A comparison of downward continuation techniques of terrestrial gravity anomalies

J. Huang[1], M. G. Sideris[2], P. Vaníček[3], I.N. Tziavos[4]

[1]Geodetic Survey Division, Natural Resources Canada, Ottawa, ON, Canada

[2]Department of Geomatics Engineering, University of Calgary, Calgary, AB, Canada

[3]Department of Geodesy and Geomatics Engineering, University of New Brunswick, Fredericton, NB, Canada

[4]Department of Geodesy and Surveying, University of Thessaloniki, Thessaloniki, Greece

Abstract

In this work, we have conducted two comparisons. Firstly, we numerically compared the point-point Poisson downward continuation with the mean-mean one by using the 5'by5' residual Helmert gravity anomalies (n>20) and mean heights over the most rugged part of the Rocky Mountains. The results indicate that the former geoidal contribution is about 10 percent smaller than the latter one that ranges from - 4 cm to 94 cm with a mean of 48 cm and a RMS of 51 cm. In practical applications, certain cautious step has to be taken in selecting a model to minimize the incompatibility between the model and data. Secondly, we compared the point-point Poison downward continuation with the analytical downward continuation numerically. It is shown that the former geoidal result is about 25 cm smaller on average than the latter one that takes the first three terms into account. However their difference is dominated by long-wavelength components with a standard deviation of about 4 cm in absolute value. A possible cause of the difference may be due to the use of the planar approximation in the computation.

1. Introduction

The use of Stokes's formula for the determination of the geoid requires that the gravity anomalies Δg represent boundary values at the geoid, which implies two requisite conditions of using it: firstly, the gravity anomalies must refer to the geoid; secondly, there must be no masses outside the geoid. Hence, gravity reduction is necessary to meet the conditions. It can be carried out with the following two steps: (1) the topographic masses outside the geoid are removed and/or shifted below or on the geoid; (2) the gravity anomalies are harmonically reduced from the Earth's surface downward to the geoid. The second step is in geodetic literature described as *downward continuation* of the gravity anomalies.

In principle, all gravity reductions are equivalent and must lead to the same geoid if they are properly applied, the indirect effect being considered (Heiskanan and Moritz, 1967). Among the proposed reduction models, Helmert's 2^{nd} condensation reduction, in which the topographic masses is condensed as a mass layer on the geoid, is widely being used in the practical determination of the geoid (Vaníček and Kleusberg, 1987; Véronneau, 1996; Smith and Milbert, 1999; Featherstone et al., 2001). Even though the same reduction model is used, the actual reductions may be evaluated by different approaches. A major difference among the approaches is related to downward continuation methods applied.

There are two classes of downward continuation methods available for the geoid determination. One is *the Poisson downward continuation* that is based on the Poisson integral formula (Heiskanan and Moritz, 1967; Bjerhammar, 1987; Vaníček, et. al, 1996; Martinec, 1996). The second one is called *the analytical downward continuation* that is based on the Taylor series expansion (Moritz, 1980; Sideris, 1987). The former can be further branched into the point-point model (Martinec, 1996) and the mean-mean model (Vaníček, et. al, 1996). Sun and Vaníček (1998) pointed out that the point-point solutions are up to five times smaller than the mean-mean solutions. It should be noted that the point-point model by Sun and Vaníček (1998) is different from the one by Martinec (1996). The difference is with the treatment for the coefficient of the central point. By the point-point Poisson model, we refer to Martinec's one since its coefficient for the central point approaches to one, and the others to zeros when the height of the central point approaches to zero. This property can be mathematically justified because the Poisson kernel function switches to a Delta function when the height of the central point goes to zero.

(Prof. Sideris and Prof. Tziavos: Do you want to add some review about the analytical downward continuation with the appropriate references here?)

In this contribution, we are particularly interested in answering the following questions: How different is the point-point Poisson downward continuation result from the mean-mean one numerically? Is the analytical downward continuation numerically equivalent to the Poisson one? If not, how different are they?

2. Methodology 1: the Poisson downward continuation formulae

The Poisson integral formula can be written as follows (Heiskanen and Moritz, 6-74, 1967)

$$\Delta g(r,\Omega) = \frac{R}{4\pi r}\int_{\Omega'} K(r,\psi,R)\Delta g(R,\Omega')\,d\Omega' \tag{1}$$

where K is the Poisson kernel function

$$K(r,\psi,R) = \frac{R(r^2 - R^2)}{l^3}\,. \tag{2}$$

$\Delta g(r,\Omega)$ is the gravity anomaly function; Ω is the geocentric angle denoting the pair (θ, λ), the spherical co-latitude and lognitude; ψ and l are the angular and spatial distances between the point (r,Ω) and the surface element $d\,\Omega'$; r is the radius of the point (r,Ω).

Given $\Delta g(R,\Omega)$ that is the gravity anomaly function on the spherical surface with a radius R, evaluating the gravity anomaly at any point outside the spherical surface is called *upward continuation*, that can be done directly by evaluating the Poisson integral. The gravity anomaly $\Delta g(r,\Omega)$ at a point above the spherical surface represents a weighted average of the gravity anomalies $\Delta g(R,\Omega)$ given on the spherical surface, thus it tends to be smoother. When the gravity anomalies are known on a surface above the spherical surface, and the gravity anomalies on the spherical surface are sought, we face the problem of downward continuation that is achieved by solving the Poisson integral equation. In mathematics, this equation is called the Fredholm integral equation of the first kind. In contrast to upward continuation, downward continuation tends to 'de-smooth' or accentuate details of the gravity anomalies.

The geoid group at UNB developed a program package (**DOWN97)** which can be used to evaluate gravity anomalies on the geoid from gravity anomalies at the Earth's surface by solving the discrete Poisson integral equation (Vaníček, et al., 1998). It is capable of performing the point-point and mean-mean downward continuations, depending on how to evaluate kernel coefficients from which the coefficient matrix of a linear system of equations is formed. Bearing in mind, the Poisson downward continuation merely gives a spherical approximation results. However the effect of the geoid flattening is generally less than 1 cm because the downward continuation contribution from the Earth's surface to the geoid is usually smaller than 3 m in the geoidal height.

The discrete Poisson integral for the point-point downward continuation can be written as (Martinec, 1996)

$$\Delta g_i^t = \frac{R}{4\pi(R+H_i)} \sum_j K_{ij}\ \Delta g_j^g + F_{\Delta g} \tag{3}$$

where subscripts t and g stands for on the Earth's surface and the geoid, respectively; indices i and j indicate the computation and integration points, respectively; H_i is the height of a computation point; K_{ij} are the kernel coefficients; F represents the contribution outside the chosen near-zone cap, called the far-zone contribution.

The discrete Poisson integral for the mean-mean downward continuation can be expressed as (Vaníček, et. al, 1996)

$$\overline{\Delta g_i^t} = \frac{R}{4\pi(R+H_i)} \sum_j \overline{\overline{K_{ij}}}\ \overline{\Delta g_j^g} + \overline{F_{\Delta g}} \tag{4}$$

where the single over-lines indicate the mean values of the corresponding variables; the doubly over-lined K_{ij} represent the doubly averaged Poisson kernel coefficients.

The Seidel iterative method is used to solve the linear system of equations. Let B represent the coefficient matrix, and b be the constant vector, and x be the unknown vector, then discrete Poisson integral equations can be written as follows

$$Bx = b \tag{5}$$

Let $A = I - B$, equation (5) becomes

$$x = Ax + b\ . \tag{6}$$

Then the Seidel iteration can be expressed as

$$\begin{gathered} x^{(1)} = Ax^{(0)} + b, \\ x^{(2)} = Ax^{(1)} + b, \\ \dots \\ x^{(k+1)} = Ax^{(k)} + b\ \ until\ \ Max(|\ x^{(k+1)} - x^{(k)}\ |) \le \varepsilon\ . \end{gathered} \tag{7}$$

The threshold value ε is set as 0.02 mGal. The tests with synthetic data showed that the software gives downward continuation results with an accuracy of better than 1 cm in the geoidal height.

3. Methodology 2: the analytical downward continuation formulae

The analytical downward continuation can be formulated as follows (Moritz, sect. 45, 1980)

$$\Delta g^{g} = \sum_{n=0}^{\infty} g_{n} \tag{8}$$

where

$$\begin{aligned} g_0 &= \Delta g^{t}, \\ g_1 &= -H \cdot L(g_0), \\ g_2 &= -H \cdot L(g_1) - H^2 \cdot L_2(g_0), \\ &\dots \\ g_n &= -\sum_{m=1}^{n} H^{m} \cdot L_m \cdot g_{n-m}. \end{aligned} \tag{9}$$

The operator L is defined as

$$L(f) = \frac{R^2}{2\pi} \int_{\Omega'} \frac{f - f_P}{l_0^3} d\Omega' - \frac{1}{R} f_P \tag{10}$$

and

$$L_k = \frac{1}{n!} L^n = \frac{1}{n} L L_{k-1.} \tag{11}$$

where l_0 is the distance between computation point an integration point.

Up to date, most practical applications have only used the g_1 term in combination with the assumption of a linear relation between the gravity anomaly and the height. It has been shown that the g_1 term is close to the Condensed Terrain Effect (CTE) in Helmert's 2nd condensation model (Véronneau, 1996; Vaníček, et al., 1999). It is this fact that provides the background for the use of the Faye anomaly in the geoid determination because the Faye anomaly approximately represents the Helmert gravity anomaly on the geoid when the g_1 term is assumed to cancel out the CTE (Véronneau, 1996). Since the linear relation between the gravity anomaly and the height can not strictly be established for the Helmert anomaly, it should be avoided when the centimeter-accuracy geoid is determined through the Stokes-Helmert approach.

The evaluation of the analytical downward is very time-consuming. When the L operator is applied in planar approximation, it can then be written in a convolution form

$$L(f) = \frac{R^2}{2\pi} \int_{\Omega'} \frac{f - f_P}{l_0^3} d\Omega' = f * d_Z, \quad L_n f = (d_Z *)^n f. \tag{12}$$

The integral above can be efficiently evaluated by FFT using either the discrete or the analytical form of the $(l_0)^{-3}$ spectrum (Sideris, 1987)

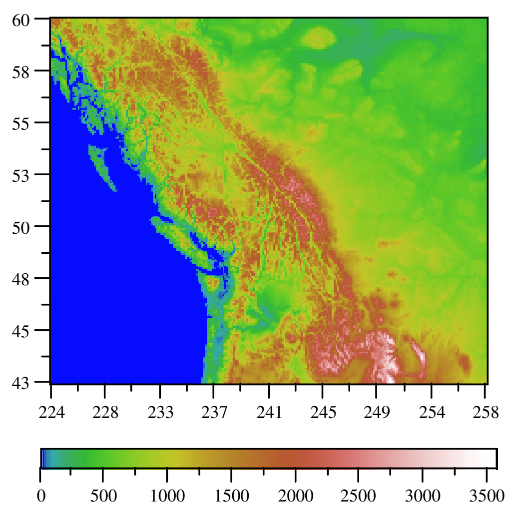
$$\frac{R^2}{2\pi}\int_{\Omega'}\frac{f-f_P}{l_0^3}d\Omega' = \frac{1}{2\pi}[\,F^{-1}\{F\{f\}F\{l_0^{-3}\}\}] - fF^{-1}\{F\{u\}F\{l_0^{-3}\}\}],\quad u=1,$$
$$\frac{R^2}{2\pi}\int_{\Omega'}\frac{f-f_P}{l_0^3}d\Omega' = f*d_z = F^{-1}\{F\{f\}(-2\pi q)\},\quad q = radial\ frequency. \qquad (13)$$

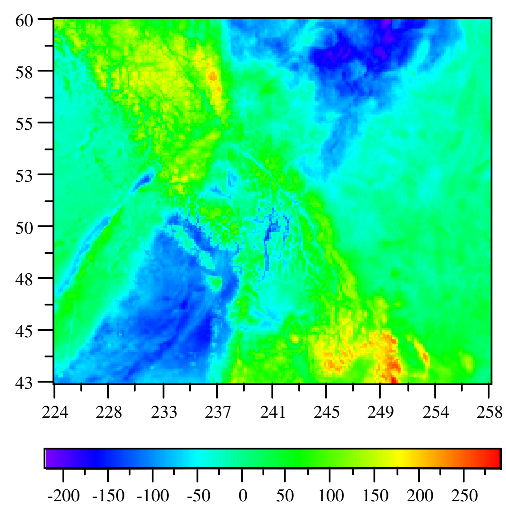
(Prof. Sideris and Prof. Tziavos: Do you want to add some text about the use of FFT here?)

4. Data description

The selected test region covers part of the Rocky Mountains (41°<φ<60°; 224°<λ<258°), which represents the most rugged area in Canada. It allows us to visualize the extreme scenarios of the differences between the point-point and mean-mean Poisson downward continuations, and between the Poisson downward continuation and the analytical downward continuation if they are not equivalent. In addition, Helmert's 2nd condensation yields a rougher gravity field than the isostatic gravity field from either of the Airy-Heiskanen and Pratt-Hayford condensation models (Martinec, 1996). Therefore, we are dealing with the comparison under the worst case in the two-fold senses.

The mean 5' by 5' heights and residual Helmert gravity anomalies (n>20) used for the comparison are shown in Figure 1. In the ocean and flat areas, the residual Helmert anomalies are more or less similar to the high-frequency free-air anomalies due to small terrain and CTE corrections, while in the mountainous area, they may differ from the high-frequency free-air anomalies by more than 100 percent.





(a) (b)

Fig. 1. (a) The topography (in meters), Min.= 0, Max.=3576, Mean=733, StdDev.=605. (b) The gravity anomalies (in mGals), Min.=-223, Max.=291, Mean=-0.8, StdDev.=75.8.

5. The Poisson downward continuation results

The same gravity data and height data have been treated as point values in one case and as mean values in the other. In the computation, the radius of the near-zone cap was chosen as 1 arc-degree. The far-zone contribution was ignored because it accounts for only about 1 cm in the test region when EGM96 is adopted. Strictly speaking, a global geopotential model must be first transformed into Helmert's space to be compatible with the Helmert anomalies. However, the far-zone contribution is predictably small considering the fact that the Poisson kernel is inversely proportional to the cubic of the distance between the computation point and the integration point.

Tables 1 and 2 show the summaries of the point-point and mean-mean Poisson downward continuation results in gravity and the geoidal height, respectively. The gravity difference between the results of the two continuation models is about 10% for mean, and more than 40% for both standard deviation and RMS of the mean-mean gravity downward continuation contribution. The geoidal difference between them is about 10% for mean, less than 20% for both standard deviation and RMS of the mean-mean geoidal downward continuation contribution. The fact that the geoidal difference is not so significant as the gravity one is due to the Stokes integral that tends to function as an averaging filter in transforming the gravity result into the geoidal one. The geoidal results from the two models are displayed in Figure 2, and the geoidal difference is displayed in Figure 3.

Depending on how one wants to carry out the numerical Stokes integration on the geoid, one may seek either point anomaly values or mean anomaly values on the geoid. If one wishes to work with mean values, one can either downward continue point values (known on the Earth's surface) and evaluate the mean values on the geoid, or one can evaluate mean values on the Earth's surface and downward continue these onto the geoid. We have tested these two scenarios in a limited way, using only the Poisson approach. We note that this raises a meaningful theoretical question as to whether mean values of a harmonic function are themselves harmonic. This question will not be explored here.

The algorithms for downward continuation of point and mean values are, of course, different, with the latter requiring a pre-computation of doubly averaged Poisson's kernels. Results obtained with the doubly averaged kernels were reported by Vaníček et al. (1996). The difference in the results using the two algorithms was analyzed by Sun and Vaníček (1998). Interestingly enough, the downward continuation algorithm for mean values was found giving rougher results than the algorithm for point values.

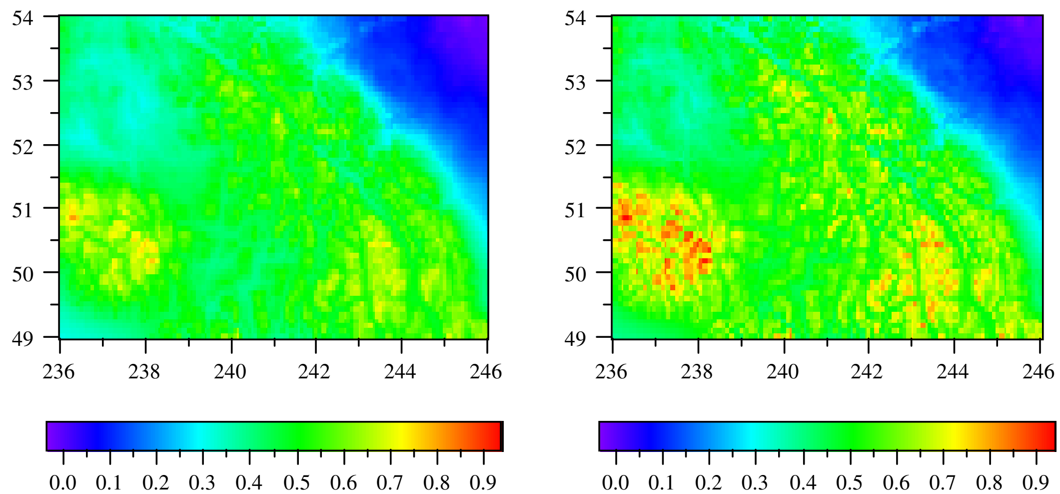
Clearly, the mean values, either on the Earth's surface or the geoid, should be expected to be somewhat smoother than the corresponding point values. This is a result of the smoothing property of the averaging operation. Unfortunately, the data on the surface of the Earth that we had at our disposal were neither the point values, nor the mean values. The data had been created by gridding the irregularly spaced observed point values for a 5 by 5 arc-minute geographical grid. Thus, at some locations, these gridded values are closer to true "point values", at other locations, where there was an abundance of observed values available, they are closer to true "mean values". It was therefore impossible properly to test the performance of the two algorithms and it seems to us likely that it may even not be possible to conduct proper tests in the foreseeable future.

Table 1. Differences of the Poisson downward continuation gravity corrections between the point-point and mean-mean models in the area of $44° < \varphi < 59°$; $226° < \lambda < 256°$ (180×360 grid, 64800 values). Unit: mGal.

Model	Mean	Min.	Max.	StdDev	RMS
(1) point-point	0.80	-23.25	64.23	4.71	4.77
(2) mean-mean	0.88	-46.20	102.49	7.63	7.68
(1) - (2)	-0.08	-38.25	24.85	3.29	3.29

Table 2. Differences of the Poisson downward continuation geoidal corrections between the point-point and mean-mean models in the area of $49° < \varphi < 54°$; $236° < \lambda < 246°$ (60×120 grid, 7200 values). Unit: meter.

Model	Mean	Min.	Max.	StdDev	RMS
(1) point-point	0.427	-0.038	0.810	0.154	0.454
(2) mean-mean	0.479	-0.042	0.942	0.175	0.510
(1) - (2)	-0.052	-0.199	0.020	0.031	0.060



(a) (b)

Fig. 2. (a) The point-point downward continuation result in the geoidal height. (b) The mean-mean downward continuation result in the geoid height. Unit: meter.

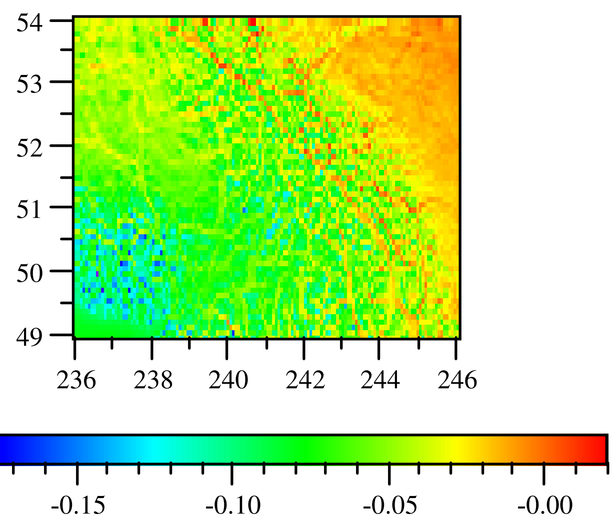


Fig. 3. The difference between the results of the point-point and mean-mean models. Unit: meter.

6. The analytical downward continuation results

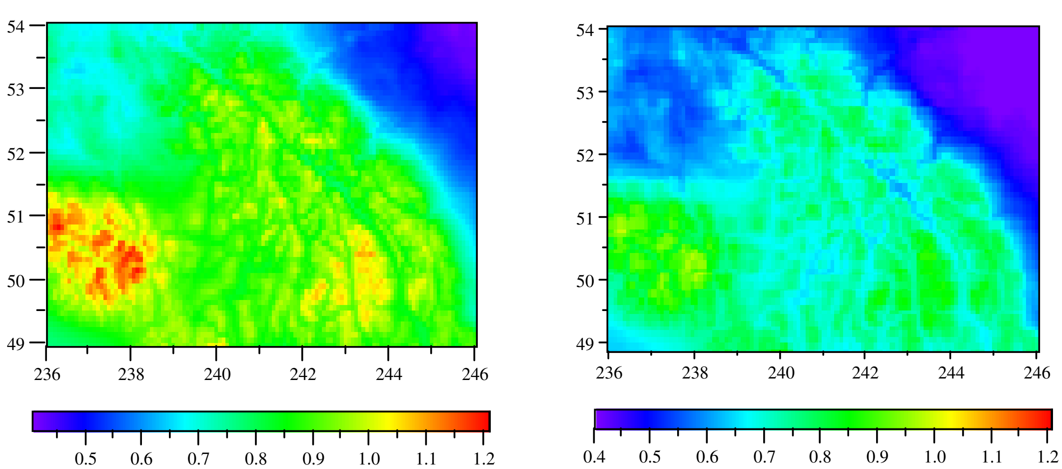
(Prof. Sideris and Prof. Tziavos: Could you please add some text here describing how the analytical DC was computed, and some remarks on the results below to complete this section?)

Table 3. The g_n term point corrections to the gravity anomalies in the area of 44º<φ<59º; 226º<λ<256º (180×360 grid, 64800 values). Unit: mGal.

Item	Mean	Min.	Max.	StdDev.	RMS
g_1	0.683	-20.784	54.126	4.204	4.259
g_2	-0.143	-17.613	10.110	1.116	1.125
g_3	0.032	-3.662	7.000	0.324	0.326
$g_1 + g_2$	0.541	-16.554	36.513	3.310	3.354
$g_1 + g_2 + g_3$	0.573	-17.032	41.000	3.482	3.529

Table 4. The g_n term point corrections to the geoidal heights in the area of 49º<φ<54º; 236º<λ<246º (60×120 grid, 7200 values). Unit: meter.

Item	Mean	Min.	Max.	StdDev.	RMS
$\zeta\,(g_1)$	0.830	0.405	1.212	0.154	0.844
$\zeta\,(g_2)$	-0.203	-0.298	-0.104	0.040	0.207
$\zeta\,(g_3)$	0.047	0.025	0.080	0.010	0.048
$\zeta\,(g_1 + g_2)$	0.627	0.298	0.954	0.118	0.638
$\zeta\,(g_1 + g_2 + g_3)$	0.675	0.323	1.011	0.125	0.686



(a) (b)

Fig. 4. Effect of the g_n terms on the geoidal height. (a) The g_1 term effect on the geoidal height. (2) The $g_1 + g_2 + g_3$ terms effect on the geoidal height. Unit: meter.

7. Comparison between the point-point Poisson and analytical downward continuations

The differences between the point-point Poisson and analytical downward continuations are statistically summarized in Tables 5 and 6. The significant differences have been shown between the results from the two methods. The range of the gravity differences reaches about 40 mGals with a standard deviation of about 1.5 mGals when the analytical downward continuation is summed up to the 3rd terms. Again, we can see that the geoidal differences are not so pronounced as the gravity ones due to the smoothing of the Stokes integration. An unacceptable fact is that the g_1 alone apparently agrees best with the Poisson result while it merely represents the first-order approximation. A further study is needed in this regard.

Even though remarkable geoidal differences (20 cm to 40 cm on average) present at individual points of the test region, their standard deviations are invariably smaller than 5 cm in absolute value. It implies that the differences primarily lie within the long-wavelength band. This character can be easily seen in Figure 5 that shows the dominant long-wavelength features. A possible interpretation to it is the uses of the 2-D FFT and the planar approximation in the computation of the analytical downward continuation.

(Do you have more explanation for the discrepancies?)

Table 5. The gravity differences between the g_n term point corrections and the point-point Poisson downward continuation corrections in the area of 44º<φ<59º; 226º<λ<256º (180×360 grid, 64800 values). Unit: mGal.

Item	Mean	Min.	Max.	StdDev.	RMS
g_1 - Poisson	0.11	-5.72	17.64	0.88	0.88
($g_1 + g_2$) - Poisson	0.26	-14.56	30.39	1.75	1.77
($g_1 + g_2 + g_3$)- Poisson	-0.22	-27.78	11.36	1.49	1.51

Table 6. The geoidal differences between the g_n term point corrections and the Poisson downward continuation geoidal corrections in the area of 49º<φ<54º; 236º<λ<246º (60×120 grid, 7200 values). Unit: mGal.

Item	Mean	Min.	Max.	StdDev.	RMS
$\zeta(g_1) - \zeta$ (Poisson)	0.404	0.488	0.245	0.038	0.405
$\zeta(g_1 + g_2) - \zeta$ (Poisson)	0.200	0.033	0.336	0.048	0.206
$\zeta(g_1 + g_2 + g_3) - \zeta$ Poisson)	0.248	0.081	0.361	0.042	0.251

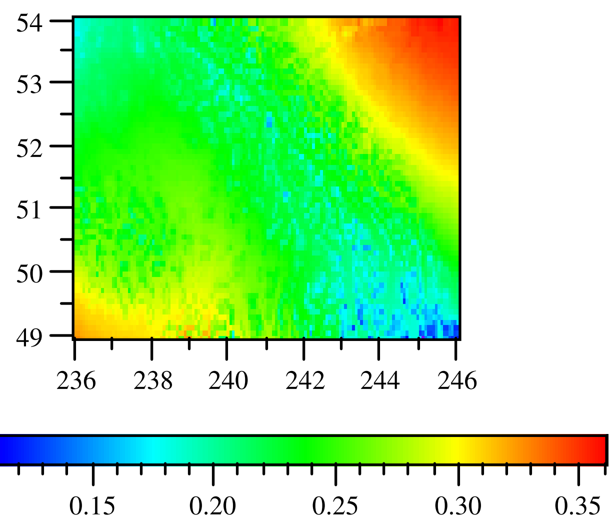


Fig. 5. The geoidal difference between the analytical and Poisson downward continuation results. Unit: meter.

8. Summary

We have numerically compared the point-point Poisson downward continuation with the mean-mean one using the 5' by 5' residual Helmert gravity anomalies (n>2) and mean height data in the Canadian Rocky Mountains. The geoidal results show that the point-point downward continuation contribution is about 10 percent smaller than that of the mean-mean one on average. The mean-mean downward continuation contribution ranges from -4 cm to 94 cm with a mean of 48 cm and a RMS of 51 cm. Since the data on the Earth's surface we have are neither the point values, nor the mean values. At some locations, they are closer to the point values, at other locations, they are closer to the mean values depending the distribution density of the observed data. It is therefore impossible properly to tell which one is better. In application, caution is needed in selecting a proper downward continuation model.

Another comparison has also been conducted between the point-point Poisson and analytical downward continuations using the same residual Helmert gravity anomalies and height data. The former geoidal contribution is about 25 cm smaller on average than the latter one that is summed up to the 3rd order term. But the standard deviation of their difference is smaller than 5 cm in absolute value. The analytical downward continuation contribution ranges from 68 cm to 101 cm with a mean of 68 cm and a RMS of 69 cm. The difference between the two methods characterized by the long-wavelength features may be attributed to the planar approximation in the computation of the analytical downward continuation, and the method itself. The comparison based on the spherical approximation is needed to more realistically reflect its performance.

Acknowledgements:

The research described in this contribution was supported by a DSS contract with the Federal Government of Canada, Geodetic Survey Division (GSD) in Ottawa. This manuscript is a reproduction of the presentation under the same title presented at XXII IUGG General Assembly, Birmingham, UK, July18-30, 1999.

References:

(Please add your references)

Bjerhammar, A. 1987. Discrete physical geodesy. OSU Rep. 380, Dept. of geodetic Science and Surveying, The Ohio State Univ., Columbus.

Featherstone, W. E., J. F. Kirby, A. H. W. Kearsley, J. F. Gilliland, G. M. Johnston, J. Steed, R. Forsberg, M. G. Sideris, 2001. The AusGeoid98 Geoid model of Australia: data treatment, computations and comparisons with GPS-levelling data. *Journal of Geodesy*, 75, 313-330.

Heiskanen, W.H., and H. Moritz, 1967. *Physical Geodesy*. W.H. Freeman and Co., San Francisco.

Martinec, Z., 1996. Stability investigations of a discrete downward continuation problem for geoid determination in the Canadian Rocky Mountains. *Journal of Geodesy*, 70, 805-828.

Moritz,H., 1980, *Advanced Physical Geodesy*, H. Wichmann Verlag, Karlaruhe.

Sideris M.G., 1987. Spectral methods for the numerical solution of Molodensky's problem. PhD dissertation, Departmetn of Surveying Engineering, University of Calgary.

Smith, D. A., D. G. Milbert 1999. The GEOID96 high resolution geoid height model for the United States. *Journal of Geodesy*, 73, 219-236.

Sun, W. and P. Vaníček, 1998. On some problems of the downward continuation of 5' x 5' mean Helmert's gravity disturbance. *Journal of Geodesy*, 72, 411- 420.

Vaníček, P. and A. Kleusberg, 1987. The Canadian geoid-Stokesian approach, *Manuscripta Geodaetica*. 12, 86-98.

Vaníček, P., W. Sun, P. Ong, Z. Martinec, M. Najafi, P. Vajda, and B. Ter Horst, 1996. Downward continuation of Helmert's gravity. *Journal of Geodesy*, 71, 21-34.

Vaníček, P., J. Huang, P. Novak, S. Pagiatakis, M. Veronneau, Z. Martinec, and W.E. Featherstone, 1999. Determination of the boundary values for the Stokes-Helmert problem. *Journal of Geodesy*, 73, 180-192.

Vaníček, P., J. Huang, and P. Novak, 1998. DOWN'97 discrete Poisson downward continuation program package for 5' by 5' data. Contract Progress Report for GSD.

Véronneau, M, 1996. The GSD95 geoid model of Canada. Gravity, geoid and marine geodesy. Proc. Int. symp., Tokyo, Springer Berlin Heidelberg New York, vol. 117, 573-580. Edited by J. Segawa, H. Fujimoto, S. Okubo.