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Construction of Green's function for the Stokes boundary-value problem with ellipsoidal corrections in the boundary condition

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Abstract. Green's function for the boundary-value problem of Stokes's type with ellipsoidal corrections in the boundary condition for anomalous gravity is constructed in a closed form. The 'spherical-ellipsoidal' Stokes function describing the effect of two ellipsoidal correcting terms occurring in the boundary condition for anomalous gravity is expressed in $O(e_0^2)$-approximation as a finite sum of elementary functions analytically representing the behaviour of the integration kernel at the singular point $\psi = 0$. We show that the 'spherical-ellipsoidal' Stokes function has only a logarithmic singularity in the vicinity of its singular point. The constructed Green function enables us to avoid applying an iterative approach to solve Stokes's boundary-value problem with ellipsoidal correction terms involved in the boundary condition for anomalous gravity. A new Green-function approach is more convenient from the numerical point of view since the solution of the boundary-value problem is determined in one step by computing a Stokes-type integral. The question of the convergence of an iterative scheme recommended so far to solve this boundary-value problem is thus irrelevant.

Key words. Spherical Stokes's function · Ellipsoidal corrections · Surface spherical harmonics · Addition theorem

1 Introduction

Prescribing the surface gravity potential, the magnitude of surface gravity, the volumetric density of topographical masses and the angular velocity of the Earth's rotation, the scalar, free boundary-value problem for determination of the geoid can be formulated. This problem, sometimes called the problem for gravimetric determination of the geoid, is non-linear with respect to the sought gravity potential since the magnitude of gravity is a non-linear function of generating potential. The problem may be linearized with respect to a reference gravity GM/r^2, which is quoted as a spherical approximation in gravity space. The error of this linearization may reach 5 mgal counted in gravity. To reduce the linearization error of spherical approximation to a level of the order of 0.2 mgal, the gravity field of a level ellipsoid, the so-called Somigliana-Pizzetti normal gravity, is to be introduced as a reference. Compared to the spherical approximation, the linearized boundary condition for anomalous gravity contains two additional terms, the so-called ellipsoidal corrections (e.g. Cruz 1985), the magnitudes of which are of the order of the squared eccentricity of the level ellipsoid.

Having linearized the boundary-value problem for gravimetric determination of the geoid in gravity space, the problem is still non-linear (and also free) in geometry space since the geoid to be determined is a non-linear function of a sought anomalous potential. Hence, an additional linearization, this time in geometry space, is convenient to employ.

Only two geometrical approximations of the geoid in the boundary condition for anomalous gravity have been introduced so far, namely the approximation of the geoid by a sphere or by an ellipsoid of revolution. Both linearize (and also fix) the problem with respect to the geometrical shape of the geoid; they may thus be termed as the spherical or ellipsoidal approximation in geometry space. The spherical approximation of the geoid produces the relative error of the order of 3×10^{-3} (Heiskanen and Moritz 1967, Sect. 2.14) which may cause an absolute error reaching 0.5 m counted in geoidal heights. Approximating the figure of the geoid by a level ellipsoid in boundary condition for the anomalous gravity reduces the linearization error in geometry space since the geoid deviates from an ellipsoid of revolution much less than from a sphere. The ellipsoidal linearization may cause relative errors of the order of 1.5×10^{-5}; the absolute error in geoidal heights does not exceed 2 mm.

Taking into account different types of linearization in gravity and geometry space, the originally non-linear, free boundary-value problem for determination of gravimetric geoid may be approximated by various types of linearized, fixed boundary-value problems. Adopting the spherical approximation in both gravity and geometry spaces results in the well-known spherical Stokes problem, first formulated and solved by Stokes (1849).

The linearization process of the boundary-value problem for gravimetric geoid determination may follow another possibility: the ellipsoidal approximation in gravity space and the spherical approximation in geometry space. This leads to a boundary-value problem of Stokes's type with two ellipsoidal corrections involved in boundary condition. These additional terms couple the spherical harmonics of degree two with spectral content of sought anomalous potential. Due to this coupling effect, there is no exact analytical solution of the Stokes boundary-value problem when the ellipsoidal corrections are involved. An iterative solution to this problem is usually recommended in geodetic literature (e.g. Cruz 1985; Heck 1991). At the initial step, the ellipsoidal correction terms are computed from a priori known potential, e.g. global gravity model, and subtracted from the right-hand side of the boundary condition for anomalous gravity. Then the solution is looked for by successive iterations evaluating the ellipsoidal corrections from the preceding iterative step. However, the convergency of this iterative scheme has not been proved.

Instead of applying the iterative solution, in this paper we shall attempt to construct Green's function associated with the problem. As already discussed, an exact Green function cannot be found due to the coupling effect between spherical harmonics of degree two and spherical harmonics of anomalous potential. That is why Green's function will be constructed approximately retaining the terms of magnitudes up the order of Earth's flattening. Neglecting higher-order terms causes relative errors of the order of 1.5×10^{-5}; the absolute error in geoidal heights does not exceed 2 mm.

Two remarks should be made in this context. The construction of Green's function allows us to avoid the use of an iterative approach in solving Stokes's boundary-value problem with ellipsoidal corrections involved in boundary condition as proposed by Cruz (1985) or Heck (1991). The iterative scheme is questionable from a numerical point of view since the convergency of iterations has not yet been proved. Martinec and Matyska (1997) demonstrated that the iterative scheme fails when it is applied to solve the Stokes pseudo-boundary-value problem arising when the heights of the Earth's topography enter the boundary-value problem for gravimetric geoid determination instead of the geopotential numbers. In this case, the ellipsoidal correction terms can be computed neither from the previous iterative step nor from a known global gravitational model of the Earth.

There is even another possibility of how to linearize the originally non-linear, free boundary-value problem; namely to use the ellipsoidal approximation in both gravity and geometry spaces. This approach seems to be the most precise compared to other ways of linearization discussed, since the linearization errors are equal to 0.2 mgal and 2 mm in gravity and geometry spaces, respectively. However, this approximation does not remove the ellipsoidal correction terms in the boundary condition for anomalous gravity. They have different analytical forms compared to the formulae introduced in Eq. (2) and depend on the type of the projection of the Earth's surface onto the geoid; their analytical forms are derived in Heck (1991).

Martinec and Grafarend (1997) made a first step in solving this type of boundary-value problem. They did not consider the ellipsoidal correction terms in the boundary condition for anomalous gravity, which corresponds to linearization in gravity space by the reference gravity GM/u^2 (u is the first ellipsoidal coordinate), and constructed Green's function for this problem to within an accuracy of the order of $O(e_0^2)$. We hope to report about the progress in solving the ellipsoidal Stokes problem when the ellipsoidal corrections are also included in the boundary condition for anomalous gravity in the near future.

2 Formulation of the boundary-value problem

We will solve the boundary-value problem of Stokes's type with the ellipsoidal corrections involved in the boundary condition in the geocentric spherical coordinates (r, Ω), where Ω stands for the pair of angular coordinates, co-latitude ϑ and longitude λ. The potential $T(r, \Omega)$ to be determined *on and outside* the reference sphere $r = R$ is governed by the following boundary-value problem:

$$\nabla^2 T = 0 \quad \text{for } r > R \tag{1}$$

$$\frac{\partial T}{\partial r} + \frac{2}{r} T - \varepsilon_h(T) - \varepsilon_\gamma(T) = -f \quad \text{for } r = R \tag{2}$$

$$T = \frac{c}{r} + O\left(\frac{1}{r^3}\right) \quad \text{for } r \to \infty \tag{3}$$

where the so-called *ellipsoidal corrections* read

$$\varepsilon_h(T) = e_0^2 \sin\vartheta \cos\vartheta \frac{1}{r} \frac{\partial T}{\partial \vartheta} \tag{4}$$

$$\varepsilon_\gamma(T) = e_0^2 (3\cos^2\vartheta - 2) \frac{T}{r} \tag{5}$$

e_0 is the first eccentricity of the reference ellipsoid of revolution (Heiskanen and Moritz 1967, Sect. 2.10). The ellipsoidal correction $\varepsilon_h(T)$ occurs due to the difference between the derivative of the potential T with respect to the plumbline of the normal gravity field and the radial derivative of T, whereas the term $\varepsilon_\gamma(T)$ arises due to the upward continuation of the normal gravity field from the reference ellipsoid of revolution to the Earth's surface. These terms have been introduced by a couple of authors to approximate the originally non-linear boundary-value problem for geoid determination by means of the linearized problem given by Eqs. (1)–(3)

with the accuracy of the order of Earth's eccentricity. The reader is referred to Moritz (1980), Jekeli (1981) or Heck (1991) for more details.

We assume that $f(\Omega)$ is a known square-integrable function, i.e. $f(\Omega) \in L_2(\Omega)$, which can be obtained from measurements of the gravity field on the Earth's surface reduced by the attraction of the reference ellipsoid of revolution and the topographical masses. The first-degree harmonics of T have been removed from the solution in order to guarantee the uniqueness of the solution [c in Eq. (3) is a constant]. Furthermore, we can easily deduce that the solution to Eqs. (1)–(3) exists only if the first-degree spherical harmonics of $f(\Omega)$ are removed by the postulate

$$\int_{\Omega_0} f(\Omega) Y^*_{1m}(\Omega) \mathrm{d}\Omega = 0 \quad \text{for } m = -1, 0, 1 \tag{6}$$

where $Y_{1m}(\Omega)$ are spherical harmonics of first-degree and order m, Ω_0 is the full solid angle, $\mathrm{d}\Omega = \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\lambda$, and the asterisk stands for complex conjugation. Throughout this paper we will assume that conditions given by Eq. (6) are satisfied.

3 Spectral form of the solution

The solution of the Laplace equation (1) can be written in terms of spherical harmonics $Y_{jm}(\Omega)$ as follows

$$T(r,\Omega) = \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \left(\frac{R}{r}\right)^{j+1} \sum_{m=-j}^{j} T_{jm} Y_{jm}(\Omega) \tag{7}$$

where the T_{jm} are expansion coefficients to be determined from the boundary condition (Eq. 2). To satisfy the asymptotic condition given by Eq. (3), the term with angular degree $j = 1$ has been excluded from summation over j's.

Substituting Eq. (7) and Eq. (A8) for harmonic representation of the ellipsoidal corrections (see Appendix A) into the boundary condition in Eq. (2), we obtain

$$\begin{aligned} &\frac{1}{R} \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} (j-1) \sum_{m=-j}^{j} T_{jm} Y_{jm}(\Omega) + \frac{e_0^2}{R} \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \sum_{m=-j}^{j} \\ &\times \left\{ \frac{j+1}{2j-1} \sqrt{\frac{[(j-1)^2 - m^2][j^2 - m^2]}{(2j-3)(2j+1)}} T_{j-2,m} \right. \\ &\quad + \left[\frac{j(j+1) - 3m^2}{(2j-1)(2j+3)} - 1 \right] T_{jm} - \frac{j}{2j+3} \\ &\quad \left. \times \sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} T_{j+2,m} \right\} \\ &\times Y_{jm}(\Omega) = -f(\Omega) \end{aligned} \tag{8}$$

Moreover, expanding function $f(\Omega)$ in a series of spherical harmonics,

$$f(\Omega) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \int_{\Omega_0} f(\Omega') Y^*_{jm}(\Omega') \mathrm{d}\Omega' Y_{jm}(\Omega) \tag{9}$$

where the first-degree spherical harmonics of $f(\Omega)$ are equal to zero due to Eq. (6), substituting Eq. (9) into Eq. (8), and comparing the coefficients at spherical harmonics $Y_{jm}(\Omega)$, we end up with the infinite system of linear algebraic equations for coefficients T_{jm}:

$$e_0^2 a_{jm} T_{j-2,m} + (1 + e_0^2 b_{jm}) T_{jm} + e_0^2 c_{jm} T_{j+2,m} = d_{jm} \tag{10}$$

$j = 0, 2, \ldots, m = -j, -j+1, \ldots, j,$ where the system matrix elements are equal to

$$a_{jm} = \frac{j+1}{(j-1)(2j-1)} \sqrt{\frac{[(j-1)^2 - m^2][j^2 - m^2]}{(2j-3)(2j+1)}} \tag{11}$$

$$b_{jm} = \frac{3}{j-1} \left[\frac{(j+1)^2 - m^2}{(2j-1)(2j+3)} - \frac{j}{2j-1} \right] \tag{12}$$

$$c_{jm} = -\frac{j}{(j-1)(2j+3)} \sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} \tag{13}$$

and the right-hand side is given by surface integrals

$$d_{jm} = \frac{R}{j-1} \int_{\Omega_0} f(\Omega') Y^*_{jm}(\Omega') \mathrm{d}\Omega' \tag{14}$$

4 The $\mathbf{O(e_0^2)}$-approximation

From a practical point of view, the spectral form (Eq. 7) of the solution of the boundary-value problem in Eqs. (1)–(3) is often inconvenient, since the construction of the spectral components of $f(\Omega)$ and solving the system of Eq. (10) for some cut-off degree $j = j_{\max}$ may become time consuming. Moreover, in the case that the magnitudes of ellipsoidal correction terms $\varepsilon_h(T)$ and $\varepsilon_\gamma(T)$ in the boundary condition of Eq. (2) are much smaller then the Stokes term $\partial T/\partial r + 2T/r$, which is the case for the Earth, the solution of our problem should be close to the solution of the spherical Stokes problem. We will thus attempt to rewrite $T(r,\Omega)$ as a sum of the well-known Stokes integral plus the contribution due to the corrections $\varepsilon_h(T)$ and $\varepsilon_\gamma(T)$. An evident advantage of such a decomposition is that existing theories as well as numerical codes for geoid height computation can simply be corrected for the ellipsoidal correction terms.

To build up the theory to be as simple as possible but still matching the requirements on geoid height accuracy, we will keep throughout the following derivations the terms of magnitudes of the order of $O(e_0^2)$ and neglect terms of higher powers of e_0^2. This approximation is justifiable because the error introduced by this approximation is at most 1.5×10^{-5}, which then causes an error of at most a few millimeters in the geoidal heights.

Inspecting Eq. (10) we can see that this system is not coupled with respect to the order m and can thus be solved separately for an individual order m. Furthermore, the equations are also separated for even and odd j, and again can be solved separately for even and odd angular degrees j. In matrix notation, the Eq. (10) has a tridiagonal form:

$$\begin{pmatrix} 1+e_0^2 b_{mm} & e_0^2 c_{mm} & 0 & \cdots \\ e_0^2 a_{m+2,m} & 1+e_0^2 b_{m+2,m} & e_0^2 c_{m+2,m} & \cdots \\ 0 & e_0^2 a_{m+4,m} & 1+e_0^2 b_{m+4,m} & e_0^2 c_{m+4,m} \; \cdots \\ & & \cdots & \end{pmatrix} \times \begin{pmatrix} T_{mm} \\ T_{m+2,m} \\ T_{m+4,m} \\ \cdots \end{pmatrix} = \begin{pmatrix} d_{mm} \\ d_{m+2,m} \\ d_{m+4,m} \\ \cdots \end{pmatrix} \tag{15}$$

$m = 0, 2, 3, \ldots$. A simple analysis of Eqs. (11)–(13) reveals that the magnitudes of matrix elements can be estimated as ($j \neq 1$)

$$|a_{jm}| < 1, \quad |b_{jm}| \leq 1, \quad |c_{jm}| < 1 \tag{16}$$

Assuming that $e_0^2 \ll 1$, the sizes of the off-diagonal elements in the system of Eq. (15) are at least $1/e_0^2$-times smaller than those of the diagonal elements. As a consequence, the solution to the system of Eq. (15) always exists and is unique, again provided that $j \neq 1$ and $e_0^2 \ll 1$.

A special structure of the matrix of Eq. (15) allows us to solve the system approximately such that the accuracy of the order of e_0^2 is maintained in the solution. In Appendix B, we present such an approach resulting in Eq. (B10). Applying this result to Eq. (15), the solution is

$$T_{jm} = d_{jm} - e_0^2 (b_{jm} d_{jm} + a_{jm} d_{j-2,m} + c_{jm} d_{j+2,m}) + O(e_0^4) \tag{17}$$

Inserting coefficients T_{jm} into Eq. (7) yields

$$\begin{aligned} T(r,\Omega) = & \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \left(\frac{R}{r}\right)^{j+1} \sum_{m=-j}^{j} d_{jm} Y_{jm}(\Omega) \\ & - e_0^2 \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \left(\frac{R}{r}\right)^{j+1} \sum_{m=-j}^{j} (b_{jm} d_{jm} + c_{jm} d_{j+2,m}) \\ & \times Y_{jm}(\Omega) - e_0^2 \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \left(\frac{R}{r}\right)^{j+3} \\ & \times \sum_{m=-j}^{j} a_{j+2,m} d_{jm} Y_{j+2,m}(\Omega) + O(e_0^4) \end{aligned} \tag{18}$$

Substituting for coefficients d_{jm} from Eq. (14), interchanging the order of summation over j and m with integration over Ω' due to the uniform convergence of the series given by Eq. (18), the solution of the boundary-value problem in Eqs. (1)–(3) within the accuracy of the order of $O(e_0^2)$ reads

$$\begin{aligned} T(r,\Omega) = & \frac{R}{4\pi} \int_{\Omega_0} f(\Omega') \Bigg[4\pi \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \frac{1}{j-1} \left(\frac{R}{r}\right)^{j+1} \\ & \times \sum_{m=-j}^{j} Y_{jm}(\Omega) Y^*_{jm}(\Omega') \\ & - e_0^2 \big(S_{00}^{\text{elco}}(r,\Omega,R,\Omega') \\ & + S^{\text{elco}}(r,\Omega,R,\Omega') \big) \Bigg] \mathrm{d}\Omega' \end{aligned} \tag{19}$$

where

$$\begin{aligned} S_{00}^{\text{elco}}(r,\Omega,R,\Omega') = 4\pi \Bigg[& -\frac{R}{r} b_{00} Y_{00}(\Omega) Y^*_{00}(\Omega') \\ & + \frac{R}{r} c_{00} Y_{00}(\Omega) Y^*_{20}(\Omega') \\ & - \left(\frac{R}{r}\right)^3 a_{20} Y_{20}(\Omega) Y^*_{00}(\Omega') \Bigg] \end{aligned} \tag{20}$$

and

$$\begin{aligned} S^{\text{elco}}(r,\Omega,R,\Omega') = & 4\pi \sum_{j=2}^{\infty} \left(\frac{R}{r}\right)^{j+1} \\ & \times \sum_{m=-j}^{j} \Bigg[\frac{b_{jm}}{j-1} Y_{jm}(\Omega) Y^*_{jm}(\Omega') \\ & + \frac{c_{jm}}{j+1} Y_{jm}(\Omega) Y^*_{j+2,m}(\Omega') \\ & + \left(\frac{R}{r}\right)^2 \frac{a_{j+2,m}}{j-1} Y_{j+2,m}(\Omega) Y^*_{jm}(\Omega') \Bigg] \end{aligned} \tag{21}$$

Using the Laplace addition theorem for spherical harmonics,

$$P_j(\cos\psi) = \frac{4\pi}{2j+1} \sum_{m=-j}^{j} Y_{jm}(\Omega) Y^*_{jm}(\Omega') \tag{22}$$

where $P_j(\cos\psi)$ is the Legendre polynomial of degree j, and ψ is the angular distance between directions Ω and Ω', the first term in the square brackets in Eq. (19) is equal to the *inhomogeneous spherical Stokes function* $S(r,\psi,R)$ (Moritz 1980, p. 367),

$$\begin{aligned} S(r,\psi,R) = & \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \frac{2j+1}{j-1} \left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) \\ = & \frac{2R}{\ell} - \frac{3R^2}{r^2} \cos\psi \ln\left(\frac{\ell + r - R\cos\psi}{2r}\right) \\ & - \frac{3R\ell}{r^2} - \frac{5R^2}{r^2} \cos\psi \end{aligned} \tag{23}$$

where

$$\ell = \sqrt{r^2 + R^2 - 2rR\cos\psi} \tag{24}$$

Furthermore, substituting for $b_{00} = 1$, $c_{00} = 0$, $a_{20} = \sqrt{4/5}$, $Y_{00}(\Omega) = 1/\sqrt{4\pi}$, and $Y_{20}(\Omega) = \sqrt{5/\pi}(3\cos^2\vartheta - 1)/4$ into Eq. (20), the zero-degree kernel $S_{00}^{\mathrm{elco}}(r, \Omega, R, \Omega')$ becomes

$$S_{00}^{\mathrm{elco}}(r, \Omega, R, \Omega') = -\frac{R}{r} - \left(\frac{R}{r}\right)^3 (3\cos^2\vartheta - 1) \tag{25}$$

Finally, the solution given by Eq. (19) of the boundary-value problem in Eqs. (1)–(3) can be written as

$$\begin{aligned} T(r, \Omega) = {} & \frac{R}{4\pi}\int_{\Omega_0} f(\Omega')S(r, \psi, R)\mathrm{d}\Omega' \\ & + e_0^2\left[\frac{R}{r} + \left(\frac{R}{r}\right)^3 (3\cos^2\vartheta - 1)\right]\frac{R}{4\pi}\int_{\Omega_0} f(\Omega')\mathrm{d}\Omega' \\ & - e_0^2\frac{R}{4\pi}\int_{\Omega_0} f(\Omega')S^{\mathrm{elco}}(r, \Omega, R, \Omega')\mathrm{d}\Omega' \end{aligned} \tag{26}$$

In particular, we are interested in finding the potential $T(r, \Omega)$ on the reference sphere $r = R$, i.e. function $T(R, \Omega)$. In this case, the general Eq. (26) reduces to

$$\begin{aligned} T(R, \Omega) = {} & \frac{R}{4\pi}\int_{\Omega_0} f(\Omega')S(\psi)\mathrm{d}\Omega' \\ & + 3e_0^2\cos^2\vartheta\frac{R}{4\pi}\int_{\Omega_0} f(\Omega')\mathrm{d}\Omega' \\ & - e_0^2\frac{R}{4\pi}\int_{\Omega_0} f(\Omega')S^{\mathrm{elco}}(\Omega, \Omega')\mathrm{d}\Omega' \end{aligned} \tag{27}$$

where

$$\begin{aligned} S(\psi) := S(R, \psi, R) = {} & \frac{1}{\sin\frac{\psi}{2}} \\ & - 3\cos\psi\,\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \\ & - 6\sin\frac{\psi}{2} - 5\cos\psi \end{aligned} \tag{28}$$

is the *homogeneous spherical Stokes function* (Heiskanen and Moritz 1967, Eq. 2.164), and

$$\begin{aligned} S^{\mathrm{elco}}(\Omega, \Omega') & := S^{\mathrm{elco}}(R, \Omega, R, \Omega') \\ & = 4\pi\sum_{j=2}^{\infty}\sum_{m=-j}^{j}\left[\frac{b_{jm}}{j-1}Y_{jm}(\Omega)Y_{jm}^*(\Omega') \right. \\ & \quad + \frac{c_{jm}}{j+1}Y_{jm}(\Omega)Y_{j+2,m}^*(\Omega') \\ & \quad \left. + \frac{a_{j+2,m}}{j-1}Y_{j+2,m}(\Omega)Y_{jm}^*(\Omega')\right] \end{aligned} \tag{29}$$

5 The 'spherical-ellipsoidal' Stokes function

We will call the function $S^{\mathrm{elco}}(\Omega, \Omega')$ the '*spherical-ellipsoidal*' *Stokes function* because it describes the effect of the ellipsoidal correction terms $\varepsilon_h(T)$ and $\varepsilon_\gamma(T)$ on the solution of spherical Stokes's boundary-value problem. The next effort will be devoted to convert the spectral form (Eq. 29) of $S^{\mathrm{elco}}(\Omega, \Omega')$ to a spatial representation.

Substituting for a_{jm}, b_{jm} and c_{jm} from Eqs. (11)–(13) into Eq. (29), we have

$$\begin{aligned} & S^{\mathrm{elco}}(\Omega, \Omega') \\ & = 4\pi\sum_{j=2}^{\infty}\sum_{m=-j}^{j}\left\{\frac{3}{(j-1)^2}\left[\frac{(j+1)^2 - m^2}{(2j-1)(2j+3)} - \frac{j}{2j-1}\right]\right. \\ & \qquad \times Y_{jm}(\Omega)Y_{jm}^*(\Omega') - \frac{j}{(j^2-1)(2j+3)} \\ & \qquad \times \sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} \\ & \qquad \times Y_{jm}(\Omega)Y_{j+2,m}^*(\Omega') + \frac{j+3}{(j^2-1)(2j+3)} \\ & \qquad \times \sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} \\ & \qquad \left. \times Y_{j+2,m}(\Omega)Y_{jm}^*(\Omega')\right\} \end{aligned} \tag{30}$$

One of the trickiest steps in finding the spatial representation of $S^{\mathrm{elco}}(\Omega, \Omega')$ is to sum up the series over m occurring in Eq. (30). Making use of the Laplace addition theorem for spherical harmonics, we present the approach for summing these series in Appendix C. Substituting from Eqs. (C22)–(C24) into Eq. (30), the 'spherical-ellipsoidal' Stokes function can be composed from seven different terms

$$S^{\mathrm{elco}}(\Omega, \Omega') = \sum_{i=1}^{7} h_i(\vartheta, \psi, \alpha)M_i(\cos\psi) \tag{31}$$

where ψ and α are polar coordinates, and

$$h_1(\vartheta, \psi, \alpha) = \sin^2\vartheta(\cos^2\alpha - \sin^2\alpha) \tag{32}$$

$$\begin{aligned} h_2(\vartheta, \psi, \alpha) = {} & \cos^2\vartheta\sin\psi - 2\sin\vartheta\cos\vartheta\cos\psi\cos\alpha \\ & - \sin^2\vartheta\sin\psi\cos^2\alpha \end{aligned} \tag{33}$$

$$\begin{aligned} h_3(\vartheta, \psi, \alpha) = {} & \cos^2\vartheta\cos\psi + 2\sin\vartheta\cos\vartheta\sin\psi\cos\alpha \\ & - \sin^2\vartheta\cos\psi\cos^2\alpha \end{aligned} \tag{34}$$

$$\begin{aligned} h_4(\vartheta, \psi, \alpha) = {} & \sin\vartheta(\cos\vartheta\sin\psi\cos\psi\cos\alpha \\ & - \sin\vartheta\cos^2\psi\cos^2\alpha + \sin\vartheta\sin^2\alpha) \end{aligned} \tag{35}$$

$$h_5(\vartheta, \psi, \alpha) = -\sin\vartheta\cos\alpha(\cos\vartheta\sin\psi - \sin\vartheta\cos\psi\cos\alpha) \tag{36}$$

$$h_6(\vartheta, \psi, \alpha) = 1 - \sin^2\vartheta\sin^2\alpha \tag{37}$$

$$h_7(\vartheta, \psi, \alpha) = -1 \tag{38}$$

The isotropic parts $M_i(\cos\psi), i = 1, \ldots, 7$, of $S^{\mathrm{elco}}(\Omega, \Omega')$ are given as infinite series of the Legendre polynomials and their derivatives:

$$M_1(\cos\psi) = \sum_{j=2}^{\infty} \frac{3}{(j^2-1)(2j+3)} \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \tag{39}$$

$$M_2(\cos\psi) = \sin\psi \sum_{j=2}^{\infty} \frac{2(j^2+3j+3)}{(j^2-1)(2j+3)} \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \tag{40}$$

$$M_3(\cos\psi) = \sum_{j=2}^{\infty} \frac{3(j+2)}{(j-1)(2j+3)} P_{j+1}(\cos\psi) \tag{41}$$

$$M_4(\cos\psi) = \sum_{j=2}^{\infty} \frac{3(2j+1)}{(j-1)^2(2j-1)(2j+3)} \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \tag{42}$$

$$M_5(\cos\psi) = \sum_{j=2}^{\infty} \frac{3(j+1)(2j+1)}{(j-1)^2(2j-1)(2j+3)} P_{j+1}(\cos\psi) \tag{43}$$

$$M_6(\cos\psi) = \sum_{j=2}^{\infty} \frac{3(j+1)^2(2j+1)}{(j-1)^2(2j-1)(2j+3)} P_j(\cos\psi) \tag{44}$$

$$M_7(\cos\psi) = \sum_{j=2}^{\infty} \frac{3j(2j+1)}{(j-1)^2(2j-1)} P_j(\cos\psi) \tag{45}$$

6 Spatial forms of functions $\mathbf{M_i}(\cos\psi)$

We now attempt to express infinite sums for $M_i(\cos\psi)$ as finite combinations of elementary functions depending on $\cos\psi$. Simple manipulations with Eqs. (39)–(45) result in fact that $M_i(\cos\psi)$ can be expressed in terms of sums

$$\sum_{j=3}^{\infty} \frac{P_j(\cos\psi)}{2j+1} \quad \text{and} \quad \sum_{j=3}^{\infty} \frac{P_j(\cos\psi)}{(j-1)^2} \tag{46}$$

The first sum is expressible in the full elliptic integrals (Pick et al. 1973, Appendix 18) which may only be evaluated approximately by some method of numerical quadrature (Press et al. 1989, Sect. 6.11). The second sum is equal to a definite integral the primitive function of which cannot be expressed in a closed analytical form (Pick et al. 1973, Appendix 18) but again only numerically. We can thus see that sums given by Eqs. (39)–(45) cannot be expressed in closed analytical forms. Therefore, our method of summation will be based on the following idea. Since the kernels $M_i(\cos\psi)$ are singular at the point $\psi = 0$, we will take out those contributions from Eqs. (39)–(45) which are responsible for the singular behaviour at the point $\psi = 0$. These contributions will be expressed in closed analytical forms. Having removed singular contributions, the remainders of sums will be represented by quickly convergent infinite series, which are bounded on the whole interval $0 \le \psi \le \pi$. Prescribing an error of computation, they can be simply summed up numerically.

As a preparatory step, we introduce a few formulae for sums of Legendre polynomials and their derivatives which will help us in the following manipulations. Pick et al. (1973, Eqs. D.18;1 and D.18;3) show that

$$\sum_{j=0}^{\infty} P_j(\cos\psi) = \frac{1}{2\sin\frac{\psi}{2}} \tag{47}$$

and

$$\sum_{j=1}^{\infty} \frac{P_j(\cos\psi)}{j} = -\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \tag{48}$$

Furthermore, Martinec and Grafarend (1997) demonstrate that

$$\sum_{j=1}^{\infty} \frac{4}{(2j-1)(2j+3)} \frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi} = \frac{1}{2\sin\frac{\psi}{2}} \tag{49}$$

$$\sum_{j=2}^{\infty} \frac{2(4j+1)}{(j^2-1)(2j-1)(2j+3)} \frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi} = -\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) + \frac{1}{10} \tag{50}$$

Let us start to sum up the infinite series of Eq. (39). Shifting the summation index to $j-1$, the fraction occurring in this series can be decomposed as

$$\frac{1}{(j-2)j(2j+1)} = \frac{4j+1}{2(j^2-1)(2j-1)(2j+3)} + \frac{3(6j^3-j^2-2j+2)}{2j(j-2)(j^2-1)(4j^2-1)(2j+3)} \tag{51}$$

By means of this last equation and Eq. (50), function $M_1(\cos\psi)$ reads

$$M_1(\cos\psi) = -\frac{3}{4}\left[\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) + \frac{6}{7}\cos\psi - \frac{1}{10}\right] + R_1(\cos\psi) \tag{52}$$

where

$$R_1(\cos\psi) = \frac{9}{2}\sum_{j=3}^{\infty} \frac{6j^3-j^2-2j+2}{j(j-2)(j^2-1)(4j^2-1)(2j+3)} \times \frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi} \tag{53}$$

Let us prove that the function $R_1(\cos\psi)$ is bounded for $\psi \in \langle 0, \pi \rangle$. For those ψs, the derivatives of Legendre polynomials can be estimated as

$$\left|\frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi}\right| \le j^2 \tag{54}$$

which yields

$$\begin{aligned}
|R_1(\cos\psi)| &\le \frac{9}{2}\sum_{j=3}^{\infty}\frac{j(6j^3-j^2-2j+2)}{(j-2)(j^2-1)(4j^2-1)(2j+3)} \\
&< \frac{9}{2}\sum_{j=3}^{\infty}\frac{6j^3-j^2-2j+2}{(j-2)(j-1)(4j^2-1)(2j+3)} \\
&< \frac{9}{2}\sum_{j=3}^{\infty}\frac{6j^3-j^2-2j+2+(j+19)}{(j-2)(j-1)(4j^2-1)(2j+3)} \\
&= \frac{9}{2}\sum_{j=3}^{\infty}\frac{3j^2-5j+7}{(j-2)(j-1)(4j^2-1)} \\
&< \frac{9}{8}\sum_{j=3}^{\infty}\frac{12j^2-20j+28+(20j-31)}{(j-2)(j-1)(4j^2-1)} \\
&= \frac{27}{8}\sum_{j=3}^{\infty}\frac{1}{(j-2)(j-1)} = \frac{27}{8}
\end{aligned} \tag{55}$$

where we have used (Mangulis 1965, p. 53)

$$\sum_{j=1}^{\infty}\frac{1}{j(j+1)} = 1 \tag{56}$$

It should be pointed out that the last estimate is rather weak; Fig. 2 will show that $|R_1(\cos\psi)| < 1.2$.

Nevertheless, it is important that the function $R_1(\cos\psi)$ is bounded at any point in the interval $0 \le \psi \le \pi$, so that the singularity of function $M_1(\cos\psi)$ at the point $\psi = 0$ is expressed analytically by the first term on the right-hand side of Eq. (52). We can see that $M_1(\cos\psi)$ has a logarithmic singularity at the point $\psi = 0$. In addition, function $R_1(\cos\psi)$ can simply be evaluated numerically since it is represented by a quickly convergent series. Figure 1 demonstrates this fact in a transparent way; for $\psi = 0$ it plots the decay of magnitudes of series terms in Eq. (53) with increasing degree j. Inspecting Fig. 1 we can estimate that it is sufficient to sum up the infinite series of Eq. (53) for $R_1(\cos\psi)$ up to $j \approx 25$ in order to achieve an absolute accuracy of the order of 0.01. This accuracy is sufficient for evaluating the 'spherical-ellipsoidal' Stokes function $S^{\mathrm{elco}}(\Omega,\Omega')$ in the frame of the $O(e_0^2)$-approximation.

The spatial forms of the other integral kernel $M_i(\cos\psi)$, $i = 2,\ldots,7$, can be expressed in a similar fashion as the kernel $M_1(\cos\psi)$; after some cumbersome but straightforward algebra we can arrive at

$$\begin{aligned}
M_2(\cos\psi) &= \frac{\cos\frac{\psi}{2}}{2\sin\frac{\psi}{2}} + \frac{\cos\frac{\psi}{2}(6+5\sin\frac{\psi}{2})}{2(1+\sin\frac{\psi}{2})} \\
&\quad - \frac{33}{8}\sin\psi\,\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right) \\
&\quad - \frac{207}{80}\sin\psi - \frac{181}{28}\sin\psi\cos\psi + R_2(\cos\psi)
\end{aligned}$$

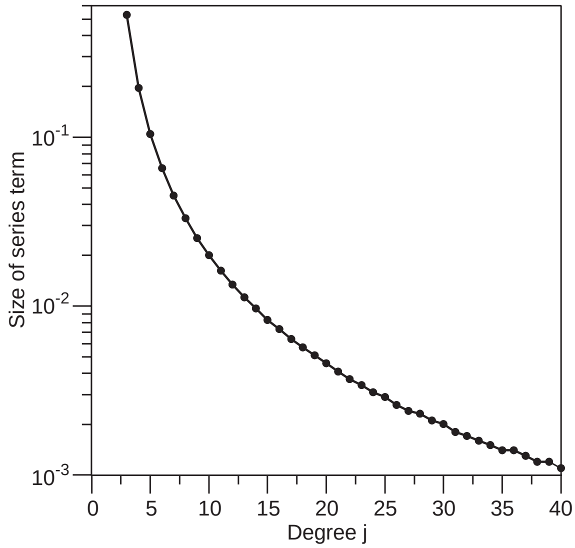


Fig. 1. The sizes of particular terms creating the infinite series for $R_1(1)$

$$\begin{aligned}
M_3(\cos\psi) &= -\frac{3}{4}(3\cos^2\psi-1)\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right) \\
&\quad -\frac{3}{2}(1+3\cos\psi)\sin\frac{\psi}{2}+\frac{9}{8}+\frac{3}{2}\cos\psi \\
&\quad -\frac{21}{8}\cos^2\psi + R_3(\cos\psi) \\
M_4(\cos\psi) &= -\frac{3}{4}\left[\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)+\frac{6}{7}\cos\psi-\frac{1}{10}\right] \\
&\quad + R_4(\cos\psi) \\
M_5(\cos\psi) &= R_5(\cos\psi) \\
M_6(\cos\psi) &= -\frac{3}{2}\left[\cos\psi\,\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)\right. \\
&\quad \left. +2\sin\frac{\psi}{2}+\cos\psi-1\right] + R_6(\cos\psi) \\
M_7(\cos\psi) &= -3\left[\cos\psi\,\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)\right. \\
&\quad \left. +2\sin\frac{\psi}{2}+\cos\psi-1\right] + R_7(\cos\psi)
\end{aligned} \tag{57}$$

where the residuals $R_i(\cos\psi)$ are of the forms

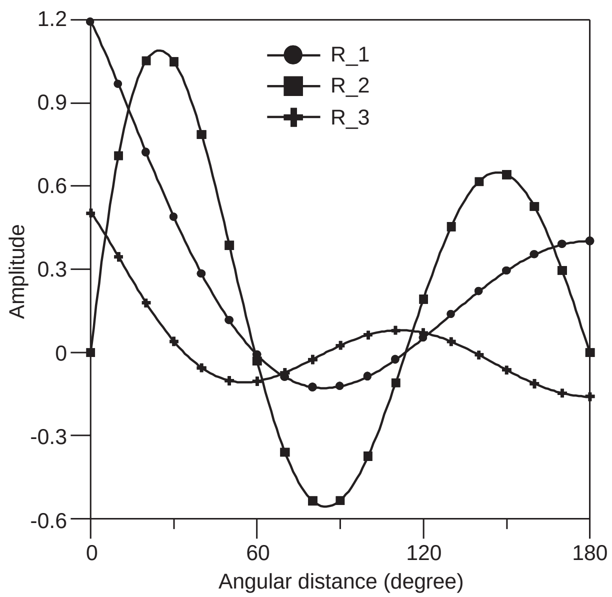
$$\begin{aligned}
R_2(\cos\psi) &= \frac{3}{4}\sin\psi\sum_{j=3}^{\infty}\frac{138j^3+17j^2-6j+16}{(j-2)j(j^2-1)(4j^2-1)(2j+3)} \\
&\quad \times\frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi} \\
R_3(\cos\psi) &= \frac{3}{2}\sum_{j=3}^{\infty}\frac{1}{(j-2)(2j+1)}P_j(\cos\psi) \\
R_4(\cos\psi) &= \frac{3}{2}\sum_{j=3}^{\infty}\frac{84j^4-132j^3-15j^2+72j+6}{(j-2)^2(j^2-1)(4j^2-1)(4j^2-9)} \\
&\quad \times\frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi}
\end{aligned}$$

$$R_5(\cos\psi) = 3\sum_{j=3}^{\infty}\frac{j(2j-1)}{(j-2)^2(2j-3)(2j+1)}P_j(\cos\psi)$$

$$R_6(\cos\psi) = \frac{3}{2}\sum_{j=2}^{\infty}\frac{10j^2+15j-1}{(j-1)^2(2j-1)(2j+3)}P_j(\cos\psi)$$

$$R_7(\cos\psi) = 3\sum_{j=2}^{\infty}\frac{4j-1}{(j-1)^2(2j-1)}P_j(\cos\psi) \tag{58}$$

Figure 2 plots the residuals $R_i(\cos\psi)$, $i = 1, \ldots, 7$, within the interval $0 \le \psi \le \pi$. We can observe that $R_i(\cos\psi)$ are 'sufficiently' smooth functions bounded for all angles ψ. This is a consequence of the fact that the magnitudes of series terms in Eq. (58) quickly decrease with increasing summation index j. In order to achieve an absolute accuracy of the order of 0.01, which is sufficient in the framework of $O(e_0^2)$-approximation, Eq. (58) may be truncated at degree of about $j \approx 25$.



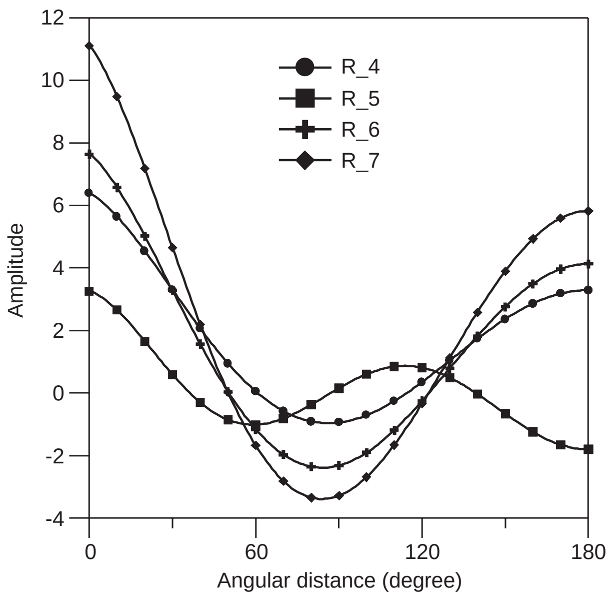


Fig. 2. Functions $R_i(\cos\psi), i = 1, \ldots, 7$, for $\psi \in <0, \pi>$

The preceding formulae make it possible to study the behaviour of functions $M_i(\cos\psi)$ in the vicinity of the point $\psi = 0$. We can readily see that

$$\lim_{\psi\to 0} M_2(\cos\psi) \sim \frac{1}{\psi} \tag{59}$$

$$\lim_{\psi\to 0} M_i(\cos\psi) \sim \ln(\psi/2) \quad \text{if } i = 1,3,4,6,7 \tag{60}$$

$$\left|\lim_{\psi\to 0} M_5(\cos\psi)\right| < A \quad (A \text{ is finite}) \tag{61}$$

Moreover, taking Eq. (C18) into account, we can see that

$$2\sin\vartheta\cos\vartheta\sin\psi\cos\psi\cos\alpha|_{\vartheta=\vartheta'} = \sin^2\psi(\cos^2\vartheta - \sin^2\vartheta\cos^2\alpha) \tag{62}$$

By this, the function $h_2(\vartheta, \psi, \alpha)$ defined by Eq. (33) behaves like ψ when $\psi \to 0$. Consequently, 'spherical-ellipsoidal' Stokes function $S^{\text{elco}}(\Omega, \Omega')$ has a logarithmic singularity at the point $\psi = 0$, i.e. $S^{\text{elco}}(\Omega, \Omega') \sim \ln(\psi/2)$ for $\psi \to 0$. Using Eqs. (C25) and (C26), we can get the proportional factor of this singularity:

$$S^{\text{elco}}(\Omega, \Omega') \sim 3\sin^2\vartheta \ln(\psi/2) \quad \text{for } \psi \to 0 \tag{63}$$

We can conclude that the singularity of $S^{\text{elco}}(\Omega, \Omega')$ at the point $\psi = 0$ is weaker than that of the spherical Stokes function.

7 Conclusion

This work was motivated by the question of whether the solution to Stokes's boundary-value problem with ellipsoidal corrections involved in boundary condition for anomalous gravity can be expressed in a closed spatial form which would be convenient for numerical computations. The objective behind this effort is to avoid applying the iterative approach usually recommended for solving this type of geodetic boundary-value problem. To answer this question, we have first constructed Green's function in terms of spherical harmonics. The spherical harmonic coefficients of the sought potential satisfy an infinite system of linear algebraic equations with a tridiagonal system matrix. We have confined ourselves to this system and found its solution with an accuracy of the order of the Earth's eccentricity. This is an acceptable approximation since it is consistent with the linearization errors hidden behind the formulated boundary-value problem. Moreover, this accuracy is still fairly good for today's requirements concerning geoid height computations. Within this accuracy, we have shown that the solution can be written as a surface integral taken over the full solid angle with integration kernel consisting of the traditional spherical Stokes function and the correction due to the appearance of the ellipsoidal corrections in the boundary conditions for anomalous gravity. We have managed to express the correcting integration kernel, originally represented in the form of an infinite spherical harmonics, as a finite combination of elementary analytical functions exactly representing the behaviour of integration kernel at the vicinity of its singular point $\psi = 0$. The important conclusion is that the correcting integration kernel has only a logarithmic singularity at $\psi = 0$ that is weaker than the singularity of the spherical Stokes function.

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Appendix A
Spectral form of ellipsoidal corrections

In this section we aim to derive the spectral representation of ellipsoidal correction terms, $\varepsilon_h(T)$ and $\varepsilon_\gamma(T)$, defined by Eqs. (4) and (5). Substituting the spherical harmonic expansion given by Eq. (7) of the disturbing potential T into these equations, taking the derivative of T with respect to ϑ for the term $\varepsilon_h(T)$, and putting $r = R$, we get

$$\varepsilon_h(T) = \frac{e_0^2}{R} \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \sum_{m=-j}^{j} T_{jm} \sin\vartheta \cos\vartheta \frac{\partial Y_{jm}(\Omega)}{\partial\vartheta} \tag{A1}$$

$$\varepsilon_\gamma(T) = \frac{e_0^2}{R} \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \sum_{m=-j}^{j} T_{jm} (3\cos^2\vartheta - 2) Y_{jm}(\Omega) \tag{A2}$$

To express the product of derivative $\partial Y_{jm}(\Omega)/\partial\vartheta$ with $\sin\vartheta\cos\vartheta$ in terms of $Y_{jm}(\Omega)$, we use the recurrence formula for the first derivative of spherical harmonics (Varshalovich et al. 1989, p. 147, Eq. 6):

$$\sin\vartheta\cos\vartheta \frac{\partial Y_{jm}(\Omega)}{\partial\vartheta} = j\sqrt{\frac{(j+1)^2 - m^2}{(2j+1)(2j+3)}} \cos\vartheta\, Y_{j+1,m}(\Omega) - (j+1)\sqrt{\frac{j^2 - m^2}{(2j-1)(2j+1)}} \times \cos\vartheta\, Y_{j-1,m}(\Omega) \tag{A3}$$

By recurrence formulae Varshalovich et al. 1989, p.145, Eq. (2),

$$\cos\vartheta\, Y_{j+1,m}(\Omega) = \sqrt{\frac{(j+2)^2 - m^2}{(2j+3)(2j+5)}} Y_{j+2,m}(\Omega) + \sqrt{\frac{(j+1)^2 - m^2}{(2j+1)(2j+3)}} Y_{jm}(\Omega) \tag{A4}$$

and

$$\cos\vartheta\, Y_{j-1,m}(\Omega) = \sqrt{\frac{(j-1)^2 - m^2}{(2j-3)(2j-1)}} Y_{j-2,m}(\Omega) + \sqrt{\frac{j^2 - m^2}{(2j-1)(2j+1)}} Y_{jm}(\Omega) \tag{A5}$$

we obtain

$$\sin\vartheta\cos\vartheta \frac{\partial Y_{jm}(\Omega)}{\partial\vartheta} = -\frac{j+1}{2j-1}\sqrt{\frac{[(j-1)^2 - m^2][j^2 - m^2]}{(2j-3)(2j+1)}} Y_{j-2,m}(\Omega) + \frac{3m^2 - j(j+1)}{(2j-1)(2j+3)} Y_{jm}(\Omega) + \frac{j}{2j+3} \times \sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} Y_{j+2,m}(\Omega) \tag{A6}$$

To make a similar arrangements for the term $\varepsilon_\gamma(T)$, we make use of the recurrence formula for spherical harmonics (Varshalovich et al. 1989, p.145, Eq. 5) and get

$$(3\cos^2\vartheta - 2) Y_{jm}(\Omega) = \frac{3}{2j-1}\sqrt{\frac{[(j-1)^2 - m^2][j^2 - m^2]}{(2j-3)(2j+1)}} Y_{j-2,m}(\Omega) - \left[\frac{2[3m^2 - j(j+1)]}{(2j-1)(2j+3)} + 1\right] Y_{jm}(\Omega) + \frac{3}{2j+3}\sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} \times Y_{j+2,m}(\Omega) \tag{A7}$$

Finally, substitution of Eqs. (A6) and (A7) into Eqs. (A1) and (A2) results in spherical harmonic representation of the sum of ellipsoidal correction terms,

$$\varepsilon_h(T) + \varepsilon_\gamma(T) = \frac{e_0^2}{R} \sum_{\substack{j=0 \\ j\neq 1}}^{\infty} \sum_{m=-j}^{j} \left\{ \frac{j+1}{2j-1} \times \sqrt{\frac{[(j-1)^2 - m^2][j^2 - m^2]}{(2j-3)(2j+1)}} T_{j-2,m} + \left[\frac{j(j+1) - 3m^2}{(2j-1)(2j+3)} - 1\right] T_{jm} - \frac{j}{2j+3} \times \sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} \times T_{j+2,m} \right\} Y_{jm}(\Omega) \tag{A8}$$

Appendix B
The approximate solution of tridiagonal system

The set of equations to be solved is

$$\begin{pmatrix} 1+b_1 & c_1 & 0 & \dots & & & \\ a_2 & 1+b_2 & c_2 & \dots & & & \\ & & & \dots & & & \\ & & & \dots & a_{n-1} & 1+b_{n-1} & c_{n-1} \\ & & & \dots & 0 & a_n & 1+b_n \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \dots \\ r_{n-1} \\ r_n \end{pmatrix} \tag{B1}$$

The solution of this system of equations with a tridiagonal matrix can be carried out by the following recurrence formulae (Isaacson and Keller 1973, Sect. 2.3.2). For $j = 2, \dots, n$, let us generate the quantities:

$$\gamma_j = \frac{c_{j-1}}{\beta_{j-1}} \tag{B2}$$

$$\beta_j = 1 + b_j - a_j\gamma_j \tag{B3}$$

$$\alpha_j = \frac{r_j - a_j\alpha_{j-1}}{\beta_j} \tag{B4}$$

with starting values

$$\beta_1 = 1 + b_1, \qquad \alpha_1 = \frac{r_1}{\beta_1} \tag{B5}$$

Having created α_j, β_j and γ_j, the solution of tridiagonal system of Eq. (B1) is

$$u_n = \alpha_n \tag{B6}$$

$$u_j = \alpha_j - \gamma_{j+1}u_{j+1} \quad \text{for } j = n-1, \dots, 1 \tag{B7}$$

Now, let us suppose that the elements a_j, b_j and c_j are small quantities such that $|a_j| \le \varepsilon$, $|b_j| \le \varepsilon$, $|c_j| \le \varepsilon$, and $\varepsilon \ll 1$. To get an approximate solution of Eq. (B1) with the accuracy of the order of ε, we can approximately put

$$\beta_j \doteq 1 + b_j + O(\varepsilon^2) \tag{B8}$$

since $|\gamma_j| \le \varepsilon$. Within the same accuracy, Eq. (B4) can be replaced by

$$\alpha_j \doteq r_j - b_j r_j - a_j r_{j-1} + O(\varepsilon^2) \tag{B9}$$

Substituting these formulae to Eqs. (B6) and (B7) leads to the approximate solution of Eq. (B1)

$$u_j \doteq r_j - b_j r_j - a_j r_{j-1} - c_j r_{j+1} + O(\varepsilon^2), \quad j = 1, 2, \dots, n \tag{B10}$$

assuming that $c_n = 0$.

Appendix C
Different forms of the addition theorem for spherical harmonics

In this section, we shall derive different forms of addition theorem for spherical harmonics. Let us start with the recurrence relation for the spherical harmonics (Varshalovich et al. 1989, p.147, Eq. 6)

$$\sqrt{\frac{2j+3}{2j+1}[(j+1)^2 - m^2]}\, Y_{jm}(\Omega) = \sin^2\vartheta \frac{\partial Y_{j+1,m}(\Omega)}{\partial \cos\vartheta} + (j+1)\cos\vartheta\, Y_{j+1,m}(\Omega) \tag{C1}$$

This recurrence formula can be rewritten in a different form:

$$\sqrt{\frac{2j+3}{2j+5}[(j+2)^2 - m^2]}\, Y_{j+2,m}(\Omega) = -\sin^2\vartheta \frac{\partial Y_{j+1,m}(\Omega)}{\partial \cos\vartheta} + (j+2)\cos\vartheta\, Y_{j+1,m}(\Omega) \tag{C2}$$

Multiplying Eq. (C1) by complex conjugate relation with Eq. (C2) taken at a point Ω', we get

$$\begin{aligned} &(2j+3)\sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}}\, Y_{jm}(\Omega)\, Y^*_{j+2,m}(\Omega') \\ &= -\sin^2\vartheta\sin^2\vartheta' \frac{\partial Y_{j+1,m}(\Omega)}{\partial\cos\vartheta}\frac{\partial Y^*_{j+1,m}(\Omega')}{\partial\cos\vartheta'} \\ &- (j+1)\cos\vartheta\sin^2\vartheta'\, Y_{j+1,m}(\Omega)\frac{\partial Y^*_{j+1,m}(\Omega')}{\partial\cos\vartheta'} \\ &+ (j+2)\sin^2\vartheta\cos\vartheta'\, Y^*_{j+1,m}(\Omega')\frac{\partial Y_{j+1,m}(\Omega)}{\partial\cos\vartheta} \\ &+ (j+1)(j+2)\cos\vartheta\cos\vartheta'\, Y_{j+1,m}(\Omega)\, Y^*_{j+1,m}(\Omega') \end{aligned} \tag{C3}$$

Summing Eq. (C3) from $m = -j-1$ up to $m = j+1$, and realizing that the factor $(j+1)^2 - m^2$ is equal to zero for $m = \pm(j+1)$, we get

$$\begin{aligned} &(2j+3)\sum_{m=-j}^{j}\sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} \times Y_{jm}(\Omega)\, Y^*_{j+2,m}(\Omega') \\ &= -\sin^2\vartheta\sin^2\vartheta' \sum_{m=-(j+1)}^{j+1} \frac{\partial Y_{j+1,m}(\Omega)}{\partial\cos\vartheta}\frac{\partial Y^*_{j+1,m}(\Omega')}{\partial\cos\vartheta'} \\ &\quad - (j+1)\cos\vartheta\sin^2\vartheta' \sum_{m=-(j+1)}^{j+1} Y_{j+1,m}(\Omega)\frac{\partial Y^*_{j+1,m}(\Omega')}{\partial\cos\vartheta'} \\ &\quad + (j+2)\sin^2\vartheta\cos\vartheta' \sum_{m=-(j+1)}^{j+1} Y^*_{j+1,m}(\Omega')\frac{\partial Y_{j+1,m}(\Omega)}{\partial\cos\vartheta} \\ &\quad + (j+1)(j+2)\cos\vartheta\cos\vartheta' \times \sum_{m=-(j+1)}^{j+1} Y_{j+1,m}(\Omega)\, Y^*_{j+1,m}(\Omega') \end{aligned} \tag{C4}$$

The sums of the products of spherical harmonics and their derivatives will be simplified by means of the Laplace addition theorem for spherical harmonics. Taking the Laplace addition theorem (Eq. 22) for index $j+1$, we have

$$\frac{2j+3}{4\pi} P_{j+1}(\cos\psi) = \sum_{m=-(j+1)}^{j+1} Y_{j+1,m}(\Omega) Y^*_{j+1,m}(\Omega') \tag{C5}$$

Differentiating Eq. (C5) with respect to $\cos\vartheta$ and $\cos\vartheta'$, and substituting the results to Eq. (C4), we get

$$\begin{aligned} 4\pi \sum_{m=-j}^{j} &\sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} Y_{jm}(\Omega) Y^*_{j+2,m}(\Omega') \\ &= -\sin^2\vartheta \sin^2\vartheta' \frac{\partial^2 P_{j+1}(\cos\psi)}{\partial\cos\vartheta\,\partial\cos\vartheta'} \\ &\quad - (j+1)\cos\vartheta \sin^2\vartheta' \frac{\partial P_{j+1}(\cos\psi)}{\partial\cos\vartheta'} \\ &\quad + (j+2)\sin^2\vartheta \cos\vartheta' \frac{\partial P_{j+1}(\cos\psi)}{\partial\cos\vartheta} \\ &\quad + (j+1)(j+2)\cos\vartheta\cos\vartheta' P_{j+1}(\cos\psi) \end{aligned} \tag{C6}$$

The partial derivatives of Legendre polynomials $P_{j+1}(\cos\psi)$ with respect to $\cos\vartheta$ and $\cos\vartheta'$ will now be expressed in terms of the ordinary derivatives of the Legendre polynomials with respect to $\cos\psi$ by making use of the well-known formulae of spherical trigonometry (e.g. Heiskanen and Moritz 1967, Eq. 2.208),

$$\cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\lambda - \lambda') \tag{C7}$$

$$\sin\psi\cos\alpha = \sin\vartheta\cos\vartheta' - \cos\vartheta\sin\vartheta'\cos(\lambda - \lambda') \tag{C8}$$

$$\sin\psi\sin\alpha = -\sin\vartheta'\sin(\lambda - \lambda') \tag{C9}$$

where α is the azimuth between directions Ω and Ω'. Realizing that

$$\frac{\partial P_{j+1}(\cos\psi)}{\partial\cos\vartheta} = \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \frac{\partial\cos\psi}{\partial\cos\vartheta} \tag{C10}$$

and taking the derivative of Eq. (C7) with respect to $\cos\vartheta$, we have

$$\begin{aligned} \frac{\partial P_{j+1}(\cos\psi)}{\partial\cos\vartheta} &= [\cos\vartheta' - \cot\vartheta\sin\vartheta'\cos(\lambda - \lambda')] \\ &\quad \times \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \end{aligned} \tag{C11}$$

Similarly, we can write

$$\begin{aligned} \frac{\partial P_{j+1}(\cos\psi)}{\partial\cos\vartheta'} &= [\cos\vartheta - \cot\vartheta'\sin\vartheta\cos(\lambda - \lambda')] \\ &\quad \times \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \end{aligned} \tag{C12}$$

Furthermore, differentiating Eq. (C11) with respect to $\cos\vartheta'$, we obtain the second-order derivative occurring in Eq. (C6),

$$\begin{aligned} &\frac{\partial^2 P_{j+1}(\cos\psi)}{\partial\cos\vartheta\,\partial\cos\vartheta'} \\ &= [1 + \cot\vartheta\cot\vartheta'\cos(\lambda - \lambda')] \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \\ &\quad + [\cos\vartheta' - \cot\vartheta\sin\vartheta'\cos(\lambda - \lambda')] \\ &\quad \times [\cos\vartheta - \cot\vartheta'\sin\vartheta\cos(\lambda - \lambda')] \frac{\mathrm{d}^2 P_{j+1}(\cos\psi)}{\mathrm{d}(\cos\psi)^2} \end{aligned} \tag{C13}$$

Substituting Eqs. (C11)–(C13) into Eq. (C6), we have

$$\begin{aligned} 4\pi \sum_{m=-j}^{j} &\sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} Y_{jm}(\Omega) Y^*_{j+2,m}(\Omega') \\ &= -\sin\vartheta\sin\vartheta'[\sin\vartheta\cos\vartheta' - \cos\vartheta\sin\vartheta'\cos(\lambda - \lambda')] \\ &\quad \times [\sin\vartheta'\cos\vartheta - \cos\vartheta'\sin\vartheta\cos(\lambda - \lambda')] \\ &\quad \times \frac{\mathrm{d}^2 P_{j+1}(\cos\psi)}{\mathrm{d}(\cos\psi)^2} - \sin\vartheta\sin\vartheta'[\sin\vartheta\sin\vartheta' \\ &\quad + \cos\vartheta\cos\vartheta'\cos(\lambda - \lambda')] \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \\ &\quad - (j+1)\cos\vartheta\sin\vartheta'[\sin\vartheta'\cos\vartheta \\ &\quad - \cos\vartheta'\sin\vartheta\cos(\lambda - \lambda')] \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \\ &\quad + (j+2)\sin\vartheta\cos\vartheta'[\sin\vartheta\cos\vartheta' \\ &\quad - \cos\vartheta\sin\vartheta'\cos(\lambda - \lambda')] \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \\ &\quad + (j+1)(j+2)\cos\vartheta\cos\vartheta' P_{j+1}(\cos\psi) \end{aligned} \tag{C14}$$

The functions in square brackets standing in front of the derivatives of Legendre polynomials will be expressed by means of the angular distance ψ and the azimuth α between directions Ω and Ω'. Using Eqs. (C7)–(C9), we can, after some algebraic manipulation, obtain:

$$\begin{aligned} &\sin\vartheta'\cos\vartheta - \cos\vartheta'\sin\vartheta\cos(\lambda - \lambda') \\ &\quad = \frac{1}{\sin\vartheta'}(\cos\vartheta - \cos\vartheta'\cos\psi) \end{aligned} \tag{C15}$$

$$\begin{aligned} &\sin\vartheta\sin\vartheta' + \cos\vartheta\cos\vartheta'\cos(\lambda - \lambda') \\ &\quad = \frac{1}{\sin\vartheta'}(\sin\vartheta - \cos\vartheta'\sin\psi\cos\alpha) \end{aligned} \tag{C16}$$

Considering Eqs. (C8), (C15) and (C16) in Eq. (C14), we have

$$\begin{aligned} 4\pi \sum_{m=-j}^{j} &\sqrt{\frac{[(j+1)^2 - m^2][(j+2)^2 - m^2]}{(2j+1)(2j+5)}} Y_{jm}(\Omega) Y^*_{j+2,m}(\Omega') \\ &= -\sin\vartheta\sin\psi\cos\alpha(\cos\vartheta - \cos\vartheta'\cos\psi) \frac{\mathrm{d}^2 P_{j+1}(\cos\psi)}{\mathrm{d}(\cos\psi)^2} \\ &\quad - \sin\vartheta(\sin\vartheta - \cos\vartheta'\sin\psi\cos\alpha) \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \\ &\quad - (j+1)\cos\vartheta(\cos\vartheta - \cos\vartheta'\cos\psi) \frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \end{aligned}$$

$$+ (j+2)\sin\vartheta\cos\vartheta'\sin\psi\cos\alpha\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+1)(j+2)\cos\vartheta\cos\vartheta' P_{j+1}(\cos\psi) \qquad \text{(C17)}$$

Multiplying Eq. (C7) by $\cos\vartheta/\sin\vartheta$ and adding the result to Eq. (C8), we get

$$\cos\vartheta' = \cos\vartheta\cos\psi + \sin\vartheta\sin\psi\cos\alpha \qquad \text{(C18)}$$

Employing the last formula and Legendre's differential equation

$$\sin^2\psi\frac{\mathrm{d}^2P_{j+1}(\cos\psi)}{\mathrm{d}(\cos\psi)^2} - 2\cos\psi\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+1)(j+2)P_{j+1}(\cos\psi) = 0 \qquad \text{(C19)}$$

the sum of the first, the second and the last term on the right- hand side of Eq. (C17) reads

$$-\sin\vartheta\sin\psi\cos\alpha(\cos\vartheta - \cos\vartheta'\cos\psi)\frac{\mathrm{d}^2P_{j+1}(\cos\psi)}{\mathrm{d}(\cos\psi)^2}$$
$$-\sin\vartheta(\sin\vartheta - \cos\vartheta'\sin\psi\cos\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+1)(j+2)\cos\vartheta\cos\vartheta' P_{j+1}(\cos\psi)$$
$$= -\sin\vartheta(\sin\vartheta + \cos\vartheta\sin\psi\cos\psi\cos\alpha$$
$$-\sin\vartheta\cos^2\psi\cos^2\alpha - \sin\vartheta\cos^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+1)(j+2)(\cos^2\vartheta\cos\psi + 2\sin\vartheta\cos\vartheta\sin\psi\cos\alpha$$
$$-\sin^2\vartheta\cos\psi\cos^2\alpha)P_{j+1}(\cos\psi) \qquad \text{(C20)}$$

Moreover, by Eq. (C18), the sum of the third and fourth term on the right-hand side of Eq. (C17) becomes

$$-(j+1)\cos\vartheta(\cos\vartheta - \cos\vartheta'\cos\psi)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+2)\sin\vartheta\cos\vartheta'\sin\psi\cos\alpha\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$= -(j+1)\sin\psi(\cos^2\vartheta\sin\psi - 2\sin\vartheta\cos\vartheta\cos\psi\cos\alpha$$
$$-\sin^2\vartheta\sin\psi\cos^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ \sin\vartheta(\cos\vartheta\sin\psi\cos\psi\cos\alpha + \sin\vartheta\sin^2\psi\cos^2\alpha)$$
$$\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi} \qquad \text{(C21)}$$

Substituting Eqs. (C20) and (C21) into Eq. (C17), we finally have

$$4\pi\sum_{m=-j}^{j}\sqrt{\frac{[(j+1)^2-m^2][(j+2)^2-m^2]}{(2j+1)(2j+5)}}Y_{jm}(\Omega)Y^*_{j+2,m}(\Omega')$$
$$= \sin^2\vartheta(\cos^2\alpha - \sin^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$-(j+1)\sin\psi(\cos^2\vartheta\sin\psi - 2\sin\vartheta\cos\vartheta\cos\psi\cos\alpha$$
$$-\sin^2\vartheta\sin\psi\cos^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+1)(j+2)(\cos^2\vartheta\cos\psi + 2\sin\vartheta\cos\vartheta\sin\psi\cos\alpha$$
$$-\sin^2\vartheta\cos\psi\cos^2\alpha)P_{j+1}(\cos\psi) \qquad \text{(C22)}$$

In an analogous way, other forms of the addition theorem for spherical harmonics can be derived. We introduce two of them without demonstrating a detailed proof:

$$4\pi\sum_{m=-j}^{j}\sqrt{\frac{[(j+1)^2-m^2][(j+2)^2-m^2]}{(2j+1)(2j+5)}}$$
$$\times Y_{jm}(\Omega')Y^*_{j+2,m}(\Omega)$$
$$= \sin^2\vartheta(\cos^2\alpha - \sin^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+2)\sin\psi(\cos^2\vartheta\sin\psi - 2\sin\vartheta\cos\vartheta\cos\psi\cos\alpha$$
$$-\sin^2\vartheta\sin\psi\cos^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$+ (j+1)(j+2)(\cos^2\vartheta\cos\psi + 2\sin\vartheta\cos\vartheta\sin\psi\cos\alpha$$
$$-\sin^2\vartheta\cos\psi\cos^2\alpha)P_{j+1}(\cos\psi) \qquad \text{(C23)}$$

and

$$\frac{4\pi}{2j+1}\sum_{m=-j}^{j}[(j+1)^2-m^2]Y_{jm}(\Omega)Y^*_{jm}(\Omega')$$
$$= \sin\vartheta(\cos\vartheta\sin\psi\cos\psi\cos\alpha - \sin\vartheta\cos^2\psi\cos^2\alpha$$
$$+ \sin\vartheta\sin^2\alpha)\frac{\mathrm{d}P_{j+1}(\cos\psi)}{\mathrm{d}\cos\psi}$$
$$-(j+1)\sin\vartheta\cos\alpha(\cos\vartheta\sin\psi$$
$$-\sin\vartheta\cos\psi\cos\alpha)P_{j+1}(\cos\psi)$$
$$+ (j+1)^2(1-\sin^2\vartheta\sin^2\alpha)P_j(\cos\psi) \qquad \text{(C24)}$$

Particularly, when $\Omega = \Omega'$, we get

$$4\pi\sum_{m=-j}^{j}\sqrt{\frac{[(j+1)^2-m^2][(j+2)^2-m^2]}{(2j+1)(2j+5)}}Y_{jm}(\Omega)Y^*_{j+2,m}(\Omega)$$
$$= \tfrac{1}{2}(j+1)(j+2)(3\cos^2\vartheta - 1) \qquad \text{(C25)}$$

and

$$\frac{4\pi}{2j+1}\sum_{m=-j}^{j}[(j+1)^2-m^2]\,|Y_{jm}(\Omega)|^2$$
$$= (j+1)^2 - \tfrac{1}{2}j(j+1)\sin^2\vartheta \qquad \text{(C26)}$$

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