manuscripta geodaetica (1995) 20:193–203

manuscripta geodaetica

The effect of lake water on geoidal height

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Received 15 December 1993; Accepted 14 July 1994

Abstract

The complete theory of topographical effects in the Stokes-Helmert technique for geoid determination is developed. New formulae for direct and indirect topographical effects consider lateral density variations of topographical masses. The formulae are further simplified for computing the topographical effects of water in a lake. Numerical values of the particular topographical terms are given for the lake Superior.

Introduction

Today's effort of geodesists is focused to compute the geoid with an absolute accuracy of 1 cm. To achieve this accuracy, the theory of solving the geodetic boundary value problem for geoid determination used up to now - to compute the geoid with an accuracy of about 50 cm - has to be improved. There are many theoretical problems that have to be resolved for computing such an accurate geoid (Vaníček and Martinec, 1994). For example: "How to continue the gravity data from the earth's surface to the geoid through the topographical masses so that the standard Stokes integral can be applied to gravity anomalies?", "How to incorporate the truncation error of the Stokes integration into the geoidal height corrections?", "How many terms of the Taylor expansion of the gravitational potential of topographical masses are to be taken into consideration when computing the gravitational effect of the topographical masses?", to spell out just a few.

In this paper we focus our attention on yet another problem, that of the influence of lateral changes of the density of the topographical masses (masses between the geoid and the earth's surface) on geoidal height computation. The gravitational effect of topographical masses on geoidal heights is described in the Stokes-Helmert technique by three terms (Martinec, 1993): the *direct topographical effect on gravity* which is the gravitational attraction of topographical masses at a point on the earth's surface; the *primary indirect topographical effect on potential* which is the gravitational potential of topographical masses at a point on the geoid, and the *secondary indirect topographical effect on gravity* which is the gravitational effect of topographical masses on the anomalous gravity on the geoid.

Up until now, all the formulations of the direct and both indirect topographical effects, have assumed that the topographical masses have a homogeneous density. The density was considered equal to a mean crustal value of $\varrho_0 = 2.67$ g/cm^3. This appears to be too coarse a model, especially in the vicinity of lakes, because of the large difference between the water density and the mean crustal density ϱ_0. It is thus natural to ask whether the density contrast between lake water and surrounding rock is significant enough, and if it is, how to modify the existing formulae to take into consideration this density inhomogeneity for a precise geoid computation.

The aim of the paper is to formulate an adequate theory for describing this phenomenon. Numerical example of the lake Superior (one of the Great Lakes in the central part of North America) will give an insight into the magnitude of corrections that must be added to geoidal heights when the density of lake water is erroneously modelled by the value of 2.67 g/cm^3.

The gravitational potential of topographical masses

Throughout the paper we assume that the geoid is computed by the Stokes-Helmert technique (Martinec at all., 1993). This section summarizes the formulae for computing the gravitational effects of the topographi-

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cal masses as required in the Stokes-Helmert technique. More details can be found in, e.g., Martinec et al. (1993), Martinec (1993), or Martinec and Vaníček (1994a,b).

An important quantity in the Stokes-Helmert technique for geoid computation is the residual topographical potential δV which is defined as the difference between the gravitational potential V^t of the topographical masses and the gravitational potential of the condensed topographical masses, i.e.,

$$\delta V = V^t - V^c \ . \tag{1}$$

Let us express the potentials V^t and V^c by means of Newton's integrals. Let the geocentric radius of the geoid be $r_g(\Omega)$ and the geocentric radius of the earth surface be $r_g(\Omega) + H(\Omega)$. This means that $H(\Omega)$ is the height of the earth surface above the geoid reckoned along the geocentric radius; this height, to a relative accuracy better then 5×10^{-6}, is equal to the ordinary orthometric height, which we shall assume throughout the paper to be positive and correctly determined. The argument Ω stands for a horizontal position given by co-latitude ϑ and longitude λ.

The gravitational potential V^t induced by the topographical masses at a point (r, Ω) is given by the Newton volume integral

$$V^t(r,\Omega) = \\ = G \int_{\Omega'} \int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \varrho(r',\Omega') L^{-1}(r,\psi,r') r'^2 \, dr' d\Omega' \ , \tag{2}$$

where G is Newton's gravitational constant, $\varrho(r,\Omega)$ is the density of the topographical masses, $L^{-1}(r,\psi,r')$ is the Newton kernel (reciprocal spatial distance between the dummy point (r',Ω') and the computation point (r,Ω)):

$$L^{-1}(r,\psi,r') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi}} \ , \tag{3}$$

ψ is the geocentric angular distance between the geocentric directions Ω and Ω', and the integration in eqn.(2) with respect to Ω' is taken over the full solid angle.

Further, we shall abbreviate the notation for orthometric heights $H(\Omega)$ leaving out the argument Ω. Therefore, we will use H instead of $H(\Omega)$ for the orthometric height of the topographical surface in direction Ω and H' instead of $H(\Omega')$ for the orthometric height of the topography in direction Ω'.

The potential of the Helmert condensation layer may be expressed by Newton's surface integral:

$$V^c(r,\Omega) = G \int_{\Omega'} \sigma(\Omega') \, L^{-1}\left(r,\psi,r_g(\Omega')\right) r_g^2(\Omega') d\Omega' \ , \tag{4}$$

where $\sigma(\Omega)$ is the surface density of the condensation layer. Later we will show how to choose this surface density appropriately.

Approximating the radius of the geoid $r_g(\Omega)$ by a mean radius of the earth, R, and the actual density of the topographical masses $\varrho(r,\Omega)$ by an average column value $\overline{\varrho}(\Omega)$,

$$\overline{\varrho}(\Omega) = \frac{1}{H} \int_{r=R}^{R+H} \varrho(r,\Omega) dr \ , \tag{5}$$

(the arguments for making such assumptions are discussed by, e.g., Martinec and Vaníček (1994a)), and removing the singularity of the Newton kernel, the potential V^t takes the form (Martinec, 1993):

$$V^t(r,\Omega) = \\ = V^B(r,\Omega) + G \int_{\Omega'} \left[\overline{\varrho}(\Omega') \, \widetilde{L^{-1}}(r,\psi,r') \Big|_{r'=R}^{R+H'} - \right. \\ \left. - \overline{\varrho}(\Omega) \, \widetilde{L^{-1}}(r,\psi,r') \Big|_{r'=R}^{R+H} \right] d\Omega' \ , \tag{6}$$

where the symbol $\widetilde{L^{-1}}(r,\psi,r')$ stands for the radial integral of the Newton kernel weighted by r'^2:

$$\widetilde{L^{-1}}(r,\psi,r') = \int_{r'} L^{-1}(r,\psi,r') r'^2 dr' \ . \tag{7}$$

The gravitational potential $V^B(r,\Omega)$ of the spherical Bouguer shell of thickness H and density $\overline{\varrho}(\Omega)$ may be expressed as (Wichiencharoen, 1982),

$$V^B(r,\Omega) = \begin{cases} 4\pi G \overline{\varrho}(\Omega) \frac{1}{r} \left[R^2 H + R H^2 + \frac{1}{3} H^3 \right] \ , & \text{if } r \geq R + H \ , \\ 2\pi G \overline{\varrho}(\Omega) \left[(R+H)^2 - \frac{2}{3}\frac{R^3}{r} - \frac{1}{3} r^2 \right] \ , & \text{if } R \leq r \leq R + H \ , \\ 4\pi G \overline{\varrho}(\Omega) \left[RH + \frac{1}{2} H^2 \right] \ , & \text{if } r \leq R \ . \end{cases} \tag{8}$$

The singularity of the potential of the condensed masses may be removed in a similar way (Martinec, 1993):

$$V^c(r,\Omega) = V^\ell(r,\Omega) + \\ + GR^2 \int_{\Omega'} \left[\sigma(\Omega') - \sigma(\Omega) \right] L^{-1}(r,\psi,R) d\Omega' \tag{9}$$

Here the symbol $V^\ell(r,\Omega)$ denotes the gravitational potential of a spherical layer with the density $\sigma(\Omega)$:

$$V^\ell(r,\Omega) = \begin{cases} 4\pi G \sigma(\Omega) \frac{R^2}{r} \ , & r > R \ , \\ 4\pi G \sigma(\Omega) R \ , & r \leq R \ . \end{cases} \tag{10}$$

The density $\sigma(\Omega)$ of condensed masses can be chosen in a variety of ways. In this paper we will choose it according to the principle of conservation of topographical masses (Wichiencharoen, 1982), i.e.,

$$\sigma(\Omega) = \overline{\varrho}(\Omega) \tau(\Omega) \ , \tag{11}$$

where

$$\tau(\Omega) = H\left(1 + \frac{H}{R} + \frac{H^2}{3R^2}\right) . \quad (12)$$

Topographical effects in the Stokes-Helmert technique

As mentioned in the Introduction, the effect of topographical masses on geoid determination is described by three terms (Martinec, 1993): the direct topographical effect on gravity at the earth's surface:

$$\delta A(\Omega) = \frac{\partial\, \delta V(r,\Omega)}{\partial r}\bigg|_{r=R+H} ; \quad (13)$$

the primary indirect topographical effect on potential at the geoid:

$$\delta V(R,\Omega) ; \quad (14)$$

and the secondary indirect topographical effect on the gravity at the geoid,

$$\delta A^{(2)}(\Omega) = \frac{2}{R}\delta V(R,\Omega) . \quad (15)$$

The first and the last terms affect the gravitation rather than the potential. To express their effect on the anomalous potential T, or, better still, on the geoidal height N, Stokes's integration has to be applied to them. The total topographical effect on geoidal height, N_{top}, is then given as

$$N_{top} = N_{dir} + N_{pri} + N_{sec} , \quad (16)$$

where the term N_{dir} is caused by the direct topographical effect on gravity:

$$N_{dir}(\Omega) = \frac{R}{4\pi\gamma}\int_{\Omega'} \delta A(\Omega') S(\psi) d\Omega' , \quad (17)$$

the term N_{pri} is caused by the primary indirect topographical effect on potential,

$$N_{pri}(\Omega) = \frac{\delta V(R,\Omega)}{\gamma}, \quad (18)$$

and the term N_{sec} is caused by the secondary indirect topographical effect on gravity,

$$N_{sec}(\Omega) = \frac{R}{4\pi\gamma}\int_{\Omega'} \delta A^{(2)}(\Omega') S(\psi) d\Omega' . \quad (19)$$

Here we have used the standard notation: $S(\psi)$ is the Stokes function (Heiskanen and Moritz, 1967, eqn. (2-164)), and γ is the normal gravity on the reference ellipsoid.

Substituting for δA and $\delta A^{(2)}$ from eqns.(13) and (15), and then for the residual potential δV from the Helmert decomposition (1), each of three terms N_{dir}, N_{pri} and N_{sec} may be split into two constituents:

$$N_{dir,pri,sec} = N^{t}_{dir,pri,sec} - N^{c}_{dir,pri,sec} , \quad (20)$$

where geoidal height increments N with superscripts 't' are induced by the topographical potential V^t and those with superscripts 'c' are induced by the condensation potential V^c. Explicitly,

$$N^{t,c}_{dir}(\Omega) = \frac{R}{4\pi\gamma}\int_{\Omega'} \frac{\partial V^{t,c}(r,\Omega')}{\partial r}\bigg|_{r=R+H'} S(\psi) d\Omega' , \quad (21)$$

$$N^{t,c}_{pri}(\Omega) = \frac{1}{\gamma} V^{t,c}(R,\Omega) , \quad (22)$$

and

$$N^{t,c}_{sec}(\Omega) = \frac{1}{2\pi\gamma}\int_{\Omega'} V^{t,c}(R,\Omega') S(\psi) d\Omega' . \quad (23)$$

Let us now discuss the 3 effects separately.

The direct topographical effect on gravity

The direct topographical effect on gravity, $\delta A(\Omega)$, is nothing but the radial derivative of the residual potential δV taken at a point $(R+H,\Omega)$ on the earth surface, cf., eqn.(13). Substituting for the residual potential δV from Helmert's decomposition (1) into eqn.(13), we can write

$$\delta A(\Omega) = A^t(\Omega) - A^c(\Omega) , \quad (24)$$

where

$$A^t(\Omega) = \frac{\partial V^t(r,\Omega)}{\partial r}\bigg|_{r=R+H} ,$$

and

$$A^c(\Omega) = \frac{\partial V^c(r,\Omega)}{\partial r}\bigg|_{r=R+H} , \quad (25)$$

are the radial components of the gravitational attraction induced by the topographical and condensed masses at the point on the earth surface.

Taking the radial derivative of eqn.(6) and then putting $r = R + H$, we obtain

$$A^t(\Omega) = $$

$$= A^B(\Omega) + G\int_{\Omega'} \left[\overline{\varrho}(\Omega') \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r}\bigg|_{r'=R}^{R+H'} - \overline{\varrho}(\Omega) \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r}\bigg|_{r'=R}^{R+H} \right]_{r=R+H} d\Omega' , \quad (26)$$

where $A^B(\Omega)$ is the radial component of the attraction of the Bouguer shell at the point $(R+H,\Omega)$, i.e.,

$$A^B(\Omega) = \frac{\partial V^B(r,\Omega)}{\partial r}\bigg|_{r=R+H} \quad (27)$$

Applying, as we should, the first of eqns.(8), $A^B(\Omega)$ becomes

$$A^B(\Omega) = -4\pi G \overline{\varrho}(\Omega) H \frac{R^2}{(R+H)^2}\left(1 + \frac{H}{R} + \frac{H^2}{3R^2}\right) . \quad (28)$$

Similarly, we can derive the attraction $A^c(\Omega)$ of the condensed masses at $(R+H,\Omega)$. Taking the radial derivative of eqn.(9), we get

$$A^c(\Omega) = A^\ell(\Omega) +$$

$$+GR^2 \int_{\Omega'} [\sigma(\Omega') - \sigma(\Omega)] \left. \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right|_{r=R+H} d\Omega' \; , \tag{29}$$

where $A^\ell(\Omega)$ is the attraction of a spherical single layer with the density $\sigma(\Omega)$:

$$A^\ell(\Omega) = \left. \frac{\partial V^\ell(r,\Omega)}{\partial r} \right|_{r=R+H} . \tag{30}$$

Considering, as we should, the first of eqns.(10), $A^\ell(\Omega)$ becomes

$$A^\ell(\Omega) = -4\pi G \sigma(\Omega) \frac{R^2}{(R+H)^2} . \tag{31}$$

Substituting eqns.(26) and (29) into (24), the direct topographical effect on gravity may be expressed in the form:

$$\delta A(\Omega) = A^B(\Omega) - A^\ell(\Omega) +$$

$$+G \int_{\Omega'} \left[\overline{\varrho}(\Omega') \left. \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \right|_{r'=R}^{R+H'} - \right.$$

$$- \overline{\varrho}(\Omega) \left. \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \right|_{r'=R}^{R+H} -$$

$$\left. -R^2 [\sigma(\Omega') - \sigma(\Omega)] \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right]_{r=R+H} d\Omega' \tag{32}$$

Provided that the condensation of the topographical masses is performed according to the principle of mass conservation, i.e., when the condensation density $\sigma(\Omega)$ is given by eqns.(11) and (12), then eqns.(28) and (31) readily show that

$$A^B(\Omega) = A^\ell(\Omega) . \tag{33}$$

This means that the attraction of the Bouguer shell at a point on the earth surface is equal to the attraction of a single layer at the same point. As a consequence, the first two terms in eqn.(32) cancel and the direct topographical effect on gravity reads

$$\delta A(\Omega) = G \int_{\Omega'} \left[\overline{\varrho}(\Omega') \left. \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \right|_{r'=R}^{R+H'} - \right.$$

$$- \overline{\varrho}(\Omega) \left. \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \right|_{r'=R}^{R+H} -$$

$$\left. -R^2 [\sigma(\Omega') - \sigma(\Omega)] \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right]_{r=R+H} d\Omega' . \tag{34}$$

The primary indirect topographical effect on potential

The formulae (6) and (9) may be used directly for determining the geoidal height increments $N^{t,c}_{pri}(\Omega)$ due to the primary indirect topographical effect on potential. All that is required is to put $r = R$ and divide them by normal gravity γ - cf. eqn.(22) - to get:

$$N^t_{pri}(\Omega) = \frac{4\pi G}{\gamma} \overline{\varrho}(\Omega) \left(RH + \frac{1}{2} H^2 \right) +$$

$$+\frac{G}{\gamma} \int_{\Omega'} \left[\overline{\varrho}(\Omega') \left. \widetilde{L^{-1}}(R,\psi,r') \right|_{r'=R}^{R+H'} - \right.$$

$$\left. -\overline{\varrho}(\Omega) \left. \widetilde{L^{-1}}(R,\psi,r') \right|_{r'=R}^{R+H} \right] d\Omega' \; , \tag{35}$$

and

$$N^c_{pri}(\Omega) = \frac{4\pi GR}{\gamma} \sigma(\Omega) +$$

$$+\frac{GR^2}{\gamma} \int_{\Omega'} [\sigma(\Omega') - \sigma(\Omega)] \, L^{-1}(R,\psi,R) d\Omega' \; , \tag{36}$$

where for the gravitational potential of the Bouguer shell we have substituted from the last of eqn.(8), and for the potential of spherical material layer from the last of eqn.(10). Taking the condensation density $\sigma(\Omega)$ as defined by eqns.(11) and (12), the geoidal height increments due to the primary indirect topographical effect of the residual potential δV becomes

$$N_{pri}(\Omega) = N^t_{pri}(\Omega) - N^c_{pri}(\Omega) =$$

$$= -\frac{2\pi G}{\gamma} \overline{\varrho}(\Omega) H^2 \left(1 + \frac{2}{3} \frac{H}{R} \right) +$$

$$+\frac{G}{\gamma} \int_{\Omega'} \left[\overline{\varrho}(\Omega') \left. \widetilde{L^{-1}}(R,\psi,r') \right|_{r'=R}^{R+H'} - \right.$$

$$-\overline{\varrho}(\Omega) \left. \widetilde{L^{-1}}(R,\psi,r') \right|_{r'=R}^{R+H} -$$

$$\left. -R^2 [\sigma(\Omega') - \sigma(\Omega)] \, L^{-1}(R,\psi,R) \right] d\Omega' . \tag{37}$$

The secondary indirect topographical effect on gravity

By definition (cf., eqn.(15)), the secondary indirect topographical effect on gravity, $\delta A^{(2)}(\Omega)$, is given by the residual topographical potential δV at a point on the geoid multiplied by $2/R$. This effect may be expressed by means of the geoidal height increment $N_{pri}(\Omega)$ as

$$\delta A^{(2)}(\Omega) = \frac{2\gamma}{R} N_{pri}(\Omega) . \tag{38}$$

Substituting from eqn.(37) into eqn.(38), the secondary indirect topographical effect on gravity takes the form:

$$\delta A^{(2)}(\Omega) = -\frac{4\pi G}{R} \overline{\varrho}(\Omega) H^2 \left(1 + \frac{2}{3} \frac{H}{R} \right) +$$

$$+\frac{2G}{R}\int_{\Omega'}\left[\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'} - \right.$$
$$-\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H} -$$
$$\left. -R^2\left[\sigma(\Omega')-\sigma(\Omega)\right]L^{-1}(R,\psi,R)\right]d\Omega' . \qquad (39)$$

Anomalous density of topographical masses

For computing the geoidal heights, the density of the topographical masses is usually modelled by the mean crustal density $\varrho_0 = 2.67$ g/cm^3 (e.g., Vaníček and Kleusberg (1987) or Sideris (1990)). In the mountains, small variations from this value become significant. Also the density of water in lakes such as the Great Lakes in North America, differs significantly from the value of 2.67 g/cm^3. How much of an error can this cause in geoidal heights?

To answer this question, we express density $\overline{\varrho}(\Omega)$, assumed to be varying only laterally, as a sum of the constant 'reference' value $\varrho_0 = 2.67$ g/cm^3 and a laterally varying 'anomalous' density $\delta\overline{\varrho}(\Omega)$, i.e.,

$$\overline{\varrho}(\Omega) = \varrho_0 + \delta\overline{\varrho}(\Omega) . \qquad (40)$$

Substituting the decomposition (40) into eqn.(11), the condensation density $\sigma(\Omega)$ is written analogously as:

$$\sigma(\Omega) = \sigma_0(\Omega) + \delta\sigma(\Omega) . \qquad (41)$$

The 'reference' value $\sigma_0(\Omega)$, that corresponds to the reference density ϱ_0, varies only with terrain height:

$$\sigma_0(\Omega) = \varrho_0\,\tau(\Omega) , \qquad (42)$$

whereas the 'anomalous' condensation density $\delta\sigma(\Omega)$ is associated with both the anomalous density $\delta\overline{\varrho}(\Omega)$ and the height as

$$\delta\sigma(\Omega) = \delta\overline{\varrho}(\Omega)\tau(\Omega) . \qquad (43)$$

Substituting for $\overline{\varrho}(\Omega)$ and $\sigma(\Omega)$ in eqn.(34) from eqns.(40) and (41), the direct topographical effect on gravity can be written as

$$\delta A(\Omega) = \delta A_0(\Omega) + \delta A_{\delta\overline{\varrho}}(\Omega) , \qquad (44)$$

where

$$\delta A_0(\Omega) = G\varrho_0\int_{\Omega'}\left[\frac{\partial\widetilde{L^{-1}}(r,\psi,r')}{\partial r}\Bigg|_{r'=R+H}^{R+H'} - \right.$$
$$\left. -R^2\left[\tau(\Omega')-\tau(\Omega)\right]\frac{\partial L^{-1}(r,\psi,R)}{\partial r}\right]_{r=R+H} d\Omega' , \quad (45)$$

and

$$\delta A_{\delta\overline{\varrho}}(\Omega) = G\int_{\Omega'}\left[\delta\overline{\varrho}(\Omega')\,\frac{\partial\widetilde{L^{-1}}(r,\psi,r')}{\partial r}\Bigg|_{r'=R}^{R+H'} - \right.$$
$$-\delta\overline{\varrho}(\Omega)\,\frac{\partial\widetilde{L^{-1}}(r,\psi,r')}{\partial r}\Bigg|_{r'=R}^{R+H} -$$
$$\left. -R^2\left[\delta\sigma(\Omega')-\delta\sigma(\Omega)\right]\frac{\partial L^{-1}(r,\psi,R)}{\partial r}\right]_{r=R+H} d\Omega' . \qquad (46)$$

Similarly, the geoidal height increments $N_{pri}(\Omega)$ caused by the primary indirect topographical effect on potential, cf. eqn.(37), may be decomposed as

$$N_{pri}(\Omega) = N_{pri,0}(\Omega) + N_{pri,\delta\overline{\varrho}}(\Omega) , \qquad (47)$$

where

$$N_{pri,0}(\Omega) = -\frac{2\pi G}{\gamma}\varrho_0 H^2\left(1+\frac{2}{3}\frac{H}{R}\right) +$$
$$+\frac{G\varrho_0}{\gamma}\int_{\Omega'}\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H}^{R+H'} - \right.$$
$$\left. -R^2\left[\tau(\Omega')-\tau(\Omega)\right]L^{-1}(R,\psi,R)\right]d\Omega' , \qquad (48)$$

and

$$N_{pri,\delta\overline{\varrho}}(\Omega) = -\frac{2\pi G}{\gamma}\delta\overline{\varrho}(\Omega)H^2\left(1+\frac{2}{3}\frac{H}{R}\right) +$$
$$+\frac{G}{\gamma}\int_{\Omega'}\left[\delta\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'} - \right.$$
$$-\delta\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H} -$$
$$\left. -R^2\left[\delta\sigma(\Omega')-\delta\sigma(\Omega)\right]L^{-1}(R,\psi,R)\right]d\Omega' . \qquad (49)$$

Finally, the secondary indirect topographical effect on gravity, cf.eqn.(39), may be decomposed as

$$\delta A^{(2)}(\Omega) = \delta A_0^{(2)}(\Omega) + \delta A_{\delta\overline{\varrho}}^{(2)}(\Omega) , \qquad (50)$$

where

$$\delta A_0^{(2)}(\Omega) = -\frac{4\pi G}{R}\varrho_0 H^2\left(1+\frac{2}{3}\frac{H}{R}\right) +$$
$$+\frac{2G\varrho_0}{R}\int_{\Omega'}\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H}^{R+H'} - \right.$$
$$\left. -R^2\left[\tau(\Omega')-\tau(\Omega)\right]L^{-1}(R,\psi,R)\right]d\Omega' , \qquad (51)$$

and

$$\delta A_{\delta\overline{\varrho}}^{(2)}(\Omega) = -\frac{4\pi G}{R}\delta\overline{\varrho}(\Omega)H^2\left(1+\frac{2}{3}\frac{H}{R}\right) +$$
$$+\frac{2G}{R}\int_{\Omega'}\left[\delta\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'} - \right.$$

$$-\delta\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H} -$$

$$-R^2\left[\delta\sigma(\Omega')-\delta\sigma(\Omega)\right]L^{-1}(R,\psi,R)\,d\Omega' \,. \qquad (52)$$

In summary, the terms $\delta A_0(\Omega)$, $N_{pri,0}(\Omega)$ and $\delta A_0^{(2)}(\Omega)$ represent the direct and both the indirect topographical effects induced by topographical masses of constant density ϱ_0, whereas terms $\delta A_{\delta\overline{\varrho}}(\Omega)$, $N_{pri,\delta\overline{\varrho}}(\Omega)$ and $\delta A_{\delta\overline{\varrho}}^{(2)}(\Omega)$ represent the effects induced by column averages of lateral anomalies $\delta\overline{\varrho}(\Omega)$ of the topographical density. The former terms are usually considered in geoid computations (e.g., Vaníček and Kleusberg (1987), or Sideris (1990)), while the latter terms are not. Since our interest here is focused on exploring the effects of lateral anomalies of topographical density on geoidal heights, we will further investigate only the terms $\delta A_{\delta\overline{\varrho}}(\Omega)$, $N_{pri,\delta\overline{\varrho}}(\Omega)$ and $\delta A_{\delta\overline{\varrho}}^{(2)}(\Omega)$.

One particular example: a lake

A lake whose surface has a non-zero topographical height represents an obvious example of lateral changes in the topographical density $\overline{\varrho}(\Omega)$ - note that a lake at the altitude of sea level, or ocean, for that matter, has zero topographical height and thus zero topographical effects. We will denote the density of water by ϱ_w, (1.0 g/cm^3), while the density of surrounding topographical masses will be denoted by ϱ_0 (2.67 g/cm^3). Let the orthometric height of the lake surface be H ($H > 0$) and the depth of the lake be $D(\Omega)$, ($D(\Omega) \geq 0$). To a high degree of approximation, we may assume that $H = H_0 = const.$ over the whole lake.

By definition (see eqn.(5)), the laterally varying density $\overline{\varrho}(\Omega)$ is evaluated from the actual 3-D density $\varrho(r,\Omega)$ by averaging along the topographical column of height H. For our example of a lake, this formula yields

$$\overline{\varrho}(\Omega) = \begin{cases} \left[\varrho_w\, D(\Omega) + \varrho_0\,(H_0 - D(\Omega))\right]/H_0 \,, & \text{if } D(\Omega) \leq H_0 \,, \\ \varrho_w \,, & \text{if } D(\Omega) > H_0 \,. \end{cases} \qquad (53)$$

The anomalous density $\delta\overline{\varrho}(\Omega)$, cf. eqn.(40), is then given by

$$\delta\overline{\varrho}(\Omega) = \begin{cases} (\varrho_0 - \varrho_w)D(\Omega)/H_0 \,, & D(\Omega) \leq H_0 \,, \\ \varrho_0 - \varrho_w \,, & D(\Omega) > H_0 \,. \end{cases} \qquad (54)$$

The first of eqns.(54) shows in particular that if $D(\Omega) = 0$, then $\delta\overline{\varrho}(\Omega) = 0$.

For a lake, the direct topographical effect on gravity given by the term $\delta A_{\delta\overline{\varrho}}(\Omega)$, cf., eqn.(46), may be further simplified. Realizing that the height of the lake surface is constant, we may put $r' = R + H_0$ (instead of $r' = R + H'$) in the first term on the right hand side. Moreover, the second term in this equation is equal to zero when the computation point lies outside the lake because the anomalous density $\delta\overline{\varrho}(\Omega)$ vanishes outside the lake. At the lake, the height H of the computation point is equal to H_0. Therefore, we may put $r' = R + H_0$ (insted of $r' = R + H$) in the second term without changing its numerical value. The term $\delta A_{\delta\overline{\varrho}}(\Omega)$ then becomes

$$\delta A_{\delta\overline{\varrho}}(\Omega) =$$

$$= G \int_{\Omega'} \left[\left[\delta\overline{\varrho}(\Omega') - \delta\overline{\varrho}(\Omega)\right] \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \Bigg|_{r'=R}^{R+H_0} - \right.$$

$$\left. -R^2\left[\delta\sigma(\Omega') - \delta\sigma(\Omega)\right] \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right]_{r=R+H} d\Omega' . \qquad (55)$$

Expressing the anomalous condensation densities in the positions Ω and Ω' according to eqn.(41), we have

$$\delta\sigma(\Omega) = \delta\overline{\varrho}(\Omega)\,\tau(\Omega) \,,$$

$$\delta\sigma(\Omega') = \delta\overline{\varrho}(\Omega')\,\tau(\Omega') \,, \qquad (56)$$

where

$$\tau(\Omega) = H\left(1 + \frac{H}{R} + \frac{H^2}{3R^2}\right) ,$$

and

$$\tau(\Omega') = H_0\left(1 + \frac{H_0}{R} + \frac{H_0^2}{3R^2}\right) . \qquad (57)$$

The condensation density $\delta\sigma(\Omega)$ vanishes if the computation point Ω is outside the lake, therefore, the function $\tau(\Omega)$ may be chosen arbitrarily for this position. At the lake, the function $\tau(\Omega)$ is equal to $\tau(\Omega')$ because of the fixed height of the lake surface. In summary, we may put

$$\tau(\Omega') = \tau(\Omega) = \tau_0 = H_0\left(1 + \frac{H_0}{R} + \frac{H_0^2}{3R^2}\right) . \qquad (58)$$

The spherical approximation (used throughout the paper) then yields

$$\tau(\Omega') = \tau(\Omega) = \tau_0 \doteq H_0 \,. \qquad (59)$$

Substituting eqns.(56) and (59) into (55), $\delta A_{\delta\overline{\varrho}}(\Omega)$ may be finally written in the following form:

$$\delta A_{\delta\overline{\varrho}}(\Omega) =$$

$$= G \int_{\Omega'} \left[\delta\overline{\varrho}(\Omega') - \delta\overline{\varrho}(\Omega)\right] \left[\frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \Bigg|_{r'=R}^{R+H_0} - \right.$$

$$\left. -R^2 H_0 \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right]_{r=R+H} d\Omega' . \qquad (60)$$

The geoidal height increment $N_{pri,\delta\overline{\varrho}}(\Omega)$, cf. eqn.(49), may be expressed analogously, getting:

$$N_{pri,\delta\overline{\varrho}}(\Omega) = -\frac{2\pi G}{\gamma}\delta\overline{\varrho}(\Omega)H_0^2\left(1 + \frac{2}{3}\frac{H_0}{R}\right) +$$

$$+\frac{G}{\gamma}\int_{\Omega'}[\delta\overline{\varrho}(\Omega')-\delta\overline{\varrho}(\Omega)]\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H_0}-\right.$$
$$\left.-R^2H_0L^{-1}(R,\psi,R)\right]d\Omega' \ . \qquad (61)$$

The secondary indirect effect $\delta A^{(2)}_{\delta\overline{\varrho}}(\Omega)$, cf. eqn.(52), is simple to derive from the last equation:

$$\delta A^{(2)}_{\delta\overline{\varrho}}(\Omega)=-\frac{4\pi G}{R}\delta\overline{\varrho}(\Omega)H_0^2\left(1+\frac{2}{3}\frac{H_0}{R}\right)+$$
$$+\frac{2G}{R}\int_{\Omega'}[\delta\overline{\varrho}(\Omega')-\delta\overline{\varrho}(\Omega)]\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H_0}-\right.$$
$$\left.-R^2H_0L^{-1}(R,\psi,R)\right]d\Omega' \ . \qquad (62)$$

For the highest lake in the world, the lake Titicaca in Peru ($H_0 = 3810$ m), the second term of $N_{pri,\delta\overline{\varrho}}(\Omega)$, the term

$$-\frac{4\pi G}{3\gamma}\delta\overline{\varrho}(\Omega)\frac{H_0^3}{R}$$

has the value of -0.4 mm. Similarly, the corresponding term of $\delta A^{(2)}_{\delta\overline{\varrho}}(\Omega)$, the term

$$-\frac{8\pi G}{3}\delta\overline{\varrho}(\Omega)\frac{H_0^3}{R^2}$$

has the value of -10^{-3} mgals. It is thus not worth considering these terms at all. Neglecting them, eqns.(61) and (62) read in their final form:

$$N_{pri,\delta\overline{\varrho}}(\Omega)\doteq-\frac{2\pi G}{\gamma}\delta\overline{\varrho}(\Omega)H_0^2+$$
$$+\frac{G}{\gamma}\int_{\Omega'}[\delta\overline{\varrho}(\Omega')-\delta\overline{\varrho}(\Omega)]\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H_0}-\right.$$
$$\left.-R^2H_0L^{-1}(R,\psi,R)\right]d\Omega' \ , \qquad (63)$$

and

$$\delta A^{(2)}_{\delta\overline{\varrho}}(\Omega)\doteq-\frac{4\pi G}{R}\delta\overline{\varrho}(\Omega)H_0^2+$$
$$+\frac{2G}{R}\int_{\Omega'}[\delta\overline{\varrho}(\Omega')-\delta\overline{\varrho}(\Omega)]\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H_0}-\right.$$
$$\left.-R^2H_0L^{-1}(R,\psi,R)\right]d\Omega' \ . \qquad (64)$$

Numerical results for the lake Superior

Lake Superior in the central part of North America was chosen to ilustrate the effect of lateral changes of topographical density caused by lake water on geoidal heights. The area under study is bounded by latitudes $\phi \doteq 46^o - 49^o$ North and longitudes $\lambda \doteq 268^o - 276^o$ East. Figure 1 shows the depth of lake Superior as provided by the Geodetic Survey of Canada. The maximum depth is 329 m, the orthometric height of the lake surface is approximately 183 m.

Figure 2 shows the plot of the term $\delta A_{\delta\overline{\varrho}}(\Omega)$ over the lake Superior. We can see that the magnitude of this term ranges from -0.14 to 0.18 mGal. The Stokes integration of the corrections $\delta A_{\delta\overline{\varrho}}(\Omega)$ provides the increment $N_{dir,\delta\overline{\varrho}}(\Omega)$ to the geoidal heights in the form:

$$N_{dir,\delta\overline{\varrho}}(\Omega)=\frac{R}{4\pi\gamma}\int_{\Omega'}\delta A_{\delta\overline{\varrho}}(\Omega')S(\psi)d\Omega' \ . \qquad (65)$$

Figure 3 shows that this increment ranges from -1.1 to 1.3 cm.

To get a better estimate of the magnitude of the individual terms making the increment $N_{pri,\delta\overline{\varrho}}(\Omega)$ to the geoidal undulations due to the primary indirect topographical effect, let us divide this term into two constituents:

$$N_{pri,\delta\overline{\varrho}}(\Omega)=N^B_{pri,\delta\overline{\varrho}}(\Omega)+N^R_{pri,\delta\overline{\varrho}}(\Omega) \ , \qquad (66)$$

where the Bouguer term $N^B_{pri,\delta\overline{\varrho}}(\Omega)$ is equal to

$$N^B_{pri,\delta\overline{\varrho}}(\Omega)=-\frac{2\pi G}{\gamma}\delta\overline{\varrho}(\Omega)H_0^2\left(1+\frac{2}{3}\frac{H_0}{R}\right) \ . \qquad (67)$$

Figure 4 shows the plot of the term $N^B_{pri,\delta\overline{\varrho}}(\Omega)$ over the lake Superior. We can observe that the magnitude of this term ranges from -0.24 to 0.0 cm, the largest negative values are encountered in the deepest parts of the lake.

Let us explain the reason why the minimal value of -0.24 cm has a 'plateau' over the deepest parts of the lake Superior. The orthometric height of the surface of the lake is 183 m, while the depth of the lake reaches the value of 329 m. This means that the deepest water masses of the lake, whose depth is more than 183 m, lie under the geoid. Inspecting equation (54) for the anomalous density $\delta\overline{\varrho}(\Omega)$ of the topographical masses, we can find that the "topographical masses" for these deepest parts of the lake have a constant density $\varrho_0 - \overline{\varrho}_w$ (=1.67 g/cm^3). Since the height H of the observer is also constant over the lake, $H = H_0$, eqn.(67) shows that the term $N^B_{pri,\delta\overline{\varrho}}(\Omega)$ is constant over the deepest parts of the lake, i.e., parts whose the depth is larger than 183 m.

The term $N^R_{pri,\delta\overline{\varrho}}(\Omega)$, an analogue of the terrain term in the indirect effect (Heiskanen and Moritz, 1967, sect.3-3) for a case when the density of topographical masses varies laterally, is given by

$$N^R_{pri,\delta\overline{\varrho}}(\Omega)=$$
$$=\frac{G}{\gamma}\int_{\Omega'}[\delta\overline{\varrho}(\Omega')-\delta\overline{\varrho}(\Omega)]\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H_0}-\right.$$
$$\left.-R^2H_0L^{-1}(R,\psi,R)\right]d\Omega' \ . \qquad (68)$$

Figure 5 plots the term $N^R_{pri,\delta\overline{\varrho}}(\Omega)$ over the lake Superior. The magnitude of this term is of the order of 4×10^{-5} m. Comparing these values with the magnitude of the term $N^B_{pri,\delta\overline{\varrho}}(\Omega)$ (Figure 4), we observe that

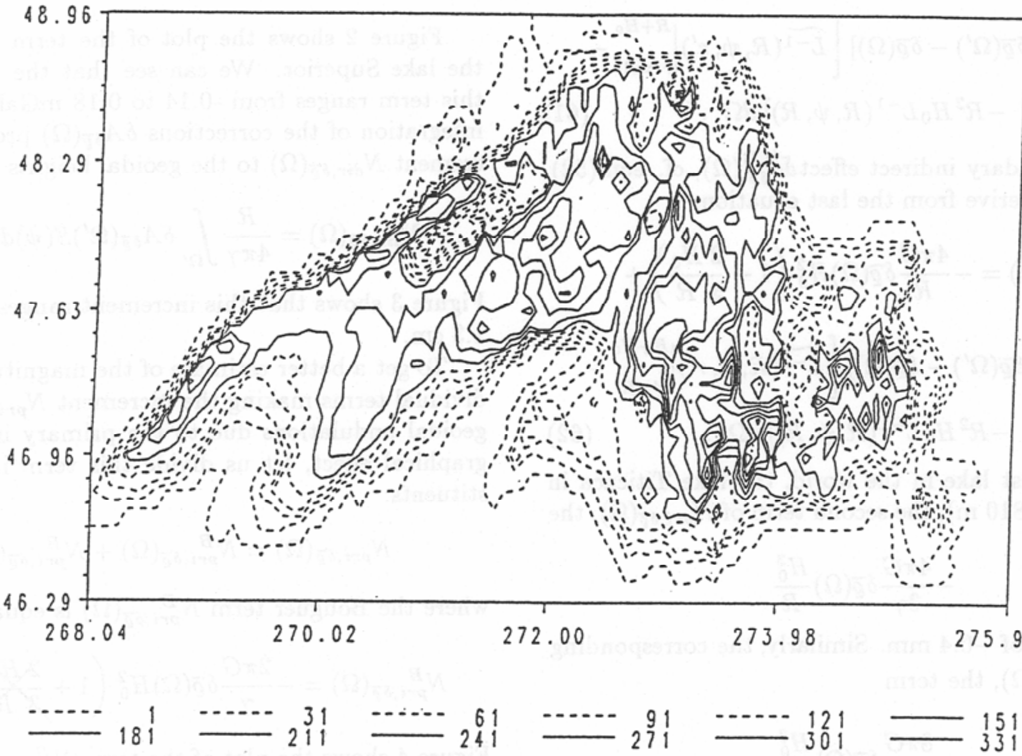

Figure 1: Depth of lake Superior (in metres).

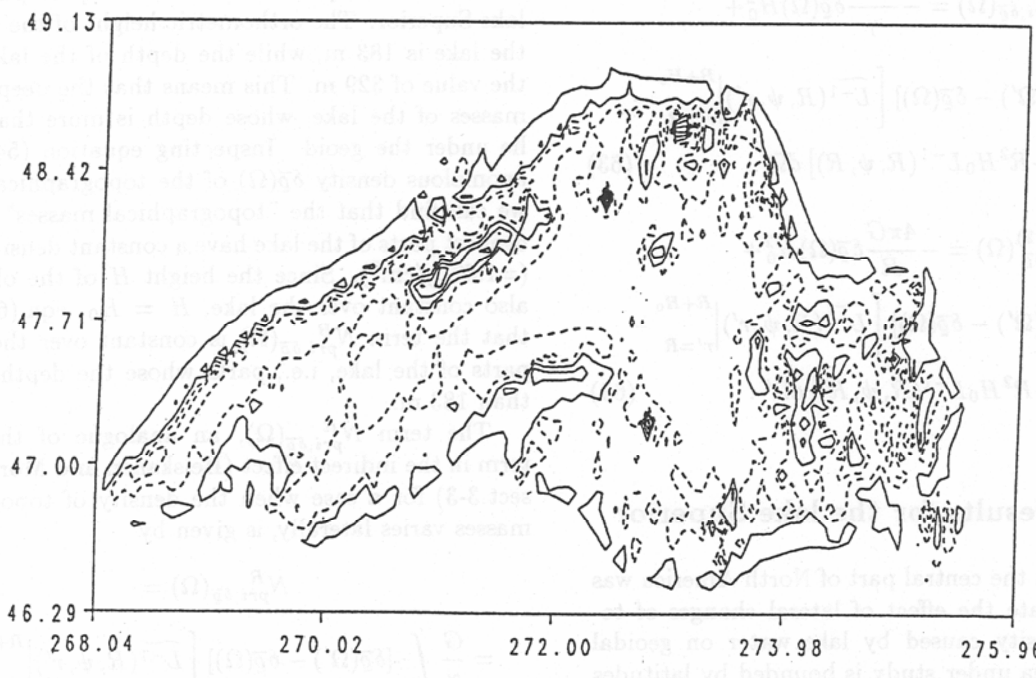

Figure 2: The direct topographical effect on gravity $\delta A_{\delta\overline{\varrho}}(\Omega)$ over lake Superior (in mGals).

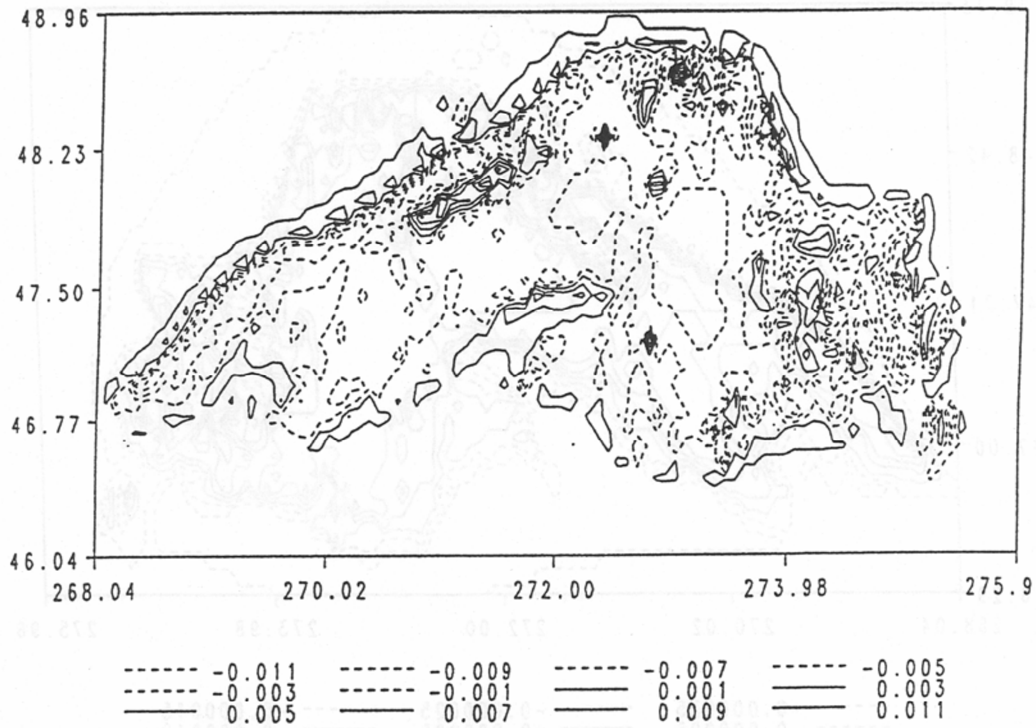

Figure 3: The geoid height increment generated by corrections $\delta A_{\delta\overline{\varrho}}(\Omega)$ over lake Superior (in metres). Truncation radius of Stokes's integration is 6°.

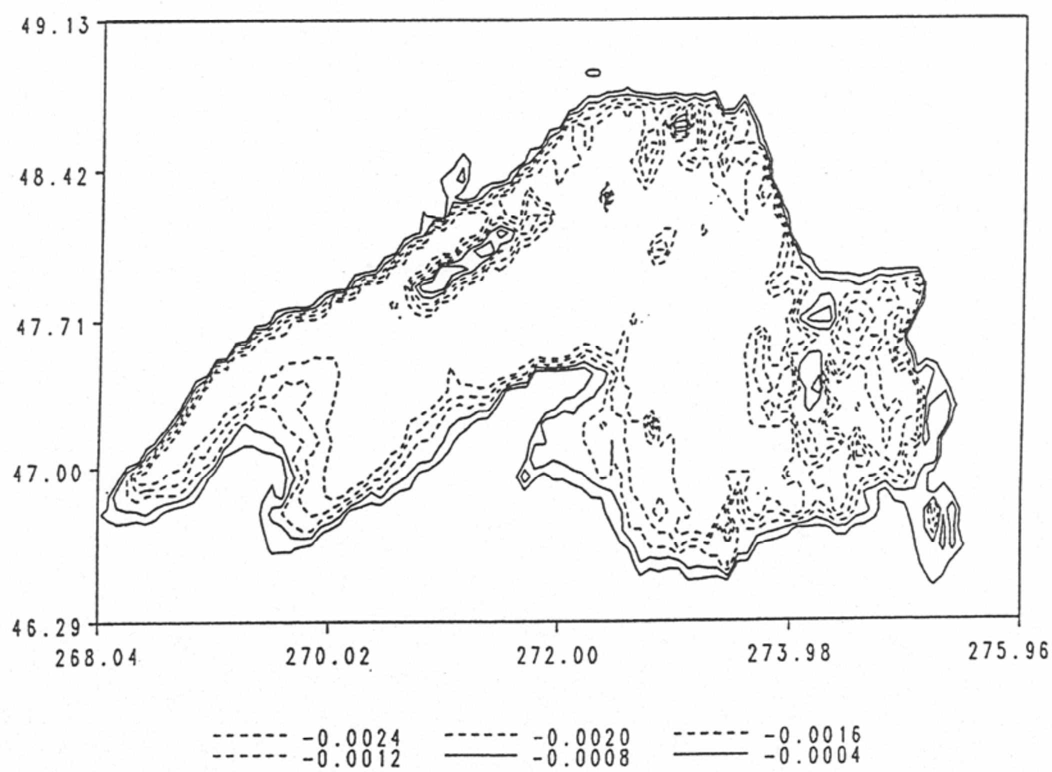

Figure 4: The geoid height increment generated by the Bouguer part of the primary indirect topographical effect $N^{B}_{pri,\delta\overline{\varrho}}(\Omega)$ over lake Superior (in metres).

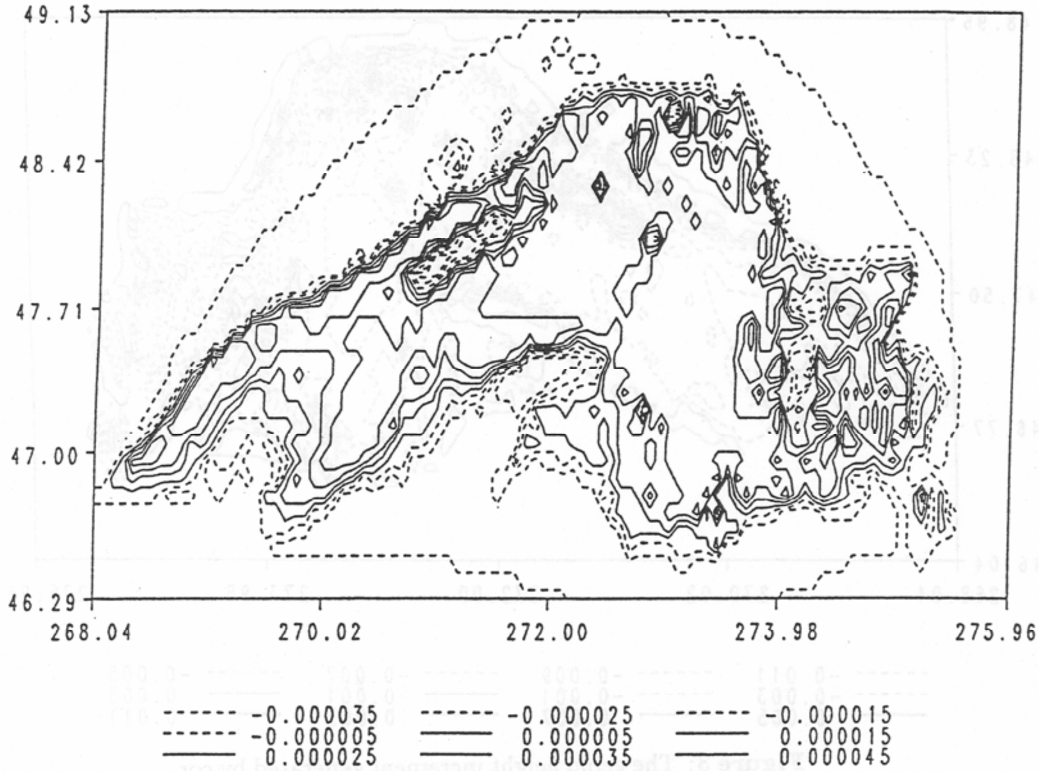

Figure 5: The geoid height increment generated by the terrain correction to the primary indirect topographical effect $N^{R}_{pri,\delta\overline{\varrho}}(\Omega)$ over lake Superior (in metres).

manuscripta geodaetica (1995) 20:204–230

manuscripta geodaetica

The theory of optimal linear estimation for continuous fields of measurements

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Received 30 December 1993; Accepted 16 November 1994

Summary

The main problem of linear estimation theory in infinite dimensional spaces is presented and its typical difficulties are illustrated.
The use of Wiener measures to represent continuous observation equations is carefully analysed in relation to the physical description of measurements and to the mathematical limit when the number of observations grows to infinity.
The overdetermined problem is solved by applying the Wiener principle of minimizing the mean square estimation error; the solution is proved to exist and to be unique under very general conditions on the observation operators.
Examples coming from space geodesy, potential theory and image analysis are presented to prove the effectiveness of the method and its applicability in different contexts.

1. Introduction

In this paragraph we would like to discuss the main differences and difficulties encountered when we try to generalize the usual least squares estimation theory to infinite dimensional spaces.
A classical scheme in linear estimation theory is the following: assume the set of measurements to be collected in a vector Y belonging to some linear vector space H_Y, also called the space of observables, with finite dimensions n_Y; the theoretical value y is constrained to belong to some linear (proper) manifold in H_Y, which for instance is described in parametric form as

$$\{ y = Ax + a \; ; \; x \in H_X \}$$

where the vector x ranges in the so-called parameter space H_X, also finite dimensional, with $n_X < n_Y$, a is a constant vector in H_Y, A is a linear operator (matrix) from H_X into H_Y.
The relation

$$y = Ax + a \qquad (1.1)$$

reflects the physics and the geometry of the observational process, i.e. a description of all what is known of the physical and geometric relations between the quantities relevant to the experiment, including those which describe the internal states of the instruments; in (1.1) the vector a represents just a fixed translation of the range of A, $\mathcal{R}_A$, away from the origin of H_Y and it can be eliminated by a coordinate shift, therefore from now on we suppose a = 0 and the manifold of the admissible values will coincide with $\mathcal{R}_A$. The vector of observations Y does not belong to the manifold of the admissible values, because the model (1.1) is imperfect in describing the measurement process, i.e. we rather have an observational model

$$Y = Ax + \nu \qquad (1.2)$$

the Bouguer term $N^B_{pri,\delta\overline{\varrho}}(\Omega)$ is about two orders larger than the terrain term $N^R_{pri,\delta\overline{\varrho}}(\Omega)$. This fact can be easily explained by the shape of the bottom of the lake. The slope of the lake banks is very steep so that a larger part of the lake has a depth greater than 183 m. Therefore, replacing real "topographical masses" by the Bouguer plate is a fairly good approximation.

We have also computed the effect of the secondary indirect topographical effect on the geoid, cf., eqn.(62). The Stokes integration of $A^{(2)}_{\delta\overline{\varrho}}(\Omega)$ yields the increment $N_{sec,\delta\overline{\varrho}}(\Omega)$. The numerical values of this contribution to the geoid are another of magnitude smaller than the term $N^R_{pri,\delta\overline{\varrho}}(\Omega)$; for a 1 cm geoid both $N^R_{pri,\delta\overline{\varrho}}(\Omega)$ and $N_{sec,\delta\overline{\varrho}}(\Omega)$ may be safely neglected.

Conclusions

This paper was motivated by a question whether lakes should be considered as lateral density inhomogeneities of the topographical masses, when the geoid is to be determined with a high accuracy. The standard way of approximating the density of topographical masses when computing their gravitational effects is to take the constant value of 2.67 g/cm^3, while the lake water density is 62% less.

Numerical computations have been caried out for lake Superior, the largest of the Great Lakes of North America. We have computed the correction to geoidal heights when the density of 1.0 g/cm^3 was used in the computations instead of the density of 2.67 g/cm^3. The numerical values of these corrections for lake Superior are fairly small; the correction to the geoidal height due to the direct topographical effect on gravity lies within the range (-1.1, 1,3) cm, the correction to the geoidal height due to the primary indirect topographical effect on potential is within (-0.24, 0.0) cm. We have shown that the dominant term in the latter corrections is the Bouguer term (67), which depends linearly on the density contrast between water and surrounding rock, and quadratically on the orthometric height of the lake surface.

To take into consideration a laterally varying density, the singularity of the Newton integral for the direct as well as primary and secondary indirect topographical effects have been removed using a different approach from that for a case of a constant topographical density (Moritz, 1968). Moreover, we have shown that for the case of a lake the general formulae may be simplified making the numerical evaluation easier.

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