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The rigorous determination of orthometric heights

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Abstract The main problem of the rigorous definition of the orthometric height is the evaluation of the mean value of the Earth's gravity acceleration along the plumbline within the topography. To find the exact relation between rigorous orthometric and Molodensky's normal heights, the mean gravity is decomposed into: the mean normal gravity, the mean values of gravity generated by topographical and atmospheric masses, and the mean gravity disturbance generated by the masses contained within geoid. The mean normal gravity is evaluated according to Somigliana–Pizzetti's theory of the normal gravity field generated by the ellipsoid of revolution. Using the Bruns formula, the mean values of gravity along the plumbline generated by topographical and atmospheric masses can be computed as the integral mean between the Earth's surface and geoid. Since the disturbing gravity potential generated by masses inside the geoid is harmonic above the geoid, the mean value of the gravity disturbance generated by the geoid is defined by applying the Poisson integral equation to the integral mean. Numerical results for a test area in the Canadian Rocky Mountains show that the difference between the rigorously defined orthometric height and the Molodensky normal height reaches $\sim$0.5 m.

Keywords Mean gravity · Normal height · Orthometric height · Plumbline

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1 Introduction

The orthometric height is the distance, measured positive outwards along the plumbline, from the geoid (zero orthometric height) to a point of interest, usually on the topographic surface (e.g., Heiskanen and Moritz 1967, chap 4; Vaníček and Krakiwsky 1986; chap 16.4). The curved plumbline is at every point tangential to the gravity vector generated by the Earth, its atmosphere and rotation. The orthometric height can be computed from the geopotential number, if available, using the mean value of the Earth's gravity acceleration along the plumbline between the geoid and the Earth's surface. Alternatively and more practically, it can be computed from spirit levelling measurements using the so-called orthometric correction, embedded in which is the mean value of gravity (cf. Strang van Hees 1992). Ignoring levelling errors and the many issues surrounding practical vertical datum definition (see, e.g., Drewes et al. 2002; Lilje 1999), the rigorous determination of the orthometric height reduces to the accurate determination of the mean value of the Earth's gravity acceleration along the plumbline between the geoid and the point of interest.

An appropriate method for the evaluation of the mean gravity has been discussed for more than a century. The first theoretical attempt is attributed to Helmert (1890). In Helmert's definition of the orthometric height, the Poincaré–Prey gravity gradient is used to evaluate the approximate value of mean gravity from gravity observed on the Earth's surface (also see Heiskanen and Moritz 1967, chap 4; Vaníček and Krakiwsky 1986; chap 16.4). Later, Niethammer (1932) and Mader (1954) took into account the mean value of the gravimetric terrain correction within the topography. Heiskanen and Moritz (1967, p 165) also mentioned a general method for calculating mean gravity along the plumbline that includes the gravitational attraction of masses above a certain equipotential surface, thus accounting for the shape of the terrain. More recently, Vaníček et al. (1995), Allister and Featherstone (2001) and Hwang and Hsiao (2003) introduced further corrections due to vertical and lateral variations in the topographical mass-density. In addition to the above theoretical developments, numerous empirical studies have been published on the orthometric height (e.g., Ledersteger 1955; Rapp 1961; Krakiwsky 1965; Strange 1982; Sünkel 1986; Kao et al. 2000; Tenzer and Vaníček 2003; Tenzer et al. 2003; Dennis and Featherstone 2003).

Asserting that the topographical density and the actual vertical gravity gradient inside the Earth could not be determined precisely, Molodensky (1945, 1948) formulated the theory of normal heights. Here, the mean actual gravity within the topography is replaced by the mean normal gravity between the reference ellipsoid and the telluroid (also see Heiskanen and Moritz 1967, chap 4). Normal heights have been adopted in some countries, whereas (usually Helmert) orthometric heights have been adopted in others. An approximate formula relating normal and orthometric heights is given in Heiskanen and Moritz (1967, Eqs. (8–103)), with a more refined version given by Sjöberg (1995). Given that the principal difference between orthometric and normal heights is governed by the effect of physical quantities (i.e., the gravitational effects of the topography and atmosphere, and the gravity disturbance generated by the masses contained within the geoid) on the mean gravity, these are investigated in this article. It can also be argued that Molodensky's objection to the orthometric height is no longer so convincing because more and more detailed information is becoming available about the shape of (i.e., digital elevation models) and mass–density distribution inside the topography (e.g., from geological maps, cross-sections, boreholes and seismic surveys).

Finally, when we claim our theory to be rigorous, this does not imply that orthometric heights determined according to this theory are error-free. There will be errors even in the proposed rigorous orthometric heights, which originate from the errors in the field process of spirit levelling as well as in the evaluation of the mean gravity along the plumbline. The errors in the mean gravity values will depend on the distribution and accuracy of gravity, digital terrain and topographical mass–density data and the accuracy of numerical methods used for a computation.

2 Mean gravity along the plumbline

Let us begin with the 'classical' definition of the orthometric height $H^{\mathrm{O}}(\Omega)$, (e.g., Heiskanen and Moritz 1967, Eq. (4–21))

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad H^{\mathrm{O}}(\Omega) = \frac{C\,[r_t(\Omega)]}{\bar{g}(\Omega)}, \tag{1}$$

where $C\,[r_t(\Omega)]$ is the geopotential number of the point of interest, which in this case will be taken on the Earth's surface $[r_t(\Omega)]$, and $\bar{g}(\Omega)$ is the mean value of the magnitude of gravity along the plumbline between the Earth's surface $\left(r_t(\Omega) \cong r_g(\Omega) + H^{\mathrm{O}}(\Omega)\right)$ and the geoid surface for which the geocentric radius is denoted by $r_g(\Omega)$. To describe a 3D position, the system of geocentric coordinates ϕ, λ and r is used throughout this paper, where ϕ and λ are the geocentric spherical coordinates $\Omega = (\phi,\ \lambda)$, $\left(-\pi/2 \le \phi \le \pi/2;\ 0 \le \lambda < 2\pi\right)$, and r is the geocentric radius $\left(r \in \Re^{+}\right)$. The unit sphere is denoted by Ω_{O}, and $\Re^{+}$ represents the real numbers at the interval $\langle 0,\ +\infty)$.

The mean gravity $\bar{g}(\Omega)$ along the plumbline in Eq. (1) is defined by

$$\forall\, \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{g}(\Omega) = \frac{1}{H^{\mathrm{O}}(\Omega)} \int\limits_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} g(r,\Omega)\cos\left(-\mathbf{g}(r,\Omega),\ \mathbf{r^o}\right)\,\mathrm{d}r, \tag{2}$$

where $\cos\left(-\mathbf{g}(r,\Omega),\ \mathbf{r^o}\right)$ is the cosine of the deflection of the plumbline from the geocentric radial direction, and $\mathbf{r^o}$ is the unit vector in the geocentric radial direction. Equation (2) is equivalent to the integral taken along the curved plumbline as given in Heiskanen and Moritz (1967, Eq. (4–20)).

In order to analyse the mean gravity along the plumbline, the actual gravity $g(r,\Omega)$ in Eq. (2) is decomposed into the normal gravity $\gamma(r,\phi)$, the gravity *disturbance* generated by masses inside the geoid $\delta g^{\mathrm{NT}}(r,\Omega)$, and the gravitational attraction of topographical and atmospheric masses $g^t(r,\Omega)$ and $g^a(r,\Omega)$, respectively, so that (Tenzer et al. 2003)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\quad r \in \Re^{+}:$$
$$g(r,\Omega) = \gamma(r,\phi) + \delta g^{\mathrm{NT}}(r,\Omega) + g^t(r,\Omega) + g^a(r,\Omega). \tag{3}$$

Applying the above decomposition to Eq. (2), the mean gravity $\bar{g}(\Omega)$ becomes

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad \bar{g}(\Omega) = \bar{\gamma}(\Omega) + \overline{\delta g}^{\mathrm{NT}}(\Omega) + \bar{g}^t(\Omega) + \bar{g}^a(\Omega). \tag{4}$$

The relation between the mean normal gravity $\bar{\gamma}(\Omega)$ within the topography in Eq. (4) and Molodensky's mean normal gravity is formulated in Appendix A.

The main problem to be discussed in the sequel is the evaluation of the mean gravity disturbance generated by the masses inside the geoid $\overline{\delta g}^{\mathrm{NT}}(\Omega)$, and the mean topography-generated gravitational attraction $\bar{g}^t(\Omega)$. The superscript NT is used here in accordance with the notation introduced in Vaníček et al. (2004) to denote a quantity reckoned in the so-called "no-topography" space, where the gravitational effect of the topographic and atmospheric masses has been removed and treated separately. The last term in Eq. (4), i.e., the mean atmosphere-generated gravitational attraction $\bar{g}^a(\Omega)$, is derived in Appendix B.

3 Mean gravity disturbance generated by masses within the geoid

The mean gravity disturbance generated by the geoid $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ in Eq. (4) is given exactly by

$$\forall\, \Omega \in \Omega_{\mathrm{O}}:$$
$$\overline{\delta g}^{\mathrm{NT}}(\Omega) = \frac{1}{H^{\mathrm{O}}(\Omega)} \times \int\limits_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \delta g^{\mathrm{NT}}(r,\Omega)\cos\left(-\mathbf{g}(r,\Omega),\mathbf{r^o}\right)\,\mathrm{d}r. \tag{5}$$

In a spherical approximation ($r_g(\Omega) \approx R$, where R is the mean radius of the Earth, see Bomford 1971), Eq. (5) reduces to

$$\forall \Omega \in \Omega_{\mathrm{O}}: \quad \overline{\delta g}^{\mathrm{NT}}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} \delta g^{\mathrm{NT}}(r, \Omega)\, \mathrm{d}r. \qquad (6)$$

Considering an accuracy of <1 mm, the spherical approximation of the geoid surface cannot be applied directly to the evaluation of the mean gravity in Eq. (2). This is because the Earth's gravity is at least 1.5×10^3 larger than the geoid-generated gravity disturbance and topography-generated gravity. Therefore, the correction to the orthometric height due to the deflection of the vertical is investigated in Appendix A. Assessing from the estimation of the maximum magnitude of the correction of mean normal gravity due to the deflection of the plumbline, the spherical approximation in Eq. (6) causes, at most, a few μGal error in the computation of $\overline{\delta g}^{\mathrm{NT}}(\Omega)$, which propagates as an error in the orthometric height of <1 mm.

In order to evaluate the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r, \Omega)$ on the right-hand-side of Eq. (6), Poisson's solution to the Dirichlet boundary value problem is used. This is described by the Poisson integral (e.g., Kellogg 1929)

$$\forall \Omega \in \Omega_{\mathrm{O}},\ r \geq R: \quad \delta g^{\mathrm{NT}}(r, \Omega) = \frac{1}{4\pi} \frac{R}{r} \times \iint_{\Omega_O} K(r, \Omega;\ R, \Omega')\, \delta g^{\mathrm{NT}}\left[r_g(\Omega')\right] \mathrm{d}\Omega', \qquad (7)$$

where $K(r, \Omega;\ R, \Omega')$ is the spherical Poisson kernel, and $\delta g^{\mathrm{NT}}\left[r_g(\Omega)\right]$ denotes the geoid-generated gravity disturbance specifically at the geoid surface. Inserting for $\delta g^{\mathrm{NT}}(r, \Omega)$ in Eq. (6) from Eq. (7), the mean gravity disturbance $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ becomes

$$\forall \Omega \in \Omega_{\mathrm{O}}: \quad \overline{\delta g}^{\mathrm{NT}}(\Omega) \cong \frac{1}{4\pi} \frac{R}{H^{\mathrm{O}}(\Omega)} \times \iint_{\Omega_{\mathrm{O}}} \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} r^{-1} K(r, \Omega;\ R, \Omega')\, \mathrm{d}r \times \delta g^{\mathrm{NT}}\left[r_g(\Omega')\right] \mathrm{d}\Omega'. \qquad (8)$$

The radially integrated Poisson's kernel $\bar{K}\left(r, \Omega;\ R, \Omega'\right)$ in Eq. (8) can be formulated as follows:

$$\forall \Omega, \Omega' \in \Omega_{\mathrm{O}},\ r \geq R: \quad \bar{K}\left(r, \Omega;\ R, \Omega'\right) = \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} r^{-1} K\left(r, \Omega;\ R, \Omega'\right) \mathrm{d}r$$

$$= \left| -2\, R\, \ell^{-1}\left(r, \Omega;\ R, \Omega'\right) + \ln \left| \frac{R - r \cos \psi + \ell\left(r, \Omega;\ R, \Omega'\right)}{r \sin \psi} \right| \right|_{r=R}^{R+H^{\mathrm{O}}(\Omega)}, \qquad (9)$$

where $\ell\left(r, \Omega;\ r', \Omega'\right)$ is the direct Euclidean distance between the computation and roving points, and the argument ψ stands for the geocentric spherical distance.

To obtain the mean gravity disturbance from Eq. (8), the gravity disturbances generated by the geoid have to be first downward continued from the Earth's surface onto the geoid. Vaníček et al. (2004) define the gravity disturbances and anomalies at the Earth's surface as generated by the masses inside the geoid, as well as the evaluation of the inverse Dirichlet's boundary value problem for the downward continuation of the geoid-generated gravity anomalies. Alternatively, the mean value of the geoid-generated gravity disturbance $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ can be obtained directly from the gravity disturbances $\delta g^{\mathrm{NT}}\left[r_t(\Omega)\right]$ at the Earth's surface, which, in turn, is obtained from the real gravity disturbances $\delta g\left[r_t(\Omega)\right]$ by subtracting the gravitational attraction of topographical and atmospheric masses from them (*ibid.*), i.e.,

$$\forall \Omega \in \Omega_{\mathrm{O}}: \quad \delta g^{\mathrm{NT}}\left[r_t(\Omega)\right] = \delta g\left[r_t(\Omega)\right] - g^t\left[r_t(\Omega)\right] - g^a\left[r_t(\Omega)\right]. \qquad (10)$$

We shall now show how this is achieved for discrete values of the gravity disturbance.

The solution to the inverse Dirichlet's boundary value problem is described by the Poisson integral equation. To define its discretized form, the surface integration domain is split into a finite number N of 'rectangular' geographical cells $\Delta\Omega_i = \cos\phi_i\, \Delta\phi_i\, \Delta\lambda_i;\ i \in \langle 1, 2, \ldots, N\rangle$, where $\Delta\phi$ and $\Delta\lambda$ represent steps of numerical discretization in latitude and longitude. For each geographical cell, the average value of the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}\left[r_t(\Omega_i)\right]$; $i \in \langle 1, 2, \ldots, N\rangle$ is evaluated at the Earth's surface. Equivalently for each corresponding geographical cell at the geoid surface, the solution of the Poisson integral equation is parameterized by discrete values of $\delta g^{\mathrm{NT}}\left[r_g(\Omega_j)\right]$; $j \in \langle 1, 2, \ldots, N\rangle$.

The functional $b\left[r_t(\Omega_i);\ R, \Omega_j\right]$ of the Poisson integral that defines the relation between $\delta g^{\mathrm{NT}}\left[r_t(\Omega_i)\right]$ and $\delta g^{\mathrm{NT}}\left[r_g(\Omega_j)\right]$ is equal to

$$\forall i, j \in \langle 1, 2, \ldots, N\rangle: \quad b\left[r_t(\Omega_i);\ R, \Omega_j\right] = \frac{1}{4\pi} \frac{R}{r_t(\Omega_i)} K\left[r_t(\Omega_i);\ R, \Omega_j\right] \Delta\Omega_j = \frac{1}{4\pi} \frac{R^2}{r_t(\Omega_i)} \frac{r_t^2(\Omega_i) - R^2}{\ell^3\left[r_t(\Omega_i);\ R, \Omega_j\right]} \cos\phi_j\, \Delta\phi_j\, \Delta\lambda_j. \qquad (11)$$

Denoting the vector of the gravity disturbances $\delta g^{\mathrm{NT}}\left[r_t(\Omega_i)\right]$ by $\boldsymbol{\delta g}^{\mathbf{NT}}\left[\mathbf{r_t(\Omega_i)}\right]$ and the vector of the gravity disturbances $\delta g^{\mathrm{NT}}\left[r_g(\Omega_j)\right]$ by $\boldsymbol{\delta g}^{\mathbf{NT}}\left[\mathbf{r_g(\Omega_j)}\right]$, the discrete form of Poisson's integral equation is expressed by (Martinec 1996; Vaníček et al. 1996; Sun and Vaníček 1998)

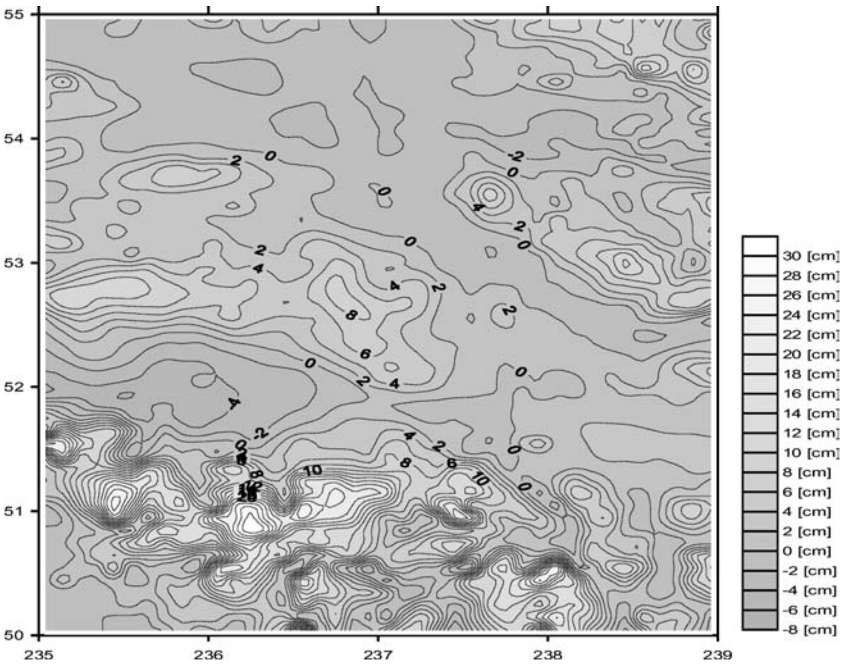

Fig. 1 Effect of the mean gravity disturbance generated by masses inside the geoid on the orthometric height (units in cm)

$$\left(\boldsymbol{\delta}\mathbf{g}^{\mathbf{NT}}\left[\mathbf{r}_{\mathbf{g}}(\boldsymbol{\Omega}_{\mathbf{j}})\right]\right)^{\mathrm{T}} = \mathbf{B}^{-1}\left[\mathbf{r}_{\mathbf{t}}(\boldsymbol{\Omega}_{\mathbf{i}});\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right]\left(\boldsymbol{\delta}\mathbf{g}^{\mathbf{NT}}\left[\mathbf{r}_{\mathbf{t}}(\boldsymbol{\Omega}_{\mathbf{i}})\right]\right)^{\mathrm{T}}, \tag{12}$$

where $\mathbf{B}\left[\mathbf{r}_{\mathbf{t}}(\boldsymbol{\Omega}_{\mathbf{i}});\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right]$ is the matrix of coefficients $b\left[r_t\left(\Omega_i\right);\ R,\Omega_j\right]$.

Consequently, the discrete form of the radially integrated Poisson's integral in Eq. (8) can be formulated as follows. The relation between the scalar value of the mean geoid-generated gravity disturbance $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ and the vector of the geoid-generated gravity disturbances referred to the geoid surface $\boldsymbol{\delta}\mathbf{g}^{\mathbf{NT}}\left[\mathbf{r}_{\mathbf{g}}\left(\boldsymbol{\Omega}_{\mathbf{j}}\right)\right]$ is introduced by

$$\forall \Omega \in \Omega_{\mathrm{O}}:$$
$$\overline{\delta g}^{\mathrm{NT}}(\Omega) = \bar{\mathbf{b}}\left[\mathbf{r},\boldsymbol{\Omega};\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right]\left(\boldsymbol{\delta}\mathbf{g}^{\mathbf{NT}}\left[\mathbf{r}_{\mathbf{g}}\left(\boldsymbol{\Omega}_{\mathbf{j}}\right)\right]\right)^{\mathrm{T}}, \tag{13}$$

where $\bar{\mathbf{b}}[\mathbf{r},\boldsymbol{\Omega};\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}]$ is the vector of radially integrated Poisson's kernels $\bar{K}(r,\Omega;\, R,\Omega_j)$ (Eq. 9). It reads

$$\bar{\mathbf{b}}\left[\mathbf{r},\boldsymbol{\Omega};\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right] = \frac{1}{4\pi}\frac{R}{H^{\mathrm{O}}(\Omega)}\bar{K}\left(r,\Omega;\ R,\Omega_j\right) \times \cos\phi_j\,\Delta\phi_j\,\Delta\lambda_j, \quad j \in \langle 1,\ 2,\ldots,\ N\rangle. \tag{14}$$

Substituting Eq. (12) into Eq. (13), the mean gravity disturbance generated by the geoid is obtained by solving the following system of linear algebraic equations

$$\overline{\delta g}^{\mathrm{NT}}(\Omega) \cong \bar{\mathbf{b}}\left(\mathbf{r},\boldsymbol{\Omega};\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right)\ \mathbf{B}^{-1}\left(\mathbf{r}_{\mathbf{t}}(\boldsymbol{\Omega}_{\mathbf{i}});\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right) \times \left(\boldsymbol{\delta}\mathbf{g}^{\mathbf{NT}}\left[\mathbf{r}_{\mathbf{t}}(\boldsymbol{\Omega}_{\mathbf{i}})\right]\right)^{\mathrm{T}}, \tag{15}$$

where the inequality shows that there remains only the discretization error.

To solve the system of linear equations for the computation of the mean gravity disturbance according to Eq. (15), the direct inversion of the matrix $\mathbf{B}\left[\mathbf{r}_{\mathbf{t}}(\boldsymbol{\Omega}_{\mathbf{i}});\, \mathbf{R},\boldsymbol{\Omega}_{\mathbf{j}}\right]$ is problematic. The iterative methods are preferably used in practice. The iterative method and the particular problems related to this topic can be found for instance in Vaníček et al. (1996) and Martinec (1996).

A numerical experiment was conducted in our test area in the Canadian Rocky Mountains, for which digital elevation and gravity data are available. This is the same test area used for previous studies (e.g., Huang et al. 2001; Martinec 1996). For the computation of the mean geoid-generated gravity disturbances $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ by solving the system of linear algebraic equations in Eq. (15), the geoid-generated gravity disturbances at the Earth's surface averaged for $5' \times 5'$ geographical grid and corresponding mean orthometric heights have been used. The number of equations has been reduced by solving Eq. (15) only for the near-zone integration sub-domain, while the far-zone contribution was estimated from a global gravity model.

The optimal size and step of the numerical integration for the near-zone depend on the shape of the topography and the variation of the gravity disturbances. Therefore, the optimal values of these parameters will vary depending on the study area, but can be deduced empirically by varying them until some predefined criterion (e.g., 1 mm) is satisfied. In this study, the $5' \times 5'$ step of the numerical integration has been used for a $7° \times 7°$ near-zone with the exclusion of the immediate neighbourhood ($15' \times 15'$) of the integration point, where a $1'' \times 1''$ step was used for the discretized numerical integration.

From Fig. 1, the contribution of the mean gravity disturbance generated by the geoid on the orthometric height varies between −8 cm and +44 cm (corresponding to heights ranging from 4 m to 2736 m, and geoid-generated gravity disturbances at the Earth's surface ranging from −153 mGal to 116 mGal).

4 Mean topography-generated gravitational attraction

By analogy with Eq. (6), the spherical approximation of the geoid surface is assumed to evaluate the mean value of the topography-generated gravitational attraction; this gives

$$\forall\,\Omega \in \Omega_O: \quad \bar{g}^t(\Omega) \cong \frac{1}{H^O(\Omega)} \int_{r=R}^{R+H^O(\Omega)} g^t(r,\Omega)\,\mathrm{d}r. \tag{16}$$

Expressing the gravitational attraction $g^t(r,\Omega)$ as a negative radial derivative of the gravitational potential of topographical masses $V^t(r,\Omega)$, Eq. (16) is rewritten as

$$\forall\,\Omega \in \Omega_O:$$
$$\bar{g}^t(\Omega) \cong -\frac{1}{H^O(\Omega)} \int_{r=R}^{R+H^O(\Omega)} \frac{\partial V^t(r,\Omega)}{\partial r}\,\mathrm{d}r. \tag{17}$$

According to the Bruns (1878) formula, the topography-generated gravitational attraction $\bar{g}^t(\Omega)$ in Eq. (17) becomes

$$\forall\,\Omega \in \Omega_O: \quad \bar{g}^t(\Omega) \cong \frac{V^t\left[r_g(\Omega)\right] - V^t\left[r_t(\Omega)\right]}{H^O(\Omega)}. \tag{18}$$

The gravitational potential of topographical masses $V^t(r,\Omega)$ is given by Newton's volume integral (e.g., Martinec 1998), which is evaluated at the points $r_g(\Omega)$ and $r_t(\Omega)$

$$\forall\,\Omega \in \Omega_O,\ r \in \Re^+:$$
$$V^t(r,\Omega) = G \iint_{\Omega_O} \int_{r'=R}^{R+H^O(\Omega')} \rho(r',\Omega')\ell^{-1}(r,\Omega;r',\Omega')r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega', \tag{19}$$

where G denotes Newton's gravitational constant, and $\rho(r,\Omega)$ is the actual density of the topographical masses.

The Newtonian integral (Eq. 19) can be rewritten as a sum (superposition) of the contributions from the spherical Bouguer shell (cf. Wichiencharoen 1982), 'terrain roughness' term (Martinec and Vaníček 1994) and anomalous topographical density distribution. For the interior of topography $r \in \langle R,\ R+H^O(\Omega)\rangle$, it reads (Wichiencharoen 1982; see also Martinec 1998, Eq. 3.14)

$$\begin{aligned} V^t(r,\Omega) &= 2\pi G\rho_o\left[R^2 + 2RH^O(\Omega) + \left[H^O(\Omega)\right]^2 - \frac{2}{3}\frac{R^3}{r} - \frac{1}{3}r^2\right] \\ &+ G\rho_o \iint_{\Omega_O} \int_{r'=R+H^0(\Omega)}^{R+H^O(\Omega')} \ell^{-1}(r,\Omega;r',\Omega')r'^2\mathrm{d}r'\mathrm{d}\Omega' \\ &+ G \iint_{\Omega_O} \int_{r'=R}^{R+H^O(\Omega')} \delta\rho(r',\Omega')\ell^{-1}(r,\Omega;r',\Omega')r'^2\,\mathrm{d}\Omega', \end{aligned} \tag{20}$$

where the topographical density $\rho(r,\Omega)$ is divided between the mean topographical density ρ_o and anomalous topographical density $\delta\rho(r,\Omega)$, such that $\rho(r,\Omega) = \rho_o + \delta\rho(r,\Omega)$.

Substitution of Eq. (20) into Eq. (18) yields

$$\begin{aligned} \forall\,\Omega \in \Omega_O: \quad \bar{g}^t(\Omega) &\cong 2\pi\, G\,\rho_o\,H^O(\Omega)\left[1 + \frac{2}{3}\frac{H^O(\Omega)}{R}\right] \\ &+ \frac{G\,\rho_o}{H^O(\Omega)} \iint_{\Omega_O} \int_{r'=R+H^O(\Omega)}^{R+H^O(\Omega')} (\ell^{-1}(R,\Omega;r',\Omega') \\ &- \ell^{-1}[r_t(\Omega);r',\Omega'])r'^2\mathrm{d}r'\,\mathrm{d}\Omega' \\ &+ \frac{G}{H^O(\Omega)} \iint_{\Omega_O} \int_{r'=R}^{R+H^O(\Omega')} \delta\rho(r',\Omega')(\ell^{-1}(R,\Omega;r',\Omega') \\ &- \ell^{-1}[r_t(\Omega);r',\Omega'])r'^2\,\mathrm{d}r'\mathrm{d}\Omega'. \end{aligned} \tag{21}$$

Treating the spherical Bouguer shell and terrain roughness terms separately, computations were performed in the same test area of the Canadian Rocky Mountains (cf. Fig. 1). The mean values of the gravitational attraction generated by the spherical Bouguer term have been computed simply according to the first term in Eq. (21), while for computation of the mean spherical terrain corrections, a detailed $3'' \times 3''$ digital terrain model has been used for the numerical integration up to 3° of spherical distance ψ around the computation point. Since the reciprocal spatial distances $\ell^{-1}(R,\Omega;\ r',\Omega')$ and $\ell^{-1}\left[r_t(\Omega);\ r',\Omega'\right]$ are practically equal for $\psi > 3^\circ$, the far-zone contribution in the second term on the right-hand-side of Eq. (21) is negligible. The effect of the spherical Bouguer shell on the orthometric height $H^O(\Omega)$, given by the first term on the right-hand-side of Eq. (21), ranges from 0 cm to −74.4 cm (Fig. 2). Likewise, the effect of terrain roughness term on the orthometric height $H^O(\Omega)$ ranges between −10 cm and +6 cm (Fig. 3). These values assume a constant topographical mass–density of 2,670 kg m^{-3}.

Disregarding water bodies, the variation of actual topographical mass density is mostly within ±300 kg m^{-3} of the commonly adopted mean value $\rho_o = 2,670$ kg m^{-3} (e.g., Martinec 1998). Therefore, the influence of laterally anomalous topographical density $\delta\rho(r,\Omega)$ amounts to about 10% of the total effect of topographical masses (Huang et al. 2001). However, larger topographical mass density variations (20–30%) are encountered in some other parts of the world (e.g., Tziavos and Featherstone 2001). Mass–density lateral variations are documented to generate centimeter to decimeter effects on the orthometric height (Vaníček et al. 1995; Tenzer et al. 2003; Tenzer and Vaníček 2003; cf. Hwang and Hsiao 2003; Allister and Featherstone 2001). In the test area used here, this effect ranges from −7 to +2 cm (Fig. 4), where the lateral topographical mass density data are the same as those used by Huang et al. (2001). At the moment, very little is known about the effect of radial variations of topographical density, which will have to be investigated in the near future. Finally, the total effect of topography (including only lateral density variations) on the orthometric height, as described

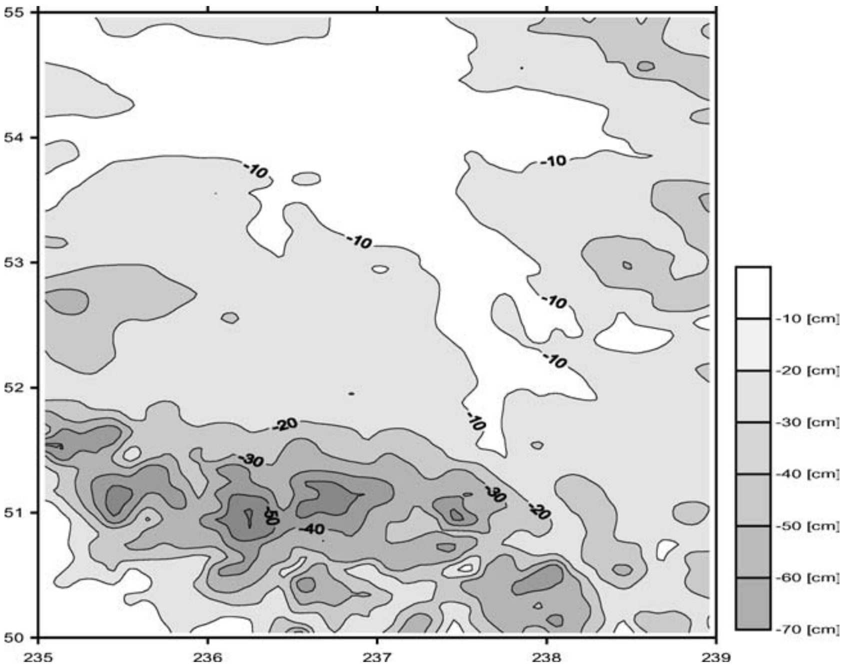

Fig. 2 Effect of the spherical Bouguer shell's gravitation on the orthometric height (units in cm)

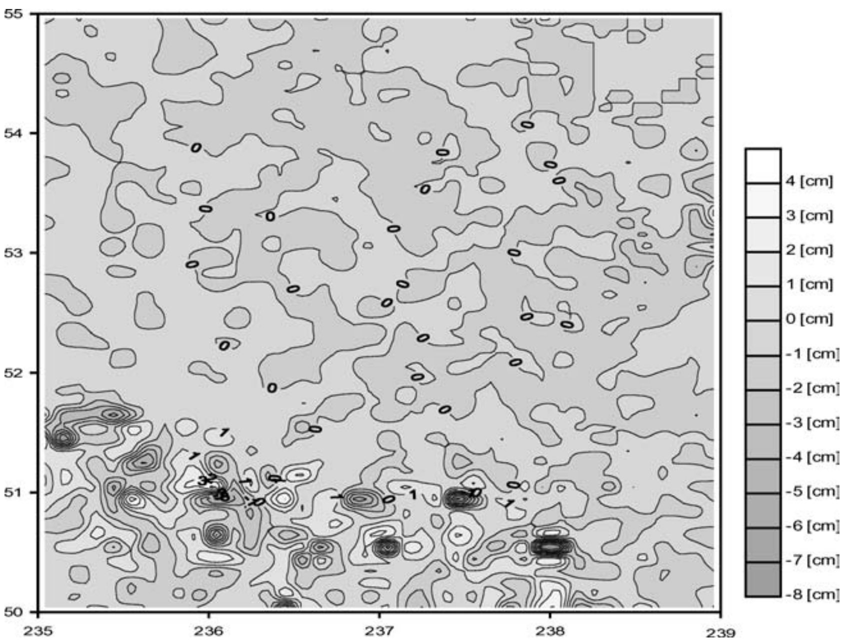

Fig. 3 Effect of terrain roughness term's gravitation on the orthometric height (units in cm)

by Eq. (21), varies between +0.1 cm and −86.5 cm in the test area (Fig. 5).

5 Discussion and conclusions

The definition of mean gravity along the plumbline in Eq. (4), which is essential to rigorously compute the orthometric height, can be considered to consist of two parts. The first part, independent of the actual gravity field, represents the mean normal gravity (Appendix A), while the second part defines the mean value of the actual gravity disturbance between the geoid and Earth's topography surface. According to Eq. (4), this mean gravity disturbance is further decomposed into the mean gravity disturbance generated by the geoid (Sect. 3) and the mean values of the gravitational attraction of topographical masses, comprising the Bouguer shell, terrain roughness and lateral density variations (Sect. 4), and the smaller valued atmospheric masses (Appendix B).

It follows from the theoretical investigation in Appendix A that the mean normal gravity between the Earth's surface

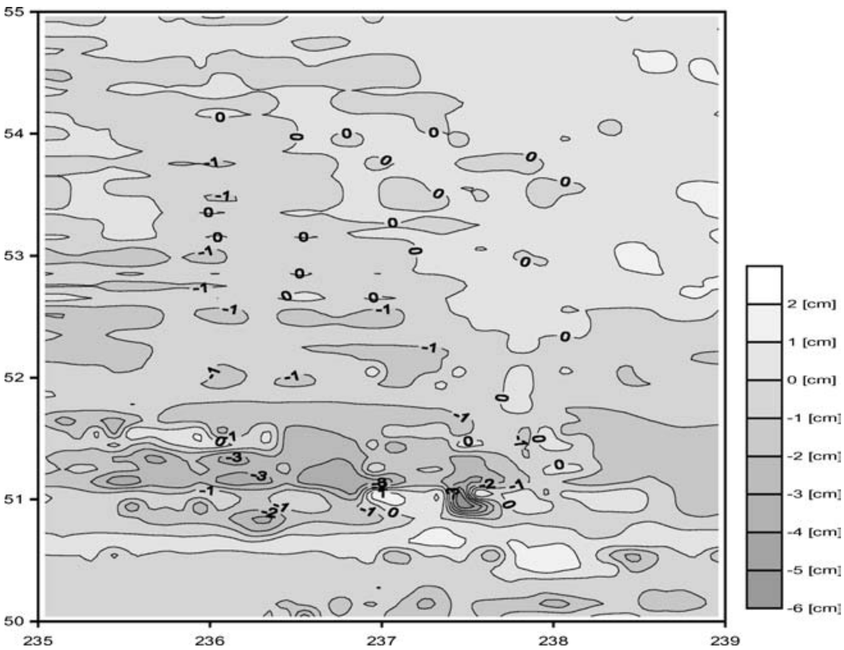

Fig. 4 Effect of anomalous topographical density distribution on the orthometric height (units in cm)

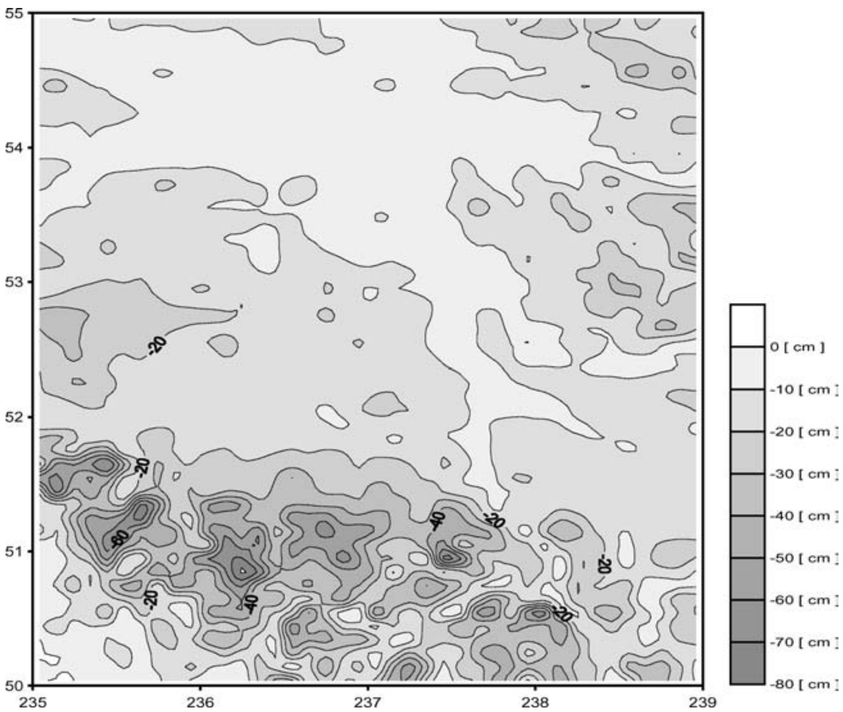

Fig. 5 Combined effect of the topography (i.e., Bouguer shell, terrain roughness term, and anomalous topographic density) on the orthometric height (units in cm)

and the geoid is defined in terms of Molodensky's mean normal gravity between the telluroid and the ellipsoid surface, plus the reductions of mean normal gravity due to the deflection and curvature of the plumbline, the height anomaly and the geoid-to-quasigeoid separation. Considering their global effects, the correction of mean normal gravity due to the height anomaly is introduced in Eq. (33). For the maximum value of the height anomaly ∓ 100.0 m, this correction reaches ± 31 mGal, which in turn corresponds to an influence on the orthometric height of up to 25 cm. Considering that the maximum vertical displacement between the geoid and quasigeoid is ~ 2 m (e.g., Sjöberg 1995), the geoid-to-quasigeoid correction to the mean normal gravity can reach up to 0.3 mGal. Based on Eq. (35), the maximum

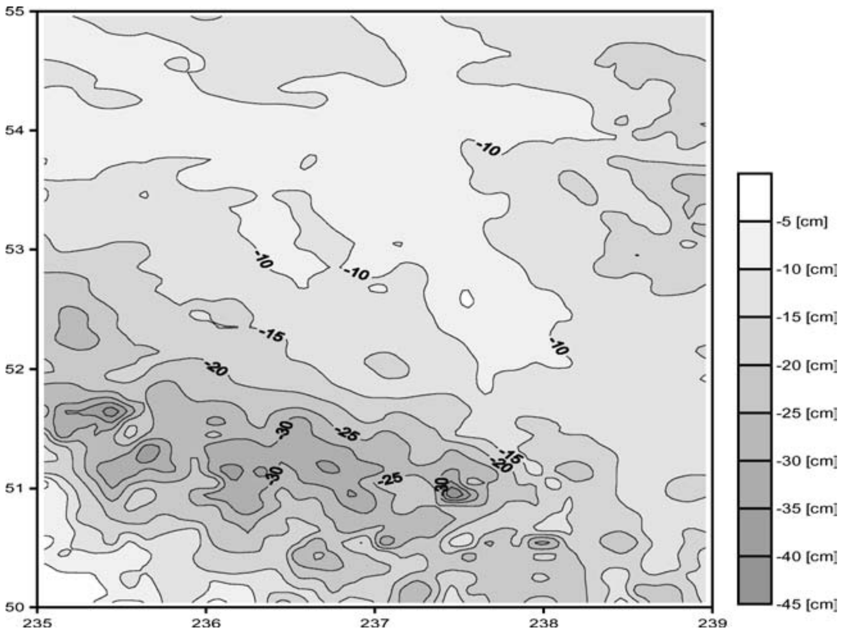

Fig. 6 Differences between the rigorous orthometric height and normal height (units in cm)

magnitude of the correction of mean normal gravity due to the deflection of the plumbline is estimated to be ~ 2.1 mGal for an extreme 2-arc-minute deflection. Hence, the geoid-to-quasigeoid correction to the mean normal gravity and the correction of mean normal gravity due to the deflection of the plumbline cause, at most, a few millimetre change in the rigorous orthometric height.

From the numerical investigations conducted in a high-elevation and rugged part of the Canadian Rocky Mountains (Sects. 3 and 4), the effect of topography and the effect due to the gravity disturbance generated by the masses inside the geoid cause up to several dm of change in the orthometric height. The total influence of these two effects on the orthometric height, which is identical to the difference between the rigorous orthometric height defined here and Molodensky normal height, varies from -0.1 cm to -45.6 cm (Fig. 6). The absence of positive difference values in this test area is because the dominant part of the influence is caused by the spherical Bouguer term (Fig. 2). On the other hand, from Appendix B and Tenzer et al. (2004), the mean atmosphere-generated gravitational attraction varies between -0.01 mGal, and -0.10 mGal, and thus has a negligible influence ($<< 1$ mm) on the orthometric height.

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Appendix A

Mean normal gravity within the topography

The mean normal gravity $\bar{\gamma}(\Omega)$ in Eq. (4) reads

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}(\Omega) = \frac{1}{H^{\mathrm{O}}(\Omega)} \times \int\limits_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \gamma\,(r,\phi)\cos\,\theta\,(r,\Omega)\cos\left(-\mathbf{g}\,(r,\Omega)\,,\,\mathbf{r}^{\mathbf{o}}\right)\,\mathrm{d}r\,, \tag{22}$$

where $\cos\,\theta(r,\Omega)$ reduces the normal gravity along the ellipsoidal normal to the plumbline.

The deflection of the plumbline from the geocentric radial direction is given by (Vaníček et al. 1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$
$$\cos\left(-\mathbf{g}\,(r,\Omega)\,,\,\mathbf{r}^{\mathbf{o}}\right) \approx 1 - \frac{(f\,\sin\,2\varphi + \xi\,(r,\Omega))^{\,2}}{2} - \frac{\eta^2\,(r,\Omega)}{2}, \tag{23}$$

where φ denotes the geodetic latitude, f is the flattening of the geocentric reference ellipsoid, and $\xi(r,\Omega)$ and $\eta(r,\Omega)$ are, respectively, the meridian and prime vertical components of the deflection of the vertical $\theta(r,\Omega)$; $\theta^2(r,\Omega) = \xi^2(r,\Omega) + \eta^2(r,\Omega)$.

The cosine of the deflection of the plumbline $\theta(r, \Omega)$ can be expressed by (Vaníček and Krakiwsky 1986)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$
$$\cos\ \theta(r, \Omega) \approx 1 - \frac{\theta^2(r, \Omega)}{2} = 1 - \frac{\xi^2(r, \Omega)}{2} - \frac{\eta^2(r, \Omega)}{2}. \tag{24}$$

Multiplying Eqs. (23) and (24) gives the following relation:

$$\forall\ \Omega \in \Omega_{\mathrm{O}}, \quad r \in \Re^{+}: \quad \cos\ \theta(r, \Omega) \cos\left(-\mathbf{g}(r, \Omega),\ \mathbf{r^o}\right)$$
$$\approx 1 - \xi(r, \Omega)\ f \sin\ 2\varphi - \theta^2(r, \Omega) - \frac{f^2 \sin^2 2\varphi}{2}. \tag{25}$$

With reference to Eq. (25), the mean normal gravity $\bar{\gamma}(\Omega)$ in Eq. (22) is rewritten as

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \times \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \gamma(r, \phi) \left(1 - \xi(r, \Omega)\ f \sin\ 2\varphi - \theta^2(r, \Omega) - \frac{f^2 \sin^2 2\varphi}{2}\right) \mathrm{d}r. \tag{26}$$

Molodensky's (1945, 1960) mean normal gravity $\bar{\gamma}^{\mathrm{N}}(\Omega)$ between the telluroid $r_o(\phi) + H^{\mathrm{N}}(\Omega)$ and the surface of the geocentric reference ellipsoid $r_o(\phi)$ reads

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}^{\mathrm{N}}(\Omega) = \frac{1}{H^{\mathrm{N}}(\Omega)} \times \int_{r=r_o(\phi)}^{r_o(\phi)+H^{\mathrm{N}}(\Omega)} \gamma(r, \phi) \cos\left(-\boldsymbol{\gamma}(r, \phi),\ \mathbf{r^o}\right)\ \mathrm{d}r, \tag{27}$$

where $\boldsymbol{\gamma}(r, \phi)$ is the vector of normal gravity, and $H^{\mathrm{N}}(\Omega)$ is the Molodensky normal height. Using the relation between geocentric and geodetic latitudes (Bomford 1971), i.e.,

$$\forall\ \phi \in \langle -\pi/2,\ \pi/2 \rangle:$$
$$\cos\left(-\boldsymbol{\gamma}(r, \phi),\ \mathbf{r^o}\right) = \cos(f \sin\ 2\varphi) \approx 1 - \frac{1}{2} f^2 \sin^2 2\varphi, \tag{28}$$

Equation (27) further takes the form

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}^{\mathrm{N}}(\Omega) = \frac{1}{H^{\mathrm{N}}(\Omega)} \times \int_{r=r_o(\phi)}^{r_o(\phi)+H^{\mathrm{N}}(\Omega)} \gamma(r, \phi) \left(1 - \frac{1}{2} f^2 \sin^2 2\varphi\right)\ \mathrm{d}r. \tag{29}$$

The first term on the right-hand-side of Eq. (26), i.e., the mean normal gravity along the radial direction, can be defined as the difference of the normal gravity potentials referred to the geoid and Earth's surface

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \gamma(r, \phi)\ \mathrm{d}r = \frac{U\left[r_g(\Omega)\right] - U\left[r_t(\Omega)\right]}{H^{\mathrm{O}}(\Omega)}. \tag{30}$$

By analogy with Eq. (30), the first term of Molodensky's mean normal gravity in Eq. (29) is defined as the difference of the normal gravity potentials referred to the ellipsoid surface and telluroid, so that

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}^{\mathrm{N}}(\Omega) \cong \frac{1}{H^{\mathrm{N}}(\Omega)} \int_{r=r_o(\phi)}^{r_o(\phi)+H^{\mathrm{N}}(\Omega)} \gamma(r, \phi)\ \mathrm{d}r = \frac{U\left[r_o(\phi)\right] - U\left[r_o(\phi) + H^{\mathrm{N}}(\Omega)\right]}{H^{\mathrm{N}}(\Omega)}. \tag{31}$$

Comparing Eqns. (30) and (31), the following relation is obtained:

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\bar{\gamma}(\Omega) - \bar{\gamma}^{\mathrm{N}}(\Omega) \cong \left.\frac{\partial\ \gamma(r, \phi)}{\partial n}\right|_{r=r_o(\phi)} \times \varsigma(\Omega) - \frac{1}{2} \left.\frac{\partial\ \gamma(r, \phi)}{\partial n}\right|_{r=r_o(\phi)} \times \left(H^{\mathrm{O}}(\Omega) - H^{\mathrm{N}}(\Omega)\right), \tag{32}$$

where $\varsigma(\Omega)$ is the height anomaly, and $\partial\gamma(r, \phi)/\partial n$ is the normal gravity gradient.

It therefore follows from Eq. (32) that two corrections are needed to reduce Molodensky's mean normal gravity $\bar{\gamma}^{\mathrm{N}}(\Omega)$ to the mean normal gravity $\bar{\gamma}(\Omega)$ between the geoid and the Earth's surface: one due to the height anomaly, and another due to the geoid-to-quasigeoid separation.

1. The correction of mean normal gravity due to the height anomaly $\varepsilon_{\bar{\gamma}}^{\varsigma}(\Omega)$ represents the shift of the integration interval from the telluroid to the Earth's surface

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\varepsilon_{\bar{\gamma}}^{\varsigma}(\Omega) \cong \left.\frac{\partial\ \gamma(r, \phi)}{\partial n}\right|_{r=r_o(\phi)} \times \varsigma(\Omega) \approx -\frac{2\gamma_o(\phi)}{a} \varsigma(\Omega), \tag{33}$$

where $\gamma_o(\phi)$ is the normal gravity on the ellipsoid surface.

2. The geoid-to-quasigeoid correction to the mean normal gravity $\varepsilon_{\bar{\gamma}}^{H}(\Omega)$ caused by a different length of the integration intervals is given by

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\varepsilon_{\bar{\gamma}}^{H}(\Omega) \cong -\left.\frac{1}{2}\frac{\partial\,\gamma\,(r,\phi)}{\partial n}\right|_{r=r_o(\phi)} \left(H^{\mathrm{O}}(\Omega) - H^{\mathrm{N}}(\Omega)\right)$$
$$\approx \frac{\gamma_o\,(\phi)}{a}\,\left(H^{\mathrm{O}}(\Omega) - H^{\mathrm{N}}(\Omega)\right). \qquad (34)$$

Comparing the second-order terms in Eqs. (26) and (29), the correction of mean normal gravity due to the deflection of the plumbline $\varepsilon_{\bar{\gamma}}^{\theta}(\Omega)$ is written finally as

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\varepsilon_{\bar{\gamma}}^{\theta}(\Omega) \cong -\frac{1}{H^{\mathrm{O}}\,(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \gamma\,(r,\phi)$$
$$\times\left(\xi\,(r,\Omega)\ f\sin\ 2\varphi + \theta^2\,(r,\Omega)\right)\,\mathrm{d}r. \qquad (35)$$

Appendix B

Mean atmosphere-generated gravitational attraction

By analogy with Eq. (18), the mean value of the atmosphere-generated gravitational attraction $\bar{g}^a(\Omega)$ in Eq. (4) reads

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\bar{g}^a(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} g^a\,(r,\Omega)\ \mathrm{d}r$$
$$= \frac{V^a\left[r_g(\Omega)\right] - V^a\left[r_t(\Omega)\right]}{H^{\mathrm{O}}(\Omega)}, \qquad (36)$$

where $V^a\,(r,\Omega)$ is the gravitational potential of the atmospheric masses.

Considering only the radially distributed atmospheric mass-density $\rho^a(r)$, the gravitational potential $V^a(r,\Omega)$ of atmospheric masses is given by (Sjöberg 1999, 2001; Novák 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\quad r \in \Re^+:$$

$$V^a(r,\Omega) \cong G \iint_{\Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\max}} \rho^a(r')\ell^{-1}(r,\Omega;r',\Omega')$$
$$\times r'^2\mathrm{d}r'\ \mathrm{d}\Omega' + G\iint_{\Omega_O} \int_{r'=R+H_{\max}}^{r_{\lim}} \rho^a(r')$$
$$\times\ell^{-1}(r,\Omega;r',\Omega')r'^2\mathrm{d}r'\ \mathrm{d}\Omega'. \qquad (37)$$

The volume integration domain within the Earth's atmosphere in Eq. (37) is divided into an 'atmospheric spherical shell' and an 'atmospheric roughness term' (analogously with the treatment of the topographic masses). The atmospheric spherical shell is defined between the upper limit of the topography $r = R + H_{\max}\left(H_{\max} = \max H^{\mathrm{O}}\,(\Omega)\right)$ and the upper limit of the atmosphere $r = r_{\lim}$. The atmospheric roughness term is enclosed by the Earth's surface and the upper limit of topography.

Since the gravitational potential of atmospheric spherical shell (given by the second integral on the right-hand side of Eq. 37) is constant in the interior $r < R + H_{\max}$ (e.g., MacMillan 1930)

$$G\iint_{\Omega_O} \int_{r'=R+H_{\max}}^{r_{\lim}} \rho^a(r')\ell^{-1}(r,\Omega;r',\Omega')r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega'$$
$$= 4\pi\ G \int_{r'=R+H_{\max}}^{r_{\lim}} \rho^a(r')\ r'\ \mathrm{d}r', \qquad (38)$$

the mean value $\bar{g}^a(\Omega)$ of the atmosphere-generated gravitational attraction in Eq. (36) reduces to

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\bar{g}^a(\Omega) \cong \frac{G}{H^{\mathrm{O}}(\Omega)} \iint_{\Omega_O} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\max}} \rho^a(r')\left(\ell^{-1}(R,\Omega;r',\Omega')\right.$$
$$\left.-\ell^{-1}\left[r_t(\Omega);r',\Omega'\right]\right)r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega'. \qquad (39)$$

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