On evaluation of Newton integrals in geodetic coordinates: exact formulation and spherical approximation

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Abstract: Newton integrals for the potential and the vertical component of the attraction vector (gravitational effect), serve for evaluating various topographical effects (negative topographical corrections) as well as for evaluating the potential and the gravitational effect of various bodies and/or models of mass density distribution (real, constant or anomalous). Here we review global evaluation of the Newton integrals in geodetic coordinates (in Gauss ellipsoidal coordinates) formulated exactly and in spherical approximation. Various topographical corrections are addressed by investigating their definitions in terms of the upper and lower topographical boundary and the used density. Numerical aspects of the evaluation of the Newton integrals, such as the weak singularity treatment, split–up into spherical shell and terrain terms, and a requirement to integrate over the entire globe are also addressed. Implications associated with regional and local evaluation of the Newton integrals are indicated. Special attention is paid to the so-called "ellipsoidal topography of constant density" ("ETC") and to NETC topo–corrections to potential and gravity. The abbreviation "NETC" stands for "No ETC" and represents the removal of the effect of "ETC" on potential or gravity.

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Introduction

In geodesy and geophysics we often face the need to remove the gravitational potential of topographical masses of real or model (constant) density from the actual and/or disturbing potential. Equivalently, we may need to remove the vertical component of the attraction vector (so-called gravitational effect) of these topographical masses from the actual (observed or synthetic) gravity (thus also from gravity disturbances or anomalies). Generally, the topographical masses are defined as density distribution (real or model) between Earth's surface (topographical surface, shortly topo–surface) and the geoid. However, there are applications (both geodetic and geophysical) that call for defining the topographical masses as the density distribution between the topo–surface and the reference ellipsoid (as opposed to the geoid). In geophysics the need of removing the effect of the latter type of topography was advocated by e.g. *Chapman and Bodine (1979); Vogel (1982); Jung and Rabinowitz (1988); Meurers (1992); Talwani (1998); Hackney and Featherstone (2003).* In order to distinguish the types of topographical masses according to their lower boundary and mass density, *Vajda et al. (2004)* proposed the following classification:
The term "ellipsoidal topography" is not to be understood as the topogra-

Tab. 1. Models of topographical masses. The upper boundary is always the topo–surface

type of topography	**lower boundary**	**density**	**notation**
topography	geoid	real	T
topography of constant density	geoid	constant (model)	TC
ellipsoidal topography	reference ellipsoid	real	ET
ellipsoidal topography of constant density	reference ellipsoid	constant (model)	ETC

phy of the ellipsoid, but as the topography reckoned from the ellipsoid.

In addition, in geophysics we need to evaluate the potential and/or gravitational effect of various bodies, or density distributions of various sub–regions of the Earth – of real, constant, or anomalous density – not mentioning horizontal and higher order derivatives of the actual and/or disturbing potential. All the above mentioned potentials and gravitational effects are evaluated by means of the Newton integrals for the potential and vertical component of the attraction vector, respectively. The significance of the Newton integral is great, as it represents the solution to the direct (forward) gravimetric problem. Below we deal with evaluating these two kinds of the Newton integral in the geodetic coordinates. Our starting point will be global and rigorous, followed by considering approximations such as neglecting deflections of the vertical, spherical approximation, or simplifications that can be adopted in regional or local studies.

The need to express the Newton integrals in the geodetic coordinates is implied by the fact, that in practice evaluation/observation points (stations) are often positioned in the geodetic coordinates. To be more specific, horizontal coordinates are given as *geodetic latitude* and *geodetic longitude* (respective to a *reference ellipsoid* as the *horizontal datum*), vertical coordinate is given as a *geodetic (ellipsoidal) height* reckoned from the same reference ellipsoid as the *vertical datum.* The geodetic height is either measured or may be evaluated (as an acceptable approximation) as the sum of a "height above sea level" (e.g., *orthometric* or *normal height*) reckoned from the "sea level" (i.e., *geoid* or *quasigeoid*), and the geoidal/quasigeoidal height (that is referred to the same reference ellipsoid).

1. Spherical and geodetic coordinates

We shall refer the discussed quantities to *geocentric* coordinate systems, namely to a geocentric *Cartesian coordinate system*, geocentric *spherical coordinate system*, and geocentric *geodetic (Gauss ellipsoidal) coordinate system* (e.g., *Pick et al., 1973*, p. 437; *Heiskanen and Moritz, 1967*, Chapter 5–3; *Vaníček and Krakiwsky, 1986*, Chapter 15.4). In the *spherical coordinate system* any point P in space is given by the *geocentric distance* r, *spherical (geocentric) latitude* $\bar{\phi}$, and *spherical (geocentric) longitude* $\bar{\lambda}$, that are related to the *Cartesian coordinates* of the point $P \equiv (x, y, z) \equiv (r, \bar{\phi}, \bar{\lambda})$ as follows, cf. Fig. 1:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_P = \begin{bmatrix} r\cos\bar{\phi}\cos\bar{\lambda} \\ r\cos\bar{\phi}\sin\bar{\lambda} \\ r\sin\bar{\phi} \end{bmatrix}_P , \qquad \begin{bmatrix} r \\ \bar{\phi} \\ \bar{\lambda} \end{bmatrix}_P = \begin{bmatrix} \sqrt{x^2+y^2+z^2} \\ 2\arctan\left(y\Big/\left[x+\sqrt{x^2+y^2}\right]\right) \\ \arcsin\left(z\Big/\sqrt{x^2+y^2+z^2}\right) \end{bmatrix}_P . \quad (1)$$

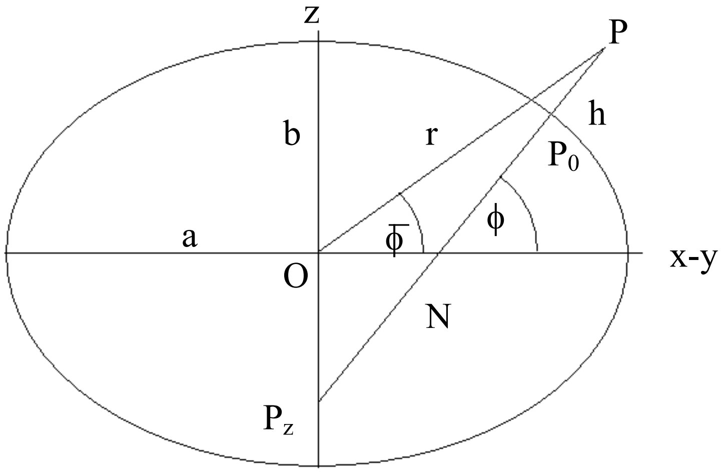

Fig. 1. The geodetic coordinate system and the relationship between spherical and geodetic coordinates.

For brevity we shall often denote the horizontal position in the geocentric spherical coordinates as $\bar{\Omega} \equiv (\bar{\phi}, \bar{\lambda})$. *Geocentric geodetic coordinates* – *geodetic height* h, *geodetic latitude* ϕ, and *geodetic longitude* λ – are defined in the geocentric *geodetic coordinate system*, which is based on the *mean earth ellipsoid* (e.g. *Heiskanen and Moritz, 1967*, Section 2–21). The mean earth ellipsoid is a reference ellipsoid that is not only geocentric and biaxial, it is also a so called "level" ellipsoid (ibid, Section 2–7). There is a unique and physically meaningful link between the level ellipsoid and the normal gravity field. The level ellipsoid is the equipotential surface of the normal gravity potential on which the normal gravity potential has the same value as the actual gravity potential on the geoid. Thus the mean earth ellipsoid (or *normal ellipsoid*) generates the normal gravity, while its surface serves as a (geocentric geodetic) coordinate surface. As a surface it is defined by a major semi–axis a, and minor semi–axis b. Thus the geodetic coordinates are inevitably associated with the mentioned two parameters of the ellipsoid. Alternatively to the minor semi–axis, the *first numerical eccentricity* e ($e^2 = [a^2 - b^2]/a^2$), *focal distance* E, or *flattening* f can be used (ibid). The geodetic coordinates are related to the Cartesian coordinates of the point $P \equiv (x, y, z) \equiv (h, \phi, \lambda)$ as follows (e.g., ibid, Section 5–3), cf. Fig. 1:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_P = \begin{bmatrix} (N(\phi) + h)\cos\phi\cos\lambda \\ (N(\phi) + h)\cos\phi\sin\lambda \\ (N(\phi)(1 - e^2) + h)\sin\phi \end{bmatrix}_P , \tag{2}$$

where

$$N(\phi) = \frac{a}{\sqrt{1 - e^2 \sin^2\phi}} \tag{3}$$

is the *prime vertical radius of curvature* (not to be confused with the geoidal height due to the same notation). The inverse transformation is not as straightforward, cf. e.g. *(Jones, 2002; Pollard, 2002; Vermeille, 2002)*. For brevity, we will often denote the horizontal position in the geodetic coordinates as $\Omega \equiv (\phi, \lambda)$. Equations (1) and (2) are used to transform the *geodetic coordinates* of the point P to its (geocentric) *spherical coordinates* or vice versa. Obviously, if the biaxial ellipsoid represents the horizontal datum, longitudes in both coordinate systems are identical, i.e. $\lambda = \bar{\lambda}$.

2. Newton integrals in geocentric spherical coordinates

In Section 2 we will review the formulation of the Newton integral for the gravitational potential and for the vertical component of the attraction vector in geocentric spherical coordinates. The integral boundaries will be thus referred in spherical coordinates, and the infinitesimal solid (volume) element will be expressed also in the mentioned coordinates.

2.1. Newton integral for the gravitational potential

The Newton integral for the gravitational potential expressed in the spherical coordinates reads

$$\forall\, (r_P, \bar{\Omega}_P): \quad V\,(r_P, \bar{\Omega}_P) = G \int\limits_{r_1(\bar{\Omega})}^{r_2(\bar{\Omega})} \iint\limits_{\Omega_0} \rho(r, \bar{\Omega}) L^{-1}(r_P, \bar{\Omega}_P, r, \bar{\Omega})\, \mathrm{d}\bar{\vartheta}\,, \tag{4}$$

where G is the universal gravitational constant, $\rho\,(r, \bar{\Omega})$ is the mass density distribution (real, constant, or anomalous), L is a 3–D Euclidean distance between the evaluation and integration (running) point,

Tab. 2. Examples of specific gravitational potentials defined by the Newton integral. The upper boundary is for all examples the topo–surface $r_2(\bar{\Omega}) \equiv r_t(\bar{\Omega})$

quantity defined by Eq. (4)	density	lower boundary
potential of topography (T) V^{GT}	real $\rho(r,\bar{\Omega})$	geoid $r_1(\bar{\Omega}) \equiv r_g(\bar{\Omega})$
potential of topography of constant density (TC) V_0^{GT}	constant ρ_0	geoid $r_1(\bar{\Omega}) \equiv r_g(\bar{\Omega})$
potential of ellipsoidal topography (ET) V^{ET}	real $\rho(r,\bar{\Omega})$	ref. ellipsoid $r_1(\bar{\Omega}) \equiv r_e(\bar{\Omega})$
potential of ellipsoidal topography of constant density (ETC) V_0^{ET}	constant ρ_0	ref. ellipsoid $r_1(\bar{\Omega}) \equiv r_e(\bar{\Omega})$
potential of real Earth's masses V	real $\rho(r,\bar{\Omega})$	geocentre $r_1(\bar{\Omega}) \equiv 0$
potential of anomalous Earth's masses δV	anomalous $\delta\rho(r,\bar{\Omega})$	geocentre $r_1(\bar{\Omega}) \equiv 0$

$$\mathrm{d}\bar{\vartheta} = r^2 \cos\bar{\phi}\ \mathrm{d}r\ \mathrm{d}\bar{\phi}\ \mathrm{d}\bar{\lambda} = r^2\ \mathrm{d}r\ \mathrm{d}\bar{\Omega} \tag{5}$$

is the infinitesimal *solid (volume) element* in the spherical coordinate system, $\Omega_0 \equiv \langle -\pi/2; \pi/2\rangle \otimes \langle 0; 2\pi)$ is the full solid angle, and $r_1(\bar{\Omega})$ and $r_2(\bar{\Omega})$ are the lower and upper integral boundaries, respectively. Examples of specific gravitational potentials defined by Eq. (4) are listed in Tab. 2.

The first four rows of Tab. 2 define, taken with the negative sign, four different (*NT*, *NTC*, *NET*, *NETC*) *topographical corrections to the potential* used for constructing the topo–corrected actual and/or disturbing potential. Note, that in Eq. (4) the evaluation point is not restricted to lie on the topo–surface or any other reference surface.

2.2. Newton integral for the vertical component of the attraction vector (gravitational effect)

The *gravitational effect* of mass density from a specific region is defined as a vertical component of the attraction vector generated by the mass density (real, constant, or anomalous) within that region. In the spherical coordinates it reads as follows

Tab. 3. Various gravitational effects defined by the Newton integral for the vertical component of the attraction vector. The upper boundary is the topo–surface $r_2\left(\bar{\Omega}\right) \equiv r_t\left(\bar{\Omega}\right)$

quantity defined by Eq. (7)	density	lower boundary
gravitational effect of topography (T) A^{GT}	real $\rho\left(r,\bar{\Omega}\right)$	geoid $r_1\left(\bar{\Omega}\right) \equiv r_g\left(\bar{\Omega}\right)$
gravitational effect of topography of constant density (TC) A_0^{GT}	constant ρ_0	geoid $r_1\left(\bar{\Omega}\right) \equiv r_g\left(\bar{\Omega}\right)$
gravitational effect of ellipsoidal topography (ET) A^{ET}	real $\rho\left(r,\bar{\Omega}\right)$	ref. ellipsoid $r_1\left(\bar{\Omega}\right) \equiv r_e\left(\bar{\Omega}\right)$
gravitational effect of ellipsoidal topography of constant density (ETC) A_0^{ET}	constant ρ_0	ref. ellipsoid $r_1\left(\bar{\Omega}\right) \equiv r_e\left(\bar{\Omega}\right)$
gravitational effect of real Earth's masses A	real $\rho\left(r,\bar{\Omega}\right)$	geocentre $r_1\left(\bar{\Omega}\right) \equiv 0$
gravitational effect of anomalous Earth's masses δA	anomalous $\delta\rho\left(r,\bar{\Omega}\right)$	geocentre $r_1\left(\bar{\Omega}\right) \equiv 0$

$$\forall\left(r_P, \bar{\Omega}_P\right) : A\left(r_P, \bar{\Omega}_P\right) = -\,G \int\limits_{r_1(\bar{\Omega})}^{r_2(\bar{\Omega})} \iint\limits_{\Omega_0} \rho(r, \bar{\Omega}) \frac{\partial L^{-1}(r_P, \bar{\Omega}_P, r, \bar{\Omega})}{\partial n}\, \mathrm{d}\bar{\vartheta}\,, \tag{6}$$

where $\partial/\partial n$ is the derivative in the direction of the outer normal to the actual equipotential surface at the evaluation point P (along actual plumbline at P). Everything else remains as defined in Eq. (4). When neglecting the deflection of the vertical at P, we can replace the $\partial/\partial n$ by the derivative in the direction of the ellipsoidal normal, i.e. by the derivative with respect to the geodetic height of the evaluation point $\partial/\partial h_P$. Thus

$$\forall\left(r_P, \bar{\Omega}_P\right) : A\left(r_P, \bar{\Omega}_P\right) \cong -\,G \int\limits_{r_1(\bar{\Omega})}^{r_2(\bar{\Omega})} \iint\limits_{\Omega_0} \rho(r, \bar{\Omega}) \frac{\partial L^{-1}(r_P, \bar{\Omega}_P, r, \bar{\Omega})}{\partial h_P}\, \mathrm{d}\bar{\vartheta}\,, \tag{7}$$

The integration domain and corresponding density distribution define the gravitational effect given by Eqs (6) and (7), cf. Tab. 3 for some examples. Taken with the negative sign, the first four rows define four different *topographical corrections to actual gravity*, cf. also Tab. 8 in *Vajda et al. (2004)*. Note, that again in Eqs (6) or (7) the point of evaluation is not restricted to lie on the topo–surface or any other reference surface.

3. Newton integrals in geocentric geodetic coordinates

In Section 3 we will present the formulation of the Newton integral for the gravitational potential and for the vertical component of the attraction vector (gravitational effect) in geocentric geodetic coordinates. This will require expressing the solid element, and the Euclidean distance with its vertical derivative, in the geodetic coordinates. Also the integral boundaries will be now referred in the geodetic coordinates.

3.1. Newton integral for the gravitational potential in the geodetic coordinates

The infinitesimal *solid element* in the geodetic coordinates reads

$$\mathrm{d}\vartheta = J\left(a, e, h, \phi\right)\ \mathrm{d}h\ \mathrm{d}\phi\ \mathrm{d}\lambda\,, \tag{8}$$

where the expression for the Jacobian $J\left(a, e, h, \phi\right)$ is derived by means of Eq. (2)

$$J(a,e,h,\phi) = \left|\frac{\partial x}{\partial h}\left(\frac{\partial y}{\partial \phi}\frac{\partial z}{\partial \lambda} - \frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \phi}\right) - \frac{\partial x}{\partial \phi}\left(\frac{\partial y}{\partial h}\frac{\partial z}{\partial \lambda} - \frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial h}\right) + \right.$$
$$\left. + \frac{\partial x}{\partial \lambda}\left(\frac{\partial y}{\partial h}\frac{\partial z}{\partial \phi} - \frac{\partial y}{\partial \phi}\frac{\partial z}{\partial h}\right)\right|\,. \tag{9}$$

For the partial derivatives we get

$$\frac{\partial x}{\partial h} = \cos\phi\ \cos\lambda\,, \quad \frac{\partial y}{\partial h} = \cos\phi\ \sin\lambda\,, \quad \frac{\partial z}{\partial h} = \sin\phi\ , \tag{10}$$

$$\frac{\partial x}{\partial \phi} = \cos\phi\ \cos\lambda\frac{\partial N(\phi)}{\partial \phi} - \left(N(\phi) + h\right)\sin\phi\ \cos\lambda\,, \tag{11a}$$

$$\frac{\partial y}{\partial \phi} = \cos\phi\ \sin\lambda\frac{\partial N(\phi)}{\partial \phi} - \left(N(\phi) + h\right)\sin\phi\ \sin\lambda\,, \tag{11b}$$

$$\frac{\partial z}{\partial \phi} = \sin\phi\left(1 - e^2\right)\ \frac{\partial N(\phi)}{\partial \phi} + \left[N(\phi)\left(1 - e^2\right) + h\right]\cos\phi\ , \tag{11c}$$

$$\frac{\partial x}{\partial \lambda} = -\left(N(\phi)+h\right)\cos\phi\ \sin\lambda, \qquad \frac{\partial y}{\partial \lambda} = \left(N(\phi)+h\right)\cos\phi\ \cos\lambda,$$

$$\frac{\partial z}{\partial \lambda} = 0, \tag{12}$$

where the curvature $N(\phi)$ is given by Eq. (3), and where

$$\frac{\partial N(\phi)}{\partial \phi} = a\ e^2\ \sin\phi\ \cos\phi\ \left(1-e^2\ \sin^2\phi\right)^{-\frac{3}{2}} = \frac{e^2 \sin\phi\ \cos\phi}{1-e^2\ \sin^2\phi}\ N(\phi)\,. \tag{13}$$

Upon the back substitution and required algebraic treatment we obtain

$$J\left(a,e,h,\phi\right) = \left(N(\phi)+h\right)\left(\frac{1-e^2}{1-e^2\ \sin^2\phi}N(\phi)+h\right)\cos\phi. \tag{14}$$

For the behaviour of the Jacobian, see Appendix. In the geodetic coordinates, the Euclidean distance L reads

$$\begin{aligned} L(h_P,\phi_P,\lambda_P,h,\phi,\lambda) = &\Big\{\left[N(\phi_P)+h_P\right]^2\cos^2\phi_P + \left[N(\phi)+h\right]^2\cos^2\phi - \\ &-2\left(N(\phi_P)+h_P\right)\left(N(\phi)+h\right)\cos\phi_P\cos\phi\cos\left(\lambda_P-\lambda\right) + \\ &+\Big[\left(N(\phi_P)+h_P\right)-e^2N(\phi_P)\Big]^2\sin^2\phi_P + \Big[\left(N(\phi)+h\right)-e^2N(\phi)\Big]^2\sin^2\phi - \\ &-2\Big[\left(N(\phi_P)+h_P\right)-e^2N(\phi_P)\Big]\Big[\left(N(\phi)+h\right)-e^2N(\phi)\Big]\sin\phi_P\sin\phi\Big\}^{\frac{1}{2}}. \end{aligned} \tag{15}$$

Thus the Newton integral for the gravitational potential reads in the geodetic coordinates

$$\forall\ \left(h_P,\Omega_P\right) : V\left(h_P,\Omega_P\right) = G\int\limits_{h_1(\Omega)}^{h_2(\Omega)}\iint\limits_{\Omega_0}\rho(h,\Omega)L^{-1}(h_P,\Omega_P,h,\Omega)\mathrm{d}\vartheta, \tag{16}$$

where the distance L is given by Eq. (15), solid element by Eq. (8), and the Jacobian by Eq. (14). The upper and lower boundaries are now also given in the geodetic coordinates. For instance, the topo–surface is given as $h_t\left(\Omega\right)$, geoid as $N\left(\Omega\right)$ and reference ellipsoid as $h\left(\Omega\right)=0$. When the integration is to be carried out over the whole Earth's interior, starting at the geocentre, the lower integration boundary will no longer be a single point. The lower boundary will become $h_1\left(\Omega\right) = h_C\left(\phi\right)$, defined as $z=0$ for $\phi \neq 0$, i.e.,

$$\forall\phi:\ 0<|\phi|\le\frac{\pi}{2}: \qquad h_C(\phi)=\ -\left(1-e^2\right)N(\phi)\,, \tag{17}$$

where $N(\phi)$ is given by Eq. (3), cf. Fig. 2.

Recently, *Novák and Grafarend (2004)* derived the Newton integral for

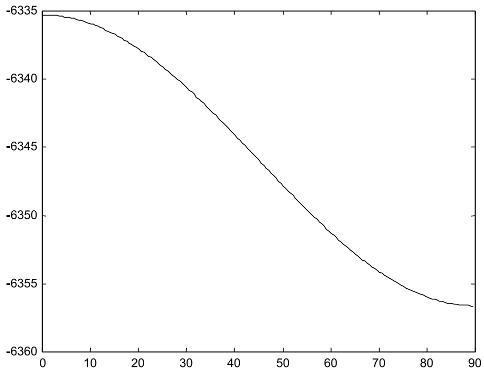

Fig. 2. The lower integration boundary $h_C(\phi)$ [km] for evaluating the Newton integral over the whole interior of the Earth (of the ellipsoid). The horizontal axis is geodetic latitude [arcdeg]. The reference ellipsoid is given by a = 6378 km and e^2 = 0.0067.

the potential in the geodetic coordinates while neglecting the terms of e^4 and higher (ibid, Section 2). They also discussed the computational aspects in detail, as well as the expression of this integral in the spectral form (ibid, Section 4).

3.2. Newton integral for the gravitational effect in the geodetic coordinates

The derivative of the inverse Euclidean distance with respect to the geodetic height reads

$$\frac{\partial L^{-1}(h_P,\phi_P,\lambda_P,h,\phi,\lambda)}{\partial h_P}=-\,L^{-3}\Big\{[N\left(\phi_P\right)+h_P]-[N\left(\phi\right)+h]\cos\psi+ \\ +e^2\sin\phi_P\left[N\left(\phi\right)\sin\phi-N\left(\phi_P\right)\sin\phi_P\right]\Big\}, \tag{18}$$

where

$$\cos\psi = \sin\phi_P \sin\phi + \cos\phi_P \cos\phi \cos(\lambda_P - \lambda), \tag{19}$$

and the distance L is given by Eq. (15). The gravitational effect given by Eq. (7) reads in the geodetic coordinates

$$\forall(h_P, \Omega_P) : A(h_P, \Omega_P) = -G \int\limits_{h_1(\Omega)}^{h_2(\Omega)} \iint\limits_{\Omega_0} \rho(h, \Omega) \frac{\partial L^{-1}(h_P, \Omega_P, h, \Omega)}{\partial h_P} \mathrm{d}\vartheta, \tag{20}$$

where $\partial L^{-1}/\partial h_P$ is given by Eq. (18).

Novák and Grafarend (2004) also derived the Newton integral for the gravitational effect in the geodetic coordinates while neglecting the terms of e^4 and higher (ibid, Section 3). They also discussed the computational aspects in detail.

4. Spherical approximation of Newton integrals in geocentric geodetic coordinates

In Section 4 we will deal with the spherical approximation of the Newton integrals presented in Section 3. The spherical approximation will imply expressing the solid element, as well as the Euclidean distance and its vertical derivative, in the spherical approximation, still using the geodetic coordinates.

4.1. Newton integral for the potential in the geodetic coordinates − spherical approximation

Under the *spherical approximation* (*Moritz, 1980*, p. 349) we shall hereafter understand neglecting the terms multiplied by e^2 and higher order terms in the expressions for the Euclidean distance, derivative of the inverse Euclidean distance with respect to the geodetic height of the evaluation point, and the Jacobian. Moreover, the major semi–axis a is replaced by the mean radius $R = \sqrt[3]{a^2 b}$. Since

$$N(\phi) = \frac{a}{\sqrt{1 - e^2 \sin^2\phi}} \cong a + \frac{a}{2} e^2 \sin^2\phi \approx a \approx R, \tag{21}$$

Equation (15) reads in the spherical approximation as follows:

$$L(h_P, \Omega_P, h, \Omega) \approx \sqrt{(R + h_P)^2 + (R + h)^2 - 2\,(R + h_P)\,(R + h)\cos\psi}, \quad (22)$$

where $\cos\psi$ is given by Eq. (19). Here ψ stands for the angular distance between the evaluation and integration point. In spherical approximation, the Jacobian reads

$$J\,(a, e, h, \phi) \approx (R + h)^2 \cos\phi. \quad (23)$$

Thus the Newton integral for the gravitational potential in spherical approximation reads

$$\forall(h_P, \Omega_P)\,:\, V\,(h_P, \Omega_P) \approx$$

$$\approx G \int\limits_{h_1(\Omega)}^{h_2(\Omega)} \iint\limits_{\Omega_0} \rho(h, \Omega) L^{-1}(h_P, \Omega_P, h, \Omega)\,(R + h)^2\,\mathrm{d}h\,\mathrm{d}\Omega, \quad (24)$$

where $\mathrm{d}\Omega = \cos\phi\ \mathrm{d}\phi\ \mathrm{d}\lambda$, and the distance L is given by Eq. (22). In spherical approximation, when integrating over the whole Earth's interior, the integration starts (as expected) at the geocentre, i.e. the lower integral boundary is given as follows, cf. Eq. (17):

$$h_C\,(\phi) \approx\ - R\,. \quad (25)$$

Novák and Grafarend (2004) showed, that the Newton integral for the potential in the geodetic coordinates (neglecting the terms of e^4 and higher) can be written as a sum of a spherical term and ellipsoidal correction to the spherical term (ibid, Eqs (13) through (15)). In the case of the topographical correction, they showed for a test area in the Canadian Rocky Mountains that the ellipsoidal correction is by three orders of magnitude smaller than the spherical term (ibid, Section 5).

4.2. Newton integral for the gravitational effect in the geodetic coordinates − spherical approximation

Equation (18) reads in spherical approximation as follows:

$$\frac{\partial L^{-1}(h_P, \Omega_P, h, \Omega)}{\partial h_P} \approx -\frac{(R + h_P) - (R + h) \cos\psi}{L^3}, \tag{26}$$

with the distance L given by Eq. (22). Thus the gravitational effect in spherical approximation reads

$$\forall (h_P, \Omega_P) \,:\, A\,(h_P, \Omega_P) \approx$$

$$\approx -\,G \int\limits_{h_1(\Omega)}^{h_2(\Omega)} \iint\limits_{\Omega_0} \rho(h, \Omega) \frac{\partial L^{-1}(h_P, \Omega_P, h, \Omega)}{\partial h_P} \,(R + h)^2 \,\mathrm{d}h \,\mathrm{d}\Omega, \tag{27}$$

where $\partial L^{-1}/\partial h_P$ is given by Eq. (26).

Novák and Grafarend (2004) showed, that the Newton integral for the gravitational effect in the geodetic coordinates (neglecting the terms of e^4 and higher) can be written as the sum of a spherical term and ellipsoidal correction to the spherical term (ibid, Eqs (36) and (37)). Again the ellipsoidal correction was three orders of magnitude smaller than the spherical term (ibid, Section 5).

5. Spherical approximation of the topographical corrections in terms of orthometric (or normal) heights with the geoid (or quasigeoid) as a vertical datum

When the Newton integral for the gravitational potential or the gravitational effect is evaluated with the geoid as the lower topo–boundary, in spherical approximation, while the topo–surface as well as the evaluation and integration points are positioned using *orthometric/normal heights* reckoned from the *geoid/quasigeoid* as a *vertical datum*, the spherical approximation applies to the geoid. Recall the relationship between geodetic (h) and orthometric (H), or normal (H^N) heights:

$$h\,(\Omega) \cong H\,(\Omega) + N\,(\Omega) \quad (a), \qquad h\,(\Omega) \cong H^N\,(\Omega) + \zeta\,(\Omega) \quad (b), \tag{28}$$

where ζ is the height anomaly reckoned from the reference ellipsoid. Everything considered in the sequel for the orthometric heights applies also to the normal heights.

The *gravitational potential of topography* reads (cf. Tab. 2)

$$V^{GT}(H_P,\Omega_P) \approx$$
$$\approx G \int\limits_0^{H_t(\Omega)} \iint\limits_{\Omega_0} \rho(H,\Omega) L^{-1}(H_P,\Omega_P,H,\Omega)\,(R+H)^2\,\mathrm{d}H\,\mathrm{d}\Omega, \tag{29}$$

where the Euclidian distance L is given by

$$L(H_P,\Omega_P,H,\Omega) \approx$$
$$\approx \sqrt{(R+H_P)^2 + (R+H)^2 - 2\,(R+H_P)\,(R+H)\cos\psi}, \tag{30}$$

and angular distance between the evaluation and integration points ψ is given by Eq. (19). The upper boundary is the topo–surface $H_t(\Omega)$ reckoned from the geoid. The lower boundary is the geoid, given as $h(\Omega) = N(\Omega)$ or $H(\Omega) = 0$. Similarly the *gravitational effect of topography* reads (cf. Tab. 3)

$$A^{GT}(H_P,\Omega_P) \approx$$
$$\approx -G \int\limits_0^{H_t(\Omega)} \iint\limits_{\Omega_0} \rho(H,\Omega) \frac{\partial L^{-1}(H_P,\Omega_P,H,\Omega)}{\partial H_P}\,(R+H)^2\,\mathrm{d}H\,\mathrm{d}\Omega, \tag{31}$$

where the vertical derivative with respect to the geodetic height at the evaluation point $\partial/\partial h_P$ was replaced by the vertical derivative with respect to the orthometric height at the evaluation point $\partial/\partial H_P$, i.e.

$$\frac{\partial L^{-1}(H_P,\Omega_P,H,\Omega)}{\partial H_P} \approx -\frac{(R+H_P)-(R+H)\cos\psi}{L^3}\ , \tag{32}$$

where the distance L is given by Eq. (30) and $\cos\psi$ by Eq. (19). The upper and lower boundaries are identical to those in Eq. (29).

6. Numerical aspects

Now we will turn our attention to some aspects of numerical evaluation of the discussed Newton integrals. We will take a look at the singularity of the integrals, we will focus on the topographical corrections in spherical approximation and discuss the splitting of the topo–correction into a spherical shell term and a terrain term. Finally, we will look at the requirement of integrating over the whole globe.

6.1. Weak singularity

Both the Newton integrals – gravitational potential and gravitational effect – discussed in Sections 2 through 5 are singular for the integration point coinciding with the evaluation point, i.e., $(h, \Omega) = (h_P, \Omega_P)$, or $(H, \Omega) = (H_P, \Omega_P)$. The singularity is encountered when evaluating the integrals on the topo–surface or below it. However, the singularity is weak, removable, cf. e.g. (*Kellogg, 1929*, p. 151).

6.2. Spherical topographical corrections – spherical shell and terrain terms

In the sequel we shall deal with topo–corrections in spherical approximation only. The spherical approximation implies that the lower topographical boundary is approximated by a sphere (not actually, only for the sake of computing the distances between the evaluation and integration points). In the case of topo–corrections, that adopt the constant model density of topographical masses, it is conventional and convenient to split the topo–correction into a *spherical shell term* and a *terrain term* (sometimes referred to as the *roughness term*)

$$\int_0^{h_t(\Omega)} \ldots = \int_0^{h_t(\Omega_P)} \ldots + \int_{h_t(\Omega_P)}^{h_t(\Omega)} \ldots \; (a), \quad \int_0^{H_t(\Omega)} \ldots = \int_0^{H_t(\Omega_P)} \ldots + \int_{H_t(\Omega_P)}^{H_t(\Omega)} \ldots \; (b), \tag{33}$$

where $h_t(\Omega_P)$ is the geodetic height and $H_t(\Omega_P)$ is the orthometric height of the topo–surface at the horizontal position of the evaluation point. The *spherical shell term* (dropping the negative sign) is the potential, or the gravitational effect, of the *spherical layer* (shell) with the inner radius equal to the mean Earth's radius R, and thickness $h_t(\Omega_P)$, or $H_t(\Omega_P)$, which can be evaluated analytically (closed-form solution), cf. e.g. (*Wichiencharoen, 1982; Blakely, 1995*, Sections 3.2.1 and 3.2.2; *Vaníček et al., 2001; 2004*).

Note, that the thickness of the spherical shell is equal to the elevation of the topo–surface at the horizontal position of the evaluation point; thus it changes from one evaluation point to another. The *terrain term* (dropping the negative sign) is the potential, or the gravitational effect, of the *terrain relative to the spherical shell*, sometimes called the *roughness term*, which is to be evaluated by numerical integration over the entire globe. The terrain term remains singular, but again the singularity is weak and removable in the same manner as already discussed in Section 6.1.

Topo–corrections (to the potential and gravity) for topography defined by the geoid as its lower boundary, referred in terms of the orthometric heights, cf. Section 5, are discussed in detail in e.g. *(Vaníček et al., 2004)*. Hereafter we shall focus on the NETC topo–corrections.

6.3. NETC topo–correction to the gravitational potential and gravity

In Section 6.3 we shall deal exclusively with the evaluation of the NETC topo–correction to the potential $(-V_0^{ET})$ and the NETC topo–correction to gravity $(-A_0^{ET})$. For more details regarding the potential and gravitational effect of the ellipsoidal topography of constant density (ETC), refer to (*Vajda et al., 2004*), particularly to Sections 3, 3.1, 3.6 and 4. The potential and the gravitational effect of the ETC can also be split (for evaluation points above, on or below the topo–surface) into the spherical shell and terrain term, cf. Fig. 3. Recall, that the surface of the inner quasi–ellipsoid is taken as the lower topo–boundary of the ETC, in order to account properly for the areas over the globe, where the topo–surface dips below the reference ellipsoid (ibid, Section 4).

For the potential of the ETC, V_0^{ET}, and the gravitational effect of the ETC, A_0^{ET}, the spherical shell term is the potential, and the gravitational effect, of the *spherical layer* (shell) of the radius of the inner sphere $(R - h^*)$ and of thickness $(h_t(\Omega_P) + h^*)$, cf. Fig. 3, which can be evaluated analytically.

For the *gravitational potential of the ETC* we can write

$$V_0^{ET}(h_P, \Omega_P) = V_0^{ET,B}(h_P, \Omega_P) + V_0^{ET,R}(h_P, \Omega_P)\,, \tag{34}$$

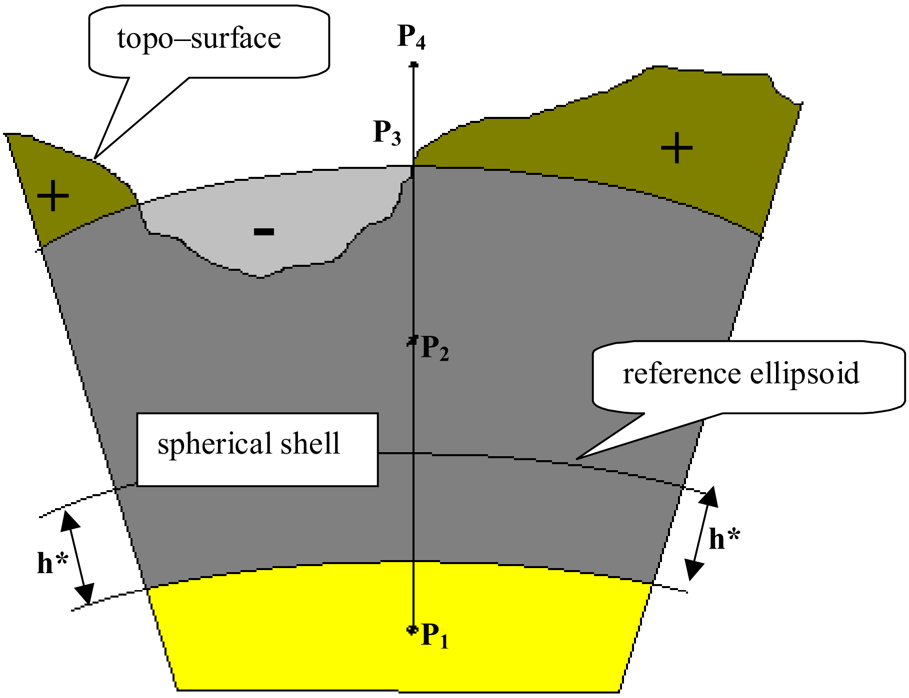

Fig. 3. Sketch illustrating the spherical shell with the upper boundary $h = h_t(\Omega_P)$ and lower boundary $h = -h^*$. The topo–surface forms the terrain relative to the spherical shell, indicated by "+" and "–" signs. Subscripts indicate four possible positions of the evaluation point $P \equiv (h_P, \Omega_P)$: 1 – below the spherical shell, 2 – inside the spherical shell, 3 – at the topo–surface, and 4 – above the topo–surface.

where $V_0^{ET,B}$ is the *gravitational potential of the spherical shell* of a constant density ρ_0 and thickness $(h_t(\Omega_P) + h^*)$, and

$$V_0^{ET,R}(h_P, \Omega_P) = G\rho_0 \int\limits_{h_t(\Omega_P)}^{h_t(\Omega)} \iint\limits_{\Omega_0} L^{-1}(h_P, \Omega_P, h, \Omega)(R+h)^2 \, \mathrm{d}h \, \mathrm{d}\Omega \tag{35}$$

is the *gravitational potential of the terrain* relative to the spherical shell. Rigorously it must be numerically evaluated over the entire globe. The terrain term of the ETC potential can be written also as a surface integral (cf. *Martinec, 1998*, Eq. (3.52); *Sjöberg, 2000*, Eqs 9–11), when the geodetic

heights replace in the formulae the orthometric heights.
For the *gravitational effect of the ETC* we can write

$$A_0^{ET}(h_P,\Omega_P) = A_0^{ET,B}(h_P,\Omega_P) + A_0^{ET,R}(h_P,\Omega_P)\,, \tag{36}$$

where $A_0^{ET,B}$ is the *gravitational effect of the spherical shell* of the constant density ρ_0 and thickness $(h_t(\Omega_P)+h^*)$. The gravitational effect of the terrain relative to the spherical shell is

$$A_0^{ET,R}(h_P,\Omega_P) = -G\rho_0 \int\limits_{h_t(\Omega_P)}^{h_t(\Omega)} \iint\limits_{\Omega_0} \frac{\partial L^{-1}(h_P,\Omega_P,h,\Omega)}{\partial h_P}(R+h)^2\,\mathrm{d}h\,\mathrm{d}\Omega \tag{37}$$

Rigorously it must also be numerically evaluated over the entire globe.

6.3.1. Evaluation point on the topo–surface

If the evaluation point is located on the topo–surface, we have $h_t(\Omega_P) = h_P$. The potential of the spherical shell becomes, following e.g. (*Blakely, 1995*, Sections 3.2.1 and 3.2.2), and also (*Jekeli and Serpas, 2003*, Eqs (9) and (15); *Vaníček et al., 2004,* Eq. (25))

$$h_P = h_t(\Omega_P) : V_0^{ET,B}(h_P,\Omega_P) = \frac{4}{3}\pi G\rho_0 \frac{(R+h_P)^3-(R-h^*)^3}{(R+h_P)}\,. \tag{38}$$

The gravitational effect of the spherical shell becomes, following e.g. (*Blakely, 1995*, Sections 3.2.1 and 3.2.2), and also (*Jekeli and Serpas, 2003*, Eqs (10) and (15); *Vaníček et al., 2004*, Eq. (40))

$$h_P = h_t(\Omega_P) : A_0^{ET,B}(h_P,\Omega_P) = -\frac{4}{3}\pi G\rho_0 \frac{(R+h_P)^3-(R-h^*)^3}{(R+h_P)^2} \tag{39}$$

The *NETC terrain correction to gravity* $\left(-A_0^{ET,R}\right)$, in the case of the evaluation point located on the topo–surface, is computed exactly (when replacing orthometric heights by geodetic heights) as the *spherical terrain correction to surface gravity* that was recently discussed by *Novák et al. (2001)*, cf. Sections 3 and 6, and Fig. 7.

A lot of material on the numerical aspects of the evaluation of topo–corrections for the evaluation point on the topo–surface, which applies also to the *NETC topo–corrections*, if replacing orthometric heights by geodetic heights, can be found in geodetic literature as part of the discussion of the *direct topographical effect on gravity.* The *direct topographical effect* consists of two terms – the effect of topographical masses ("*removal of topographical masses*") and the effect of a condensed layer ("*restoration of topographical masses*") according to a specific scheme, such as the first or second Helmert condensation method (e.g., *Heck, 2003; Martinec, 1998; Vaníček and Martinec, 1994; Jekeli and Serpas, 2003*). By considering just the terms reflecting the removal of the topographical masses, the same numerical procedures can be applied also to the computation of the NETC topo–corrections to the potential and gravity. Let us quote *Martinec and Vaníček (1994), Sjöberg (1994), (1996), (2000), Nahavandchi and Sjöberg (1998), Nahavandchi (2000), Novák et al. (2001), Jekeli and Serpas (2003).* For more details on the numerical aspects of evaluating the terrain effects (negative terrain corrections) the reader is referred also to Section 3.2.3 in (*Hackney and Featherstone, 2003*) and literature cited therein.

6.3.2. Evaluation point below the topo–surface

In the case of the evaluation point below the topo–surface, $h_P < h_t\left(\Omega_P\right)$, the terrain terms will remain as given by Eqs (35) and (37). Only the spherical shell terms will read differently, following e.g. *(Vaníček et al., 2004*, Eqs (25) and (40)); namely, the gravitational potential is

$$\forall(h_P,\Omega_P); -h^* \le h_P < h_t(\Omega_P) : V_0^{ET,B}\left(h_P,\Omega_P\right) = \\ = \frac{2}{3}\pi G\rho_0 \left[3\left(R+h_t(\Omega_P)\right)^2 - 2\frac{(R-h^*)^3}{R+h_P} - \left(R+h_P\right)^2\right], \tag{40}$$

$$\forall\left(h_P,\Omega_P\right); h_P < -h^* : \\ V_0^{ET,B}\left(h_P,\Omega_P\right) = 2\pi G\rho_0\Big[\left(R+h_t(\Omega_P)\right)^2 - \left(R-h^*\right)^2\Big] = \text{const}, \tag{41}$$

and the gravitational effect

$$\forall\left(h_P,\Omega_P\right); -h^* \le h_P < h_t\left(\Omega_P\right) : \\ A_0^{ET,B}\left(h_P,\Omega_P\right) = \frac{4}{3}\pi G\rho_0\left[\frac{\left(R-h^*\right)^3}{\left(R+h_P\right)^2} - \left(R+h_P\right)\right], \tag{42}$$

$$\forall\left(h_P,\Omega_P\right);h_P<-h^*:A_0^{ET,B}\left(h_P,\Omega_P\right)=0. \tag{43}$$

6.3.3. Evaluation point above the topo–surface

Again, the terrain terms will remain as given by Eqs (35) and (37). The spherical shell terms will now read, following e.g. (*Blakely, 1995*, Sections 3.2.1 and 3.2.2),

$$\forall\left(h_P,\Omega_P\right);h_P>h_t\left(\Omega_P\right):$$
$$V_0^{ET,B}\left(h_P,\Omega_P\right)=\frac{4}{3}\pi G\rho_0\frac{\left(R+h_t(\Omega_P)\right)^3-\left(R-h^*\right)^3}{\left(R+h_P\right)}, \tag{44}$$

$$\forall\left(h_P,\Omega_P\right)\;;\;\;h_P>h_t\left(\Omega_P\right):$$
$$A_0^{ET,B}\left(h_P,\Omega_P\right)=-\frac{4}{3}\pi G\rho_0\frac{\left(R+h_t(\Omega_P)\right)^3-\left(R-h^*\right)^3}{\left(R+h_P\right)^2}. \tag{45}$$

6.4. Integration domain – entire globe

The discussed Newton integrals, or at least the terrain terms in the case of topo–corrections, must be evaluated over the entire globe. This is a numerically demanding requirement. If the data coverage is not global, this requirement even cannot be met. Often, in spherical approximation, the integration is split into two zones: a *near zone* (inner zone) and a *far zone* (outer zone), or even more zones. The evaluation strategy and numerical procedure may then differ from zone to zone. Such a split–up into zones is possible if the integral is expressed in local polar coordinates of the evaluation point (e.g., *Vaníček and Krakiwsky, 1986*), where the horizontal position of the evaluation point (Ω_P) becomes the origin, and the horizontal position of the integration point (Ω) is given by means of (ψ,α), where ψ is the *angular distance* from the origin, and α is the *azimuth* reckoned from an arbitrary direction such as the north. In spherical approximation we then have

$$V\left(h_P,\Omega_P\right)\approx$$
$$\approx G\int\limits_{h_1(\psi,\alpha)}^{h_2(\psi,\alpha)}\int\limits_0^{\pi}\int\limits_0^{2\pi}\rho(h,\psi,\alpha)L^{-1}(h_P,h,\psi)\left(R+h\right)^2\sin\psi\;\mathrm{d}h\mathrm{d}\psi\,\mathrm{d}\alpha, \tag{46}$$

$$A\left(h_P,\Omega_P\right)\approx$$
$$\approx -G\int\limits_{h_1(\psi,\alpha)}^{h_2(\psi,\alpha)}\int\limits_0^{\pi}\int\limits_0^{2\pi}\rho(h,\psi,\alpha)\frac{\partial L^{-1}(h_P,h,\psi)}{\partial h_P}\left(R+h\right)^2\sin\psi\;\mathrm{d}h\mathrm{d}\psi\,\mathrm{d}\alpha,\qquad(47)$$

since $\sin\psi\;\mathrm{d}\psi\,\mathrm{d}\alpha\;=\;\cos\phi\;\mathrm{d}\phi\mathrm{d}\lambda$, and where $L\left(h_P,h,\psi\right)$ is given by Eq. (22), $\cos\psi$ by Eq. (19), and $\partial L^{-1}\left(h_P,h,\psi\right)/\partial h_P$ by Eq. (26). The *spherical cap* of a pre–selected radius ψ_0, in terms of the angular distance from the evaluation point, constitutes the near zone $\langle 0,\psi_0\rangle$, while the integration domain beyond this radius represents the far zone $(\psi_0,\,\pi\rangle$. The upper and lower integral boundaries remain the same, only are positioned in the new coordinates $h_i\left(\psi,\alpha\right)\equiv h_i\left(\phi,\lambda\right)$, $i=1,\;2$. In the case of splitting the topo–corrections into the spherical shell and terrain terms, the split–up into the zones applies either to both terms or just to the terrain term.

The integration must be carried out over the whole globe, that is in angular distance from 0 to π. Often the contribution from the far zone is neglected, resulting in a *truncation error.* The radius of the near zone may vary by a scholar and by application. Instead of neglecting the far- zone contribution, global digital elevation models (DEMs) in a spherical harmonic representation may be used in this zone (*Sjöberg, 1994; 1996; Nahavandchi and Sjöberg, 1998; Novák et al., 2001; Sun, 2002*), or a combined approach may be adopted, as suggested by *Nahavandchi (2000).* Approximations additional to spherical approximation may be adopted, when developing the reciprocal Euclidean distance into a series expansion, as classified recently by Jekeli and Serpas (2003) – planar approximation, flat–Earth approximation, and linear approximation.

7. Discussion and concluding remarks

Newton integrals play an important role in geodesy and geophysics, as they represent the solution to the direct (forward) gravimetric problem. They provide the means for evaluating a gravitational potential, its gradient (vertical and horizontal components of the attraction vector), as well as higher order derivatives of the potential (actual or disturbing) of various bodies or density distributions of real, model (such as constant), or anomalous density. Among most common examples we can find the potential or gravitational effect of a body that may be the subject of the gravimetric inverse problem, or the potential or gravitational effect of the topographical masses. Here we have discussed exclusively the Newton integral for the potential, as defined in Section 2.1, and the Newton integral for the gravitational effect, i.e., for the vertical component of the attraction vector, as defined in Section 2.2. The points of interest (evaluation points) and the integration (running) points are nowadays commonly positioned in the geodetic coordinates. That is why we have dealt with evaluating the Newton integrals in the geodetic coordinates, as given by Sections 3.1 and 3.2. The evaluation of the Newton integrals in the geodetic coordinates calls for evaluating the solid (volume) element, thus the Jacobian, in the geodetic coordinates, as well as evaluating the 3D reciprocal Euclidean distance and its vertical derivative in the geodetic coordinates.

In most applications, spherical approximation to the exact evaluation of the Newton integrals in the geodetic coordinates, as given in Sections 4.1, 4.2, and 5 will be acceptable, since the ellipsoidal correction to spherical approximation of the Newton integral for the potential and gravitational effect is by three orders of magnitude smaller than the spherical term *(Novák and Grafarend, 2004)*.

Several aspects of numerical evaluation of the discussed Newton integrals were addressed in Section 6. The Newton integrals may become singular, but this singularity at the coincidence of the integration point with the evaluation point is weak and removable. We focused on the topographical corrections in spherical approximation that adopt the constant (model) density of the topo–masses, and discussed the splitting of the topo–correction into the spherical shell and terrain (roughness) term, cf. Sections 6.2 and 6.3. We paid a particular attention to the NETC topo–correction to the potential and gravity; in other words, to the potential and gravitational effect of the ellipsoidal topography of a constant density. Globally the "ETC" is defined as topographical masses of constant density bounded by the surface of the inner quasi–ellipsoid as the lower topo–boundary (which plays the role of the reference ellipsoid and originates from the occurrence of negative ellipsoidal topography in some areas over the globe) and the topo–surface as the upper topo–boundary. The reasons for defining and using the ETC topo–masses and their potential and gravitational effect are discussed in more detail by *Vajda et al. (2004)*. Here we looked in detail at evaluating the ETC spherical shell term for evaluation points above, on, and below the topo–surface, cf. Sections 6.3 through 6.3.3. The advantage of splitting the ETC potential and ETC gravitational effect (or NETC topo–corrections to the potential and gravity) into the spherical shell and terrain term dwells in the analytical evaluation of the (N)ETC spherical shell term and in the fact, that the *NETC terrain correction* (negative ETC terrain term) is equal to the *spherical terrain correction* (to the potential or gravity) as generally known and used in geodesy and geophysics. Finally, we looked at the requirement of integrating in the horizontal coordinates over the whole globe.

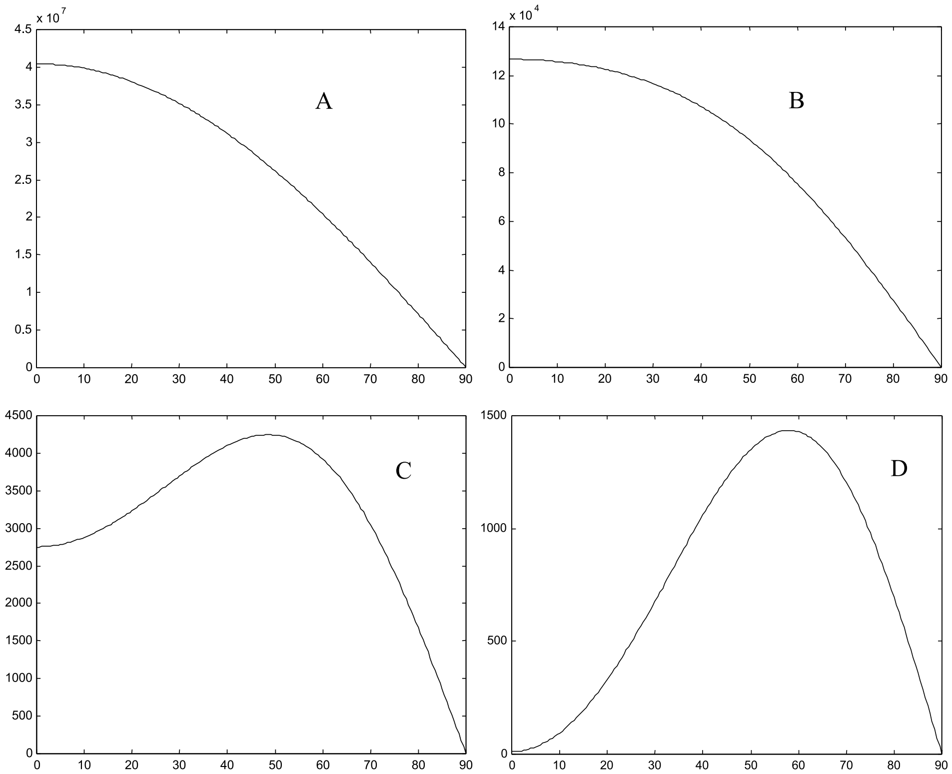

Fig. 4. The behaviour of the Jacobian. The horizontal axis is geodetic latitude [arcdeg.]. Reference ellipsoid is given by $a = 6378$ km, $e^2 = 0.0067$. (A) $h = 0$ km, (B) $h =$ -6000 km, (C) $h =$ -6300 km, (D) $h =$ -6335 km.

In Section 6.4 we mentioned the truncation of the integration domain to a spherical cap, and different approaches and numerical procedures that may be adopted in near- and far-zone contribution evaluation, providing references to published work on this topic.

Appendix

To show the behaviour of the Jacobian given by Eq. (14) in the interior of the reference ellipsoid, we portray it, cf. Fig. 4, as a function of the geodetic latitude (in the first quadrant) for several values of the (negative) geodetic height: (A) h = 0 km, (B) h = -6000 km, (C) h = -6300 km, (D) h = -6335 km. Notice, that to stay in the first quadrant, in terms of the geodetic height and geodetic latitude, the geodetic height must comply with the following inequality, cf. Eq. (17) and Fig. 2:

$$\forall\phi : 0 < \phi \leq \frac{\pi}{2} : \; h \geq h_C(\phi). \tag{48}$$

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