The use of smooth piecewise algebraic approximation in the determination of vertical crustal movements in Eastern Canada

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Abstract. The objective of this study is to compile a physically meaningful map of vertical crustal movements (VCM) for Eastern Canada. Average vertical velocities over the past century are determined by repeated precise levelling and monthly mean sea level observations from 17 tide gauges. The spatial vertical velocities may be mathematically expressed in any number of ways.

In this study, the uplift rate is calculated using smooth piecewise algebraic polynomial approximation. The mathematical model of the approximation for the geodetic data is given. We show how a vertical velocity surface is approximated using piecewise algebraic polynomials and what conditions should be satisfied to guarantee the smoothness of the surface.

First, we divide Eastern Canada into zones. The vertical movement is represented by a different polynomial surface in each zone. The polynomials are joined together at nodal points along the border of adjacent zones in such a way that a certain degree of smoothness (differentiability) of the resulting function is guaranteed.

This study shows that piecewise polynomial surfaces can represent the available data in a unified map. The pattern of a northwest to southeast gradient of crustal movements is consistent with the existing Glacial Isostatic Adjustment (GIA) models. Present-day radial displacement predictions due to postglacial rebound over North America computed using VM2 Earth model and ICE-4G adopted ice history show a zero line (hinge line) very similar to ours along the St. Lawrence River.

The main advantage of the presented technique is its capability of accommodating in one model, different kinds of information when the re-levelled segments are scattered not only in time but also in space. Piecewise approximations make it easier to get the physically meaningful details of the map, without increasing the degree of polynomials.

Keywords. Least Square Approximation, crustal movements, geodynamics, Glacial Isostatic Adjustment

1 Introduction

It has been recognized for several decades that the determination of a Vertical Crustal Motion model is of importance in geosciences. In geophysics, for example, it is of primary interest in the study of the rheology of the mantle and lithosphere which is crucial in understanding geodynamical processes. In geodesy, they are important in the definition of vertical datum which is in turn, required in many application areas such as navigation, mapping, and environmental studies. The first VCM model in Canada was compiled by Vaníček and Christodulides (1974) using scattered geodetic relevelled segments and the first study which covered the whole of Canada was carried out by Vaníček and Nagy (1981) using precise re-leveled segments and tide gauge records. The country was divided into regions and polynomial surfaces of order 2, 3 and 4 were calculated by the method of least squares for each region to obtain representations of the vertical movements. A considerably larger database has been gathered since then, and this, together with additional insight into the nature of the data, led to the recompilation of the map of vertical crustal movement of Canada by Carrera et al. (1994) in which a vertical polynomial was fitted to the data.

In order to infer a physically meaningful VCM, it is necessary to combine the geodetic and geophysical data, theories, methodologies and techniques that are somehow linked together. Hence, finding the best approach to reconcile geodetic data with geological phenomena is required.

In North America, the most significant geophysical process that has an evident effect on the shape of the viscoelastic earth is postglacial rebound or Glacial Isostatic Adjustment (GIA). During the last major glaciation event, immense masses of ice accumulated over regions of North America, causing subsidence of the Earth's crust in these ice covered regions, and uplift in peripheral regions. When this ice has melted during the last 20,000 years, the viscoelastic rebounding of the crust in the ice covered regions started and has been ongoing since (Peltier, 1996).

In this paper, some ideas are exploited in an effort to infer a more physically meaningful VCM model for Canada. Using smooth piecewise approximation method, the velocity surfaces are computed in pieces, and then they are tied together to guarantee their continuity across the zone boundaries.

2 Sea Level data and re-levelled segments

The data used in this study are of two kinds: sea-level records and relevelled segments of the first-order levelling network. A number of 17 permanent tide gauge stations with long enough records were selected in the area of interest (Figure 1). The subset of 17 sites was then selected to include all stations for which continuous records of at least 10 years duration are available. In the studies of vertical crustal motion, tide gauge records with longer time span are considered more reliable. Sea level records with duration of less than 50 years may not be taken as representative for the secular trends sought, if they are studied individually. However, when they are treated in pairs, the secular variations can be accurately estimated. There is a well documented feature of tide gauge records: their striking similarity when they are obtained at two close-by locations. (Vaníček and Carrera, 1993). This spatial coherence is caused by common atmospheric and oceanic noise. Clearly, a large portion of these variations disappears when the records are differenced. This behaviour offers an alternative way of treating sea level trends in close-by tide gauges. In this study, a straight--forward trend analysis was carried out on monthly mean values for all stations. Then, it was decided to use the differencing technique to treat the sea level records. The regional correlation matrices and correlation coefficient confidence interval is used to select the optimum pairing of sites, i.e., a tree diagram for optimum differencing, that gives the most precise and accurate velocity differences to be used in the modelling. Figure 1 demonstrates the optimum pairing of tide gauges in Eastern Canada.

A total of 14168 relevelled segments from Maritimes and southern Quebec were chosen for this study. They were observed during the period between 1909 and 2002. The distribution of data is shown in Figure 1.

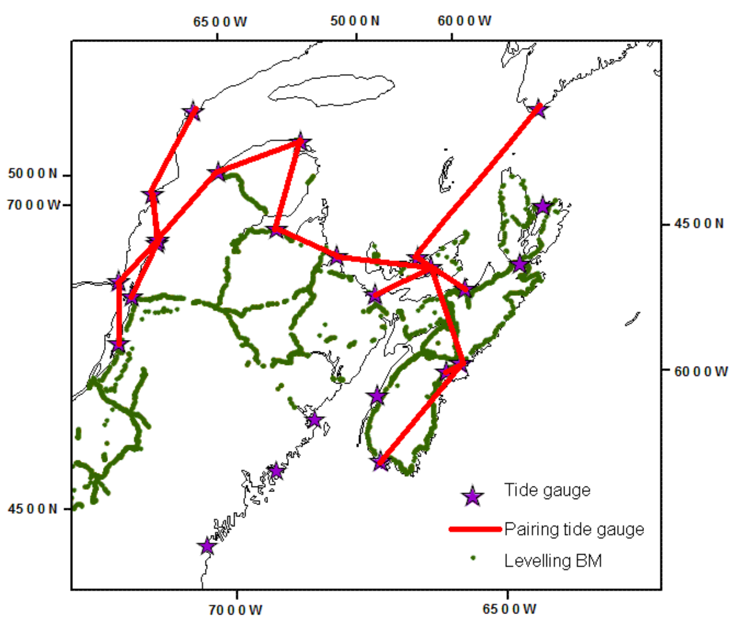

Fig 1: Data distribution used in computations. The optimum tree diagram of tide-gauges for differencing is shown by red lines.

3 Mathematical Model

In order to predict the spatial vertical velocities, or uplift rates, a vertical velocity surface should be fitted to the sea level linear trends and levelling height difference differences data reviewed in the previous section. Therefore, the main concern is to provide an approximation to a function *V(x,y)* from geodetic data. The assumptions underlying this approach are that the uplift rates are linear in time and that they vary smoothly with location.

The velocity surface is first obtained in the form of

$$V(x,y) = \sum_{i,j=0}^{n} c_{ij} x^i y^j \quad , \qquad (1)$$

where *(x,y)* is the location of the points in an arbitrary selected local horizontal coordinate system, n is the degree of polynomials, and c_{ij} are the sought coefficients. Here, the algebraic functions are the simplest functions to deal with numerically and are adequate when the solution is confined to the regions where sufficient data exits; the poor behaviour appears only when the solution is used in an extrapolation mode (Vaníček and Nagy, 1981). The procedure of fitting a surface to the geodetic data involves the use of both the point rates and the gradients simultaneously, together with their proper weights. The point rates are determined from some of the tide gauge data which were selected to be used in the point velocity mode, and the gradients come from relevelled segments and tide gauge pairs.

To get the details needed for the map to be meaningful, the order of the velocity surface would have to be too high to be numerically manageable. A practical way to avoid this is to divide the area of study into zones, and seek the velocity surface piecewise.

In general, if we divide the area of study into m zones and the degree of all the algebraic polynomials is n, the resulting function is a polynomial function of degree n with m zones. A given polynomial in the m-th zone looks as follows:

$$V_m(x,y) = \sum_{i,j=0}^{n} c_{ij,m} (x - x_{m,k})^i (y - y_{m,k})^j \ , \qquad (2)$$

where V_m is the algebraic least squares velocity surface for zone m, fitted to the desired data *(x,y)*. The pair $(x_{m,k}, y_{m,k})$ for $k=1,2,\ldots,q$ represents the position of each node ($P_{m,k}$) located in the predefined border between two zones (zones m and $m+1$) . Here, q, represents the maximum number of the nodal points in each border.

In order to piece the polynomials together, the following conditions should be satisfied:

$$V_m(x_{m,k}, y_{m,k}) = V_{m+1}(x_{m,k}, y_{m,k}) \qquad \forall k = 1,2,\ldots,q \qquad (3.a)$$

$$\left.\frac{\partial V_m(x,y)}{\partial x}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}} = \left.\frac{\partial V_{m+1}(x,y)}{\partial x}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}}$$
$$\left.\frac{\partial V_m(x,y)}{\partial y}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}} = \left.\frac{\partial V_{m+1}(x,y)}{\partial y}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}} \qquad \forall k = 1,2,\ldots,q \qquad (3.b)$$

$$\left.\frac{\partial^2 V_m(x,y)}{\partial x^2}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}} = \left.\frac{\partial^2 V_{m+1}(x,y)}{\partial x^2}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}}$$
$$\left.\frac{\partial^2 V_m(x,y)}{\partial y^2}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}} = \left.\frac{\partial^2 V_{m+1}(x,y)}{\partial y^2}\right|_{\substack{x=x_{m,k}\\ y=y_{m,k}}} \qquad \forall k = 1,2,\ldots,q \qquad (3.c)$$

Conditions (3.a) make sure that the piecewise polynomial fits to the nodal points ($P_{m,1}$, $P_{m,2}$, …, $P_{m,k=q}$). These conditions imply that the function is continuous everywhere in the region. Conditions (3.b) and (3.c) ensure that the polynomials are continuous in slope and curvature respectively throughout the region spanned by the points *(x,y)*. Assuming the velocity to be constant in time, the difference of the two levelled height differences divided by the time span between the two levellings gives the velocity difference between the two levelling segment's ends. These 'observations' are used to compute the coefficients by means of least-squares method.

The main mathematical model is equation (2) while all the conditions under (3) show the existence of constraints on the main model. To find the least square solutions, equations (2) and (3) can be simplified in a general form:

$$f(\boldsymbol{c}, \boldsymbol{l}) = 0, \qquad (4.a)$$

$$f_c(\boldsymbol{c}) = 0. \qquad (4.b)$$

Here, $\boldsymbol{l}$ is the vector of observations and $\boldsymbol{c}$ is the vector of unknown coefficients. It will be assumed that it is possible to solve for $\boldsymbol{c}$, using only the main model (4.a). The auxiliary model f_c consists of some constraint functions that enforce the conditions which should be guaranteed. The above models are next linearized to yield:

$$\begin{aligned} \boldsymbol{A\delta} + \boldsymbol{Br} + \boldsymbol{w} &= 0, \\ \boldsymbol{D\delta} + \boldsymbol{w}_c &= 0. \end{aligned} \qquad (5)$$

In equations (5), $\boldsymbol{r}$ is the vector of expected residuals. Matrices $\boldsymbol{A}$ and $\boldsymbol{D}$ are the Jacobian matrices of transformation from parameter space to the two model spaces, valid for a small neighborhood of $\boldsymbol{c}^{(0)}$. Matrix $\boldsymbol{B}$ is the Jacobian matrix of transformation from observation space to the main model space. It is observed that equations (5) are merely the differentional form of the original non-linear mathematical model equations (4.a) and (4.b) and describe the relations of quantities in the neighborhoods of $\boldsymbol{c}^{(0)}$, the point of expansion in the parameter space, and $\boldsymbol{w}^{(0)}$, the misclosure vector, where,

$$\begin{aligned} \boldsymbol{\delta} &= \boldsymbol{c} - \boldsymbol{c}^{(0)}, \\ \boldsymbol{w}^{(0)} &= f(\boldsymbol{l}^{(0)}, \boldsymbol{c}^{(0)}). \end{aligned} \qquad (6)$$

The variation function for finding the least-squares solution is written as,

$$\phi = \boldsymbol{r}^T \boldsymbol{C}_r^{-1} \boldsymbol{r} + 2\boldsymbol{k}^T (\boldsymbol{A\delta} + \boldsymbol{Br} + \boldsymbol{w}) + 2\boldsymbol{k}_c^T (\boldsymbol{D\delta} + \boldsymbol{w}_c), \qquad (7)$$

where $C_r \equiv C_l$ is the covariance matrix of the observations. Here, there are two sets of Lagrange correlates: $\boldsymbol{k}$, $\boldsymbol{k}_c$, reflecting the fact that two models are present. The minimum with respect to $\boldsymbol{r}$ is found by the Lagrange approach (Vaníček and Krakiwsky, 1986) as

$$\hat{\boldsymbol{\delta}} = \boldsymbol{\delta}^{(1)} - \boldsymbol{N}^{-1} \boldsymbol{D}^T (\boldsymbol{D N}^{-1} \boldsymbol{D}^T)^{-1} (\boldsymbol{w}_c + \boldsymbol{D\delta}^{(1)}), \qquad (8)$$

where

$$\boldsymbol{N} = (\boldsymbol{A}^T (\boldsymbol{B C}_r \boldsymbol{B}^T)^{-1} \boldsymbol{A})^{-1} \qquad (9)$$

$$\boldsymbol{u} = \boldsymbol{A}^T (\boldsymbol{B C}_r \boldsymbol{B}^T)^{-1} \boldsymbol{w} \qquad (10)$$

$$\boldsymbol{\delta}^{(1)} = -\boldsymbol{N}^{-1} \boldsymbol{u}\ . \qquad (11)$$

Equation (11) represents the solution from the main model f alone, and the corrective term $\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}^{(1)}$ in equation (8), arises from the enforcement of the constraints.

The next task is to obtain the covariance matrix of the parameters. It is given by Vaníček and Krakiwsky (1986) as:

$$\hat{\boldsymbol{C}}_{\hat{\delta}} = \boldsymbol{N}^{-1} - \boldsymbol{N}^{-1} \boldsymbol{D}^T (\boldsymbol{D N}^{-1} \boldsymbol{D}^T)^{-1} \boldsymbol{D N}^{-1}. \qquad (12)$$

The appropriate degree of the velocity surface is determined by testing the estimated accuracy, or the ‘a posteriori standard deviation’. This is computed from

$$\hat{\sigma}_0^2 = \frac{\hat{\boldsymbol{r}}^T \boldsymbol{C}_l^{-1} \hat{\boldsymbol{r}}}{v}, \qquad (13)$$

where $\hat{\boldsymbol{r}}$ is the vector of least square residuals and v denotes the number of degree of freedom.

Due to the geophysical diversity in Eastern Canada, for example, different geological characteristics and different rate of seismicity, Eastern Canada was divided into two zones: the Maritimes zone, and the zone containing the southern part of St. Lawrence River (Figure 2). The border of these two zones is dictated by the actual data distribution and the present knowledge of the geodynamics of the area. For example, the estuary of the St. Lawrence River is an area where 50 to 100 earthquakes are detected yearly. The region, known as the Lower St. Lawrence Seismic Zone, was originally defined by spatial clustering of magnitude (M) <5 earthquakes (Basham et al., 1982 from M. Lamontagne et al., 2003). This information was used to select the zone boundaries. The vertical movement was then represented by a different polynomial surface in each zone. The polynomials were joined together at the nodal points along the zone border in such a way that the desired degree of smoothness (differentiability) of the resulting function was guaranteed.

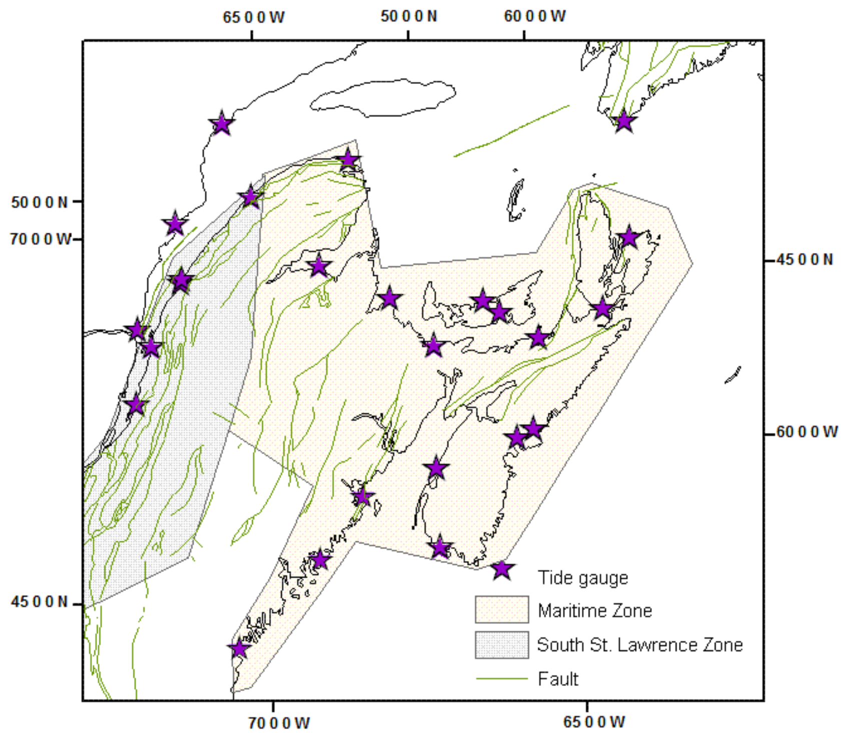

Fig 2: The polygonal subdivision used to compute the partial solutions describing the trends of VCM.

4 Results

Several tests were made to determine the appropriate degree of the velocity surface to be computed. Table 1 shows the a posteriori variance factors for the degrees 2, 3 and 4. All degrees of the polynomials yielded the a posteriori variance factors between 8.1-8.5. The value n=3 was finally selected as the highest degree compatible with data distribution.

Table1. The a posteriori variance factors of polynomial surfaces of degree 2, 3 and 4.

Degree of polynomials	Degree 2	Degree 3	Degree 4
a posterior variance factor	8.4	8.1	8.3

The map of vertical crustal movements in Eastern Canada produced by smooth piecewise algebraic polynomials is shown in Figure 3. The standard deviation for the area of interest is typically 1.4 mm/a. The solution is evidently much generalized. This is due to the sparseness of data which imposes the use of smooth functions. The map depicts clearly the zero line of the postglacial rebound. The zero line follows the St. Lawrence River. Present-day radial displacement predictions of postglacial rebound over North America computed using VM2 Earth model and ICE-5G adopted ice history (Figure 4, Peltier 2004) show a zero line very similar to ours along St. Lawrence River.

The general Northwest Southeast trend of vertical crustal movements is consistent with the predictions of Glacial Isostatic Adjustment models of Mitrovica et al. (1994), Peltier(1994), Wu(2002) and Peltier(2004).

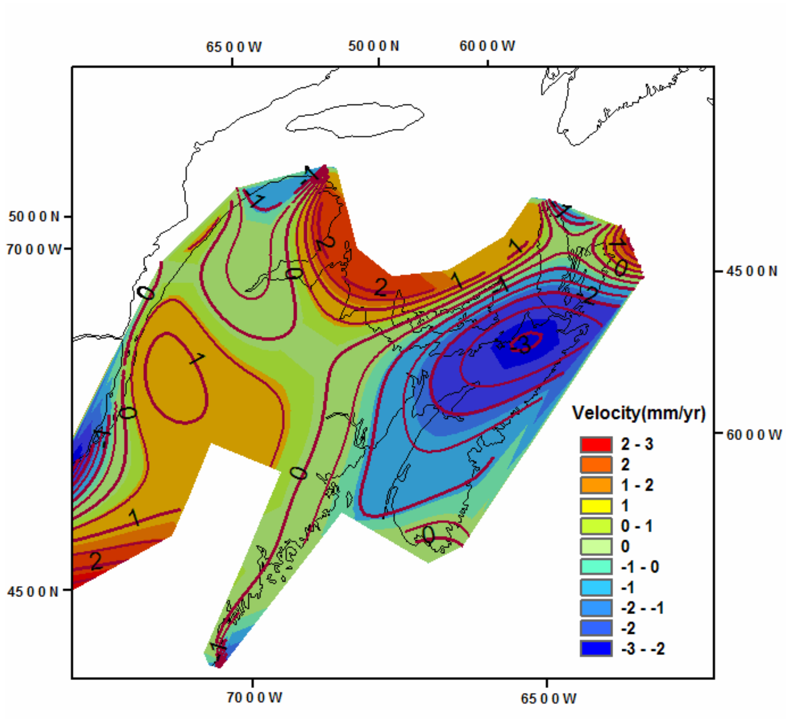

Fig 3:.Pattern of vertical crustal movements in Eastern Canada. Contours are in millimetre/year

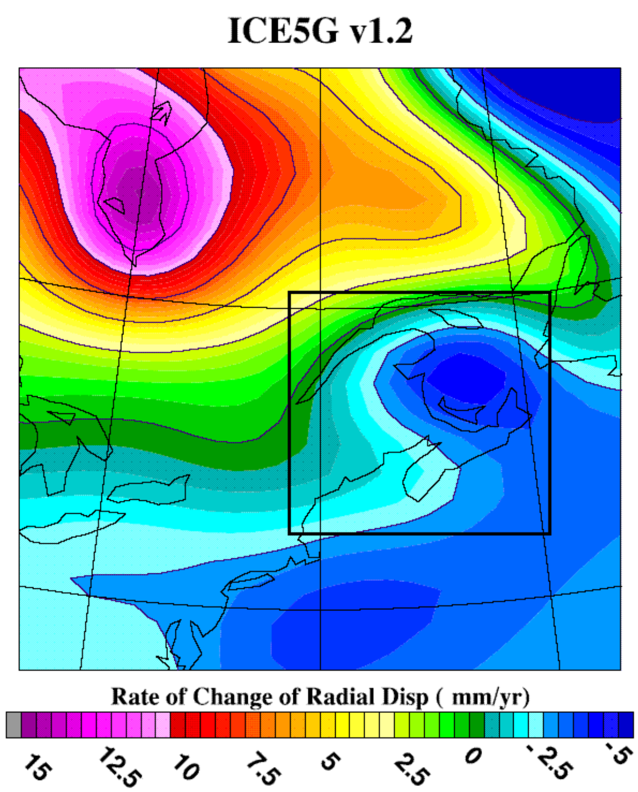

Fig 4. ICE-5G (VM2) prediction of the present-day rate of vertical movement (rate of radial displacement) of Earth's crust over the Eastern North American continent.(Adopted from Peltier, 2004)

With respect to the individual features, caution is required when interpreting the map. The technique used here was designed to model only linear vertical movements, i.e., movements with velocities constant in time. It is unlikely that all parts of Eastern Canada are undergoing such a steady vertical movements. The subsidence in Maritimes predominantly in Nova Scotia and eastern New Brunswick is due to postglacial rebound. This area lies immediately outside of the region that was covered by the Laurentide Ice Sheet at the last glacial maximum (see Peltier, 1994 for maps of surface ice cover from LGM to present). As the Laurentian ice started to decay, leading to the postglacial rebound of the crust in the once ice covered region, the forebulge began to collapse to accommodate the uplift in the central region (Peltier, 1996). The map of VCM in this area reflects this phenomenon and is also compatible with the recent map of gravity changes (See Pagiatakis, 2003 for the map of gravity changes).

The pattern shown in the north eastern margin of the former Laurentide ice sheet (the border of which has been postulated to have been parallel to St.Lawrence river) is complicated due to the probable fragmentation of the crust in this zone. The map seems to justify the concentration of seismicity in Lower St. Lawrence Zone (See Lamontagne et al., 2003 for the definition of Lower St.Lawrence Siesmic Zone), which opens new doors into the study of geodynamics of this complex area. The earlier reported uplift of the northern New Brunswick and the subsidence of the south St. Lawrence River (Carrera, G. and P. Vaníček 1994) are here more sharply defined.

5 Conclusions

The technique of smooth piecewise polynomial approximation is capable of accommodating in one model, different kinds of information when the re-levelled segments are scattered not only in time but also in space. Piecewise approximations make it easier to get the details of the map to be physically meaningful, without increasing the degree of polynomials.

The general pattern of the map reflects the main geophysical phenomenon in the region, postglacial rebound. The local pattern of the map gives more details of the South St. Lawrence River, compared to the previous maps. This is mainly due to the improvements in the methodology that enables us to define different surfaces for geophysically different areas and still maintain the continuity and smoothness throughout the region of interest. However, the computed value of 8.1 for the a posteriori variance factor indicates the probability of the existence of some shorter wavelength features that could not be modelled by a surface of such a low degree. Increasing the number of intervals (zones) in the area of computation, might be a solution for representing shorter wavelength features of VCM which would be the next step in our studies.

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