The Stokes-Helmert Scheme for the Evaluation of a Precise Geoid

Petr Vaníček
Department of Surveying Engineering,
University of New Brunswick, Fredericton, N.B., Canada
and
Zdenek Martinec
Department of Geophysics
Charles University, Prague, Czech Republic

Abstract

In this paper, we explore the theoretical properties of Stokes's solution to the geodetic boundary value problem in Helmert's modification. We show that the formulation embodied in Helmert's "second condensation method" should remove the widespread objection to Stokes's approach – that topographical density has to be known very accurately if an accurate geoid is to ensue – by reducing the effect of topographical masses by several orders of magnitude. The study draws heavily on several papers of ours on partial aspects of the Stokes-Helmert scheme that have been recently published.

Introduction

Whether the geoid can be determined to a "sufficient" accuracy has been vehemently discussed in geodetic circles for many decades. The main objection, by those who do not think it can, has always been that the mass density distribution "within the earth" will never be known accurately enough to allow us to compute the geoid to any reasonable level of accuracy. This was the main reason that Molodenskij's quasi-geoid and the theory of its determination were in vogue for several decades, while the potential accuracy of Stokes's approach was questioned.

With the arrival of GPS and its capability to measure ellipsoidal height differences fairly accurately, the interest in this debate has been renewed. In recent years, different groups have been trying to compute either an accurate geoid or an accurate quasi-geoid. The ultimate goal of these efforts is the determination of a geoid/quasi-geoid with an error of one centimetre or less.

Here we discuss those aspects of Stokes's approach that represent to many people the theoretical limitations to a higher accuracy obtainable with this approach. We put aside the questions of gravity data accuracy and density requirements; naturally, these requirements would have to be considered seriously in any computational attempt to compile a "centimetre geoid". We also leave out the question of orthometric height accuracy. We are convinced, however, that if the theory cannot guarantee the centimetre accuracy, there would be little point even in trying to collect gravity data to the satisfaction of these requirements.

The biggest theoretical obstacle in Stokes's approach to the geodetic boundary value problem solution is the presence of topography. Stokes formulated his famous solution [Stokes, 1849] at a time when gravity data were sparse, and its practical application could not even be contemplated. Thus accuracy was not foremost in Stokes's mind. The fact that the presence of earth topographical masses violated the basic assumption behind his solution, which is that of the harmonicity of disturbing potential outside the boundary, the geoid, was not perceived as an important flaw in the theory.

The first serious attempt to remedy this flaw can be attributed to Helmert [1884]. Helmert suggested that the earth's topography can be replaced by an infinitesimally thin layer of an areal density equal to the product of the mean real topographical density and height. This layer, which he called the "condensation layer," could be located anywhere on or beneath the geoid without violating the required assumption of harmonicity. In the "second condensation method" that Helmert formulated, the condensation layer is placed right on the geoid.

We elected to analyse this second Helmert condensation technique used in conjunction with Stokes's approach as the most straightforward method to solve the geodetic boundary value problem. We shall be referring to the combination of these two ideas as the "Stokes-Helmert scheme". To be sure, there have been other notable methods advanced by other geodesists, such as Bjerhammar [1963], Krarup [1969], Sansó [1977], Moritz [1980], not to mention Molodenskij et al. [1960] who, as mentioned above, really defined and then solved a slightly different boundary value problem. That we opted for the Stokes-Helmert scheme should not be seen as a judgement passed on the other methods. As we shall see below, Helmert's second condensation technique is probably the most natural approach, and it lends itself to an easy physical interpretation. The Stokes-Helmert scheme, when presented in a proper manner, is thus readily understood even by non-specialists.

Formulation of the problem

Let us begin by denoting the earth's gravity potential, composed of gravitational (attraction) W^g and centrifugal acceleration W^c potentials, by W. One of the equipotential surfaces of W, the one that approximates the mean sea level most closely, is given a special significance. We denote it by

$$W = W_g = \text{const.} \tag{1}$$

and call it the <u>geoid</u>. We are interested in describing as precisely as possible both W^g outside the earth and the geoid.

To this end, we introduce an analytical reference field U, called the normal field, with one of its equipotential surfaces; namely,

$$U = W_g \, , \tag{2}$$

having the shape of a geocentric biaxial ellipsoid. We shall refer to this ellipsoid also as the reference ellipsoid. The normal field is selected so that it satisfies the following Poisson equation outside the generating ellipsoid:

$$\nabla^2 U = 2\omega^2 \quad . \tag{3}$$

Normal gravity $\vec{\gamma}$ is simply the gradient of U

$$\vec{\gamma} = \text{grad } U \tag{4}$$

with its magnitude equal to γ.

The difference between the gravity and normal potentials is called the disturbing potential T, and we can write

$$T = W - U \tag{5}$$

with T being about five orders of magnitude smaller than W. If it were not for the presence of the earth's atmosphere, the disturbing potential T would satisfy the Laplace equation outside the earth.

We can now attempt to solve for T and obtain, at any given point, the potential W^g by adding U and subtracting W^c, both of which we can compute to an arbitrarily high accuracy since their analytical prescriptions are known. Also, once T on the geoid (T_g) becomes known, we can use Bruns's formula [Bruns, 1878]

$$N = \frac{T_g}{\gamma_0} \quad , \tag{6}$$

where γ_0 is the normal gravity on the reference ellipsoid, to obtain the separation N between the normal field generating (reference) ellipsoid and the geoid. Thus the problem is reduced to that of the determination of T on and outside the geoid.

Bruns's formula is valid only when the normal potential U is selected so that its value U_0 on the reference ellipsoid equals to the value W_0 of the actual potential W on the geoid. If this is not satisfied, then a constant value correction has to added to all computed geoidal heights N. As can be readily shown, Bruns's formula is accurate to better than $1.5 {*} 10^{-7}$ [m^{-1}] N^2. The inaccuracy stems from the derivation of the formula, which reads as follows:

$$T_g = W_0 - U_g = U_0 - U_g = \left.\frac{\partial U}{\partial H}\right|_g N + \frac{1}{2}\left.\frac{\partial^2 U}{\partial H^2}\right|_g N^2 + \dots \quad . \tag{7}$$

The above accuracy estimate is arrived at through two realizations: that the first derivative of U along the plumbline, i.e., with respect to height H, is equal, to a high enough degree of accuracy (see below), to negative g_g, and that the second derivative is equal, again with a sufficient degree of accuracy, to the vertical gradient of normal gravity at the geoid which equals approximately to 0.3 mGal/m. Since the largest geoidal height (in absolute value) is about 100 m, we will consider this error, which may reach up to 1.5 mm, negligible when the geoid is sought to an accuracy of 1 cm.

Gravity observed at the surface of the earth provides the information needed to solve the new problem. Since gravity g is merely the magnitude of the gradient of the gravity potential W, we can write

$$g = |\mathrm{grad}\ W| = |\mathrm{grad}\ U + \mathrm{grad}\ T| \quad . \tag{8}$$

The observed values g can now be corrected for the effect of atmospheric attraction. This may be done at least one order of magnitude more accurately than is the accuracy of gravity observations [Ecker and Mittermayer, 1969]. It remains to be seen whether this accuracy is sufficient to give the geoid to the desired accuracy. The detailed investigation of this point was considered outside the scope of this paper. In the following, we will assume that the correction for atmospheric attraction effectively removes the influence of the atmosphere in our problem; and that the hypothetical disturbing potential T^* that corresponds to the corrected gravity g^* can be taken as being harmonic outside the earth:

$$\nabla^2 T^*(r) = 0 \quad \text{, for } r \text{ outside the earth} \quad . \tag{9}$$

If we wanted to solve only for T outside the earth, we could use g^* at the earth's surface as a boundary condition, solve the Laplace problem for T^* (eqn. (9)) outside the earth, correct the resulting T^* for the effect of atmospheric attraction, and be done. What we also need, however, is the actual T at the geoid, i.e., inside the topographical masses. This would imply a formulation involving Poisson's equation and thus an adequate knowledge of the density distribution within the topographical masses. Such knowledge, unfortunately, does not exist. A more hopeful way is to introduce Helmert's idea of a condensation layer on the geoid, which relaxes the accuracy requirement for the density distribution.

Helmert's model

Let us introduce the model of the real earth that corresponds to Helmert's second condensation technique and call it Helmert's model. Helmert's model consists of the same distribution of masses inside the geoid as that of the real earth and the model has no atmosphere. On the one hand, to this mass distribution we add the condensation layer on the geoid, whose areal density σ at any point is given by

$$\sigma = \bar{\rho} H \quad , \tag{10}$$

where $\bar{\rho}$ is the mean actual density of the topographical column of (orthometric) height H above the point. On the other hand, we subtract the topographical density distribution. This model is not meant to approximate the real case; rather, it is meant to provide the appropriate tool that would allow us to solve the Laplace problem everywhere above the geoid.

Let us now denote the potential of the topographical masses by V^t and the potential of the condensation layer by V^c. The difference

$$V = V^t - V^c \tag{11}$$

is the potential that essentially distinguishes the real earth from the Helmert model of the earth. As we shall see in this paper, we are required to evaluate this potential only above and below the topographical masses.

It is important to stress here that the condensation should be carried out with the real mass density considered in eqn. (10). Only when real density is considered do we get the total effect of topography modelled correctly. This is of course the crux of the problem: any inaccuracy in modelling the mass density will have an effect on the resulting geoid as we will see from the pertinent equations towards the end of this paper. It should be also mentioned at this stage that the condensation prescribed by eqn. (10) changes slightly the total mass of the earth, as well as its centre of mass and its moments of inertia. To ensure that the centre of mass is left undisturbed by the condensation — an important condition for the validity of Stokes's solution — a little different condensation scheme would have to be used. Detailed discussion of this point, however, is considered beyond the scope of this paper.

Helmert's earth obviously has a gravity potential W^h given by

$$W^h = W - \delta T^* - V \, , \tag{12}$$

where δT^* is the difference between T and T^*, i.e., the gravity potential of the atmosphere. Once again, we shall assume that δT^*, which is about one order of magnitude smaller than V, can be evaluated accurately enough from existing models of atmospheric density. By analogy with the real earth, we can now define <u>Helmert's disturbing potential</u> as

$$T^h = W^h - U \tag{13}$$

and substituting from eqns. (12) and (5) we also have

$$T^h = T^* - V = T - \delta T^* - V \ . \tag{14}$$

Now, the Helmert disturbing potential is harmonic outside the geoid, i.e.,

$$\nabla^2 T^h = 0 \ . \tag{15}$$

This can be seen from the last two terms in eqn. (14); these terms make T, originally harmonic outside the atmosphere, successively harmonic outside the topography and then outside the geoid. To solve for T^h, we can solve the Laplace problem (outside the geoid) for which we must have some boundary conditions on the geoid.

Considering eqn. (13), we can write

$$g_g^h = -\left.\frac{\partial W^h}{\partial H}\right|_g = -\left.\frac{\partial U}{\partial H}\right|_g - \left.\frac{\partial T^h}{\partial H}\right|_g \ . \tag{16}$$

Here, the derivative of U along the actual plumbline can be replaced by a derivative with respect to the corresponding normal plumbline, with an error of $(1 - \cos\theta)\,\gamma$, where θ is the angle between the two plumblines , i.e., the deflection of the vertical. At the geoid, the deflection of the vertical is at most 10 arcsec, and the error caused by the replacement is thus at most 1 µGal. We shall consider negligible any effect on gravity smaller than 1 µGal, since it may contribute, under the worst circumstances, only about 1 mm to the geoidal height. Hence, with a negligible error of less than 1 µGal, we get

$$g_g^h = \gamma_g - \left.\frac{\partial T^h}{\partial H}\right|_g \ . \tag{17}$$

We can now express the normal gravity on the geoid γ_g by means of normal gravity on the reference ellipsoid γ_0, again using the Taylor series expansion

$$\gamma_g = \gamma_0 - \left.\frac{\partial^2 U}{\partial n^2}\right|_0 N + \ldots \ , \tag{18}$$

where n stands for the direction of normal plumbline; and the derivative is taken on the reference ellipsoid, hence subscript 0. The higher order terms amount to less than 0.3 µGal and again can be safely neglected. Replacing N by T_g / γ_0 from eqn. (6) and substituting for T_g from eqn. (14), we get

$$\gamma_g = \gamma_0 - \left.\frac{\partial^2 U}{\partial n^2}\right|_0 \frac{\left(T_g^h + V_g + \delta T_g^*\right)}{\gamma_0} \ . \tag{19}$$

The term $\partial^2 U / \partial n^2|_0$, the second derivative of the normal potential along the normal plumbline, may be replaced by the negative vertical gradient of normal gravity at the reference ellipsoid. This gradient can be derived from Bruns's equation [Bruns, 1878], which is exact and reads

$$\frac{\partial \gamma}{\partial n} = -2\left(\gamma J + \omega^2\right) \ , \tag{20}$$

where J is the mean curvature of the normal equipotential surface passing through the point of interest, and ω is the earth's angular spin velocity. Realizing that on the reference ellipsoid, J is the mean curvature of the ellipsoid itself, we can write

$$J_0 = 2 / (R_M + R_N) \ , \tag{21}$$

where R_M and R_N are the radii of curvature of the reference ellipsoid in the meridian and prime vertical directions [Vaníček and Krakiwsky, 1986; eqns. (7.14) (15.58)]. Substituting in eqn. (20), after some development we get

$$\left.\frac{\partial^2 U}{\partial n^2}\right|_0 = -\left.\frac{\partial \gamma}{\partial n}\right|_0 = \frac{2\gamma_0}{a}\left[1 + \frac{\omega^2 a}{\gamma_0} + f\cos 2\phi + \frac{f^2}{16}(11 + 12\cos 2\phi + \cos 4\phi)\right] \ , \tag{22}$$

where a, and f are the major semi-axis and flattening of the reference ellipsoid, respectively, and ϕ is the geodetic latitude of the point of interest. This equation is correct to the order of f^3. This means that it differs from the exact form only by a term of the order of 10^{-5} µGal/m, which, when applied to a maximum H can cause an error smaller than 0.1 µGal.

Substituting back to eqn. (17), we obtain finally

$$g_g^h - \gamma_0 + \frac{1}{\gamma_0}\left.\frac{\partial^2 U}{\partial n^2}\right|_0 \left(V_g + \delta T_g^*\right) = -\left.\frac{\partial T^h}{\partial H}\right|_g - \frac{1}{\gamma_0}\left.\frac{\partial^2 U}{\partial n^2}\right|_0 T_g^h \ . \tag{23}$$

Let us denote the left-hand side of this expression by $\Delta g^h{}_g$, and call it Helmert's gravity anomaly. The first term on the right-hand side can be written as

$$-\left.\frac{\partial T^h}{\partial H}\right|_g = -\left.\frac{\partial T^h}{\partial r}\right|_g \ . \tag{24}$$

This replacement is accurate to the order of $1 - \cos\beta$, where β is the difference between geodetic and geocentric latitudes [Bomford, 1971] given as

$$\beta = f\sin^2 2\phi \ , \tag{25}$$

and the error of the approximation is $5.5*10^{-6} \sin^2 2\phi$.

The second term on the right-hand side of eqn. (23) can be approximated as

$$\frac{1}{\gamma_0} \frac{\partial^2 U}{\partial n^2}\bigg|_0 T_g^h = \frac{2}{R} T_g^h \quad , \tag{26}$$

where R is the mean radius of the earth. This approximation is much coarser than the one in eqn. (24). The error, introduced by this “spherical approximation,” can be evaluated as the difference between the correct expression and its spherical approximation, which can be obtained from eqn. (22) as

$$\left\{ \frac{1}{\gamma_0} \frac{\partial^2 U}{\partial n^2}\bigg|_0 - \frac{2}{R} \right\} T_g^h = D^S = \left\{ \frac{2}{a} \left[1 + \frac{\omega^2 a}{\gamma_0} + f \cos 2\phi - O\left(f^2\right) \right] - \frac{2}{R} \right\} T_g^h \quad . \tag{27}$$

Here, the second term in the brackets is known as the “geodetic parameter” m [International Association of Geodesy, 1981], which is of the order of f. The second order term $O(f^2)$ is smaller than $1.6*10^{-5}$ and can again be safely disregarded. Defining the mean radius R of the earth as

$$R = \left(a^2 b\right)^{1/3} = a(1-f)^{1/3} = a\left[1 - \frac{f}{3} + O\left(f^2\right)\right] \quad , \tag{28}$$

after a few elementary operations, we arrive at the following expression for the above difference D^S:

$$D^S = \frac{2}{R}\left[m + f\left(-\frac{1}{3} + \cos 2\phi\right)\right] T_g^h \quad . \tag{29}$$

The maximum value of this difference is about 160 μGal. We will show later how to correct for this approximation and how the accuracy of the solution can be arbitrarily improved.

Using approximations (24) and (26), we can now rewrite eqn. (23) in a more familiar form, i.e., as

$$\Delta g^h = -\frac{\partial T^h}{\partial r}\bigg|_g - \frac{2}{R} T_g^h \quad . \tag{30}$$

which represents the version of the <u>fundamental gravimetric equation</u> [Heiskanen and Moritz, 1967] valid for the Helmert model.

Evaluation of Helmert's gravity anomaly

Helmert's gravity on the geoid g_g^h, needed to assemble the Helmert anomaly in eqn. (23), is obtained by first transforming the observed surface gravity into the Helmert model and then reducing it (still in the Helmert model) onto the geoid. Mathematically, the transformation of actual gravity, corrected for the atmospheric attraction effect, to Helmert's gravity g^h is achieved by taking the gradient of eqn. (12); namely,

$$g^h = \left|\mathrm{grad}\left(W - \delta T^* - V\right)\right| \quad . \tag{31}$$

This equals to

$$g^h = g^* - \frac{\partial V}{\partial H} + O\left(\frac{1}{2}\mathrm{grad}^2 \frac{V}{g^*}\right) \quad , \tag{32}$$

where the last term on the right-hand side is going to be definitely smaller than 1 μGal because |grad(V)| is certainly smaller than 40 mGal. The last term then can be safely neglected. For surface gravity transformation, the derivative in this equation is obviously evaluated at the earth's surface; it is then called the <u>direct topographical effect</u> on gravity [Heiskanen and Moritz, 1967].

We have shown [Martinec and Vaníček, 1993b] that the leading term of the direct topographical effect can be written as

$$\begin{aligned}\left.\frac{\partial V(H,\Omega)}{\partial r}\right|_t = & -4\pi G\bar{\rho}H^2(\Omega) + \frac{G}{2R}\left\{\int_{\Omega'}\bar{\rho}(\Omega')\left[H^2(\Omega') - H^2\right]K(H,\psi) + \right.\\ & \left. + H^2\int_{\Omega'}\left[\rho(\Omega') - \bar{\rho}\right]K(H,\psi)\right\} \quad ,\end{aligned} \tag{33}$$

where G is the gravitational constant, Ω stands for a horizontal position given by latitude ϕ and longitude λ, and the integration kernel K is equal to

$$K(H,\psi) = -\sum_{j=0}^{\infty}(j+2)\left(\frac{R}{R+H}\right)^{j+2} P_j(\cos\psi) \quad , \tag{34}$$

where ψ is the angular distance between points Ω and Ω', and P_j is the Legendre polynomial of degree j. Equation (33) is accurate to $5.5 \cdot 10^{-6}\sin^2\phi$ (cf. eqn. (24) and the text after it). We reiterate that eqn. (33) describes only the leading term in the development and higher order terms may have to be considered if an accuracy of 1 μGal is required. The direct topographical effect can also be readily expressed in a spatial form as shown in Martinec and Vaníček [1993b]. All the integrals here, as well as in the higher order terms, are regular.

The reduction of the Helmert gravity at the surface g_t^h onto the geoid (done in Helmert's model) can be accomplished by developing the surface gravity into a Taylor series

$$g_g^h = g_t^h + \left.\frac{\partial^2 W^h}{\partial H^2}\right|_g H + \left.\frac{\partial^3 W^h}{\partial H^3}\right|_g + \ldots \quad . \tag{35}$$

This can be done because Helmert's potential W^h above the geoid satisfies a Poisson equation with a constant right-hand side (equal to $2\omega^2$) and all the derivatives with respect to height thus exist. This process is called in the literature the <u>downward continuation</u> of (Helmert's) gravity.

The evaluation of the second and higher derivatives in eqn. (35) is not all that straightforward. Substituting for W^h from eqn. (13), we can write

$$g_g^h - g_t^h = \left[\left.\frac{\partial^2 U}{\partial H^2}\right|_g + \left.\frac{\partial^2 T^h}{\partial H^2}\right|_g\right] H + \frac{1}{2}\left[\left.\frac{\partial^3 U}{\partial H^3}\right|_g + \left.\frac{\partial^3 T^h}{\partial H^3}\right|_g\right] H^2 + \ldots \quad . \tag{36}$$

The second derivative of normal potential U along the actual plumbline can be replaced by the second derivative with respect to normal plumbline to yield the negative vertical gradient of normal gravity given by eqn. (22) with an error that can be easily evaluated as $-\gamma\theta^2$ (cf. the text after eqn. (16)), small enough to be neglected. The third derivative may be written as

$$\left.\frac{\partial^3 U}{\partial H^3}\right|_g = \frac{1}{a}\left.\frac{\partial \gamma}{\partial n}\right|_g = -\frac{48.4 * 10^{-6} \mu Gal}{m^2} \tag{37}$$

with a sufficient accuracy. Higher order derivatives of U contribute even less and can be safely neglected.

It should be noted here that, in the literature, the quantity

$$g_g = g_t + \left(\left.\frac{\partial^2 U}{\partial n^2}\right|_g + \frac{1}{2}\left.\frac{\partial^3 U}{\partial n^3}\right|_g H\right) H \tag{38}$$

is referred to as the "free-air gravity" on the geoid. When normal gravity γ_0 is subtracted from it, the difference is called the <u>free-air gravity anomaly</u>, denoted by Δg_g^F. Clearly, we can write Helmert's anomaly (eqn. (23)) also as

$$\Delta g_g^h = \Delta g_g^F - \delta g_t^* - \left.\frac{\partial V}{\partial H}\right|_t - \frac{1}{\gamma_0}\left.\frac{\partial^2 U}{\partial n^2}\right|_0 \left(V_g + \delta T_g^*\right) + \left(\left.\frac{\partial^2 T^h}{\partial H^2}\right|_g + \frac{1}{2}\left.\frac{\partial^3 T^h}{\partial H^3}\right|_g H + \ldots\right) H \quad , \tag{39}$$

where $\delta g^* = g - g^*$.

The direct topographical effect $\partial V / \partial H|_t$ has been shown to contribute at the very most only about 2 metres, and usually much less, to the resulting geoid [Martinec and Vaníček, 1993b]. Thus, to evaluate the geoid to the accuracy of 1 cm, it is necessary to know the topographical effect to an accuracy of at most 0.5%. This implies that lateral changes of vertically averaged density in topography have to be known to this accuracy. This is likely to pose a problem in high mountains, where the topographical effect is larger. In flat areas, where the topographical effect reaches at most only a few decimetres, density variations do not have to be known better than to some 5%, which should be routinely achievable. We would like to repeat that this is the real accomplishment of Helmert's modelling. The introduction of the condensation layer with a realistic density model reduces the classical problem of inadequate knowledge of topographical density, alluded to in the introduction, by 3 orders of magnitude, and gives us a real possibility to determine the geoid to the desired 1 cm accuracy.

The fourth term on the right-hand side is sometimes called the <u>secondary indirect effect of topography</u> [Heiskanen and Moritz, 1967] on gravity. At most, it can contribute some 0.3 mGal, and it can be written to a sufficient accuracy as

$$\frac{1}{\gamma_0}\left.\frac{\partial^2 U}{\partial n^2}\right|_0 V_g = \frac{2}{R} V_g \quad . \tag{40}$$

The necessary accuracy of the density model in the evaluation of this term is of the same order of magnitude as that in the direct effect evaluation.

The most troublesome term in eqn. (39) is the last one, which we call the downward continuation of anomalous gravity, or simply *Dg*. In this term, the vertical derivatives of T^h cannot be evaluated before the Laplace problem is solved; the Helmert disturbing potential is not known beforehand. They have to be evaluated from (the first iteration of) the Laplace solution. This would then start the iterations, which should converge since the *Dg* correction to free-air anomaly is sufficiently smaller than the anomaly itself. This, however, is not the only problem with *Dg*. It turns out that the Taylor series itself oscillates (succesive terms have opposite signs) and may converge, in the absolute sense, only slowly. Thus possibly higher than third derivatives would have to be taken in this series to ensure sufficient accuracy of the solution. Therefore, a better approach is necessary as we shall discuss below.

The downward continuation of anomalous gravity may be written in a more straightforward manner as

$$Dg = \left.\frac{\partial T^h}{\partial H}\right|_g - \left.\frac{\partial T^h}{\partial H}\right|_t \quad . \tag{41}$$

Since T^h is harmonic between the geoid and the earth's surface so is $r\ \partial T^h/\partial H$ [Heiskanen and Moritz, 1967] and we can solve for $r\ \partial T^h/\partial H|_t$ as a function of $r\,\partial T^h/\partial H|_g$. The functional relation is given by a Poisson integral

$$r\left.\frac{\partial T^h}{\partial H}\right|_t = \frac{R}{4\pi}\int_{\Omega'}\left.\frac{\partial T^h(\Omega')}{\partial r}\right|_g \kappa(H,\psi)d\Omega' \quad , \tag{42}$$

where the replacement of H by r in the derivative is, once again, justified. The Poisson kernel κ is equal to

$$\kappa(H,\psi) = \sum_{j=2}^{\infty}(2j+1)\left(\frac{R}{R+H}\right)^{j+1} P_j(\cos\psi) \quad . \tag{43}$$

From this equation we can write directly the final (exact) expression for Dg in a closed form as shown in Martinec and Vaníček [1993c].

The two remaining terms, $-\ \delta g^*_t$ and $-\ 2/R\ \delta T^*_g$, reflect the atmospheric attraction. In the second term, we use the same approximation as we did in the secondary topographical indirect effect. Again, these terms are small, and we assume that they can be evaluated to a sufficient accuracy from existing atmospheric models.

Stokes's solution of the Laplace problem

We are now ready to discuss the solution of the boundary value problem posed above. We want to solve the Laplace eqn. (15) for Helmert's disturbing potential outside the geoid with the boundary condition prescribed by eqn. (30). The two remaining hindrances that have to be addressed before the solution can be formulated are:

1) Helmert's gravity anomaly in eqn. (30) can be evaluated to the required accuracy of 1 µGal except for the downward continuation of anomalous gravity from the earth's surface to the geoid, that is, the Dg term.
2) The fundamental gravimetric equation, which spells out the relation between the boundary data Δg^h on the geoid and the unknown function T^h, is only approximate, with D^S being the approximation error.

Let us discuss the second hindrance first.

As shown above, the first term of eqn. (30) is accurate to the relative order of f^2. Since the anomaly will be at most of the order of 10^2 mGal, this inaccuracy translates to about 0.5 µGal on the left-hand side. The error in the first term thus can be safely neglected. The error D in the second term is much larger (up to 160 µGal) but still considerably smaller than the anomaly itself. It thus can be treated as a correction to Helmert's anomaly and, since it is a (linear) function of the unknown potential T^h at the geoid, it can be solved for iteratively in the same way as the Dg term under 1).

Let us now summarize the above. Denoting the part of Helmert's anomaly that can be evaluated without the knowledge of the downward continuation of anomalous gravity by $\tilde{\Delta}g^h$ and transferring the effect D^S (eqn. (29)) of spherical approximation on the left-hand side, eqn. (30) can be rewritten as

$$\Delta g^h - D^S = \tilde{\Delta}g^h - Dg - D^S = \tilde{\Delta}g^h + \delta\Delta g^h\left(T^h\right) = -\left.\frac{\partial T^h}{\partial r}\right|_g - \frac{2}{R}T_g^h \quad . \tag{44}$$

Here, the correction $\delta\Delta g^h(T^h)$ to the approximate Helmert anomaly, being a function of Helmert's disturbing potential and being much smaller than the Helmert anomaly itself, can be evaluated iteratively. Under the assumption that the direct topographical effect and the secondary indirect effect can be evaluated to a sufficient accuracy, eqn. (44) provides a good enough tool to yield a T^h accurate enough to get geoidal heights to the required 1 cm accuracy.

The boundary condition (30) on the geoid (for the Laplace eqn. (15)) is of the usual mixed type, and the Laplace problem can be solved exactly by the classical Stokes technique [Stokes, 1849]. We note that the first degree spherical harmonic of T^h is missing if an appropriate condensation is selected as noted earlier, and the mixed type Laplace problem gives the correct solution up to the zero-order term (scale). The zero order term has to be evaluated separately, as usual. The resulting first iteration $T^{h(1)}$ is obtained either in a spectral, or in a spatial (closed) form. There should be no problem in obtaining the (iterated) solution T^h to a sufficient accuracy, probably after only one iteration.

Let us point out here that, if the solution is sought in the spatial form (in terms of the Stokes integral), then the effect DT^h of the Dg on T^h, can be obtained from the same Stokes integral applied to Dg instead of the gravity anomaly. The Stokes integral over Dg can be relatively easily evaluated to give a more convenient formula to use compared to equation for Dg. Since the Stokes integration represents a numerical smoothing process, the integration kernel tapers off to zero (with the distance from the point of interest) much more rapidly than its counterpart in eqn. (42). In other words, the actual integration does not have to be carried out too far away from the point of interest.

A similar treatment can be devised for the effect D^S of the spherical approximation. This treatment, however, is considered beyond the scope of this paper.

Evaluation of the geoid

Once the Helmert disturbing potential is obtained to the desired accuracy, it has to be transformed back from the Helmert model into the original model that depicts the real earth. In other words, T_g^h has to be transformed back into T_g to allow us to apply the Bruns formula (6) and obtain the geoid. This transformation is easily done using first eqn. (14) to get

$$T_g^* = T_g^h + V_g \quad , \tag{45}$$

and then transforming the disturbing potential reduced for atmospheric attraction, T^*, to the real disturbing potential T by

$$T_g = T_g^* + \delta T_g^* \quad . \tag{46}$$

Here V_g is called the <u>primary indirect effect of topography</u> on potential. Its magnitude is comparable to that of the direct topographical effect introduced earlier, and it can be evaluated to the same degree of accuracy. In spatial form, the leading terms of the indirect effect read [Martinec and Vaníček, 1993a]:

$$\begin{aligned} V_g(\Omega) = 2\pi G\bar{\rho}H^2 + \frac{3}{4}\frac{G}{R}\Bigg[\int_{\Omega'} \bar{\rho}(\Omega')\frac{H^2(\Omega') - H^2}{\ell_0} R^2\, d\Omega' \\ + H^2(\Omega)\int_{\Omega'} \frac{\bar{\rho}(\Omega') - \bar{\rho}}{\ell_0} R^2\, d\Omega'\Bigg] \quad , \end{aligned} \tag{47}$$

where

$$\ell_0 = 2R\sin\frac{\psi}{2} \quad . \tag{48}$$

Higher order terms may have to be considered here, to get the geoid solution to a sufficient accuracy. All the integrals in eqn. (47) and the higher order terms, are regular and, once more, a limited knowledge of density is acceptable because the effect is small.

The term δT_g^* may be called the <u>indirect effect of atmosphere</u> (at the geoid). It is about one order of magnitude smaller than the indirect effect of topography, and thus hopefully can be evaluated to a sufficient degree of accuracy from existing atmospheric models.

Once the real disturbing potential at the geoid is known, it can be converted to geoidal heights by the Bruns formula (6). We note in passing that Bruns's formula applied to T^h, instead to T, yields what is referred to in the literature as a <u>co-geoid</u>; it should, in our context, be called more appropriately the Helmert geoid. Real disturbing potential outside the atmosphere is obtained as a harmonic continuation of T_g. Actual gravity potential W for any given point outside the atmosphere is then obtained simply from eqn. (5).

Conclusions

From our analysis, it seems clear that the theory for geoid computation using the Stokes-Helmert approach is good enough to give geoidal heights to a centimetre accuracy everywhere, except in the highest mountains where 5 to 10 cm probably would be a more realistic goal. Using the Helmert idea of condensation of topography onto the geoid, the need to have an accurate knowledge of topographical mass density is replaced by a requirement for an approximate mean lateral density model good to about 5%. This, of course, makes the scheme viable.

In all of the discussion above, we have not addressed the question of gravity and height data accuracy. Neither have we dealt with the data density required for the numerical integration of Stokes's convolution integral. Yet, in any practical application of the theory described here, the effects of data distribution (irregularity and sparseness) and data accuracy are likely to be very significant. Systematic errors in orthometric heights of gravity points (mainly due to the oversimplified model for the vertical gradient of gravity used in the evaluation of these heights), represent a serious source of distortion in the resulting geoid. These errors can reach more than one metre and their effect on the geoid will be almost unabated; this effect may prove very difficult to model. On a more optimistic note, from our earlier experience with the compilation of the geoid for Canada [Vaníček and Kleusberg, 1987], we have learned that the effect of gravity data inaccuracy is mostly between 3 cm and 10 cm.

The other problem we have not investigated here is the effect of numerical errors in the actual computation due to truncated integration and the pre-ordained sizes of surface cells. This effect, once again, can reach up to the one decimetre level and has to be dealt with carefully in any practical application. The investigation of these aspects, however, was considered beyond the scope of this paper.

Finally, we realize that a practical evaluation of the geoid will have to use a combination of terrestrial gravity (as described here) and satellite determined long wavelength field features. The inclusion of this type of information will have an impact on all the formulae given here; most of the corrections and reductions will be somewhat smaller, and potential problems with their applications will be somewhat attenuated. This investigation too will have to wait for another paper.

Acknowledgments

We would like to acknowledge that our collaboration on this research started when we were both staying at the Geodetic Institute of the University of Stuttgart as "Senior Distinguished Scientist" and "Fellow" of the Alexander von Humboldt Foundation.

Discussions with our host, Prof. E.W. Grafarend, as well as his staff and students, provided the initial inspiration and encouragement. Most of the actual research behind this paper was carried out while the second author was at the University of New Brunswick, supported by a National Science and Engineering Research Council of Canada International Cooperation grant. Thoughtful criticism by Prof. A. Kleusberg has been much appreciated by the authors.

Literature

Bjerhammar, A. (1963). A new theory of gravimetric geodesy. Geodesy Division of the Royal Institute of Technology Report, Stockholm.

Bomford, G. (1971). *Geodesy*. 3rd ed., Oxford University Press.

Bruns, H. (1878). *Die Figur der Erde*. Publikation des Koniglichen Preussischen Geodetischen Institutes, Berlin.

Ecker, E., and E. Mittermayer (1969). Gravity correction for the influence of theatmosphere. *Boll. Geoph. Teor.Appl.* XI, 70-79.

Heiskanen, W.H., and H. Moritz (1967). *Physical Geodesy*. W.H.Freeman & Co., San Francisco.

Helmert, F.R. (1884). *Die mathematische und physikalische Theorien der hoheren Geodasie.* Vol. 2, B.G.Teubner, Leipzig (reprinted in 1962 by Minerva GMBH, Frankfurt/Main)

International Association of Geodesy (1981). The Geodesist's Handbook. *Bull. Geod.* 54 (3).

Krarup, T.A. (1969). A contribution to the mathematical foundation of physical geodesy. Publ. 44, Dan. Geod. Inst., Copenhagen.

Martinec, Z., and P. Vaníček (1993a). The indirect effect of topography in the Stokes-Helmert technique for a spherical approximation of the geoid. *Manuscripta Geodaetica* (in press).

Martinec, Z., and P. Vaníček (1993b). The direct topographical effect for a spherical geoid. *Manuscripta Geodaetica*, (submitted).

Martinec, Z., and P. Vaníček (1993c). Downward continuation of anomalous gravity in the Helmert condensation technique. *Manuscripta Geodaetica* (submitted).

Molodenskij, M.S., V.F. Eremeev, and M.I. Yurkina (1960). *Methods for Study of the External Gravitational Field and Figure of the Earth.* Translated from Russian by the Israel Program for Scientific Tranlations for the Office of Technical Services, U.S.Department of Commerce, Washington, D.C., U.S.A., 1962.

Moritz, H. (1980). *Advanced Physical Geodesy*. Wichmann, Karlsruhe.

Sanso, F. (1977). The geodetic boundary value problem in gravity space. *Mem. Accad. Naz. Lincei* 14, 39-97.

Stokes, G.G. (1849). On the variation of gravity at the surface of the earth. *Trans. Cambridge Philos. Soc.* VIIII, 672-695.

Vaníček, P., and A. Kleusberg (1987). The Canadian geoid—Stokesian approach. *Manuscripta Geodaetica* 12, 86-98.

Vaníček, P., and E.J. Krakiwsky (1986). *Geodesy: The Concepts*. 2nd ed., North Holland, Amsterdam.