MEAN GRAVITY ALONG PLUMBLINE

Beth-Anne Martin [1], Chris MacPhee [2], Robert Tenzer [1], Petr Vaníček [1] and Marcelo Santos [1]

[1] University of New Brunswick, Department of Geodesy and Geomatics Engineering,
Fredericton, N.B., E3B 5A3, Canada

[2] Advanced Computational Research Laboratory, Faculty of Computer Science,
University of New Brunswick, Fredericton, N.B., E3B 5A3, Canada

1. Introduction

The mean value of gravity along the plumbline between the geoid and the earth surface depends on the mass density distribution within the Earth and the shape of the Earth. Since the actual values of gravity along the plumbline cannot be measured, the mean value of gravity along the plumbline has to be computed from the gravity observed at the surface of the Earth. This can be done by reducing the observed gravity according to some accepted physical model.

2. Mean gravity along plumbline

According to the theorem of mean integral values, the "*mean value* $\bar{g}(\Omega)$ *of gravity along the plumbline between the geoid and the earth surface*" reads (Heiskanen and Moritz, 1967, Eq. 4-20)

$$\bar{g}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} g(r,\Omega)\, \mathrm{d}r \,, \tag{1}$$

where $g(r,\Omega)$ is the gravity at a point of which geocentric position is $(r,\Omega) \equiv (r,\phi,\lambda)$, and the geocentric radius $r_t(\Omega)$ of the earth surface is given (with an accuracy of at most a few millimeters) by the geocentric radius of the geoid $r_g(\Omega)$ plus the orthometric height $H^{\mathrm{O}}(\Omega)$.

The gravity $g(r,\Omega)$ can be expressed as (Tenzer and Vaníček, 2003)

$$g(r,\Omega) \cong \gamma(r,\phi) + \delta g^{\mathrm{NT}}(r,\Omega) - \left.\frac{\partial V^t(r,\Omega)}{\partial r}\right|_r \,, \tag{2}$$

where $\gamma(r,\phi)$ is the normal gravity of the geocentric reference ellipsoid, $\delta g^{\mathrm{NT}}(r,\Omega)$ is the gravity disturbance in the No Topography gravity space (i.e., the gravity disturbance generated by the mass within the geoid itself, Vaníček et al., 2003), and $-\partial V^t(r,\Omega)/\partial r$ is the gravitational attraction of topographical masses.

Substituting Eq. (2) back into Eq. (1), the mean gravity $\bar{g}(\Omega)$ takes the following form

$$\bar{g}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \gamma(r,\phi) + \delta g^{\mathrm{NT}}(r,\Omega) - \left.\frac{\partial V^t(r,\Omega)}{\partial r}\right|_r \mathrm{d}r \,. \tag{3}$$

3. Mean normal gravity along plumbline

The first term on the right-hand-side of Eq. (3) defines the "*mean value* $\bar{\gamma}(\Omega)$ *of normal gravity along the plumbline between the geoid and the earth surface*"

$$\bar{\gamma}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \gamma(r,\phi)\, \mathrm{d}r \,. \tag{4}$$

Neglecting the deflection of plumbline and the correction caused by different lengths of the plumbline (between the geoid and the earth surface) and the ellipsoidal normal (between the geocentric reference ellipsoid and the telluroid), the mean normal gravity along the plumbline can be rewritten as (Tenzer and Vaníček, 2003)

$$\bar{\gamma}(\Omega) \cong \frac{1}{H^{\mathrm{N}}(\Omega)} \int_{r=r_o(\phi)}^{r_o(\phi)+H^{\mathrm{N}}(\Omega)} \gamma(r,\phi)\,\mathrm{d}r + \left.\frac{\partial\,\gamma(r,\phi)}{\partial\,\mathrm{n}}\right|_{r=r_o(\phi)} N(\Omega), \tag{5}$$

where $r_o(\phi)$ is the geocentric radius of the geocentric reference ellipsoid, $H^{\mathrm{N}}(\Omega)$ is the normal height (Molodensky, 1945), and $N(\Omega)$ is the geoidal height.

The mean value of normal gravity along the normal between the reference (geocentric) ellipsoid and the telluroid is evaluated by the following formula (Heiskanen and Moritz, 1967, Eq. 4-42)

$$\bar{\gamma}(\Omega) = \frac{1}{H^{\mathrm{N}}(\Omega)} \int_{r=r_o(\phi)}^{r_o(\phi)+H^{\mathrm{N}}(\Omega)} \gamma(r,\phi)\,\mathrm{d}r \cong \gamma_o(\phi)\left[1 - \left(1 + \mathrm{f} + \frac{\omega^2 \mathrm{a}^2 \mathrm{b}}{\mathrm{GM}} - 2\mathrm{f}\sin^2\varphi\right)\frac{H^{\mathrm{N}}(\Omega)}{\mathrm{a}} + \left(\frac{H^{\mathrm{N}}(\Omega)}{\mathrm{a}}\right)^2\right], \tag{6}$$

where a, b are the semi-axes and $\mathrm{f} = (\mathrm{a} - \mathrm{b})/\mathrm{a}$ is the first numerical flattening of the geocentric reference ellipsoid, ω is the mean angular velocity of Earth's rotation, φ is the geodetic latitude, GM is the geocentric gravitational constant, and $\gamma_o(\phi)$ is the normal gravity at the surface of the geocentric reference ellipsoid.

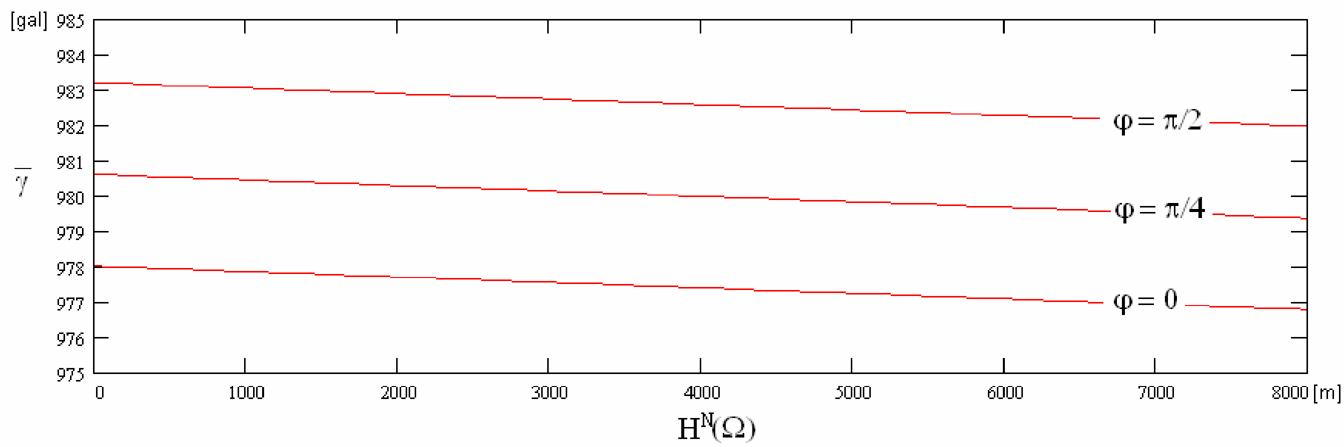

Fig. 1: Relation between the mean value $\bar{\gamma}(\Omega)$ of normal gravity (reference ellipsoid GRS-80) and the normal height $H^{\mathrm{N}}(\Omega)$.

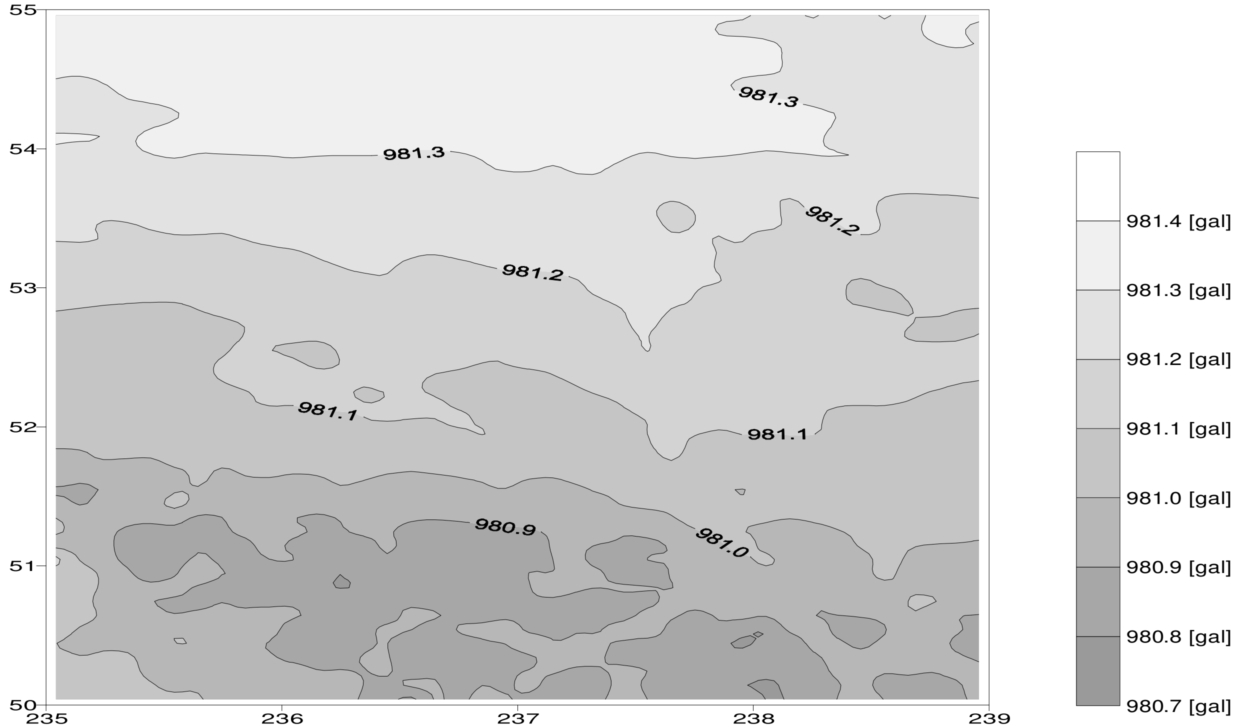

Fig. 2: Mean values of normal gravity.

The "*correction to normal gravity* $\varepsilon_{\bar{\gamma}}^{N}(\Omega)$ *due to the geoid undulation*", i.e., correction for a shift of the integration interval from the reference ellipsoid to the geoid, is given by (Tenzer and Vaníček, 2003)

$$\varepsilon_{\bar{\gamma}}^{N}(\Omega) \approx \gamma(r_g(\Omega)) - \gamma_o(\phi) = \left.\frac{\partial\, \gamma(r,\phi)}{\partial\ \mathrm{n}}\right|_{r=r_o(\phi)} N(\Omega) \approx -\frac{2\gamma_o(\phi)}{\mathrm{a}} N(\Omega). \tag{7}$$

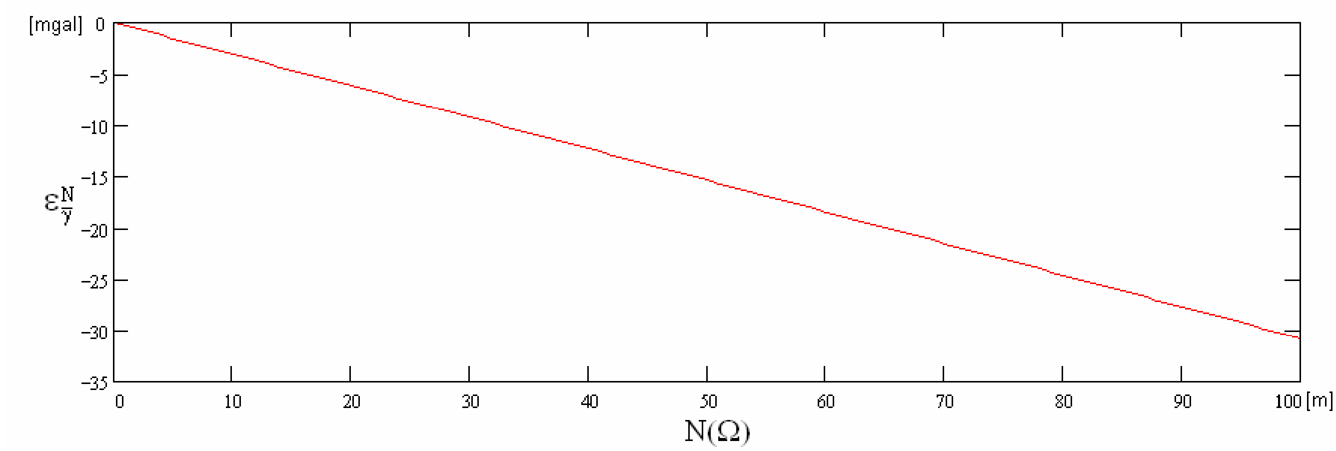

Fig. 3: Relation between the correction $\varepsilon_{\bar{\gamma}}^{N}(\Omega)$ to normal gravity due to the geoid undulation and the geoidal height $N(\Omega)$.

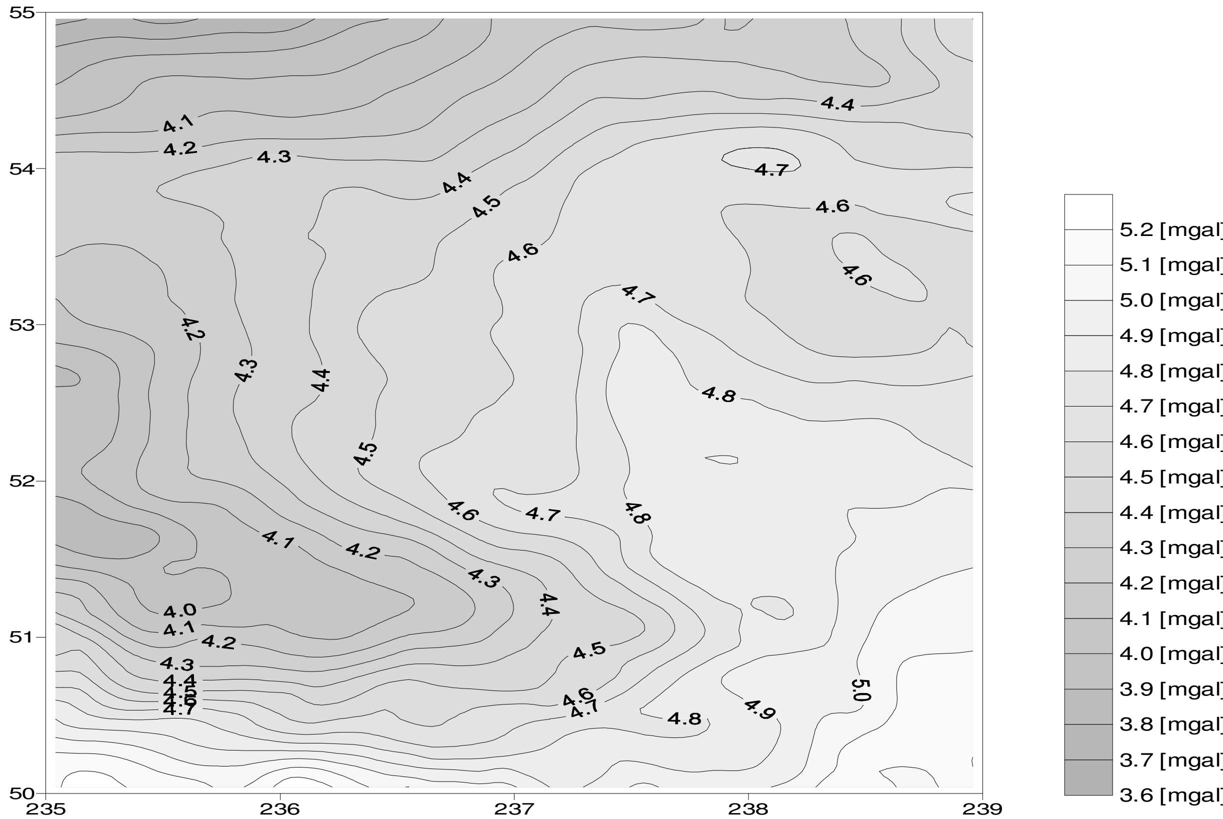

Fig. 4: Correction to normal gravity due to the geoid undulation.

4. Mean geoid-generated gravity disturbance

The "*mean value* $\overline{\delta g}^{\,\mathrm{NT}}(\Omega)$ *of geoid-generated gravity disturbance along the plumbline between the geoid and the earth surface*" is given by the second term on the right-hand-side of Eq. (3), i.e.,

$$\overline{\delta g}^{\,\mathrm{NT}}(\Omega) \cong \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)} \delta g^{\mathrm{NT}}(r,\Omega)\,\mathrm{d}r\,. \tag{8}$$

Since the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ multiplied by a geocentric radial distance r is harmonic above the geoid in the No Topography gravity space, i.e., satisfies the Laplace equation, the gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ is evaluated by solving Dirichlet's boundary value problem (Kellogg, 1929)

$$\delta g^{\mathrm{NT}}(r,\Omega) = \frac{1}{4\pi}\frac{\mathrm{R}}{r} \iint_{\Omega'\in\Omega_{\mathrm{O}}} \mathrm{K}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right] \delta g^{\mathrm{NT}}(\mathrm{R},\Omega')\,\mathrm{d}\Omega'\,, \tag{9}$$

where the gravity disturbance $\delta g^{\mathrm{NT}}(\mathrm{R},\Omega) \equiv \delta g^{\mathrm{NT}}(r_g(\Omega))$ is referred on the co-geoid, R is the mean radius of the Earth which approximates the geocentric radius of the geoid (or co-geoid) surface, and Poisson's integral kernel $\mathrm{K}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]$ is given by

$$\mathrm{K}\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right] = \mathrm{R}\,\frac{r^2-\mathrm{R}^2}{l^3\left[r,\psi(\Omega,\Omega'),\mathrm{R}\right]}\,. \tag{10}$$

Substituting Eqns. (10) and (9) back into Eq. (8) and performing the radial integration with respect to r, Eq. (8) takes the following form (Tenzer and Vaníček, 2003)

$$\overline{\delta g}^{\,\mathrm{NT}}(\Omega) = \frac{1}{4\pi}\frac{\mathrm{R}^2}{H^{\mathrm{O}}(\Omega)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Bigg| - \frac{2}{l[r, \psi(\Omega,\Omega'), \mathrm{R}]} +$$

$$+ \frac{1}{\mathrm{R}} \operatorname{argsinh}\left(\frac{\mathrm{R}}{r \sin\psi(\Omega,\Omega')} + \frac{1}{\tan\psi(\Omega,\Omega')} \right) \Bigg|_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)} \delta g^{\mathrm{NT}}(\mathrm{R},\Omega')\, \mathrm{d}\Omega' . \tag{11}$$

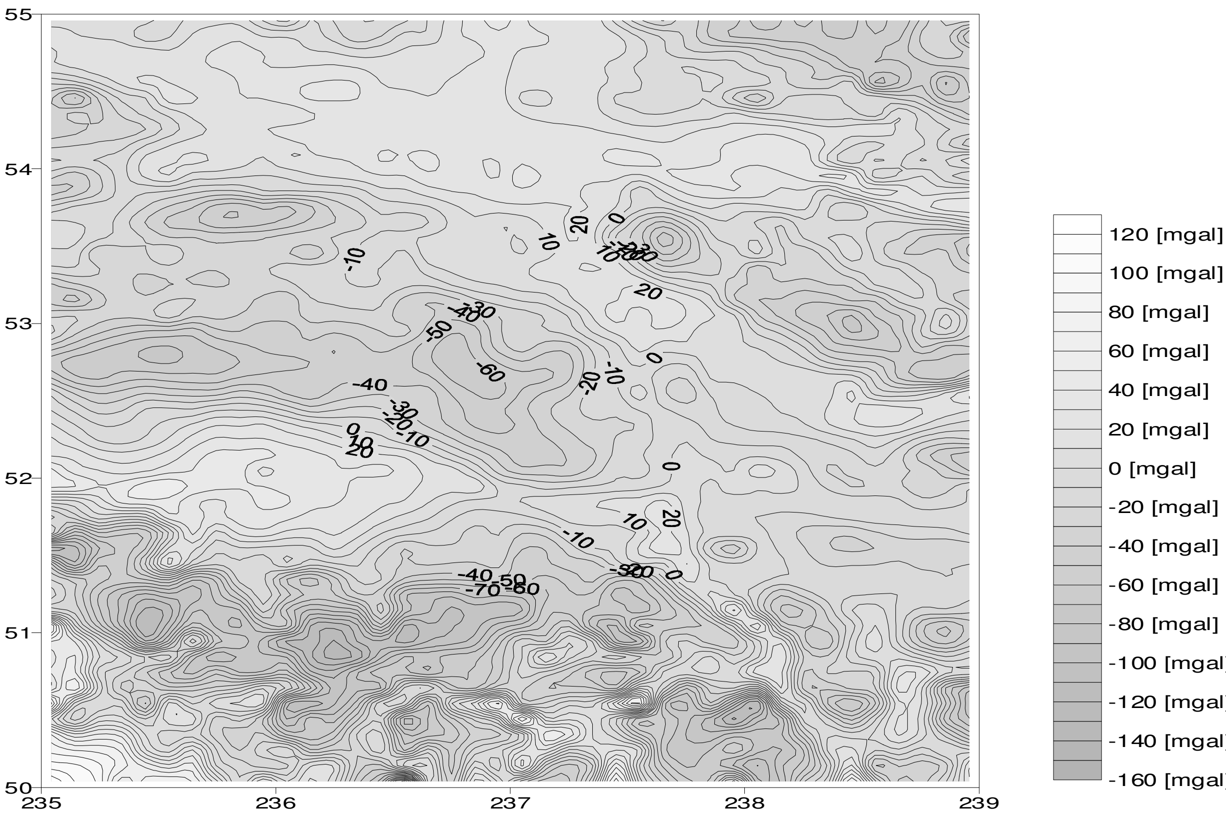

Fig. 5: Mean geoid-generated gravity disturbances $\overline{\delta g}^{\,\mathrm{NT}}(\Omega)$.

5. Mean topography-generated gravitational attraction

The "*mean value of topography-generated gravitational attraction along the plumbline between the geoid and the earth surface*" (given by the third term on the right-hand-side of Eq. 3) can be derived as (Tenzer and Vaníček, 2003)

$$-\frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)} \frac{\partial\, V^t(r,\Omega)}{\partial\, r}\bigg|_r \,\mathrm{d}r = -\frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)} \frac{\partial\, V^t(r,\Omega)}{\partial\, r}\bigg|_r \,\mathrm{d}r$$

$$= \frac{V^t(\mathrm{R},\Omega) - V^t(r_t(\Omega))}{H^{\mathrm{O}}(\Omega)} , \tag{12}$$

where $V^t(\mathrm{R},\Omega)$ and $V^t(r_t(\Omega))$ are the gravitational potentials of topographical masses as a referred on the geoid and at the earth surface.

According to Martinec (1998), the gravitational potential $V^t(r,\Omega)$ of the topographical masses for a point inside the topographical mass $\mathrm{R} \le r(\Omega) \le \mathrm{R} + H^{\mathrm{O}}(\Omega)$ reads

$$V^{t}(r,\Omega) = 2\pi\,\mathrm{G}\,\rho_{o}\left[\mathrm{R}^{2} + 2\mathrm{R}H^{\mathrm{O}}(\Omega) + \left[H^{\mathrm{O}}(\Omega)\right]^{2} - \frac{2}{3}\frac{\mathrm{R}^{3}}{r} - \frac{1}{3}r^{2}\right] +$$
$$+\,\mathrm{G}\,\rho_{o}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right] r'^{2}\,\mathrm{d}r'\,\mathrm{d}\Omega' +$$
$$+\,\mathrm{G}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \delta\rho(r',\Omega')\, l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right] r'^{2}\,\mathrm{d}r'\,\mathrm{d}\Omega', \tag{13}$$

where G is Newton's gravitational constant, $l[r,\psi(\Omega,\Omega'),r']$ is the spatial distance between (r,Ω) and (r',Ω'), and $\psi(\Omega,\Omega')$ is the spherical distance between the geocentric direction to Ω and Ω'.

The first term on the right-hand-side of Eq. (13) is the gravitational potential of the spherical Bouguer shell (of the mean topographical density ρ_o and thickness of $H^{\mathrm{O}}(\Omega)$) inside the topographical masses (Wichiencharoen, 1982). The second term stands for the gravitational potential of the terrain roughness term of density ρ_o (Martinec, 1998), and the third term represents the effect of anomalous topographical density $\delta\rho(r,\Omega)$ distribution on the gravitational potential.

Substituting Eq. (13) back into Eq. (12), the mean value of topography-generated gravitational attraction along the radial direction (which approximates the plumbline) becomes (Tenzer and Vaníček, 2003)

$$-\frac{1}{H^{\mathrm{O}}(\Omega)}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)} \left.\frac{\partial V^{t}(r,\Omega)}{\partial r}\right|_{r} \mathrm{d}r = \frac{2}{3}\pi\,\mathrm{G}\,\rho_{o}\,H^{\mathrm{O}}(\Omega)\left[\frac{3\mathrm{R} + H^{\mathrm{O}}(\Omega)}{r_{t}(\Omega)}\right] +$$
$$+\,\frac{\mathrm{G}\,\rho_{o}}{H^{\mathrm{O}}(\Omega)}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}+H^{\mathrm{O}}(\Omega)}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')}\left(l^{-1}\left[\mathrm{R},\psi(\Omega,\Omega'),r'\right] - l^{-1}\left[r_{t}(\Omega),\psi(\Omega,\Omega'),r'\right]\right) r'^{2}\,\mathrm{d}r'\,\mathrm{d}\Omega' +$$
$$+\,\frac{\mathrm{G}}{H^{\mathrm{O}}(\Omega)}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')}\delta\rho(r',\Omega')\left(l^{-1}\left[\mathrm{R},\psi(\Omega,\Omega'),r'\right] - l^{-1}\left[r_{t}(\Omega),\psi(\Omega,\Omega'),r'\right]\right) r'^{2}\,\mathrm{d}r'\,\mathrm{d}\Omega'. \tag{14}$$

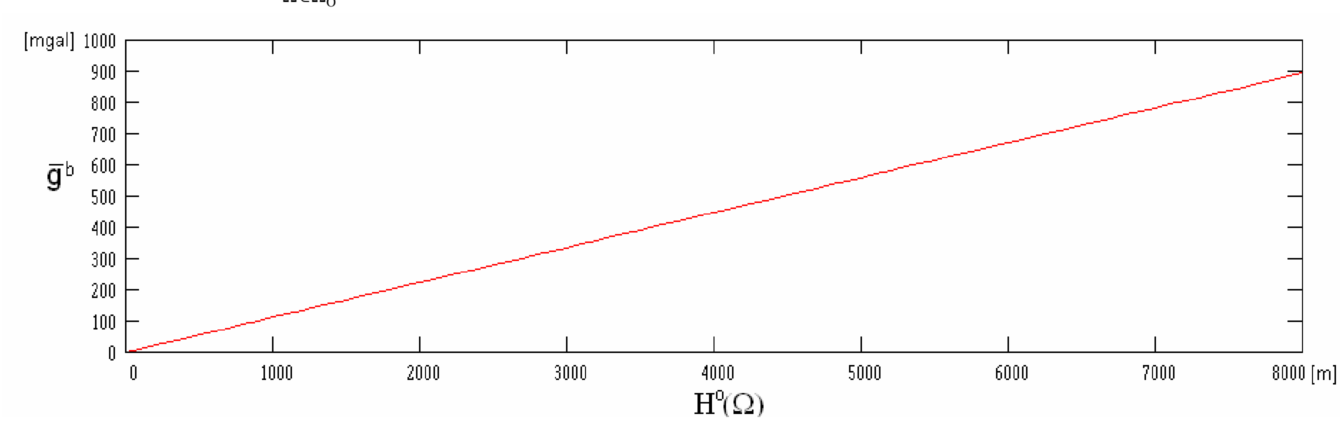

Fig. 6: Relation between the mean gravity $\bar{g}^{\mathrm{b}}(\Omega)$ generated by the spherical Bouguer shell and the orthometric height $H^{\mathrm{O}}(\Omega)$.

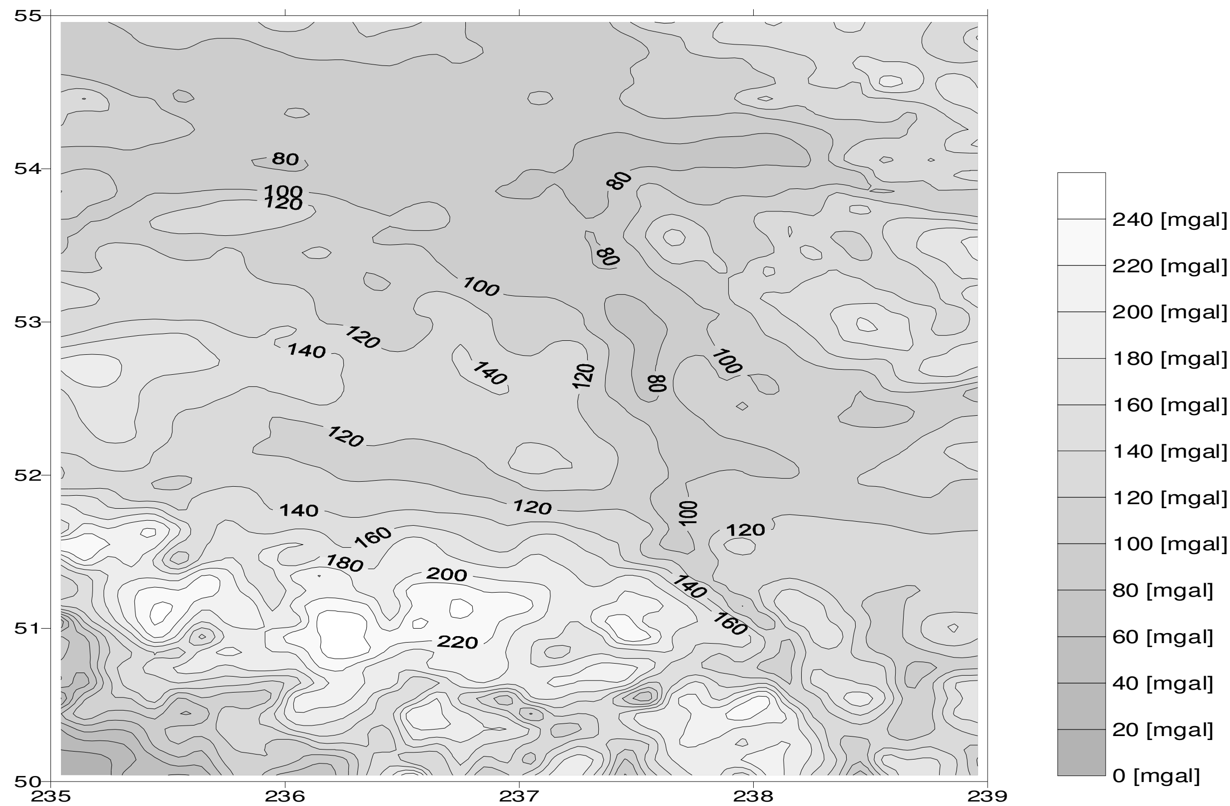

Fig. 7: Mean values of the gravitational attraction caused by the spherical Bouguer shell of density ρ_o.

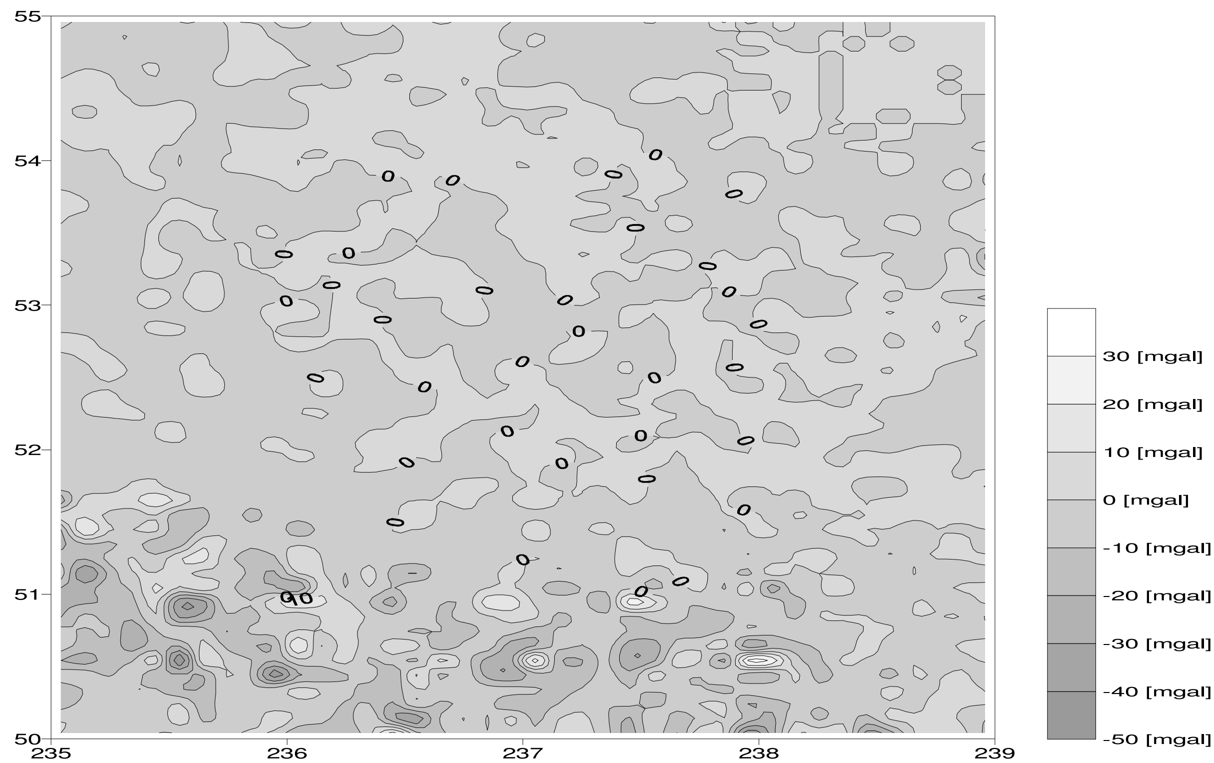

Fig. 8: Mean values of the gravitational attraction caused by the terrain roughness term.

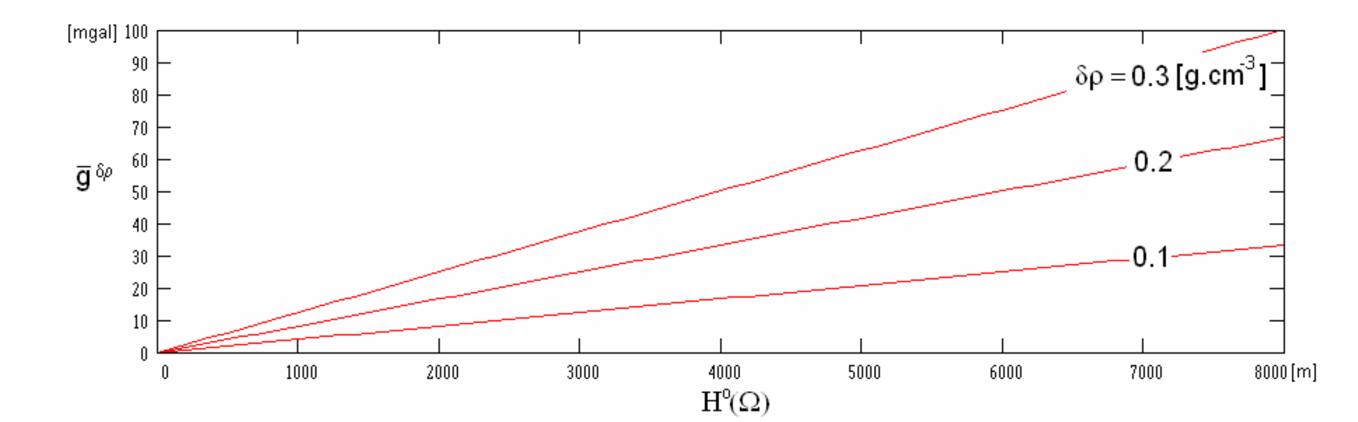

Fig. 9: Relation between the mean gravitational attraction $\bar{g}^{\delta\rho}(\Omega)$ of the anomalous (lateral) topographical density distribution and the orthometric height $H^O(\Omega)$.

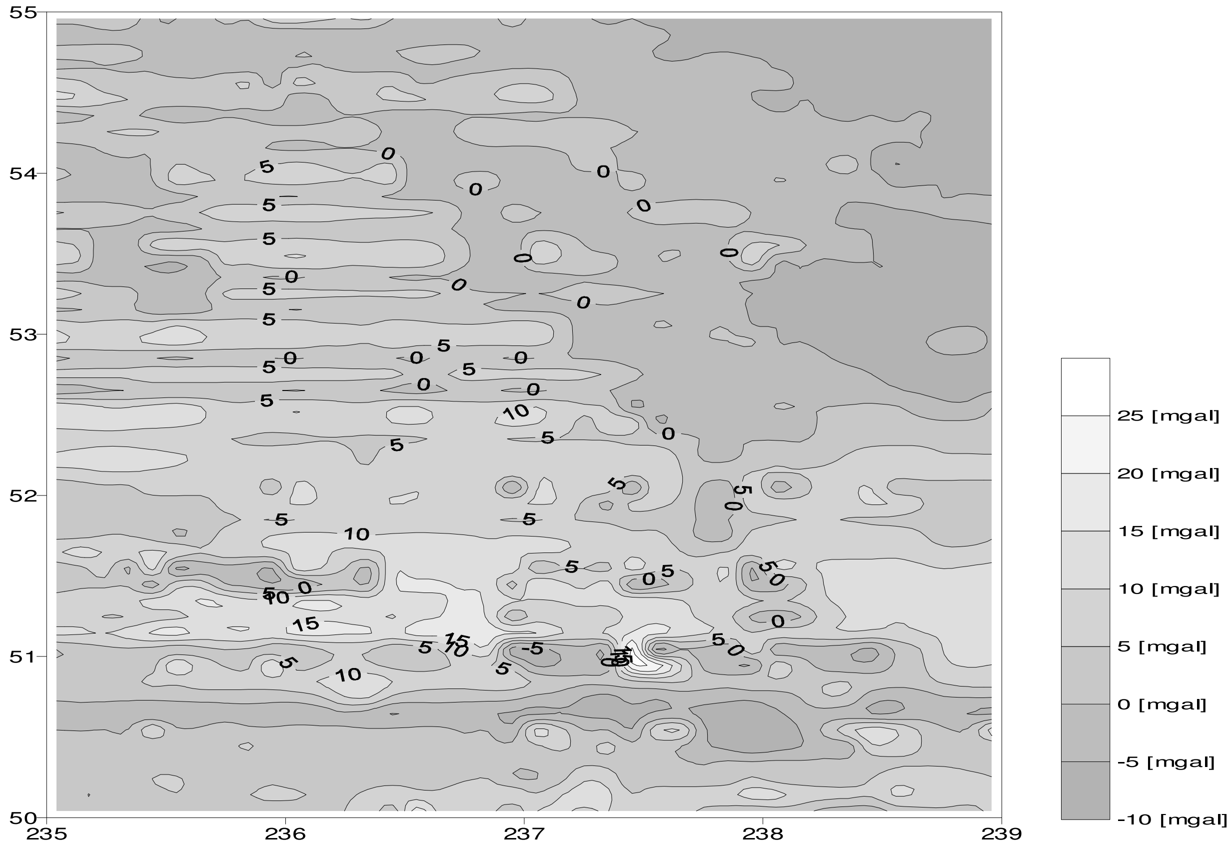

Fig. 10: Mean values of the gravitational attraction caused by the anomalous (laterally varying) topographical density distribution.

6. Conclusions

The mean value of gravity (generated by the solid Earth without the atmosphere) along the plumbline between the geoid and the earth surface (Fig. 11) is described as a sum of the mean normal gravity between the ellipsoid and the telluroid (Eq. 6), the correction to normal gravity for geoid undulation (Eq. 7), the mean geoid-generated gravity disturbance (Eq. 11), and the mean topography-generated gravitational attraction (Eq. 14). The mean topography-generated gravitational attraction consists of the mean gravitational attractions caused by the spherical Bouguer shell of density ρ_o (first term in Eq. 14), the terrain roughness term (second term in Eq. 14), and anomalous topographical density distribution (third term in Eq. 14).

Minimum, maximum, and average values of orthometric heights (Fig. 12), geoidal heights (Fig. 13) and the lateral variation of the topographical density (Fig. 14) at the testing area $\varphi \in \langle 50^{\circ}, 55^{\circ} \rangle$, $\lambda \in \langle 235^{\circ}, 239^{\circ} \rangle$ in part of the Canadian Rocky Mountains are in Tab. 1. The results of the numerical investigation of all components to the mean gravity are summarized in Tab. 2.

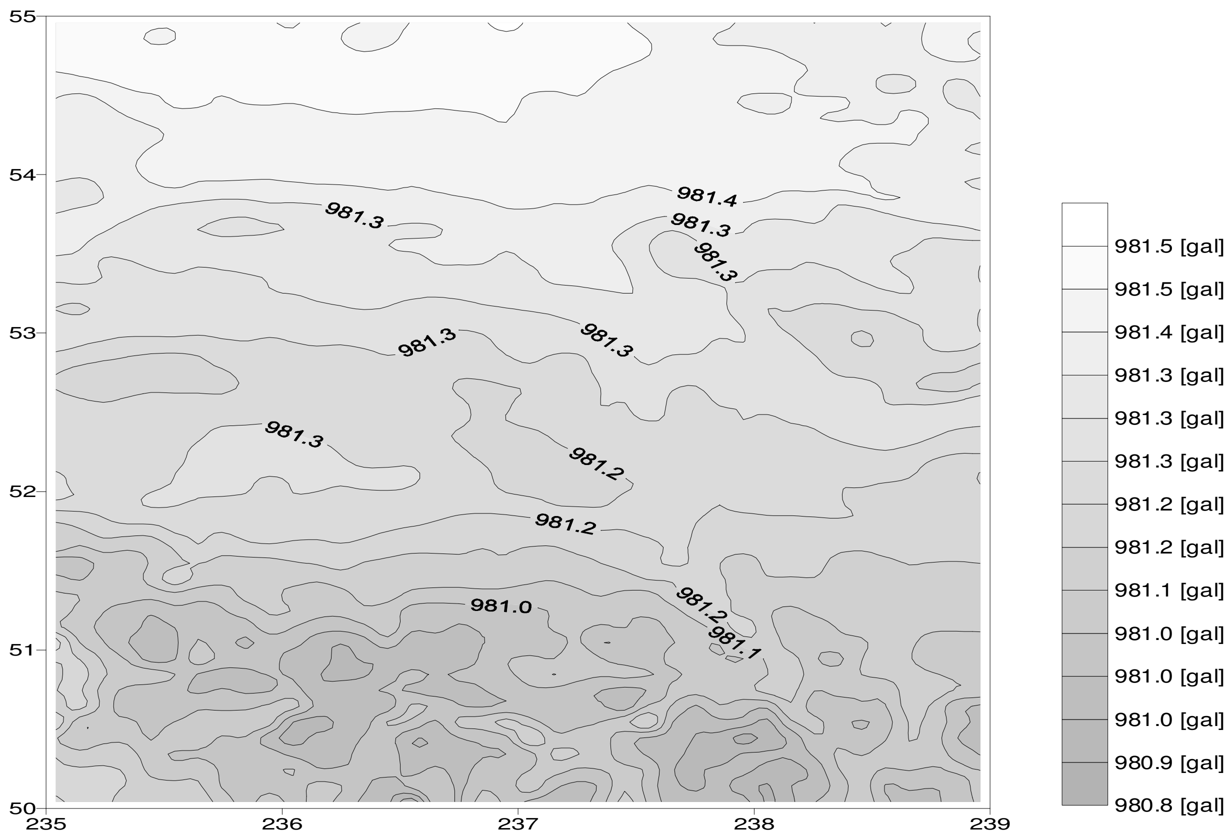

Fig. 11: Mean values of gravity along the plumbline between the geoid and the earth surface at a part of the Canadian Rocky Mountains.

Tab. 1:

	Min.	Max.	Average
Orthometric Heights [m] (Fig. 12)	4	2736	1166
Geoidal Heights [m] (Fig. 13)	-17.17	-11.90	-14.63
Anomalous Lateral Density [$g.cm^{-3}$] (Fig. 14)	-0.18	0.31	0.04

Tab. 2:

	Min. [mgal]	Max. [mgal]	Average [mgal]
Mean Normal Gravity (Fig. 2)	980751.5	981392.1	981111.0
Correction to Normal Gravity for Geoid Undulation (Fig. 4)	3.7	5.3	4.6
Mean Geoid-Generated Gravity Disturbance (Fig. 5)	-169.2	128.2	-14.6
Mean Gravitation Attraction of Spherical Bouguer Shell (Fig. 7)	0.0	286.1	130.4
Mean Gravitational Attraction of Terrain Roughness Term (Fig. 8)	-64.3	35.0	-2.1
Mean Gravitational Attraction of Anomalous Topographical Density (Fig. 10)	-10.1	29.0	2.3
Mean Gravity (Fig. 11)	**980858.5**	**981517.1**	**981232.2**

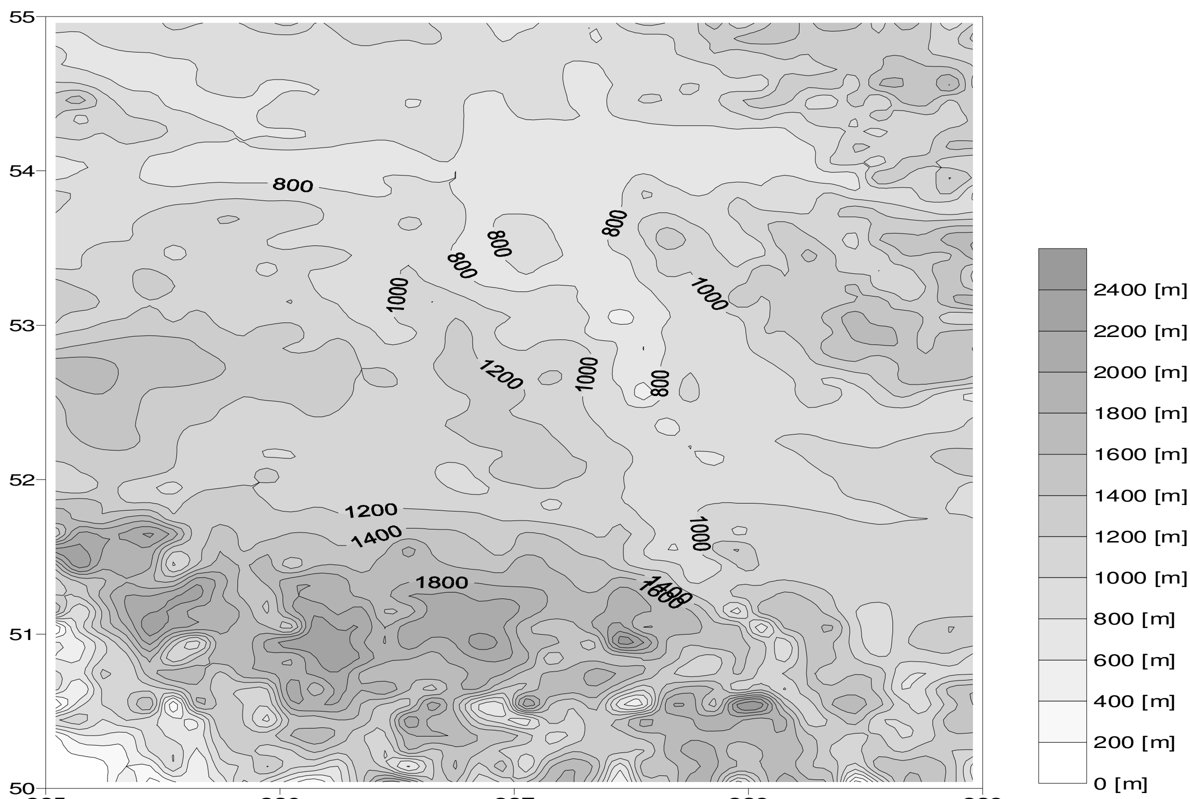

Fig. 12: Terrain at a part of the Canadian Rocky Mountains.

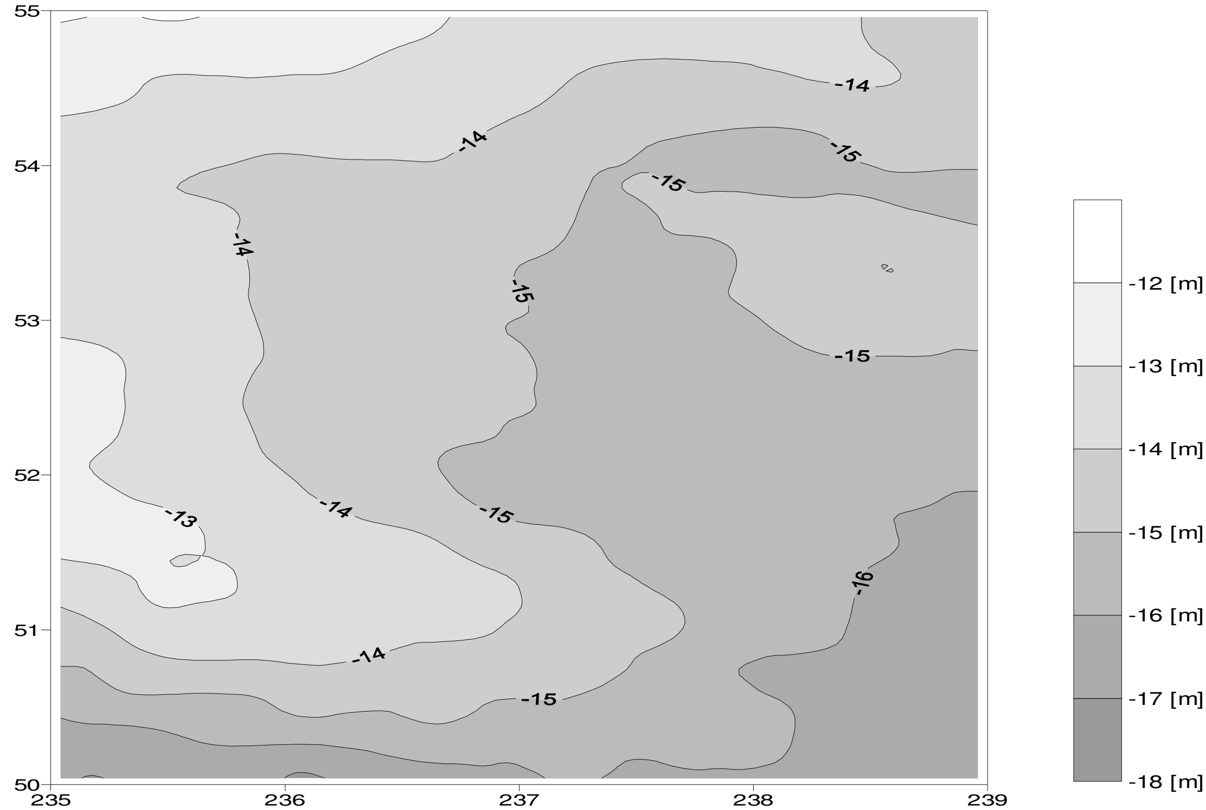

Fig. 13: Geoid at a part of the Canadian Rocky Mountains.

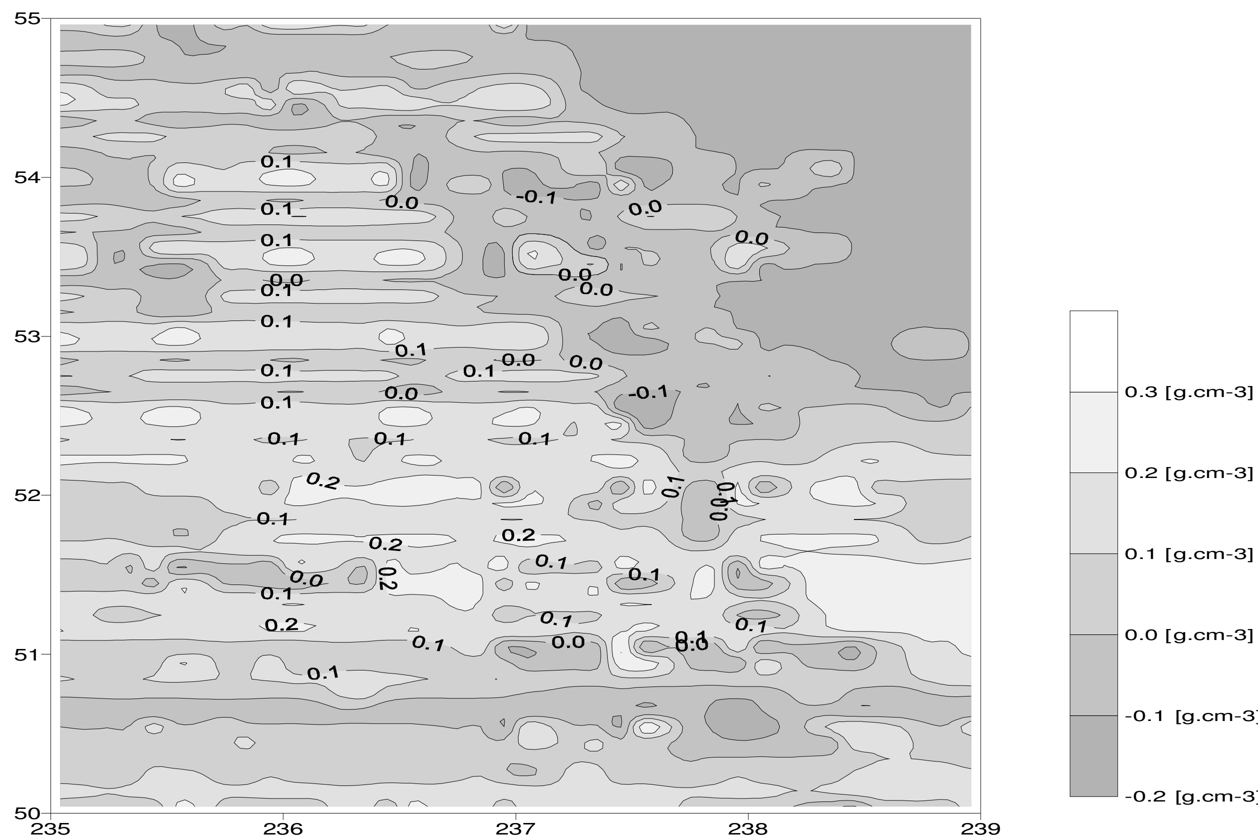

Fig: 14: Lateral variation of topographical density at a part of the Canadian Rocky Mountains.

References:

Heiskanen W. A., Moritz H., 1967: Physical geodesy. W.H. Freeman and Co., San Francisco.

Kellogg O.D., 1929: Foundations of potential theory. Springer. Berlin.

Martinec Z., 1998: Boundary value problems for gravimetric determination of a precise. Lecture notes in earth sciences, Vol. 73, Springer.

Molodensky M.S., 1945: Fundamental problems of Geodetic Gravimetry (in Russian). TRUDY Ts NIIGAIK 42, Geodezizdat, Moscow.

Tenzer R., Vaníček P., 2003: Rigorous definition of mean gravity in theory of height. Journal of Geodesy, Springer. (in preparation)

Vaníček P., Tenzer R., Sjöberg L.E., Martinec Z., Featherstone W.E., 2003: New views of the spherical Bouguer gravity anomaly. Geophysical Journal International. (submitted)

Wichiencharoen C., 1982: The indirect effects on the computation of geoid undulations. Dept. of Geod. Sci. Report No.336, Ohio State University.