Numerical Investigation of Downward Continuation Techniques for Gravity Anomalies

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Summary.- In this work, we made two comparisons. Firstly, we compared the values of the point-to-point Poisson downward continuation with the values of the mean-to-mean one, by using the 5'by5' mean residual Helmert gravity anomalies (for spherical harmonic degrees higher than 20) and mean heights, over the most rugged part of the Canadian Rocky Mountains. The results indicate that the former geoidal contribution is about 10 percent smaller than the latter one that ranges from - 4 cm to 94 cm with a mean of 48 cm and a RMS of 51 cm. In practical applications, certain caution has to be taken in selecting a model (point-to-point or mean-to-mean) to minimize its incompatibility with the data. Secondly, we compared the point-to-point Poison downward continuation with the analytical downward continuation numerically. It was shown that the former geoidal results are in average about 25 cm smaller than the latter when the first three terms are taken into account. However their difference is dominated by long-wavelength components with a standard deviation of about 4 cm. A possible cause of the difference may be due to the use of the planar approximation in the analytical downward continuation computation.

1. INTRODUCTION

The use of Stokes's formula for the determination of the geoid requires that the gravity anomalies Δg represent boundary values at the geoid, which implies two requisite conditions of using it: firstly, the gravity anomalies must refer to the geoid; secondly, there must be no masses outside the geoid. Hence, gravity reduction is necessary to meet the conditions. It can be carried out with the following two steps: (1) the gravity anomalies are transformed into a harmonic space such as Helmert's space (Vaníček et al., 1999); (2) the gravity anomalies are harmonically reduced from the Earth's surface downward to the geoid. The second step is in geodetic literature described as a *downward continuation* of the gravity anomalies.

In principle, all gravity reductions should be equivalent and should lead to the same geoid if they are properly applied, and when the indirect effect is also considered (Heiskanan and Moritz, 1967). Among the proposed reduction models, Helmert's 2nd condensation reduction, in which the topographic masses are condensed as a mass layer on the geoid, is widely used in the practical determination of the geoid (Vaníček and Kleusberg, 1987; Véronneau, 1996; Smith and Milbert, 1999; Featherstone et al., 2001). Even though the same reduction model is used, the actual reductions may be evaluated by different approaches. A major difference among the approaches is due to the downward continuation methods applied.

There are two classes of downward continuation methods available for the geoid determination. One is the class of the *Poisson downward continuations* that are based on the Poisson integral formula (e.g., Heiskanan and Moritz, 1967; Schwarz, 1978; Bjerhammar, 1987; Wang, 1988; Vaníček, et. al, 1996; Martinec, 1996; Sun and Vaníček, 1998). The second is the class of the *analytical downward continuations* that are based on the Taylor series expansion (Moritz, 1980; Sideris, 1987; Wang, 1988). Martinec et al.(1996) show that the Taylor series becomes divergent when the integration point is very close to the computation point. The former can be further divided into the point-to-point values model (Martinec, 1996) and the mean-to-mean values model (Vaníček, et. al, 1996). Sun and Vaníček (1998) pointed out that the point-to-point solutions are up to five times smaller than the mean-mean solutions in the extreme case.

In this contribution, we are particularly interested in answering the following questions: How different is the point-to-point Poisson downward continuation result from the mean-to-mean one numerically? Is the analytical downward continuation numerically equivalent to the Poisson one? If not, how different are they?

2. METHODOLOGY 1: THE POISSON DOWNWARD CONTINUATION FORMULAE

The Poisson integral formula can be written as follows (Heiskanen and Moritz, 6-74, 1967)

$$\Delta g(r,\Omega)=\frac{R}{4\pi r}\int_{\Omega'} K(r,\psi,R)\Delta g(R,\Omega')d\Omega' \tag{1}$$

where K is the Poisson kernel function

$$K(r,\psi,R)=\frac{R(r^2-R^2)}{l^3}. \tag{2}$$

$\Delta g(r,\Omega)$ is the gravity anomaly; Ω is the geocentric angle denoting the pair (θ, λ), the spherical co-latitude and lognitude; ψ and l are the angular and spatial distances between the point (r,Ω) and the surface element $d\Omega'$; r is the radius of the point (r,Ω).

Given $\Delta g(R,\Omega)$, the gravity anomaly on the spherical surface of a radius R, evaluating the gravity anomaly at any point outside the spherical surface is called *upward continuation*, that can be done in spatial sense directly by evaluating the Poisson integral. The gravity anomaly $\Delta g(r,\Omega)$ at a point above the spherical surface represents a weighted average of the gravity anomalies $\Delta g(R,\Omega)$ given on the spherical surface, thus it tends to be smoother. When the gravity anomalies are known on a surface above the spherical surface, and the gravity anomalies on the spherical surface are sought, we face the problem of downward continuation that is achieved by solving the Poisson integral equation. In mathematics, this equation is called the Fredholm integral equation of the first kind. In contrast to upward continuation, downward continuation tends to 'de-smooth' or accentuate details of the gravity anomalies. Numerical tests related to the stable downward continuation of gravity anomalies using the previously mentioned integral equation are given by different authors and especially in applications of airborne gravity data to gravity field modeling (see, e.g., Novák et al., 2001).

The geoid group at UNB developed a program package (**DOWN97)** which can be used to evaluate gravity anomalies on the geoid from gravity anomalies at the Earth's surface by solving the discrete Poisson integral equation (Vaníček, et al., 1998). It is capable of performing the point-to-point and mean-to-mean downward continuations, depending on how the kernel coefficients (from which the coefficient matrix of a linear system of equations is formed) are evaluated. We note that, the Poisson downward continuation merely gives results only to the spherical approximation. However the effect of the geoid flattening is generally less than 1 cm because the downward continuation contribution from the Earth's surface to the geoid is usually smaller than 3 m in the geoidal height.

The discrete Poisson integral for the point-to-point downward continuation can be written as (Martinec, 1996)

$$\Delta g_i^t = \frac{R}{4\pi(R+H_i)} \sum_j K_{ij}\ \Delta g_j^g + F_{\Delta g} \tag{3}$$

where subscripts t and g stands for on the Earth's surface and the geoid, respectively; indices i and j indicate the computation and integration points, respectively; H_i is the height of a computation point; K_{ij} are the kernel coefficients; $F_{\Delta g}$ represents the contribution outside the chosen near-zone cap, called the far-zone contribution.

The discrete Poisson integral for the mean-to-mean downward continuation can be expressed as (Vaníček, et. al, 1996)

$$\overline{\Delta g_i^t} = \frac{R}{4\pi(R+H_i)} \sum_j \overline{\overline{K_{ij}}}\ \overline{\Delta g_j^g} + \overline{F_{\Delta g}} \tag{4}$$

where the single over-bars indicate the mean values of the corresponding variables; the doubly over-bared K_{ij} represent the doubly averaged Poisson kernel coefficients.

The Seidel iterative method is used to solve the linear system of equations. Let B represent the coefficient matrix, and b be the constant vector (gravity anomalies on the Earth's surface), and x be the unknown vector (gravity anomalies on the geoid), then discrete Poisson integral equations can be written as follows

$$Bx = b \tag{5}$$

Let $A = I - B$; equation (5) then becomes

$$x = Ax + b\,. \tag{6}$$

Then the Seidel iteration can be written as

$$\begin{aligned} x^{(1)} &= Ax^{(0)} + b, \\ x^{(2)} &= Ax^{(1)} + b, \\ &\ldots \\ x^{(k+1)} &= Ax^{(k)} + b \;\; until \;\; Max(|\, x^{(k+1)} - x^{(k)} |) \le \varepsilon\,. \end{aligned} \tag{7}$$

The threshold value ε is set as 0.02 mGal. Numerical tests with synthetic data showed that the software gives downward continuation results with an accuracy of better than 1 cm (when converted into an effect on the geoid). The iterative model described before proves to be numerically an efficient method although small biases are possible to be detected in the final results (see, e.g., Novák et al., 2001).

3. METHODOLOGY 2: THE ANALYTICAL DOWNWARD CONTINUATION FORMULAE

The analytical downward continuation can be formulated as follows (Moritz, sect. 45, 1980)

$$\Delta g^g = \sum_{n=0}^{\infty} g_n \tag{8}$$

where

$$\begin{aligned} g_0 &= \Delta g^t, \\ g_1 &= -H \cdot L(g_0), \\ g_2 &= -H \cdot L(g_1) - H^2 \cdot L_2(g_0), \\ &\ldots \\ g_n &= -\sum_{m=1}^{n} H^m \cdot L_m \cdot g_{n-m}. \end{aligned} \tag{9}$$

The operator L is defined as

$$L(f) = \frac{R^2}{2\pi}\int_{\Omega'} \frac{f - f_P}{l_0^3} d\Omega' - \frac{1}{R} f_P \tag{10}$$

and

$$L_k = \frac{1}{n!} L^n = \frac{1}{n} L L_{k-1.} \tag{11}$$

where l_0 is the distance between computation point an integration point.

Up to date, most practical applications have only used the g_1 term in combination with the assumption of a linear relation between the gravity anomaly and the height. It has been shown that the g_1 term is close to the Condensed Terrain Effect (CTE) in Helmert's 2nd condensation model (Véronneau, 1996; Vaníček, et al., 1999). It is this fact that provides the background for the use of the Faye anomaly in the geoid determination because the Faye anomaly approximately represents the Helmert gravity anomaly on the geoid when the g_1 term is assumed to cancel out the CTE (Véronneau, 1996). Since the linear relation between the gravity anomaly and the height can not strictly be established for the Helmert anomaly, it should be avoided when the centimeter-accuracy geoid is determined through the Stokes-Helmert approach.

The evaluation of the analytical downward is very time-consuming. When the L operator is applied in planar approximation, it can then be written in a convolution form

$$L(f) = \frac{R^2}{2\pi}\int_{\Omega'} \frac{f - f_P}{l_0^3} d\Omega' = f * d_Z, \quad L_n f = (d_Z *)^n f. \tag{12}$$

The integral above can be efficiently evaluated by FFT using either the discrete or the analytical form of the $(l_0)^{-3}$ spectrum (Sideris, 1987)

$$\begin{aligned} &\frac{R^2}{2\pi}\int_{\Omega'} \frac{f - f_P}{l_0^3} d\Omega' = \frac{1}{2\pi}[F^{-1}\{F\{f\}F\{l_0^{-3}\}\}] - fF^{-1}\{F\{u\}F\{l_0^{-3}\}\}], \quad u = 1, \\ &\frac{R^2}{2\pi}\int_{\Omega'} \frac{f - f_P}{l_0^3} d\Omega' = f * d_z = F^{-1}\{F\{f\}(-2\pi q)\}, \quad q = radial\ \ frequency. \end{aligned} \tag{13}$$

The use of the analytically defined spectrum saves significant computer time required by the rigorous determination of the kernel function. However, it introduces additional errors to the results (see, e.g., Li, 1993; Tziavos, 1996). Optimal result can be obtained by using a zero-padding technique and computing the discrete kernel function in both the area covered by gravity data and the zero-padded area (Li, 1993).

4. DATA DESCRIPTION

The selected test region covers part of the Rocky Mountains (41º<φ<60º; 224º<λ<258º), which represents the most rugged area in Canada. It allows us to visualize the extreme scenarios of the differences between the point-to-point and mean-to-mean Poisson downward continuations, and between the Poisson downward continuation and the analytical downward continuation if they are not equivalent. In addition, Helmert's 2nd condensation yields a rougher gravity field than the isostatic gravity field from either of the Airy-Heiskanen and Pratt-Hayford condensation models (Martinec, 1996). Therefore, we are dealing with the comparison under the worst case in both senses.

The mean 5' by 5' heights and residual Helmert gravity anomalies (for spherical harmonic degrees higher than 20) used for the comparison are shown in Figure 1. In the ocean and flat areas, the residual Helmert anomalies are more or less similar to the residual free-air anomalies because the terrain and CTE corrections are relatively small. In the mountainous area, these corrections may differ from the high-frequency free-air anomalies by more than 100 percent.

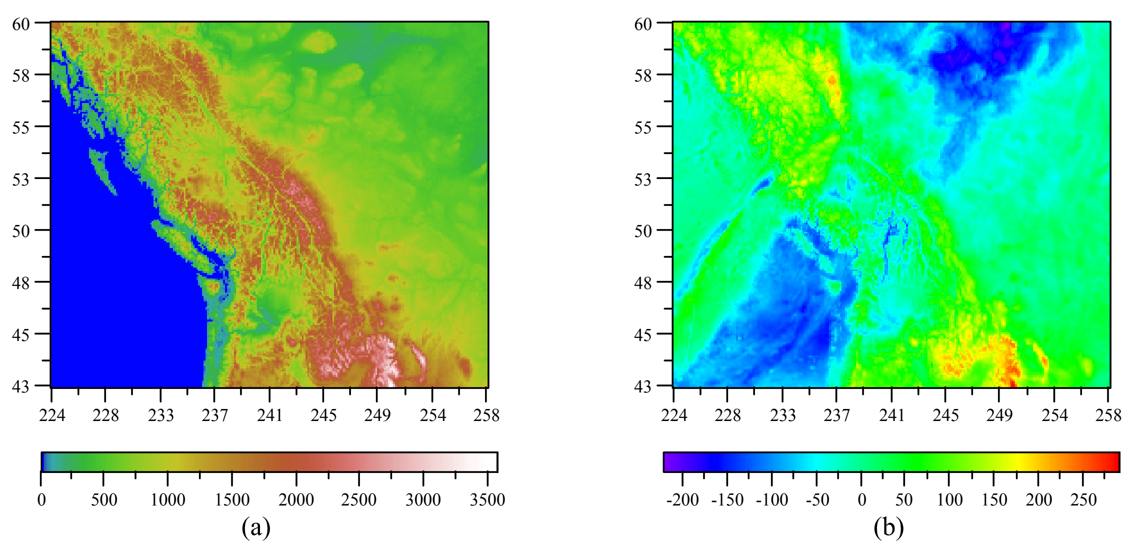

Fig. 1. (a) The topography (in meters), Min.= 0, Max.=3576, Mean=733, StdDev.=605. (b) The gravity anomalies (in mGals), Min.=-223, Max.=291, Mean=-0.8, StdDev.=75.8.

5. THE POISSON DOWNWARD CONTINUATION RESULTS

The same gravity data and height data have been treated as point values in one case and as mean values in the other. This introduces a bias into the comparison! The mean data will, in reality, be smoother and will thus give different results. (The neglect of this difference was responsible for the seemingly unreasonable findings by Sun and Vaníček (1998) as mentioned above.) In the computation, the radius of the near-zone cap was chosen as 1 arc-degree. The far-zone contribution was ignored because it accounts for only about 1 cm effect on the geoid in the test region when EGM96 is adopted as source of input data. Strictly speaking, a global geopotential model must be first transformed into Helmert's space to represent Helmert anomalies. However, the far-zone contribution is predictably small considering the fact that the Poisson kernel is inversely proportional to the cubic of the distance between the computation point and the integration point.

Tables 1 and 2 show the summaries of the point-to-point and mean-to-mean Poisson downward continuation results in gravity and the geoidal height, respectively. The gravity difference between the results of the two downward continuation models is about 10% for mean, and more than 40% for both standard deviation and RMS of the mean-to-mean gravity downward continuation contribution. The geoidal correction difference between them is about 10% for mean, and less than 20% for both the standard deviation and the RMS of the mean-mean geoidal downward continuation contribution. The fact that the geoidal difference is not so significant as the gravity one is due to the action of the Stokes integral that tends to function as an averaging filter in transforming the gravity result into the geoidal one. The geoidal results from the two models are displayed in Figure 2, and the geoidal difference is displayed in Figure 3.

Depending on how one wants to carry out the numerical Stokes integration on the geoid, one may seek either point anomaly values or mean anomaly values on the geoid. If one wishes to work with mean values, one can either downward continue point values (known on the Earth's surface) and evaluate the mean values on the geoid, or one can evaluate mean values on the Earth's surface and downward continue these onto the geoid. We have tested these two scenarios in a limited way, using only the Poisson approach. We note that this raises a meaningful theoretical question as to whether mean values of a harmonic function are themselves harmonic. This question will not be explored here.

The algorithms for downward continuation of point and mean values are, of course, different, with the latter requiring a pre-computation of doubly averaged Poisson's kernels or some other approach. Results obtained with the doubly averaged kernels were reported by Vaníček et al. (1996). The difference in the results using the two algorithms was analyzed by Sun and Vaníček (1998). Interestingly enough, the downward continuation algorithm for mean values was found giving rougher results than the algorithm for point values.

Clearly, the mean values, either on the Earth's surface or the geoid, should be expected to be somewhat smoother than the corresponding point values. This is a result of the smoothing property of the averaging operation. Unfortunately, the data on the surface of the Earth that we had at our disposal were neither the point values, nor the mean values. The data had been created by regular gridding of the irregularly spaced observed point values for a 5 by 5 arc-minute geographical grid. Thus, at some locations, these gridded values are closer to true "point values", at other locations, where there was an abundance of observed values available, they are closer to true "mean values". It was therefore impossible properly to test the performance of the two algorithms and it seems to us likely that it may even not be possible to conduct proper tests in the foreseeable future.

Table 1. Differences of the Poisson downward continuation gravity corrections between the point-to-point and mean-to-mean models in the area of $44^\circ<\varphi<59^\circ$; $226^\circ<\lambda<256^\circ$ (180×360 grid, 64800 values). Unit: mGal.

Model	Mean	Min.	Max.	StdDev	RMS
(1) point-to-point	0.80	-23.25	64.23	4.71	4.77
(2) mean-to-mean	0.88	-46.20	102.49	7.63	7.68
(1) - (2)	-0.08	-38.25	24.85	3.29	3.29

Table 2. Differences of the Poisson downward continuation geoidal corrections between the point-to-point and mean-to-mean models in the area of 49º<φ<54º; 236º<λ<246º (60×120 grid, 7200 values). Unit: meter.

Model	Mean	Min.	Max.	StdDev	RMS
(1) point-to-point	0.427	-0.038	0.810	0.154	0.454
(2) mean-to-mean	0.479	-0.042	0.942	0.175	0.510
(1) - (2)	-0.052	-0.199	0.020	0.031	0.060

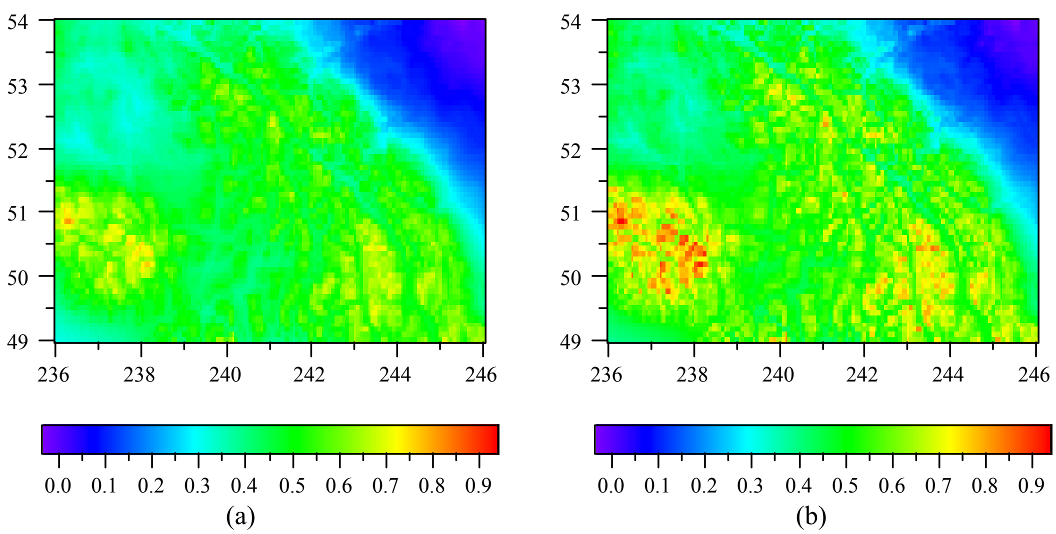

Fig. 2. (a) The point-to-point downward continuation result in the geoidal height. (b) The mean-to-mean downward continuation result in the geoid height. Unit: meter.

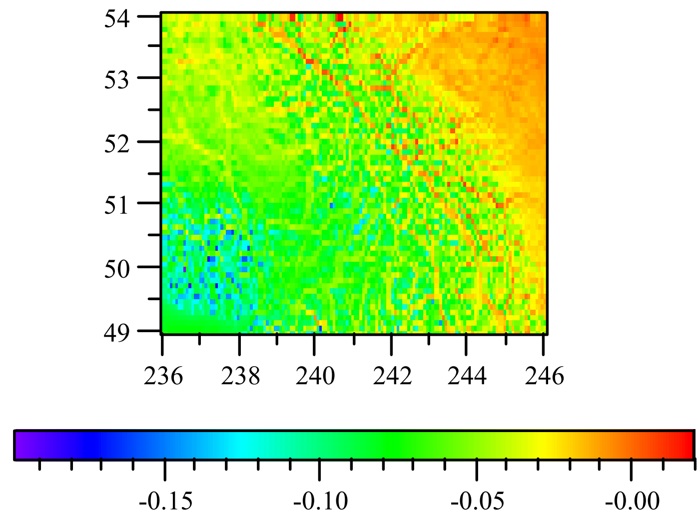

Fig. 3. The differences between the results of the point-to-point and mean-to-mean models. Unit: meter.

6. THE ANALYTICAL DOWNWARD CONTINUATION RESULTS

The Molodensky series terms given in equations (9) can be computed by an analytical continuation to a level surface on which the potential is equal to the potential of the computational point. In fact, the gravity anomalies can be first downward-continued to the geoid, where Stokes's (1849) formula can be easily evaluated, and then gravity anomalies can be computed, as a functional of the disturbing potential, on the surface of the Earth (actually, on the telluroid) by a corresponding upward-continuation methodology via a Taylor series expansion. It must be mentioned that the Molodensky series terms in equations (9), computed at any arbitrary point, are dependent on the computational point. This fact renders them rather impractical, as a new set of g_n-values has to be computed for the computation of gravity anomalies at another point. This difficulty can be overcome by performing the analytical continuation in two steps, according to the previously described methodology. This methodology results in formulas for the computation of the g_n-terms at the geoid, which do not depend anymore on the computation point and, consequently, it is a lot easier to evaluate simultaneously for all data points by use of the FFT technique (Sideris, 1987). Then, the second step of upward-continuing the results to point level can be performed by a Taylor series expansion. The formulas for the analytical downward continuation can be derived and used either in spherical or planar approximation (Sideris, 1987).

Table 3. The g_n term point corrections to the gravity anomalies in the area of 44º<φ<59º; 226º<λ<256º (180×360 grid, 64800 values). Unit: mGal.

Item	Mean	Min.	Max.	StdDev.	RMS
g_1	0.683	-20.784	54.126	4.204	4.259
g_2	-0.143	-17.613	10.110	1.116	1.125
g_3	0.032	-3.662	7.000	0.324	0.326
$g_1 + g_2$	0.541	-16.554	36.513	3.310	3.354
$g_1 + g_2 + g_3$	0.573	-17.032	41.000	3.482	3.529

The results in Table 3 show that the contribution of the first three g-terms to gravity anomalies have a rapidly decreasing effect for the 5' by 5' data. It is important to note that the contribution of the g_3-term is significant in mountainous test areas and should be taken into account in the final corrections, especially when a sub-mGal accuracy is necessary.

Table 4. The g_n term point corrections to the geoidal heights in the area of 49º<φ<54º; 236º<λ<246º (60×120 grid, 7200 values). Unit: meter.

Item	Mean	Min.	Max.	StdDev.	RMS
$\zeta\,(g_1)$	0.830	0.405	1.212	0.154	0.844
$\zeta\,(g_2)$	-0.203	-0.298	-0.104	0.040	0.207
$\zeta\,(g_3)$	0.047	0.025	0.080	0.010	0.048
$\zeta\,(g_1 + g_2)$	0.627	0.298	0.954	0.118	0.638
$\zeta\,(g_1 + g_2 + g_3)$	0.675	0.323	1.011	0.125	0.686

The results in Table 4 show a considerably decreasing effect to height anomalies, which range from 1.2 m (g_1-term) to 1 cm (g_3-term). For data available on denser than 5'-grids, the analytical continuation of gravity and height anomalies could need the computation of higher order terms (g_4, g_5) in order to obtain accuracies at the level of sub-mGal, respectively (see, e.g. Li et al., 1996).

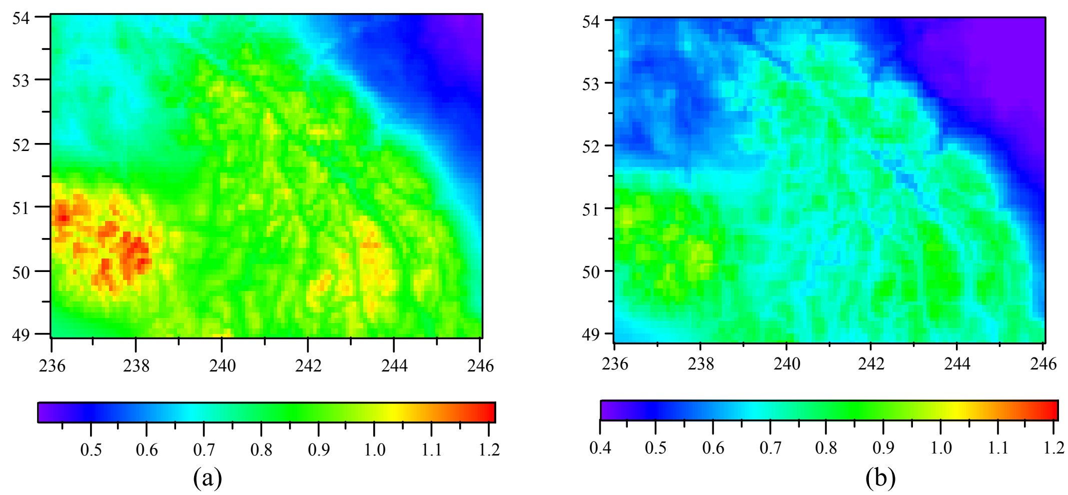

Fig. 4. Effect of the g_n terms on the geoidal height. (a) The g_1 term effect on the geoidal height. (2) The $g_1 + g_2 + g_3$ terms effect on the geoidal height. Unit: meter.

7. COMPARISON BETWEEN THE POINT-TO-POINT POISSON AND ANALYTICAL DOWNWARD CONTINUATIONS

The differences between the point-to-point Poisson and analytical downward continuations are statistically summarized in Tables 5 and 6. The significant differences exit between the results from the two methods. The range of the gravity differences reaches about 40 mGals with a standard deviation of about 1.5 mGals when the analytical downward continuation is summed up to the 3rd terms. Again, we can see that the geoidal differences are not so pronounced as the gravity ones due to the smoothing of the Stokes integration. An unacceptable fact is that the g_1 term alone apparently agrees best with the Poisson result while it merely represents the first-order approximation. A further study is needed in this regard.

Even though significant geoidal differences (20 cm to 40 cm on average) are present at individual points of the test region, their standard deviations are invariably smaller than 5 cm. It implies that the differences primarily lie within the long-wavelength band. This character can be easily seen in

Figure 5 that shows the dominant long-wavelength features. A possible interpretation to it is the use of the 2-D FFT and the planar approximation in the computation of the analytical downward continuation.

Table 5. The gravity differences between the g_n term point corrections and the point-to-point Poisson downward continuation corrections in the area of 44º<φ<59º; 226º<λ<256º (180×360 grid, 64800 values). Unit: mGal.

Item	Mean	Min.	Max.	StdDev.	RMS
g_1 – Poisson	0.11	-5.72	17.64	0.88	0.88
($g_1 + g_2$) – Poisson	0.26	-14.56	30.39	1.75	1.77
($g_1 + g_2 + g_3$) – Poisson	-0.22	-27.78	11.36	1.49	1.51

Table 6. The geoidal differences between the g_n term point corrections and the Poisson downward continuation geoidal corrections in the area of 49º<φ<54º; 236º<λ<246º (60×120 grid, 7200 values). Unit: mGal.

Item	Mean	Min.	Max.	StdDev.	RMS
ζ (g_1)- ζ (Poisson)	0.404	0.488	0.245	0.038	0.405
ζ ($g_1 + g_2$) - ζ (Poisson)	0.200	0.033	0.336	0.048	0.206
ζ ($g_1 + g_2 + g_3$) - ζ (Poisson)	0.248	0.081	0.361	0.042	0.251

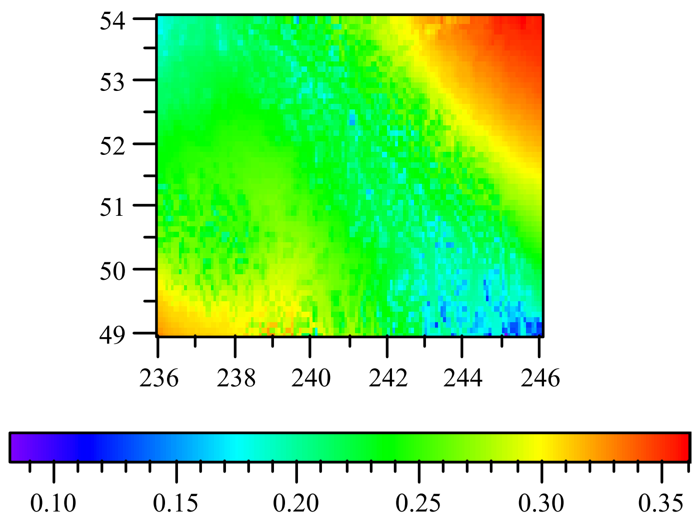

Fig. 5. The geoidal differences between the analytical and Poisson downward continuation results. Unit: meter.

8. CONCLUDING REMARKS

We have numerically compared the point-to-point Poisson downward continuation with the mean-to-mean one using the 5' by 5' residual Helmert gravity anomalies (for spherical harmonic degree higher than 20) and mean height data in the Canadian Rocky Mountains. The geoidal results show that the point-to-point downward continuation result is about 10 percent smaller than that of the mean-mean one on average when the same gravity anomalies on the Earth's surface are used as input. The mean-to-mean downward continuation contribution ranges from -4 cm to 94 cm with a mean of 48 cm and a RMS of 51 cm. Since the data on the Earth's surface we have are neither the point values, nor the mean values. At some locations, they are closer to the point values, at other locations, they are closer to the mean values depending the distribution density of the observed data. It is therefore impossible properly to tell which one is better. In applications, caution is needed in selecting a proper downward continuation model.

Another comparison has also been conducted between the point-to-point Poisson and analytical downward continuations using the same residual Helmert gravity anomalies and height data. The former geoidal contribution is about 25 cm smaller on average than the latter one (that is summed up to the 3rd order term). But the standard deviation of their difference is smaller than 5 cm. The analytical downward continuation contribution ranges from 32 cm to 101 cm with a mean of 68 cm and a RMS of 69 cm. The difference between the two methods is characterized by long-wavelength features which may be attributed to the planar approximation in the computation of the analytical downward continuation, and the method itself. The comparison based on the spherical approximation is needed to more realistically assess its performance.

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