# THE FAR ZONE CONTRIBUTION IN SPHERICAL STOKES'S INTEGRATION

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**Abstract**

The solution of the geodetic boundary value problem requires the evaluation of the Stokes integral all over the Earth. Since the distribution of gravity observations on the surface of our planet is not homogenous, the terrestrial sphere is divided into two areas of integration, the near zone and the far zone. Point observations are used in the first zone and a geopotencial model in spectral form in the second. In this paper the mathematical formulation for evaluating the contribution of the far zone with Stokes's spherical kernel is shown, considering Mexican territory as the application area .

**Resumen**

La solución de problema geodésico de valor frontera requiere la evaluación de la integral de Stokes sobre toda la Tierra. Ya que la distribución de observaciones de gravedad sobre la superficie de nuestro planeta es irregular y no tiene un cubrimiento homogéneo, se divide a la esfera terrestre en dos áreas de integración, la zona cercana y la lejana, empleándose observaciones puntuales en la primera y un modelo geopotencial en la segunda. En este documento se muestra la formulación matemática para evaluar la contribución de la zona lejana en el kernel esférico de Stokes minimizando los coeficientes de truncación de Molodenskij, tomando como área de aplicación el territorio mexicano.

**Introduction**

Stokes's classical approach is based on the solution of the external boundary value problem for the disturbing potential T. The famous Stokes integral reads [Vaníček and Krakiwsky, 1986, eq. 22.16]:

$$T(\Omega) = \frac{R}{4\pi}\int_{\Omega'} \Delta g(\Omega) S(\psi) d\Omega' \tag{1}$$

where:

- $T(\Omega)$: is the disturbing potential at $\Omega$
- $\Delta g(\Omega)$: is the gravity anomaly at $\Omega$
- $S(\psi)$: is the spherical Stokes' function
- $\Omega$: is the pair of angular spherical coordinates

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ψ: is the angular distance between two points
R: is the mean Earth radius

The spherical Stokes kernel (function) is an isotropic and homogeneous function. This means that the function depends neither on direction nor on the position of the integration point. Its value is only a function of the spherical distance between the integration point and the dummy point.
The integration expressed in equation (1) must be carried out over the whole earth (sphere), and the approximate equality sign in this equation is because the expression is correct only to the order of $e^2$ (the square of eccentricity of the reference ellipsoid).
The Stokes kernel may be represented in spatial form [Heiskanen and Moritz, 1967, eq. 2-164] as:

$$S(\psi) = 1 + \frac{1}{\sin\frac{\psi}{2}} - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \quad (2)$$

or in spectral form [Vaníček and Krakiwsky, 1986, eq. 22.15] as:

$$S(\psi) = \sum_{j=2}^{\infty} \frac{2j+1}{j-1} P_j(\cos\psi) \quad (3)$$

where:

$P_j$: are the Legendre's functions

In Figure 1 the shape of the Stokes kernel is shown.

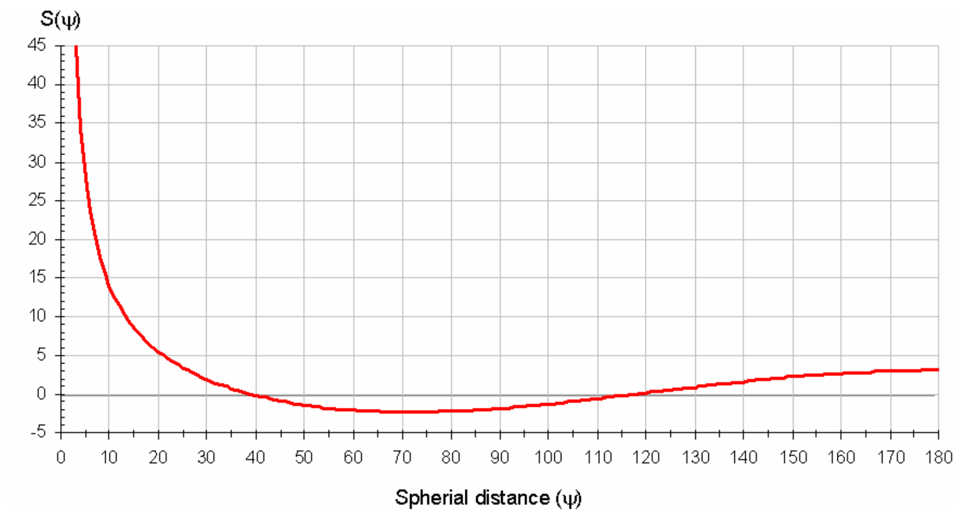

Figure 1. Stokes's function

The disturbing potential is related to the geoidal height through well known Bruns's formula [Vaníček and Krakiwsky, 1986, eq. 21.4]:

$$N(\Omega) = \frac{T(\Omega)}{\gamma_0(\Omega)} \quad (4)$$

where:

$N(\Omega)$: is the geoidal height at $\Omega$

$\gamma_0(\Omega)$: is the normal gravity at $(r, \Omega)$

From a rigorous point of view, all the above formulas and expressions must be expressed in the Helmert or some other harmonic space [see Martinec et al, 1933; Novak, 2000].

**Splitting the Integration Domain**

As pointed out above, Stokes's integral must be applied over the whole sphere. The incomplete global coverage of terrestrial gravity requires the use of a geopotential model in the determination of the geoid combined with regional gravity data, the integration domain is split into two integration zones, far and near. The input for the near zone are the observed terrestrial gravity data, while for the far zone, gravity derived from a geopotential model is used. This splitting is done to reduce the effect incomplete global coverage of terrestrial gravity data and it also reduces the impact of the spherical approximation inherent in the Stokes kernel [Heiskanen and Moritz, 1967, p. 97]. The latter is achieved because most of the geoid's power is contained in the lower frequencies.
The integral expressed in equation (1) could be written as [Vaníček and Janák, 2000]:

$$T(\Omega) = \frac{R}{4\pi}\int_C \Delta g(\Omega) S(\psi) d\Omega' + \frac{R}{4\pi}\int_{\Omega'-C} \Delta g(\Omega) S(\psi) d\Omega' \qquad (5)$$

where C denotes the near zone, a spherical cap of an arbitrary radius $\psi_0 \leq \pi$, , and the second integral on the right hand side of equation (5) represent the far zone contribution; it is sometimes called the "truncation error". In Figure 2 the graphical representation of the near and far zones is shown.

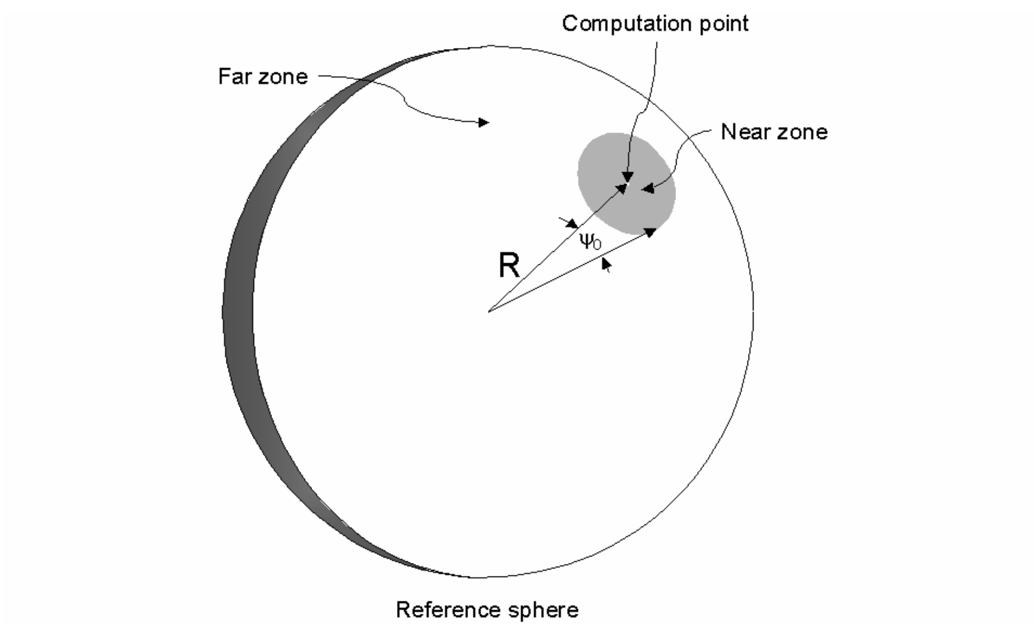


Figure 2. Near and far zones

We can now define a new kernel:

$$S(\psi)^{near} = \begin{cases} S(\psi), & \text{for } \psi \leq \psi_0 \\ 0, & \text{for } \psi_0 \leq \psi \leq \pi \end{cases} \qquad (6)$$

which can be expressed also in spectral form as:

$$S(\psi)^{near} = \sum_{j=2}^{\infty} \frac{2j+1}{2} s_j(\psi_0) P_j(\cos\psi) \tag{7}$$

where for all j=0, 1, 2, …: $\psi_0 \le \pi$ we get:

$$s_j(\psi_0) = \int_0^{\psi_0} S(\psi)^{near} P_j(\cos\psi)\sin\psi d\psi = \int_{\psi_0}^{\pi} S(\psi) P_j(\cos\psi)\sin\psi d\psi \tag{8}$$

and $s_n(\psi_0)$, called "Molodenskij's modification coefficients", must be determined.
Similarly, we can define a complementary kernel as:

$$S(\psi)^{far} = \begin{cases} 0, & \text{for } \psi \le \psi_0 \\ S(\psi), & \text{for } \psi_0 \le \psi \le \pi \end{cases} \tag{9}$$

which, in the same way as above, can be expressed also in spectral form:

$$S(\psi)^{far} = \sum_{j=2}^{\infty} \frac{2j+1}{2} q_j(\psi_0) P_j(\cos\psi) \tag{10}$$

where for all j=0, 1, 2, …: $\psi_0 \le \psi \le \pi$ we get:

$$q_j(\psi_0) = \int_0^{\pi} S(\psi)^{far} P_j(\cos\psi)\sin\psi d\psi = \int_{\psi_0}^{\pi} S(\psi) P_j(\cos\psi)\sin\psi d\psi \tag{11}$$

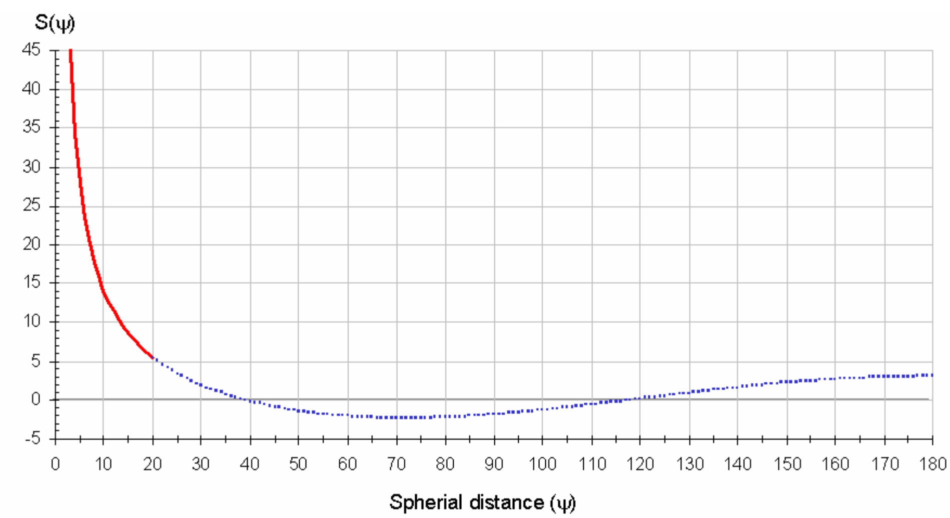

Figure 3.Splitting the integration domain

Both equations (6) and (9) are shown in Figure 3 for a spherical cap of radius equal to 20 arc-degrees. From Figure 3, we note among other things, that the division of the integration area Ω' into a spherical cap and the rest of the sphere, does not correspond to a separation of the two partial contributions: each partial integral must be expresed as a series containing all frecuencies [Vaníček and Janák, 2000] (we hope this paragraph will be more clear). If the radius of the spherical cap is small, then the kernel (9) has a great power and the far zone contribution could be very significant. For this reason, we really want to have a quite different kernel, that has all the power in the vicinity of the computation point and very little in the rest of the world

Equation (4) and equation (5) could be re-written as:

$$N(\Omega) = \frac{R}{4\pi\gamma_0(\Omega)} \int_C \Delta g(\Omega) S(\psi) d\Omega' + \frac{R}{4\pi\gamma_0(\Omega)} \int_{\Omega'-C} \Delta g(\Omega) S(\psi) d\Omega' \quad (12)$$

Or, in a compact form, as:

$$N(\Omega) = N(\Omega)^{near} + N(\Omega)^{far} \quad (13)$$

***Far zone contribution***

From equation (12) and (13) we have:

$$N(\Omega)^{far} = \frac{R}{4\pi\gamma_0(\Omega)} \int_{\Omega'-C} \Delta g(\Omega) S(\psi)^{far} d\Omega' \quad (14)$$

Expressing the Stokes's kernel in a polar coordinate system $(\psi,\alpha)$ on the unit sphere surface, the solid angle element $d\Omega'$ becomes:

$$d\Omega' = \sin\psi d\psi d\alpha \quad (15)$$

and the integral expressed in equation (4) changes to:

$$N(\Omega)^{far} = \frac{R}{4\pi\gamma_0(\Omega)} \int_{\alpha=0}^{2\pi} \int_{\psi=\psi_0}^{\pi} \Delta g(\Omega) S(\psi)^{far} \sin\psi d\psi d\alpha \quad (16)$$

So, substituting equation (10) in (16) we get:

$$N(\Omega)^{far} = \frac{R}{4\pi\gamma_0(\Omega)} \int_{\alpha=0}^{2\pi} \int_{\psi=\psi_0}^{\pi} \Delta g(\Omega) \sum_{j=2}^{\infty} \frac{2j+1}{2} q_j(\psi_0) P_j(\cos\psi) \sin\psi d\psi d\alpha \quad (17)$$

Because the series in (10) is absolutely convergent, it is possible to interchange the integral of the summation by the summation of the integrals:

$$N(\Omega)^{far} = \frac{R}{4\pi\gamma_0(\Omega)} \sum_{j=2}^{\infty} \frac{2j+1}{2} q_j(\psi_0) \int_{\alpha=0}^{2\pi} \int_{\psi=\psi_0}^{\pi} \Delta g(\Omega) P_j(\cos\psi) \sin\psi d\psi d\alpha \quad (18)$$

But, according to Heiskanen and Moritz [1969, eq. 1-71] we have:

$$\int_{\alpha=0}^{2\pi} \int_{\psi=\psi_0}^{\pi} \Delta g(\Omega) P_j(\cos\psi) \sin\psi d\psi d\alpha = \frac{4\pi\Delta g_j}{2j+1} \quad (19)$$

and (16) becomes:

$$N(\Omega)^{far} = \frac{R}{2\gamma_0(\Omega)} \sum_{j=2}^{\infty} q_j(\psi_0) \Delta g_j(\Omega) \tag{20}$$

where:

$\Delta g_j$: is the $n^{th}$-degree Laplace harmonic of $\Delta g$

***The coefficients*** $q_j(\psi_0)$

To obtain the coefficients $q_j(\psi_0)$ explicitly as functions of radius $\psi_0$, we must evaluate the integral (11). Introducing:

$$z = \sin\frac{\psi}{2} \tag{21}$$

$$\cos\psi = 1 - 2\sin^2\frac{\psi}{2} = 1 - 2z^2 \tag{22}$$

equation (11) can be written as [Heiskanen and Moritz, 1969, p.262]:

$$q_j(\psi_0) = 4\int_{z=t}^{1} S(1-2z^2)P_j(1-2z^2)zdz \tag{23}$$

where:

$$t = \sin\frac{\psi_0}{2}$$

$$\sin\psi d\psi = 4zdz$$

and the integral can thus be evaluated by any conventional method of integration.

Now, we are interested in keeping the contribution of the far zone as small as possible. This can be done by modifying the integration kernel. The purpose of keeping the influence of the far zone minimum is two-fold. First, the differences between available geopotential models [Wenzel, G.; 2000] are significant; second, to minimize the truncation error to ensure that the available global models are accurate enough, i.e, that they give essentially the same results with reasonable limits. Figure 4 shows the effect of coefficients $q_j(\psi_0)$ for the far zone contribution in spherical Stokes's kernel integration for a spherical cap equal to 5 degrees and 0 degree. Note that a cap of zero degrees is not a cap at all. It is a point. We can see that the convergence of the series to zero is very slow, and its truncation effect could be significant.

***Modified spherical Stokes's kernel***

Let us now introduce a modification of Stokes's kernel in the form of:

$$S^*(\psi) = S(\psi) - M(\psi) \tag{24}$$

where:

$$M(\psi) = \sum_{j=2}^{k} \frac{2j+1}{2} t_j P_j(\cos\psi)$$

(where the factors (2j+1)/2 are introduced for computational convenience [Vaníček and Sjöberg, 1991]). The original spherical Stokes's kernel is:

$$S(\psi) = S(\psi)^* + M(\psi) \tag{25}$$

Substituting equation (25) into (1) and splitting it into near zone and far zone contributions, we get:

$$T(\Omega) \doteq \frac{R}{4\pi} \int_{\psi_0} \Delta g[S^*(\psi) + M(\psi)] d\Omega' + \frac{R}{4\pi} \int_{\Omega-\psi_0} \Delta g[S^*(\psi) + M(\psi)] d\Omega' \tag{26}$$

$$\doteq \frac{R}{4\pi} \int_{\psi_0} \Delta g S^*(\psi) d\Omega' + \frac{R}{4\pi} \int_{\psi_0} \Delta g M(\psi) d\Omega' + \frac{R}{4\pi} \int_{\Omega-\psi_0} \Delta g S^*(\psi) d\Omega' + \frac{R}{4\pi} \int_{\Omega-\psi_0} \Delta g M(\psi) d\Omega'$$

$$\doteq \frac{R}{4\pi} \int_{\psi_0} \Delta g S^*(\psi) d\Omega' + \frac{R}{4\pi} \int_{\Omega-\psi_0} \Delta g S^*(\psi) d\Omega' + \frac{R}{4\pi} \int_{\Omega} \Delta g M(\psi) d\Omega' \tag{27}$$

In our approach, the third term on the right hand side equation (27), will be identically equal to zero because we use only the residual gravity anomalies. As these have only higher frequencies then k, the integral disappears due to global orthogonality of spherical harmonics [Heiskanen and Moritz, 1969]. Then, of course, we have to add the low order harmonics, $k \geq j$, in the form of the reference fiels.

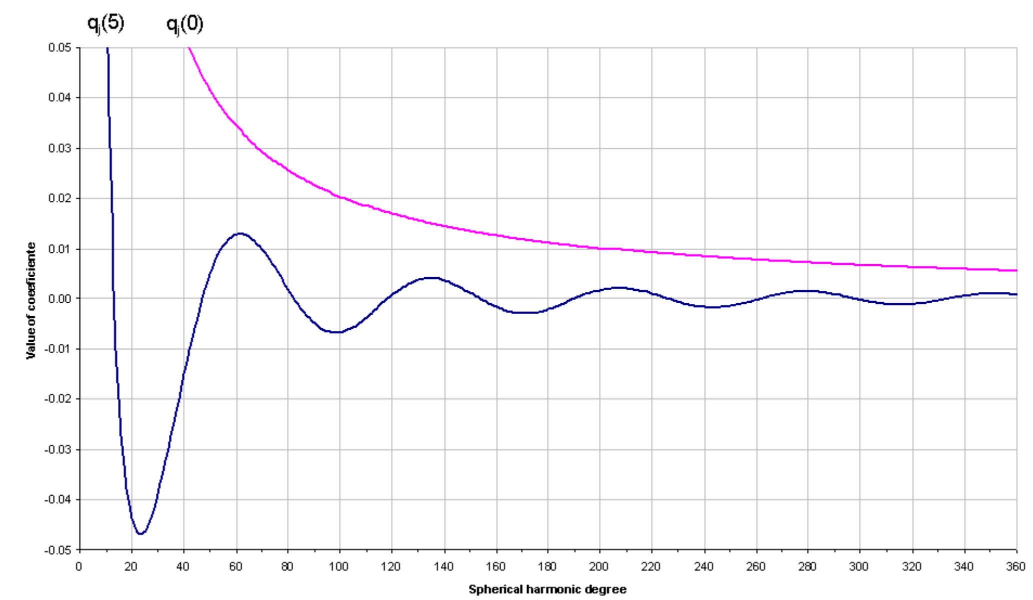


Figure 4. The spherical truncation coefficients $q_j(\psi_0)$ for $10 \leq j \leq 360$. Values for coefficients between 0 and 9 are not shown because they are too large.

The first term on the right-hand side represents a new approximation of the near zone contribution for T(Ω) and the last two terms are the new correction due to the far zone, to be minimized. From Schwarz's inequality applied to (27) it follows that

$$(\delta T(\psi))^2 \le k^2 \|\Delta g\|^2 \|S^*(\psi)\|^2 \tag{28}$$

where:

$\delta T(\psi)$: is the contribution of the far zone to the anomalous potential,

$k^2 = \dfrac{R}{4\pi}$,

$\|\bullet\|^2 = \displaystyle\int_{\Omega-\psi_0} (\bullet)^2 d\Omega'$.

For a given $\Delta g$, the norm $\|\Delta g\|$ is constant, while $\|S^*(\psi)\|$ varies with the choice of $t_n$ ($\forall\ n \ge 2$) [Vaníček and Sjöberg, 1991]. Applying the minimum condition to the latter norm leads to the following system of normal equations:

$$\forall t_j \in \Re^j : \min\left\{ \int_{\Omega-\psi_0} [S^*(\psi)]^2 d\Omega' \right\} \tag{29}$$

$$\begin{aligned} \forall i \in \Re : \frac{\partial}{\partial t_i} \int_{\Omega-\psi_0} [S^*(\psi)]^2 d\Omega' = 0 \\ \int_{\Omega-\psi_0} \frac{\partial}{\partial t_i} [S^*(\psi)]^2 d\Omega' = 0 \end{aligned} \tag{30}$$

Inserting the expression for $S^*(\psi)$

$$\int_{\Omega-\psi_0} \frac{\partial}{\partial t_i} [S(\psi) - M(\psi)]^2 d\Omega' = 0$$

$$\int_{\Omega-\psi_0} \frac{\partial}{\partial t_i} \left[ (S(\psi))^2 - 2S(\psi)M(\psi) + (M(\psi))^2 \right] d\Omega' = 0$$

$$\int_{\Omega-\psi_0} \left[ 2M(\psi)\frac{\partial M(\psi)}{\partial t_i} + 2S(\psi)\frac{\partial M(\psi)}{\partial t_i} \right] d\Omega' = 0$$

$$\int_{\Omega-\psi_0} M(\psi)\frac{\partial M(\psi)}{\partial t_i} d\Omega' = -\int_{\Omega-\psi_0} S(\psi)\frac{\partial M(\psi)}{\partial t_i} d\Omega' \tag{31}$$

Considering the spectral form of M(ψ), its derivatives with respect to $t_i$ will be:

$$\frac{\partial M(\psi)}{\partial t_i} = \frac{\partial}{\partial t_i} \sum_{l=2}^{\infty} \frac{2i+1}{2} t_i P_i(\cos\psi) = \frac{2i+1}{2} P_i(\cos\psi) \tag{32}$$

Introducing this last expression into (31) we get

$$\int_{\Omega-\psi_0} M(\psi)P_i(\cos\psi)d\Omega' = -\int_{\Omega-\psi_0} S(\psi)P_i(\cos\psi)d\Omega' \qquad (33)$$

The right-hand side of equation (33) is exactly the same as (11), and taking again the spectral form of M(ψ), the left-hand will be:

$$\int_{\Omega-\psi_0} M(\psi)P_i(\cos\psi)d\Omega' = \int_{\Omega-\psi_0} \sum_{j=2}^{\infty} \frac{2j+1}{2} t_j P_j(\cos\psi)P_i(\cos\psi)d\Omega'$$
$$= \sum_{j=2}^{\infty} \frac{2j+1}{2} t_j \int_{\Omega-\psi_0} P_j(\cos\psi)P_i(\cos\psi)d\Omega' \qquad (34)$$

Denoting the integral over the far zone:

$$e_{ij}(\psi) = \int_{\Omega-\psi_0} P_i(\cos\psi)P_j(\cos\psi)d\Omega' \qquad (35)$$

by $e_{ij}$ ,the expression (29) could be written as

$$\int_{\Omega-\psi_0} M(\psi)P_i(\cos\psi)d\Omega' = \sum_{j=2}^{\infty} \frac{2j+1}{2} e_{ij}(\psi)t_j \qquad (36)$$

Substituting (36) and (11) into (31), we finally obtain the linear equations for $t_j$, which can be solved for any given $\psi_0$.

$$\forall j : \sum_{j=2}^{\infty} \frac{2j+1}{2} e_{ij}(\psi)t_j = -q_j(\psi) \qquad (37)$$

Figure 5 shows the effect of coefficients $t_j$ for the far zone contribution with a spherical cap of 5 degrees.

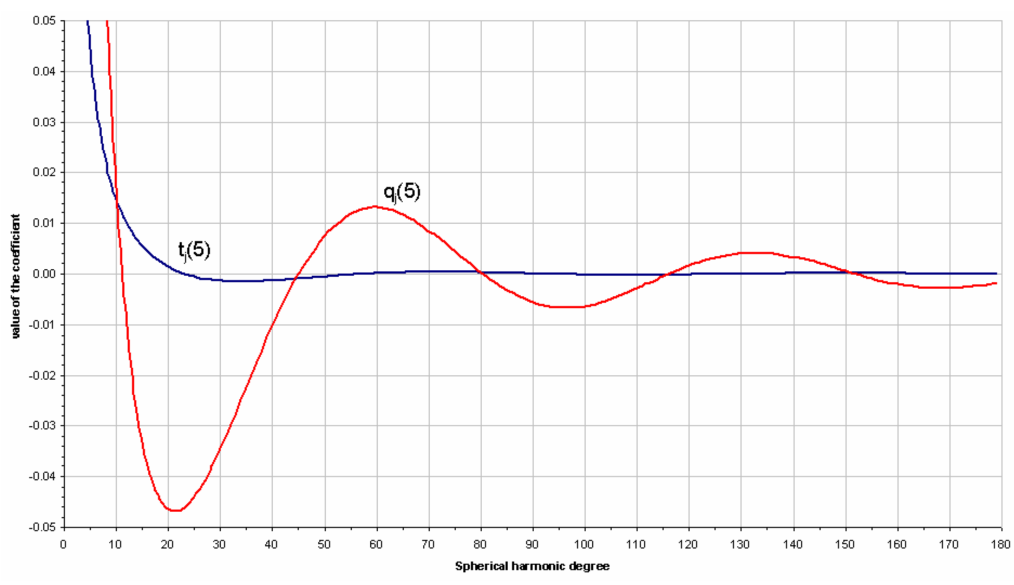

Figure 5. The spherical truncation coefficients $t_j(\psi_0)$ for $10 \le j \le 180$. Values of coefficients between 0 and 9 are not shown because they are too large.

## The Far Zone contribution for the Mexico

### ***The Earth Geopotential Model EGM360***

In order to show the size of the contribution of the far zone in spherical Stokes's integration in Mexico , we use the joint NASA, GSFC and NIMA Earth Geopotencial Model EGM96 complete from degree and order (2,0) to (360,360) [http://cddisa.gsfc.nasa.gov/926/egm96/egm96.html]. Its fully normalised, unitless spherical coefficients (and their standard deviations) are consistent with scaling values of GM and a.

The NASA Goddard Space Flight Center (GSFC), the National Imagery and Mapping Agency (NIMA), and the Ohio State University (OSU) have collaborated to develop this improved spherical harmonic model of the Earth's gravitational potential. The model incorporates improved surface gravity data, altimeter-derived anomalies from the ERS-1 and from GEOSAT Geodetic Mission (GM), extensive satellite tracking data - including new data from Satellite Laser Ranging (SLR), the Global Positioning System (GPS), NASA's Tracking and Data Relay Satellite System (TDRSS), the French DORIS system, and the US Navy TRANET Doppler tracking system - as well as direct altimeter ranges from TOPEX/POSEIDON (T/P), ERS-1, and GEOSAT. The model gives geoid undulations accurate to better than one meter (with the exception of areas void of dense and accurate surface gravity data).

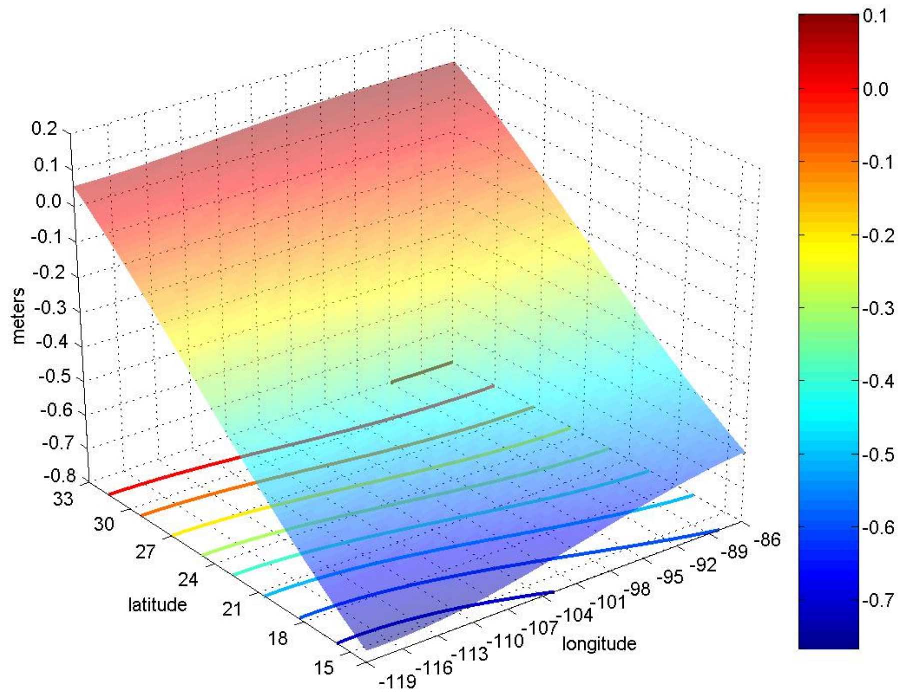


Figure 6. Contribution to the geoid of the far zone. (values in meters)

### ***Covered area and numerical investigation***

The area of study is delimited by latitudes 14° and 33° N and longitudes 86° and 119° W, which includes the whole of Mexico. The intention is to determine the contribution to geoidal height of the far integration zone, using spherical Stokes's integration with a spherical cap of 5 arc-degrees, and also to establish the maximum degree and order of the global model that has a significant influence in the test area. As a preliminary step to this computation we evaluated the coefficients $q_j(\psi_0)$ numerically as functions of radius $\psi_0=5°$ for degree 360, and later the spherical truncation coefficients $t_j(\psi_0)$ up to degree 180, the reason for computing the truncation coefficients at a lower degree than the Molodenskij's coefficients was that they converge faster than the q coefficients.
The test area was divided into a grid of 15' in latitude by 15' in longitude. Figure 6 shows the contribution of the spherical Stokes's kernel for degree and order equal to 50, the contour lines are in meters. Table 1 summarizes the statistics of this contributions.

**Deleted:** ¼???°

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| | |
|---|---|
| Mean | -0.307 m |
| Standard deviation | 0.236 m |
| Maximum | 0.098 m |
| Minimum | -0.773 m |
| Range | 0.871 m |

Table 1. Statistics of the contribution of the far zone up to degree and order 50

The reason for considering a degree and order equal to 50 was that the differences with respect to the previous degree and order were less than 0.001 mm.

### **Conclusions**

From Figure 5, we can easily appreciate that the contribution of the far zone is really minimized by taking into account the spherical truncation coefficients $t_j(\psi_0)$, Table 1 shows that the range of the contribution of the far zone using the EGM96 model is less than one meter.

We consider that this is enough to compute the expansion of any geopotential model up to a degree and an order 50, because the differences with respect to previous degree and order were less than one millimeter.

| | 10-20 | 20-30 | 30-40 | 40-50 |
|---|---|---|---|---|
| Mean | 0.001 m | 0.000 m | 0.000 m | 0.000 m |
| Standard deviation | 0.004 m | 0.000 m | 0.000 m | 0.000 m |
| Maximum | 0.009 m | 0.002 m | 0.001 m | 0.000 m |
| Minimum | -0.007 m | -0.001 m | -0.001 m | -0.000 m |
| Range | 0.016 m | 0.003 m | 0.002 m | 0.000 m |

Table 2. Statistics of contribution of the far zone for different degree and order

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