Mean Vertical Gradient of Gravity

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Abstract. The Stokes-Helmert scheme for precise geoid determination requires that Helmert's gravity anomalies are first evaluated on the earth surface. Subsequently, these anomalies must be continued downward onto geoid, where they make the boundary values for solving the geodetic boundary value problem. The anomalies are continued downward using the Poisson integral; this can be done because the Helmert disturbing potential is harmonic everywhere above the geoid. Thus, the difference between Helmert's gravity on the earth surface and on the geoid can be computed and the mean vertical gradient of gravity between the earth surface and the geoid can be obtained.

In this contribution we show a map of the mean gravity gradient for one particularly interesting area of the Rocky Mountains. We also speculate if these values can be used to make orthometric heights more precise.

Keywords. Vertical gradient of gravity, downward continuation, direct topographical effect, Helmert's gravity anomaly

1 Introduction

This investigation had started as a by-product of our work on the Stokes-Helmert theory of precise geoid determination. In this theory the Helmert's gravity anomalies are evaluated on the surface of the earth from the expression (Vaníček et al., 1999, eqn. 37)

$$\Delta g^h = \Delta g^{FA} + DTE + corrective\_terms , \quad (1)$$

where Δg^{FA} is the free-air gravity anomaly and DTE is the direct topographical effect. We can regard the addition of the DTE and corrective terms as a transformation of the free-air anomaly from the real space to Helmert's space. It has been shown that the product $r\Delta g^h$ is a harmonic function above the geoid in the Helmert's space, i.e., above the Helmert co-geoid. In the geoid evaluation the Helmert anomaly is needed on the Helmert co-geoid, thus it has to be "continued downward" from the surface of the earth. As the product $r\Delta g^h$ is harmonic (it satisfies the Laplace equation) above the co-geoid in Helmert's space, its downward continuation can be carried out by means of Poisson's solution of the Dirichlet boundary value problem (Vaníček et al., 1996). From this downward continuation it is possible to evaluate the change of Δg^h between the co-geoid and the surface of the earth, i.e., within the topography. The vertical gradient of Δg^h within the topography is then obtained by dividing this change by topographical height.

2 Vertical gradient of actual gravity within topography

First, we realize that the free-air anomaly is defined by (Vaníček et al., 1999, eqn. 37)

$$\Delta g^{FA} = g_t - \gamma_{t-z} , \quad (2)$$

where g_t is the observed gravity at the earth surface, γ_{t-z} is the normal gravity at the appropriate normal equipotential surface. The "appropriate" normal equipotential surface (surface of the same normal potential as the actual potential at the earth surface) is vertically displaced by Z from the earth surface. Substituting eqn. (2) into eqn. (1), omitting the small corrective terms and differentiating with respect to r we obtain

$$\frac{\partial \Delta g^h}{\partial r} = \left(\frac{\partial g}{\partial r}\right)_{r=r_t} - \left(\frac{\partial \gamma}{\partial r}\right)_{r=r_{t-Z}} + \frac{\partial DTE}{\partial r}. \quad (3)$$

Now, in eqn. (3) the expression on the left hand side is estimated by the Poisson downward continuation mentioned above and the second term on the right hand side can be replaced by the normal gravity gradient at the reference ellipsoid which can be calculated from the mathematical expression for normal gravity. The last term on the right hand side has to be evaluated from topographical masses which represents the main challenge here. The first term on the right hand side is the vertical gradient of actual gravity within topography that we wish to compute.

3 Evaluation of the vertical gradient of direct topographical effect

The direct topographical effect is a radial derivative of the residual potential δV, which is defined in (Martinec and Vaníček, 1994, eqn. 1) as follows

$$\delta V = V^t - V^c , \quad (4)$$

where V_t is the gravitational potential of the topographical masses and V_c is the gravitational potential of the topographical masses condensed into infinitely thin layer placed on the geoid. According this definition we can write (Martinec and Vaníček, 1994, eqn. 2)

$$DTE \equiv \frac{\partial \delta V}{\partial r} = \frac{\partial V^t}{\partial r} - \frac{\partial V^c}{\partial r}. \quad (5)$$

After applying the Newtonian potential theory and assuming the spherical approximation of the geoid one can derive analytical expression for DTE (Martinec, 1998, eqn. 3.45)

$$DTE = G \int_{\Omega'} \Big[\ \bar{\rho}(\Omega') \frac{\partial \tilde{L}^{-1}(r,\psi,r')}{\partial r} \Bigg|_{r'=R}^{R+H(\Omega')} -$$

$$- \bar{\rho}(\Omega) \frac{\partial \tilde{L}(r,\psi,r')}{\partial r} \Bigg|_{r'=R}^{R+H(\Omega)} - \quad (6)$$

$$- R^2 \left[\sigma(\Omega') - \sigma(\Omega)\right] \frac{\partial L^{-1}(r,\psi,r')}{\partial r} \Bigg] d\Omega ,$$

where G is the Newton's gravitational constant, $\bar{\rho}$ is the mean topographical density between the geoid and the earth surface along the column of height H and σ is a surface density of the condensed layer. The expression L^{-1} is a reciprocal distance between the point of computation and point of integration. Symbol $\tilde{L}^{-1}$ is an abbreviation for an indefinite radial integral (Martinec, 1998, eqn. 3.35)

$$\tilde{L}^{-1}(r,\psi,r') = \int_{r'} \frac{r'^2}{L(r,\psi,r')} dr' . \quad (7)$$

The expressions L^{-1}, $\tilde{L}^{-1}$ as well as their radial derivatives are possible to evaluate analytically (see eqn. 3.53, 3.54 in Martinec, 1998). If $r=R+H$ in equation (6), the DTE on the earth surface is computed. If $r=R$, the DTE on the geoid is computed. Then the average gradient of the DTE between the geoid and the earth surface is possible to estimate by dividing the difference between DTE on the earth surface and DTE on the geoid by the topographical height H. This process yields questionable results when topographical heights are small because of the division by a small number. The limit of the vertical gradient for H $\rightarrow$ 0 in undefined.

4 Numerical results

For one particularly interesting area in the Rocky Mountains, bounded by parallels 42° < latitude < 61° and meridians 225°< longitude < 257°, the following maps are shown in an appendix:

- mean heights $5' \times 5'$
- mean Poisson downward continuation of Helmert's gravity anomaly
- mean gradient of the Helmert's gravity anomaly

- mean gradient of the Helmert's gravity
- mean gradient of the actual gravity

The minimum and maximum values are shown in tab.1.

Tab.1. Minimum and maximum values of particular quantities

	1	2	3	4	5
min	-232.2	-0.09	-0.23	-0.40	-0.180
max	133.7	0.07	-0.11	-0.23	-0.010
mean	-0.4	-	-	-	-0.087

In tab.1 the particular columns mean the following quantities:

1. mean Poisson downward continuation of Helmert's gravity anomaly (mGal),
2. mean gradient of the Helmert's gravity anomaly (mGal/m),
3. gradient of the direct topographical effect (mGal/m),
4. mean gradient of the Helmert's gravity (mGal/m),
5. mean gradient of the actual gravity (mGal/m).

Notice: the gradients were computed only for those points where the topographical height was over 100 meters. This threshold value was chosen empirically.

The expected average value of the real gravity gradient is Poincare-Pray gradient 0.0848 mGal/m (Vaníček, Krakiwsky, 1986, eqn. 21.37), computed for average topographical density 2670 kg/m^3. Our result fits the expected value in average very well as it can be seen in tab.1. From the Poincare-Pray gradient we see that larger local density results in smaller (in absolute value) vertical gradient. Moreover between the density and the vertical gradient of the gravity exists an approximately linear relation. Therefore as the next step in the future we intend to compare our results with the topographical density model.

5 Conclusions

Described in this contribution are just the first results. Clearly, the whole process can be further refined if the circumstances would require it. But even these initial results look encouraging and seem to make a physical sense. The applicability of this technique in geophysical investigations should be fairly obvious. In geodesy, the resulting vertical gradients can be used to derive more realistic Helmert orthometric heights.

References

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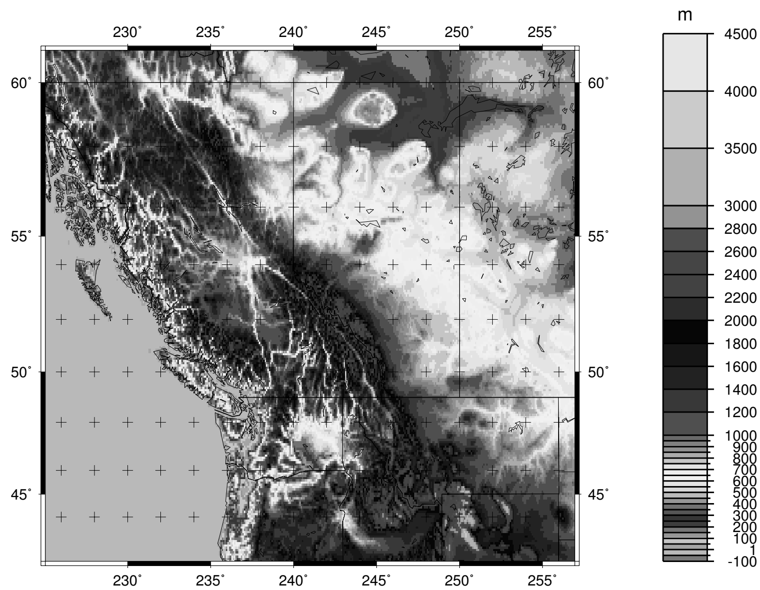
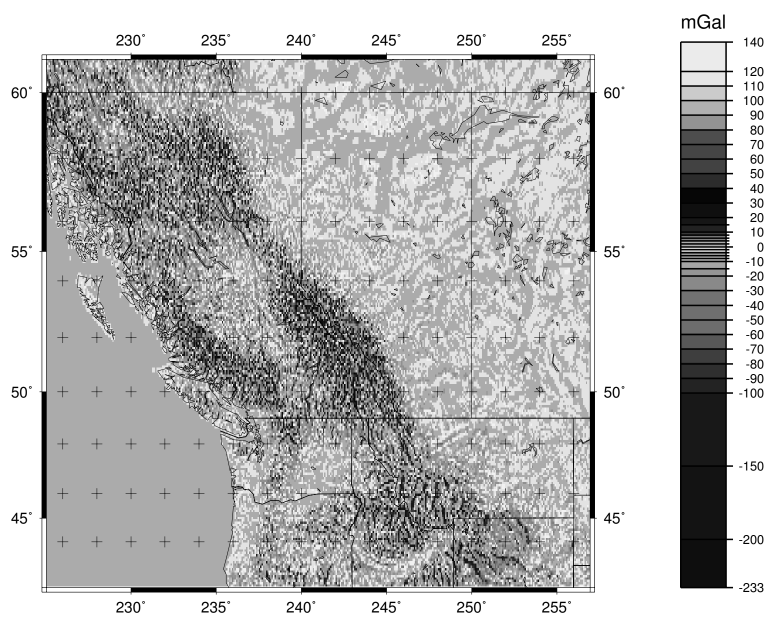
Appendix

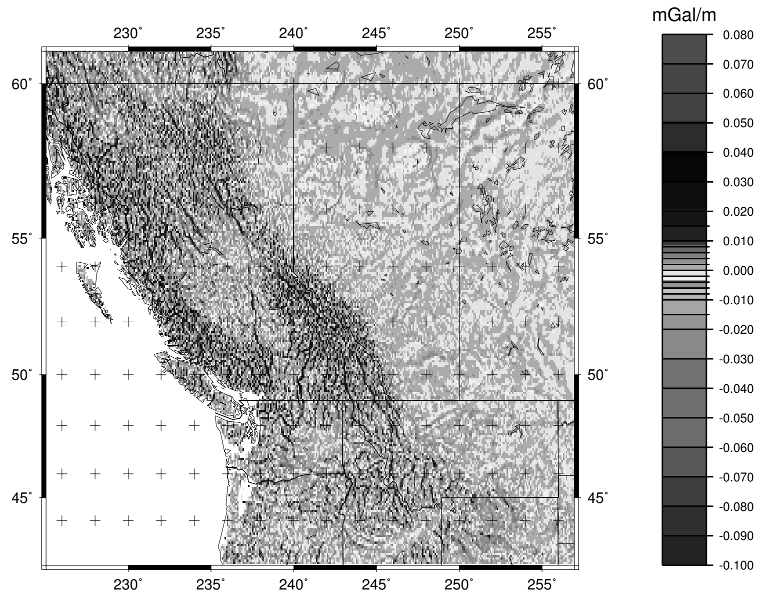
Mean heights 5' x 5'



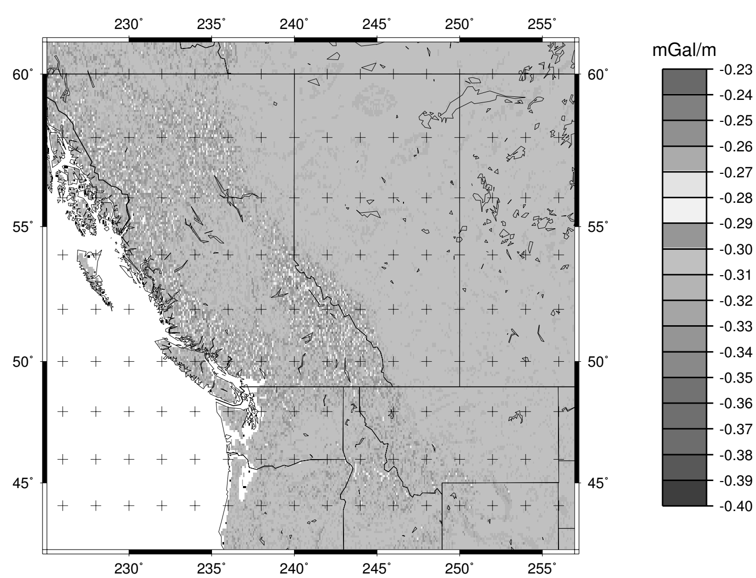
Mean Poisson downward continuation of Helmert's gravity anomaly

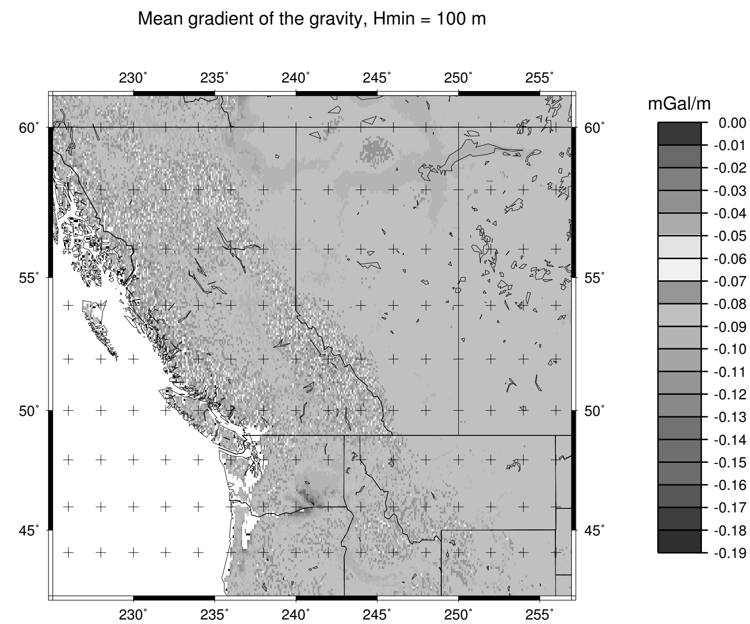


Mean gradient of the Helmert's gravity anomaly, Hmin = 100 m



Mean gradient of the Helmert's gravity, Hmin = 100 m




Mean gradient of the gravity, Hmin = 100 m
230˚
235˚
240˚
245˚
250˚
255˚
60˚
55˚
50˚
45˚
mGal/m
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
-0.10
-0.11
-0.12
-0.13
-0.14
-0.15
-0.16
-0.17
-0.18
-0.19