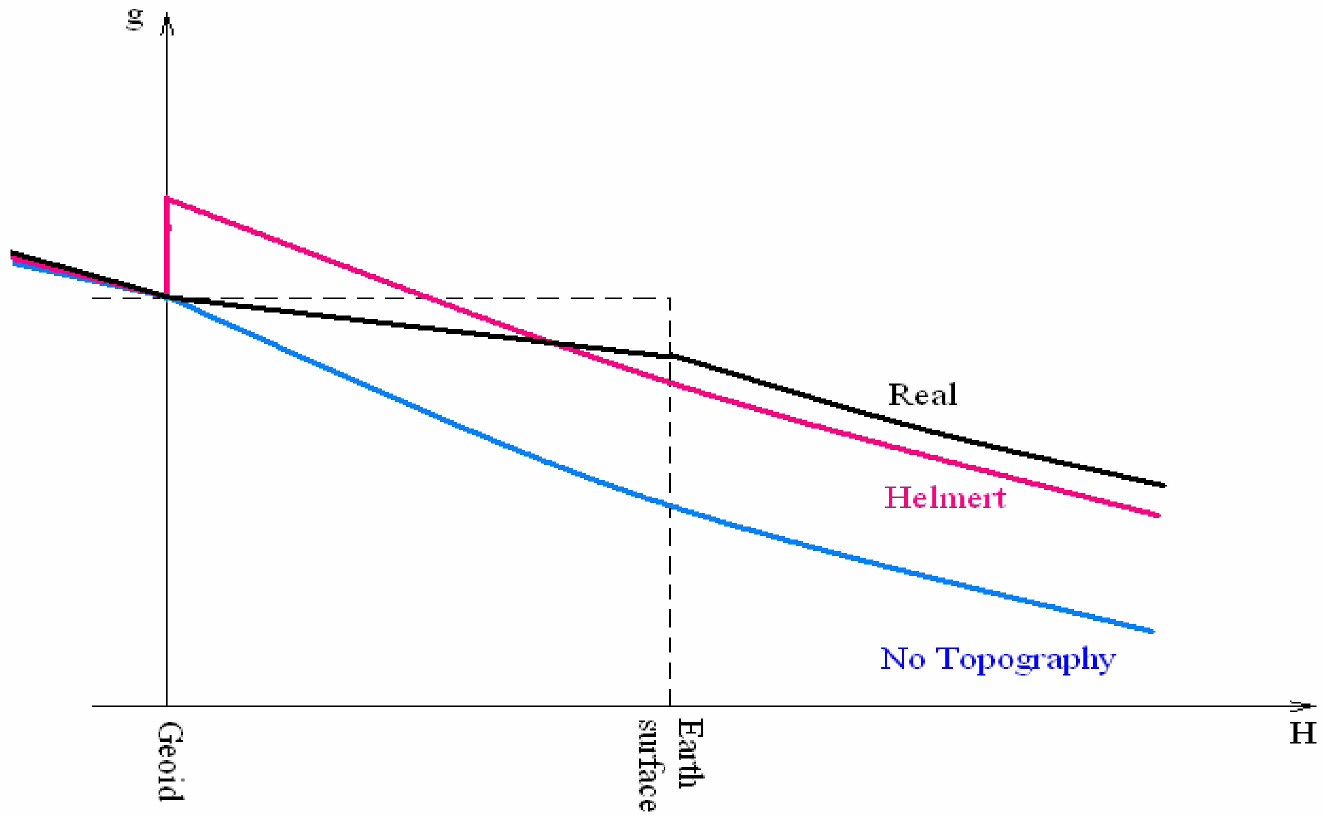
We have been formalizing the Stokes solution of the geodetic boundary value problem in Helmert's form (second Helmert's condensation method) for almost 10 years. Our first paper about this formalization was [Vaníček P. and Martinec Z., 1994: *The Stokes-Helmert scheme for the evaluation of a precise geoid*. Manuscripta Geodaetica, No.19, Springer]. The scheme used in this formalization is shown graphically on the next slide.

Molodenskij opined that it would be impossible to determine geoid precisely because of the unknown topographical density. Yet we are getting good results, both at UNB and at GSD, and the approach seems to make a good physical sense. We take this as a vindication of Stokes's and Helmert's ideas.


Real Space
Helmert Space
g(r_t,Ω)
g^h(r_t,Ω)
topography
ξ
telluroid
H^O
H^N
geoid
N
ellipsoid
ξ^h
topography
Helmert –
telluroid
SIE
(H^O)^h
(H^N)^h
co-geoid
PIE
N^h
ellipsoid

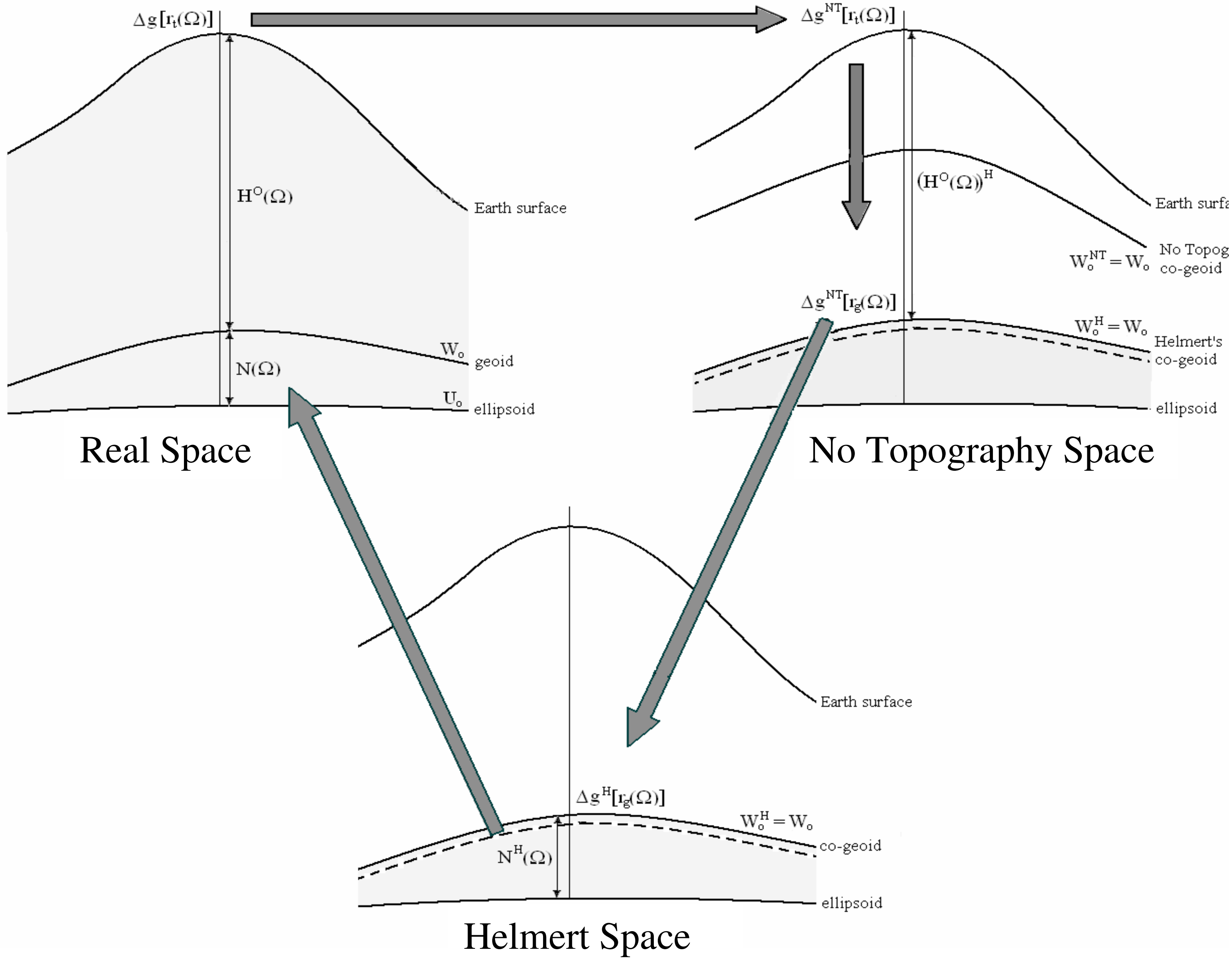
Heck pointed out in 1993, and recent theoretical work by Heck and Novák confirmed that Helmert's anomalies are quite rough due to the presence of the condensation layer on the geoid. Also, Helmert's anomalies are discontinuous on the geoid - see the next slide. These facts led some people to question our approach to the downward continuation of Helmert's anomalies from the earth surface to the geoid, formulated as Poisson's solution to the inverse Dirichlet's BVP.

Véronneau (GSD) elected to continue downward the complete Bouguer anomalies instead of Helmert's anomalies, with good numerical results. Hence we decided also to have a good look at some such alternatives.


g
Real
Helmert
No Topography
H
Geoid
Earth surface

Following Heck and Novák's theoretical suggestion and Véronneau's experimental results, we have decided to investigate the inverse Dirichlet's problem formulated in the "No Topography" space rather than the "Helmert" space. The NT-space is characterized by having the real mass density within the geoid and zero mass density everywhere else – cf., Santos et al. paper at this meeting. The gravity field in the NT-space is smoother than Helmert's field and NT-gravity is continuous everywhere.

Applying this approach – see the next slide – we get good numerical results – cf., Tenzer et al. poster at this meeting – with an additional bonus: the smoother field makes the task of Poisson's downward continuation for denser gravity data easier (needed in the future?).


Δg[r_t(Ω)]
H^O(Ω)
Earth surface
W_o
geoid
N(Ω)
U_o
ellipsoid
Real Space
Δg^NT[r_t(Ω)]
(H^O(Ω))^H
Earth surfa
W_o^NT = W_o
No Topog
co-geoid
Δg^NT[r_g(Ω)]
W_o^H = W_o
Helmert's
co-geoid
ellipsoid
No Topography Space
Earth surface
Δg^H[r_g(Ω)]
W_o^H = W_o
co-geoid
N^H(Ω)
ellipsoid
Helmert Space

The Theory

The gravity anomaly $\Delta g[r_t(\Omega),\Omega]$ on the surface of the earth in the real space is given by the well known formula (the "fundamental gravimetric equation"):

$$\Delta g\left[r_t(\Omega),\Omega\right] = -\left.\frac{\partial T(r,\Omega)}{\partial r}\right|_{r=r_t(\Omega)} + \varepsilon_{\delta g}\left[r_t(\Omega),\Omega\right] - \frac{2}{r_t(\Omega)} T\left[r_t(\Omega),\Omega\right] - \varepsilon_{\mathrm{n}}\left[r_t(\Omega),\Omega\right]$$

$$\varepsilon_{\delta g}\left[r_t(\Omega),\Omega\right] \cong -\frac{f\sin 2\varphi}{r_t(\Omega)} \left.\frac{\partial T(r,\Omega)}{\partial \varphi}\right|_{r=r_t}$$

$$\varepsilon_{\mathrm{n}}\left[r_t(\Omega),\Omega\right] \cong 2\left[\mathrm{m} + \mathrm{f}\left(\cos 2\varphi - \frac{1}{3}\right)\right] \frac{T\left[r_t(\Omega),\Omega\right]}{r_t(\Omega)}$$

The transformation of $\Delta g(r,\Omega)$ from the real space to the NT-space then takes place: $\Delta g[r_t(\Omega),\Omega] \rightarrow \Delta g^{NT}[r_t(\Omega),\Omega]$. It is given by the following formula ($V^t(r,\Omega)$ is the potential of topography and $V^a(r,\Omega)$ is the potential of the atmosphere):

$$\Delta g^{\mathrm{NT}}\left[r_t(\Omega),\Omega\right] \cong \Delta g\left[r_t(\Omega),\Omega\right] + \left.\frac{\partial V^t(r,\Omega)}{\partial r}\right|_{r=r_t(\Omega)} + \left.\frac{\partial V^a(r,\Omega)}{\partial r}\right|_{r=r_t(\Omega)} +$$

$$+\frac{2}{r_t(\Omega)} V^t\left[r_t(\Omega),\Omega\right] + \frac{2}{r_t(\Omega)} V^a\left[r_t(\Omega),\Omega\right]$$

$$+\frac{f \sin 2\varphi}{r_t(\Omega)} \left.\frac{\partial V^t\left[r,\Omega\right]}{\partial \varphi}\right|_{r=r_t} + 2\left[\mathrm{m} + \mathrm{f}\left(\cos 2\varphi - \frac{1}{3}\right)\right] \frac{V^t\left[r_t(\Omega),\Omega\right]}{r_t(\Omega)}$$

where the topographical potential is given by:

$$V^t[r_t(\Omega),\Omega] = 4\pi G\rho_o \frac{R^2}{r_t(\Omega)} H^O(\Omega)\left[1+\frac{H^O(\Omega)}{R}+\frac{1}{3}\left(\frac{H^O(\Omega)}{R}\right)^2\right]+$$

,

$$+G\rho_o \iint_{\Omega'\in\Omega_O} \int_{r'=R+H^O(\Omega)}^{R+H^O(\Omega')} l^{-1}[r_t(\Omega),\psi(\Omega,\Omega'),r']\, r'^2\, dr'\, d\Omega' +$$

$$+G \iint_{\Omega'\in\Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} l^{-1}[r_t(\Omega),\psi(\Omega,\Omega'),r']\, r'^2\, dr'\, d\Omega'$$

And the topographical attraction is given by:

$$\left.\frac{\partial V^t(r,\Omega)}{\partial r}\right|_{r=r_t} = -4\pi G\rho_o \frac{R^2}{r_t^2(\Omega)} H^O(\Omega)\left[1+\frac{H^O(\Omega)}{R}+\frac{1}{3}\left(\frac{H^O(\Omega)}{R}\right)^2\right]+$$

$$+G\rho_o \iint_{\Omega'\in\Omega_O} \int_{r'=R+H^O(\Omega)}^{R+H^O(\Omega')} \left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\right|_{r_t(\Omega)} r'^2\, dr'\, d\Omega' +$$

$$+G \iint_{\Omega'\in\Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} \left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\right|_{r_t(\Omega)} r'^2\, dr'\, d\Omega'$$

Is the NT-anomaly, $\Delta g^{NT}(r,\Omega)$, the same as the standard complete Bouguer anomaly $\Delta g^{CB}(r,\Omega)$ as we have been using it? **NO**! The $\Delta g^{NT}(r,\Omega)$ is the same as the "spherical complete Bouguer anomaly"

$$\Delta g^{CB;S}(r,\Omega).$$

This spherical anomaly has been systematically investigated by Vaníček P., Tenzer R., Sjöberg L.E., Martinec Z., Featherstone W.E. [2003: *New views of the spherical Bouguer gravity anomaly*, submitted to Geophysical Journal International]. It has been shown that it is:

defined in the whole 3D space (i.e., it has a well defined disturbing potential $T^{CB;S}(r,\Omega)$ associated with it) and that it is indeed harmonic everywhere above the geoid.

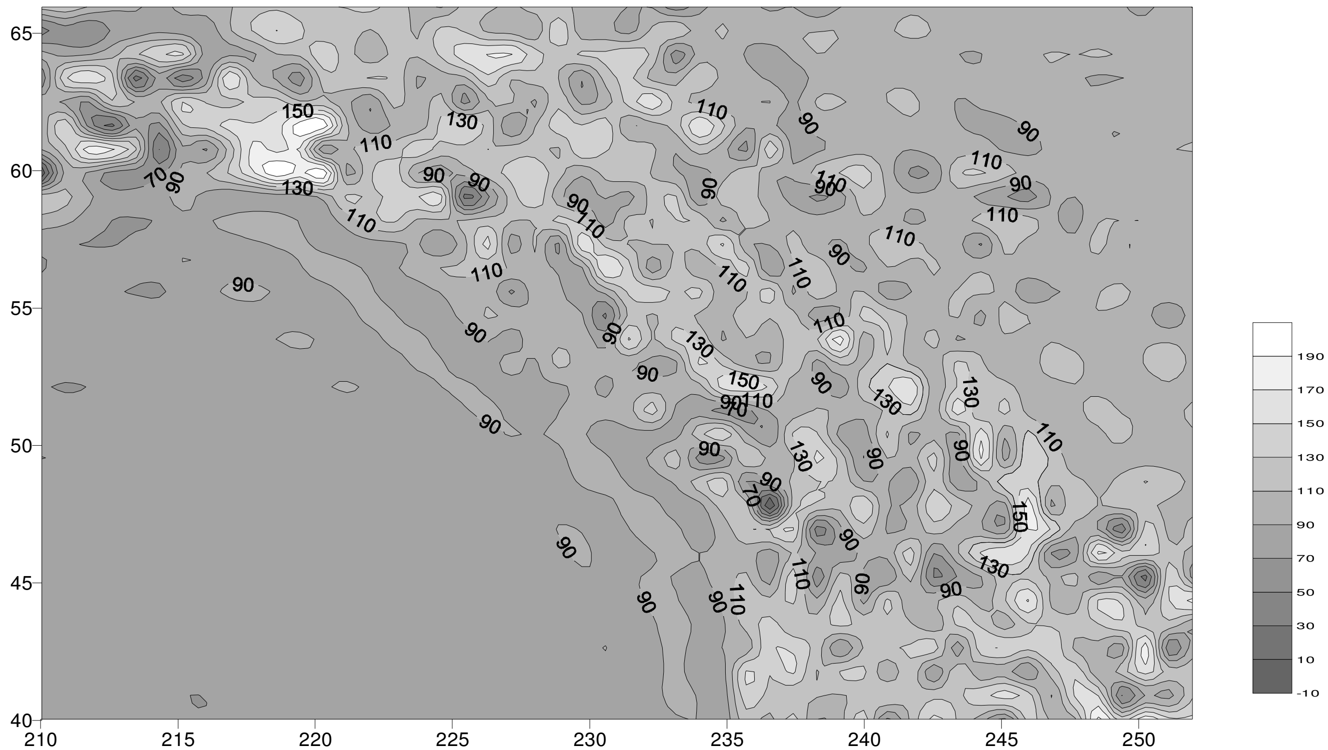
This is NOT the case with the standard Bouguer anomaly.

The spherical complete Bouguer anomaly is defined by the following equation on the earth surface:

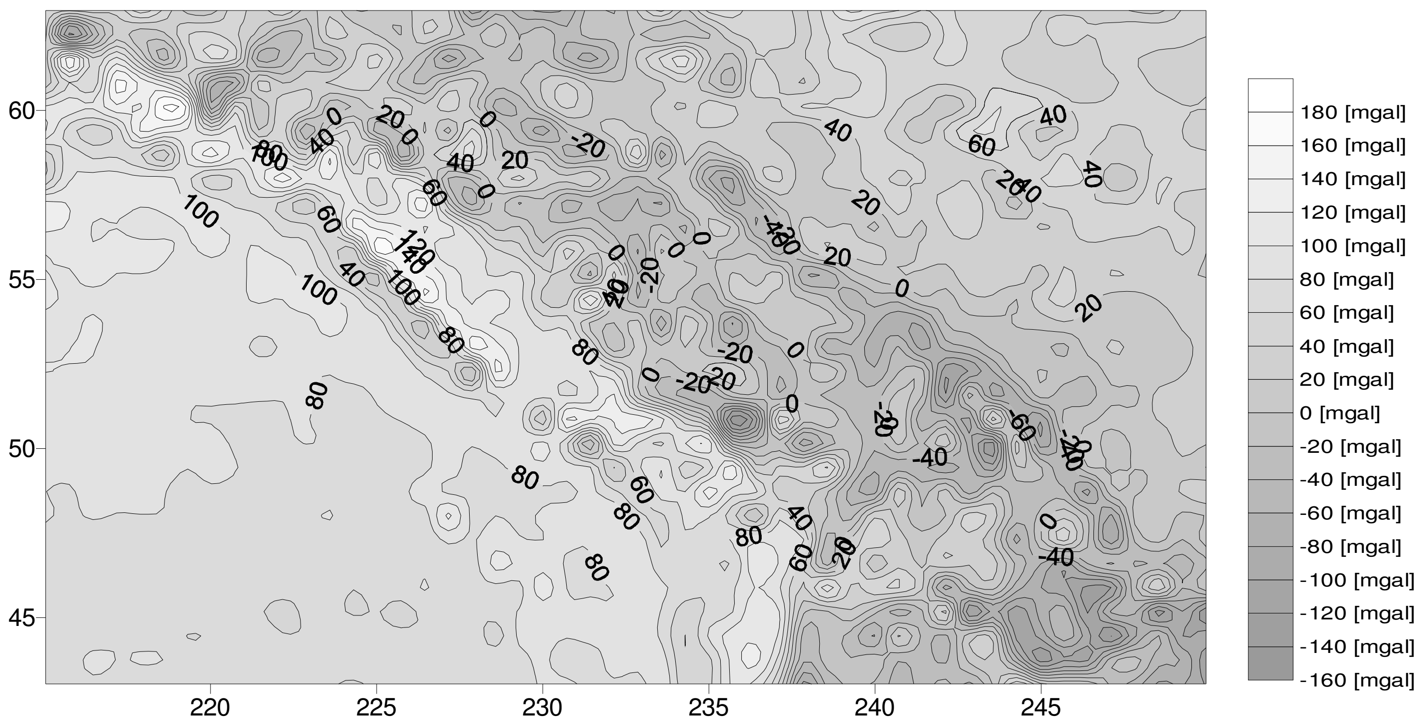
$$\forall\Omega \in \Omega_0 : \Delta g^{CB;S}[H(\Omega)] = \Delta g[H(\Omega)] - 4\pi G\rho_0 H(\Omega) + $$
$$+ TopoC^{B}[\delta\rho; H(\Omega)] + TC^{S}[\rho_0; H(\Omega)] + \frac{2}{R} V^{T}[H(\Omega)]$$

where Δg is the generic gravity anomaly in the real space, $TopoC^B$ is the correction for anomalous topographical density δρ, TC^S is the spherical terrain correction evaluated for constant topographical density ρ_0.

The difference between spherical (NT) and planar complete Bouguer anomalies



The NT-anomalies continued downward on the geoid



Once we have the (NT) anomalies on the surface of the earth in the NT-space, we continue them downward in the NT-space: they are defined everywhere and are harmonic above the geoid. This is done by means of solving the Poisson integral equation.

Note that $\Delta g^{NT}[r_t(\Omega),\Omega] \equiv \Delta g^{CB;S}[r_t(\Omega),\Omega]$ are continued downward to the Helmert co-geoid to get $\Delta g^{NT}[r_g(\Omega),\Omega]$ where we want to have them – see slide #7. Thus orthometric heights from Helmert's space must be used for the downward continuation in the NT-space. These differ from orthometric heights in the real space by the geoid - co-geoid separation.

The NT-anomalies on Helmert's co-geoid, $\Delta g^{NT}[r_g(\Omega),\Omega]$, (obtained as the Poisson solution to the downward continuation problem), are then transformed to Helmert's space. We get $\Delta g^{H}[r_g(\Omega),\Omega]$, which are then used as boundary values for the solution of the BVP of the third kind in Helmert's space as originally envisaged. In spherical approximation we get ($V^{ct}(r,\Omega)$ is the potential of topography condensed on the geoid, $V^{ca}(r,\Omega)$ is the potential of the atmosphere condensed on the geoid):

$$\Delta g^{\mathrm{H}}(\mathrm{R},\Omega) = \Delta g^{\mathrm{NT}}(\mathrm{R},\Omega) - \frac{3}{2\mathrm{R}} V^{ct}(\mathrm{R},\Omega) - \frac{3}{2\mathrm{R}} V^{ca}(\mathrm{R},\Omega)$$

where

$$V^{ct}(\mathrm{R},\Omega) = \mathrm{G}\,\mathrm{R}^2 \iint_{\Omega'\in\Omega_0} \sigma(\Omega')\, l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),\mathrm{R}]\,\mathrm{d}\Omega'$$

$$\sigma(\Omega) = \frac{\rho(\Omega)}{\mathrm{R}^2}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)} r^2\,\mathrm{d}r = \rho(\Omega)\, H^{\mathrm{O}}(\Omega)\left[1+\frac{H^{\mathrm{O}}(\Omega)}{\mathrm{R}}+\frac{1}{3}\left(\frac{H^{\mathrm{O}}(\Omega)}{\mathrm{R}}\right)^2\right]$$

$$V^{ca}(R,\Omega) = \mathrm{G}\iint_{\Omega'\in\Omega_0}\int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega')}^{\mathrm{r_{lim}}} \rho^a(r')\,r'^2\,\mathrm{d}r'\, l^{-1}[\mathrm{R},\psi(\Omega,\Omega'),\mathrm{R}]\,\mathrm{d}\Omega'$$

and ρ^a is the atmospheric density.

The final solution $N^H(\Omega)$ is transformed into the

geoidal height $N(\Omega)$

(in the real space) by adding to $N^H(\Omega)$ the "primary indirect effect" $\delta N^H(\Omega)$ (in Helmert's space). The solution $N^H(\Omega)$ is again sought in the Helmert space in order to minimize the "primary indirect topographical effect", i.e., the geoid – co-geoid separation. (The corresponding quantity $\delta N^{NT}(\Omega)$ in the NT-space is very large and cannot be computed accurately enough.)

CONCLUSIONS

1) This approach seems to work as well as the approach that uses downward continuation in Helmert's space.

2) It may prove beneficial when denser gravity data become available for geoid determination.

That's all, folks!