AN EVALUATION OF THREE TECHNIQUES FOR THE PREDICTION OF GRAVITY ANOMALIES IN CANADA

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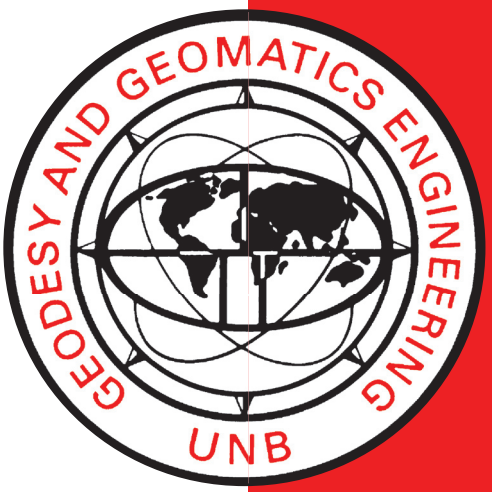
September 1980



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PREFACE

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AN EVALUATION OF THREE TECHNIQUES FOR THE PREDICTION OF GRAVITY ANOMALIES IN CANADA

by

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ABSTRACT

Recent studies have shown that the Canadian height control network, which was defined on the basis of normal gravity, suffers from the influence of gravity anomalies that can introduce significant systematic regional distortions. Proposals have been made for a new definition of heights for Canada which would be based on observed gravity values. Since, observed gravity is not now available at all points along levelling paths, (as required by the new definition), techniques suitable for the prediction of point gravity values at bench marks,say,are, therefore, required.

The performances of three prediction techniques — least-squares surface fit, least-squares collocation, and weighted mean methods — in three terrain situations in Canada were evaluated. The terrain situations considered correspond to the flat, gently rolling, and mountainous terrain types. Test points were selected randomly from each terrain type considered, and the nine samples generated by using each technique to predict for point gravity anomalies at the selected points were vigorously tested statistically. The

method of weighted means performed well in the three different types of terrain. It was the fastest of the three techniques, and the most economical in terms of computer time. The other two techniques gave good results in the flat, and rolling terrains, but did not perform so well in the mountainous terrain.

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CHAPTER 1

INTRODUCTION

At the time of the first continental adjustment of the Canadian levelling network, which took place in 1928 (Lachapelle, 1978b), a detailed knowledge of geoidal undulations and gravity field was non-existent (Vaníček et al, 1972). Hence, heights – those in the levelling network of 1928 and additional ones – which now constitute the present Canadian vertical control system were defined on the basis of normal gravity. The question of how the neglect of gravity anomalies affects heights has been debated by practicing geodesists and theoreticians alike (Nassar and Vaníček, 1975), and in 1948 the International Association of Geodesy adopted a resolution recommending that height systems be based on actual gravity (I.A.G., 1950). Recent studies in Canada have shown the influence of the neglect on heights in precise levelling networks to be significant (Nassar, 1977); and the need for a re-definition of the Canadian reference system was realized after due considerations had been given to other problems affecting the present networks (Lachapelle, 1978b).

According to proposals made for investigation into a new definition of heights for Canada (Vaníček et al, 1972), the systems proposed included dynamic and orthometric heights, both based on geopotential numbers reflecting

the observed gravity in the area. Whichever system is finally adopted will require a continuous knowledge of observed gravity in the area; and if only the bench marks are considered, the system will require that gravity values be available for every bench mark in Canada. Such information is not available now. However, the present gravity coverage in the country is sufficiently dense for one to predict gravity at bench marks other than those where it has been directly observed (Nassar, 1977; Lachapelle, 1978b). It becomes necessary, therefore, to resort to interpolating techniques for the densification of the gravity network in Canada.

The purpose of this work, therefore, is to evaluate existing methods suitable for gravity (or gravity anomaly) predictions at bench marks using gravity data from Earth Physics Branch gravity data files.

In the works of Kearsley (1977) and Tchnerning (1980) various techniques which have been used to extend the gravity field from its discrete form to a continuous one were highlighted. These include the methods of least-squares surface fitting, simple and least-squares plane fitting, minimum curvature surface, solids of revolution, least squares collocation, series fitting, and splines. Two of these techniques – least-squares collocation and least-squares surface fitting methods – which are most widely used are chosen for evaluation. A third method chosen for evaluation is not included in the list above. It is the method of weighted means which has been used

by Kearsley (1977) to estimate geoidal heights. It was chosen because of the very simple concept upon which it is based, and out of curiosity to see how it would perform compared to the other two methods.

It is necessary, at this stage, to clarify the need for the prediction ot gravity values at bench marks. The height definition presently in use in Canada has been said to be based on normal gravity. This implies that instead of using the observed gravity value g at a point on the earth's surface to define its height, an approximate value – the normal gravity value – which is computed (Nassar,1977) for the terrain point, is adopted. The intention, therefore, is to account for the difference in height due to the neglected difference $(g-\gamma)$ between observed and normal gravity values. This height difference has been expressed by Nassar (1977) as a correction which could be added to existing height differences $\Delta h'$ such that the corresponding height difference, appropriately based on actual gravity, are obtained. In Vaníček et al, (1980), the difference is expressed in terms of observed elevation difference.

The expressions were obtained by first expressing actual geopotential number difference between points A and B, say, in terms of both normal and observed gravity (Nassar,1977) as:

$$\widetilde{\Delta C}_{AB} \simeq \sum_i \bar{\gamma}^*_{ij} \Delta h_{ij} \qquad (1.1)$$

and $$\Delta C_{AB} \simeq \sum_i \bar{g}_{ij} \Delta h_{ij} \qquad (1.2)$$

where

$j = i+1$

$\bar{\gamma}^*_{ij} = \frac{1}{2} (\gamma^*_i + \gamma^*_j)$

$\bar{g}_{ij} = \frac{1}{2} (g_i + g_j)$

$\Delta h_{ij} = h_j - h_i$

and γ^* is the normal gravity computed from a normal gravity formula developed by the United States Coast and Geodetic Survey (USCGS) in 1907 (Nassar,1977). Height differences $\widetilde{\Delta h}_{AB}$, and Δh_{AB} are next obtained from equations (1.1), and (1.2) respectively under the dynamic system of heights, say, as:

$$\widetilde{\Delta h}^D_{AB} = \Delta C_{AB} / G = \frac{1}{G} \sum_i \bar{\gamma}^*_{ij} \Delta h_{ij} = \Delta h_{AB} + DC_{AB} \qquad (1.3)$$

and $$\Delta h^D_{AB} = \Delta C_{AB} / G = \frac{1}{G} \sum_i \bar{g}_{ij} \Delta h_{ij} = \Delta h_{AB} + DC_{AB} \qquad (1.4)$$

where G (Vaníček et al, 1980) is the reference gravity computed for dynamic heights taken for the United States, and Canada as the normal gravity on the ellipsoid at latitude 45°, and DC is the dynamic correction to the observed height differences. The difference between the corrective terms $\widetilde{DC}_{AB}$, and DC_{AB} is the required gravity correction GC^D_{AB} to existing height difference $\Delta h'_{AB}$. GC^D_{AB} is expressed as

$$GC^D_{AB} = \frac{\Delta h_{AB}}{G} \sum_i (\bar{g}_{ij} - \bar{\gamma}^*_{ij}) \qquad (1.5)$$

Its values between pairs of bench marks can be obtained if observed gravity values are available for the bench marks.

The observed gravity at a point i can be expressed in terms of free-air or Bouguer anomalies (Vaníček and Krakiwsky, in prep; Nassar, 1977) as:

$$g_i = \Delta g_i^F - 0.3086h_i + \gamma_{0,i} \qquad (1.6)$$

or

$$g_i = \Delta g_i^B - 0.1967h_i + \gamma_{0,i} \qquad (1.7)$$

where g_i is the observed gravity of the terrain point i in mgals, Δg_i^F and Δg_i^B are respectively free-air and Bouguer anomalies in mgals, h_i is the levelled height of point i in metres, 0.3086 and 0.1967 are respectively free-air and Bouguer gradients of gravity in mgal per metre, and $\gamma_{0,i}$ is the normal gravity value of terrain point i. Free-air and Bouguer anomalies are smoother than the corresponding absolute values of the observed gravity. They can, therefore, be predicted to better accuracies. Hence, once they have been predicted for, the corresponding observed values could be obtained through equations (1.6) and (1.7).

The approach decided upon for the evaluation is to test the chosen prediction methods in three terrain situations in Canada which correspond to flat, gently rolling and mountainous terrain types.

In this work, the second chapter will be devoted to the presentation of the various mathematical models for

the three techniques. The third chapter will deal with the identification of terrain types, while the fourth chapter will dicuss the various tests and test statistics needed for the evaluations, and the fifth chapter will discuss the results. The conclusions reached and the recommendations made will be presented in Chapter Six.

CHAPTER 2

PREDICTION METHODS

The mathematical models and other characteristics of the three prediction techniques chosen for evaluation – least-squares surface fitting, least-squares collocation, and weighted means – are presented. A fourth technique (a combination of the surface fitting and collocation methods) experimented with for the Rocky Mountains data is also described.

2.1 LEAST-SQUARES SURFACE FITTING METHOD

If there is sufficient gravity data in an area, the gravity field in the area can be predicted by regressions, the mathematical model for which is (Vaníček et al, in prep (ch. 22))

$$\underset{\sim}{\overline{\Delta g}}(\phi,\lambda) = \underset{\sim}{\Phi}^{T}(\phi,\lambda)\ \underset{\sim}{c} \qquad (2.1)$$

where $\underset{\sim}{\overline{\Delta g}}$ is the predicted value of gravity (or gravity anomaly) while $\underset{\sim}{\Phi}^{T}(\phi,\lambda)$ is the Vandermonde's matrix composed of selected base functions, and $\underset{\sim}{c}$ is a vector of coefficients to be determined using available gravity data. The selection of the base functions for the Vandermonde's matrix and the method by which the vector $\underset{\sim}{c}$ is determined decide the nature of the approximation method.

The base functions are usually to be selected (Vaníček et al, in prep (Ch. 14)) with due consideration given to the nature of the observables and their measuring

process. The selection may be achieved through the modelling of some natural phenomena affecting the observable or the measuring process (e.g. the determination of local mean sea level), or it may reflect the behavior of the observable as predicted by a law of physics or geometry (e.g. the development of gravitational potential into spherical harmonics). Otherwise, the selection has to be made arbitrarily. The base functions in this work are selected arbitrarily (Vaníček et al, in prep. (Ch. 22))the simplest choice being the mixed algebraic functions 1, x, y, xy, x^2, ..., with x, y indicating local cartesian coordinates. The gravity field in the area is, in effect, being represented by a surface described by a mixed algebraic polynomial P_n (x, y) of order n, the general form of which is given as:

$$P_n(x,y) = c_{0,0} + c_{0,1}y + c_{0,2}\,y^2 + \ldots + c_{0,(n-k)}\,y^{n-k} + c_{1,0}\,x + c_{1,1}\,xy + \ldots + c_{1,2}\,xy^2 + \ldots + c_{k,(n-k)}\,x^k\,y^{(n-k)} \qquad (2.2)$$

where:

$k = \text{int}\left(\frac{n}{2}\right)$

n = order of polynomial (positive integer), and

$c_{k,(n-k)}$ = coefficients $\underset{\sim}{c}$ to be determined.

Powers of x and y present in some specific orders of polynomial are shown in figure 1. In a shorter form equation

(2.2) can be written as:

$$P_n(x,y) = \sum_{i=0}^{k} \sum_{j=0}^{(n-k)} c_{ij}\, x^i\, y^j \qquad (2.3)$$

The source of the gravity data used in this work provides a pair of geographical coordinates (ϕ, λ) for every point value. The coordinates are transformed into cartesian coordinates (x,y) by (Nassar et al, 1975)

$$\begin{aligned} x &= R\,(\phi-\phi_o) \\ y &= R\,(\lambda-\lambda_o)\cos\phi \end{aligned} \qquad (2.4)$$

where the point (ϕ_o, λ_o) is taken for convenience as the prediction point. In this way, the prediction point is the origin of the cartesian coordinate system. The mean radius of curvature R is computed at the point (ϕ_o, λ_o) as (Nassar and Vaníček, 1975)

$$R = \sqrt{(M_o\, N_o)} \qquad (2.5)$$

where

$$M_o = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi_o)^{3/2}}$$

$$N_o = \frac{a}{\sqrt{(1-e^2 \sin^2\phi_o)}} \quad , \quad \{e^2=(a^2-b^2)/a^2\}$$

and a, b = major and minor semi-axes of reference ellipsoid.

The order n of the polynomial determines the dimensions of the vector of coefficients $\underset{\sim}{c}$. It is, therefore, chosen such that the dimension of $\underset{\sim}{c}$ is smaller than the number of available data. In this way, when trying to solve for $\underset{\sim}{c}$, one is faced with an overdetermined problem-

x \ y	0	1	2	3	4
0	x	x	x		
1	x	x	x		
2					
3					
4					

3rd order
(n = 3)

x \ y	0	1	2	3	4
0	x	x	x		
1	x	x	x		
2	x	x	x		
3					
4					

4th order
(n = 4)

x \ y	0	1	2	3	4
0	x	x	x	x	
1	x	x	x	x	
2	x	x	x	x	
3	x	x	x	x	
4					

6th order
(n = 6)

x \ y	0	1	2	3	4
0	x	x	x	x	x
1	x	x	x	x	x
2	x	x	x	x	x
3	x	x	x	x	x
4	x	x	x	x	x

8th order
(n = 8)

Figure 1

Tables Displaying the Powers of X and Y
Present in Specific Orders of Polynomial

the problem of approximation - which is solvable by the least-squares technique.

The least-squares technique has been developed in great detail in the literature (Vaníček and Wells, 1972; Vaníček and Krakiwsky, in prep (Ch. 12)) , and will be stated here (without proofs) in the context of the problem at hand. A function F defined on a discrete set $M\{X_1,..,X_{N_r}\}$ in terms of observed point gravity anomaly values is to be approximated by the polynomial $P_n(x,y)$ given in equation (2.3) namely:

$$P_n(x,y) = \sum_{i=0}^{k} \sum_{j=0}^{(n-k)} c_{ij}\, x^i\, y^j = \sum_{L=1}^{m} c_L\, \Phi_L \qquad (2.6)$$

where:

k, n are as defined above

$m = (k + 1)\ (n-k+1)$

$C_{ij} = C_L$ = element of vector of coefficients for the polynomial

$\underset{\sim}{x} = (x,y)$

N_r = number of point data available

and $\Phi \equiv \{\Phi_1,\Phi_2,\Phi_3,..\Phi_m\}$ are m linearly independent base functions. In matrix form equation (2.6) can be written as:

$$P_n = \underset{\sim}{\Phi}\ \underset{\sim}{c} \qquad (2.7)$$

where

$$\underset{\sim}{\Phi} = \{\Phi_1,\ \Phi_2,\ \Phi_3,\ldots,\ \Phi_m\}$$

and

$$\underset{\sim}{c} = \{c_1, c_2, c_3, \ldots, c_m\}^T$$

If, in addition, weight functions W(x) are defined on M, the least-squares approximation problem can be said to be the determination of the vector of coefficients $\underset{\sim}{c}$ which minimizes the distance $\rho(F,P_n)$ with weight function W(x), the distance being defined as (Vaníček and Wells, 1972)

$$\rho(F,P_n) = \sqrt{\left\{ \sum_{X\in M} W(X)\{F(X)-P_n(X)\}^2 \right\}}_{M \quad \text{Discrete}} \tag{2.8}$$

The required solution is obtained by solving the normal equations.

$$\sum_{j=1}^{m} \langle \phi_i, \phi_j \rangle c_j = \langle F, \phi_i \rangle \quad i=1,2,\ldots,m \tag{2.9}$$

where

$$\langle \phi_i, \phi_j \rangle \equiv \sum_{X\in M} W(X)\phi_i(X)\ \phi_j(X)$$

and

$$\langle F, \phi_i \rangle \equiv \sum_{X\in M} W(X)F(X)\ \phi_i(X)$$

In matrix form and according to the development in Vaníček and Krakiwsky, (in.prep,(Ch. 12)) (2.9) can be written as:

$$\underset{\sim}{A}\underset{\sim}{c} = \underset{\sim}{U} \tag{2.10}$$

where

$$\underset{\sim}{A} = \underset{\sim}{\phi}^T \underset{\sim}{W}\ \underset{\sim}{\phi}$$

$$\underset{\sim}{U} = \underset{\sim}{\phi}^T \underset{\sim}{W}\ \underset{\sim}{F}$$

$\underset{\sim}{W}$ = weight matrix

$\underset{\sim}{F}$ = vector of functions F

and $\underset{\sim}{\Phi}$, $\underset{\sim}{c}$ are as defined above

The solution is given as:

$$\underset{\sim}{c} = \underset{\sim}{A}^{-1} \underset{\sim}{U} \qquad (2.11)$$

The matrix $\underset{\sim}{A}$ of normal equations called the Gram matrix is regular non-singular, if its determinant (Gram's determinant) is different from zero. This requirement is met if the base functions are linearly independent. The problem is further simplified if orthogonal base functions are used; then, the Gram matrix – which is a matrix of the scalar products of all possible pairs of vectors of the base functions – becomes a diagonal matrix which can be easily inverted and the solution for $\underset{\sim}{c}$ is straightforward. Any system of base functions can be transformed into an orthogonal system by the process of orthogonalization. The best known of the many processes of orthogonalization is (Vaníček and Krakiwsky, in prep) the Gram Schmid process.

Weights $\underset{\sim}{W}$ are assigned inversely proportional to the variances of the individual observations as:

$$W_i = 1/S_i^2 \qquad (2.12)$$

where:

S_i^2 = variance of i^{th} observation

The estimate of the variance-covariance matrix of the coefficients $\underset{\sim}{c}$ is given as:

$$\underset{\sim}{\Sigma}_c = \underset{\sim}{\sigma}_0^2 \underset{\sim}{A}^{-1} = \underset{\sim}{Q} \qquad (2.13)$$

where: $\underset{\sim}{\sigma}_0^2$(estimated variance factor) $= \dfrac{\langle r, r \rangle}{df}$

r = residuals

and df = degrees of freedom

The residuals are obtained by resubstituting the coefficients $\underset{\sim}{c}$ from(2.11) into equation (2.6) for every point data used and by comparing the computed function P_n with the observed value F. The residual is given as:

$$\underset{\sim}{r} = \underset{\sim}{F} - \underset{\sim}{P}_n \qquad (2.14)$$

Since we now have a solution for the vector $\underset{\sim}{c}$ we can approximate the gravity field in the area by equation (2.1). The accuracies of approximations are obtained through the variance-covariance matrix $\underset{\sim}{Q}$ (Equation 2.13). The predicted gravity anomaly at a prediction point (the centre of cell) and the accuracy of prediction $\sigma_{\Delta g_p}$(standard deviation of predicted value) are given as:

$$\Delta g_p = c_{0,0} \qquad (2.15)$$

$$\sigma_{\Delta g_p} = \sqrt{Q_{1,1}} \qquad (2.16)$$

The estimated variance factor σ_0^2 is used in (2.13) to scale the variance-covariance matrix of the coefficients. The mathematical models used together with the available data and weighting scheme are assumed to be ideal. Hence, the apriori variance factor is unity. If the assumptions made are indeed correct, the estimated variance factor should be unity and (John, 1976) the standard deviations of predicted values should have the

same order of magnitude as the residuals at the observation points. Large residuals at observation points express not only the uncertainties in the observations, but also (Vaníček et al., in prep. (Ch. 14)) the uncertainties in the mathematical model (that is, the selected base functions) and perhaps those due to a poor weighting scheme.

2.2 LEAST-SQUARES COLLOCATION

The method of least squares was first used by C.F. Gauss and A.M. Legendre to process astronomical observations in 1795 (Balmino, 1978). It has since developed considerably through the works of mathematicians and other scientists who succeeded in generalizing the original method of static and dynamic applications, and for sequential solutions of problems (Balmino, 1978). It was further modified in the works of Krarup (1969) and Moritz (1972) such that in addition to solving for unknown parameters, it can also predict for signal quantities (statistically dependent residual component $\underset{\sim}{s}$) at points other than the observation points. This modified method is known as least-squares collocation.

The purpose in this section is to present the mathematical model for the prediction of gravity anomalies by least-squares collocation following the approach of, and using the notation of, Vaníček and Krakiwski, (in prep.)

The implicit linear mathematical model for a one component adjustment is given as (Vaníček and Krakiwski, in prep(Ch. 12))

$$\underset{\sim}{A}\delta + \underset{\sim}{B}r + \underset{\sim}{w} = \underset{\sim}{0} \qquad (2.17)$$

where

$\underset{\sim}{A}$ = first design matrix

$\underset{\sim}{\delta}$ = vector of corrections to unknown parameter approximations

$\underset{\sim}{B}$ = second design matrix

$\underset{\sim}{r}$ = vector of corrections to observations

$\underset{\sim}{w}$ = vector of constants

and the equivalence of the covariance matrices $\underset{\sim}{C}_l \equiv \underset{\sim}{C}_r = \mathrm{diag}(\sigma^2_{l_i})$ is assumed to be true. It is seen that $\underset{\sim}{r}$ is not decomposed here into its two components $\underset{\sim}{s}$ and $\underset{\sim}{v}$ (to be defined later) which are lumped together in it ($\underset{\sim}{r}$)

In a simple adjustment one would be interested only in obtaining estimates for (a) the unknown parameters and (b) the corrections to the observations to make them consistent within the framework of an overdetermined model. The relevant expressions for the one-component adjustment as developed in Vaníček and Krakiwski, (in prep), are:

$$\hat{\underset{\sim}{\delta}} = -\{ \underset{\sim}{A}^T (\underset{\sim}{B}\underset{\sim}{C}_r\underset{\sim}{B}^T)^{-1}\underset{\sim}{A} \}^{-1}\underset{\sim}{A}^T(\underset{\sim}{B}\underset{\sim}{C}_r\underset{\sim}{B}^T)^{-1}\ \underset{\sim}{w} \qquad (2.19a)$$

$$\hat{\underset{\sim}{x}} = \overset{0}{\underset{\sim}{x}} + \underset{\sim}{\delta} \qquad (2.19b)$$

$$\underset{\sim}{C}_{\hat{x}} = \{\underset{\sim}{A}^T(\underset{\sim}{B}\underset{\sim}{C}_r\underset{\sim}{B}^T)^{-1}\underset{\sim}{A}\}^{-1} = \underset{\sim}{C}_{\hat{\delta}} \qquad (2.19c)$$

$$\hat{\underset{\sim}{K}} = (\underset{\sim}{B}\underset{\sim}{C}_r\underset{\sim}{B}^T)^{-1}\ (\underset{\sim}{A}\delta + \underset{\sim}{w}) \qquad (2.19d)$$

$$\hat{\underset{\sim}{r}} = -\underset{\sim}{C}_r\underset{\sim}{B}^T\underset{\sim}{K} \qquad (2.19e)$$

$$\underset{\sim}{C}_{\hat{r}} = \underset{\sim}{C}_r\underset{\sim}{B}^T\underset{\sim}{M}^{-1}\ \underset{\sim}{B}\underset{\sim}{C}_r - \underset{\sim}{C}_r\underset{\sim}{B}^T\underset{\sim}{M}^{-1}\ \underset{\sim}{A}\underset{\sim}{N}^{-1}\ \underset{\sim}{A}^T\underset{\sim}{M}^{-1}\ \underset{\sim}{B}\underset{\sim}{C}_r \qquad (2.19f)$$

$$\hat{\underset{\sim}{l}} = \underset{\sim}{l} + \underset{\sim}{r} \qquad (2.19g)$$

$$\underset{\sim}{C}_{\hat{l}} = \underset{\sim}{C}_r - \underset{\sim}{C}_{\hat{r}} \qquad (2.19h)$$

where

$$\underset{\sim}{M} = \underset{\sim}{B}\underset{\sim}{C}_r\underset{\sim}{B}^T \quad ; \quad \underset{\sim}{N} = \underset{\sim}{A}^T\underset{\sim}{M}^{-1}\underset{\sim}{A}$$

$\hat{\underset{\sim}{\delta}}$ = estimates of the solution vector

$\hat{\underset{\sim}{x}}, \underset{\sim}{C}_{\hat{x}}$ = estimates of the unknown parameters and their covariances.

$\hat{\underset{\sim}{K}}$ = estimates of the vector of correlates

$\hat{\underset{\sim}{r}}, \underset{\sim}{C}_{\hat{r}}$ = estimates of residuals and their covariances

and $\hat{\underset{\sim}{l}}, \underset{\sim}{C}_{\hat{l}}$ = estimates of actual observations and their covariance.

The implicit linear mathematical model and the variation function for a two-component adjustment of observations are given as (Vaníček and Krakiwsky, in prep)

$$\underset{\sim}{A}\underset{\sim}{\delta} + \underset{\sim}{B}_s\underset{\sim}{s} + \underset{\sim}{B}_v\underset{\sim}{v} + \underset{\sim}{w} = \underset{\sim}{0} \qquad (2.20)$$

and

$$\underset{\sim}{\Phi} = \underset{\sim}{s}^T\underset{\sim}{C}_s\underset{\sim}{s} + \underset{\sim}{v}^T\underset{\sim}{C}_v^{-1}\underset{\sim}{v} + 2\underset{\sim}{K}^T(\underset{\sim}{A}\underset{\sim}{\delta} + \underset{\sim}{B}_s\underset{\sim}{s} + \underset{\sim}{B}_v\underset{\sim}{v} + \underset{\sim}{w}) \qquad (2.21)$$

where $\underset{\sim}{s}$, $\underset{\sim}{v}$, (assumed to be uncorrelated) are respectively, the statistically dependent and statistically independent components of the residuals $\underset{\sim}{r}$, and the matrices $\underset{\sim}{C}_s$ and $\underset{\sim}{C}_v$ are respectively fully populated and diagonal. It is seen that the residual $\underset{\sim}{r}$ has been decomposed into its two components $\underset{\sim}{s}$ and $\underset{\sim}{v}$. The component $\underset{\sim}{s}$ does not in the general case belong in the space $\mathcal{L}$ of observations, but in the space $\mathcal{S}$ of statistically dependent observations (see figure 2). This fact has been considered, and the necessary transformations made in equation (2.20)

Similar expressions to equations (2.19 (a-h)) can be

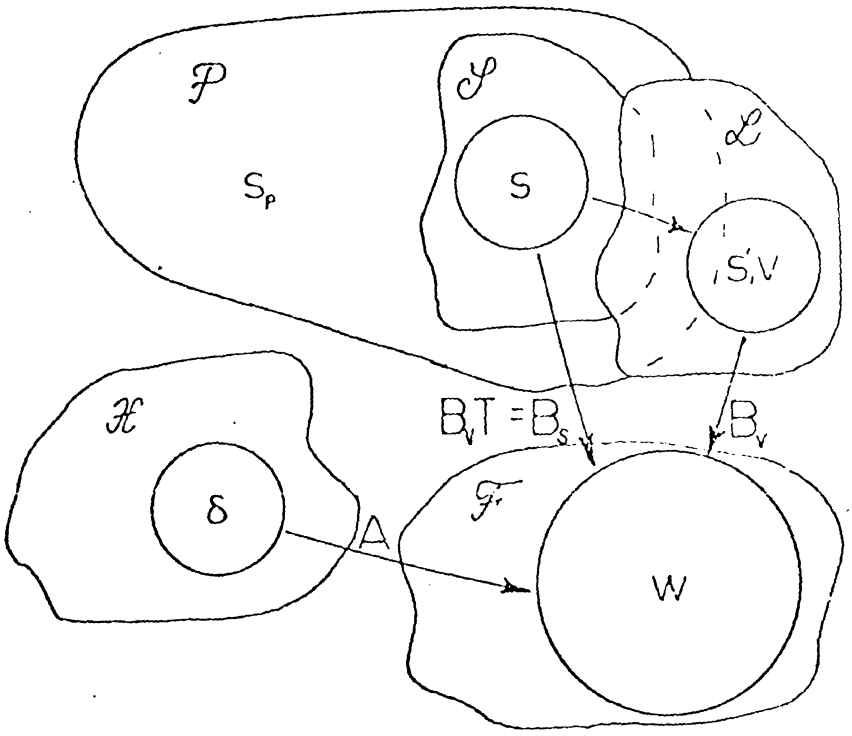

Figure 2:

Spaces Used for Least-Squares Collocation Prediction

derived for the two-component case by replacing the matrices $\underset{\sim}{B}$, $\underset{\sim}{C}_r$, and $\underset{\sim}{r}$ in the equation by their two-component equivalents $\underset{\sim}{B}'$, $\underset{\sim}{C}_r'$, and $\underset{\sim}{r}'$ given as:

$$\underset{\sim}{B}' = \left[\underset{\sim}{B}_s \; : \; \underset{\sim}{B}_v \right] \qquad (2.22a)$$

$$\underset{\sim}{C}_r' = \left[\begin{array}{c:c} \underset{\sim}{C}_s & 0 \\ \hdashline 0 & \underset{\sim}{C}_v \end{array} \right] \qquad (2.22b)$$

$$\underset{\sim}{r}' = \left[\underset{\sim}{s}^T \; : \; \underset{\sim}{v}^T \right]^T \qquad (2.22c)$$

In addition, the prediction of the residual component $\underset{\sim}{s}$, at points other than the observation points, is made possible by the method of least-squares collocation, which combines the two-component adjustment with this prediction feature. It views $\underset{\sim}{s}$ as a sample of an effect that can be modelled in a prediction space $\mathcal{P}$ wider than the space $\mathcal{S}$, where $\mathcal{S} \subset \mathcal{P}$ (figure 2). The prediction feature is built into the mathematical model of the two-component adjustment by introducing a null vector into the hyper-matrix $\underset{\sim}{B}'$, and expanding the hyper-vector $\underset{\sim}{r}'$ to include the required signal $\underset{\sim}{s}_p$ at prediction points. $\underset{\sim}{s}_p$ is stipulated to have the same stochastic characteristics as the signal $\underset{\sim}{s}$ at observation points. The covariance matrix C_r' is also modified to reflect the correlation between $\underset{\sim}{s}$ and $\underset{\sim}{s}_p$. The resulting matrices B'', C_r'', and vector r'' are given by the expressions:

$$\underset{\sim}{B}'' = [\ \underset{\sim}{0}\ :\underset{\sim}{B}_s\ :\underset{\sim}{B}_v\ ] \qquad (2.23a)$$

$$\underset{\sim}{r}'' = [\ \underset{\sim}{s}_p^T\ :\underset{\sim}{s}^T\ :\underset{\sim}{v}^T\ ] \qquad (2.23b)$$

$$\underset{\sim}{C}_r'' = \begin{bmatrix} \underset{\sim}{C}_{s_p} & \underset{\sim}{C}_{s_p s} & \underset{\sim}{0} \\ \underset{\sim}{C}_{s s_p} & \underset{\sim}{C}_s & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{C}_v \end{bmatrix} \qquad (2.23c)$$

When these expressions replace the matrices $\underset{\sim}{B}$, $\underset{\sim}{C}_r$ and vector $\underset{\sim}{r}$ in equations (2.19) accordingly, and if we consider a condition model explicit in the observations $\underset{\sim}{l}$ (that is, $\underset{\sim}{B}_s = \underset{\sim}{B}_v = -\underset{\sim}{I}$; $\underset{\sim}{A} = \underset{\sim}{0}$ and $\underset{\sim}{w} = -\underset{\sim}{l}$), the predicted signal $\underset{\sim}{s}_p$ and its covariance matrix are given as:

$$\hat{\underset{\sim}{s}}_p = \underset{\sim}{C}_{s_p s}(\underset{\sim}{C}_s + \underset{\sim}{C}_v)^{-1}\underset{\sim}{l} \qquad (2.24)$$

$$\underset{\sim}{C}_{\hat{s}_p} = \underset{\sim}{C}_s - \underset{\sim}{C}_{s_p s}(\underset{\sim}{C}_s + \underset{\sim}{C}_v)^{-1}\underset{\sim}{C}_{s_p s}^T \qquad (2.25)$$

The predicted signal of equation (2.24) is obtained as a correction to observations which are set to zero initially through the null-vector in (2.23a). Hence, realising that the observations $\underset{\sim}{l}$ represent the observed anomalies $\underset{\sim}{\Delta g}$, the required value of gravity anomaly at a prediction point and its variance are given as:

$$\underset{\sim}{\Delta g}_p = \underset{\sim}{C}_{s_p s}(\underset{\sim}{C}_s + \underset{\sim}{C}_v)^{-1}\ \underset{\sim}{\Delta g} \qquad (2.26)$$

and

$$\underset{\sim}{\sigma}^2_{\Delta g_p} = \underset{\sim}{C}_s - \underset{\sim}{C}_{s_p s}(\underset{\sim}{C}_s + \underset{\sim}{C}_v)^{-1}\underset{\sim}{C}_{s_p s}^T \qquad (2.27)$$

where: $\Delta g_p, \sigma_{\Delta g_p}$ = predicted gravity anomaly at a prediction point, and its variance

$\underset{\sim}{\Delta g}$ = vector of observed gravity anomalies

$\underset{\sim}{C}_v$ = covariance matrix of measuring errors

$\underset{\sim}{C}_s$ = covariance matrix of the residual component s

and $\underset{\sim}{C}_{s_p s}, \underset{\sim}{C}^T_{s_p s}$ = cross covariance matrices between s at observation points and $\underset{\sim}{s}_p$ at prediction points; T denotes matrix transpose.

It is seen in equation (2.26) that the errors in the observations are taken into account for the prediction. The vector $\underset{\sim}{s}_p$ in (2.24) may consist of one component (as the predicted Δg_p at a prediction point) or it may consist of several components. The result for the same signal quantity will always be the same since (Moritz, 1972) the prediction is dependent only on the observations, and each component is determined independent of the others.

Local covariance functions computed for Canada by Schwarz and Lachapelle (1979) are used in the sequel to compute the covariance matrices $\underset{\sim}{C}_s$ and $\underset{\sim}{C}_{s_p s}$. The functions were computed using available gravity anomaly data for Canada. The model function used is an isotropic harmonic function, the general form of which is given as: (Schwarz and Lachapelle, 1979)

$$C(P,Q) = A' \sum_{N=0}^{\infty} c_N \left\{ \frac{R_B^2}{r_p r_q} \right\}^{N+2} P_N(\cos\psi) = C(\psi) \qquad (2.28)$$

where P, and Q are two points in space with radius vectors

r_p and r_q, ψ is the angle between the radius vectors, $P_N(\cos\psi)$ are legendre's polynomials, R_B is the radius of the Bjerhammar sphere, and C_N are anomaly degree variances defined as:

$$C_N = \frac{N - 1}{(n-2)(n+D)} \qquad (2.29)$$

where D is a constant integer describing the structure of the anomaly degree variances. Its value is chosen such that the summation of C_N for N = 3 to some arbitrarily large number (Delikaraoglou, 1976) would yield the point anomaly variance C_0 which is the covariance $C(\psi)$ when $\psi = 0$.

Two covariance functions were computed — one for the Rocky Mountains and another for the rest of Canada. In the Rocky Mountains, linear correlation with height was removed (Lachapelle and Schwarz, 1979) from the free-air gravity anomalies, prior to the numerical evaluation of the covariance function, by computing for regression parameters (slope and intercept) through a least squares fit of the data in the area. "Height independent" gravity anomalies Δg^* are given as:

$$\Delta g^* = \Delta g_i - bH_i \qquad (2.30)$$

where: Δg^* = "height independent" gravity anomaly of i^{th} observation point

Δg_i = observed free-air gravity anomaly of i^{th} observation point

b = slope of regression line.

and H_i = Height of the i^{th} observation point.

The "height independent" anomalies Δg^* are used as data for the prediction. The free-air anomaly at the prediction point is, therefore, obtained as:

$$\Delta g_p = \Delta g_p^* + bH \qquad (2.31)$$

where Δg_p = predicted free-air gravity anomaly at a prediction point
Δg_p^* = predicted "height independent" gravity anomaly at a prediction point
H = height of prediction point
and b is as defined above.

2.2.1 COVARIANCE FILTER

The prediction of gravity anomalies at computation points can be regarded as a filtering process where the predicted values represent the filtered data. The relevant expression for the filtered data is equation (2.26) namely:

$$\underset{\sim}{\Delta g}_p = \underset{\sim}{C}_{s_p s}(\underset{\sim}{C}_s+\underset{\sim}{C}_v)^{-1}\underset{\sim}{\Delta g} \qquad (2.32)$$

or

$$\underset{\sim}{\Delta g}_p = \underset{\sim}{F}^* \underset{\sim}{\Delta g} \qquad (2.33)$$

Where $\underset{\sim}{F}^*=\underset{\sim}{C}_{s_p s}(\underset{\sim}{C}_s+\underset{\sim}{C}_v)^{-1}$ is the filter which may be called a covariance filter (Vanicek et al, in prep, Ch. 14). The filtered data of equation (2.32) consists of only one element. If predictions were made, say, at three points 1, 2, 3; and assuming that the original data $\underset{\sim}{\Delta g}$ consists of five anomalies corresponding to the points a - e in Figure 3, the expression for the covariance filter is obtained in matrix form as follows:

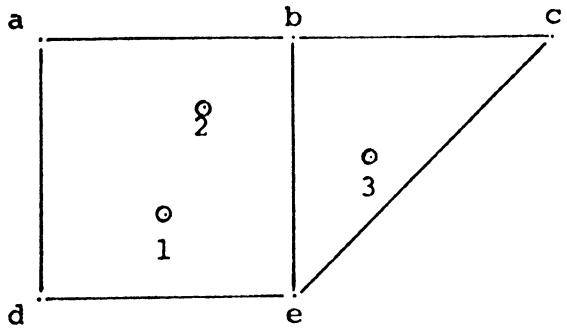

Figure 3

Predictions at three points 1, 2, 3

from equation (2.32)

$$\begin{bmatrix}\Delta g_1\\ \Delta g_2\\ \Delta g_3\end{bmatrix} = \begin{bmatrix}C_{1a} & C_{1b} & C_{1c} & C_{1d} & C_{1e}\\ C_{2a} & C_{2b} & C_{2c} & C_{2d} & C_{2e}\\ C_{3a} & C_{3b} & C_{3c} & C_{3d} & C_{3e}\end{bmatrix} \begin{bmatrix}P_{aa} & P_{ab} & P_{ac} & P_{ad} & P_{ae}\\ P_{ba} & P_{bb} & P_{bc} & P_{bd} & P_{be}\\ P_{ca} & P_{cb} & P_{cc} & P_{cd} & P_{ce}\\ P_{da} & P_{db} & P_{dc} & P_{dd} & P_{de}\\ P_{ea} & P_{eb} & P_{ec} & P_{ed} & P_{ee}\end{bmatrix} \begin{bmatrix}\Delta g_a\\ \Delta g_b\\ \Delta g_c\\ \Delta g_d\\ \Delta g_e\end{bmatrix}$$

$$\underset{\sim}{\Delta g}_p = \underset{\sim}{C}_{s_p s} \left(\underset{\sim}{C}_s + \underset{\sim}{C}_v\right)^{-1} \underset{\sim}{\Delta g}$$

$$\underset{\sim}{\Delta g}_p = \underset{\sim}{F}^* \underset{\sim}{\Delta g}$$

(2.34)

where:

$C_{ij}(i=1,2,3;j=a,b..)$ = elements of $\underset{\sim}{C}_{s_p s}$ matrix expressing correlations between prediction and data points

$P_{jk}(j,k=a,b,\ldots,e)$ = elements of the inverted matrix $(\underset{\sim}{C}_s+\underset{\sim}{C}_v)^{-1}$ expressing correlations between pairs of data points.

It is seen that the filter $\underset{\sim}{F}^{*}$ is composed of two matrices — one $\underset{\sim}{C}_{s_p s}$ expressing correlations between prediction and data points, and the other $(\underset{\sim}{C}_s+\underset{\sim}{C}_v)^{-1}$ expressing correlations between pairs of data points in terms of inverses of covariances. The dimensions of the former are dependent on the numbers of prediction and data points; while those of the latter are dependent on the number of data points alone. Hence, the matrix $(\underset{\sim}{C}_s+\underset{\sim}{C}_v)^{-1}$ remains the same for predictions regardless of how many prediction points there are, for as long as the data set is the same.

2.3 WEIGHTED MEANS

The method of weighted means has been used to some extent for height predictions (Davis, 1973). In constructing contour maps for instance, the first step usually is to produce regular grid data from a data set not regularly distributed. Estimates of heights at the grid points are known to have been derived from the nearest observations or from trend surfaces. The former approach (Davis, 1973) is likely to produce erratic values which often results in discontinuities in the maps, while the latter approach though it produces a smooth and continuous surface, has the disadvantage that none of the original observations are likely to be on the surface. The method of weighted means has, therefore, been recommended for its simplicity and it has been so used successfully. Kearsley (1977) has used the same technique in the estimation of geoidal undulations, while Sjoberg (Kearsley, 1977) used it for

gravity anomalies' prediction.

The method predicts a gravity anomaly at a prediction point by taking a weighted mean of the nearest observations surrounding the point. Weights are assigned to the observations inversely proportional to the distances d of the observations from the prediction points, raised to some power ν . The mathematical model is given by (Kearsley, 1977)

$$\Delta g_p = \frac{\sum_{i=1}^{n} (\Delta g_i / d_{Pi}^{\nu})}{\sum_{i=1}^{n} (1/ d_{Pi}^{\nu})} \qquad (2.33)$$

where

Δg_p = predicted gravity anomaly of a prediction point
Δg_i = gravity anomaly of the i^{th} observation point
d_{Pi} = distance of i^{th} observation point from the prediction point
ν = exponent of d for the weights
and n = number of observations

Kearsley (1977) estimated the errors of prediction from two uncorelated sources – (i) the error due to the observations (σ_e), and (ii) the error of representation of the weighted mean (σ_r) – the total prediction error being given by:

$$\sigma^2_{\Delta g_p} = \sigma^2_e + \sigma^2_r \qquad (2.34)$$

where:

$\sigma^2_{\Delta g_p}$ = variance of predicted gravity anomaly

The error due to the observations (σ_e) was obtained through the variance of the mean Δg_p with the observations used to generate it;

$$\sigma^2_e = \frac{\sum_{i=1}^{n} (1/d_{Pi}^{\nu})(\Delta g_p - \Delta g_i)^2}{(n-1)\sum_{i=1}^{n} (1/d_{Pi}^{\nu})} \qquad (2.35)$$

While the error of the representation of the mean σ_r was obtained as a function of the distances d

$$\sigma_r^2 = c' \sum_{i=1}^{n} (1/d_{Pi}) \qquad (2.36)$$

where the value of c'had to be experimentally determined. Equation (2.35) will succeed in giving us information about the ruggedness of the data used, but neither of the two contributing error functions considers the errors in the observations which are known to us. It is felt, therefore, that it would be adequate enough for our purpose to propagate the errors in the observations through the mathematical model equation (2.33) rather than use (2.34). The error of prediction is, therefore, obtained by propagating the variances of the observed values as:

$$\sigma^2_{\Delta g_P} = \left(\frac{\partial \Delta g_P}{\partial \Delta g_i} \right)^2 \sigma^2_{\Delta g_i} \qquad (2.37)$$

where:

$\sigma^2_{\Delta g_i}$ = variance of i^{th} observed gravity anomaly.

If equation (2.33) is substituted into (2.37) and if the partial derivatives are taken with respect to the observed values, equation (2.37) becomes:

$$\sigma^2_{\Delta g_P} = \frac{\sum (1/d^{\nu}_{Pi})^2 \sigma^2_{\Delta g_i}}{(\sum 1/d^{\nu}_{Pi})^2} \qquad (2.38)$$

The method of weighted means has been known to perform very well (Kearsley, 1977) in areas where data is comparatively dense. When the data coverage is sparse, it has sometimes given unacceptable results. It is, however, a simple prediction method which requires very little in terms of programming skill.

2.3.1 PREDICTION BY WEIGHTED MEANS AS A FILTERING PROCESS

If one is prepared to accept as filtered data the values of gravity anomalies obtained at prediction points from using the method of weighted means, the prediction process can, then, be called a filtering process. The expression for the filter can be obtained from equation (2.33) namely:

$$\Delta g_P = \frac{\sum_{i=1}^{n} (\Delta g_i / d_{Pi}^{\nu})}{\sum_{i=1}^{n} (1/d_{Pi}^{\nu})} \qquad (2.39)$$

Let the weight $(1/d_{Pi}^{\nu})$ be represented by W_{pi}; then, (2.39) becomes:

$$\Delta g_P = \frac{\sum_{i=1}^{n} (W_{Pi}\ \Delta g_i)}{\sum_{n} W_{Pi}} \qquad (2.40)$$

and in matrix form:

$$\Delta g_P = \frac{1}{\sum_i W_{Pi}}\ \underset{\sim}{W}\ \underset{\sim}{\Delta g} \qquad (2.41)$$

or:

$$\Delta g_P = \underset{\sim}{F}^{**}\ \underset{\sim}{\Delta g} \qquad (2.42)$$

where: $\underset{\sim}{F}^{**} = \frac{1}{\Sigma W_{Pi}}\ \underset{\sim}{W}$ is the required linear filter

$\underset{\sim}{\Delta g}$ = original data at observation points

and: Δg_P = filtered data - predicted at a prediction point

If the example used in Section 2.2.1 is used here, equation (2.41) when written in full becomes: (see figure 3)

$$\begin{bmatrix} \Delta g_1 \\ \Delta g_2 \\ \Delta g_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\Sigma W_{1k}} & 0 & 0 \\ 0 & \frac{1}{\Sigma W_{2k}} & 0 \\ 0 & 0 & \frac{1}{\Sigma W_{3k}} \end{bmatrix} \begin{bmatrix} W_{1a} W_{1b} W_{1c} W_{1d} W_{1e} \\ W_{2a} W_{2b} W_{2c} W_{2d} W_{2e} \\ W_{3a} W_{3b} W_{3c} W_{3d} W_{3e} \end{bmatrix} \begin{bmatrix} \Delta g_a \\ \Delta g_b \\ \Delta g_c \\ \Delta g_d \\ \Delta g_e \end{bmatrix} \qquad (2.43)$$

or $\underset{\sim}{\Delta g}_P = \underset{\sim}{W^*} \quad \underset{\sim}{W} \quad \underset{\sim}{\Delta g} \qquad (2.44)$

or $\underset{\sim}{\Delta g}_P = \underset{\sim}{F^{**}} \quad \underset{\sim}{\Delta g} \qquad (2.45)$

where:

$\underset{\sim}{\Delta g}_P$ = vector of filtered data

$\underset{\sim}{\Delta g}$ = original data

$\underset{\sim}{W^*}$ = diagonal matrix with inverses of weights' summation as elements

$\underset{\sim}{W}$ = matrix of weights assigned to each data with respect to each prediction point

and $\underset{\sim}{F^{**}} = \underset{\sim}{W^*}\ \underset{\sim}{W}$ is the linear filter.

The filter is seen, therefore, to be composed of two matrices $\underset{\sim}{W^*}$ and $\underset{\sim}{W}$. The matrix $\underset{\sim}{W^*}$ is diagonal with elements which are inverses of summations of weights, where the element in the (1,1) position is the summation of the weights assigned to each data (a-e) with respect to the prediction point 1. The second matrix $\underset{\sim}{W}$ is a matrix of weights, each row containing the weights assigned to each data point with respect to a corresponding prediction point. A weight is assigned inversely proportional to the distance between a prediction point and a data point raised to some power ν :

$$W_{Pi} = 1/d_{Pi}^{\nu} \qquad (2.46)$$

If this filter is compared with the covariance filter (equation (2.34)) it can be seen that a matrix similar in content to the $(\underset{\sim}{C}_s + \underset{\sim}{C}_v)^{-1}$ matrix is absent here. The functions of the two filters are however similar, since they both produce the filtered data $\Delta\underset{\sim}{g}_p$ from the original data $\Delta\underset{\sim}{g}$. The filter from the weighted mean method assigns weights to data such that the closest points are required to contribute most to the filtering process.

2.4 COMBINATION OF LEAST-SQUARES SURFACE FITTING AND LEAST-SQUARES COLLOCATION METHODS

Gravity anomalies' prediction in the Rocky Mountains is problematic due to the ruggedness of the gravity field (free-air anomaly values range between - 180 mgals and + 125 mgal) and its strong correlation with heights. When a smoother gravity field is required, the corresponding Bouguer values are used. Hence, these are the values used with the least-squares surface fitting technique and the method of weighted means. In order for a successful application of the collocation method in this terrain, the correlation between the free-air anomalies and heights had to be removed from the data through the use of a regression line (Section 2.2). The parameters (slope and intercept) of the line are then used to reduce the data to "height independent" gravity anomalies before a prediction is effected. This is equivalent to fitting a plane to the gravity data and treating the residuals from this as "height independent" anomalies, which are found to be

much smoother than the free air values.

A similar approach to this, but more rigorous, replaces the plane by a higher order surface and treats the residuals as above. This approach uses the least-squares surface fitting technique (section 2.1) to fit a surface to the data surrounding the prediction point, takes the resulting residuals and inputs them as data into a least-squares collocation method (Section 2.2). The latter prediction is, therefore, used to refine that from the former. This technique is used on the smoother Bouguer anomaly values.

The covariance functions used to compute the necessary covariance matrices for the collocation method (to refine the first prediction) are computed from available data by using a polynomial approximation, to the analytical covariance functions:

$$C(d) = c_0 \{ 1+(\frac{d}{s}) \}^{-1} \quad (2.3)$$

which is given as (Merry, 1979).

$$C(d_1) = \sum_{i=1}^{3} c_i \, d^i \quad (2.39)$$

where:

$C(d), C(d_1)$ = covariance expressed as functions of distances d, d_1

c, s = coefficients to be determined

and d, d_1 = separation (distance) between two points in space.

The covariance function is isotropic, and its value does not depend on the position of points in the region but only on the separation d between pairs of points.

The data for covariance computations are residual anomalies $\Delta\underset{\sim}{g}^{**}$ obtained from fourth order polynomial surface fit to 2^{o} X 2^{o} gravity data blocks.

The covariance value $C(d_1)$ for a distance d_1 is obtained as a mean of the products of all possible pairs of residual anomalies $\Delta\underset{\sim}{g}^{**}$at distance d_1 apart, and can be expressed as:

$$C(d_1) = E\{\Delta g_1^{**}, \Delta g_2^{**}\} \quad (2.40)$$

where:

$\Delta g_1^{**}, \Delta g_2^{**}$ are residual anomalies at any two points in space at distance d_1 apart and E is mathematical expectation (Linear operator). Thirteen covariance values are obtained in this way for thirteen distances from 0' to 60' both inclusive, and the corresponding observation equations are written according to (2.39). Since only four coefficients are required, we are again faced by an overdetermined problem. The coefficients are obtained by using the method of least-squares to solve the thirteen equations.

CHAPTER 3

TERRAIN TYPES AND GRAVITY DATA

3.1 TERRAIN TYPES

The prediction methods chosen for the evaluation are to be tested in three kinds of terrain in Canada which correspond to flat, gently rolling and mountainous terrain types. It is necessary, therefore, to use some criterion to help identify the different kinds of terrain. The criterion used is as follows:

A flat terrain is considered as any area where the height range is 0-1500 meters; while a rolling terrain is considered as any area with heights ranging from 1500 meters (or less) to 2500 meters; and a mountainous terrain as any area with heights above 2500 meters. This is only used as a guide since these areas do overlap. Hence, the Maritime Provinces, Quebec, Ontario, Manitoba and Saskatchewan are considered to have flat terrain type; while Alberta and part of British Columbia are considered to have the rolling terrain type; and the Rocky Mountains in British Columbia considered to have the mountainous terrain type. Gravity data exist on tapes and disks for all the three terrain situations. However, smaller files which would hold data in smaller blocks ($5^{o} \times 5^{o}$, say) need to be created from the tapes or disks for easy handling.

3.2 GRAVITY DATA

The gravity data used in this work are those extracted from the gravity file supplied by the Gravity Division of the Department of Energy, Mines, and Resources, Ottawa (Nassar and Vaníček, 1975). They consist of gravity values for about 110,000 stations in Canada, and are residing on seven files (for geographically overlapping areas). Each file consists of records filed sequentially; and each record (for one gravity station) contains, geographical coordinates (ϕ,λ), observed gravity, free-air gravity anomaly, elevation, and elevation accuracy of the station (Nassar and Vaníček, 1975(Appendix II)).

Free-air anomalies, from the files, are used in the least-squares collocation method since, the covariance functions for the technique were derived using free-air anomalies (Schwarz and Lachapelle, 1979). However, since Bouguer anomaly values are smoother than free-air anomalies, they are used in the least-squares surface fit, and weighted mean methods. Bouguer anomalies and their accuracies are obtained, from the gravity data described above, as (Vaníček et al, 1972; Vaníček and Krakiwsky, in prep)

$$\Delta g^B = \Delta g^F - 0.1119\ h \qquad (3.1)$$

and

$$\sigma^2_{\Delta g^B} = (0.05)^2 + (0.1967\ \sigma_h)^2 \qquad (3.2)$$

where:

Δg^B ; $\sigma^2_{\Delta g}B$ = Bouguer anomaly and its variance

Δg^F ; $\sigma^2_{\Delta g}F$ = free-air anomaly and its variance

h ; σ^2_h = height (of station in metres) and its variance

and the gravity gradient (0.1119) is in mgal/metre. The corresponding expression for the variance of free-air anomaly is,

$$\sigma^2_{\Delta g}F = (0.05)^2 + (0.3086\ \sigma_h)^2 \qquad (3.3)$$

which is seen to be made up of two terms; the first accounting for measurement error, and the second being a function of the error in height.

CHAPTER 4

STATISTICAL TESTING OF PERFORMANCES OF INDIVIDUAL TECHNIQUES

The approach followed for evaluating the performances of individual techniques is equivalent to measuring a quantity (the difference between observed and predicted anomaly) whose value is known (mathematical expectation equals zero) with a technique of unknown accuracy (Vaníček and Krakiwsky, in prep, Ch. 13). One can, therefore, determine and compare estimates of accuracies of the techniques for the purpose of the evaluation.

In this chapter, the test statistics needed for the evaluation are defined. The statistical tests used to assess the observations and to analyse the results of predictions are described.

4.1 TEST STATISTICS

Test statistics are the various measures that will be used to evaluate the performances of the prediction methods based on the differences obtained between predicted and observed gravity anomalies. In this section the test statistics to be defined include (1) the difference $\delta\Delta g$ between the predicted and observed gravity anomaly at a prediction point (2) the standard deviation $\sigma_{\delta\Delta g}$ of the difference, (3) the mean of differences $\delta\underset{\sim}{\Delta g}$

in a sample and (4) the variance S^2 of the mean.

4.1.1 THE DIFFERENCE ($\delta\Delta g$) BETWEEN OBSERVED AND PREDICTED ANOMALIES

The difference $\delta\Delta g$ between the observed and predicted gravity anomalies at a prediction point is given by:

$$\delta\Delta g = \Delta g_{obs} - \Delta g_P \qquad (4.1)$$

where

Δg_{obs} = observed gravity anomaly

Δg_P = predicted gravity anomaly

If the prediction technique were perfect, the difference $\delta\Delta g$ would ideally be zero. But, it would seldom be so because the prediction method is not perfect, and the observed value is not without error. It is, however, the best measure we have of how well the prediction method has performed in predicting the gravity anomaly at one point.

4.1.2 THE STANDARD DEVIATION $\sigma_{\delta\Delta g}$ OF THE DIFFERENCE

The standard deviation $\sigma_{\delta\Delta g}$ of the difference $\delta\Delta g$ is obtained as a quadratic sum of the standard deviations $\sigma_{\Delta g_P}, \sigma_{\Delta g_{obs}}$ of the predicted and observed anomalies respectively. It is given as:

$$\sigma_{\delta\Delta g} = \sqrt{(\sigma^2_{\Delta g_P} + \sigma^2_{\Delta g_{obs}})} \qquad (4.2)$$

where:

$\sigma_{\Delta g_P}$ = standard deviation of predicted anomaly

$\sigma_{\Delta g}$ = standard deviation of observed anomaly.

More will be said about this measure in Section 4.3.5.

4.1.3 MEAN DIFFERENCE $\overline{\delta\Delta g}$

The mean difference $\overline{\delta\Delta g}$ is the mean of the differences $\underset{\sim}{\delta\Delta g}$ generated by a prediction method in a particular terrain, and is given by the expression:

$$\overline{\delta\Delta g} = \frac{1}{n} \sum_{i=1}^{n} \delta\Delta g_i \qquad (4.3)$$

where

$\delta\Delta g_i$ = difference between predicted and observed anomaly of the ith sample point.

This measure is discussed in greater detail in 4.3.4.

4.1.4 STANDARD DEVIATION S OF OBSERVATIONS ABOUT THE MEAN

The standard deviation S of the differences $\delta\Delta g$ about the mean difference $\overline{\delta\Delta g}$ is given by:

$$S = \sqrt{\left\{\frac{1}{(n-1)} \sum_{i=1}^{n} (\delta\Delta g_i - \delta\Delta g_{obs})^2\right\}} \qquad (4.4)$$

Since the mean difference $\overline{\delta\Delta g}$ is an estimate of the population mean , one degree of freedom is given up for using the same observations again. Hence, the use of the factor $1/(n-1)$ in (4.4). The population mean μ of the differences is known to be zero. If this is used to compute, equation (4.4) becomes:

$$S^* = \sqrt{\left\{\frac{1}{n} \sum \delta\Delta g_i^2\right\}} \qquad (4.5)$$

The two standard deviations S, S* are the same quantities. The expression for S is used when the population mean is

not known and has to be estimated from the observations.

The standard deviation S is a measure of the dispersion of the differences about the mean. A small value of S indicates that the variables are clustered tightly around the mean, while a large value indicates a wide scatter about the mean. A prediction method with the smallest $\overline{\delta\Delta g}$ and S values is considered to have given the best performance in whatever terrain situation is being considered.

4.2 THE DIFFERENCE $\delta\Delta g$ AS A STATISTICAL SAMPLE

The purpose in this work as stated in Chapter 1 is to evaluate the performances of the chosen prediction methods; and the differences $\delta\Delta g$ between observed and predicted anomalies provide the necessary information for the evaluation (see 4.1.1). The quantity $\delta\Delta g$ is generated every time a prediction method is used to predict at a sample point. If the sample points are selected randomly (see 5.1.1), the quantities $\delta\Delta g$ can be regarded as random variables, and the sample of these quantities obtained from using a prediction method in a particular terrain as a random sample.

4.2.1 BASIC POSTULATE

Any random sample is considered to belong to a population infinite in size (Vaníček, 1974), and the distribution of the population is usually postulated for the random sample. A very large number of random

variables observed in nature are known to possess a normal probability distribution (Mendenhall, 1979). Hence, the PDF for the population of most random variables is usually the normal. The postulated PDF for the sample of differences $\delta\Delta g$ is, therefore, the normal PDF. It is necessary to test this postulate for statistical validity, so that any inferences that may eventually be made based on it would be justifiable. But, before going into that, one has to ensure that the observations are free of outliers.

4.2.2 TESTING FOR OUTLIERS

This test examines each observation $\delta\Delta g$ separately and tries to check how statistically compatible it is with the remaining observations. The null hypothesis (H_0) to be tested, therefore, is that the observation $\delta\Delta g_i$ belongs to a sample with PDF $N(\delta\Delta g;\mu,S^2)$, where μ is the population mean and S^2 is the sample variance. (Vaníček and Krakiwsky, in prep (Ch. 13)).

The test computes a statistic y which is later compared with a t-distribution (t_{df}) with degrees of freedom df at some level of significance (usually 5%). The statistic y can also be used with the t-distribution to obtain a confidence interval within which the observation should be. The statistic y and the (1-α) confidence interval are given by:

$$y = \frac{\delta g_i - \mu}{S} \qquad (4.6)$$

$$\mu - St_{n-1,\alpha/2} < \delta\Delta g_i < \mu + St_{n-1,1-\alpha/2} \qquad (4.7)$$

where

S = sample standard deviation

μ = population mean

$t_{n-1,\frac{\alpha}{2}}$ = t-distribution with (n-1) degree of freedom and at $\frac{\alpha}{2}$ level of significance

and n = sample size.

If the observation falls outside the confidence interval, it is rejected and removed from the sample.

The test described so far is for "out of context" testing in which an individual observation is tested while the existence of the other members of the series is disregarded. The "within context" testing on the other hand tests each observation within the context of the observation series. The necessary confidence interval is obtained by multiplying the standard deviation S by a factor obtained through a new statistic - max y which (Vanicek et al, in prep (Ch. 13)) accounts for the interplay of the individual members of the series. The factor can be scaled from a graph of multiples of standard deviation against degrees of freedom (Vaníček and Krakiwsky, in prep) prepared using the max y statistic. The confidence interval corresponding to (4.7) for 100 degrees of freedom is, therefore, given by:

$$\mu - 3.4S < \delta\Delta g_i < \mu + 3.4S \qquad (4.8)$$

where the factor scaled from the graph is 3.4.

4.2.3 STANDARDIZATION OF $\delta\Delta g$ ($\delta\Delta g \rightarrow t$)

The outcome t of the standardization of the difference $\delta\Delta g$ is given by:

$$t = \frac{\delta\Delta g}{\sigma_{\delta\Delta g}} \qquad (4.9)$$

where:

$\delta\Delta g, \sigma_{\delta\Delta g}$ = difference between observed and predicted anomaly and its standard deviation

If $\delta\Delta g$ was a random variable with mean value and variance ($\overline{\delta\Delta g}, \sigma^2_{\delta\Delta g}$), the outcome t would also be a standardized random variable with mean value and variance (μ_t, σ_t^2) (Vaníček, 1974), However, the difference $\delta\Delta g$ is postulated to be a normal variable. Hence, the outcome t should be a standard normal variable with the PDF $N(t; 0,1)$. This can also be tested.

The mean $\bar{t}$ and variance s_t^2 of the standardized differences are given by:

$$\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i \qquad (4.10)$$

and

$$s_t^2 = \frac{1}{n} \sum_{i=1}^{n} (t_i - \mu_t)^2 \qquad (4.11)$$

where:

$\bar{t}, s_t^2$ = mean and variance of t

μ_t = population mean of t

and n = sample size

4.3 TESTING THE BASIC POSTULATE

4.3.1 TEST FOR THE MEAN $\overline{\delta\Delta g}$ OF DIFFERENCES $\delta\Delta g$ (or $\bar{t}$)

The mean $\overline{\delta\Delta g}$ of differences $\delta\underset{\sim}{\Delta}g$ is given by equation (4.3) namely:

$$\overline{\delta\Delta g} = \frac{1}{n} \sum_{i=1}^{n} (\delta\Delta g)_i \qquad (4.12)$$

where:

$(\delta\Delta g)_i$ = difference between observed and predicted anomalies for the i^{th} sample point.

It is an unbiased estimator of the population mean which is known to be zero. Hence, ideally the mean $\overline{\delta\Delta g}$ should be equal to zero. But, it is seldom so.

The test on $\overline{\delta\Delta g}$ (or $\bar{t}$), therefore, compares it with the population mean and tries to establish whether the difference between them ($\overline{\delta\Delta g}$, μ) is statistically significant or not. The null hypothesis (H_o) being tested is that the sample has the PDF-$N(\delta\Delta g; \mu, S^2)$.

The test procedure is similar to that described in section 4.2.2, but with y statistic and $(1-\alpha)$ confidence interval given as: (Vaníček and Krakiwsky, in prep, Ch. 13)

$$y = \frac{\overline{\delta\Delta g} - \mu}{S/\sqrt{n}} \qquad (4.13)$$

and

$$\overline{\delta\Delta g} - \frac{S}{\sqrt{n}} t_{n-1,\frac{\alpha}{2}} < \mu < \overline{\delta\Delta g} + \frac{S}{\sqrt{n}} t_{n-1,1-\frac{\alpha}{2}} \qquad (4.14)$$

where:

S^2 = sample variance
μ = population mean

$t_{n-1,\frac{\alpha}{2}}$ = t-distribution with (n-1) degree of freedom and $\frac{\alpha}{2}$ level of significance.

and n = sample size.

If the population mean falls outside the confidence interval (4.14), H_o is rejected, and the mean $\overline{\delta\Delta g}$ is statistically different from the population mean.

If $\bar{t}$ were the quantity to be tested, we only need to replace $\overline{\delta\Delta g}$ and S in (4.13) and (4.14) by $\bar{t}$ and S_t respectively.

4.3.2 TEST FOR STANDARD DEVIATION $\sigma_{\delta\Delta g}$ OF THE DIFFERENCE $\delta\Delta g$

The standard deviation $\sigma_{\delta\Delta g}$ of the difference $\delta\Delta g$ is given by equation (4.2) namely:

$$\sigma_{\delta\Delta g} = \sqrt{\left(\sigma^2_{\Delta g_{obs}} + \sigma^2_{\Delta g} \right)} \qquad (4.15)$$

where:

$\sigma_{\Delta g_{obs}}$ = standard deviation of observed anomaly

and $\sigma_{\Delta g}$ = standard deviation of predicted anomaly

It reflects the errors of both the predicted and observed values, and should be statistically compatible with the difference $\delta\Delta g$ if the prediction method is estimating its precision properly. The precision estimate may be optimistic or pessimistic. This can be determined through a test on the standard deviation $\underset{\sim}{\sigma}_{\delta\Delta g}$.

The test uses the outcomes $\underset{\sim}{t}$ of the standardization of the differences $\underset{\sim}{\delta\Delta g}$ by the corresponding standard deviations $\underset{\sim}{\sigma}_{\delta\Delta g}$. The outcomes $\underset{\sim}{t}$ are expected to have a standard normal PDF-n(t;0,1) (see 4.2.3); hence, the

sample mean $\bar{t}$ should be zero, and the sample variance should be unity if the standard deviations $\sigma_{\delta\Delta g}$ are properly computed. The null hypothesis (H_0'), therefore, is that the mean $\bar{t}$ of outcomes $\underset{\sim}{t}$ be not statistically different from zero, and that the sample variance S_t^2 be not statistically different from unity $(\sigma^2)^0$.

The mean $\bar{t}$ and variance S_t^2 are computed using equations (4.10) and (4.11). The test on mean $\bar{t}$ is carried out as in section 4.3.4. The y statistic and the $(1-\alpha)$ confidence interval for the test on the variance S_t^2 are given as: (Vaníček and Krakiwsky, in prep (Ch. 13))

$$y = \frac{nS_t^2}{(\sigma^2)^0} = \sum_{i=1}^{n} \left(\frac{t_i - \mu}{\sigma} \right)^2 \qquad (4.16)$$

and

$$\frac{nS_t^2}{\chi^2_{n,1-\frac{\alpha}{2}}} < (\sigma^2)^0 < \frac{nS_t^2}{\chi^2_{n,\frac{\alpha}{2}}} \qquad (4.17)$$

where:

$(\sigma^2)^0$ = unity (one)

If the variance $(\sigma^2)^0$ falls within the interval (4.17), the test passes, and the standard deviations $\sigma_{\delta\Delta g}$ are considered to be computed properly. If the test fails because the variance S_t^2 is too large, this is an indication that the precision estimates $\sigma_{\delta\Delta g}$ are optimistic. That is, they are much smaller than they should be as indicated by the value of the difference $\delta\Delta g$. If, on the other hand, the test fails because S_t^2 is too small, this is an indication that the precision estimates $\sigma_{\delta\Delta g}$ are pessimistic. The

values are much larger than they should be, and the method is in effect saying the predicted value is not good enough when, in actual fact, it is good as indicated by the difference $\delta\Delta g$.

Another approach to the test on $\sigma_{\delta\Delta g}$ is to determine a measure T as in the following section.

4.3.2.1 PERCENTAGE (T) OF ABSOLUTE STANDARDIZED DIFFERENCES $\underset{\sim}{t}$ LESS THAN UNITY

If the PDF of a variable is known, areas under the distribution curve within any specified range (Davis, 1973) can be calculated precisely. In the case of a standard normal PDF, the area under the curve within one standard deviation on either side of the mean line (Davis, 1973; Klugh, 1970(p. 50)) is approximately 68% of the total area (figure 4). Hence, for an observation series which is normally distributed, about 68% of the observations should have values greater than or less than the mean value by one standard deviation.

The standardized differences $\underset{\sim}{t}$ from 4.2.3 are expected to have a standard normal PDF. If the PDF is indeed the standard normal - $N(t; 0, 1)$ the percentage T of the absolute standardized differences $\underset{\sim}{t}$ with values less than unity should be approximately 68%. If T was less than or greater than 68% this is an indication that the standard deviations $\sigma_{\delta\Delta g}$ used to standardize $\delta\Delta g$ are not being computed properly. If T was less than 68%, $\sigma_{\delta\Delta g}$ are said to be optimistic; and if T was greater than 68%, they

are said to be pessimistic.

T is computed from the expression:

$$T = (\frac{n_1}{n})100\% \qquad (4.18)$$

where:

n_1 = number of absolute t less than unity

n = sample size

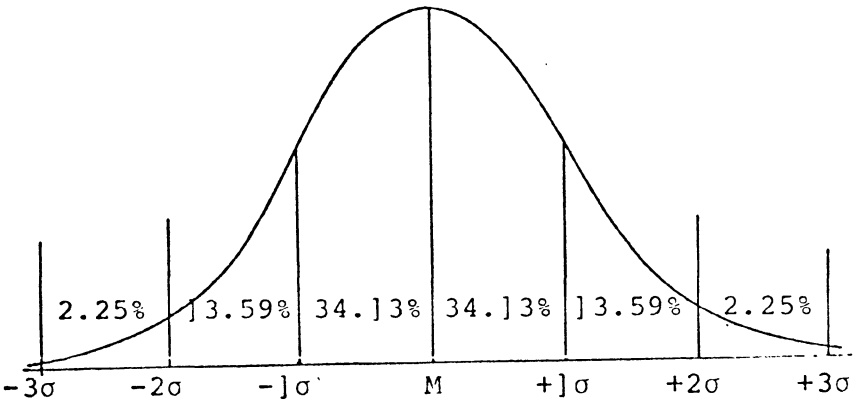

Figure 4

A normal distribution showing relative frequencies within given standard deviation distances from the mean (Klugh, 1970 (p. 50))

4.3.3 TEST FOR NORMALITY

The postulated PDF for the differences $\delta\Delta g$ is (4.2.1) the normal PDF - $N(\delta\Delta g;\mu,S^2)$. In order to test the correctness of this postulate, the differences $\delta\Delta g$ are standardized using the estimated population standard deviation ($S/\sqrt{n}$) (Snedecor et al, 1973 (p. 70)); and the outcomes Ξ are expected to have a standard normal PDF-$N(\Xi;0,1)$. Hence the chi-square goodness of fit test can be performed on the standardized differences Ξ given by:

$$z = \frac{\delta\Delta g - \mu}{S/\sqrt{n}} \qquad (4.19)$$

where n = is the sample size.

The null hypothesis (H_0) to be tested, therefore, is that the histogram of the quantities z is compatible with the standard normal PDF.

The test essentially computes a statistic y which is later compared with a chi-square (χ^2_{df})distribution with degrees of freedom df. The statistic y is computed according to the expression: (Vanicek et al, in prep. (Ch. 13))

$$y = \sum_{i=1}^{n_2} \frac{(a_i - e_i)^2}{e_i} \qquad (4.20)$$

where:

a_i = actual count for the i^{th} class/segment

e_i = theoretical count for the i^{th} class/segment

n_2 = number of class/segments

and y = chi-square statistic

One approach for obtaining the counts a_i, e_i is to group the standardized differences z into different classes and (a) to count the differences z in each class - actual count a_i; then, (b) to multiply the area of each class, as represented under the standard normal curve, into the total number of observations - theoretical count e_i. The approach used here (Davis, 1973) is to divide the area under the standard normal curve into equal segments - two,

three, four or ten equal segments - so that the theoretical count is the same for each. The corresponding limits for each segment are determined from a table of the standard normal distribution, and the actual counts are found using the limits.

The (1-α) confidence interval for the tested quantity is given by (Vanicek et al. in prep (Ch. 13)).

$$0 < y < \chi^2_{n_2-1,1-\alpha} \qquad (4.21)$$

The degrees of freedom is $(n_2 - 2)$ because the population mean of the differences is known. Otherwise, it would have had to be estimated along with the population standard deviation from the corresponding sample statistics ($\overline{\delta\Delta g}$, S) , and the degrees of freedom would have been $(n_2 - 3)$. The 95% confidence level is used in the test. If y does not fall within the interval defined in (4.21), the test fails, and the null hypothesis is rejected. According to Snedecor and Cochran (1973), this test may be described as a non-specific test, in that the test criterion is directed against no particular type of departure from normality. Hence, the observations may be skewed and χ^2 test may still pass. It is necessary, therefore, to supplement the χ^2 test with the tests for skewness and kurtosis.

4.3.4 TEST FOR KURTOSIS

If observations are grouped mostly around the mean, the distribution is said to be peaked or kurtic. A measure of peakedness or kurtosis (α_4) is computed according to the expression: (Freund, 1975; Snedecor and Cochran, 1973)

$$\alpha_4 = \frac{m_4}{(m_2)^2} \qquad (4.22)$$

where m_k (k=1, 2, 3, ...) is the k^{th} moment of the observations about their mean. m_k is given by:

$$m_k = \frac{\sum_{i=1}^{n} (\delta\Delta g_i - \overline{\delta\Delta g})^k}{n} \qquad (4.23)$$

If the observations were normally distributed the value of α_4 should be 3 (Snedecor and Cochran, 1973). If the computed value is greater than 3, the distribution is leptokurtic; and if less than 3 the distribution is platykurtic. These criteria are accurate for large sample sizes (greater than 200). In sample sizes less than 200, an alternative test criterion for kurtosis is (Snedecor and Cochran, 1973)

$$a_4 = \frac{\sum_{i=1}^{n} |\delta\Delta g_i - \overline{\delta\Delta g}|}{n\sqrt{m_2}} \qquad (4.24)$$

where a_4 is a new measure of kurtosis.

If $\delta\Delta g$ were normally distributed, the value of a_4 computed for the whole population should be (Snedecor and Cochran, 1973) 0.7979. Values lower than this indicate leptokurtosis while higher values indicate platykurtosis. Values of α_4 and a_4 are computed and compared (see Section 5.2.3). In very large samples which are normally distributed, $(\alpha_4 - 3)$ should be normally distributed with zero mean and standard deviation of $(24/n)$ (Snedecor et al, 1973).

4.3.4 TEST FOR SKEWNESS

If the distribution of the observations is not

symmetrical but has a pronounced tail on one side, it is said to be skewed. The distribution is negatively skewed if the lower (left side) tail is pronounced, and positively skewed if the upper (right hand) tail is pronounced. A measure of skewness α_3 is given by (Freund, 1973; Snedecor and Cochran, 1973).

$$\alpha_3 = \frac{m_3}{(\sqrt{m_2})^3} \qquad (4.25)$$

where m_k (k=1,2,3,...) is defined in equation (4.23). If the sample comes from a normal population, α_3 is approximately normally distributed with zero mean and standard deviation $\sqrt{(6/n)}$ (Snedecor and Cochran, 1973). This is accurate enough for sample sizes greater than 150. For sample sizes between 25 and 200, there are tables that could be consulted (at significance levels 5% and 1%) for critical values of α_3 (Snedecor et al, 1973 (Table A6)), called α_3^* in this work.

CHAPTER 5

COMPUTATIONAL CONSIDERATIONS AND RESULTS

5.1 COMPUTATIONAL CONSIDERATIONS

The intention in this section is to outline some computational details about the prediction methods which should be considered for effective applications. The first section, therefore, deals with such considerations which apply to all the methods. Subsequent sections treat such considerations which vary from one method to another.

5.1.1 SELECTION OF POINTS

It is essential to make the test situations the same or nearly so for all prediction methods. In this way, each method is given a fair chance, and bias is removed from the process of evaluation..

The samples for the evaluation are selected from three kinds of terrain in Canada - flat, gently rolling and mountainous. Hence, there are three samples in all. The sample for a particular terrain is made up of 100-150 points selected randomly (for the appropriate terrain type) from gravity files which, for each kind of terrain, are set up in (5^{o} X 5^{o}) blocks with a 30' overlap around each block. The files, when set up this way, are easier to handle. One has easier access to a small set of data

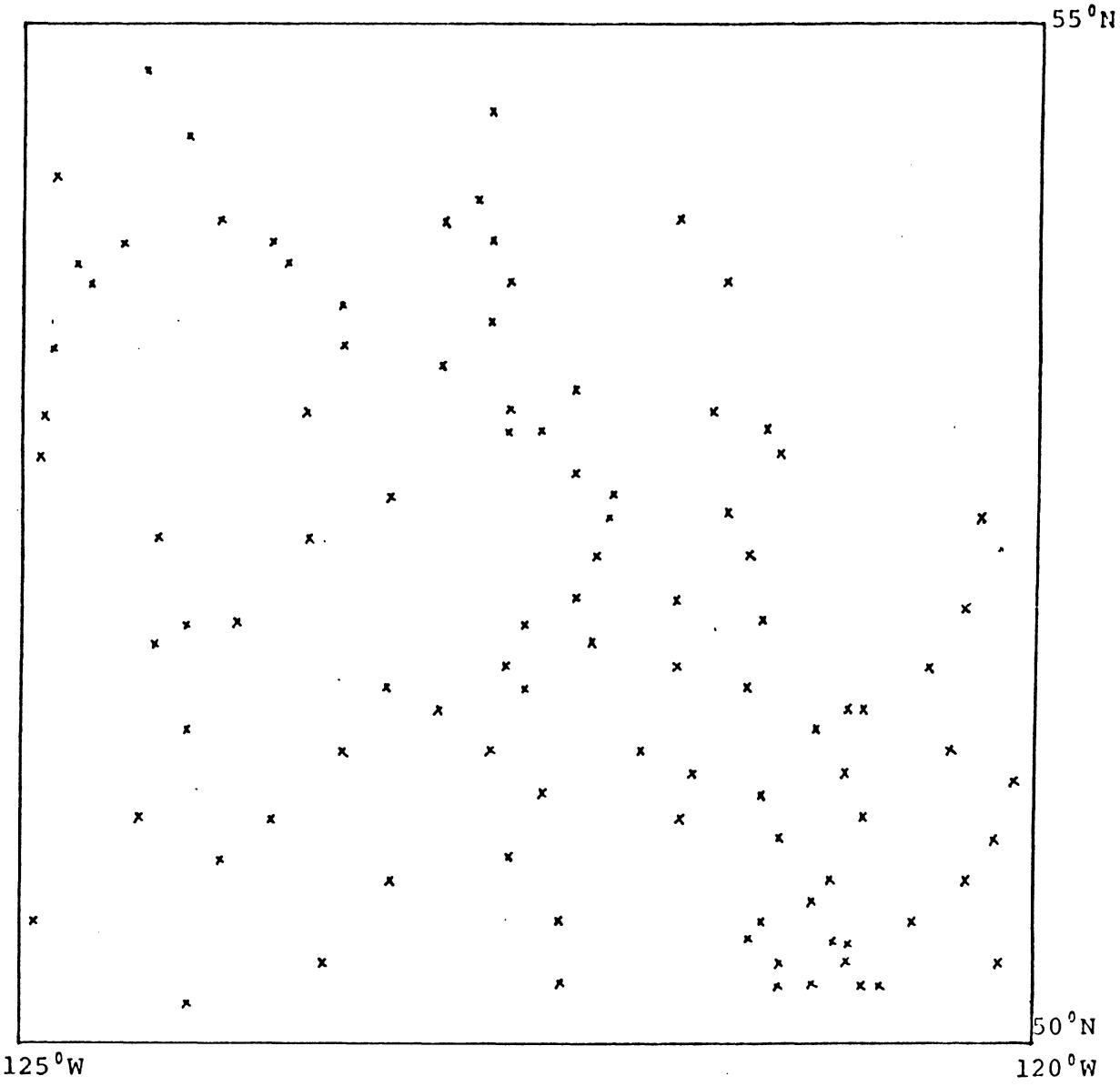

Figure 5

Distribution of Sample Data
within Mountainous Terrain

than if the Master file were to be used; and the prediction of a sample point at the edge of a block is made possible by the 30' overlap. The distribution of sample points in one terrain is shown in Figure 5.

The data for the predictions is assembled just prior to the prediction. The procedure followed for each method is to set up a 30' x 30' (or 1^{o} x 1^{o} in areas of sparse data) block with the prediction point at the centre, and to extract from file all available data for the block. Then, starting from the centre, points within a 5' ring are selected from the extracted data. The ring size is increased in steps of 5' each time (See figure 6) until enough points have been selected according to the maximum number of data points (50) specified, or alternatively, according to some other requirement depending on the method. In this way, the same set of data (or part of) are used to predict for the same sample point by all the methods.

The distribution of data around a prediction point is controlled by imposing the conditions that there be at least one data point in any three of the four quadrants around it; and that if there are clusters of points, to replace each by a weighted mean of the points constituting the cluster.

5.1.2 SURFACE FITTING TECHNIQUE

The choice of the overdetermined case is made for this technique (section 2.1) so that the method of least-squares can be used to solve for the vector of coefficients

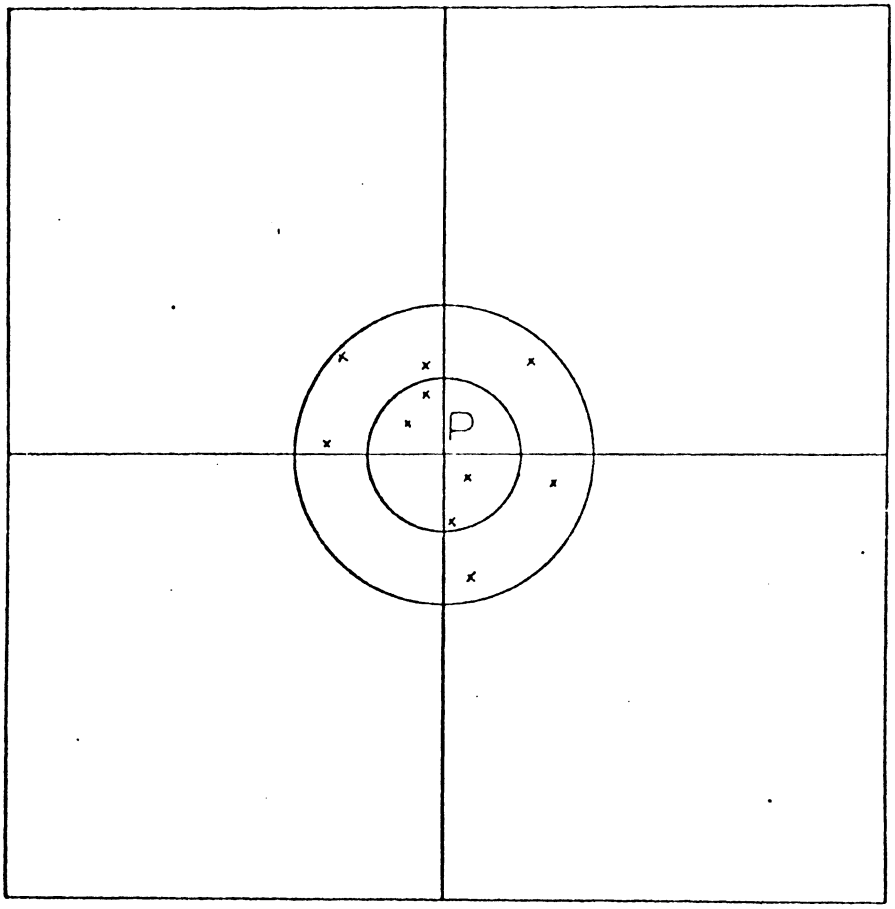

Figure 6
Selection of data for prediction

1. P is the prediction point. In the first 5' ring there are only four data points in two quadrants.
2. When ring size is increased by 5', six more points are picked up and the distribution improved.

$\underset{\sim}{c}$ in equation (2.7) namely:

$$\underset{\sim}{P}_n = \underset{\sim}{\Phi}\ \underset{\sim}{c} \qquad (5.1)$$

It is essential, therefore, that enough data be selected to give adequate degrees of freedom for the prediction based on the order of polynomial chosen, and, thus, the number of coefficients to be determined. The criterion used is that the degrees of freedom be not less than half the number of coefficients to be determined. The data selection, therefore, continues until this condition is fulfilled.

Three polynomial orders - 4th, 6th and 8th orders - are tested for this technique in the three kinds of terrain using a sample size of 50 points for each terrain. The results summarized in Table 1, indicate that the fourth order polynomial is quite adequate for our purpose. The results obtained for it are reasonable. While the higher orders give marginal improvements on the results, the extra CPU times required for the improvements are too high. It would be uneconomical to use them.

5.1.3 COLLOCATION

Two covariance functions - one for the flat and rolling terrain, and one for the mountainous terrain - are used for this technique. The covariance functions were computed by Schwarz and Lachapelle (1979), the model function being an isotropic harmonic function (see section 2.2). A program COVAX was adapted (COVASX) to compute the functions. It takes as input, some essential parameters and the spherical

Table 1

COMPARISON OF RESULTS FOR DIFFERENT ORDERS OF POLYNUMIAL

(SAMPLE SIZE = 50)

PREDICTION TECHNIQUE	FLAT (PRAIRIE)				ROLLING				MOUNTAINOUS (ROCKIES)			
	Mean Diff. mgal	RMS Diff mgal	T % < 1σ	CPU Time sec	Mean Diff. mgal	RMS Diff. mgal	T % <1σ	CPU Time sec	Mean Diff. mgal	RMS Diff. mgal	T % < 1 σ	CPU Time sec
9-term polynomial (4th order)	0.41	1.77	70	3.94	-0.86	3.92	51	2.38	0.22	6.30	47	2.50
16-term polynomial (6th order)	0.40	1.64	79	9.44	-1.47	4.59	40	8.00	0.15	5.81	43	8.45
25-term polynomial (8th order)					-0.61	3.95	51	20.58	1.03	6.36	48	20.71

distance between the pair of points, and outputs a covariance value as required. By using the appropriate essential parameters, either of the covariance functions may be so computed (Schwarz and Lachapelle, 1979).

5.1.4 COMBINED METHOD

The approximate function used to compute the covariance functions for this technique is defined in equation (2.39) namely:

$$C(d_1) = \sum_{i=0}^{3} C_i d_1^i \qquad (5.2)$$

Four coefficients constituting the vector $\underset{\sim}{C}$ are required. Two data blocks (see figure 7) are used to compute the coefficients. The results are summarized in Table 2, and the plots of the functions are shown in Figure 8.

5.2 RESULTS

The gravity anomaly values of selected sample points together with their standard deviations are extracted from gravity files to be used as observed quantities.

The gravity anomaly at a sample point is predicted using the data assembled as described in section 5.1.1. Using the same technique, this process is repeated until gravity anomalies have been predicted for every point in the sample. The process above is repeated for the remaining prediction methods, and that completes the test predictions for one sample. All of the above are, then, repeated for each sample to complete all test predictions.

Table 2

COEFFICIENTS FOR TWO COVARIANCE FUNCTIONS

IN THE ROCKY MOUNTAINS

$C(d) = c_0 + c_1 d + c_2 d^2 + c_3 d^3$					Sample Size = 105		
Covariance Function	c_0	c_1	c_2	c_3	Mean Difference	S	T % < 1σ
Block							
1	51.05	-228.81	375.10	-188.06	0.52	11.88	49
2	111.91	-625.62	1018.45	-497.89	0.50	11.82	63

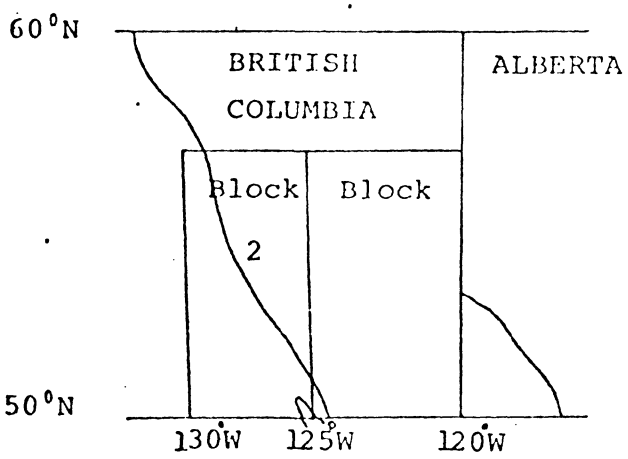

Figure 7

DATA BLOCKS IN THE ROCKIES

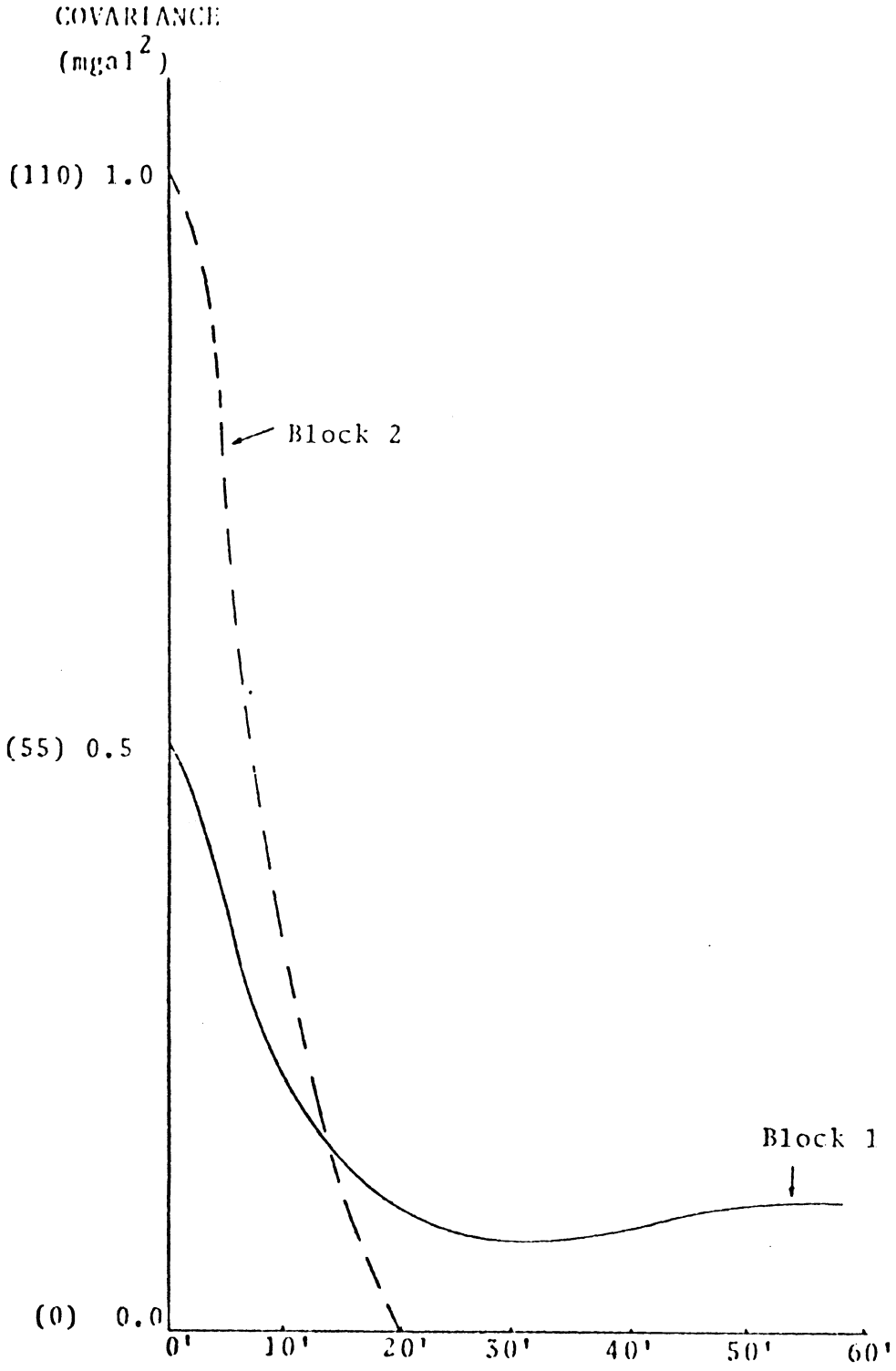

FIGURE 8

COVARIANCE FUNCTIONS FOR THE ROCKY MOUNTAINS

5.2.1 PREDICTIONS

The results of all predictions are summarized in Table 4. on page 64. The test statistics $-\overline{\delta\Delta g}$, S, and T - tabulated have been defined in Chapter 4. (4.1.3, 4.1.4, and 4.3.5.1) The CPU (Central Processing Unit) time spent actually computing the predicted anomaly,assuming that all information required was available is also computed. It does not include time spent selecting data for the prediction, nor the time spent computing covariance functions in the case of the collocation technique.

5.2.1.1 TESTING FOR OUTLIERS

The highest percentage of outliers detected using the "within context" test is less than 2% of the observations. The outliers are removed from the sample and the test statistics - $\overline{\delta\Delta g}$, S, T - recomputed before other tests are performed.

5.2.1.2 TESTING FOR BIAS

The null hypothesis in this test (Section 4.3.1) is that the mean difference $\overline{\delta\Delta g}$ is not significantly different from the population mean $\mu(\mu=0)$. The test passed at the 95% significance level for each sample and prediction method, except for the sample generated by the surface fit method in the rolling terrain which passed at the 99% significance level. The results are summarized in Table 3. The mean differences are, therefore, not statistically different from zero.

Table 3

RESULTS OF TEST FOR BIAS
AT 95% SIGNIFICANCE LEVEL

PREDICTION TECHNIQUES	FLAT(PRAIRIES)		ROLLING		MOUNTAINOUS(ROCKIES)	
	Mean Diff.	Test for Bias	Mean Diff.	Test for Bias	Mean Diff.	Test for Bias
	mgal		mgal		mgal	
1 Least-squares surface fit	0.01	Pass	-0.73	Fail	1.85	Pass
2 Least-squares Collocation	-0.32	Pass	0.20	Pass	0.58	Pass
3 Weighted Means	0.17	Pass	-0.68	Pass	0.54	Pass
4 Combination of 1 and 2					1.05	Pass

5.2.1.3 SURFACE FIT METHOD

The performance of the least-squares surface fit technique in the Prairies, with mean difference and standard deviation values of 0.01 and 2.40 mgals (Table 4), is reasonable. Its T value of 74% is greater than 68%. Hence, some of the technique's precision estimates ($\sigma_{\delta\Delta g}$) are pessimistic. That is, they are larger than expected as indicated by individual differences $\delta\Delta g$.

In the rolling terrain, the mean difference failed the test for bias at the 95% significance level, but passed at the 99% level. The standard deviation is, however, reasonable compared to the performances of the other techniques in this terrain. The T value of 40% is less than 68%. Some of the precision estimates are, therefore, optimistic.

In the Rocky Mountains, the mean difference obtained is not significantly different from zero, and the standard deviation compares very well with that of the collocation method (Table 4). The T value again indicates that the technique has computed some optimistic precision estimates in this terrain.

The trend shown in these results is that the performance of the technique worsened as we go across the table (Table 4) from the flat terrain to the mountainous terrain. This is attributed first to the fact that the density of gravity data coverage, in relative terms, is best in the Prairies and worst in the Rocky Mountains; and second, to the

fact that the degree of ruggedness of the data increases as we approach the mountainous terrain from the flat terrain. It is felt that the former contributes more than the latter, to the declining performance of the technique since there is not enough data to work with. It is also noticed that while the individual precision estimates in the Prairies are pessimistic, those in the rolling and mountainous terrains are optimistic. The estimates in the Prairies are, in fact, closer to reality than those in the other two kinds of terrain. This is again attributed to the two factors mentioned above.

The time of computation for the Prairies is seen to be about twice as high as in the other kinds of terrain. This is due to the data coverage too, and the maximum of 50 points specified (Section 5.1.1) is usually obtained in this terrain. The technique does not take much time in computing.

5.2.1.4 COLLOCATION METHOD

The least-squares collocation method gave a good performance in the flat terrain. The mean difference is not statistically different from zero, and the standard deviation is better than that obtained with the surface fit technique. Its T value indicates that its precision estimates are pessimistic. The time of computation in this terrain is higher than either of the times in the other kinds of terrain. The reason for this is as explained above. (Section 5.2.1.3)

In the rolling terrain, the technique gave the best performance, compared to the other two techniques. Its mean difference and standard deviation values are 0.20 and 3.94 mgals respectively. Its T value also indicates that it computes pessimistic precision estimates.

The performance of this technique in the Rocky Mountains is comparable with that of the surface fit method in the same terrain, though with a better mean difference of 0.58. Its T value is 57% indicating a little optimism in its precision estimates.

It is noted that, of the three techniques under study in this work, the least-square collocation takes the greatest CPU time for computations in any kind of terrain. It takes about twice as much time as the surface fit method and four times as much time as the weighted mean technique. Most of the excess time is spent in the inversion of the covariance matrix $(\underset{\sim}{C}_S + \underset{\sim}{C}_V)$ in equation (2.26) namely:

$$\Delta g_P = \underset{\sim}{C}_{s_P s} (\underset{\sim}{C}_s + \underset{\sim}{C}_v)^{-1} \underset{\sim}{\Delta g} \qquad (5.3)$$

The dimensions of this matrix are the same as the number of data (maximum = 50) used for the prediction. The only matrix inverted in the surface fit technique has dimensions equal to the number of coefficients (9 for 4th order polynomial) to be determined, while the weighted mean technique, if expressed in matrix form, has only a diagonal matrix to invert (Section 2.3.1, equation (2.43)).

Least-squares collocation technique has consistently performed well in the Prairies and rolling terrain. This

is attributed to the covariance function used which is considered very appropriate for Canada excluding the Rocky Mountains.

The trend noted above for the surface fit method, with regards to the decline in the performance of the prediction technique as we move across the table from flat to mountainous terrains, is also apparent here.

The collocation method is observed to be more realistic in estimating its precision.

5.2.1.5 WEIGHTED MEANS TECHNIQUE

The best performance in the Prairies is given by the weighted mean method with a mean difference and standard deviation values of 0.17 and 1.19 mgals. A T value of 82% indicates that some of the technique's precision estimates ($\sigma_{\delta\Delta g}$) are pessimistic.

In the rolling terrain, the performance of the technique is comparable with that of the surface fit technique and is quite reasonable. The T value indicates optimistic precision estimates.

In the Rockies, the technique gave the best performance with a mean difference of 0.54 mgals, and a standard deviation of 7.73 mgals. Again the precision estimates are optimistic.

The performances given by this technique, therefore, is surprising considering that the concept it uses is a simple one requiring no modelling of the gravity field.

It is the fastest of the three techniques being investigated. The decline in performance as the data coverage becomes more scarce, and the terrain (or data) becomes more rugged is apparent here too. It is seen to have the worst performance in estimating its precision in the rolling and mountainous terrains.

5.2.1.6 COMBINATION OF SURFACE FIT AND COLLOCATION METHODS

The mean difference and standard deviation values given by this technique which is applied only in the Rocky Mountains are reasonable. The mean difference is not significantly different from zero, and the standard deviation is improved compared to those given by the surface fit and collocation methods. The T value also indicates that the technique's precision estimates are very realistic. The time of computation, is, however, rather high, and the question which comes to mind is whether the improved performance is really worth the extra time spent, especially when the weighted mean method took one eighth of the time and gave better results with the same sample.

5.2.2 TESTING FOR OPTIMISM AND PESSIMISM

The test for the standard deviation $\sigma_{\delta\Delta g}$ is described in section 4.3.4. It uses the standardized differences $\underset{\sim}{t}$ expected to have a standard normal PDF – $N(t;0,1)$, and checked their means $\bar{t}$ and variances S_t^2 for statistical compatibility with the corresponding standard normal

Table 4

RESULTS OF ALL PREDICTIONS

PREDICTION TECHNIQUES	FLAT (PRAIRIES)				ROLLING				MOUNTAINOUS (ROCKIES)			
	Sample Size = 150				150				110			
	MEAN DIFF $\overline{\delta\Delta g}$	S S*	T	CPU TIME	MEAN DIFF $\overline{\delta\Delta g}$	S S*	T	CPU TIME	MEAN DIFF $\overline{\delta\Delta g}$	S S*	T	CPU TIME
	mgal	mgal	%	sec	mgal	mgal	%	sec	mgal	mgal	%	sec
LEAST-SQUARES SURFACE FIT	0.01	2.40 2.40	74	11	-0.73	4.42 4.47	40	6	1.85	10.76 10.87	37	5
LEAST-SQUARES COLLOCATION	-0.32	1.88 1.88	76	19	0.20	3.94 3.95	82	13	0.58	10.85 10.86	57	10
WEIGHTED MEANS	0.17	1.19 1.19	82	6	-0.68	4.44 4.47	21	3	0.54	7.73 7.71	18	2
COMBINATION OF 1 AND 2									1.05	8.94 8.89	63	16

parameters – 0,and 1. The results of the tests are summarized in Table 5. T values (Section 4.3.2.1) are also included.

The tests on means $\bar{t}$ passed at the 95% significance level except for two samples generated in the rolling terrain by the surface fit method and the weighted mean technique. Hence, but for these two samples, the means of outcomes t are not significantly different from zero. On the other hand, the tests on the variances S_t^2 failed for each sample and prediction method at the 95% level of significance.

From the standard deviation (S_t) values in table 5, it is seen how the outcomes $\underset{\sim}{t}$ are distributed around the mean value. A small value of S_t shows that they are grouped tightly around the mean. In effect, the quantities $\underset{\sim}{t}$ are closer to zero in absolute value. This indicates that the precision estimates $\sigma_{\delta\Delta g}$ are pessimistic. If on the other hand the standard deviation S_t is large, this shows that the quantities $\underset{\sim}{t}$ are larger, indicating that $\sigma_{\delta\Delta g}$is optimistic - that is, $\sigma_{\delta\Delta g}$ is smaller than expected. These indications are also seen in the T values which are greater than or lesser than 68% as $\sigma_{\delta\Delta g}$ are pessimistic or optimistic.

While the collocation method gave pessimistic precision estimates in the flat and rolling terrains, the estimates seem to be more realistic. The method performed best in estimating its precision in the three kinds of terrain. The method of weighted means gave the largest values of S_t in the rolling and mountainous terrains. Its precision

Table 5

RESULTS OF TESTS FOR OPTIMISM

AND PESSIMISM AT 95% SIGNIFICANCE LEVEL

PREDICTION TECHNIQUES	FLAT (PRAIRIES)			ROLLING			MOUTAINOUS (ROCKIES)		
	Sample size= 150			150			110		
	$\bar{t}$ Test on $\bar{t}$	S_t Test on S_t^2	T	$\bar{t}$ Test on $\bar{t}$	S_t Test on S_t^2	T	$\bar{t}$ Test on $\bar{t}$	S_t Test on S_t^2	T
	mgal	mgal	%	mgal	mgal	%	mgal	mgal	%
LEAST SQUARES SURFACE FIT	0.03 Pass	1.13 Fail	74	-0.49 Fail	2.61 Fail	40	0.24 Pass	3.60 Fail	37
LEAST SQUARES COLLOCAT-ION	-0.02 Pass	1.25 Fail	76	-0.03 Pass	1.74 Fail	82	0.01 Pass	2.05 Fail	57
WEIGHTED MEANS	0.31 Pass	1.23 Fail	82	-2.89 Fail	17.56 Fail	21	1.73 Pass	24.97 Fail	18

estimates are too optimistic.

5.2.3 TESTING FOR NORMALITY

The basic postulate for the differences $\delta\underset{\sim}{\Delta g}$, which is the normal PDF-$N(\delta\Delta g;\ \mu, S^2)$, is being tested through the standardized differences $\underset{\sim}{z}$, by checking the statistical compatibility of the histogram of $\underset{\sim}{z}$ with the standard normal PDF. The test failed for each sample and prediction method at the 95% significance level. The distributions may, therefore, be kurtic or skewed. Hence, measures of kurtosis and skewness are computed (see sections 4.3.4, 4.3.5). The results of these computations are summarized in Table 6.

There is no obvious trend that could be read into the values shown in Table 6 with regards to whether the observations become more skewed or kurtic as the terrain becomes more rugged or as data becomes more scarce. The highest skewness measure recorded is - 2.41, which is significant at the 5% level. The remaining skewness measures but one are also significant at the 5% level. However, the observations are generally not badly skewed. The two measures of kurtosis computed (α_4 , a_4) both compare well except for the observations generated in the mountainous terrain by the weighted mean and surface fit techniques. While α_4 indicates that the distributions are only slightly leptokurtic (compare 3.86 and 4.50 to the value of 3.00 for observations with normal PDF) the other measure a_4

indicates that they are more so (the values 0.07 and 0.071 are much less than 0.7979 for observations with normal PDF). The latter measure is, however, expected to be more accurate for small sample sizes. Hence, the nine samples are leptokurtic. This situation is not expected especially since outliers have been removed from the samples. It is believed that there are some other errors in the samples which are not accounted for, and which affect the values of the standard deviations of the samples.

Table 6

MEASURES OF SKEWNESS AND KURTOSIS

PREDICTION TECHNIQUES	FLAT (PRAIRIES) Sample Size = 150				ROLLING 150				MOUNTAINOUS (ROCKIES) 150			
	SKEWNESS α_3	α_3^* $S_{\alpha_3^*}$	α_4 a_4	α_4^* a_4^*	SKEWNESS α_3	α_3^* $S_{\alpha_3^*}$	α_4 a_4	α_4^* a_4^*	SKEWNESS α_3	α_3^* $S_{\alpha_3^*}$	α_4 a_4	α_4^* a_4^*
LEAST-SQUARES SURFACE FIT	0.79	0.321 0.196	11.08 0.050	3.00 0.7979	-2.41	0.321 0.196	12.53 0.049	3.00 0.7979	0.88	0.389 0.238	4.50 0.071	3.00 0.7979
LEAST-SQUARES COLLOCATION	-1.10	0.321 0.196	9.66 0.055	3.00 0.7979	-0.14	0.321 0.196	10.26 0.051	3.00 0.7979	-1.80	0.389 0.238	11.28 0.64	3.00 0.7979
WEIGHTED MEANS	0.88	0.321 0.196	10.21 0.052	3.00 0.7979	-1.02	0.321 0.196	13.43 0.048	3.00 0.7979	0.390	0.389 0.238	3.86 0.07	3.00 0.7979
COMBINATION OF 1 AND 2												

CHAPTER 6

CONCLUSIONS

The performances of three prediction methods suitable for gravity anomaly predictions at bench marks have been evaluated for three terrain situations in Canada. The test situations were set up such that each technique was given a fair chance (Section 5-1).

It was found for the least-squares surface fitting technique, that a fourth order polynomial was adequate for predictions in the three terrain situations (section 5.1.2). The method was found to have performed well in the flat and rolling terrains but, not so well in the Rocky Mountains (Table 4). It was simple to use and did not take much computer time in an overall sense; and it performed better than the method of weighted means in estimating its precision (Section 5.2.2).

The least-squares collocation method gave consistently good results in the flat and rolling terrains (Table 4). The covariance function used in these terrains was felt to be quite adequate for Canada excluding the Rocky Mountains. Its performance in the Rockies was about the same as that of the least-squares surface fitting method. The technique required twice and four times as much time for computations as the surface fitting and weighted means techniques respectively (Table 4). It, however, performed

best in estimating its precision (Section 5.2.2).

The method of weighted means was the simplest of the three techniques conceptually. It performed well in all the three terrain situations and gave the best results in the flat and mountainous terrains (Table 4). The estimates of its precision were the poorest (Section 5.2.2). It was, however, the fastest of the three techniques (Table 4).

The combination of surface fitting and collocation methods applied only in the mountainous terrain gave improved results compared to either the surface fitting technique or the collocation method. It, however, took eight times as much time as the method of weighted means which gave better results (Table 4).

6.1 RECOMMENDATIONS

In the evaluation of the performances of the three most common prediction methods, it was seen that the method of weighted means gave the best performances - in an overall sense - for point anomaly predictions in three kinds of terrain (Table 4). However, the technique performed poorly in estimating its precision compared to the collocation method (Section 5.2.1.5). In an attempt made to discover the reason for the superior performances of the technique over those of the others, the weighted mean method was expressed as a filter and compared with the covariance filter (Sections 2.2.1, 2.3.1). The two filters were set up such that the matrices

constituting them might be compared. The main difference between them was that while the corelations of gravity anomalies at observation and prediction points were modelled through a covariance function for the covariance filter, the weighted mean method did not model corelations, but it assigned weights to observed anomalies such that the closest to the prediction point contributed most to the predicted value (Section 2.3.1). The exact structures of the two filters have not been found yet. However, the weighted mean method was seen to be a better filter than the covariance filter.

It is recommended, therefore, that the weighted means method be used where point predictions of gravity anomalies are required. It is a fast and economical technique to use. The technique's precision estimations must, however, be improved upon.

It is also recommended that the possibility of designing a covariance function for point predictions of gravity anomalies from the concept used in the weighted means method be investigated.

6.2 CONTRIBUTIONS

(i) The most hopeful techniques for point predictions of gravity anomalies have been evaluated giving the best possible chance to each method.

(ii) The mathematical models for the collocation and weighted means techniques were compared with the intent to discover why the latter was a better filter.

(iii) The results of predictions by the three techniques chosen for evaluation were vigorously tested and assesed statistically.

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