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Formulation of the boundary-value problem for geoid determination with a higher-degree reference field

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SUMMARY

In this paper we formulate the boundary-value problem for the determination of the gravimetric geoid considering a satellite gravitational model as a reference. We show that the long-wavelength part of the gravitational field generated by topographical masses must be added to the satellite model in order to prescribe a reference gravitational potential for a partly internal and partly external problem for geoid determination. We choose a reference potential that does not depend on the way topographical masses are compensated or condensed, but only on the satellite reference model and on the difference of gravitational potentials induced by topographical masses in the spaces outside the Earth and below the geoid. The latter contribution to the reference potential is expressed in the form of an ellipsoidal harmonic series, and the expansion coefficients are tabulated numerically up to degree 20.

Key words: boundary-value problem, ellipsoidal harmonics, geoid determination, gravitational potential.

1 INTRODUCTION

Two techniques have recently been employed for the determination of the geoid over continental areas. Combining GPS positioning with orthometric heights results in the *geometrical* geoid, whose undulations with respect to the level ellipsoid are given as the ellipsoidal (GPS-determined) heights minus the orthometric heights (e.g. Hofmann-Wellenhof, Lichtenegger & Collins 1992). On the other hand, surface gravity observations supplemented by geodetic levelling can be used to construct the so-called *gravimetric* geoid (e.g. Vaníček *et al.* 1987). As a matter of fact, these techniques are not independent, since both make use of a density hypothesis within the Earth. Only the ellipsoidal heights resulting from GPS positioning on the one hand and gravimetric data on the other hand are independently determined. The geoid, therefore, can be determined in two ways, both of which depend on the density distribution within the Earth. Hence, there is hope that in the near future these two techniques can be combined, which will lead to an improvement of density distribution modelling in the uppermost part of the Earth. In order to reach this goal, the geoid should be determined with an accuracy of 1 dm (or better), since a change in the density model for the uppermost part of the Earth from a commonly accepted constant value of 2.67 g cm^{-3} to a more realistic 3-D density model changes the geoid by a few decimetres only (Martinec 1993).

The magic accuracy of 1 dm in the determination of the gravimetric geoid—not yet realizable in mountainous terrain—requires not only highly accurate surface gravity observations but also accurate theories and corresponding numerical codes for geoidal height computations. The last requirement has not yet been resolved satisfactorily since existing theories for geoid computations still contain some assumptions which do not allow the desired accuracy of 1 dm to be reached.

In this paper, our aim is to formulate the boundary-value problem for the determination of the gravimetric geoid with an accuracy of 1 dm. Throughout the paper, we will call this problem the *boundary-value problem for geoid determination* (BVPGD). Besides the usual data of surface gravity measurements and heights of the Earth's surface above the geoid, we shall assume that the low-degree potential harmonic expansion obtained from analysis of satellite orbit perturbations, truncated approximately at degree 20, is known *a priori*, so that it can serve as the reference potential. Let us note that it is the spherical harmonic coefficients of the Earth's gravitational field that are derived from the analysis of satellite orbit perturbations; to obtain the ellipsoidal harmonic coefficients of this field, the transformation relations between spherical and ellipsoidal harmonics derived by Jekeli (1988) are used. Formulating the BVPGD for a higher-degree reference potential has several advantages. For instance, the truncation error of Stokes' integral applied to observed gravity data reduced to the reference gravity field is significantly smaller than in the case where the classical Stokes theory is applied to original gravity data that contain low- as well as high-frequency components (Vaníček & Sjöberg 1991). This reduced truncation error can be evaluated numerically using a global gravity model truncated at degree 120 at most (Martinec 1993).

However, the formulation of the BVPGD with a reference potential given *a priori* may encounter some difficulties, since it is not as easy as in, for instance, the case of Molodensky's boundary-value problems (Heck 1991). Whereas Molodensky's problems are governed by the Laplace equation in the external space, and a reference satellite potential represents low-degree components of the solution in the whole space of interest, the reference satellite harmonics used in a partly internal and partly external BVPGD represent a solution only in the external space. The gravitational potential induced by topographical masses must be added to the satellite gravity model in order to construct the low-degree part of the solution within the topographical masses.

Vaníček *et al.* (1995) have made a first attempt to use satellite potential harmonics in the BVPGD as the reference. They defined the reference potential for Helmert's disturbing potential as the difference between the satellite model and the low-degree components of the direct topographical effect on the potential. This means that the reference harmonics of the sought potential depend on the way the topographical masses are condensed. In this paper, we will show that the reference potential for the BVPGD can be introduced differently, such that it does not depend on the way the topographical masses are compensated or condensed. This model better reflects the physical and mathematical background of the BVPGD, because the long-wavelength part of its solution is uniquely determined by the boundary conditions on the geoid and the Earth's surface and, of course, by the Laplace–Poisson equation. Note that the uniqueness and stability of the short-wavelength part of the solution of the BVPGD is influenced by the way the topographical masses are condensed (Engels *et al.* 1993). Another question not answered by Vaníček *et al.* (1995) is how to reduce the observed surface gravity to the reference field. Again, we will show that such a reduction can be performed without specifying the model of compensation of topographical masses.

2 FORMULATION OF THE BVPGD

The BVPGD will be formulated in ellipsoidal coordinates, as, later on, after linearization, the geoid will be approximated by a level ellipsoid—the ellipsoidal coordinates are most suitable to introduce this approximation. The 3-D ellipsoidal coordinates $\{u, \beta, \lambda\}$ can, for instance, be introduced by their relation to Cartesian coordinates $\{x, y, z\}$ (e.g. Heiskanen & Moritz 1967, Sect. 1-19; Thong & Grafarend 1989):

$$x = \sqrt{u^2 + E^2} \sin\beta \cos\lambda,$$
$$y = \sqrt{u^2 + E^2} \sin\beta \sin\lambda, \tag{1}$$
$$z = u \cos\beta,$$

where the parameter E is constant and defines the common focal distance of the family of confocal ellipsoidal coordinate surfaces $u = \text{const}$.

Let the geoid S_g be described by a function $u = u_g(\Omega)$, where Ω stands for the pair of angular coordinates (β, λ), i.e. $(u_g(\Omega), \Omega)$ are points on the geoid. We will assume that the function $u_g(\Omega)$ is not known. Let $H(\Omega)$ (≥ 0) be the height of the Earth's surface S_t above the geoid reckoned along a coordinate line $\beta = \text{const.}$, $\lambda = \text{const.}$ We will assume that $H(\Omega)$ is a known function. Finally, let the following quantities be given: the gravity $g(\Omega)$ measured on the Earth's surface, the density $\varrho(u, \Omega)$ of the topographical masses (the masses between the geoid and the Earth's surface), and the gauge value W_0 of the gravity potential on the geoid.

Since it is our intention to deal with the gravity field generated by the Earth's internal masses, we make a few simplifying assumptions. First, we assume that the observations of g are corrected for the attraction of the atmosphere and the direct gravitational effect of the other bodies, mainly the moon and the sun. Second, we assume the Earth is a rigid, undeformable body, uniformly rotating (with a constant angular frequency ω) around a fixed axis passing through its centre of mass. This assumption excludes consideration of the indirect gravitational effect of other celestial bodies, such as the tidal deformation of the Earth. Third, as already mentioned, we assume the height $H(\Omega)$ of the Earth's surface S_t above the geoid reckoned along a coordinate line $\Omega = \text{const.}$ to be known. It can be defined analogously to the usual orthometric heights, $H = C/\bar{g}_u$, where C is the geopotential number and $\bar{g}_u$ is the mean value of the u-component of gravity along a coordinate line $\Omega = \text{const.}$ between the geoid and the Earth's surface. Values of $\bar{g}_u$ can be estimated by a procedure similar to that used for $\bar{g}$ (Heiskanen & Moritz 1967, Sect. 4-4). However, $\bar{g}$ determined in this way is a rough estimate of the actual value. Only after finding the geoid with a high accuracy (better than 1 dm) will we be able to improve both $\bar{g}_u$ and $\bar{g}$.

The question we pose is: how do we determine the gravity potential $W(u, \Omega)$ *inside* and *outside* the topographical masses and the radius $u_g(\Omega)$ of the geoid? The problem is governed by Poisson's equation with the boundary conditions given on the free boundaries S_t and S_g coupled by means of height $H(\Omega)$:

$$\nabla^2 W = -4\pi G\varrho + 2\omega^2 \qquad \text{outside } S_g, \tag{2}$$

$$|\text{grad } W| = g \qquad \text{on } S_t, \tag{3}$$

$$W = W_0 \qquad \text{on } S_g, \tag{4}$$

$$W = \frac{1}{2}\omega^2(x^2 + y^2) + \frac{GM}{r} + O\left(\frac{1}{r^3}\right) \qquad r \to \infty, \tag{5}$$

where GM is the geocentric gravitational constant, ϱ is equal to zero outside the Earth, and r is the distance from the geocentre, $r = (x^2 + y^2 + z^2)^{1/2}$. The first-degree harmonics are left out of the potential W because of the geocentric coordinate system.

The gravity potential W can now be split into the normal (known) gravity potential U and a disturbing (unknown) gravitational potential T:

$$W = U + T, \tag{6}$$

where the normal potential U is generated by a level ellipsoid (of minor semi-axis b_0, say) spinning with the same angular velocity as the Earth (Heiskanen & Moritz 1967, Sect. 2-7). Throughout the paper, we will assume that the mass of the level ellipsoid is equal to the mass of the Earth, and that the mass-centre of the level ellipsoid coincides with that of the Earth. Then the zero- and first-degree harmonics of the potential U are equal to those of the potential W, and thus they vanish in the disturbing potential T. Moreover, we will assume that the normal gravity potential U on the level ellipsoid is equal to the actual gravity potential W_0 on the geoid. The free, non-linear boundary-value problem (2)–(5) can be reformulated for the disturbing potential T:

$$\nabla^2 T = -4\pi G\varrho \qquad \text{outside } S_g, \tag{7}$$

$$|\mathrm{grad}(U+T)| = g \qquad \text{on } S_t, \tag{8}$$

$$U + T = W_0 \qquad \text{on } S_g, \tag{9}$$

$$T \sim O\left(\frac{1}{r^3}\right) \qquad r \to \infty. \tag{10}$$

The unknowns to be determined by solving the problem (7)–(10) are the disturbing potential T in the space outside the geoid, and the ellipsoidal u-coordinate of S_g. Clearly, the boundary value of potential U cannot be subtracted from W_0 in boundary condition (9) because the surface S_g is not known, and thus the normal potential U cannot be directly evaluated on S_g. The asymptotic condition (10) imposed on T at infinity follows from (5), the fact that zero- and first-degree harmonics of the normal potential U are equal to those of the gravity potential W, and that both fields, W and U, are related to the same angular velocity.

The non-linear boundary-value problem (7)–(10) with a free boundary will be treated in a linearized form. Let us define points P, P_g, and Q on the Earth's surface, the geoid, and the level ellipsoid $u = b_0$, respectively, such that they lie on the same coordinate line $\Omega = \text{const}$. Then the boundary condition (8) can be linearized and written in the form

$$\left.\frac{\partial T}{\partial u}\right|_P + \frac{2}{b_0}\left.T\right|_{P_g} - \varepsilon_h(T_P) - \varepsilon_\gamma(T_{P_g}) = -\Delta g^F \tag{11}$$

(Martinec 1990; Heck 1991; Martinec *et al.* 1993), where we have introduced the free-air gravity anomaly Δg^F,

$$\Delta g^F = g_P - \gamma_Q + F, \tag{12}$$

where F is the free-air reduction (Heiskanen & Mortiz 1967), $\varepsilon_h(T_P)$ and $\varepsilon_\gamma(T_{P_g})$ are the ellipsoidal corrections,

$$\varepsilon_h(T_P) = -\frac{e^2}{2}\sin^2\beta \left.\frac{\partial T}{\partial u}\right|_P, \tag{13}$$

$$\varepsilon_\gamma(T_{P_g}) = e_0^2 \cos^2\beta \frac{T_{P_g}}{b_0}, \tag{14}$$

e is the first numerical eccentricity,

$$e = \frac{E}{\sqrt{u^2 + E^2}}, \tag{15}$$

and e_0 is the first eccentricity of the level ellipsoid $u = b_0$, $e_0 = E/\sqrt{b_0^2 + E^2}$. It can be simply shown that leaving out the non-linear terms in eq. (11) results in a relative error of 10^{-7} and an absolute error of the order of 0.1 mgal. The bias in the geoidal heights induced by this linearization error is at most 15 mm (Seitz, Schramm & Heck 1994).

By the above linearization, the free, non-linear boundary-value problem (7)–(10) can be reduced to a fixed, linear boundary-value problem [described by eqs (7), (10) and (11)] for determining the disturbing potential T outside the surface S_g. In contrast to in the problem (7)–(10), S_g is now considered to be known and fixed. The easiest and most often used way to approximate the geoid is by a mean sphere. The relative error introduced by this spherical approximation is of the order of 3×10^{-3} (Heiskanen & Moritz 1967, Sect. 2-14), which then causes an error of at most 0.5 m in the geoidal heights.

To reach a better accuracy in the determination of the disturbing potential T, we will approximate the geoid in the problem (7), (10) and (11) by a level ellipsoid, i.e. we put

$$u_g(\Omega) \doteq b_0. \tag{16}$$

In fact, the actual shape of the geoid deviates from a level ellipsoid by 100 m at most. Therefore, if we treat the geoid as the level ellipsoid in the formulae relating to the disturbing potential T, this causes a relative error of up to 1.5×10^{-5}; the absolute error in geoidal heights then does not exceed 2 mm.

After solving the fixed, linear boundary-value problem described by eqs (7), (10) and (11), and finding the disturbing potential T outside the geoid, the height N of the geoid above the level ellipsoid is obtained from boundary condition (9); its linearized form turns out to be the well-known Bruns' formula for height N (Heiskanen & Moritz 1967, eq. 2-144):

$$N = \frac{T_{P_g}}{\gamma_Q}. \tag{17}$$

3 COMPENSATION OF TOPOGRAPHICAL MASSES

Now, let us look for a particular solution to the Laplace–Poisson equation (7). The gravitational potential $V^t(u, \Omega)$ induced by the topographical masses,

$$V^t(u,\Omega) = G\int_{\Omega_0}\int_{u'=b_0}^{b_0+H(\Omega')} \frac{\varrho(u',\Omega')}{L(u,\Omega,u',\Omega')} w(u',\beta')\,du'\,d\Omega', \tag{18}$$

where

$$w(u,\beta) = u^2 + E^2\cos^2\beta, \tag{19}$$

Ω_0 is the full solid angle, $L(u, \Omega, u', \Omega')$ is the distance between the computation point (u, Ω) and an integration point (u', Ω'), and $d\Omega' = \sin\beta'\,d\beta'\,d\lambda'$, is a quantity that satisfies eq. (7). [Note that the ellipsoidal approximation (16) of the geoid has already been used in the formula for the potential $V^t(u, \Omega)$.] However, it is a well-known fact that the equipotential surfaces of V^t undulate by several hundreds of metres with respect to a level ellipsoid. Thus, it is not very advantageous to consider only V^t as a particular solution of eq. (7): another solution of this equation must be added to the potential V^t in order to reduce its large magnitude.

The fact that the known undulations of the geoid are significantly smaller than those induced by the potential V^t indicates that a compensation mechanism must exist which reduces the gravitational effect of topographical masses. This mechanism is probably mainly connected with lateral mass heterogeneities of the crust (Martinec 1994a) but also partly with deep dynamical processes (Martinec 1994b; Matyska 1994). To describe the compensation mathematically, a number of more or less idealized compensation models have been proposed. For the purpose of geoid computation, we can, in principle, employ any compensation model generating a harmonic gravitational field outside the geoid. For instance, the topographic–isostatic compensation models (e.g. Rummel *et al.* 1988; Moritz 1990) are based on compensation by the anomalies of density distribution $\varrho_c(u, \Omega)$ in a shell between the geoid and the compensation level $b_0 - D(\Omega)$, $D(\Omega) > 0$; i.e.;

the gravitational potential

$$V^{\text{isost.}}(u,\Omega) = G\int_{\Omega_0}\int_{u'=b_0-D(\Omega')}^{b_0}\frac{\varrho_c(u',\Omega')}{L(u,\Omega,u',\Omega')}w(u',\beta')\,du'\,d\Omega' \tag{20}$$

reduces the gravitational effect of topographical masses.

In the limiting case, the topographical masses can be compensated by a mass layer located on the geoid. This kind of compensation, called the Helmert second condensation (Helmert 1884), produces a potential described by the surface Newton integral:

$$V^{\text{conden.}}(u,\Omega) = G\int_{\Omega_0}\frac{\sigma(\Omega')}{L(u,\Omega,b_0,\Omega')}w(b_0,\beta')\,d\Omega', \tag{21}$$

where $\sigma(\Omega)$ is the density of the condensation layer. The 2-D condensation density $\sigma(\Omega)$ can be chosen in various ways depending on the approximation used for fitting the topographical potential V^{t} with the condensation potential $V^{\text{conden.}}$ (Martinec 1993).

Having introduced a compensation mechanism for the topographical masses, the associated compensation potential V^{c} 'approximating' the topographical potential V^{t} reads

$$V^{\text{c}} = V^{\text{isost.}} \quad \text{or} \quad V^{\text{c}} = V^{\text{conden.}} \tag{22}$$

for the isostatic compensation and Helmert's condensation of topographical masses respectively. Finally, a particular solution of the Laplace–Poisson equation (7) can be chosen as

$$\delta V = V^{\text{t}} - V^{\text{c}}, \tag{23}$$

where δV is the so-called *residual topographical potential*, i.e. the misfit of V^{t} and V^{c}.

4 A HIGHER-DEGREE REFERENCE GRAVITATIONAL POTENTIAL

Now, let us assume that some low-degree harmonics of the gravitational potential T have been determined from satellite orbit analyses. The question arises of how to reformulate the fixed boundary-value problem described by eqs (7), (10) and (11) so that a low-degree satellite gravity model can be considered as a reference gravitational potential.

Since the disturbing potential T is harmonic outside the Earth and it vanishes at infinity, it can be represented as a series of ellipsoidal harmonics $e^{j+1}q_{jm}(e)Y_{jm}(\Omega)$, $j=0,1,\ldots$, and $|m|\le j$, which all vanish at infinity (Heiskanen & Moritz 1967, p. 43, eq. 1-111b):

$$T(u,\Omega) = \sum_{j=2}^{\infty}\sum_{m=-j}^{j}T_{jm}\left(\frac{e}{e_0}\right)^{j+1}\frac{q_{jm}(e)}{q_{jm}(e_0)}Y_{jm}(\Omega), \tag{24}$$

where $q_{jm}(e)$ are defined by eq. (A4) in the Appendix. This series is convergent outside the bounding ellipsoid $u=b_1$, but it may be divergent in the space between the Earth's surface and this bounding ellipsoid (Sjöberg 1977; Jekeli 1983; Grafarend & Engels 1994). To define a higher-degree reference potential for the above boundary-value problem, however, we are interested only in the low-degree part of the potential T. Let us thus split the disturbing potential T into the (known) low-degree reference potential T_ℓ and a (unknown) higher-degree gravitational potential T^ℓ:

$$T = T_\ell + T^\ell. \tag{25}$$

The set of potential coefficients T^{e}_{jm} (where superscript e stands for external) determined by analyses of perturbations of satellite orbits will be taken as the reference. The reference gravitational potential T_ℓ in the space outside the bounding ellipsoid $u=b_1$ is then represented as

$$T_\ell(u,\Omega) = \sum_{j=2}^{\ell}\sum_{m=-j}^{j}T^{\text{e}}_{jm}\left(\frac{e}{e_0}\right)^{j+1}\frac{q_{jm}(e)}{q_{jm}(e_0)}Y_{jm}(\Omega), \tag{26}$$

where ℓ is the cut-off degree of the reference potential coefficients, say $\ell=20$. Moreover, since $|T^{\text{e}}_{jm}|<\infty$, the finite series (26) has finite values not only outside the bounding ellipsoid $u=b_1$, but also in the space between the Earth's surface and the ellipsoid $u=b_1$. Therefore, T_ℓ represented by the finite series (26) can also be considered as the reference gravitational potential for gravity observations performed on the Earth's surface.

Analogously, the residual gravitational potential δV can also be split into low-degree and high-degree parts:

$$\delta V = \delta V_\ell + \delta V^\ell. \tag{27}$$

The low-degree part δV_ℓ is given by the low-degree parts of expansions (A16) derived in the Appendix:

$$\delta V_\ell(u,\Omega) = \begin{cases} \displaystyle\sum_{j=0}^{\ell}\sum_{m=-j}^{j}(V^{\text{t,e}}_{jm}-V^{\text{c}}_{jm})\left(\frac{e}{e_0}\right)^{j+1}\frac{q_{jm}(e)}{q_{jm}(e_0)}Y_{jm}(\Omega) & \text{for } u\ge b_0+H(\Omega),\\ \displaystyle\sum_{j=0}^{\ell}\sum_{m=-j}^{j}(V^{\text{t,i}}_{jm}-V^{\text{c}}_{jm})Y_{jm}(\Omega) & \text{for } u=b_0. \end{cases} \tag{28}$$

Applying the same argument as in the preceeding paragraph, we represent δV_ℓ by the first series not only above $u=b_1$ but also between the Earth's surface and the bounding ellipsoid $u=b_1$. Note that formulae (28) do not determine the potential $\delta V_\ell(u,\Omega)$ within the topographical masses.

5 REFERENCE GRAVITY ANOMALY

The crucial point of the problem of a higher-degree reference field is determining the (low-frequency) part Δg^{F}_ℓ of the free-air gravity anomaly Δg^{F} that is generated by the reference potential T_ℓ. Obviously, Δg^{F}_ℓ is given by boundary condition (11) applied to T_ℓ:

$$\Delta g^{\text{F}}_\ell = -\left.\frac{\partial T_\ell}{\partial u}\right|_{u=b_0+H(\Omega)} - \frac{2}{b_0}T_\ell\Big|_{u=b_0} + \varepsilon_h(T_\ell)|_{u=b_0+H(\Omega)} + \varepsilon_\gamma(T_\ell)|_{u=b_0}. \tag{29}$$

The first and third terms of Δg^{F}_ℓ can easily be computed employing the representation (26) of T_ℓ. Unfortunately, the same representation cannot be used for evaluating the second and fourth terms of Δg^{F}_ℓ, because formula (26) is valid only outside the Earth (not on the geoid). Hence, the next step will be devoted to deriving the ellipsoidal harmonic expansion of T_ℓ at a point on the geoid.

The gravity potential W can be considered to be a sum of the gravitational potential V^{g} generated by the masses below the geoid, the topographical potential V^{t}, and the centrifugal potential V^{∞}:

$$W = V^{g} + V^{t} + V^{\infty}. \tag{30}$$

The two different decompositions (6) and (30) of the gravity potential W can now be put together so that the disturbing potential T reads

$$T = V^{t} + V^{g} + V^{\infty} - U. \tag{31}$$

This equation is valid everywhere inside and outside the Earth. Outside the geoid, in particular, the gravitational potential $V^{g} + V^{\infty} - U$ is harmonic, and it can be represented by an ellipsoidal harmonic series of the form (valid also for $u = b_0$)

$$V^{g} + V^{\infty} - U = \sum_{k=0}^{\infty} \sum_{m=-j}^{j} (V^{g} + V^{\infty} - U)_{jm} \left(\frac{e}{e_0}\right)^{j+1} \times \frac{q_{jm}(e)}{q_{jm}(e_0)} Y_{jm}(\Omega) \quad \text{for } u \geq b_0, \tag{32}$$

where $(V^{g} + V^{\infty} - U)_{jm}$ are expansion coefficients. Substituting the last formula together with the expansions (A9) and (26) into eq. (31), comparing the coefficients by ellipsoidal harmonics $e^{j+1} q_{jm}(e) Y_{jm}(\Omega)$ up to degree ℓ, and considering the continuation property of harmonic functions, in particular the unique extension of the region of definition of a harmonic function (Kellogg 1953, Theorem V, Chap. X), we obtain

$$(V^{g} + V^{\infty} - U)_{jm} = T^{e}_{jm} - V^{t,e}_{jm} \quad \text{for } u \geq b_0, \tag{33}$$

where $j = 0, 1, \ldots, \ell$, $|m| \leq j$, and $V^{t,e}_{jm}$ are given by integrals (A10).

On the geoid, $u = b_0$, formula (31) together with expansions (A11) and (32), yields

$$T(b_0, \Omega) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} [V^{t,i}_{jm} + (V^{g} + V^{\infty} - U)_{jm}] Y_{jm}(\Omega). \tag{34}$$

The low-degree part T_ℓ of potential T [see decomposition (25)] at a point on the geoid then reads

$$T_\ell(b_0, \Omega) = \sum_{j=0}^{\ell} \sum_{m=-j}^{j} [V^{t,i}_{jm} + (V^{g} + V^{\infty} - U)_{jm}] Y_{jm}(\Omega), \tag{35}$$

or, on substituting for $(V^{g} + V^{\infty} - U)_{jm}$ from eq. (33), we have

$$\boxed{T_\ell(b_0, \Omega) = \sum_{j=0}^{\ell} \sum_{m=-j}^{j} T^{i}_{jm} Y_{jm}(\Omega),} \tag{36}$$

where

$$\boxed{T^{i}_{jm} = T^{e}_{jm} + V^{t,i}_{jm} - V^{t,e}_{jm},} \tag{37}$$

$j = 0, 1, \ldots, \ell$, and $|m| \leq j$.

Finally, we are ready to evaluate the reference free-air gravity anomaly Δg^{F}_{ℓ}. Substituting eqs (26) and (36) into (29), we obtain

$$\boxed{\begin{aligned} \Delta g^{F}_{\ell}(\Omega) = {} & \left(1 + \frac{e^2}{2} \sin^2 \beta\right) \sum_{j=0}^{\ell} \sum_{m=-j}^{j} \left(\frac{e}{e_0}\right)^{j+1} \\ & \times \left[-(j+1) \frac{u}{u^2 + E^2} \frac{q_{jm}(e)}{q_{jm}(e_0)} + \frac{1}{q_{jm}(e_0)} \frac{dq_{jm}(e)}{du} \right]_{u=b_0+H(\Omega)} \\ & \times T^{e}_{jm} Y_{jm}(\Omega) + \frac{1}{b_0}(2 - e_0^2 \cos^2 \beta) \sum_{j=0}^{\ell} \sum_{m=-j}^{j} T^{i}_{jm} Y_{jm}(\Omega). \end{aligned}} \tag{38}$$

It is important that the reference free-air gravity anomaly Δg^{F}_{ℓ} does not depend on the way the topographical masses are compensated or condensed, but only on the reference satellite harmonics T^{e}_{jm}, the density distribution of topographical masses via differences, $V^{t,i}_{jm} - V^{t,e}_{jm}$, and on the topographical height $H(\Omega)$.

6 BVPGD WITH A HIGHER-DEGREE REFERENCE FIELD

Subtracting eq. (29) from eq. (11) and using the decomposition (25), we have

$$\begin{aligned} & \left.\frac{\partial T^{\ell}}{\partial u}\right|_{u=b_0+H(\Omega)} + \left.\frac{2}{b_0} T^{\ell}\right|_{u=b_0} \\ & - \varepsilon_h(T^{\ell})|_{u=b_0+H(\Omega)} - \varepsilon_\gamma(T^{\ell})|_{u=b_0} = -\Delta g^{F,\ell}, \end{aligned} \tag{39}$$

where T^{ℓ} is the high-degree part of the potential T, and $\Delta g^{F,\ell}$ is the high-degree part of the free-air gravity anomaly,

$$\Delta g^{F,\ell} = \Delta g^{F} - \Delta g^{F}_{\ell}. \tag{40}$$

On the strength of assumption (16), the residual topographical potential δV, and thus also its high-degree part δV^{ℓ}, can be considered as known quantities at points on the Earth's surface and the geoid. The latter quantity can readily be determined from formula (27), where δV_ℓ is given by the ellipsoidal harmonic expansion (28). This makes it possible to introduce a new unknown potential $T^{h,\ell}$:

$$T^{h,\ell} = T^{\ell} - \delta V^{\ell}. \tag{41}$$

By noting that the function $T - \delta V$ is harmonic outside the geoid, its high-degree part $T^{h,\ell}$, the high-degree part of Helmert's so-called *disturbing gravitational potential* (Martinec *et al.* 1993) when the topographical masses are compensated according to Helmert's second condensation technique, satisfies the boundary-value problem of the form

$$\boxed{\begin{aligned} & \nabla^2 T^{h,\ell} = 0 & & u > b_0, & (42) \\ & \left.\frac{\partial T^{h,\ell}}{\partial u}\right|_{u=b_0+H(\Omega)} + \left.\frac{2}{b_0} T^{h,\ell}\right|_{u=b_0} & & & \\ & \quad - \varepsilon_h(T^{h,\ell})|_{u=b_0+H(\Omega)} - \varepsilon_\gamma(T^{h,\ell})|_{u=b_0} & & & \\ & \quad = \Delta g^{F,\ell} \quad \delta A^{\ell} \quad \delta S^{\ell} & & & \\ & \quad + \varepsilon_h(\delta V^{\ell})|_{u=b_0+H(\Omega)} + \varepsilon_\gamma(\delta V^{\ell})|_{u=b_0}, & & & (43) \\ & T^{h,\ell} \sim O\left(\frac{1}{u^{\ell+1}}\right) & & u \to \infty, & (44) \end{aligned}}$$

where the boundary condition (43) follows immediately from the substitution of eq. (41) into eq. (39). The high-degree parts of the direct and secondary indirect topographical effects on gravity (Martinec & Vaníček 1994a, b) read

$$\begin{aligned} \delta A^{\ell} = {} & \left.\frac{\partial \delta V(u, \Omega)}{\partial u}\right|_{u=b_0+H(\Omega)} - \sum_{j=0}^{\ell} \sum_{m=-j}^{j} (V^{t,e}_{jm} - V^{c}_{jm}) \left(\frac{e}{e_0}\right)^{j+1} \\ & \times \left[-(j+1) \frac{u}{u^2 + E^2} \frac{q_{jm}(e)}{q_{jm}(e_0)} + \frac{1}{q_{jm}(e_0)} \frac{dq_{jm}(e)}{du} \right]_{u=b_0+H(\Omega)} \\ & \times Y_{jm}(\Omega), \end{aligned} \tag{45}$$

and

$$\delta S^{\ell}=\frac{2}{b_0}\delta V(b_0,\Omega)-\frac{2}{b_0}\sum_{j=0}^{\ell}\sum_{m=-j}^{j}(V^{\mathrm{t,i}}_{jm}-V^{\mathrm{c}}_{jm})Y_{jm}(\Omega). \quad (46)$$

The asymptotic constraint (44) reflects condition (10) and the fact that only the high-degree part of the gravitational potential is sought, while the low-degree component T_ℓ is assumed to be completely known *a priori*. Not even the best satellite gravity model is error-free, however, leaving an unmodelled low-degree residual in the boundary data. The following question may arise: should we try to determine this low-frequency residual or should we be satisfied with the low-frequency part of the solution as defined by a satellite gravity model and a low-degree model of topography, and only search for the high-frequency part of the solution? Nowadays, we tend to accept the latter point of view, since the coverage of gravimetric data over the Earth's surface is still not accurate, dense and homogeneous enough to be used to determine low-frequency components of the Earth's gravity field more precisely. This concept is also supported by the fact that the modelling of the long-wavelength term δV_ℓ of the residual topographical potential is not error-free because of an insufficient knowledge of the density of topographical masses. Thus, some long-wavelength residual of the gravitational effect of topographical masses affects the surface gravity data. Without knowledge of the 3-D density structure of topographical masses, it is impossible to distinguish this residual from long-wavelength errors of a satellite gravity model.

Accepting this concept, i.e. assuming that the solution is sought only for high-degree components (even though the reduced boundary data contain some low-frequency noise), the solution to the problem (42)–(44) may be affected by a low-degree aliasing effect. Whether or not the high-frequency part of a solution T^ℓ we are looking for is distorted by low-frequency noise in the boundary data depends on how the problem is solved. For instance, if the above boundary-value problem is solved by means of Poisson's and Stokes' integrals, the integration kernels must be constructed such that they are 'blind' to low-frequency components of the boundary data. Then the low-degree harmonics do not affect the high-frequency solution. The construction of Poisson's and Stokes' integration kernels with such a desired property can be found in Vaníček *et al.* (1987; 1996).

It is not the intention of this paper to construct the solution to the problem (42)–(44). For a long time, geodesists have been solving more or less similar problems. In most cases, they have tried to transform geodetic boundary-value problems similar to the problem (42)–(44) to Stokes' problem and to solve them by Stokes' integration (Heiskanen & Moritz 1967, Sect. 2-26). However, a satisfactory solution to the problem (42)–(44) matching today's accuracy requirements has not yet been presented. Perhaps the most crucial problems are that (1) the existence of a solution to the problem (42)–(44) cannot be guaranteed, and (2) the solution is unstable (e.g. Engels *et al.* 1993; Martinec & Matyska 1996). What conditions guarantee the existence of the solution and how to reasonably stabilize the solution are the open questions which have not yet been answered satisfactorily.

After finding $T^{\mathrm{h},\ell}$ by solving the fixed boundary-value problem (42)–(44), the solution to the free boundary-value problem (2)–(5) can be found by giving the undulations N of the geoid with respect to the reference ellipsoid. Substituting eqs (25) and (41) into Bruns' formula (17), we obtain

$$N=N_\ell+\frac{1}{\gamma_Q}(T^{\mathrm{h},\ell}+\delta V^\ell)|_{P_g}, \quad (47)$$

where we have introduced low-degree geoidal undulations N_ℓ as

$$N_\ell=\frac{1}{\gamma_Q}T_\ell|_{P_g}. \quad (48)$$

From eqs (36) and (37), we obtain

$$\boxed{N_\ell=\frac{1}{\gamma_Q}\sum_{j=0}^{\ell}\sum_{m=-j}^{j}(T^{\mathrm{e}}_{jm}+V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm})Y_{jm}(\Omega).} \quad (49)$$

It should be emphasized that neither $\Delta g^{\mathrm{F}}_\ell$ nor N_ℓ depends on the way topographical masses are compensated or condensed, but only on the reference harmonics T^{e}_{jm} and on the differences $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$ of ellipsoidal harmonics induced by topographical masses. On the other hand, the boundary-value problem (42)–(44) for $T^{\mathrm{h},\ell}$ depends on the way the topographical masses are compensated. There is an open question, not addressed here, whether Helmert's now popular condensation technique (e.g. Martinec *et al.* 1993) is the best way to compensate the gravitational effect of topographical masses when the problem (42)–(44) is to be solved.

7 NUMERICAL RESULTS FOR $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$

Let us try now to estimate the effect of the gravitational field generated by topographical masses on the reference free-air gravity anomaly $\Delta g^{\mathrm{F}}_\ell$ and the reference geoidal undulations N_ℓ. To do this, we have evaluated the differences $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$ for low degrees, $j=0, 1, \ldots, 20$, by a numerical quadrature applied to integrals (A10) and (A12) found in the Appendix. As a first approximation of $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$, we have assumed that the density $\varrho(u,\Omega)$ of topographical masses is constant and equal to the mean crustal density of $\varrho_0=2.67\ \mathrm{g\,cm^{-3}}$. The actual density of topographical masses is expected to vary around ϱ_0 by 10 to 20 per cent. Later on, we will estimate the effect of such topographical density variations on the differences $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$.

Table 1 gives the differences $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$, $j=0, 1, \ldots, 20$, $|m|\le j$, for the TUG87 global spherical harmonic terrain model (Wieser 1987) complete up to degree and order 180. We have found that the contributions of $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$, $j=0, 1, \ldots, 20$, $|m|\le j$, to geoidal heights lie within the interval $(-2.80; 0)$ m; the minimum -2.80 m is located in the Himalayas. Note that this result is in an agreement with that obtained by Sjöberg (1994). A plot of this effect for the territory of Canada is shown in Fig. 1. As expected, the minimum value, which reaches -0.43 m, is connected with the highest part of the Canadian Rocky Mountains. Since the density of topographical masses enters Newton's integral linearly, in order to achieve the 1 dm accuracy of geoidal heights, the regional density of the Rocky Mountains massif should be known with a relative accuracy of better than 25 per cent.

Inspecting eq. (38), we can observe that the part of the reference free-air gravity anomaly $\Delta g^{\mathrm{F}}_\ell(\Omega)$ that originates from differences $V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}$ is

$$\Delta g^{\mathrm{F}}_{\ell,\mathrm{topo}}(\Omega)\doteq-\frac{2}{b_0}\sum_{j=0}^{\ell}\sum_{m=-j}^{j}(V^{\mathrm{t,i}}_{jm}-V^{\mathrm{t,e}}_{jm}). \quad (50)$$

Table 1. Spherical harmonic coefficients $V^{t,i}_{jm} - V^{t,e}_{jm}$ (in $m^2\,s^{-2}$) for the TUG87 global terrain model (Wieser 1987) and for $\varrho_0 = 2.67\ g\,cm^{-3}$.

j	m	$V^{t,i}_{jm} - V^{t,e}_{jm}$	
0	0	-0.50	0.00
1	0	0.02	0.00
1	1	-0.10	-0.23
2	0	-0.32	0.00
2	1	0.02	-0.23
2	2	0.26	-0.04
3	0	0.48	0.00
3	1	0.00	-0.23
3	2	0.32	-0.09
3	3	0.04	0.06
4	0	-0.18	0.00
4	1	0.01	0.36
4	2	0.27	-0.01
4	3	-0.06	0.30
4	4	-0.13	-0.08
5	0	0.41	0.00
5	1	-0.07	0.07
5	2	0.00	0.05
5	3	-0.03	0.26
5	4	-0.25	0.14
5	5	-0.01	-0.14
6	0	-0.47	0.00
6	1	0.05	0.39
6	2	-0.21	0.05
6	3	0.02	0.05
6	4	-0.26	0.06
6	5	0.08	-0.16
6	6	0.09	0.03
7	0	0.06	0.00
7	1	-0.14	-0.25
7	2	-0.30	0.07
7	3	-0.06	-0.11
7	4	-0.12	0.00
7	5	-0.01	-0.25
7	6	0.18	-0.06
7	7	0.05	0.12
8	0	-0.24	0.00
8	1	0.02	0.08
8	2	0.00	0.08
8	3	0.00	-0.17
8	4	0.09	0.03
8	5	-0.01	-0.18
8	6	0.15	-0.04
8	7	-0.09	0.10
8	8	0.01	0.01
9	0	0.24	0.00
9	1	-0.11	-0.40
9	2	0.08	0.04
9	3	-0.01	-0.10
9	4	0.22	-0.03
9	5	0.05	0.01
9	6	0.13	-0.04
9	7	-0.02	0.22
9	8	-0.18	0.03
9	9	0.00	-0.06
10	0	0.06	0.00
10	1	0.05	0.27
10	2	0.24	-0.02
10	3	0.06	0.11
10	4	0.17	-0.09
10	5	0.03	0.12
10	6	0.04	-0.03
10	7	0.01	0.18
10	8	-0.08	0.01
10	9	-0.02	-0.04
10	10	-0.04	0.00
11	0	0.15	0.00
11	1	-0.11	0.01
11	2	0.02	0.04
11	3	-0.01	0.20
11	4	0.02	0.02
11	5	0.01	0.12
11	6	-0.11	0.01
11	7	-0.01	0.06
11	8	-0.10	0.01
11	9	0.03	-0.18
11	10	0.07	0.00
11	11	-0.03	0.05
12	0	-0.02	0.00
12	1	0.10	0.30
12	2	-0.04	0.02
12	3	0.13	0.15
12	4	-0.09	0.02
12	5	0.03	0.05
12	6	-0.15	0.04
12	7	0.01	-0.02
12	8	-0.07	0.02
12	9	-0.02	-0.15
12	10	0.04	-0.03
12	11	0.03	-0.02
12	12	0.02	0.02
13	0	-0.16	0.00
13	1	-0.07	-0.13
13	2	-0.23	0.08
13	3	-0.03	-0.03
13	4	-0.09	0.05
13	5	0.00	-0.09
13	6	-0.05	0.02
13	7	0.00	-0.11
13	8	0.07	0.02
13	9	-0.05	-0.06
13	10	0.06	0.02
13	11	-0.04	0.11
13	12	-0.03	-0.07
13	13	0.04	-0.03
14	0	-0.01	0.00
14	1	0.06	-0.06
14	2	-0.04	0.03
14	3	0.05	-0.09
14	4	0.01	0.02
14	5	-0.04	-0.14
14	6	0.05	0.01
14	7	-0.03	-0.12
14	8	0.11	-0.03
14	9	-0.02	-0.02
14	10	0.04	0.02
14	11	0.03	0.09
14	12	-0.01	0.06
14	13	-0.03	-0.03
14	14	0.00	-0.03
15	0	-0.01	0.00
15	1	-0.06	-0.20
15	2	0.04	0.08
15	3	-0.08	-0.10
15	4	0.13	0.06
15	5	-0.01	-0.07
15	6	0.10	-0.01
15	7	-0.02	-0.01
15	8	0.06	-0.04
15	9	-0.02	0.06
15	10	-0.03	-0.02
15	11	0.03	0.05
15	12	-0.02	-0.03
15	13	0.05	0.00
15	14	0.00	0.05
15	15	-0.02	0.00
16	0	0.17	0.00
16	1	0.06	-0.02
16	2	0.17	-0.06
16	3	0.09	0.07
16	4	0.15	-0.06
16	5	0.03	0.05
16	6	0.10	-0.04
16	7	0.01	0.08
16	8	0.00	-0.03
16	9	0.01	0.11
16	10	-0.07	0.01
16	11	0.00	0.01
16	12	-0.04	-0.04
16	13	-0.03	-0.06
16	14	0.02	-0.02
16	15	0.05	0.05
16	16	0.00	0.03
17	0	0.02	0.00
17	1	-0.01	0.12
17	2	0.06	0.02
17	3	-0.04	0.12
17	4	0.06	0.00
17	5	0.05	0.08
17	6	0.01	-0.06
17	7	0.03	0.07
17	8	-0.07	-0.03
17	9	0.05	0.03
17	10	-0.05	0.03
17	11	0.03	-0.04
17	12	0.00	0.00
17	13	-0.04	-0.03
17	14	-0.02	0.03
17	15	-0.04	-0.03
17	16	0.05	-0.02
17	17	0.01	0.03
18	0	0.03	0.00
18	1	0.05	0.12
18	2	-0.02	-0.06
18	3	0.11	0.16
18	4	-0.05	-0.04
18	5	0.05	0.06
18	6	-0.08	-0.03
18	7	0.03	0.00
18	8	-0.12	0.00
18	9	0.02	-0.05
18	10	-0.02	0.01
18	11	0.01	-0.07
18	12	0.06	-0.03
18	13	0.02	0.02
18	14	0.01	0.04
18	15	0.02	0.04
18	16	-0.06	0.01
18	17	-0.02	-0.04
18	18	0.01	0.00
19	0	-0.10	0.00
19	1	0.01	0.04
19	2	-0.12	0.04
19	3	-0.03	0.00
19	4	-0.07	0.07
19	5	0.05	-0.05
19	6	-0.07	0.00
19	7	-0.01	-0.06
19	8	-0.05	0.04
19	9	-0.02	-0.06
19	10	0.04	0.01
19	11	-0.03	-0.04
19	12	0.04	-0.03
19	13	0.00	0.02
19	14	0.03	0.02
19	15	0.01	0.00
19	16	0.05	-0.02
19	17	-0.03	0.05
19	18	-0.05	-0.01
19	19	-0.01	-0.01
20	0	-0.06	0.00
20	1	0.01	-0.12
20	2	-0.08	-0.05
20	3	0.05	-0.07
20	4	-0.05	0.00
20	5	0.02	-0.09
20	6	0.03	0.00
20	7	-0.01	-0.11
20	8	0.03	0.02
20	9	-0.02	-0.02
20	10	0.08	0.02
20	11	-0.01	0.02
20	12	0.02	0.01
20	13	-0.01	0.05
20	14	-0.02	0.04
20	15	-0.01	-0.02
20	16	0.00	-0.04
20	17	0.01	-0.03
20	18	0.03	0.02
20	19	-0.02	0.01
20	20	-0.02	0.00

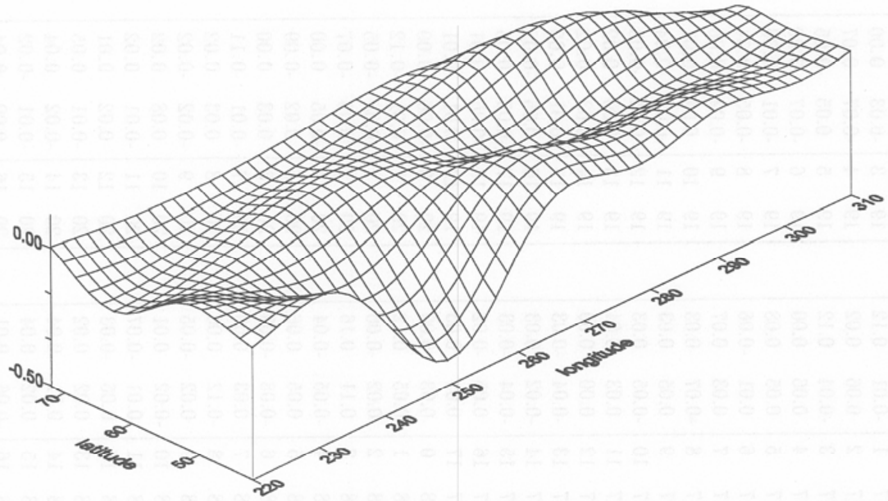

Figure 1. The part of the reference geoidal heights N_ℓ (in metres) that originates from differences $V^{t,i}_{jm} - V^{t,e}_{jm}$ over the territory of Canada.

We have evaluated $\Delta g^{F}_{\ell,\mathrm{topo}}(\Omega)$ globally for $\ell = 20$ and found that it reaches at most 0.09 mgal.

8 CONCLUSION

To employ a reference potential in the computation of the gravimetric geoid has an evident advantage: it reduces the magnitude of quantities we work with and thus enables us to linearize the originally non-linear boundary-value problem for geoid determination. There are certainly a lot of possible ways to bring the reference potential into the geoid computation. Here, a reference potential has been considered to consist of the harmonics derived from the analyses of satellite orbits. We have assumed that such *a priori* information on long-wavelength components of the gravitational field is fixed and should not be corrected from surface gravity observations. This fact is expressed by an asymptotic condition (44), which says that only the short-wavelength part of the gravitational potential is sought from surface gravity data. On the other hand, introducing a reference potential into the problem of geoid determination requires that the reference potential harmonics be accurate enough and that they contain meaningful information. Therefore, we have assumed that reference satellite harmonics are taken up to degree about 20.

It should be emphasized that satellite potential harmonics define the reference gravitational potential in the space external to the Earth. In order to construct a reference potential for a partly internal and partly external boundary-value problem of geoid determination, the low-degree part of the gravitational potential induced by topographical masses must be taken into account. Vaníček *et al.* (1995) have already formulated the BVPGD for the case where a satellite reference potential is taken as the reference. They confined themselves to the so-called Stokes–Helmert technique for geoid computation, and introduced the reference potential for Helmert's disturbing potential as a satellite gravitational potential minus the direct topographical effect on the potential. Evidently, such a reference field depends on the way topographical masses are condensed.

Here, we were motivated by whether the BVPGD could be formulated in such a way that a higher-degree reference potential would be independent of the way the topographical masses are compensated. We have shown that such a formulation exists; the reference free-air gravity anomaly as well as the reference geoidal height are determined by a satellite gravitational model and by the differences of the external and internal gravitational fields generated by topographical masses. The reference potential for geoid determination is insensitive to the way the topographical masses are compensated. Numerically, the magnitude of that part of the reference geoidal heights which comes from low-degree components of the topographical potential is approximately three times larger than the corresponding direct topographical effect on the potential in the Stokes–Helmert technique (Vaníček *et al.* 1995).

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APPENDIX A: ELLIPSOIDAL HARMONIC REPRESENTATION OF δV

In this Appendix, we express the residual topographical potential δV in a spectral form. Because of the ellipsoidal approximation (16) of the geoid, it is advantageous to represent δV in terms of ellipsoidal harmonics.

The expansion of the reciprocal distance in terms of ellipsoidal coordinates can be found in Hobson (1965, pp. 424–430); for $u > u'$,

$$\frac{1}{L(u,\Omega,u',\Omega')} = 4\pi \frac{i}{E} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} (-1)^m \frac{(j-m)!}{(j+m)!} Q_j^m\left(i\frac{u}{E}\right) \times P_j^m\left(i\frac{u'}{E}\right) Y_{jm}(\Omega) Y_{jm}^*(\Omega'), \tag{A1}$$

where $i = \sqrt{-1}$, $P_j^m(iu'/E)$ and $Q_j^m(iu/E)$ are the Legendre functions of the first and second kinds normalized according to Hobson (1965), $Y_{jm}(\Omega)$ are fully normalized spherical harmonics normalized according to Varshalovich, Moskalev & Khersonskii (1989), and the asterisk denotes the complex conjugate.

Thong (1993) has shown that Legendre's functions $P_j^m(iu/E)$ and $Q_j^m(iu/E)$ can be written as infinite power series of the first eccentricity e, defined by eq. (15), as

$$P_j^m\left(i\frac{u}{E}\right) = (-1)^{j/2} \frac{(2j-1)!!}{(j-m)!} e^{-j} p_{jm}(e), \tag{A2}$$

$$Q_j^m\left(i\frac{u}{E}\right) = (-1)^{m-(j+1)/2} \frac{(j+m)!}{(2j+1)!!} e^{j+1} q_{jm}(e), \tag{A3}$$

where

$$p_{jm}(e) = \sum_{k=0}^{\infty} a_{jmk} e^{2k}, \qquad q_{jm}(e) = \sum_{k=0}^{\infty} b_{jmk} e^{2k}. \tag{A4}$$

The coefficients a_{jmk} and b_{jmk} can be defined, for instance, by the following recurrence relations:

$$a_{jmk} = \frac{(-j+2k-2)^2 - m^2}{2k(-2j+2k-1)} a_{jm,k-1} \qquad \text{for } k \geq 1 \tag{A5}$$

with $a_{jm0} = 1$, and

$$b_{jmk} = \frac{(j+2k-1)^2 - m^2}{2k(2j+2k+1)} b_{jm,k-1} \qquad \text{for } k \geq 1, \tag{A6}$$

again with $b_{jm0} = 1$. For points lying on the Earth's surface or close to the Earth's surface, the eccentricities e and e' are of the order of 7×10^{-3} at most, and the series over k in eq. (A4) quickly converge for low degrees j. Later on, we will restrict ourselves to ellipsoidal harmonics of degrees $j \leq 20$; in such a case, it will be sufficient to sum the series in eq. (A4) only up to $k = 2$ and still keep the relative accuracy of the order of

10^{-5}. Such an accuracy ensures the determination of the geoid to an accuracy of the order of 1 mm. Substituting eqs (A2) and (A3) into the expansion (A1), we get

$$\frac{1}{L(u,\Omega,u',\Omega')} = 4\pi \frac{e}{E} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{1}{2j+1} \left(\frac{e}{e'}\right)^j \times q_{jm}(e) p_{jm}(e') Y_{jm}(\Omega) Y_{jm}^*(\Omega), \tag{A7}$$

valid for $u > u'$, or, equivalently for $e < e'$.

Let us now turn our attention to the topographical potential V^t. Substituting the above expansion of the reciprocal distance into Newton's integral (18), the gravitational potential V^t at a point outside the ellipsoid (of minor semi-axis b_1, say) completely enclosing the Earth reads

$$V^t(u,\Omega) = 4\pi G \frac{e}{E} \int_{\Omega_0} \int_{u'=b_0}^{b_0+H(\Omega')} \varrho(u',\Omega') \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{1}{2j+1} \left(\frac{e}{e'}\right)^j \times q_{jm}(e) p_{jm}(e') Y_{jm}(\Omega) Y_{jm}^*(\Omega') w(u',\beta')\, du'\, d\Omega'. \tag{A8}$$

For $u \geq b_1$, it is admissible to interchange the order of summation over j and m with the integration over u' and Ω' because of the uniform convergence of the series. We get

$$V^t(u,\Omega) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} V_{jm}^{t,e} \left(\frac{e}{e_0}\right)^{j+1} \frac{q_{jm}(e)}{q_{jm}(e_0)} Y_{jm}(\Omega) \qquad \text{for } u \geq b_1, \tag{A9}$$

where the expansion coefficients $V_{jm}^{t,e}$ read

$$V_{jm}^{t,e} = \frac{4\pi G}{2j+1} \frac{e_0}{E} q_{jm}(e_0) \int_{\Omega_0} \int_{u'=b_0}^{b_0+H(\Omega')} \varrho(u',\Omega') \left(\frac{e_0}{e'}\right)^j \times p_{jm}(e') Y_{jm}^*(\Omega') w(u',\beta')\, du'\, d\Omega'. \tag{A10}$$

Here, we have introduced factors $e_0^{j+1} q_{jm}(e_0)$ to normalize the expansion coefficients $V_{jm}^{t,e}$.

Analogously, for $u \leq b_0$, the gravitational potential V^t induced by the topographical masses can be expressed in terms of ellipsoidal harmonics that are regular at the origin, i.e. as

$$V^t(u,\Omega) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} V_{jm}^{t,i} \left(\frac{e_0}{e}\right)^j \frac{p_{jm}(e)}{p_{jm}(e_0)} Y_{jm}(\Omega) \qquad \text{for } u \leq b_0, \tag{A11}$$

with coefficients

$$V_{jm}^{t,i} = \frac{4\pi G}{2j+1} \frac{e_0}{E} p_{jm}(e_0) \int_{\Omega_0} \int_{u'=b_0}^{b_0+H(\Omega')} \varrho(u',\Omega') \left(\frac{e'}{e_0}\right)^{j+1} \times q_{jm}(e') Y_{jm}^*(\Omega') w(u',\beta')\, du'\, d\Omega'. \tag{A12}$$

As we have discussed, the compensation of topographical masses plays an important role in geoid determination. Employing expansion (A7) of the Newton kernel, the compensation potential V^c, eq. (22), at points outside/on the geoid can be expressed as a series of ellipsoidal harmonics:

$$V^c(u,\Omega) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} V_{jm}^c \left(\frac{e}{e_0}\right)^{j+1} \frac{q_{jm}(e)}{q_{jm}(e_0)} Y_{jm}(\Omega) \qquad \text{for } u \geq b_0, \tag{A13}$$

where the coefficients V_{jm}^c are equal to

$$V_{jm}^c = \frac{4\pi G}{2j+1} \frac{e_0}{E} q_{jm}(e_0) \int_{\Omega_0} \int_{u'=b_0-D(\Omega')}^{b_0} \varrho_c(u',\Omega') \left(\frac{e_0}{e'}\right)^j \times p_{jm}(e') Y_{jm}^*(\Omega') w(u',\beta')\, du'\, d\Omega', \tag{A14}$$

for the isostatic compensation of topographical masses, and

$$V_{jm}^c = \frac{4\pi G}{2j+1} \frac{e_0}{E} q_{jm}(e_0) \int_{\Omega_0} \sigma(\Omega') p_{jm}(e_0) Y_{jm}^*(\Omega') w(b_0,\beta')\, d\Omega', \tag{A15}$$

for Helmert's second condensation.

Finally, we are ready to write the ellipsoidal harmonic representation of the residual topographical potential $\delta V = V^t - V^c$. Using expansions (A9), (A11), and (A13), the potential δV reads

$$\delta V(u,\Omega) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \sum_{m=-j}^{j} (V_{jm}^{t,e} - V_{jm}^c) \left(\frac{e}{e_0}\right)^{j+1} \frac{q_{jm}(e)}{q_{jm}(e_0)} Y_{jm}(\Omega) & \text{for } u \geq b_1, \\ \displaystyle\sum_{j=0}^{\infty} \sum_{m=-j}^{j} (V_{jm}^{t,i} - V_{jm}^c) Y_{jm}(\Omega) & \text{for } u = b_0, \end{cases} \tag{A16}$$

where the coefficients $V_{jm}^{t,e}$, $V_{jm}^{t,i}$, and V_{jm}^c are given by eqs (A10), (A12), and (A14), respectively.