AN ALTERNATIVE ALGORITHM TO FFT FOR THE NUMERICAL EVALUATION OF STOKES'S INTEGRAL

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Summary: *Stokes's kernel used for the evaluation of a gravimetric geoid is a function of the spherical distance between the point of interest and the dummy point in the integration. Its values thus are obtained from the positions of pairs of points on the geoid. For the integration over the near integration zone (near to the point of interest), it is advantageous to pre-form an array of kernel values where each entry corresponds to the appropriate locations of the two points, or equivalently, to the latitude and the longitude-difference between the point of interest and a dummy point. Thus, for points of interest on the same latitude, the array of the Stokes kernel values remains the same and may only be evaluated once. Also, only one half of the array need be evaluated because of its longitudinal symmetry: the near zone can be folded along the meridian of the point of interest.*

Numerical tests show that computation speed improves significantly after this algorithm is implemented. For an area of 5 by 10 arc-degrees with the grid of 5 by 5 arc-minutes, the computation time reduces from half an hour to about 1 minute. To compute the geoid for the whole of Canada (20 by 60 arc-degrees, with the grid of 5 by 5 arc-minutes), it takes only about 17 minutes on a 400MHz PC computer.

Compared with the Fast Fourier Transform algorithm, this algorithm is easier to implement including the far zone contribution evaluation that can be done precisely, using the (global) spectral description of the gravity field.

Keywords: Stokes's integral, geoid, gravity

1. INTRODUCTION

Stokes's integral for evaluating the geoid undulation at a point on the geoid can expressed as (*Stokes, 1849*; *Vaníček and Krakiwsky, 1986*)

$$N = \frac{R}{4\pi\gamma} \int_\sigma \Delta g S(\psi)\, d\sigma \ , \tag{1}$$

where R is the mean earth radius, γ the normal gravity, σ the sphere of integration, Δg the gravity anomalies reduced to the geoid, and $S(\psi)$ the Stokes function of spherical distance ψ, which is

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a function of the latitudes of and longitude difference between the point of interest and a dummy point:

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi) \ . \tag{2}$$

In this context, by a point of interest we mean the point where the geoid height is evaluated, and by a dummy point, the running point in the integration.

In practice, the global integration domain is divided into two parts: the near zone and the far zone. The near zone contribution is evaluated from regional gravity anomalies, and if it is chosen as a spherical cap then the far zone contribution can be estimated from a global geopotential model. Equation (1) becomes (*Molodenskii et al., 1962*)

$$N = \frac{R}{4\pi\gamma} \int_{\sigma_0} \Delta g S(\psi)\, d\sigma + \frac{R}{2\gamma} \sum_{n=2}^{\infty} Q_n(\psi_0)\, \Delta g_n \ , \tag{3}$$

where σ_0 stands for the near zone cap, and

$$\forall n = 2, \ldots : Q_n(\psi_0) = \int_{\psi_0}^{\pi} S(\psi)\, P_n(\cos\psi) \sin\psi \, d\psi \tag{4}$$

are the Molodenskii truncation coefficients. The second term in Eq. (3) represents the far zone contribution; it is sometimes called the truncation error.

Furthermore, it has become a standard practice to use a global geopotential model up to a certain degree L as a reference field. This reference field generates the reference spheroid to which the residual geoid is referred. The geoidal heights above the reference spheroid are called residual geoidal heights. The far zone contribution should be reduced as much as possible to eliminate the dependency of the residual geoidal height on the global geopotential model. Equation (4) shows that the far zone contribution is a function of the truncation coefficients which are functions of Stokes's kernel. To reduce the effect of the far zone, various modification techniques to the Stokes kernel have been suggested and used (*Molodenskii et al., 1962*; *de Witte, 1967*; *Wong and Gore, 1969*; *Meissl, 1971*; *Jekeli, 1981*; *Sjöberg, 1984, 1986, 1991*; *Vaníček and Kleusberg, 1987*; *Vaníček and Sjöberg, 1991*; *Vaníček and Featherstone, 1998*; *Featherstone et al., 1998*).

The near zone contribution can be evaluated as precisely as needed by numerical integration. From the programming point of view, numerical integration is easily implemented. The only drawback is its slowness. In this contribution, a faster algorithm is developed to improve the speed of the numerical Stokes integration over the near zone.

2. A FASTER ALGORITHM FOR THE NUMERICAL STOKES INTEGRATION

Stokes's kernel $S(\psi)$ is a function of the spherical distance which can be expressed as

$$\psi = \arccos[\sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda' - \lambda)] \ , \tag{5}$$

where ϕ and λ are the latitude and longitude of the point of interest, ϕ' and λ' are the latitude and longitude of the dummy point. It can be seen that the spherical distance does not depend on longitudes of the point of interest and dummy point: it only depends on the longitude difference between the two points. This means that kernel values computed at one point of interest can be used at other points of interest on the same latitude. In other words, all points of interest at the same latitude use the same set of kernel values. Further, the kernel values are symmetrical with respect to the meridian of the point of interest and only one half of the kernel values need be thus evaluated. In addition, for fixed grid steps $d\lambda'$, $d\phi'$, the surface element $d\sigma = \cos\phi' d\lambda' d\phi'$ depends evidently only on the dummy latitude and it needs be computed only once for each dummy latitude.

Any modified Stokes's kernel, like the original Stokes kernel, is still a function of the spherical distance between the point of interest and the dummy point, and it thus possesses the same isotropy and symmetry with respect to longitude as the original Stokes kernel. For instance, the modified spheroidal Stokes kernel (*Vaníček and Sjöberg, 1991*)

$$\forall M, L > 1 : S^{M,L}(\psi) - \sum_{n=2}^{L} \frac{2n+1}{2} t_n(\psi_0) P_n(\cos\psi) \ , \tag{6}$$

shows quite clearly its isotropy resulting in longitudinal symmetries. The ellipsoidal Stokes kernel with a relative error of the order of the square of the flattening of the Earth can be written in the following form (*Martinec and Grafarend, 1997*):

$$\begin{aligned} S^{ell}(\Omega, \Omega') = {} & \sin\vartheta \left(\cos\vartheta \sin\psi \cos\psi \cos\alpha - \sin\vartheta \cos^2\psi \cos^2\alpha + \right. \\ & \left. + \sin\vartheta \sin^2\alpha\right) K_1(\cos\psi) + \left(1 - \sin^2\vartheta \sin^2\alpha\right) K_2(\cos\psi) - \\ & - \sin\vartheta \cos\alpha \left(\cos\vartheta \sin\psi - \sin\vartheta \cos\psi \cos\alpha\right) K_3(\cos\psi) - K_4(\cos\psi) \ , \end{aligned} \tag{7}$$

where the azimuth

$$\alpha = \arccos\left[\csc\psi \left(\sin\vartheta \cos\vartheta' - \cos\vartheta \sin\vartheta' \cos(\lambda - \lambda')\right)\right] \ . \tag{8}$$

Ω is the full solid angle, $\Omega = (\vartheta, \lambda)$. ϑ is the complement of the reduced latitude. This kernel, once again, shows that the isotropy applies. It is thus easy to see that the longitudinal symmetries present in the original Stokes kernel are also present in all the kernels used in practical computations. These symmetries can now be exploited to speed up the integration process.

A new algorithm has been designed to exploit the kernel symmetrical properties. The main difference between the new and the standard algorithms is in the evaluation of kernel values $S(\psi) = S(\phi, \lambda; \phi', \lambda')$: for the standard algorithm, the evaluation of the kernel is placed within the 'longitude loop'; in the new algorithm it is placed outside the 'longitude loop' and inside the 'latitude loop'. Therefore, the kernel values are only computed once for any latitude of interest. For example, Canada spans about 60 arc-degrees in longitude. Assuming the geoid to be computed in 5 arc-minute step in longitude, the kernel values have to be evaluated 720 times by the standard algorithm compared to only once by the new algorithm. This change significantly reduces the time of kernel evaluation, which operation takes most of the CPU time. Furthermore, by making use of the longitudinal symmetry of the Stokes kernel, only about one half of the kernel values are evaluated in the new algorithm; the integration area is 'folded' along the meridian of the point of interest.

A simple time analysis can show the theoretical efficiency of the new algorithm with respect to the standard algorithm. The minimal number of computational steps to evaluate a kernel value is about 20. This count varies with the type of the Stokes kernel. Assuming n and m to be the number of points of interest along the meridian and along the parallel, and $2k$ and $2l$ the number of rows and columns of the matrix of dummy points in the near zone, then the computational step count is $22*n*m*2k*2l$ for the standard algorithm and $20*n*2k*\ (l+1)+n*m*2k*\ (l+1)\ *3$ for the new algorithm. The ratio of computational steps between the standard and the new algorithms can be written as

$$\frac{88nmkl}{40nk(l+1)+6nmk(l+1)} \doteq \frac{44m}{20+30m} \ . \tag{9}$$

The efficiency is thus increased about 14 times for $m > 120$. This represents a lower bound estimate: if the modified Stokes kernel is used, the efficiency is higher than 14. On the other hand, for the geoid evaluation at one point only, or at a string of points in different latitudes, both the standard and the new algorithms have the same computation speed.

Numerical tests have been carried out to measure the actual efficiency of the new algorithm. They were run on the Dell Optiplex GS1p PC with 400MHz CPU and the Linux operating system. The programming language was Fortran77. The spherical and the Molodenskii modified spheroidal Stokes kernels (*Vaníček and Kleusberg, 1987*) were used in the tests. Table 1 shows the CPU time of the near zone integration (a spherical cap of 6 arc-degrees) for the standard and the new algorithms. The speed of the new algorithm is about 20 times faster when using the spherical Stokes kernel, and 45 times faster when using the modified spheroidal Stokes kernel. For the whole of Canada, it takes about 5 hours of the CPU time using the spherical Stokes kernel, and 14 hours of the CPU time using the modified spheroidal Stokes kernel to compute the geoid by the standard integration algorithm, and only about 17 minutes by the new algorithm. Further, the speed of the new algorithm depends less on the complexity of the used kernel than that of the standard algorithm since the kernel evaluation accounts only for a smaller portion of the CPU time in the new algorithm.

Table 1. The CPU time needed when using the standard and new algorithms.

Area	Standard (min.)	New (min.)	Kernel
5° x 10°	11.60	0.57	spherical
23° x 60° (Canada)	319.29	14.63	spherical
5° x 10°	30.16	0.72	modified
23° x 60° (Canada)	841.11	17.75	modified

3. A COMPARISON OF THE NEW INTEGRATION ALGORITHM AND THE FFT METHOD

A particularly popular method for the Stokes integration is the Fast Fourier Transform (*Schwarz et al., 1990*; *Strang van Hees, 1990*). It is the most time efficient procedure to evaluate convolution integrals with. But for both the planar and the spherical 2-D FFT methods, approximations are introduced in the kernel function, and consequently only approximate results can be obtained (*Haagmanns et al., 1992*; *She, 1993*). *Grafarend and Krumm (1996)* show that the planar approximation with only the first order terms in the Stokes function used may introduce a relative error of the order of 50% into the Stokes integral. For the 1-D FFT method, however, the exact kernel can be used and the accuracy of this method is thus comparable with the numerical integration that we have introduced here (*Haagmanns et al., 1992*; *Novák et al., 1999*).

As far as the computational speed is concerned, *Haagmanns et al. (1993)* show that the standard algorithm is about ten times slower than the 1-D FFT method for a region of 14 by 23 arc-degrees with 6 by 10 arc-minute grid. This implies that the presented new algorithm possesses a comparable or even higher computational speed than the 1-D FFT method.

In Table 2, three approaches for the evaluation of Stokes's integral are compared in terms of accuracy, far zone contribution, speed and implementation. Among these approaches, the new algorithm appears quite competitive.

Table 2. A comparison of the new algorithm and the FFT integration.

Attribute	**New direct summation**	**1-D FFT**	**2-D FFT**
Accuracy	exact	exact	approximate
Far zone	optimal	optimal	approximate
Speed	fast	fast	very fast
Implementation	very simple	simple	complicated

4. SUMMARY

In this work, a new algorithm for numerical Stokes integration is suggested. The new algorithm is based on the isotropy and longitudinal symmetry of the Stokes kernel function. For areas of 10° (and more) in longitude, its speed is about 20 times higher when using the spherical Stokes kernel and 45 times higher when using the modified spheroidal Stokes kernel than that of the standard algorithm. Using the new algorithm, it takes a fast PC about 17 minutes to compute the geoid for the whole of Canada. The new algorithm retains the attributes of the standard 2-D numerical integration, the exactness of results, flexibility of combination with global geopotential models, simplicity of implementation, and shows a satisfactory speed for the determination of geoid over large regions.

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