STRAIN INVARIANTS

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Abstract

In the past, the application of strain in geodesy was more or less limited to horizontal geodetic networks, i.e., the applications were distinctly two-dimensional (2D). Strain of three-dimensional (3D) nature has been used in many non-geodetic applications but it is relatively new in geodesy. Three-dimensional networks are a recent phenomenon in geodesy whose introduction has been predicated on the appearance of three-dimensional satellite positioning techniques. With the introduction of three-dimensional strain descriptors there arose a question as how to relate the three-dimensional descriptors to the two-dimensional ones so they could be easily compared and put in similar schemes for example for specification purposes. It has been long established that for the strain descriptors to be useful in studying geodetic networks, for instance in the context of Robustness analysis, they must be invariant under any rotation of coordinate systems. They are by nature invariant in any shift of the origin of coordinate systems. The requirements of invariance originate from the overall explicit requirement that the descriptors be sensitive only to the shape of the network and the quality of the observations.

Keywords: Robustness analysis, network reliability, strain, invariants, displacement.

1 Introduction

The aim of this contribution is two-fold:

1) To show some meaningful strain invariants in 3D, i.e., those that have natural physical meaning by themselves.
2) To show how the 2D invariants are related to the 3D invariants and by showing this to demonstrate how 2D invariants can be obtained from their 3D counterparts.

If we succeed in this venture, then we will have proved that investigation of strain of geodetic networks can always be done in 3D, in the most appropriate coordinate system, and the 2D counterparts can be obtained from the 3D invariants by some hopefully simple mathematical manipulations. The main application on our mind here is the investigation of strain of geodetic networks for Robustness (Vaníček et al., 2001), i.e., when looking for virtual, rather than real deformations. In fact, the idea of getting 2D strain invariants from 3D invariants can, of course, be used while investigating either real or virtual deformations of any structure.

To investigate the relations we are after, we shall be using Euclidean spaces only. We shall assume that the 3D geometrical space in which the strain is determined is a Euclidean space complete with a Cartesian coordinate system (X, Y, Z). To investigate these relations, we shall use 3D Euclidean geometry, which is the embedding space of 2D Riemann geometry. In plain language, the 2D space will be a plane with a 2D Cartesian coordinate system (x, y) defined in it.

Let us begin by denoting the displacements of a 3D position ${}^{3}\mathbf{r} = (X, Y, Z)^T$ by

$$
{}^{3}\Delta\mathbf{r} = (U, V, W)^T, \tag{1}
$$

and the corresponding 2D displacement ${}^2\Delta\mathbf{r} = (u,\ v)^T$ of a 2D position ${}^2\mathbf{r} = (x,\ y)^T$. In this Introduction, we shall, for better or worse, follow the terminology and the symbolism introduced by Vaníček et al. (2001). As the cited paper appeared relatively recently in this journal, we keep the explanations here at a minimal level, trusting that if need be, the reader will have an easy access to the cited paper. The strain matrix **E** is composed of partial derivatives

$$
{}^3\mathbf{E}({}^3\mathbf{r}) = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & \frac{\partial U}{\partial Z} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & \frac{\partial V}{\partial Z} \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \frac{\partial W}{\partial Z} \end{bmatrix} . \tag{2}
$$

Similarly, the 2D strain matrix is given by

$$
{}^2\mathbf{E}({}^2\mathbf{r}) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \tag{3}
$$

To obtain the appropriate descriptors of strain, expressed as different linear combinations of the elements of the above strain matrices, these matrices are split into symmetric (**S**) and anti-symmetric (**A**) parts, by the standard well known operations:

$$
\begin{aligned} \mathbf{S} &= \tfrac{1}{2}\left(\mathbf{E} + \mathbf{E}^{\mathbf{T}}\right) \\ \mathbf{A} &= \tfrac{1}{2}\left(\mathbf{E} - \mathbf{E}^{\mathbf{T}}\right) \end{aligned} \tag{4}
$$

These operations are applied the same way to both the 3D and the 2D cases. The 3D symmetrical part, for instance, becomes

$$
{}^{3}\mathbf{S}({}^{3}\mathbf{r}) = \begin{bmatrix} \dfrac{\partial U}{\partial X} & \tfrac{1}{2}\left(\dfrac{\partial U}{\partial Y}+\dfrac{\partial V}{\partial X}\right) & \tfrac{1}{2}\left(\dfrac{\partial U}{\partial Z}+\dfrac{\partial W}{\partial X}\right) \\ \tfrac{1}{2}\left(\dfrac{\partial V}{\partial X}+\dfrac{\partial U}{\partial Y}\right) & \dfrac{\partial V}{\partial Y} & \tfrac{1}{2}\left(\dfrac{\partial V}{\partial Z}+\dfrac{\partial W}{\partial Y}\right) \\ \tfrac{1}{2}\left(\dfrac{\partial W}{\partial X}+\dfrac{\partial U}{\partial Z}\right) & \tfrac{1}{2}\left(\dfrac{\partial W}{\partial Y}+\dfrac{\partial V}{\partial Z}\right) & \dfrac{\partial W}{\partial Z} \end{bmatrix}, \tag{5}
$$

where the diagonal elements $\dfrac{\partial U}{\partial X}$, $\dfrac{\partial V}{\partial Y}$, $\dfrac{\partial W}{\partial Z}$ reflect the dilation along the *X*-, *Y*- and *Z*-axes. These dilations are usually denoted by special symbols σ_1, σ_2 and σ_3. Analogously, in 2D where there would, of course, be only two such dilations along *x*- and *y*- axes. The 3-D anti-symmetric part can be written similarly as

$$
{}^{3}\mathbf{A}({}^{3}\mathbf{r}) = \begin{bmatrix} 0 & \tfrac{1}{2}\left(\dfrac{\partial U}{\partial Y}-\dfrac{\partial V}{\partial X}\right) & \tfrac{1}{2}\left(\dfrac{\partial U}{\partial Z}-\dfrac{\partial W}{\partial X}\right) \\ \tfrac{1}{2}\left(\dfrac{\partial V}{\partial X}-\dfrac{\partial U}{\partial Y}\right) & 0 & \tfrac{1}{2}\left(\dfrac{\partial V}{\partial Z}-\dfrac{\partial W}{\partial Y}\right) \\ \tfrac{1}{2}\left(\dfrac{\partial W}{\partial X}-\dfrac{\partial U}{\partial Z}\right) & \tfrac{1}{2}\left(\dfrac{\partial W}{\partial Y}-\dfrac{\partial V}{\partial Z}\right) & 0 \end{bmatrix}. \tag{6}
$$

It is customary to denote the off-diagonal elements $\tfrac{1}{2}\left(\dfrac{\partial U}{\partial Y}-\dfrac{\partial V}{\partial X}\right), \tfrac{1}{2}\left(\dfrac{\partial U}{\partial Z}-\dfrac{\partial W}{\partial X}\right)$ and $\tfrac{1}{2}\left(\dfrac{\partial V}{\partial Z}-\dfrac{\partial W}{\partial Y}\right)$ by $-\omega_{1,2}$, $-\omega_{1,3}$ and $-\omega_{2,3}$. In 2D, the anti-symmetric part reads

$$
{}^{2}\mathbf{A}({}^{2}\mathbf{r}) = \begin{bmatrix} 0 & \tfrac{1}{2}\left(\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}\right) \\ \tfrac{1}{2}\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right) & 0 \end{bmatrix}, \tag{7}
$$

where the off-diagonal element $\tfrac{1}{2}\left(\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}\right)$ is customarily denoted by $-\omega$. Thus ${}^{2}\mathbf{A}({}^{2}\mathbf{r})$ is often written as

$$ {}^{2}\mathbf{A}({}^{2}\mathbf{r}) = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8} $$

For the strain descriptors to be of any use, they have to be independent of the coordinate systems used in their evaluation. In other words, the descriptors have to be invariant in coordinate translations and rotations. The invariance in translations is assured automatically as the partial derivatives in ***S*** are blind to translations of coordinates and thus to translations of displacement as well. The real problem to be addressed is the invariance in rotation, one rotation for the 2D case and three rotations in the 3D case.

Invariance is a standard mathematical property that has been studied by mathematicians and others for centuries. Let us quote here one textbook for all, (Kaye and Wilson, 1998), where the interested reader can get the theoretical background as well as the historical perspective. A good discussion of strain invariants in mechanics can be found in (Love 1944, p. 43), some geodetic applications are listed by Krumm and Grafarend (2002; Table 3).

Some 2D strain invariants were studied by Vaníček et al. (2001), where three meaningful strain invariants, "dilation invariant", "differential rotation magnitude invariant" and "total shear magnitude invariant" were identified. It made sense then to try to find 3D counterparts to these 2D invariants. It soon became clear though that the third strain descriptor, total shear, was an invariant in 2D but not in 3D (Berber 2006), and it had to be replaced by a different descriptor. The selection fell on the "maximum shear" used by Grafarend and Voosoghi (2003) in their 2D applications. It turns out that this invariant can be very simply generalised into 3D as we have done here . Maximum shear is simply defined as the difference between the largest and the smallest eigenvalues of the ***S*** matrix.

We shall demonstrate numerically the validity of the derived transformations between 3D and 2D invariants using two standard geodetic coordinate systems. The 3D coordinate system will be the (geocentric) Geodetic coordinate system (G), used naturally in most satellite positioning work. The 2D coordinate system will be the x,y-plane of the local geodetic (LG) coordinate system in which most traditional horizontal positions are given. For the exact definition of these two coordinate systems the reader is advised to consult Vaníček and Krakiwsky (1986).

2.1 Dilation

The **dilation invariant** is the only linear invariant. It is given by

$$\forall n = 2,3:\ {}^{n}\sigma = \frac{1}{n}\sum_{i=1}^{n} \sigma_i \ , \tag{9}$$

where n is the dimensionality of the problem. The relation between the 2D and 3D cases is

$$\Sigma = \tfrac{1}{3}\left[2\,\sigma + \sigma_3\right] \ , \tag{10}$$

where we denote the 3D dilation invariant ,${}^{3}\sigma$, simply by Σ and the two dimensional ,${}^{2}\sigma$, by σ. The dilation σ does not depend on the selection of the coordinate system in the 2D manifold (Vaníček et al. 2001). On the other hand, the value of σ_3, the magnitude of dilation along the third coordinate axis (z), is clearly connected to the direction of the third axis, which is perpendicular to the 2D manifold. We note that once the 2D manifold is selected, we do NOT have the freedom to select the direction of the third axis in the 3D coordinate system arbitrarily as it is perpendicular to that manifold.

2.2 Differential rotation

The **differential rotation magnitude invariant** is a quadratic invariant. It is given by the following expression in 3D

$$\Omega^2 = (\omega_{1,2})^2 + (\omega_{1,3})^2 + (\omega_{2,3})^2 , \tag{11}$$

while in 2D the invariant is simply a scalar quantity ω^2 (Vaníček et al. 2001). This 3D invariant can be interpreted as the square of the length of the "differential rotation vector" $(\Omega_1, \Omega_2, \Omega_3)^T$, where the components are equal to the 2D scalar invariants ω_1, ω_2, ω_3 in the coordinate planes 1,2; 1,3 and 2,3.

We know from the investigation of the 2D invariants ω that these are nothing else but magnitudes of vectors $\boldsymbol{\omega}_i$ $i = 1,2,3$ that are perpendicular to the coordinate planes 1,2; 1,3 and 2,3 in this order [ibid.]. Using the more mundane notation of x,y,z, instead of x_1, x_2, x_3, the argument goes as follows:

1) The displacement $\Delta\mathbf{r} = (u, v)^T$ of a 2D position $\mathbf{r} = (x, y)^T$ due to the differential rotation ω is given by

$$\Delta\mathbf{r} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \mathbf{r} = \begin{bmatrix} \omega y \\ -\omega x \end{bmatrix} = \omega \begin{bmatrix} y \\ -x \end{bmatrix} . \tag{12}$$

2) Equation (12) can be written equivalently using the 3D symbolism as

$$\Delta\mathbf{r} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \omega y \\ -\omega x \\ 0 \end{bmatrix} = \omega \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} . \tag{13}$$

3) Another, equivalent way of writing Eq.(12) is

$$\Delta\mathbf{r} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \omega y \\ -\omega x \\ 0 \end{bmatrix} , \tag{14}$$

where $\times$ denotes the vector product.

4) In Eq.(14), the differential rotation vector $[0,0,\ \omega]^T$ is a vector of magnitude ω, perpendicular to the (x,y)-plane, or $\boldsymbol{\omega}_1$ in our alternative notation, so that

$$\Delta\mathbf{r} = -\boldsymbol{\omega}_1 \times \mathbf{r}\ . \tag{15}$$

5) Similarly, we can show that $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ are perpendicular to planes (x,z) and (y,z) respectively.

For clarity, we shall denote these vectors $\boldsymbol{\omega}_i$ and the corresponding coordinates ω_i by two subscripts, referring them to the coordinate planes used in their computations rather than the coordinate axes. Thus we shall write ω_1 as $\omega_{2,3}$, ω_2 as $\omega_{1,3}$ and ω_3 as $\omega_{1,2}$. This means that the 2D vectors $\boldsymbol{\omega}_i$ i = 1,2,3, are nothing else but components of the 3D vector $\boldsymbol{\Omega}$. Thus

$$\boldsymbol{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} \omega_{2,3} \\ \omega_{1,3} \\ \omega_{1,2} \end{bmatrix} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3\ , \tag{16}$$

which vector is fixed in the space implied by the strain matrix, but its magnitude is invariant in any coordinate transformation. This can be seen by realising that under any rotation of the original coordinate system, neither the magnitudes, nor the configuration of the triad ($\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$) change.

The simplest way of computing the projection of $\boldsymbol{\Omega}$ onto the normal to the 2D manifold (to get the value of $\omega = \omega_3$, which we seek) is to first rotate the 3D coordinate system into a position where the (x,y) plane coincides with the 2D manifold. The rotation can be done by the standard formula:

$$\boldsymbol{\Omega}^* = \mathbf{R}\boldsymbol{\Omega}\ , \tag{17}$$

where **R** is the rotation matrix that transforms coordinates from the original 3D coordinate system into the coordinate system – denoted by an asterisk - in which the (x,y) plane is identical to the 2D space. The z-coordinate of $\boldsymbol{\Omega}^*$ is the scalar ω we seek.

2.3 Maximum shear strain

The eigenvalues of the symmetric part **S** of the 3D strain matrix **E** are, of course invariants. As the matrix **S** is positive definite, the eigenvalues are all real and positive (Boresi et al. 1993). Let us denote them by $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)^T$ and agree to order them always from the largest, Λ_1, to the smallest, Λ_3. Thus the maximum shear M will be always given by

$$M = \Lambda_1 - \Lambda_3 \quad . \tag{18}$$

The eigenvalues are obtained as a solution of the standard cubic equation

$$\det\left(\mathbf{\Lambda I} - \mathbf{S}\right) = \Lambda^3 - I_1\Lambda^2 - I_2\Lambda - I_3 \tag{19}$$

where I_1, I_2, I_3 are the linear, quadratic and cubic invariants in 3D. The linear invariant is equal to $I_1 = Tr(^3\mathbf{S})$ and similarly the other two invariants. The solution for cubic equations is outlined in the Appendix.

For the 2D case, we compute the vector of eigenvalues $(\lambda_1, \lambda_2)^T = \boldsymbol{\lambda}$ from the standard quadratic equation

$$\det\left(\boldsymbol{\lambda}\mathbf{I} - \mathbf{S}\right) = \lambda^2 - i_1\lambda - i_2 \tag{20}$$

where i_1 and i_2 are the usual linear and quadratic invariants of $^2\mathbf{S}$.

Let us adopt a similar convention here that we adopted for the 3D case: let us always have $\lambda_1 > \lambda_2$, and

$$\mu = \lambda_1 - \lambda_2 . \tag{21}$$

Solution for this quadratic equation is fairly straightforward. The roots of Eq. (20) are

$$\lambda_1 = \frac{i_1 + \sqrt{i_1^2 + 4i_2}}{2} \text{ and } \lambda_2 = \frac{i_1 - \sqrt{i_1^2 + 4i_2}}{2} . \qquad (22)$$

Substituting for λ_1 and λ_2 in Eq. (21) we arrive at

$$\mu = \sqrt{i_1^2 + 4i_2} . \qquad (23)$$

The last step that should be performed here is the transformation of M into μ. This transformation is, unfortunately, more involved than the straight calculation of μ. Thus our recommendation is that this transformation not be performed. Instead, the ${}^3\mathbf{S}$ matrix in the 3D coordinate system can be transformed into the 2D manifold coordinate system by the following rotation

$${}^3\mathbf{S}^* = {}^3\mathbf{R}\,{}^3\mathbf{S}\,{}^3\mathbf{R}^{\mathrm{T}} , \qquad (29)$$

where ${}^3\mathbf{S}^*$ is reckoned in the 2D coordinate system augmented by z-axis perpendicular to the plane. The $\mathbf{R}$ matrix is the rotation matrix mediating the rotation from the (X, Y, Z) coordinate system to the augmented (x,y) coordinate system. Taking then the upper left 2 by 2 submatrix of ${}^3\mathbf{S}^*$, which corresponds to the (x, y) coordinate system, μ can be calculated from Eq.(23).

3 Numerical results

In order to be able to show the numerical behaviour of the 2D and 3D invariants, a geodetic network from Northwest Territories, Canada is used. This is a relatively recent 3D network, observed with GPS; its horizontal projection is shown in Fig.1. The strain computed for this network is the virtual strain needed for the Robustness analysis therefore we do not show the initial data needed for the analysis. (These data consist of 3D position differences and their standard deviations.)

Table 1 shows the numerical values for all the 33 points in that network. The most interesting phenomenon is that the 3D invariants are several orders of magnitude larger than their 2D counterparts. This can be understood as being caused by the fact that the strains in the vertical direction are much larger than the strains in the horizontal direction; perhaps the behaviour of dilation can be intuitively understood here *in lieu* of any strain. Thus the strain invariants in 3D, which are functions of strain in all three directions, are here also much larger than the 2D invariants.

The physical reason for this phenomenon is the geometrical shape of the analysed configuration, i.e., the 3D geodetic network from Northwest Territories observed with GPS. Its shape is essentially two-dimensional, meaning that the variations in positions in the horizontal (xy-) plane are much larger than the variations in the vertical (z-) direction. We can think as if we were dealing with a physical body that looks a lot like a membrane stretched in the horizontal plane. The strain across the membrane will, of course, appear much larger than the strains within the membrane; consequently, the strain invariants in 3D will be much larger than those in two dimensions.

We pointed this phenomenon out in (Vaníček et al. 1991; page 28) already. In this publication we argued that it did not make much sense to look at the robustness of even the so called three-dimensional geodetic networks in three dimensions and we proceeded to study robustness only in two dimensions. Later on, Berber pointed out the same thing in his work (2006; page 11). Clearly, the problem of inherent two-dimensionality of geodetic networks (even 3D network) does not disappear when 3D strain invariants are used. While for other structures three-dimensional strains may be used to an advantage, it seems to us that this is not the case with geodetic networks, even when these are in three dimensions.

4 Conclusions

According to our investigations, there are at least three meaningful invariants in 3D. These are "dilation invariant", "differential rotation magnitude invariant" and "maximum shear strain invariant". "Total shear" is not invariant in 3D. Compared to differential rotation and maximum shear strain, the relation between 3D and 2D in terms of dilation is straightforward. The differential rotation in 2D may be understood as the length of a rotation vector $\vec{\omega}$ that "sticks out" of the 2D manifold. In 3D the differential rotation is a vector fixed in space implied by the strain matrix, but its magnitude is invariant in any coordinate transformation. This can be seen by realizing that under any rotation of the original coordinate system, neither the magnitudes, nor the configuration of the triad ($\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$) change. The simplest way of computing the projection of the differential rotation in 3D onto the normal to the 2D manifold is to first rotate the 3D coordinate system into a position where the (x,y) plane coincides with the 2D manifold and then the rotation can be done by the standard formula.

It has been found out that defined as the difference between the largest and the smallest eigenvalues of the $\boldsymbol{S}$ matrix, the maximum shear strain can be simply generalised into 3D. Computing the maximum shear strain in 2D from the maximum shear strain in 3D is a very cumbersome operation. That is why instead of computing the maximum shear strain in 2D from the maximum shear strain in 3D, the maximum shear strain in 2D should be calculated directly from the 2D sub-matrix of the properly rotated symmetrical part of the 3D strain matrix. In this paper these findings have been both shown theoretically and confirmed numerically.

As a by-product of our computations we have confirmed that geodetic networks are inherently almost two-dimensional configurations. Consequently, it does not make much sense to look at their strain in the vertical direction. We have shown that even the usage of 3D strain invariants cannot overcome the difficulties caused by this fact.

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Appendix

As the solution of a cubic equation is a fairly mundane task in mathematics, described in any standard mathematical textbook, we shall give only the main steps here as given by Dickson (1914). These are:

$$\Lambda_1 = \eta_1 + \frac{I_1}{3}$$

$$\Lambda_2 = \eta_2 + \frac{I_1}{3} \qquad \text{(A1)}$$

$$\Lambda_3 = \eta_3 + \frac{I_1}{3}$$

so that we get for the maximum shear

$$M = \eta_1 - \eta_3 . \tag{A2}$$

In these formulae, the pertinent η – terms are equal to

$$\eta_1 = 2\sqrt[3]{-\frac{q}{2} + \sqrt{R}} \text{ and } \eta_3 = -\sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt{3}\sqrt[3]{-\frac{q}{2} - \sqrt{R}} \tag{A3}$$

so that

$$M = 3\sqrt[3]{-\frac{q}{2} + \sqrt{R}} - \sqrt{3}\sqrt[3]{-\frac{q}{2} - \sqrt{R}} , \tag{A4}$$

where

$$\mathrm{R} = \left(\frac{\mathrm{p}}{3}\right)^3 + \left(\frac{\mathrm{q}}{2}\right)^2 , \ \mathrm{p} = -\mathrm{I}_2 - \frac{\mathrm{I}_1^2}{3} , \ \mathrm{q} = -\mathrm{I}_3 - \frac{\mathrm{I}_1\mathrm{I}_2}{3} - \frac{2\mathrm{I}_1^3}{27} . \tag{A5}$$

Table 1. Values of strain invariants for Northwest Territories network times 10^8. (Here Σ, σ are dilations, Ω, ω are differential rotation magnitudes and M, μ are maximum shear strains in 3D and 2D.)

	Σ	σ	Ω	ω	M	μ
1	1711.3	35.2	13230.9	-61.8	16041.4	113.7
2	957.9	-7.8	6853.5	-18.4	8432.8	52.2
3	124.7	-6.2	1188.9	36.7	1364.3	37.1
4	333.3	65.9	2424.9	41.5	2986.8	105.4
5	138.9	-7.9	1263.2	37.9	1438.0	23.5
6	-13.8	-27.3	1029.9	4.8	1039.5	10.6
7	-521.8	-15.9	2749.0	1.1	3619.9	80.3
8	-637.3	-2.5	1510.8	-15.2	2678.8	30.8
9	-666.5	4.1	3601.5	-56.8	4654.3	416.7
10	200.7	13.2	1251.0	19.6	1564.6	58.4
11	-459.5	-6.0	3350.0	-9.4	4097.2	122.2
12	-1171.7	-30.4	8006.9	52.9	9931.9	190.7
13	-233.7	-11.5	2007.4	42.9	2400.8	53.2
14	-424.6	-6.6	1937.6	-3.3	2597.5	45.7
15	354.8	1.7	1211.5	-37.8	1865.5	68.1
16	670.0	-12.1	2023.8	-2.3	3241.2	20.1
17	457.9	19.9	2141.9	-5.1	2927.0	46.7
18	1741.5	-35.0	4856.0	15.8	8096.8	84.0
19	-2297.6	-14.7	8336.5	17.0	12414.7	182.1
20	-665.0	1.4	1805.2	-5.4	3026.4	144.5
21	-1882.3	-17.2	7424.2	7.0	10744.6	311.1
22	1625.0	-18.5	4222.3	-17.8	7325.9	4.3
23	1615.2	-61.0	5426.3	9.8	8335.2	189.5
24	-1171.7	-30.4	8006.9	52.9	9931.9	190.7
25	-1773.1	-17.9	5301.8	-30.9	8576.4	117.5
26	339.3	28.9	2373.7	27.2	2907.1	79.3
27	-99.1	7.4	2174.8	-1.5	2317.8	51.7
28	486.5	12.8	809.4	38.9	1827.1	31.3
29	187.8	29.0	1081.6	30.6	1469.7	315.8
30	-804.2	8.6	6823.9	46.8	8077.3	134.4
31	1625.0	-18.5	4222.3	-17.8	7325.9	4.3
32	-147.5	12.5	670.6	4.7	948.4	7.2
33	497.7	9.8	2825.1	45.0	3690.1	111.8

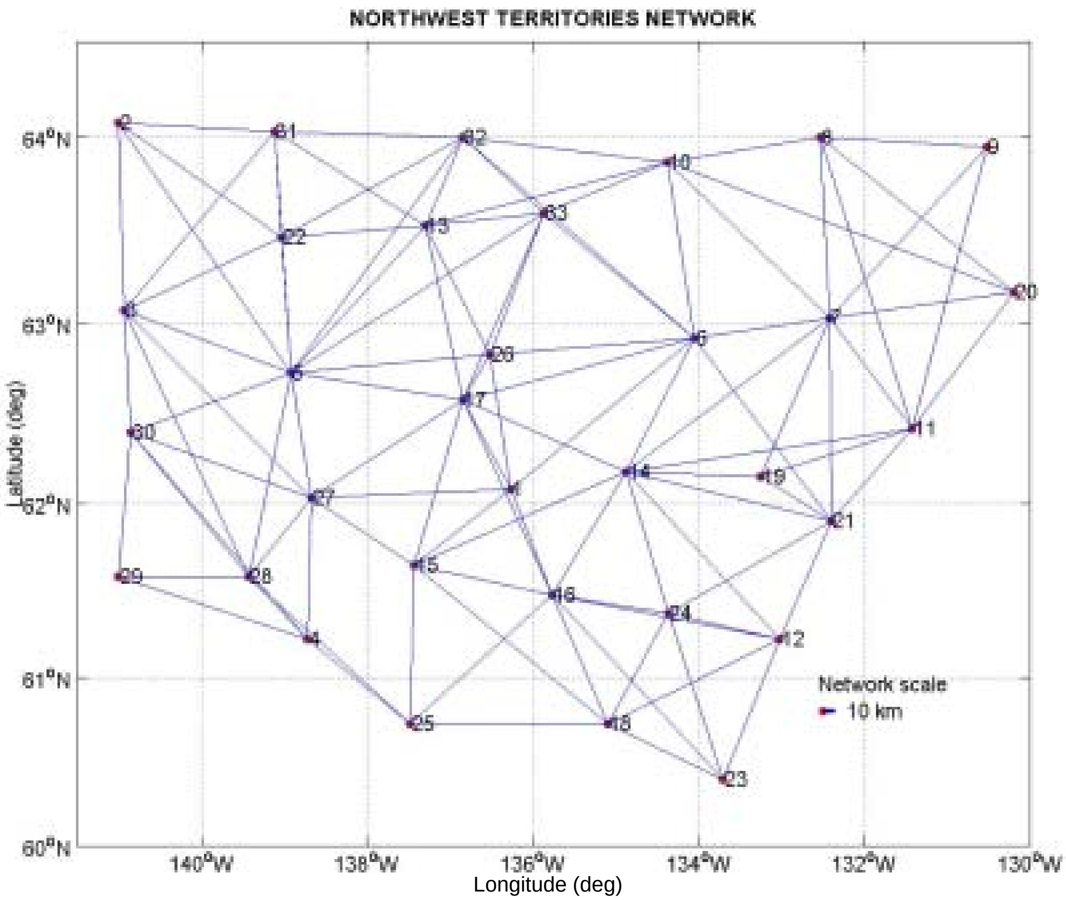

Figure 1. Northwest Territories network