

# INTERACTIVE NETWORK DESIGN

**B. G. NICKERSON**

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## PREFACE

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# **INTERACTIVE NETWORK DESIGN**

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## PREFACE

This report is an unaltered version of the author's thesis entitled "Horizontal Network Design Using Interactive Computer Graphics", which was submitted to this department in April, 1979.

The thesis advisor was Dr. E.J. Krakiwsky. Details of the assistance provided are contained in the ACKNOWLEDGEMENTS.

## ABSTRACT

The availability of relatively inexpensive and powerful computer graphics terminals has inspired their application to horizontal survey network design. This thesis develops the naturally applicable sequential adjustment model for horizontal network design. The mathematical models for network observables, statistical considerations for error ellipse computation, and a comparison of directions and angles from the network design viewpoint are presented. An example using program NETDESIGN, which was developed to test the feasibility of interactive network design, is outlined. Examples using most of the mathematical models developed herein are given in the Appendices. The interactive design of horizontal networks is concluded to be an indispensable tool for surveying engineers, and it is recommended that it be fully developed and utilized.

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## 1. INTRODUCTION

### 1.1 Background and Motivation

The design of horizontal networks in the past has been done mainly by batch methods. The main drawback to these procedures is that the results of a design are not immediately available. If the desired accuracy is not attained, then a new design is prepared and the batch program is executed again. The lack of continuity in this design process is the basic stumbling block for its effectual use.

The motivation for this thesis stems from the lack of dynamic interaction in network design. The advent of computer graphics terminals, which can plot diagrams very rapidly, has made the dynamic interactive design procedure possible. Besides being used solely for network design, this process can adapt itself readily to the task of maintenance of survey methods, i.e. the design of replacement procedures for destroyed monuments.

## 1.2 Methodology

The rigorous design of survey networks originates from the theory of least squares adjustment. From sources such as Mikhail [1976] or Krakiwsky [1975], the variance covariance matrix (accuracy estimate) of the parameters - specifically the coordinates of points in a survey network - is known to be

$$C_x = [A^T C_L^{-1} A]^{-1} \quad , \quad (1-1)$$

where  $A$  = design matrix for the network evaluated at the approximate values of the parameters  $X^0$ ,

$C_L$  = variance covariance matrix of the observations.

If some of the parameters have already been determined to a certain accuracy, and it is desired to include these existing accuracy estimates in the present adjustment, then these so-called weighted parameters can be considered to have observation equations

$$L_x = X \quad ,$$

with the corresponding variance-covariance matrix of these observations  $L_x$  being the accuracy estimates of the weighted parameters. This then changes the  $C_L$  matrix in equation (1-1) to

$$C_L^* = \left[ \begin{array}{c|c} C_L & 0 \\ \hline 0 & C_{L_x} \end{array} \right] \quad ,$$

and the design matrix to

$$A^* = \begin{bmatrix} A \\ I \end{bmatrix} .$$

Thus, the variance covariance matrix of the parameters is

$$C_x = [A^{*T} C_L^{-1} A^*]^{-1} ,$$

$$C_x = [A^T C_L^{-1} A + C_{L_x}^{-1}]^{-1} ,$$

or, realizing that  $C_{L_x}^{-1} = P_x$  (i.e. inverse of the variance-covariance matrix equals the weight matrix, which is the normal equations matrix pertaining to the weighted parameters)

$$C_x = [A^T C_L^{-1} A + P_x]^{-1} \quad (1-2)$$

The above exercise also indicates why the degrees of freedom of a network with weighted parameters is computed as

$$df = r - u + u_x , \quad (1-3)$$

where  $df$  = degrees of freedom,

$r$  = number of equations,

$u$  = total number of parameters,

$u_x$  = number of weighted parameters.

From equation (1-2) it is apparent that the accuracy estimate of the parameters  $C_x$  can be computed without making any actual observations. The  $A$  matrix is constructed knowing the approximate coordinates of the unknown points  $X^o$  along with some proposed observations (observables) among them.  $C_L$  is also part of the a priori knowledge of an adjustment; it simply represents the accuracies of the proposed observations. Finally,  $P_x$  is known from some previous accuracy estimate of the parameters. The computation of  $C_x$  based on this a priori knowledge is the essence of network design.

The nature of the design process (i.e. changing and rearranging the proposed observations, their weights and the network geometry until the desired  $C_x$  is achieved) points to the use of some sort of step by step procedure which utilizes the already existing  $C_x$ . Any changes in the design are treated as additional information to be added to (or deleted from) the design as it existed in the previous step. Methods which perform this task for adjustments are known as sequential adjustment techniques, and they are readily adapted to network design.

Sequential adjustment techniques such as Kalman filtering, Bayes filtering, Tienstra phase, summation of normal equations, sequential, and phase all possess the capability of utilizing previous computations (notably the matrix inverse) to compute  $C_x$  for the present step. From the comparisons made amongst these techniques in Krakivsky [1975], the sequential and Tienstra phase methods are shown to be computationally efficient when the number of new observables added is less than the total number of parameters. This is usually the case when dealing with network design as the number of observables added (deleted) when refining a design is usually much less than the total number of parameters in the network. Even if this is not the case, the observables can be further broken down into smaller sets such that this argument is still valid (see Chapter 3). Neither Kalman or Bayes filtering are considered as they are concerned with time varying parameters, which is not the case for network points in the scope of this work. Because both the phase and summation of normal equations methods



require an inversion of a matrix the size of the number of parameters (instead of the number of new observables), it is the sequential or Tienstra phase method which is used as the basic mathematical model for the network design procedure.

## 2. THE SEQUENTIAL ADJUSTMENT

The notion of computing an adjustment in steps or phases was introduced by Tienstra [1956]. Schmid and Schmid [1965] extended the sequential adjustment procedure into a generalised notation. Other authors [e.g. Krakiwsky, 1968; Kouba, 1970] have expanded the basic sequential model to include different types of observables and parameters. A synthesis of least squares methods done by Krakiwsky [1975] includes comparisons amongst the various step by step procedures mentioned in Chapter 1.

### 2.1 Original Model

The original sequential model (e.g. Schmid and Schmid [1965]) has the following form:

$$F_1 (X, L_1) = 0 \quad , \quad (2-1)$$

$$F_2 (X, L_2) = 0 \quad , \quad (2-2)$$

where  $F_1$  is the original mathematical model relating some parameters  $X$  and observations  $L_1$ , and  $F_2$  is the new or additional model used to link the new observations  $L_2$  to the solution for the parameters  $X$ . The observations  $L_1$  and  $L_2$  are considered to be independent, and have

weight matrices  $P_1$  and  $P_2$ , respectively. The parameters  $X$  have the weight matrix  $P_x$  (see section 1.2). Although the original model is well known, its derivation is reviewed here to set the foundation for the notation and methods which are employed in other sections of this chapter.

The linearization of equations (2-1) and (2-2) by means of a linear Taylor series expansion gives

$$A_1 X + B_1 V_1 + W_1 = 0 \quad , \quad (2-3)$$

$$A_2 X + B_2 V_2 + W_2 = 0 \quad , \quad (2-4)$$

where

$$A_1 = \left. \frac{\partial F_1}{\partial X} \right|_{X = X^\circ, L_1} \quad ,$$

( $r_1, u$ )

$$A_2 = \left. \frac{\partial F_2}{\partial X} \right|_{X = X^\circ, L_2} \quad ,$$

( $r_2, u$ )

$$B_1 = \left. \frac{\partial F_1}{\partial L_1} \right|_{X = X^\circ, L_1} \quad ,$$

( $r_1, n_1$ )

$$B_2 = \left. \frac{\partial F_2}{\partial L_2} \right|_{X = X^\circ, L_2} \quad ,$$

( $r_2, n_2$ )

$$V_1, V_2 = \text{residuals corresponding to } L_1, L_2 \quad ,$$

( $n_1, 1$ ) ( $n_2, 1$ )

$$W_1 = F_1 (X^\circ, L_1) \quad ,$$

( $r_1, 1$ )

$$W_2 = F_2 (X^\circ, L_2) \quad .$$

( $r_2, 1$ )

The above elements of equations (2-3) and (2-4) are matrices of dimension as indicated in brackets beneath them. The row and column dimensions are defined as

$$r_1 = \text{number of equations in } F_1,$$

$$u = \text{number of parameters},$$

$$n_1 = \text{number of observations in } F_1,$$

$$r_2 = \text{number of equations in } F_2,$$

$$n_2 = \text{number of observations in } F_2.$$

The variation function

$$\phi = [V_1^T \quad V_2^T X^T] \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_x \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ X \end{bmatrix} + 2[K_1^T \quad K_2^T] \left[ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \right], \quad (2-5)$$

where  $K_1$  and  $K_2$  are the vectors of Lagrange correlates, is the function to be minimized. The differentiation of equation (2-5) for the least squares minimum condition yields

$$\frac{\partial \phi}{\partial V_1} = 2V_1^T P_1 + 2K_1^T B_1 = 0, \quad \text{or} \quad V_1^T P_1 + K_1^T B_1 = 0, \quad (2-6)$$

$$\frac{\partial \phi}{\partial V_2} = V_2^T P_2 + K_2^T B_2 = 0, \quad (2-7)$$

$$\frac{\partial \phi}{\partial X} = X^T P_x + K_1^T A_1 + K_2^T A_2 = 0. \quad (2-8)$$

The transpose of the above three equations plus the constraint equations (2-3) and (2-4) make up the least squares normal equations system. In its most expanded form, it is

$$\begin{bmatrix} P_1 & 0 & B_1^T & 0 & 0 \\ 0 & P_2 & 0 & B_2^T & 0 \\ B_1 & 0 & 0 & 0 & A_1 \\ 0 & B_2 & 0 & 0 & A_2 \\ 0 & 0 & A_1^T & A_2^T & P_x \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ K_1 \\ K_2 \\ X \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ W_1 \\ W_2 \\ 0 \end{bmatrix} = 0. \quad (2-9)$$

The next steps in the derivation are devoted to finding a solution for  $X$  based on the information contained in equation (2-9).

To find a solution for  $X$ , the unknowns  $V_1$ ,  $V_2$ ,  $K_1$  and  $K_2$  are eliminated algebraically by using the following technique [e.g. Thompson, 1969]. Given a matrix equation system

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z \\ Y \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} = 0, \quad (2-10)$$

then the  $Z$  vector is eliminated by forming the following system:

$$[D - CA^{-1}B] Y + [V - CA^{-1}U] = 0, \quad (2-11)$$

where  $A$  must be non-singular.

This technique is first applied to equation (2-9) to eliminate

$V_1$ ; i.e.

$$\begin{bmatrix} P_2 & 0 & B_2^T & 0 & 0 \\ 0 & 0 & 0 & A_1 - B_1 & 0 \\ B_2 & 0 & 0 & A_2 & 0 \\ 0 & A_1^T & A_2^T & P_x & 0 \end{bmatrix} P_1^{-1} \begin{bmatrix} 0 & B_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ K_1 \\ K_2 \\ X \end{bmatrix} + \begin{bmatrix} 0 \\ W_1 \\ W_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} P_1^{-1} \begin{bmatrix} 0 \end{bmatrix} = 0. \quad (2-12)$$

Equation (2-12) reduces to

$$\begin{bmatrix} P_2 & 0 & B_2^T & 0 \\ 0 & -B_1 P_1^{-1} B_1^T & 0 & A_1 \\ B_2 & 0 & 0 & A_2 \\ 0 & A_1^T & A_2^T & P_x \end{bmatrix} \begin{bmatrix} V_2 \\ K_1 \\ K_2 \\ X \end{bmatrix} + \begin{bmatrix} 0 \\ W_1 \\ W_2 \\ 0 \end{bmatrix} = 0. \quad (2-13)$$

The same technique is again applied to equation (2-13) to eliminate  $V_2$  and give

$$\begin{bmatrix} -B_1 P_1^{-1} B_1^T & 0 & A_1 \\ 0 & -B_2 P_2^{-1} B_2^T & A_2 \\ A_1^T & A_2^T & P_x \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ X \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ 0 \end{bmatrix} = 0. \quad (2-14)$$

Denoting  $B_1 P_1^{-1} B_1^T$  as  $M_1$  and  $B_2 P_2^{-1} B_2^T$  as  $M_2$ ,  $K_1$  is removed to leave

$$\begin{bmatrix} -M_2 & A_2 \\ A_2^T & A_1^T M_1^{-1} A_1 + P_x \end{bmatrix} \begin{bmatrix} K_2 \\ X \end{bmatrix} + \begin{bmatrix} W_2 \\ A_1^T M_1^{-1} W_1 \end{bmatrix} = 0. \quad (2-15)$$

Finally, after eliminating  $K_2$ , the expression for  $X$  is

$$[A_1^T M_1^{-1} A_1 + P_x + A_2^T M_2^{-1} A_2] X + A_1^T M_1^{-1} W_1 + A_2^T M_2^{-1} W_2 = 0. \quad (2-16)$$

If all of the matrices referring to the  $F_2$  model are omitted, equation (2-16) reverts to the usual least squares expression for the parameters, i.e.

$$[A_1^T M_1^{-1} A_1 + P_x] X + A_1^T M_1^{-1} W_1 = 0. \quad (2-17)$$

It is readily seen from equation (2-16) that the normal equations matrix (the coefficient matrix of  $X$ ) is updated by adding the normal equations resulting from  $F_2$  (i.e.  $A_2^T M_2^{-1} A_2$ ). The same holds for the constant

vector  $A_2^T M_2^{-1} W_2$ . It is clear that if the information in  $F_2$  was to be subtracted, then the matrix terms resulting from  $F_2$  would be subtracted instead of being added. This leads to the expression

$$X = -[N_1 \pm A_2^T M_2^{-1} A_2]^{-1} [U_1 \pm A_2^T M_2^{-1} W_2] , \quad (2-18)$$

where  $(A_1^T M_1^{-1} A_1 + P_x)$  has been replaced by  $N_1$  and  $A_1^T M_1^{-1} W_1$  by  $U_1$ .

It now remains to form the sequential expression for equation (2-18). Equation (2-18) is the summation of normal equations approach, and it is clear that the entire normal equations would have to be inverted each time new information is added. This is a costly process, and the sequential expressions below bypass this large inversion.

The first step in deriving the sequential expressions is the expansion of equation (2-18), which gives

$$X = -[N_1 \pm A_2^T M_2^{-1} A_2]^{-1} U_1 \mp [N_1 \pm A_2^T M_2^{-1} A_2]^{-1} A_2^T M_2^{-1} W_2 , \quad (2-19)$$

where the top signs refer to the addition, and the bottom signs the deletion of the information from  $F_2$ . Two matrix identity equations [e.g. Morrison, 1969; Mikhail, 1976] are now introduced which make the sequential expressions readily apparent. They are

$$[S^{-1} \pm T^T R^{-1} T]^{-1} = S \mp S T^T [R \pm T S T^T]^{-1} T S , \quad (2-20)$$

and

$$[S^{-1} \pm T^T R^{-1} T]^{-1} T^T R^{-1} = S T^T [R \pm T S T^T]^{-1} , \quad (2-21)$$

where it is assumed that the matrices are conformable for multiplication, and that the required inverses exist. The application of equation (2-20) to the first part of equation (2-19) gives

$$[N_1 \pm A_2^T M_2^{-1} A_2]^{-1} = N_1^{-1} + N_1^{-1} A_2^T [M_2 \pm A_2 N_1^{-1} A_2^T]^{-1} A_2 N_1^{-1}. \quad (2-22)$$

Equation (2-21) is applied to the second part of equation (2-19) to give

$$[N_1 \pm A_2^T M_2^{-1} A_2]^{-1} A_2^T M_2^{-1} = N_1^{-1} A_2^T [M_2 \pm A_2 N_1^{-1} A_2^T]^{-1}. \quad (2-23)$$

Substituting equations (2-22) and (2-23) back into equation (2-19) leads to

$$X = -N_1^{-1} U_1 \pm N_1^{-1} A_2^T [M_2 \pm A_2 N_1^{-1} A_2^T]^{-1} A_2 N_1^{-1} U_1 + N_1^{-1} A_2^T [M_2 \pm A_2 N_1^{-1} A_2^T]^{-1} W_2. \quad (2-24)$$

Realizing that  $-N_1^{-1} U_1$  is the solution for the parameters  $X'$  resulting from the  $F_1$  model, the final sequential expression for the parameters is

$$X = X' + N_1^{-1} A_2^T [M_2 \pm A_2 N_1^{-1} A_2^T]^{-1} [A_2 X' + W_2], \quad (2-25)$$

or,

$$X = X' - N_1^{-1} A_2^T [\pm M_2 + A_2 N_1^{-1} A_2^T]^{-1} [A_2 X' + W_2]. \quad (2-26)$$

Equation (2-26) is the final sequential expression for the parameters  $X$ . It is evident that only one inversion of the size of the number of new observations  $L_2$  is required. As has already been stated in section 1.2, the number of new observations is usually much smaller than the number of parameters, and thus equation (2-26) requires less computations than equation (2-18).

From e.g. Mikhail [1976], it is known that the variance covariance matrix of the parameters is given by



$$C_x = N^{-1} , \quad (2-27)$$

where N is the normal equations matrix for the parameters X. Applying this result to equation (2-18) (or, equivalently, equation (2-22)) it is seen that

$$C_x = [N_1^{-1} - N_1^{-1} A_2^T [+M_2 + A_2 N_1^{-1} A_2^T]^{-1} A_2 N_1^{-1}]. \quad (2-28)$$

This equation is the basic mathematical model for horizontal network design.

In this section, the basic and rather simple sequential adjustment expression, where there are no eliminated nuisance parameters or common nuisance parameters between steps, has been derived. In the following sections, the models for more complex situations are developed.

## 2.2 Addition (Deletion) of Parameters

The addition or deletion of parameters to a solution is very useful, especially at the network design stage where new unknown points can be added or subtracted to optimise the design. This situation is characterized by the following two models:

$$F_1 (X_1, L_1) = 0 , \quad (2-29)$$

$$F_2 (X_1, X_2, L_2) = 0 . \quad (2-30)$$

The first model is exactly the same as that in section 2.1 (cf. eq. (2-1)), but  $F_2$  now contains new parameters  $X_2$ .

Linearization of equations (2-29) and (2-30) leads to

$$\begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = 0, \quad (2-31)$$

where  $A_{21} = \frac{\partial F_2}{\partial X_1}$  and  $A_2 = \frac{\partial F_2}{\partial X_2}$ . After setting up the variation function and taking the partials derivatives of it, as done for the original model in section 2.1, the system of normal equations becomes

$$\begin{bmatrix} P_1 & 0 & B_1^T & 0 & 0 & 0 \\ 0 & P_2 & 0 & B_2^T & 0 & 0 \\ B_1 & 0 & 0 & 0 & A_1 & 0 \\ 0 & B_2 & 0 & 0 & A_{21} & A_2 \\ 0 & 0 & A_1^T & A_{21}^T & P_{x_1} & 0 \\ 0 & 0 & 0 & A_2^T & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ K_1 \\ K_2 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ W_1 \\ W_2 \\ 0 \\ 0 \end{bmatrix} = 0. \quad (2-32)$$

Again, the observations  $L_1$  and  $L_2$  are assumed to be uncorrelated, and the parameters  $X_1$  and  $X_2$  both have weight matrices  $P_{x_1}$  and  $P_{x_2}$ , respectively. The algebraic elimination equations (2-10) and (2-11) are again employed here to algebraically eliminate all but the desired parameters. After  $V_1$  and  $V_2$  have been eliminated, the expression remains as

$$\begin{bmatrix} -M_1 & 0 & A_1 & 0 \\ 0 & -M_2 & A_{21} & A_2 \\ A_1^T & A_{21}^T & P_{x_1} & 0 \\ 0 & A_2^T & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ 0 \\ 0 \end{bmatrix} = 0. \quad (2-33)$$

where  $M_1 = B_1 P_1^{-1} B_1^T$  and  $M_2 = B_2 P_2^{-1} B_2^T$ . Elimination of  $K_1$  leaves

$$\begin{bmatrix} -M_2 & A_{21} & A_2 \\ A_{21}^T & A_1^T M_1^{-1} A_1 + P_{x_1} & 0 \\ A_2^T & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} K_2 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} W_2 \\ A_1^T M_1^{-1} W_1 \\ 0 \end{bmatrix} = 0. \quad (2-34)$$

Finally, elimination of  $K_2$  leaves the desired expression for the parameters as

$$\begin{bmatrix} N_1 + A_{21}^T M_2^{-1} A_{21} & A_{21}^T M_2^{-1} A_2 \\ A_2^T M_2^{-1} A_{21} & P_{x_2} + A_2^T M_2^{-1} A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} U_1 + A_{21}^T M_2^{-1} W_2 \\ A_2^T M_2^{-1} W_2 \end{bmatrix} = 0, \quad (2-35)$$

where  $N_1 = (A_1^T M_1^{-1} A_1 + P_{x_1})$  and  $U_1 = (A_1^T M_1^{-1} W_1)$ . Solving for the parameters gives

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = - \begin{bmatrix} \tilde{N}_1 & N_{12} \\ N_{12}^T & N_2 \end{bmatrix} \begin{bmatrix} \tilde{U}_1 \\ U_2 \end{bmatrix}, \quad (2-36)$$

where the following substitutions have been made to simplify the notation:

$$\begin{aligned} \tilde{N}_1 &= N_1 + A_{21}^T M_2^{-1} A_{21}, \\ N_{12} &= A_{21}^T M_2^{-1} A_2, \\ N_2 &= P_{x_2} + A_2^T M_2^{-1} A_2, \\ \tilde{U}_1 &= U_1 + A_{21}^T M_2^{-1} W_2, \\ U_2 &= A_2^T M_2^{-1} W_2. \end{aligned}$$

The development of equation (2-36) into the sequential expressions is enabled by the inversion by partitioning process [Fadeev and Fadeeva, 1963]. The first step is to make the following substitution for the normal equations of equation (2-36):

$$\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} = \begin{bmatrix} N_1 & N_{12} \\ N_{12}^T & N_2 \end{bmatrix}^{-1} .$$

Therefore

$$\begin{bmatrix} \tilde{N}_1 & N_{12} \\ N_{12}^T & N_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} , \quad (2-37)$$

where  $I_1$  and  $I_2$  are identity matrices of the same size as the number of parameters in  $X_1$  and  $X_2$ , respectively. Performing the multiplication in equation (2-37), the following four relationships are arrived at:

$$\tilde{N}_1 Q_1 + N_{12} Q_{12}^T = I_1 , \quad (2-38)$$

$$\tilde{N}_1 Q_{12} + N_{12} Q_2 = 0 , \quad (2-39)$$

$$N_{12}^T Q_1 + N_2 Q_{12}^T = 0 , \quad (2-40)$$

$$N_{12}^T Q_{12} + N_2 Q_2 = I_2 . \quad (2-41)$$

From equation (2-39), the expression for  $Q_{12}$  is

$$Q_{12} = -\tilde{N}_1^{-1} N_{12} Q_2 , \quad (2-42)$$

and upon substitution of this into equation (2-41) the result is

$$-N_{12}^T \tilde{N}_1^{-1} N_{12} Q_2 + N_2 Q_2 = I_2 ,$$

or

$$[N_2 - N_{12}^T \tilde{N}_1^{-1} N_{12}] Q_2 = I_2 ,$$

and

$$Q_2 = [N_2 - N_{12}^T \tilde{N}_1^{-1} N_{12}]^{-1} . \quad (2-43)$$

Solving for  $Q_{12}^T$  from equation (2-40) leaves

$$Q_{12}^T = -N_2^{-1} N_{12}^T Q_1 .$$

The substitution of this into equation (2-38) yields

$$[\tilde{N}_1 - N_{12} N_2^{-1} N_{12}^T] Q_1 = I_1 ,$$

and

$$Q_1 = [\tilde{N}_1 - N_{12} N_2^{-1} N_{12}^T]^{-1} . \quad (2-44)$$

Application of the matrix identity given by equation (2-20) to equation (2-44) yields

$$Q_1 = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} N_{12} [N_2 - N_{12}^T \tilde{N}_1^{-1} N_{12}]^{-1} N_{12}^T \tilde{N}_1^{-1} ,$$

or, noting equation (2-43),

$$Q_1 = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} N_{12} Q_2 N_{12}^T \tilde{N}_1^{-1} . \quad (2-45)$$

As well, from equations (2-35) and (2-36), the expression for  $\tilde{N}_1$  is

$$\tilde{N}_1 = N_1 + A_{21}^T M_2^{-1} A_{21} . \quad (2-46)$$

Again employing the identity equation (2-20) to equation (2-46) yields

$$\tilde{N}_1^{-1} = N_1^{-1} - N_1^{-1} A_{21}^T [M_2 + A_{21} N_1^{-1} A_{21}^T]^{-1} A_{21} N_1^{-1} . \quad (2-47)$$

All of the expressions necessary for the sequential evaluation of the normal equations inverse in equation (2-36) are now available. There are, however, two distinct situations for this case. When adding parameters, the size of the vectors and matrices increases, but when deleting them, the vector and matrix sizes decrease. It is not a simple matter of plus or minus signs as it was in section 2.1. The two different cases are dealt with separately.

### 2.2.1 Addition of New Parameters

For only the addition of new parameters, the equations developed in the preceding section are directly applied. There already exists  $N_1^{-1}$ , or the old normal equations inverse, as well as the present design matrices  $A_{21}$ ,  $A_2$  and  $B_2$  and the weight matrices  $P_2$  and  $P_{x_2}$ . The first step is to form  $\tilde{N}_1^{-1}$  from equation (2-47).  $N_{12}$  is evaluated next along with  $N_2$  so that  $Q_2$  can be computed using equation (2-43). Equations (2-42) and (2-45) are next employed to obtain  $Q_{12}$  and  $Q_1$  respectively. This now completes formation of the normal equations inverse. The constant vector is evaluated using equation (2-35), and the resultant parameter vector  $X$  is obtained by direct matrix multiplication as in equation (2-36). This method has the usual advantages of the sequential expressions. From equations (2-47) and (2-43) it is seen that only two matrix inverses, the first of the size of the number of additional observables, and the second of size equal to the number of additional parameters, are required. This

method is particularly advantageous for adding small numbers of new parameters to a solution already containing many parameters.

### 2.2.2 Deletion of Parameters

The deletion of parameters is dealt with separately from the addition case because the matrix and vector sizes decrease. This case can be thought of as the reverse of addition of parameters; the new inverse of the normals  $N^{-1}$  (consisting of  $Q_1$ ,  $Q_{12}$  and  $Q_2$ ) and constant vector  $U$  already exist, and the desired quantities are the old inverse  $N_1^{-1}$  and constant vector  $U_1$ .

From equation (2-46), the expression for  $N_1$  is

$$N_1 = \tilde{N}_1 - A_{21}^T M_2^{-1} A_{21} .$$

Applying identity equation (2-20) again yields the expression for  $N_1^{-1}$  as

$$N_1^{-1} = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} A_{21}^T [M_2 - A_{21} \tilde{N}_1^{-1} A_{21}^T]^{-1} A_{21} \tilde{N}_1^{-1} . \quad (2-48)$$

Taking the inverse of equation (2-44) yields the expression for  $\tilde{N}_1$  as

$$\tilde{N}_1 = Q_1^{-1} + N_{12} N_2^{-1} N_{12}^T .$$

The use of identity equation (2-20) again results in

$$\tilde{N}_1^{-1} = Q_1 - Q_1 N_{12} [N_2 + N_{12}^T Q_1 N_{12}]^{-1} N_{12}^T Q_1 . \quad (2-49)$$

For the constant vector  $U_1$ , equation (2-35) immediately yields

$$U_1 = \tilde{U}_1 - A_{21}^T M_2^{-1} W_2 . \quad (2-50)$$

Thus, the original (in this case desired) parameters  $X_1$  are now given by

$$X_1 = -N_1^{-1} U_1 ,$$

with  $N_1^{-1}$  being computed using equation (2-48) and  $U_1$  coming from equation (2-50). It should be noted that when computing  $\tilde{N}_1^{-1}$  from equation (2-49), the  $Q_1$  matrix is not always the upper left block of the full normal equations inverse. It must be understood that it is the sub-matrix of the normal equations inverse corresponding to the parameters which are not being deleted.

The above expressions for the addition and deletion of desired parameters efficiently utilize the already computed matrices from previous steps. The greatest saving computationally is in the size of matrix inversions necessary. As can be seen in equations (2-44), (2-47), (2-48) and (2-49), the only two inversions necessary {one of size  $r_2$  (the number of new (old) equations to be added (deleted)), the other of size  $u_2$  (the number of parameters to be added (deleted))} are usually of considerably smaller size than that of full normal equations matrix, especially when designing a network.



### 2.3 Addition (Deletion) of Nuisance Parameters

It may be desirable to eliminate some parameters introduced in the  $F_2$  model as nuisance parameters. One such instance of this situation is when adding a set of directions. The introduction of an orientation unknown (see section 3.3.1) means another parameter to be solved for unnecessarily if it is not eliminated. One can think of this situation as an expansion of the model treated in section 2.2. The parameter vector is now a hypervector containing two types of new parameters; the desired parameters  $X_2$  and the nuisance parameters  $Y$ . This results in the functional relationships

$$F_1 (X_1, L_1) = 0 \quad , \quad (2-51)$$

$$F_2 (X_1, X_2, Y, L_2) = 0 \quad . \quad (2-52)$$

The resulting system of normal equations is identical to those of section 2.2, except for the matrices  $A_2$  and  $P_{x_2}$ . These become hypermatrices corresponding to the hypervector of parameters.

Specifically, the  $A_2$  matrix in equation (2-31) is partitioned to arrive at the hypermatrix

$$A_2^* = [A_{2_x} \ ; \ A_{2_y}] = \left[ \frac{\partial F_2}{\partial X_2} \ ; \ \frac{\partial F_2}{\partial Y} \right] \quad , \quad (2-53)$$

and similarly for  $P_{x_2}$  to give

$$P_{x_2}^* = \left[ \begin{array}{c|c} P_{x_2} & 0 \\ \hline 0 & P_{y_2} \end{array} \right], \quad (2-54)$$

where it is assumed that the weights for the two different parameter types are uncorrelated. For the case of network design,  $P_{y_2}$  is generally zero, but for other sequential adjustment types, it may be desirable to weight the nuisance parameters. One such instance is the adjustment of Doppler satellite observations. The satellite position vector, even though it is a nuisance parameter, is sometimes weighted [cf., e.g. Wells, 1974].

The reduction of the normal equations system up to equation (2-35) is exactly the same here as for the case of no nuisance parameters in section 2.2. At equation (2-35), the derivation is easily adapted to this situation. Substituting the expressions (2-53) and (2-54) in place of  $A_2$  and  $P_{x_2}$  respectively, as well as performing simple row and column transformations leaves

$$\left[ \begin{array}{cc|cc} P_{y_2} & 0 & \begin{bmatrix} A_{2y}^T \\ A_{2x}^T \end{bmatrix} M_2^{-1} [A_{2y} \quad A_{2x}] & \begin{bmatrix} A_{2y}^T \\ A_{2x}^T \end{bmatrix} M_2^{-1} A_{21} \\ \hline 0 & P_{x_2} & \begin{bmatrix} A_{2y}^T \\ A_{2x}^T \end{bmatrix} & \begin{bmatrix} A_{2y}^T \\ A_{2x}^T \end{bmatrix} M_2^{-1} A_{21} \end{array} \right] \begin{bmatrix} Y \\ X_2 \\ X_1 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} A_{2y}^T \\ A_{2x}^T \end{bmatrix} M_2^{-1} W_2 \\ \hline U_1 + A_{21}^T M_2^{-1} W_2 \end{bmatrix} = 0. \quad (2-55)$$

This arrangement facilitates the elimination of the nuisance parameters  $Y$  by means of equations (2-10) and (2-11). First, carrying out the multiplications implied in (2-55) yields

$$\begin{bmatrix}
P_{y_2} + A_{2y}^T M_2^{-1} A_{2y} & A_{2y}^T M_2^{-1} A_{2x} & A_{2y}^T M_2^{-1} A_{21} \\
A_{2x}^T M_2^{-1} A_{2y} & P_{x_2} + A_{2x}^T M_2^{-1} A_{2x} & A_{2x}^T M_2^{-1} A_{21} \\
A_{21}^T M_2^{-1} A_{2y} & A_{21}^T M_2^{-1} A_{2x} & N_1 + A_{21}^T M_2^{-1} A_{21}
\end{bmatrix}
\begin{bmatrix}
Y \\
X_2 \\
X_1
\end{bmatrix}
+
\begin{bmatrix}
A_{2y}^T M_2^{-1} W_2 \\
A_{2x}^T M_2^{-1} W_2 \\
U_1 + A_{21}^T M_2^{-1} W_2
\end{bmatrix}
= 0.
\tag{2-56}$$

To simplify the notation, (2-56) is rewritten as

$$\begin{bmatrix}
N_{y_2} & N_{yx_2} & N_{yx_1} \\
N_{yx_2}^T & N_2 & N_{12}^T \\
N_{yx_1}^T & N_{12} & \tilde{N}_1
\end{bmatrix}
\begin{bmatrix}
Y \\
X_2 \\
X_1
\end{bmatrix}
+
\begin{bmatrix}
U_{y_2} \\
U_2 \\
\tilde{U}_1
\end{bmatrix}
= 0,
\tag{2-57}$$

where it is noted that the lower right 2x2 submatrix is exactly the same as in equation (2-36). The elimination of Y by use of equations (2-10) and (2-11) leaves

$$\begin{bmatrix}
N_2 - N_{yx_2}^T N_{y_2}^{-1} N_{yx_2} & N_{12}^T - N_{yx_2}^T N_{y_2}^{-1} N_{yx_1} \\
N_{12} - N_{yx_1}^T N_{y_2}^{-1} N_{yx_2} & \tilde{N}_1 - N_{yx_1}^T N_{y_2}^{-1} N_{yx_1}
\end{bmatrix}
\begin{bmatrix}
X_2 \\
X_1
\end{bmatrix}
+
\begin{bmatrix}
U_2 - N_{yx_2}^T N_{y_2}^{-1} U_{y_2} \\
\tilde{U}_1 - N_{yx_1}^T N_{y_2}^{-1} U_{y_2}
\end{bmatrix}
= 0,
\tag{2-58}$$

or, transforming rows and columns and again simplifying the notation,

$$\begin{bmatrix}
\tilde{N}_1 & \tilde{N}_{12} \\
\tilde{N}_{12}^T & \tilde{N}_2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
+
\begin{bmatrix}
\tilde{U}_1 \\
\tilde{U}_2
\end{bmatrix}
= 0.
\tag{2-59}$$

After solving for the desired parameters  $X_1$  and  $X_2$ , the expression (2-59) is of the same form as equation (2-36); thus, similar steps are necessary

to arrive at the sequential solution. The solution for  $Q_2$  as given by equation (2-43) is

$$Q_2 = [\tilde{N}_2 - \tilde{N}_{12}^T \tilde{N}_1^{-1} \tilde{N}_{12}]^{-1}, \quad (2-60)$$

and

$$Q_{12} = -\tilde{N}_1^{-1} \tilde{N}_{12} Q_2, \quad (2-61)$$

with

$$Q_1 = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} \tilde{N}_{12} Q_2 \tilde{N}_{12}^T \tilde{N}_1^{-1}. \quad (2-62)$$

The main difference between this notation and that of section 2.2 is in the computation of  $\tilde{N}_1^{-1}$  instead of  $\tilde{N}_1^{-1}$ . The expression for  $\tilde{N}_1$  from equation (2-58) is

$$\tilde{N}_1 = \tilde{N}_1 - N_{yx_1}^T N_{y_2}^{-1} N_{yx_1},$$

and the application of equation (2-20) yields the sequential expression as

$$\tilde{N}_1^{-1} = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} N_{yx_1}^T [N_{y_2} - N_{yx_1} \tilde{N}_1^{-1} N_{yx_1}^T]^{-1} N_{yx_1} \tilde{N}_1^{-1}. \quad (2-63)$$

The expression for  $\tilde{N}_1^{-1}$  is given by equation (2-47). Once  $\tilde{N}_{12}$ ,  $\tilde{N}_2$  and  $\tilde{N}_1^{-1}$  are computed, the normal equations inverse matrix is solved by the successive application of equations (2-63), (2-60), (2-61) and (2-62). The computation of the elements of the constant vector in equation (2-58) is straightforward and needs no further elaboration. The desired parameters are given by

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = - \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix}. \quad (2-64).$$

It is interesting to compare the expressions developed here with those of Krakowsky [1968]. The case of adding nuisance parameters only is considered, which in this instance would be the situation where there are no desired parameters  $X_2$ . The matrices  $A_{2x}$  and  $P_{x_2}$  are null matrices. This leads to the matrices  $Q_{12}$  and  $Q_2$  being zero as well, and the parameters  $X_1$  are given by

$$X_1 = -Q_1 \tilde{U}_1,$$

or

$$X_1 = -\tilde{N}_1^{-1} \tilde{U}_1. \quad (2-65)$$

Equation (2-63) gives the expression for  $\tilde{N}_1^{-1}$ . Noting that

$$\tilde{N}_1^{-1} = N_1^{-1} - N_1^{-1} A_{21}^T S^{-1} A_{21} N_1^{-1},$$

for

$$S = [M_2 + A_{21} N_1^{-1} A_{21}^T],$$

and

$$N_{yx_1} = A_{2y}^T M_2^{-1} A_{21},$$

the expression from equation (2-63) under the inverse sign becomes

$$\begin{aligned} [N_{y_2} \quad -N_{yx_1} \quad \tilde{N}_1^{-1} \quad N_{yx_1}^T]^{-1} &= [P_{y_2} + A_{2y}^T M_2^{-1} A_{2y} \quad -A_{2y}^T M_2^{-1} A_{21} (N_1^{-1} N_1^{-1} A_{21}^T S^{-1} A_{21} N_1^{-1}) A_{21} M_2^{-1} A_{2y}]^{-1} \\ &= [P_{y_2} + A_{2y}^T [M_2^{-1} - M_2^{-1} A_{21} (N_1 + A_{21}^T M_2^{-1} A_{21})^{-1} A_{21}^T M_2^{-1}] A_{2y}]^{-1} \end{aligned} \quad (2-66)$$

where  $\tilde{N}_1^{-1}$  has been replaced by its non-sequential form (see equation (2-46)). By inspection it is seen that the inner term of equation (2-66) in square brackets is nothing but the sequential expression for  $S^{-1}$ . Thus,

$$\left[ \begin{matrix} N_{y_2} & -N_{yx_1} \\ -N_{yx_1} & \tilde{N}_1^{-1} N_{yx_1}^T \end{matrix} \right]^{-1} = \left[ \begin{matrix} P_{y_2} + A_{2y}^T & \\ & (M_2 + A_{21} N_1^{-1} A_{21}^T)^{-1} A_{2y} \end{matrix} \right]. \quad (2-67)$$

The term  $\tilde{N}_1^{-1} N_{yx_1}^T$  is given by

$$\tilde{N}_1^{-1} N_{yx_1}^T = \left[ N_1 + A_{21}^T M_2^{-1} A_{21} \right]^{-1} A_{21}^T M_2^{-1} A_{2y},$$

or, applying equation (2-21),

$$\tilde{N}_1^{-1} N_{yx_1}^T = N_1^{-1} A_{21}^T \left[ M_2 + A_{21} N_1^{-1} A_{21}^T \right]^{-1} A_{2y}. \quad (2-68)$$

Substituting equations (2-67) and (2-68) back into equation (2-63)

yields

$$\tilde{N}_1^{-1} = N_1^{-1} - N_1^{-1} A_{21}^T S^{-1} A_{21} N_1^{-1} + N_1^{-1} A_{21}^T S^{-1} A_{2y} \left[ P_{y_2} + A_{2y}^T S^{-1} A_{2y} \right]^{-1} A_{2y}^T S^{-1} A_{21} N_1^{-1} \quad (2-69)$$

which is identical to the expression derived by Krakiwsky [1968] for the new normal equations inverse (equation 2.1 - 36), except for the difference in notation. The total expression for  $X_1$  is not derived here, but the development is similar to that for the  $\tilde{N}_1^{-1}$  term.

Returning to the problem of deletion of nuisance parameters, the desired expression is again

$$X_1 = -N_1^{-1} U_1. \quad (2-70)$$

$N_1^{-1}$  is computed from equation (2-48) as

$$N_1^{-1} = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} A_{21}^T [M_2 - A_{21} \tilde{N}_1^{-1} A_{21}]^{-1} A_{21} \tilde{N}_1^{-1} ,$$

and similarly  $\tilde{N}_1^{-1}$  is calculated from equation (2-58) and the identity equation (2-20) as

$$\tilde{N}_1^{-1} = \tilde{N}_1^{-1} - \tilde{N}_1^{-1} N_{yx_1}^T [N_{y_2} + \tilde{N}_{yx_1} N_1^{-1} N_{yx_1}^T]^{-1} N_{yx_1} \tilde{N}_1^{-1} .$$

Using the same strategy as in section 2.2.2, the expression for  $\tilde{N}_1^{-1}$  is computed as

$$\tilde{N}_1^{-1} = Q_1 - Q_1 \tilde{N}_{12} [\tilde{N}_2 + \tilde{N}_{12}^T Q_1 \tilde{N}_{12}]^{-1} \tilde{N}_1^T Q_1 , \quad (2-71)$$

where again  $Q_1$  is the portion of the normal equations inverse corresponding to the parameters  $X_1$ . The computation of the constant vector  $U_1$  comes from equations (2-56) and (2-58). From equation (2-58), the expression for  $\tilde{U}_1$  is

$$\tilde{U}_1 = U_1 + A_{21}^T M_2^{-1} W_2 - N_{yx_1}^T N_{y_2}^{-1} U_{y_2} , \quad (2-72)$$

and

$$U_1 = \tilde{U}_1 - A_{21}^T M_2^{-1} W_2 + N_{yx_1}^T N_{y_2}^{-1} U_{y_2} . \quad (2-73)$$

$X_1$  is now computed using equation (2-70).

In the event that there are no desired parameters  $X_2$ , then again  $A_{2x}$  and  $P_{x_2}$  are null matrices as are  $\tilde{N}_2$  and  $\tilde{N}_{12}$ .  $X_1$  is given by equation (2-70), but in the computation of  $\tilde{N}_1^{-1}$ , the equation is

$$\tilde{N}_1^{-1} = Q_1 , \quad (2-74)$$

with the second term in equation (2-71) being zero.

The expressions derived in this section have covered the cases of addition and deletion of both desired and/or nuisance parameters. This situation arises frequently when stations are added to a network design with a set of directions, or when a station is subtracted from the design, having previously been linked to the network by a single set of directions. Section 3.4 deals more fully with the problem of directions.

#### 2.4 Addition (Deletion) of Parameters with Common Nuisance Parameters between Steps

This section deals with the situation where certain nuisance parameters  $Y$  are common to two successive steps in the sequential adjustment. This condition is characterized by

$$F_1 (X_1, Y, L_1) = 0 , \quad (2-75)$$

and

$$F_2 (X_1, X_2, Y, L_2) = 0 . \quad (2-76)$$

This corresponds to the case where one direction is to be added to or subtracted from a set of directions in the previous step. The common nuisance parameter is the orientation unknown for that set of directions. The inclusion of new parameters  $X_2$  makes the model more general and thus allows the addition (deletion) of new (old) points (along with the directions).



The linearization of equations (2-75) and (2-76) yields

$$\begin{bmatrix} \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial X_1} & 0 \\ \frac{\partial F_2}{\partial Y} & \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} \end{bmatrix} \begin{bmatrix} Y \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial F_1}{\partial L_1} & 0 \\ 0 & \frac{\partial F_2}{\partial L_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = 0, \quad (2-77)$$

or, denoting the partial derivatives in their usual matrix notation,

$$\begin{bmatrix} A_{1y} & A_{1x_1} & 0 \\ A_{2y} & A_{2x_1} & A_{2x_2} \end{bmatrix} \begin{bmatrix} Y \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = 0. \quad (2-78)$$

The variation function which is to be minimized has the form

$$\begin{aligned} \phi = & [V_1^T \quad V_2^T] \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + [X_1^T \quad X_2^T] \begin{bmatrix} P_{x_1} & 0 \\ 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \\ & + 2[K_1^T \quad K_2^T] \left[ \begin{bmatrix} A_{1y} & A_{1x_1} & 0 \\ A_{2y} & A_{2x_1} & A_{2x_2} \end{bmatrix} \begin{bmatrix} Y \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \right] \end{aligned} \quad (2-79)$$

where the weight matrices  $P_1$ ,  $P_2$  correspond to the observables  $L_1$ ,  $L_2$  and  $P_{x_1}$ ,  $P_{x_2}$  are a priori weight matrices for the desired parameters  $X_1$  and  $X_2$ . The differentiation of equation (2-79) with respect to the variables leads to five equations, which, when combined with equation (2-78) make up the most expanded form of the normal equations, i.e.

$$\begin{bmatrix} P_1 & 0 & B_1^T & 0 & 0 & 0 & 0 \\ 0 & P_2 & 0 & B_2^T & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & A_{1y} & A_{1x_1} & 0 \\ 0 & B_2 & 0 & 0 & A_{2y} & A_{2x_1} & A_{2x_2} \\ 0 & 0 & A_{1y}^T & A_{2y}^T & 0 & 0 & 0 \\ 0 & 0 & A_{1x_1}^T & A_{2x_1}^T & 0 & P_{x_1} & 0 \\ 0 & 0 & 0 & A_{2x_2}^T & 0 & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ K_1 \\ K_2 \\ Y \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ W_1 \\ W_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0. \quad (2-80)$$

Proceeding in the same fashion as in previous sections, equations (2-10) and (2-11) are employed to first eliminate  $V_1$  and  $V_2$ ; this yields

$$\begin{bmatrix} -M_1 & 0 & A_{1y} & A_{1x_1} & 0 \\ 0 & -M_2 & A_{2y} & A_{2x_1} & A_{2x_2} \\ A_{1y}^T & A_{2y}^T & 0 & 0 & 0 \\ A_{1x_1}^T & A_{2x_1}^T & 0 & P_{x_1} & 0 \\ 0 & A_{2x_2}^T & 0 & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ Y \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0, \quad (2-81)$$

where  $M_1 = B_1 P_1^{-1} B_1^T$  and  $M_2 = B_2 P_2 B_2^T$  as before. The elimination of  $K_1$  from equation (2-81) results in

$$\begin{bmatrix} -M_2 & A_{2y} & A_{2x_1} & A_{2x_2} \\ A_{2y}^T & N_{1y} & N_{yx_1} & 0 \\ A_{2x_1}^T & N_{yk_1}^T & N_{1x_1} & 0 \\ A_{2x_2}^T & 0 & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} K_2 \\ Y \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} W_2 \\ U_{1y} \\ U_{1x_1} \\ 0 \end{bmatrix} = 0, \quad (2-82)$$

where the newly introduced quantities are defined as follows:

$$N_{ly} = A_{ly}^T M_1^{-1} A_{ly} , \quad (2-83)$$

$$N_{yx_1} = A_{ly}^T M_1^{-1} A_{lx_1} , \quad (2-84)$$

$$N_{lx_1} = P_{x_1} + A_{lx_1}^T M_1^{-1} A_{lx_1} , \quad (2-85)$$

$$U_{ly} = A_{ly}^T M_1^{-1} W_1 , \quad (2-86)$$

$$U_{lx_1} = A_{lx_1}^T M_1^{-1} W_1 . \quad (2-87)$$

Before eliminating  $K_2$  and  $Y$ , equation (2-82) is first rearranged so that the nuisance parameters  $Y$  are eliminated before the correlates resulting from the  $F_2$  model (i.e.  $K_2$ ). This arrangement facilitates the grouping of the resulting matrix expressions into meaningful subsets. Exchanging  $K_2$  with  $Y$ , as well as their corresponding rows and columns, gives

$$\begin{bmatrix} N_{ly} & A_{2y}^T & N_{yk_1} & 0 \\ A_{2y} & -M_2 & A_{2x_1} & A_{2x_2} \\ N_{yk_1}^T & A_{2x_1}^T & N_{lx_1} & 0 \\ 0 & A_{2x_2}^T & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} Y \\ K_2 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} U_{ly} \\ W_2 \\ U_{lx_1} \\ 0 \end{bmatrix} = 0 . \quad (2-88)$$

The nuisance parameters  $Y$  are now eliminated, yielding

$$\begin{bmatrix} -\tilde{M}_2 & \tilde{A}_{2x_1} & A_{2x_2} \\ \tilde{A}_{2x_1}^T & \tilde{N}_{1x_1} & 0 \\ A_{2x_2}^T & 0 & P_{x_2} \end{bmatrix} \begin{bmatrix} K_2 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \tilde{W}_2 \\ \tilde{U}_{1x_1} \\ 0 \end{bmatrix} = 0, \quad (2-89)$$

where the newly introduced notation is denoted by

$$\tilde{M}_2 = M_2 + A_{2y} N_{1y}^{-1} A_{2y}^T, \quad (2-90)$$

$$\tilde{A}_{2x_1} = A_{2x_1} - A_{2y} N_{1y}^{-1} N_{yx_1}, \quad (2-91)$$

$$\tilde{N}_{1x_1} = N_{1x_1} - N_{yx_1}^T N_{1y}^{-1} N_{yx_1}, \quad (2-92)$$

$$\tilde{W}_2 = W_2 - A_{2y} N_{1y}^{-1} U_{1y}, \quad (2-93)$$

and 
$$\tilde{U}_{1x_1} = U_{1x_1} - N_{yx_1}^T N_{1y}^{-1} U_{1y}. \quad (2-94)$$

Finally, the elimination of  $K_2$  leaves the normal equations expression for the desired parameters as

$$\begin{bmatrix} N_{x_1} & N_{12} \\ N_{12}^T & N_{x_2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} U_{x_1} \\ U_{x_2} \end{bmatrix} = 0, \quad (2-95)$$

where

$$N_{x_1} = \tilde{N}_{1x_1} + \tilde{A}_{2x_1}^T \tilde{M}_2^{-1} \tilde{A}_{2x_1}, \quad (2-96)$$

$$N_{12} = \tilde{A}_{2x_1}^T \tilde{M}_2^{-1} A_{2x_2}, \quad (2-97)$$

$$N_{x_2} = P_{x_2} + A_{2x_2}^T \tilde{M}_2^{-1} A_{2x_2}, \quad (2-98)$$

$$U_{x_1} = \tilde{U}_{1x_1} + \tilde{A}_{2x_1}^T \tilde{M}_2^{-1} \tilde{W}_2, \quad (2-99)$$

and

$$U_{x_2} = A_{2x_2}^T \tilde{M}_2^{-1} \tilde{W}_2. \quad (2-100)$$

Now, the solution for the desired parameters resulting from additional parameters  $X_2$  and observable  $L_2$  is

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = - \begin{bmatrix} N_{x_1} & N_{12} \\ N_{12}^T & N_{x_2} \end{bmatrix}^{-1} \begin{bmatrix} U_{x_1} \\ U_{x_2} \end{bmatrix}, \quad (2-101)$$

and the sequential solution is found as in previous sections; i.e., the normal equations inverse or variance covariance matrix  $C_x$  of the parameters is first written as

$$C_x = \begin{bmatrix} N_{x_1} & N_{12} \\ N_{12}^T & N_{x_2} \end{bmatrix}^{-1} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad (2-102)$$

where the elements  $Q_1$ ,  $Q_{12}$  and  $Q_2$  are of the same form as equations (2-45), (2-42) and (2-43), respectively. Thus,

$$Q_2 = [N_{x_2} - N_{12}^T N_{x_1}^{-1} N_{12}]^{-1}, \quad (2-103)$$

$$Q_{12} = -N_{x_1}^{-1} N_{12} Q_2, \quad (2-104)$$

and

$$Q_1 = N_{x_1}^{-1} + N_{x_1}^{-1} N_{12} Q_2 N_{12}^T N_{x_1}^{-1}. \quad (2-105)$$

From equation (2-96), the expression for  $N_{x_1}$  is

$$N_{x_1}^{-1} = \tilde{N}_{1x_1}^{-1} + \tilde{A}_{2x_1}^T M_2^{-1} \tilde{A}_{2x_1}^{-1},$$

and applying the matrix identity equation (2-20) to it yields

$$N_{x_1}^{-1} = \tilde{N}_{1x_1}^{-1} - \tilde{N}_{1x_1}^{-1} \tilde{A}_{2x_1}^T [\tilde{M}_2 + \tilde{A}_{2x_1} \tilde{N}_{1x_1}^{-1} \tilde{A}_{2x_1}^T]^{-1} \tilde{A}_{2x_1} \tilde{N}_{1x_1}^{-1}. \quad (2-106)$$

Equation (2-92) gives the expression for  $\tilde{N}_{1x_1}^{-1}$  as

$$\tilde{N}_{1x_1}^{-1} = [N_{1x_1} - N_{yx_1}^T N_{ly}^{-1} N_{yx_1}]^{-1}.$$

This expression for  $\tilde{N}_{1x_1}^{-1}$  is identically the resulting normal equations inverse for the case where only  $F_1$  is considered and the nuisance parameters  $Y$  are eliminated from  $F_1$ . Thus, when adding desired parameters  $X_2$  and nuisance parameters  $Y$  which have already been eliminated in the previous step,  $\tilde{N}_{1x_1}^{-1}$  is the old normal equations inverse or  $C_x$  matrix existing before consideration of  $F_2$ . It ( $\tilde{N}_{1x_1}^{-1}$ ) is used in evaluating  $N_{x_1}^{-1}$  in equation (2-106), which in turn is used to compute  $Q_2$ ,  $Q_{12}$  and  $Q_1$  via equations (2-103), (2-104) and (2-105), respectively. Appendix III contains a further illustration of addition of unknowns with common nuisance parameters between steps.

For the constant vector  $U$ , the sequential expression is

$$U_{x_1} = \tilde{U}_{1x_1} + \tilde{A}_{2x_1}^T M_2^{-1} \tilde{W}_2, \quad (2-107)$$

and  $\tilde{U}_{1x_1}$  is the constant vector resulting from  $F_1$ .

It is worth noting that if any of the parameters are missing from the  $F_1$  and  $F_2$  models, then their respective matrices are null, and their contributions to the desired parameters disappear. For instance, considering that in equation (2-75),  $Y$  did not exist in  $F_1$ , then  $A_{1y}$  is null, and the expressions for the desired parameters revert to those of section 2.3.

### 2.4.1 Detailed Deletion of Parameters

As for the two previous sections, it is worth describing in detail the deletion of parameters due to the elimination of certain observables. The covariance matrix of the parameters  $C_x$  composed of matrices  $Q_1$ ,  $Q_{12}$ , and  $Q_2$  already exists along with the constant vectors  $U_{x_1}$  and  $U_{x_2}$ . It is required to find  $\tilde{N}_{1x_1}^{-1}$  and  $\tilde{U}_{x_1}$  which correspond to the equations  $F_1$  only, i.e. considering the elimination of  $X_2$  and  $L_2$  and accounting for the common nuisance parameters  $Y$ .

From equation (2-96), the expression for  $\tilde{N}_{1x_1}$  is given as

$$\tilde{N}_{1x_1} = N_{x_1} - \tilde{A}_{2x_1}^T \tilde{M}_2^{-1} \tilde{A}_{2x_1}, \quad (2-108)$$

or, applying the identity equation (2-20),

$$\tilde{N}_{1x_1}^{-1} = N_{x_1}^{-1} + N_{x_1}^{-1} \tilde{A}_{2x_1}^T [\tilde{M}_2 - \tilde{A}_{2x_1} N_{x_1}^{-1} \tilde{A}_{2x_1}^T]^{-1} \tilde{A}_{2x_1} N_{x_1}^{-1}. \quad (2-109)$$

Using the same approach as in section 2.2, equation (2-44) yields

$$Q_1^{-1} = [N_{x_1} - N_{12} N_{x_2}^{-1} N_{12}^T], \quad (2-110)$$

or

$$N_{x_1} = Q_1^{-1} + N_{12} N_{x_2}^{-1} N_{12}^T. \quad (2-111)$$

Identity equation (2-20) is used on equation (2-111) to leave

$$N_{x_1}^{-1} = Q_1 - Q_1 N_{12} [N_{x_2} + N_{12}^T Q_1 N_{12}]^{-1} N_{12}^T Q_1. \quad (2-112)$$

The constant vector  $\tilde{U}_{1x_1}$  is immediately given by equation (2-107) as

$$\tilde{U}_{1x_1} = U_{x_1} - \tilde{A}_{2x_1}^T \tilde{M}_2^{-1} \tilde{W}_2. \quad (2-113)$$

Finally, the parameters  $X_1$  are computed from

$$X_1 = -\tilde{N}_{1x_1}^{-1} \tilde{U}_{1x_1} . \quad (2-114)$$

$\tilde{N}_{1x_1}^{-1}$  is computed from equation (2-109), with  $N_{x_1}^{-1}$  coming from equation (2-112), and  $\tilde{U}_{1x_1}$  is computed via equation (2-113). Equation (2-114) expresses the solution for the parameters  $X_1$  after observables  $L_2$  and parameters  $X_2$  have been eliminated subject to the common nuisance parameters  $Y$ .

Again it must be noted that  $Q_1$  is the submatrix of the entire  $Q$  matrix (see equation (2-102)) corresponding to the desired parameters  $X_1$  which are not being deleted. Details of the actual computations involved with some of these foregone expressions are explained in the appendices.

It is beyond the scope of this work to treat subjects such as sequential statistical testing or problems encountered with iteration of nonlinear models. The equations derived here are primarily for use in the interactive design of networks, and their application to this end is discussed in the next chapter.



### 3. APPLICATION OF THE SEQUENTIAL MODEL

#### TO HORIZONTAL NETWORK DESIGN

This chapter deals with the application of the mathematical models developed in Chapter 2. As already discussed in section 1.2, only the expressions for the variance-covariance matrix of the desired parameters  $C_x$  (i.e. normal equations inverse) is needed when designing networks. The constant vector  $U$  is not needed because no observations are made to obtain the misclosure vector  $W$ .

The models are applied to the different types of observables usually considered in horizontal networks, i.e. distances, azimuths and directions or angles. For each type of observable, the most general model is considered, and other models are treated as special cases of this general model. As well, some simple numerical examples are given to further illustrate the procedures involved for different types of observables.

Once the methods for arriving at the variance covariance matrix  $C_x$  of the desired parameters have been discussed for the various observable types, the procedure for extracting the confidence ellipses, which characterize the accuracy attained from a design, is described. Both the point and relative ellipses are discussed, as well as the

method for increasing the confidence level of the ellipses from standard (e.g. 39.4%) to a higher confidence level (e.g. 95%). Finally, the notion of changing the weight of an observable is discussed briefly.

Before beginning any practical considerations, it is necessary to mention a few general facts which are common to all of the observable types. First, there is the point about the B design matrix. Because each of the observable types being considered can be expressed as a function of the parameters (i.e.  $L = F(X)$ , cf. equation (3-1), then  $B (\partial F/\partial L)$  is always a negative identity matrix  $-I$ . This leads to the M matrix ( $M = BP^{-1}B^T$ ) being equal to  $P^{-1}$ , and  $M^{-1} = P = C_L^{-1}$ . Anywhere an expression similar to  $A^T M^{-1}$  appears, it is therefore, replaced by  $A^T P A$ . Note that this simplification only occurs when  $L = F(X)$ , and for different types of observables or adjustments, the B matrix may not be  $-I$ . The only necessary elements of the various expressions derived in Chapter 2 for  $C_x$  are the different A design matrices and the P or weight matrices. Thus, once these are determined, all that remains to determine  $C_x$  is the correct multiplication and inversion of matrices.

### 3.1 Distances

The mathematical model for a distance  $S_{ij}$  from  $i$  to  $j$  on a plane (in the form of equation (2-1)) is well known to be

$$F_{S_{ij}} = [(x_j - x_i)^2 + (y_j - y_i)^2]^{1/2} - S_{ij} = 0, \quad (3-1)$$

where the coordinates of the respective points  $i$  and  $j$  are as in Figure 3.1.

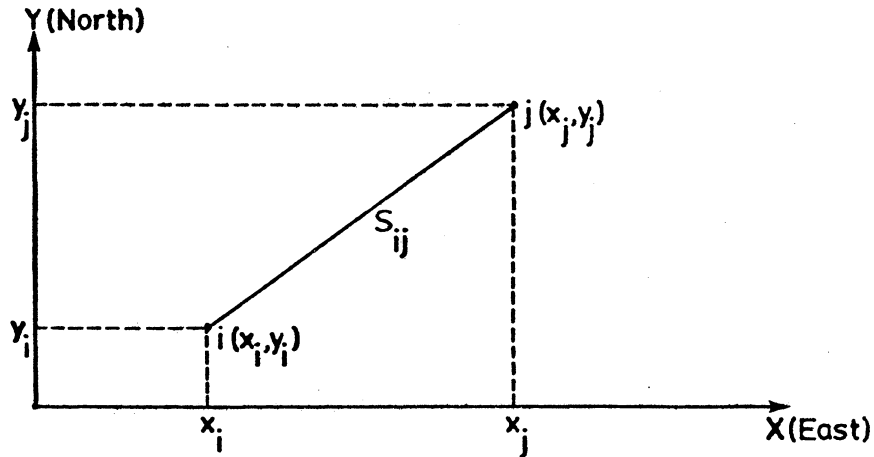


Figure 3.1. Distance on the Mapping Plane

The most general model which is applicable to the sequential addition (deletion) of distance observables is that of section 2.2. There are no nuisance parameters in the model (3-1), and the possibility exists that either point  $i$  or point  $j$  may be a new point (i.e.  $X_2$ ) in the network.

Considering first the case of addition of parameters, the assumption is that a design has already been made (for details of

initial design procedures see Appendix I) for which there exists an old normal equations inverse  $N_1^{-1}$  (cf. eq. (2-35)). The new normal equations inverse resulting from addition of parameters  $X_2$  and observables  $L_2$  is (cf. eq. (2-36))

$$C_x = \begin{bmatrix} \tilde{N}_1 & N_{12} \\ N_{12}^T & N_2 \end{bmatrix}^{-1} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad (3-2)$$

where

$$Q_2 = [N_2 - N_{12}^T \tilde{N}_1^{-1} N_{12}]^{-1} \quad (\text{cf. eq. (2-43)}),$$

$$Q_{12} = -\tilde{N}_1^{-1} N_{12} Q_2 \quad (\text{cf. eq. (2-42)}),$$

$$Q_1 = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} N_{12} Q_2 N_{12}^T \tilde{N}_1^{-1} \quad (\text{cf. eq. (2-45)}),$$

in which

$$\tilde{N}_1^{-1} = N_1^{-1} N_1^{-1} A_{21}^T [C_{L_2}^{-1} + A_{21} N_1^{-1} A_{21}^T] A_{21} N_1^{-1} \quad (\text{cf. eq. (2-47)}),$$

$$N_{12} = A_{21}^T P_2 A_2 \quad (\text{cf. eq. (2-36)}),$$

$$N_2 = P_{x_2} + A_2^T P_2 A_2 \quad (\text{cf. eq. (2-36)}).$$

Besides the already existing normal equations inverse  $N_1^{-1}$ , the elements necessary to evaluate this expression are the design matrices  $A_{21}$  and  $A_2$ , as well as the weight matrices  $P_2(C_{L_2}^{-1})$  and  $P_{x_2}$ .

The general form for one row of the A design matrix is

$$A_{s_{ij}} = \left. \frac{\partial F_{s_{ij}}}{\partial X} \right|_{X = X^0} = \left[ \frac{\partial F_{s_{ij}}}{\partial x_i}; \frac{\partial F_{s_{ij}}}{\partial y_i}; \frac{\partial F_{s_{ij}}}{\partial x_j}; \frac{\partial F_{s_{ij}}}{\partial y_j} \right],$$

or, evaluating the partial derivatives from equation (3-1) at the approximate coordinates  $X^0$

$$A_{S_{ij}} = \left[ \frac{-(x_j^\circ - x_i^\circ)}{S_{ij}^\circ}; \frac{-(y_j^\circ - y_i^\circ)}{S_{ij}^\circ}; \frac{(x_j^\circ - x_i^\circ)}{S_{ij}^\circ}; \frac{(y_j^\circ - y_i^\circ)}{S_{ij}^\circ} \right]. \quad (3-3)$$

It must be remembered that the coordinates  $x$  and  $y$  in equation (3-3) are approximate coordinates (represented by superscript $^\circ$ ) of the unknown points and that  $S_{ij}^\circ$  is the distance computed from these approximate coordinates. If either point  $i$  or point  $j$  is considered fixed, then its known coordinates are used.

From equation (2-31), the A design matrices are given as

$$A_{21} = \frac{\partial F_2}{\partial X_1} \quad \text{and} \quad A_2 = \frac{\partial F_2}{\partial X_2}.$$

Considering an already existing design for five points ( $N_1^{-1} = 10 \times 10$ ), and that a sixth point is to be added by means of a distance observation from an "old" point  $i$  to the new point  $j$ , then  $A_{21}$  is of size  $1 \times 10$  (with zeroes everywhere except in the two columns for point  $i$ ) and  $A_2$  is  $1 \times 2$ . The non-zero elements of  $A_{21}$  are the first two elements of  $A_{S_{ij}}$  (eq. (3-3)), and  $A_2$  contains the second two elements. If  $F_2$  contains  $r_2$  observations, then both  $A_{21}$  and  $A_2$  have  $r_2$  rows.

The weight matrix  $P_2$  is diagonal with  $1/\sigma_{S_{ij}}^2$  on the diagonal, where  $\sigma_{S_{ij}}^2$  is the variance of the distance  $S_{ij}$ . For the example discussed above,  $P_2$  is of size  $1 \times 1$ . The weight matrix for the new parameters  $P_{x_2}$  takes the form

$$P_{x_2} = \begin{bmatrix} \sigma_{x_j}^2 & \sigma_{x_j y_j} \\ \sigma_{x_j y_j} & \sigma_{y_j}^2 \end{bmatrix}^{-1},$$

for a single point  $j$ . This is the configuration of  $P_{x_2}$  for the above example of one distance and one new point  $j$  in  $F_2$ . For more than one new point in  $F_2$ ,  $P_{x_2}$  is of size  $u_2 \times u_2$  where  $u_2$  is twice the number of new points. The elements of  $P_{x_2}$  are assumed to come from some independent source (e.g. a previous adjustment) and are extracted from the inverse  $C_x^{-1}$  of the variance covariance matrix of this previous solution. If there is no previous solution or a priori knowledge of  $X_2$ , then  $P_{x_2}$  is assumed to be zero.

Now that all the necessary matrices are available, the evaluation of the new variance covariance matrix  $C_x$  can proceed as outlined above. If there are no new parameters  $X_2$ , then  $A_2$ ,  $P_{x_2}$ ,  $N_2$ ,  $N_{12}$ ,  $Q_{12}$  and  $Q_2$  are all zero, and the new variance covariance matrix  $C_x$  is given simply by equation (2-47). A numerical example of this case is presented in Appendix I.

When the parameters  $X_2$  and observables  $L_2$  are deleted from the existing design, then the  $C_x$  matrix composed of  $Q_1$ ,  $Q_{12}$  and  $Q_2$  already exists, and it is required to compute  $N_1^{-1}$  corresponding to the parameters not being deleted. As outlined in section 2.2.2, the expressions are

$$N_1^{-1} = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} A_{21}^T [C_{L_2} - A_{21} \tilde{N}_1^{-1} A_{21}^T]^{-1} A_{21} \tilde{N}_1^{-1} \quad (\text{cf. eq. (2-48)}),$$

where

$$\tilde{N}_1^{-1} = Q_1^{-1} - Q_1^{-1} N_{12} [N_2 + N_{12}^T Q_1 N_{12}]^{-1} N_{12}^T Q_1^{-1} \quad (\text{cf. eq. (2-49)}),$$

in which

$$N_{12} = A_{21}^T P_2 A_2 \quad (\text{see eq. (2-36)}),$$

$$N_2 = P_{x_2} + A_2^T P_2 A_2 \quad (\text{see eq. (2-36)}).$$

As was the case for addition of a new station by a distance observation, the  $A_{21}$ ,  $A_2$ ,  $P_2$  and  $P_{x_2}$  matrices are first computed to enable the evaluation of  $N_2$ ,  $N_{12}$ , and  $N^{-1}$  and  $N_1^{-1}$  above. It is important that the same weight matrices  $P_2$  and  $P_{x_2}$  used to add the distance originally are used when subtracting it. Continuing with the "addition of distance" example, if station  $j$  is deleted by subtracting a distance from  $i$  to  $j$ , then  $A_{21}$  is of size  $1 \times 10$ ,  $A_2$  is  $1 \times 2$  and  $P_2$  is  $1 \times 1$  with  $1/\sigma_{s_{ij}}^2$  on the diagonal, exactly as in the case of addition. The desired variance covariance matrix  $N_1^{-1}$  corresponding to all points except  $j$  is, however,  $10 \times 10$ , with the two rows and columns corresponding to point  $j$  eliminated.

### 3.2 Azimuths

The mathematical model for an azimuth observable  $\alpha_{ij}$  from point  $i$  to point  $j$  as depicted in Figure 3.2 is given as

$$F_{\alpha_{ij}} = \arctan \left[ \frac{x_j - x_i}{y_j - y_i} \right] - \alpha_{ij} = 0. \quad (3-4)$$

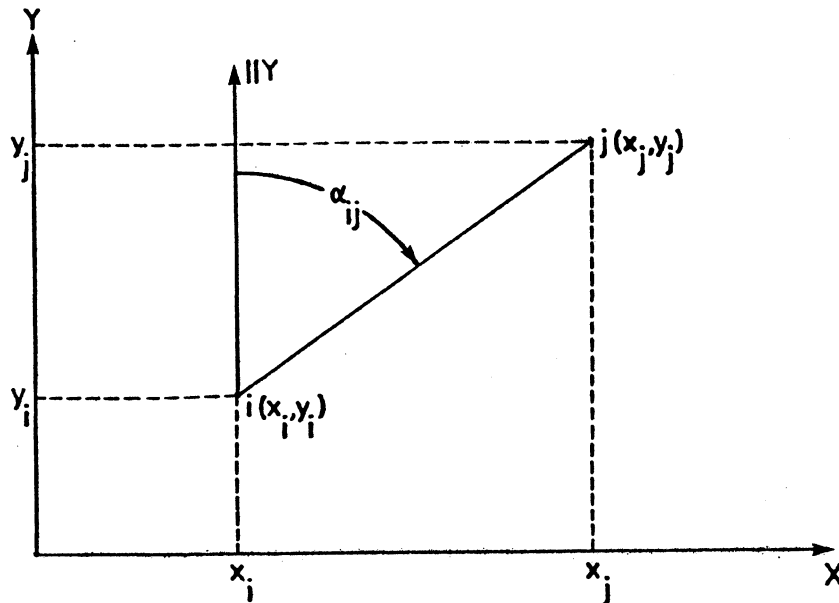


Figure 3.2. Azimuth on the Mapping Plane

The azimuth is an angle measured at point  $i$  from grid north clockwise to the desired point  $j$ . It is assumed that proper reductions are made so that the azimuth is measured from grid north, or parallel to the  $y$ -axis of the mapping plane. Proper reduction procedures for all types of observables are covered, for example, in Bomford [1975].

Since the azimuth observable does not contain any nuisance parameters, the mathematical models of section 2.2 are applicable here exactly as they were for the distance observable in section 3.1. The only differences are in the elements of the  $A$  and  $P$  matrices. One row of the  $A$  design matrix for an azimuth observable takes the form

$$A_{\alpha_{ij}} = \left. \frac{\partial F_{\alpha_{ij}}}{\partial X} \right|_{X = X^0} = \left[ \frac{-(y_j^0 - y_i^0)}{(S_{ij}^0)^2}; \frac{(x_j^0 - x_i^0)}{(S_{ij}^0)^2}; \frac{(y_j^0 - y_i^0)}{(S_{ij}^0)^2}; \frac{-(x_j^0 - x_i^0)}{(S_{ij}^0)^2} \right], \quad (3-5)$$



where the elements occupy their respective column positions for the from and to stations  $i$  and  $j$ . This form applies to both the  $A_{21}$  and  $A_2$  design matrices. Diagonal elements  $1/\sigma_{\alpha_{ij}}^2$  are used in forming the  $P_2$  (and  $C_{L_2}$ ) matrices. The  $P_{x_2}$  matrix has the same form as that discussed for distances in section 3.1.

Both addition and deletion of azimuths are carried out exactly as in section 3.1 for distances. An example of addition of azimuths along with a new station is contained in Appendix II.

### 3.3 Directions or Angles

The question of whether to use angles or directions when adjusting observations made with a theodolite has been debated for quite some time. Some authors [e.g. Bomford, 1975 and Rainsford, 1966] maintain that angles are preferable for adjustment whereas others [e.g. Allman and Bennett, 1966] would opt for directions. Reasons for these differing opinions are discussed in section 3.3.3 below.

This section covers the general adjustment procedures for both angles and directions as well as some of the reasons for conflicting conclusions about the subject. The final part of the section compares the two observable types from a network design point of view.

### 3.3.1 Directions

The general mathematical model for an observed direction is given as

$$d_{ij} = \arctan \left( \frac{x_j - x_i}{y_j - y_i} \right) - Z_i, \quad (3-6)$$

where  $Z_i$  = orientation unknown at station  $i$  (see Figure 3.3). The direction model is similar to that of the azimuth, except for the appearance of the orientation unknown which is the azimuth of the

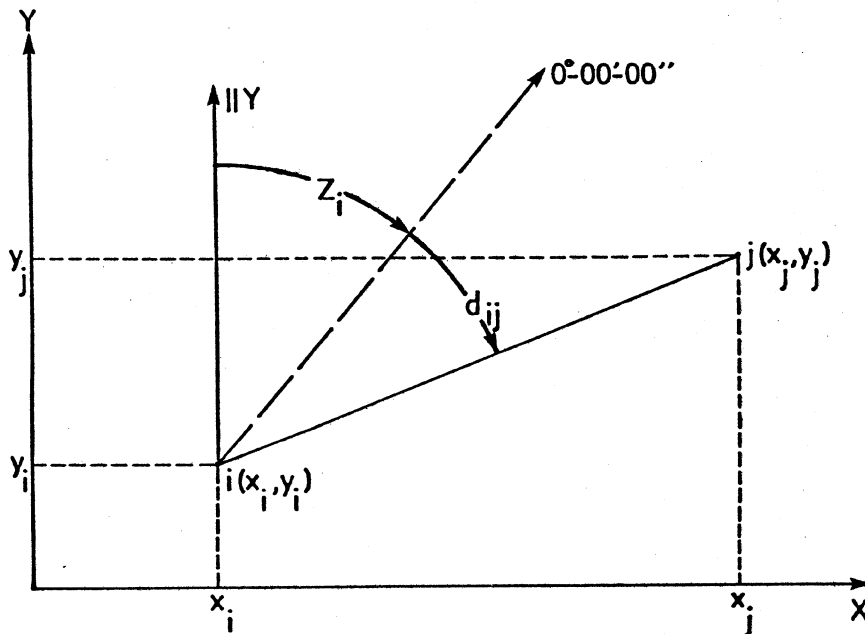


Figure 3.3. Direction on the Mapping Plane

zero reading on the horizontal circle of the theodolite.  $Z_i$  is common to all directions at station  $i$  which have this azimuth of the zero reading (commonly called a set of directions). Since the azimuth of the zero reading is unknown,  $Z_i$  is introduced to the design as a

nuisance parameter  $Y$ , and the models developed in sections 2.3 and 2.4 are applicable in this case. The form of the design matrices  $A_{1x_1}$ ,  $A_{2x_1}$  and  $A_{2x_2}$  (cf. eqs. (2-77) and (2-78)) is identical to that for azimuths; i.e.

$$A_{d_{ij}} = \frac{\partial F_{d_{ij}}}{\partial X} \Big|_{X = X^0} = \left[ \frac{-(y_j^0 - y_i^0)}{(S_{ij}^0)^2}; \frac{(x_j^0 - x_i^0)}{(S_{ij}^0)^2}; \frac{(y_j^0 - y_i^0)}{(S_{ij}^0)^2}; \frac{-(x_j^0 - x_i^0)}{(S_{ij}^0)^2} \right]. \quad (3-7)$$

The design matrices  $A_{1y}$  and  $A_{2y}$  corresponding to the orientation unknowns consist of -1's in the appropriate positions to correspond to the sets of directions being added or deleted. Once  $P_2$  (diagonal with  $1/\sigma_{d_{ij}}^2$  on the diagonal) and  $P_{x_2}$  are computed, then the addition or deletion of observables and stations proceeds as outlined in sections 2.3 and 2.4. A detailed illustrative example of addition of directions with common nuisance parameters between steps is contained in Appendix III.

### 3.3.2 Angles

The general equation for an angle observation is given as

$$a_{ijk} = \arctan \left( \frac{x_k - x_i}{y_k - y_i} \right) - \arctan \left( \frac{x_j - x_i}{y_j - y_i} \right). \quad (3-8)$$

As can be seen from Figure 3.4, an angle is essentially the difference between the two azimuths  $\alpha_{ik}$  and  $\alpha_{ij}$ .

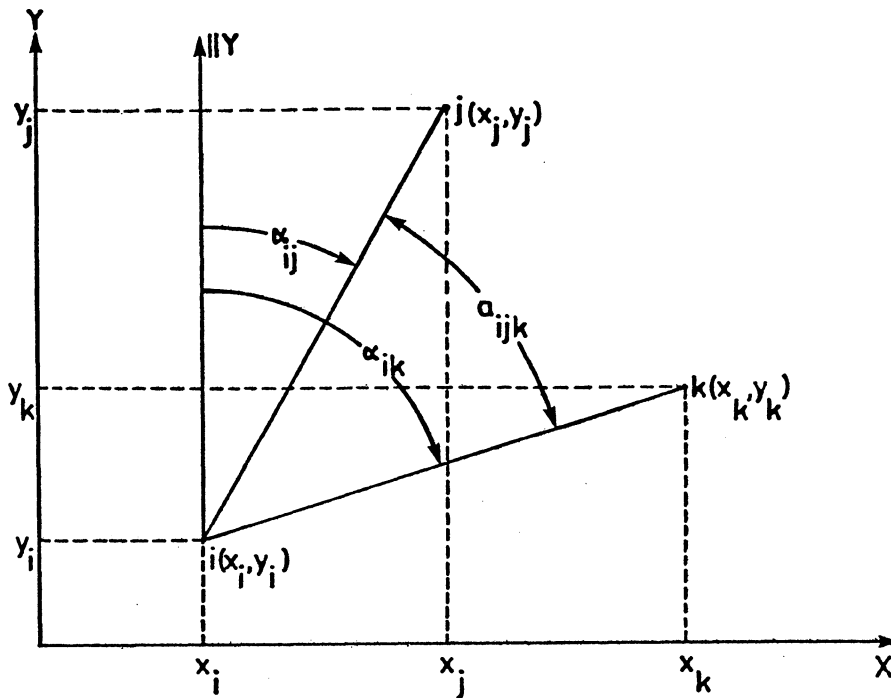


Figure 3.4. Angle on the Mapping Plane

The angle model has one notable difference from the other three observable types in that it involves the coordinates of three stations  $i$ ,  $j$  and  $k$  instead of  $i$ ,  $j$  as before. Because of this, the row of the  $A$  design matrix corresponding to an angle observation takes the following shape:

$$A_{\alpha_{ijk}} = \left[ \frac{-(y_k^{\circ} - y_i^{\circ})}{S_{ik}^{\circ 2}} + \frac{(y_j^{\circ} - y_i^{\circ})}{S_{ij}^{\circ 2}}; \frac{(x_k^{\circ} - x_i^{\circ})}{S_{ik}^{\circ 2}} - \frac{(x_j^{\circ} - x_i^{\circ})}{S_{ij}^{\circ 2}}; \frac{-(y_j^{\circ} - y_i^{\circ})}{S_{ij}^{\circ 2}}; \frac{(x_j^{\circ} - x_i^{\circ})}{S_{ij}^{\circ 2}}; \frac{(y_k^{\circ} - y_i^{\circ})}{S_{ik}^{\circ 2}}; \frac{-(x_k^{\circ} - x_i^{\circ})}{S_{ik}^{\circ 2}} \right], \quad (3-9)$$

where the above columns are for  $x_i$ ,  $y_i$ ,  $x_j$ ,  $y_j$ ,  $x_k$  and  $y_k$ , respectively.

Since there are no nuisance parameters in this case, the model developed in section 2.2 will suffice. The appropriate design ( $A_{21}$  and  $A_2$ ) and weight ( $P_2$  and  $P_{x_2}$ ) matrices are formed and combined to give the new or updated variance covariance matrix of the parameters  $C_x$ , exactly as was done for distances in section 3.1. The example in Appendix IV serves as an illustration of the details behind subtracting angles from an already existing solution.

### 3.3.3 Comparison of Angles and Directions

At first sight, it seems as though an angle is merely the difference between two directions in a set of directions. One would have

$$a_{ijk} = d_{ik} - d_{ij} = \arctan \left( \frac{x_k - x_i}{y_k - y_i} \right) - Z_i - \arctan \left( \frac{x_j - x_i}{y_j - y_i} \right) + Z_i,$$

$$a_{ijk} = \arctan \left( \frac{x_k - x_i}{y_k - y_i} \right) - \arctan \left( \frac{x_j - x_i}{y_j - y_i} \right),$$

which is identical to the expression given for angles in section 3.3.2. This is indeed the case when only one angle (i.e. two directions) is considered; however, complications arise as soon as angles are taken as the difference between successive directions in a set containing more than two directions. The successive angles then are dependent as each angle depends on one of the previous angles included directions. For instance, consider the situation depicted in Figure 3.5.

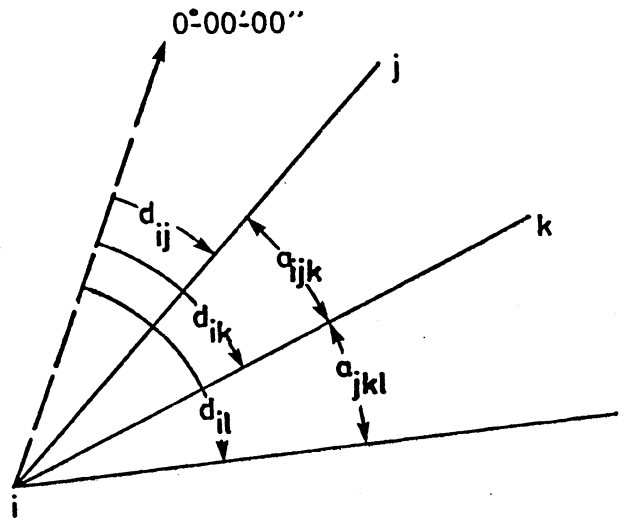


Figure 3.5. Angles and Directions

If the angles  $a_{ijk}$  and  $a_{ikl}$  are derived from a set of directions in the following fashion:

$$\begin{aligned} a_{ijk} &= d_{ik} - d_{ij} \\ a_{ikl} &= d_{il} - d_{ik} \end{aligned} \quad (3-10)$$

then the propagation of errors in the directions into the errors in the angles follows the basic covariance law; i.e.

$$C_a = BC_d B^T, \quad (3-11)$$

where

$$B = \frac{\partial a}{\partial d} \quad \text{and} \quad C_d = \begin{bmatrix} \sigma_{d_{ij}}^2 & 0 & 0 \\ 0 & \sigma_{d_{ik}}^2 & 0 \\ 0 & 0 & \sigma_{d_{il}}^2 \end{bmatrix}.$$

It must be noted that the directions in a set are considered uncorrelated or independent as is indicated by a diagonal  $C_d$  matrix. In situations where all of the directions of a set were measured at the same

time and there are no missing directions, the  $C_d$  matrix is indeed diagonal as in this case. For inconsistent sets where directions measured at many separate times are combined into one set,  $C_d$  is usually not diagonal and there will exist covariance terms in  $C_d$ . This topic is more fully discussed in Mepham [1976]. The case considered here, then, assumes that the direction sets have no missing pointings.

Returning to the derivation of  $C_a$ , the partial derivative matrix B is given by

$$B = \begin{bmatrix} \frac{\partial a_{ijk}}{\partial d_{ij}} & \frac{\partial a_{ijk}}{\partial d_{ik}} & \frac{\partial a_{ijk}}{\partial d_{il}} \\ \frac{\partial a_{ikl}}{\partial d_{ij}} & \frac{\partial a_{ikl}}{\partial d_{ik}} & \frac{\partial a_{ikl}}{\partial d_{il}} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad (3-12)$$

Applying the covariance law (3-11), the final result is

$$C_a = \begin{bmatrix} \sigma_{d_{ij}}^2 + \sigma_{d_{jk}}^2 & -\sigma_{d_{ik}}^2 \\ -\sigma_{d_{ik}}^2 & \sigma_{d_{ik}}^2 + \sigma_{d_{il}}^2 \end{bmatrix}. \quad (3-13)$$

This is the variance covariance matrix of successive angles derived from a complete set of three directions. If  $C_a$  is for more than two angles, then only adjacent angles are correlated.

The above  $C_a$  (eq. (3-13)) must be used to get results identical to those given by directions. If, however, one considers the angles as being independent, then a different result is obtained. It is shown in Appendix V that the variance covariance matrix of a network

design by independent angles is significantly larger than that computed by the direction or correlated angle method. Thus, from the preanalysis point of view, the direction or correlated angle method seems more expedient.

The discussion of which method should be used in an adjustment is worth considering briefly. Rainsford [1954] comes up with a coefficient of correlation  $\gamma$  between real errors and least squares corrections which is computed as

$$\gamma^2 = \frac{r}{n}, \quad (3-14)$$

where  $r$  = number of conditions in the condition adjustment (represented

as  $F(L) = 0$  [e.g. Mikhail, 1976; Krakiwsky, 1975]),

$n$  = number of observations  $L$  in the condition adjustment.

Using some hypothesized examples with known observational errors and adjusting them by both the angle and direction methods, he arrives at a fairly good agreement between the estimated correlation coefficient (eq. (3-14)) and the true correlation coefficient. Since  $\gamma$  is always less for directions than for angles (because  $n$  is always greater), Rainsford concludes that the angles are better suited for adjustment as the computed least squares corrections are more highly correlated with the real errors.

Also basing their work on the condition adjustment, Allman and Bennett [1966] use the minimum sum of weighted squared residuals ( $V^T P V$ ) as the testing criteria. Their numerical example shows that directions and correlated angles give a smaller  $V^T P V$  than do uncorrelated



angles. Thus, they conclude that directions are more correct as they fulfill the least squares condition (i.e.  $V^T PV = \text{minimum}$ ) better.

It is the author's opinion that any least squares adjustment procedure must try to interpret the actual observation procedure correctly so that the adjustment corresponds to what has been done in the field. If theodolite observations are made by measuring two directions per set and choosing a new zero setting for each set, then use of the uncorrelated angle method is justified. If, however, the same zero setting is used for observing three or more directions in the same set, then the direction or correlated angle method corresponds closer to the observing procedure than does the uncorrelated angle method. As well, for three or more directions per set, the results of Appendix V indicate that directions are preferable to uncorrelated angles from the preanalysis point of view.

### 3.4 Confidence Ellipses

The main objective of a preanalysis is to determine the accuracy obtained from a proposed network and its observables. This accuracy is best characterized by confidence regions called confidence ellipses. It is the purpose of this section to outline the procedures used to obtain both the point and relative ellipses as well as the methods of increasing the confidence level from standard (about 39%) to a higher level (e.g. 95%).

### 3.4.1 Computation of the Standard Ellipse

The standard error ellipse is obtained directly from the variance covariance matrix  $C_x$  of the unknown points being considered. As explained below, this procedure is based on the theory of multivariate statistics, and is an important part of the method of preanalysis.

From Hogg and Craig [1970], it is known that the quadratic form for the parameters (which have a multivariate normal distribution) is given as

$$(X - \mu_x)^T C_x^{-1} (X - \mu_x) = \chi^2_{q, 1-\alpha} \quad (3-15)$$

where  $q$  = the degrees of freedom for the  $\chi^2$  distribution, which in this case may be interpreted as the dimension of the parameter space,

$X$  = vector of  $n$  parameters,

$\mu_x$  = vector of  $n$  constants,

$C_x$  = variance covariance matrix of  $X$ ,

$1-\alpha$  = probability associated with the  $\chi^2$  distribution.

For the two dimensional case being considered here,  $X$  is the true value of the parameters ( $x, y$ ),  $\mu_x$  is the estimated (from the adjustment) values of the parameters, and  $C_x$  is the variance covariance matrix coming from the adjustment or preanalysis. Assuming now that the  $\mu_x$  vector has been subtracted from the true values, equation (3-15) becomes

$$[x \quad y] \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \chi_{2,1-\alpha}^2 \quad (3-16)$$

Through the process of diagonalization of the variance covariance matrix in (3-16), a new system is obtained whereby the transformed variables  $(x', y')$  are uncorrelated, and  $C_x$  is diagonal. This process is called the eigenvalue problem [see e.g. Kreyszig, 1972], and when applied to the problem at hand yields

$$[x' \quad y'] \begin{bmatrix} \sigma_x'^2 & 0 \\ 0 & \sigma_y'^2 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \chi_{2,1-\alpha}^2 \quad (3-17)$$

This process is interpreted in two dimensional space as being the rotation of the axes of the coordinate system  $(x, y)$  into a system  $(x', y')$  which corresponds to the direction of maximum and minimum variance in the coordinate space (see Figure 3.6).

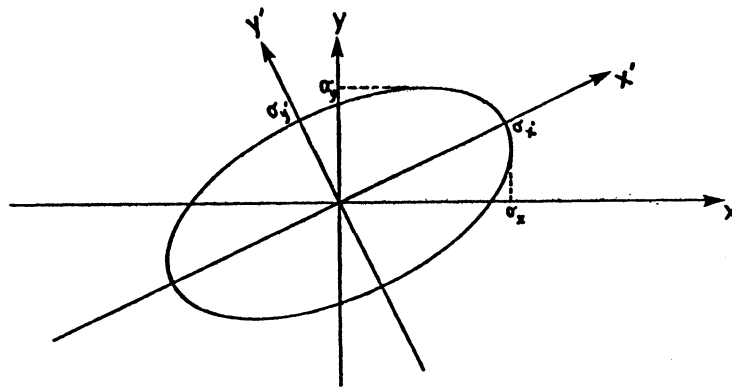


Figure 3.6. Relationship of Grid and Confidence Ellipse Coordinate Frames.

Following the development of Krakiwsky and Thomson [1978], the expansion of equation (3-17) leads to

$$\frac{x'^2}{\chi_{2, 1-\alpha}^2 \sigma_x'^2} + \frac{y'^2}{\chi_{2, 1-\alpha}^2 \sigma_y'^2} = 1, \quad (3-18)$$

which is obviously the equation of an ellipse with semimajor and semiminor axes of  $\sqrt{\chi_{2, 1-\alpha}^2 \sigma_x'^2}$  and  $\sqrt{\chi_{2, 1-\alpha}^2 \sigma_y'^2}$  respectively. Taking  $\chi_{2, 1-\alpha}^2 = 1$ , it is seen that the ellipse which results directly from the  $C_x$  matrix corresponds to an  $\alpha$  of 0.6065, or a probability level of  $1-\alpha = 0.3935$ . Thus, the standard ellipse (ellipse resulting directly from  $C_x$  with  $\chi_{2, 1-\alpha}^2 = 1$ ) is associated with a confidence level of 39.35%. Increasing this confidence to a higher level depends on various assumptions, and is not simply a case of choosing a  $\chi^2$  value for a higher probability level. This problem is further discussed in section 3.4.2.

#### 3.4.1.1 Point Ellipses

The accuracy of each station in a horizontal network is usually depicted by the so-called point ellipse. This ellipse is characterized by its semimajor and semiminor axes as well as its orientation in the (x, y) plane. As already discussed in section 3.4.1, the semimajor and semiminor axes of the standard error ellipse are the square roots of the eigenvalues of the variance covariance matrix for the particular station. This station variance covariance matrix  $C_{x_s}$  is a two by two submatrix of the variance covariance matrix  $C_x$  for all of the stations in the network; i.e.

$$C_{x_s} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} . \quad (3-19)$$

Performing an eigenvalue problem on  $C_{x_s}$  yields the eigenvalues

$$\sigma_{x'}^2 = \frac{(\sigma_x^2 + \sigma_y^2) + \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2}}{2} , \quad (3-20)$$

$$\sigma_{y'}^2 = \frac{(\sigma_x^2 + \sigma_y^2) - \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2}}{2} .$$

In accordance with equation (3-18), the semimajor and semiminor axes of the standard station ellipse (see also Figure 3.6) are the square roots of the above eigenvalues.

The orientation of the axes of the station ellipse is found by determining the orthogonal rotation matrix  $R$  which produces the diagonalized variance covariance matrix in the following relationship:

$$R^T C_{x_s} R = \begin{bmatrix} \sigma_{x'}^2 & 0 \\ 0 & \sigma_{y'}^2 \end{bmatrix} . \quad (3-21)$$

The columns of  $R$  are the normalized eigenvectors corresponding to the eigenvalues in equation (3-20) [Mikhail, 1976]. Computing the normalized eigenvectors (see e.g. Kreysig, 1972) and substituting them for the columns of  $R$  yields

$$R = \begin{bmatrix} \frac{\sigma_{xy}}{(\sigma_{xy}^2 + (\sigma_{x'}^2 - \sigma_x^2)^2)^{1/2}} & \frac{\sigma_{xy}}{(\sigma_{xy}^2 + (\sigma_{y'}^2 - \sigma_x^2)^2)^{1/2}} \\ \frac{(\sigma_{x'}^2 - \sigma_x^2)}{(\sigma_{xy}^2 + (\sigma_{x'}^2 - \sigma_x^2)^2)^{1/2}} & \frac{(\sigma_{y'}^2 - \sigma_x^2)}{(\sigma_{xy}^2 + (\sigma_{y'}^2 - \sigma_x^2)^2)^{1/2}} \end{bmatrix} \quad (3-22)$$

Regarding R as an orthogonal rotation matrix of the form

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad (3-23)$$

then the orientation angle  $\theta$  (measured positive counterclockwise from the +x axis (see Figure 3.7)) of the semimajor axis of the standard error ellipse at a station is

$$\theta = \text{sign}(\sin \theta) \cdot \arccos \left[ \frac{\sigma_{xy}}{(\sigma_{xy}^2 + (\sigma_{x'}^2 - \sigma_x^2)^2)^{1/2}} \right], \quad (3-24)$$

where  $\text{sign}(\sin \theta) = \text{sign}(\sigma_{x'}^2 - \sigma_x^2)$ .

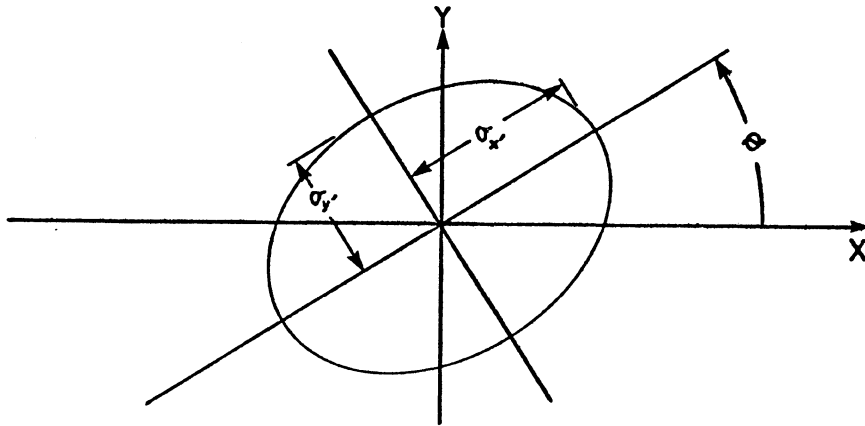


Figure 3.7. The Standard Error Ellipse.

The usual expression for the orientation [e.g. Mikhail, 1976; Richardus, 1974] is

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} . \quad (3-25)$$

Equivalence of the two formulae for orientation  $\theta$  (i.e. eqs. (3-24) and (3-25)) is proved as follows. Writing  $\tan \theta$  from equations (3-23) and (3-22) as

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{(\sigma_x^2, -\sigma_x^2)}{\sigma_{xy}} \quad (3-26)$$

and using the identity  $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ , yields

$$\tan 2\theta = \frac{2\sigma_{xy}(\sigma_x^2, -\sigma_x^2)}{\sigma_{xy}^2 - (\sigma_x^2, -\sigma_x^2)^2} . \quad (3-27)$$

Substituting  $\sigma_x^2$ , from equation (3-20) gives

$$(\sigma_x^2, -\sigma_x^2) = \frac{(\sigma_y^2 - \sigma_x^2) + \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2}}{2} \quad (3-28)$$

and

$$(\sigma_x^2, -\sigma_x^2)^2 = \frac{2(\sigma_y^2 - \sigma_x^2)^2 + 2(\sigma_y^2 - \sigma_x^2) \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2} + 4\sigma_{xy}^2}{4} . \quad (3-29)$$

Substitution of equations (3-28) and (3-29) into equation (3-27) results in

$$\tan 2\theta = \frac{4\sigma_{xy} [(\sigma_y^2 - \sigma_x^2) + \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2}]}{-2(\sigma_y^2 - \sigma_x^2) [(\sigma_y^2 - \sigma_x^2) + \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2}]} , \quad (3-30)$$

or

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} . \quad (3-31)$$

Thus the usual formula (eq. (3-25)) is equivalent to equation (3-24).

#### 3.4.1.2 Relative Ellipses

Whereas the station ellipses represent the error in an unknown station with respect to the fixed stations, relative ellipses are necessary to determine the accuracy of one unknown station with respect to another unknown station. This is expressed as in Krakiwsky and Thomson [1978], where the relative ellipses are taken to represent the coordinate differences

$$\begin{aligned} \Delta x_{ij} &= x_j - x_i , \\ \Delta y_{ij} &= y_j - y_i . \end{aligned} \quad (3-32)$$

Application of the covariance law to equation (3-32) yields

$$C_{\Delta x, \Delta y} = BC_{x, y} B^T , \quad (3-33)$$

where

$$B = \frac{\partial F}{\partial x, y} ,$$

and F is given as equations (3-32).

Expansion of equation (3-33) yields



$$C_{\Delta x, \Delta y} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{x_i}^2 & \sigma_{x_i y_i} & \sigma_{x_i x_j} & \sigma_{x_i y_j} \\ \sigma_{x_i y_i} & \sigma_{y_i}^2 & \sigma_{y_i x_j} & \sigma_{y_i y_j} \\ \sigma_{x_i x_j} & \sigma_{y_i x_j} & \sigma_{x_j}^2 & \sigma_{x_j y_j} \\ \sigma_{x_i y_j} & \sigma_{y_i y_j} & \sigma_{x_j y_j} & \sigma_{y_j}^2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (3-34)$$

or

$$C_{\Delta x \Delta y} = \begin{bmatrix} \sigma_{x_i}^2 - 2\sigma_{x_i x_j} + \sigma_{x_j}^2 & \sigma_{x_i y_i} - \sigma_{y_i x_j} - \sigma_{x_i y_j} + \sigma_{x_j y_j} \\ \sigma_{x_i y_i} - \sigma_{y_i x_j} - \sigma_{x_i y_j} + \sigma_{x_j y_j} & \sigma_{y_i}^2 - 2\sigma_{y_i y_j} + \sigma_{y_j}^2 \end{bmatrix}. \quad (3-35)$$

To compute the standard relative error ellipse, the formulae developed in section 3.4.1.1 (i.e. eqs. (3-20) and (3-24)) are used directly with the substitutions

$$\begin{aligned} \sigma_x^2 &= \sigma_{x_i}^2 - 2\sigma_{x_i x_j} + \sigma_{x_j}^2, \\ \sigma_{xy} &= \sigma_{x_i y_i} - \sigma_{y_i x_j} - \sigma_{x_i y_j} + \sigma_{x_j y_j}, \\ \sigma_y^2 &= \sigma_{y_i}^2 - 2\sigma_{y_i y_j} + \sigma_{y_j}^2. \end{aligned} \quad (3-36)$$

The resulting ellipse represents the relative precision of two unknown points.

One other point which should be mentioned is that the matrix  $C_{x, y}$  in equations (3-33) and (3-34) is a submatrix of the  $C_x$  matrix for the whole network. One must ensure that the correct elements of  $C_x$  are used in formulae (3-36).

### 3.4.2 Increasing the Confidence Level

As previously discussed in section 3.4.1, the above expressions are for the standard confidence ellipse at a 39.35% confidence level. The semimajor and semiminor axes of an ellipse of any other confidence level are usually denoted as

$$a = C\sigma_{x'} ,$$

and

$$b = C\sigma_{y'} ,$$

(3-37)

respectively. The so-called "C factor" in (3-37) is determined depending on various circumstances.

One of the circumstances is whether or not the a priori variance factor  $\sigma_0^2$  is assumed to be known or unknown. Although not previously mentioned, this factor is a scalar number which is used to obtain the weight matrix P of the observables if the variance covariance matrix  $C_L$  is known only on a relative basis. The a priori variance factor is then multiplied into  $C_L$  before inverting it to determine P. This is expressed as

$$P = \sigma_0^2 C_L^{-1} .$$

Even though in most instances of horizontal network computation  $\sigma_0^2$  is assumed to be 1, it is the assumption as to whether this factor is known or is to be estimated from the adjustment which affects computations of the C factor. If  $\sigma_0^2$  is assumed known (i.e. the scale of the observables is known), then the C factor is computed as outlined in section 3.4.1; i.e.

$$C = \sqrt{\chi_{2, 1-\alpha}^2} \quad (3-38)$$

If the a priori variance factor is to be estimated from the adjustment, then its estimate is called the a posteriori variance factor and is denoted as  $\hat{\sigma}_0^2$ . This results in the variance covariance matrix of the parameters being estimated as

$$\hat{C}_x = \hat{\sigma}_0^2 N^{-1} \quad (3-39)$$

It is shown in Hamilton [1964] that in this case the quadratic form of the parameters is distributed as

$$(X - \mu_x)^T \hat{C}_x^{-1} (X - \mu_x) = rF_{r, n-u, 1-\alpha} \quad (3-40)$$

where  $r$  = dimensionality of the coordinate space,

$n-u$  = degrees of freedom of the adjustment or preanalysis, and

$F$  = Fischer distribution.

Following the same procedure as outlined in section 3.4.1, the  $C$  factor is computed as

$$C = \sqrt{2F_{2, n-u, 1-\alpha}} \quad (3-41)$$

for the case where  $\sigma_0^2$  is assumed unknown. The  $C$  factor computed by equation (3-41) is always larger than those from equation (3-38) (for the same network preanalysis), but for large degrees of freedom  $n-u$ , they are practically identical.

One other point to be made, if  $\sigma_0^2$  is assumed unknown, is that the standard confidence ellipse resulting from a  $C$  factor of 1 is not at a 0.3935 probability level, but depends on the degrees of freedom  $n-u$ . From equation (3-41), it is seen that for  $C = 1$  and  $n-u = 5$ ,  $1-\alpha$  is

0.3661, and thus the standard ellipse has an associated confidence level of 36.6% instead of 39.35%. Again, for large degrees of freedom  $n-u$ , the standard ellipse confidence level approaches 39.35%.

The C factor for the point error ellipses already considered characterize the error of an unknown station with respect to the fixed stations. If all that is required is that the confidence ellipse of this one station be at the desired probability level of  $1-\alpha$ , then the C factors derived so far are sufficient. If, however, it is required that all of the unknown station confidence ellipses be at a probability level of  $1-\alpha$  simultaneously, then, as detailed in Vanicek and Krakiwsky [in prep.], the probability level of the individual station ellipses must be increased. This result stems from the so-called Bonferroni's inequality, which is stated as

$$\Pr \left\{ \prod_{k=1}^N (X-\mu_x)^T C_{x_k}^{-1} (X-\mu_x) \leq \chi^2_{q, 1-\frac{\alpha}{N}} \right\} \geq 1 - \sum_{k=1}^N \frac{\alpha}{N} = 1-\alpha, \quad (3-42)$$

where Pr = probability, and

$N$  = number of stations in the network.

This shows that the C factor required to compute simultaneous confidence ellipses is

$$C = \sqrt{\chi^2_{2, 1-\frac{\alpha}{N}}}, \quad (3-43)$$

where the  $\alpha$  in the nonsimultaneous C factor formula (eq. (3-38)) has been replaced by  $\frac{\alpha}{N}$ . Note that the probability is at least  $1-\alpha$  (by virtue of the  $\geq$  sign in equation (3-42)). This inequality is a result of omission of the cross covariance terms of  $C_x$  in equation (3-42) [Vanicek and Krakiwsky, in prep.; Krakiwsky, 1978].

The above argument also holds for simultaneous ellipses when  $\sigma_o^2$  is assumed unknown, and the C factor becomes

$$C = \sqrt{2F_{2, n-u, 1-\frac{\alpha}{N}}} \quad (3-44)$$

Formulae (3-43) and (3-44) are the most general for computation of the C factor. For nonsimultaneous confidence ellipses, N is simply assumed to be 1.

The above considerations concerning simultaneous probability of  $1-\alpha$  for the point ellipses are also applicable to the relative error ellipses. If it is required that N relative confidence ellipses have the probability  $1-\alpha$  simultaneously, then the simultaneous C factors (eqs. (3-43) and (3-44)) should be used. Since, however, relative confidence ellipses are concerned with only two points at once, the occurrence of nonsimultaneous relative error ellipses is most frequent.

### 3.5 Changing the Weight of an Observable

The facility to change the weight of an existing observable in the design is useful in many circumstances. This is accomplished simply by subtracting the observable with the old or previous weight, and adding it back again with the required new weight. For example, if for a design of six points which includes an azimuth with a standard deviation of 3"0, it is required to change the standard deviation to 2"0, then the following steps are performed.

1. Compute the design and weight matrices for deletion. Since no parameters or nuisance parameters between steps are involved, equation (2-28) is directly applicable, and the  $A_2$  matrix is a (1x12) matrix with the four nonzero elements of equation (3-5) in the appropriate columns. The matrix  $M_2$  is simply a (1x1) matrix of the variance; i.e.  $(9''0)^2$ .
2. Equation (2-28) is evaluated using the -ve sign for deletion of an observable, the  $A_2$  and  $M_2$  matrices above, and the existing normal equations inverse  $N_1^{-1}$ .
3. The  $A_2$  and  $M_2$  matrices are formed for addition of the azimuth. The  $A_2$  matrix is exactly the same as in step 1 above, but  $M_2$  is now  $(4''0)^2$  reflecting the required new standard deviation of 2''0.
4. Equation (2-28) is again evaluated using  $A_2$  and  $M_2$  from step 3, the normal equations inverse  $C_x$  resulting from step 2, and the +ve sign to add the azimuth back again.

The procedure is identical (except for the different  $A_2$  elements) for all other observable types except directions. To change the weight of a direction, the models of section 2.4 are employed with all matrices referring to  $X_2$  being zero. Appendix III contains a detailed explanation of the formation of the appropriate matrices.

#### 4. COMPUTER IMPLEMENTATION OF NETWORK DESIGN MODELS

An interactive program to implement the mathematical models of chapters 2 and 3 has been developed at U.N.B. It was designed for use both by students of the Department of Surveying Engineering and practicing surveying engineers and land surveyors. It is presently limited to the addition and deletion of only observables; no parameter addition or deletion is possible with the present version. The mathematical model for these more complex cases have been developed in chapter 2.

##### 4.1 Basic Program Features

This section describes the main features of program NETDESIGN as well as the hardware used to implement it. A more detailed documentation of the program is contained in Crawford [1978].

##### 4.1.1 Hardware

The hardware configuration is shown in Figure 4.1. The basic elements are a Tektronix 4015-1 graphics terminal which is interfaced as an APL terminal to an IBM 370/3032 general purpose computer. The disk shown in Figure 4.1 is used to store parts of the program so that

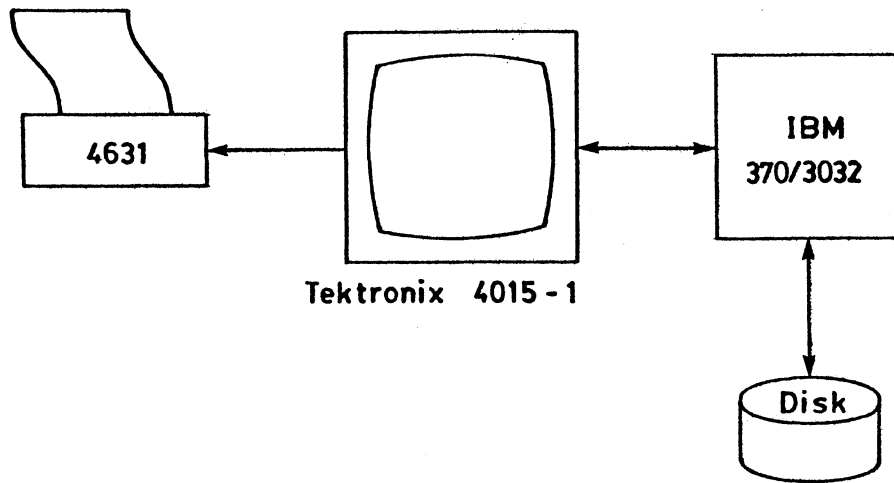


Figure 4.1 Hardware Configuration

the limited workspace size of 128 Kilobytes available at U.N.B. is efficiently utilized.

The Tektronix 4015 is the terminal where all input and output is performed. The terminal screen is the face of a storage type CRT and has a viewable screen size of about 38 by 28 cm with a resolution of 1024 by 780 addressable points. This makes for a reasonably good resolution as can be seen in the example shown in section 4.2. All input is presently done by means of the keyboard attached to the 4015. A Tektronix 4631 hard copy unit is used to make prints of information contained on the screen of the 4015. Hard copies can be made at any time during the design session. Figure 4.2 shows the 4015-1 and the 4631 hard copy unit.



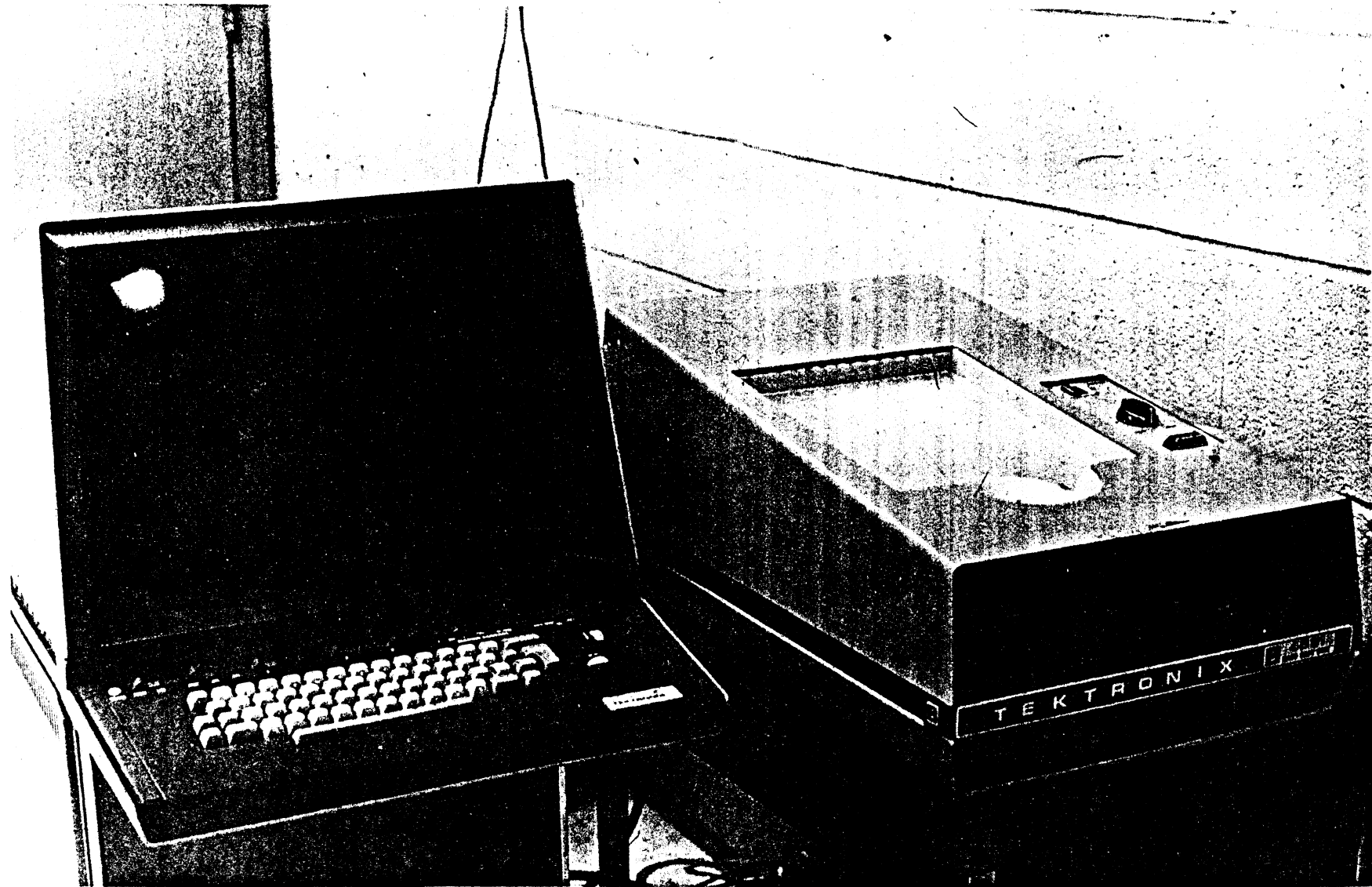


Figure 4.2. Tektronix 4015-1 Graphics Terminal and 4631 Hard Copy Unit

#### 4.1.2 Software

The programming language used by NETDESIGN is APL (A Programming Language; see e.g. Polivka and Pakin [1975]). The primary reason for using APL was that it was the only available programming language at the U.N.B. Computing Centre which supported the interactive plotting facilities of the Tektronix 4015. As well, APL is a good language for computing in an interactive fashion (i.e. prompting for and accepting input from a keyboard or other input device in real time). One minor disadvantage to this system was that the limited workspace size necessitated the storing of functions (subroutines) used in different parts of the program on disk, and calling them from disk when needed. This process is, however transparent to the user. Files of known or fixed points are also kept on disk so that one may search for known points within a given area (see section 4.2).

Figure 4.3 gives an overall description of the flow of the program which involves 3 basic steps:

1. Input initial data (limits of region where the network is to exist, known and unknown point coordinates and desired accuracy),
2. Input initial observables (as many directions, distances and azimuths as are needed to at least uniquely define the network),
3. Input update observables (if not satisfied with the present design, then add (subtract) new (old) observables in order to optimize).

These basic steps are better illustrated in the following example.

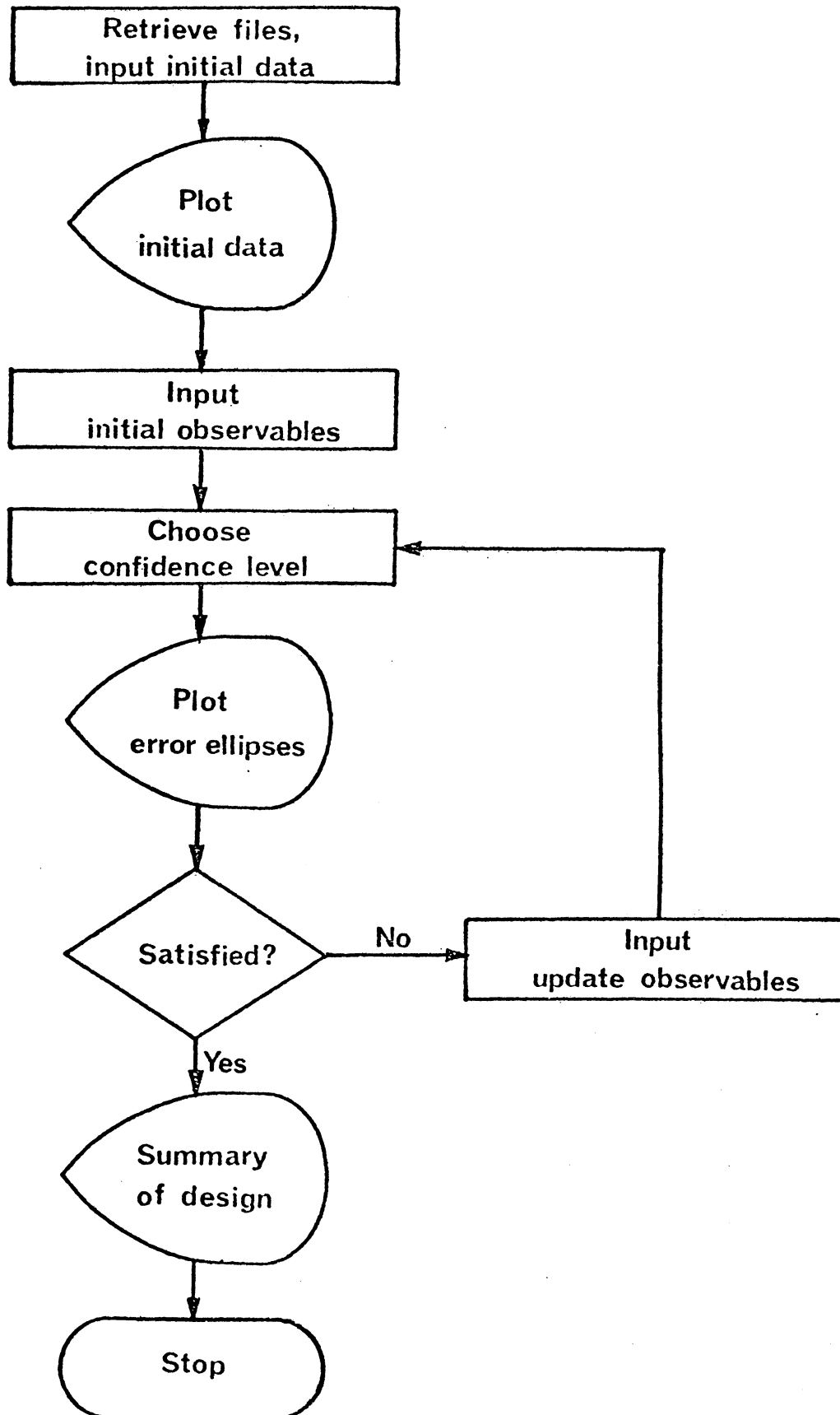


Figure 4.3. Basic Flowchart of NETDESIGN

#### 4.2 Example

The following example depicts the establishment of 3 unknown points in the form of a traverse. After typing in the word NETDESIGN, the initial stages of retrieving both program and data files begins, as well as the input of the initial data. As can be seen from Figure 4.4, this initial data includes the area in which the design is to be carried out (i.e. the design region), the approximate coordinates, desired accuracy and station numbers of the unknown points, and finally the coordinates and station numbers of the fixed or known points. The option also exists for the user not to define the known points, but to let the program search a file of known points to find those which are inside the design region.

```
FILE OPENED IR DSN=5351 APLKNOWN,DISP=OLD,RECFM=F,BLKSIZE=2320
FILE OPENED IR DSN=4919 NETFUNCS,DISP=OLD
FNS READ

DEFINE A SQUARE REGION
MAXIMUM X AND Y?295300 226500
MINIMUM X AND Y?293000 224200
ALREADY HAVE UNKNOWN COORDS. ASSIGNED FOR THIS REGION?N
TO END INPUT TYPE '/'
ALL INPUT IS TO BE IN METERS
RANGE OF STATION NUMBERS=1 TO 98
UNKNOWN PTS: X Y DESIRED ACCURACY STATION NUMBER?293682 225293 0.05 1
POINT WAS 293682 225293 0.05 1 OK?Y
UNKNOWN PTS: X Y DESIRED ACCURACY STATION NUMBER?293976 225607 0.05 2
POINT WAS 293976 225607 0.05 2 OK?Y
UNKNOWN PTS: X Y DESIRED ACCURACY STATION NUMBER?294421 225284 0.05 3
POINT WAS 294421 225284 0.05 3 OK?Y
UNKNOWN PTS: X Y DESIRED ACCURACY STATION NUMBER?/
X Y ACC. STATIONS ARE AS FOLLOWS:
  2.936820000E5    2.252930000E5    5.000000000E-2    1.000000000E0
  2.939760000E5    2.256070000E5    5.000000000E-2    2.000000000E0
  2.944210000E5    2.252840000E5    5.000000000E-2    3.000000000E0

TICK INTERVALS X AXIS?500
TICK INTERVALS Y AXIS?500
DO YOU WISH TO GIVE THE KNOWN STATIONS?Y
TO END INPUT TYPE '/'
ALL INPUT IS TO BE IN METERS
RANGE OF STATION NUMBERS=100 TO 99999
X Y STATION NUMBER?293054.171 225214.674 1102
POINT WAS 293054.171 225214.674 1102 OK?Y
X Y STATION NUMBER?293571.011 225598.373 1116
POINT WAS 293571.011 225598.373 1116 OK?Y
X Y STATION NUMBER?295267.233 225419.706 1135
POINT WAS 295267.233 225419.706 1135 OK?Y
X Y STATION NUMBER?295004.038 225951.144 1106
POINT WAS 295004.038 225951.144 1106 OK?Y
X Y STATION NUMBER?/
```

Figure 4.4. Input Initial Data

After this initial data has been entered, then as shown in Figure 4.5, the initial data is plotted. The known points are marked with  $\Delta$ 's, and the unknown points are plotted as o's. The circles around the unknowns stations are the design circles representing the desired accuracy, which, in this case, have a radius of 5 cm. and a probability level of 0.99. It is these design circles which enable the user to see if his confidence ellipses meet the design criteria.

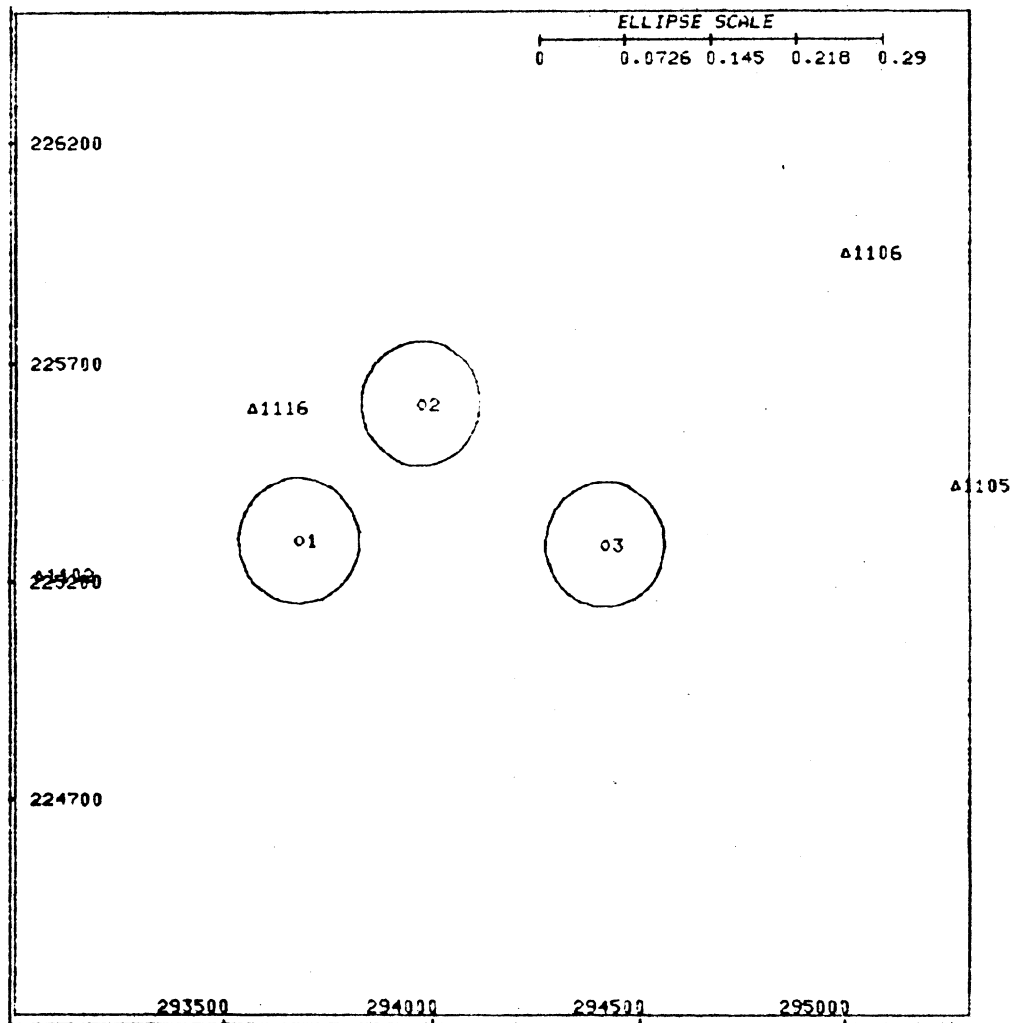


Figure 4.5. Plot of Initial Data

Figures 4.6 and 4.7 indicate the next step of the process; i.e. the input and plot of the initial observables. As can be seen in Figure 4.6, the observables are defined by their from station number, to station number, observables type and standard deviation. The observables are plotted as they are entered and are indicated on the plot by different symbols (see Figure 4.7). Enough observables must be given at this stage to at least uniquely define every unknown point. Once the initial observables have been defined, the user then chooses the confidence level desired (i.e. standard (~ 39%), 95% or 99%), and whether or not the a priori variance factor  $\sigma_0^2$  is assumed known (see section 3.4.2). The option to choose simultaneous confidence ellipses does not yet exist.

```

FNS READ
ADJUSTMENT OR FREANALYSIS?P
ALL INPUT IS TO BE IN METERS AND SECONDS
INPUT OBSERVATIONS:
TO END TYPE '/'
OBSERVATION TYPES ARE:
  1 FOR DIRECTION
  2 FOR DISTANCE
  3 FOR AZIMUTH
FROM STATION NO.: TO STATION NO.:
OBSERVATION TYPE; STANDARD DEV.
FROM TO (1,2,3) SIG?1116 1102 1 3
INPUT WAS 1116 1102 1 3 OK?Y
FROM TO (1,2,3) SIG?1116 1 1 3
INPUT WAS 1116 1 1 3 OK?Y
FROM TO (1,2,3) SIG?1116 1 2 0.02
INPUT WAS 1116 1 2 0.02 OK?Y
FROM TO (1,2,3) SIG?1 1116 1 2
INPUT WAS 1 1116 1 2 OK?Y
FROM TO (1,2,3) SIG?1 2 1 2
INPUT WAS 1 2 1 2 OK?Y
FROM TO (1,2,3) SIG?2 1 1 2
INPUT WAS 2 1 1 2 OK?Y
FROM TO (1,2,3) SIG?2 3 1 2
INPUT WAS 2 3 1 2 OK?Y
FROM TO (1,2,3) SIG?2 1 2 0.02
INPUT WAS 2 1 2 0.02 OK?Y
FROM TO (1,2,3) SIG?2 3 2 0.02
INPUT WAS 2 3 2 0.02 OK?Y
FROM TO (1,2,3) SIG?/
FNS READ
STANDARD ERROR ELLIPSES?N
CONFIDENCE LEVEL - 95 OR 99?99
A PRIORI VARIANCE FACTOR KNOWN?Y

```

Figure 4.6. Input Initial Observables

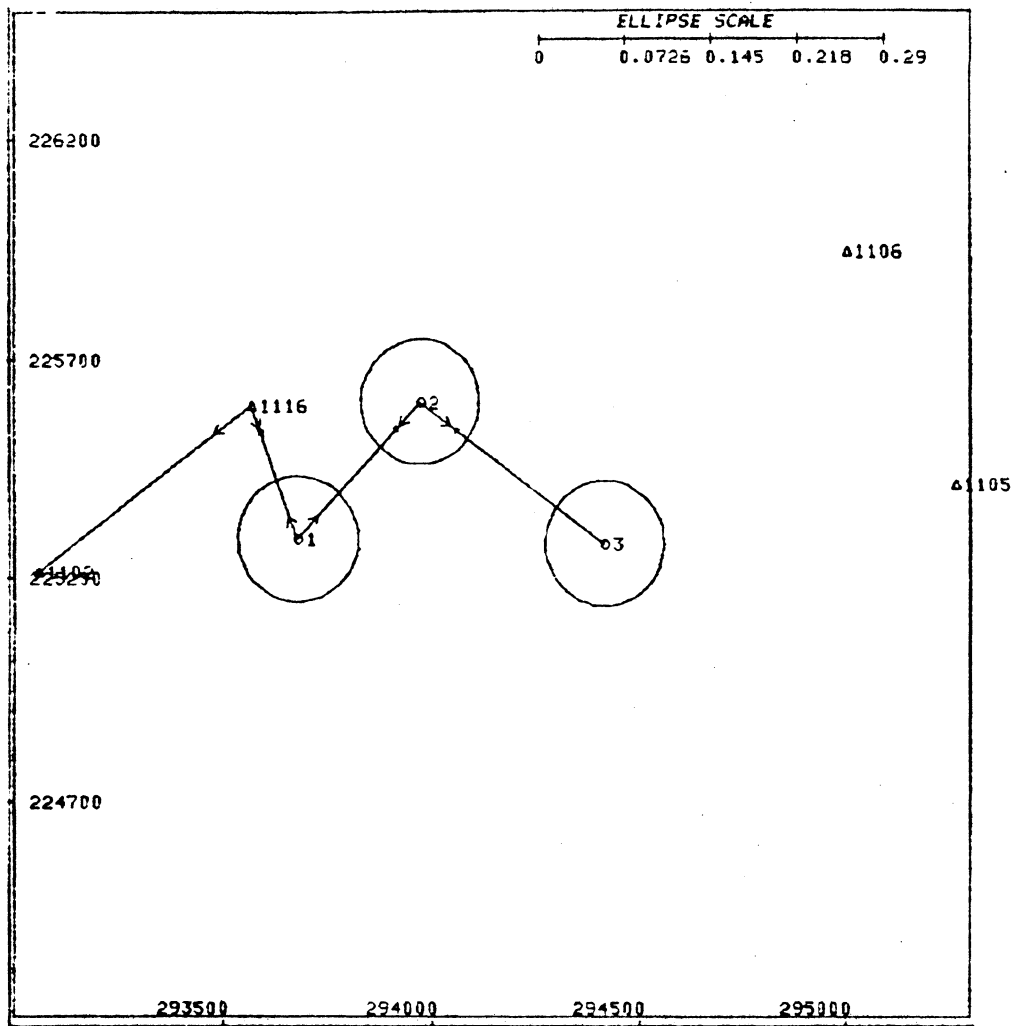


Figure 4.7. Plot of Initial Observables

The program now takes over and computes the  $C_x$  matrix from the given observables. Figure 4.8 shows the 99% confidence ellipses resulting from the initial observables of Figure 4.6. The figure clearly shows that all three points fail to meet the design circle accuracy of 5 cm. The relative confidence ellipses appear between the unknown stations.

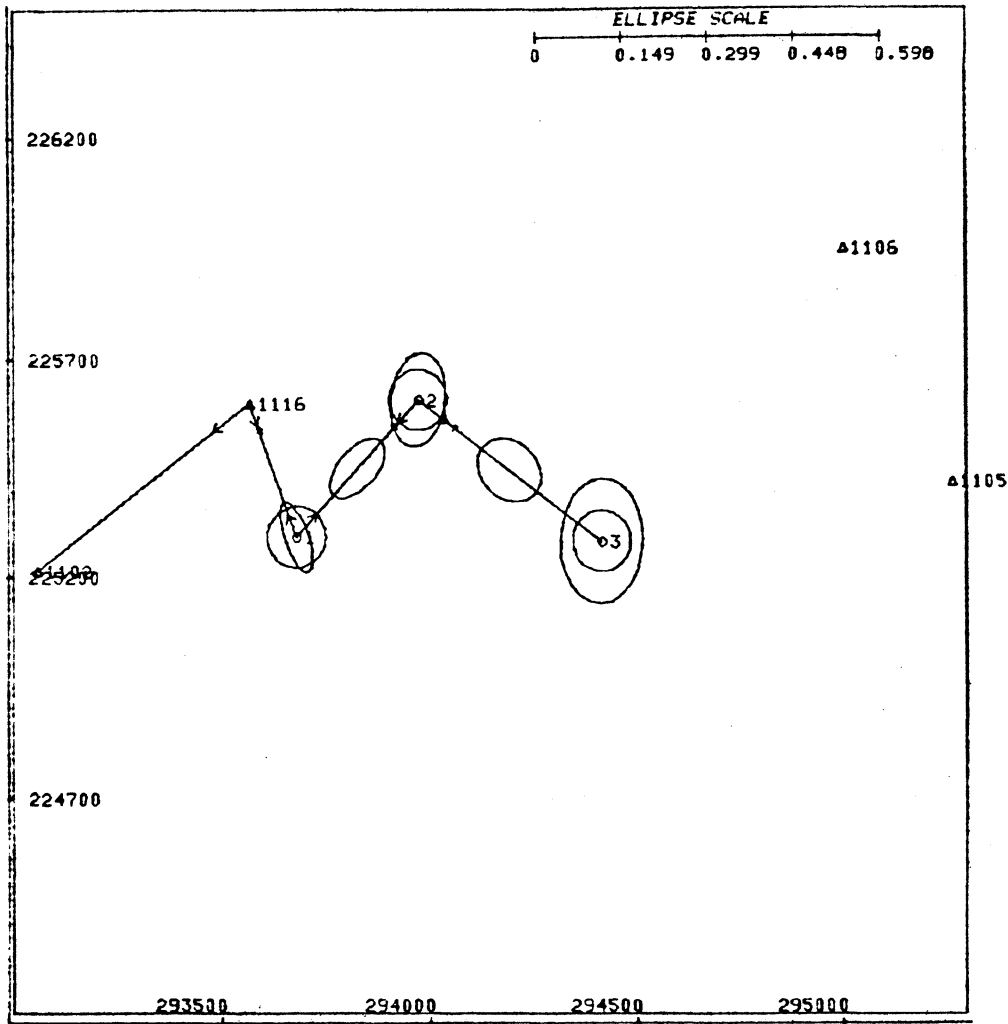


Figure 4.8. Confidence Ellipses from Initial Observables

Because the design criteria have not been met, the so-called update observables in Figure 4-9 are added. These observables amount to essentially typing the traverse to the known points 1105 and 1106. Figure 4.10 demonstrates the 99% confidence ellipses resulting from the addition of these update observables. This time the confidence ellipses do meet the design criteria of 5 cm radius error circles and the design is essentially complete. One could at this point further optimize the design by trying to get all of the station ellipses to take the



```

IS THIS SATISFACTORY?N
FNS READ
ALL INPUT IS TO BE IN METERS AND SECONDS
INPUT OBSERVATIONS:
TO END TYPE '/'
OBSERVATION TYPES ARE:
  1 FOR DIRECTION
  2 FOR DISTANCE
  3 FOR AZIMUTH
FROM STATION NO.: TO STATION NO.:
OBSERVATION TYPE; STANDARD DEV.
ADD OBSERVATIONS=1, SUBTRACT=0
FROM TO (1,2,3) SIG (1,0)3 2 1 1.5 1
INPUT WAS 3 2 1 1.5 1 OK?Y
FROM TO (1,2,3) SIG (1,0)3 1105 1 1.5 1
INPUT WAS 3 1105 1 1.5 1 OK?Y
FROM TO (1,2,3) SIG (1,0)1105 3 1 1.5 1
INPUT WAS 1105 3 1 1.5 1 OK?Y
FROM TO (1,2,3) SIG (1,0)1105 1106 1 1.5 1
INPUT WAS 1105 1106 1 1.5 1 OK?Y
FROM TO (1,2,3) SIG (1,0)1105 3 2 0.02 1
INPUT WAS 1105 3 2 0.02 1 OK?Y
FROM TO (1,2,3) SIG (1,0)1/
FNS READ
STANDARD ERROR ELLIPSES?N
CONFIDENCE LEVEL - 95 OR 99?99
APRIORI VARIANCE FACTOR KNOWN?Y
Figure 4.9. Update Observables
    
```

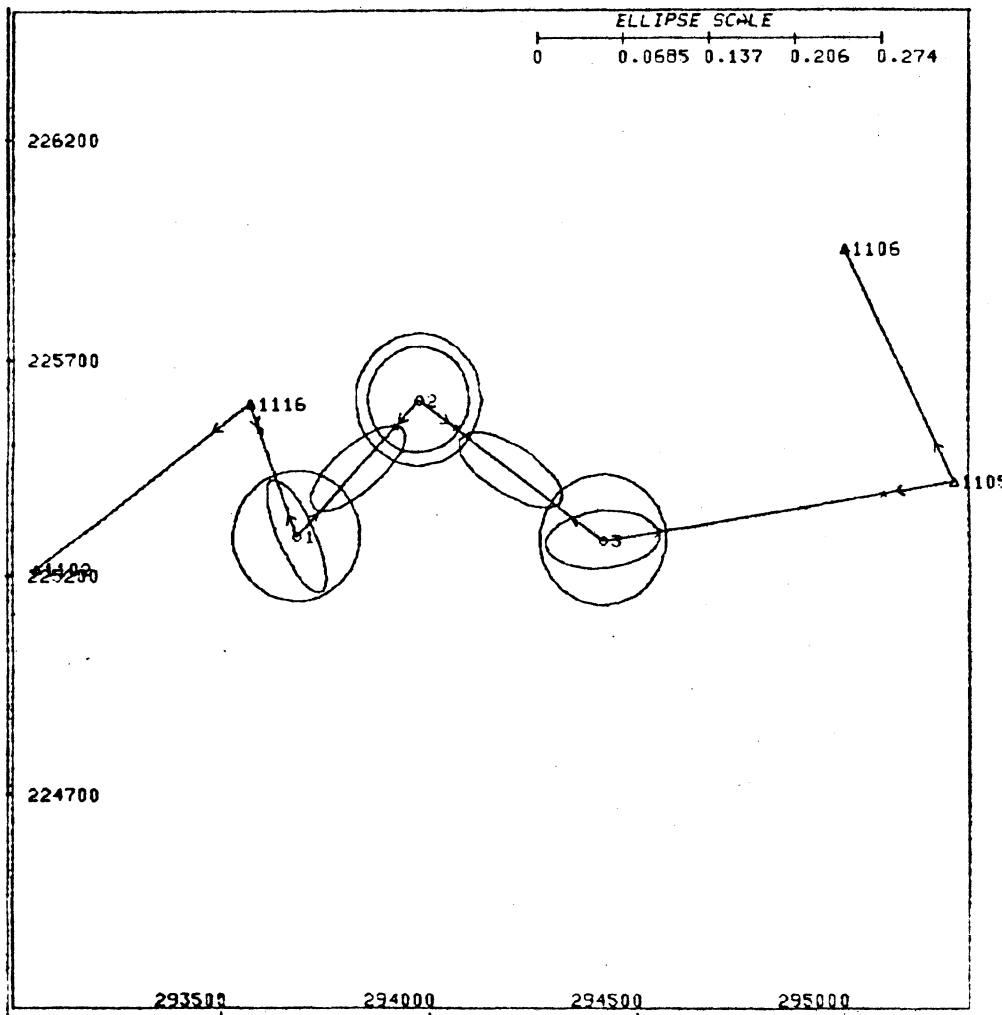


Figure 4.10. Updated Design

optimum shape of that at station 2. This would involve subtracting observables and adding others at different standard deviations. One can see by the shape of the relative error ellipses that the distances are less accurate than the directions (in a relative sense).

Assuming that the user is now satisfied with the design, he indicates this to the program, and the final step is to produce a summary of the design as shown in Figure 4.11. The summary includes the coordinates of both the known and unknown points, all observations that have been made, and the final confidence ellipses.

FIXED STATIONS:			UNKNOWN POINTS:			
STATION	X COORD.	Y COORD.	STATION	X COORD.	Y COORD.	DES.ACC.
1102	293054.171	225214.674	1	293682.000	225293.000	.050
1116	293571.011	225598.373	2	293976.000	225607.000	.050
1105	295267.293	225419.706	3	294421.000	225284.000	.050
1106	295004.038	225951.144				

OBSERVATIONS MADE:

1116	1102	1	3.0000
1116	1	1	3.0000
1116	1	2	.0200
1105	3	1	1.5000
1105	1106	1	1.5000
1105	3	2	.0200
1	1116	1	2.0000
1	2	1	2.0000
2	1	1	2.0000
2	3	1	2.0000
2	1	2	.0200
2	3	2	.0200
3	2	1	1.5000
3	1105	1	1.5000

RESULTS ACHIEVED AT 99 PERCENT CONFIDENCE LEVEL:

STATION ELLIPSES:

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
1	.047	.014	-65.7
2	.041	.040	39.1
3	.045	.022	5.0

RELATIVE ELLIPSES:

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
2	1	.046	.018	-38.9
3	2	.047	.019	-30.7

Figure 4.11. Summary of the Design

### 4.3 Results

The example shown in section 4.2 is a simple case of three unknown points tied to four known points in a traverse fashion. Other examples (see Nickerson et al. [1978]) have been done in order to illustrate the application of the network design ideas to other common networks encountered by surveying engineers. These examples have been investigated as to their actual cost at U.N.B., and as Table 4.1 shows, the approximate cost for designing a network is about \$3.50 per unknown point at the U.N.B. Computing Centre. This is a reasonable cost considering it may save time in the field where the establishment of stations is very costly, especially if unnecessary measurements are made.

Example	No. of unknown points	No. of known points	Total CPU used (sec)	Total connect time (min.)	Cost @ U.N.B.
1	3	4	50.3	20.8	\$11.85
2	5	2	67.6	24.7	\$15.47
3	10	6	165.6	47.0	\$36.08

Table 4.1. Results of Three Different Design Examples

The results have shown that the present program NETDESIGN is an economical method by which to design horizontal survey networks. It is also instructive for students to use this program in order to understand how different survey network geometries and observables affect the precision of their surveys. Although the program is by no means at the production stage, it has developed the simpler models to a workable level. At present, one drawback is that the Tektronix terminal must share the IBM 370 resources with all other APL users who happen to be signed on at the same time. At times, this causes considerable delays and the turnaround time (i.e. time for the computer to respond) is quite long. This problem is brought up again in Chapter 5.

## 5. CONCLUSIONS AND RECOMMENDATIONS

The following conclusions result from the research performed on the integration of the method of preanalysis of survey networks and interactive computer graphics.

1. The sequential model is most suitable for survey network design as it utilizes the already existing  $C_x$  matrix, which means that less computations are required than for designs done in a batch mode.
2. Directions more closely represent the usual observing procedure for theodolite observations and give a smaller variance covariance matrix for the unknown stations than do similarly weighted uncorrelated angles.
3. Interactive computer graphics is well suited for the process of preanalysis of survey networks. The near real time response and graphics capabilities result in a useful design tool which is also economically viable.
4. The present design system at U.N.B. has some limitations. The 128 Kbyte workspace size limits the number of unknown stations which can be dealt with at one time to about 20. The response or turn-around time of the system is sometimes inadequate due to a large user load on the IBM 370 computer system. Entering the data (i.e. by typing on the keyboard) is tedious when many points and observables are entered. If more than about 15 points are displayed

at once along with their associated station and relative confidence ellipses, the screen tends to be too cluttered to clearly see the detail of the design.

In light of these conclusions, and in looking ahead at future possibilities, the following are the main recommendations that can be made.

1. The present program NETDESIGN should be expanded to include all of the mathematical models developed herein which have not yet been implemented. These include allowing the known stations to have weights (section 1.2), permitting the addition or deletion of unknown points during the design (section 2.2), allowing the use of angles as observables, and incorporating the models for common nuisance parameters between steps (section 2.4). This would make the program better able to respond to the dynamic requirements of network design.
2. The possibility to design a survey network to a specified relative accuracy should be included. Since relative accuracy is usually the yardstick used to measure the precision of networks, this would give the program a greater degree of reality.
3. The ability to input both known and unknown points by means of a light pen or some such similar device would speed the data entry. This technique could also be used to enter observables by indicating the from and to stations with the light pen.

4. The addition of some local processing power to the graphics terminal (e.g. Tektronix OPTION 40, an 8 bit microprocessor and up to 32 K bytes RAM memory board combination) would decrease the turnaround time significantly.
5. Large networks could be designed by the use of "windows" on the display screen. This would entail defining regions within large networks and then plotting this desired region at a larger scale. One could then optimize the design in regions of a manageable (i.e. screen not cluttered) size.
6. The capability to perform adjustments as well as designs is a desirable capability. The graphics terminal would plot the adjusted network with its confidence ellipses as well as draw bar graphs or other graphical interpretations of the statistical analysis of the network.

Clearly, interactive computer graphics is a powerful tool and its use in surveying engineering can only continue to grow as people become aware of its immense potential.

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## INTRODUCTION TO APPENDICES

The following appendices are included to illustrate the various models developed in this thesis. The program used to compute these examples is not the program discussed in Chapter 4. Rather, it is a development program used by the author to implement the models in APL before transferring them to be compatible with the Tektronix 4015. This is the reason why the computer generated graphics is not identical to the example in Chapter 4. The precision of the results included here is 5 significant figures. The reason for this is to enable the inclusion of matrices which would be too large if printed with greater precision. One should note that all of the computations have been carried out in double precision (approximately 16 significant digits), and that the 5 significant digits are used only for printing purposes.

The results presented here have all been checked by doing a preanalysis as if the observables and stations of the final iteration or design were the initial case. As well, the results were compared to answers obtained from a program for nonsequential batch preanalysis [Steeves, 1978] and the results compare identically.

APPENDIX I. Example of Addition of Distances

The small network depicted in Figure AI.1 has been used to do a preanalysis. The points are actually representative of part of a network measured in Fredericton during 1975. The  $P_x$  or weight information is taken from an actual adjustment of data measured at that time. As can be seen from Figure AI.1, points 50 and 70 are weighted, there are no fixed points, and the observables considered at the initial stage are shown. The approximate coordinates of the points are listed in Table AI.1.

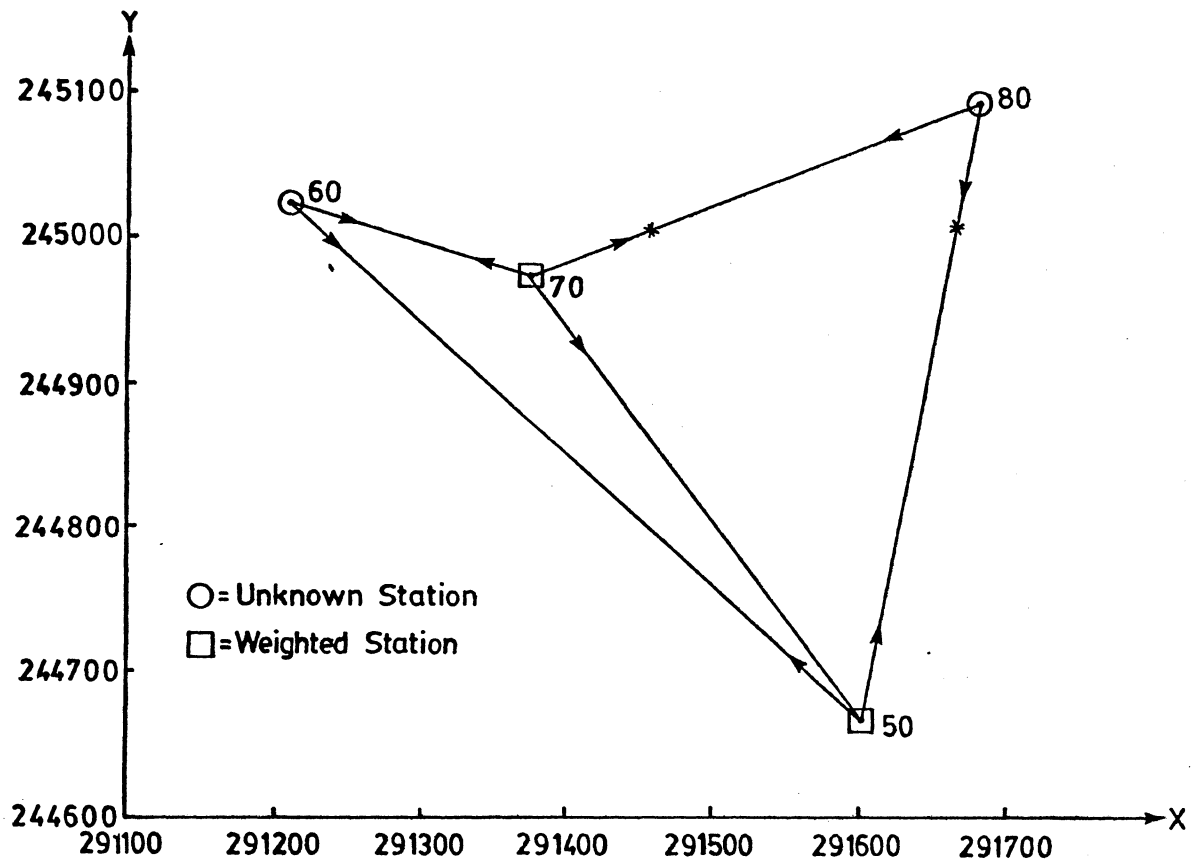


Figure AI.1. Example Network for Distance Addition

STATION	X COORD.	Y COORD.
50	291603.473	244665.211
60	291212.951	245022.422
70	291374.683	244972.597
80	291684.665	245091.121

Table AI.1. Approximate Coordinates of Points for Appendix I-IV  
Examples.

At the initial stage, the observables are added sequentially to build up the normal equations matrix  $N$ . The orientation unknowns are not eliminated as discussed in section 2.3 because we are not dealing with the sequential formation of  $C_x$ , but with  $N$ . The procedure required to eliminate  $Z$ 's when forming  $N$  has been developed in Steeves [1974], and essentially involves transforming the  $P$  or weight matrix for the set of directions into a matrix  $S$  defined as

$$S = \begin{bmatrix} P_1 + P_1^2 C & P_1 P_2 C & \dots & P_1 P_n C \\ & P_2 + P_2^2 C & \dots & P_2 P_n C \\ & & \ddots & \vdots \\ \text{symmetric} & & & P_n + P_n^2 C \end{bmatrix}, \quad (\text{AI-1})$$

where

$$C = \frac{-1}{\sum_{i=1}^n P_i},$$

and  $P_i$  = diagonal elements of the weight matrix  $P$ . Expression (AI-1) results from considering the model  $F(X_1, X_2, L) = 0$ , where  $X_2$  are the nuisance parameters, solving for  $X_2$  and substituting the result back in to solve for  $X_1$ .

Considering the sequential formation of the normal equations for Figure AI.1, the N matrix starts out as an 8x8 null matrix. The first set of directions added are those at station 50, which are to stations 50 and 80 with standard deviations of 2"0 and 1"0 respectively. The partial A matrix resulting from this set of directions is

$$A = \begin{array}{c} d_{50,60} \\ d_{50,80} \end{array} \begin{array}{c|c|c} \hline 50 & 60 & 80 \\ \hline \begin{bmatrix} -263.04 & -287.57 & 263.04 & 287.57 & 0 & 0 \\ -467.31 & 89.084 & 0 & 0 & 467.31 & -89.084 \end{bmatrix} \\ \hline \end{array},$$

and the S matrix resulting from equation (AI-1) is

$$S = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix}.$$

Note that each element of A is multiplied by  $\rho''$  (206264.8062) to make it compatible with the units of P. The matrix  $A^T S A$  is formed, and because the N matrix has rows and columns corresponding to the station sequence (50, 60, 70, 80),  $A^T S A$  is added to rows and columns (1, 2, 3, 4, 7, 8) of the normal equations matrix N to result in

$$N = \begin{bmatrix} 8345 & -15388 & 10746 & 11748 & 0 & 0 & -19091 & 3639.4 \\ & 28374 & -19815 & -21663 & 0 & 0 & 35203 & -6710.8 \\ & & 13838 & 15129 & 0 & 0 & -24584 & 4686.6 \\ & & & 16540 & 0 & 0 & -26877 & 5123.6 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ \text{symmetric} & & & & & & 43676 & -8326.0 \\ & & & & & & & 1587.2 \end{bmatrix},$$

after adding the first set of directions (to 5 significant figures). The other sets of directions, as well as the two distances shown in Figure AI.1 and listed in Table AI.4 (excepting the two distances from station 60) are added to N. After the observables have been added, the normal equations (again to 5 significant digits) is:

$$N = \begin{bmatrix} 93197 & 10052 & 25059 & 66482 & -80272 & -889.75 & -37984 & -75644 \\ & 90545 & -8673.8 & 23549 & -40025 & -62966 & 38647 & -51128 \\ & & 35809 & 89461 & -29646 & -102830 & -31222 & 22047 \\ & & & 285420 & -107520 & -370440 & -48423 & 61474 \\ & & & & 161910 & 119140 & -51988 & 28400 \\ & \text{symmetric} & & & & 601750 & -15422 & -168350 \\ & & & & & & 121190 & 25198 \\ & & & & & & & 158000 \end{bmatrix} .$$

After computing N, the weights for points 50 and 70 must now be added. As mentioned before, these weights were extracted from a previous adjustment, and the  $P_x$  matrix takes the form

$$P_x = \begin{array}{c|cc} & \text{---} 50 \text{---} & \text{---} 70 \text{---} \\ \hline 50 & 0.20406 \times 10^6 & -0.20588 \times 10^5 & -0.78524 \times 10^5 & 0.47842 \times 10^4 \\ \hline & & 0.14289 \times 10^6 & -0.38114 \times 10^5 & -0.56763 \times 10^5 \\ \hline 70 & & & 0.39262 \times 10^6 & 0.56543 \times 10^5 \\ \hline & \text{symmetric} & & & 0.64855 \times 10^5 \end{array} .$$

The zeroes for stations 60 and 80 have been omitted, and thus the above  $P_x$  matrix is added to the (1, 2, 5, 6) rows and columns of the N matrix to give

$$(N+P_x) = \begin{bmatrix} 297260 & -10536 & 25059 & 66482 & -158800 & 3894.5 & -37984 & -75644 \\ & 233440 & -8673.8 & 23549 & -78139 & -119730 & 38647 & -51128 \\ & & 35809 & 89461 & -29646 & -102830 & -31222 & 22047 \\ & & & 285420 & -107520 & -370440 & -48423 & 61474 \\ & & & & 554530 & 175690 & -51988 & 28400 \\ & \text{symmetric} & & & & 666610 & -15422 & -168350 \\ & & & & & & 121190 & 25198 \\ & & & & & & & 158000 \end{bmatrix} .$$

The inverse of  $(N+P_x)$  is taken to give the first  $C_x$  matrix, and from this  $C_x$  matrix the confidence ellipses are computed as discussed in section 3.4. The resulting error ellipses are listed in Table AI.2 and plotted in Figure AI.2, and arise from assuming that the confidence level is 95%, the a priori variance factor is known, and the station ellipses are of the nonsimultaneous type. This implies, then, that the C factor for the confidence ellipses is

$$C = \sqrt{\chi_{2,1-\alpha}^2} \sqrt{\chi_{2,0.95}^2} = \sqrt{5.99} = 2.45 .$$

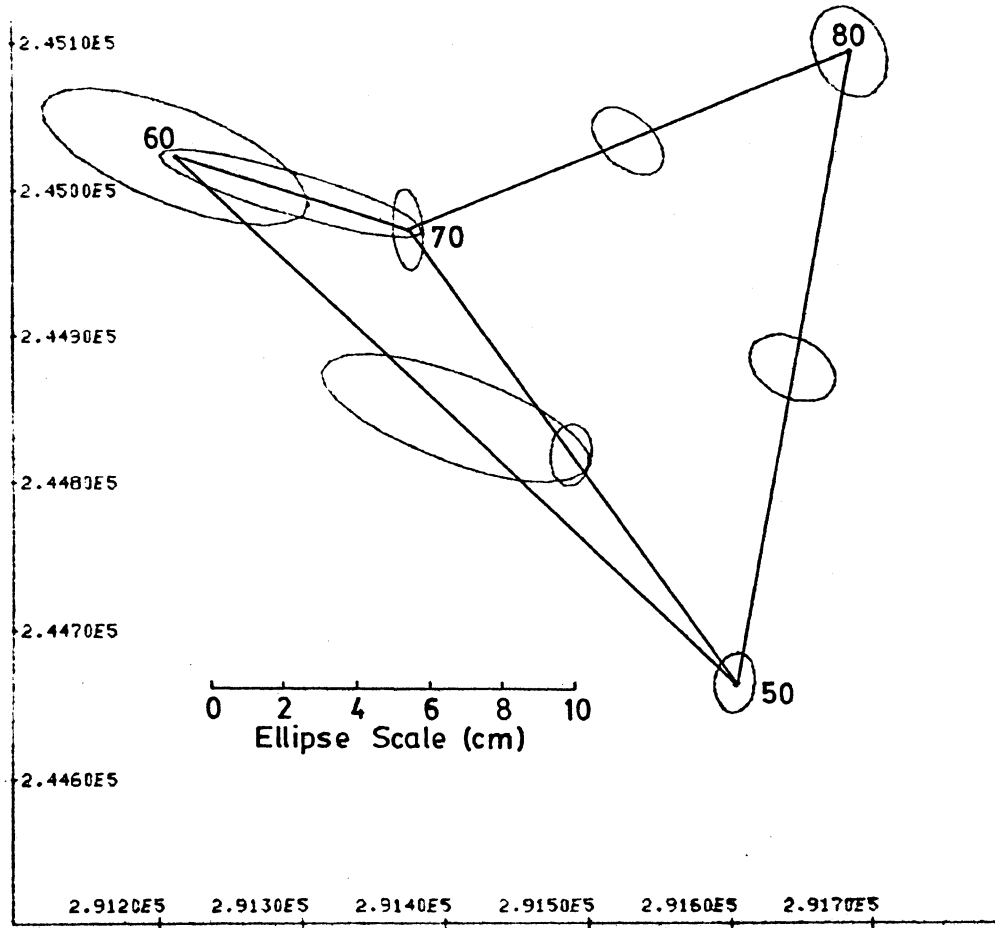


Figure AI.2. Plot of Initial 95% Confidence Ellipses for Distance Addition Example

STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	.0081	.0057	81.552
60	.0394	.0128	-21.046
70	.0112	.0041	-84.672
80	.0129	.0097	-60.855

RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	60	.0396	.0121	-19.301
50	80	.0126	.0081	-24.784
50	70	.0085	.0057	83.930
60	70	.0378	.0058	-16.186
70	80	.0115	.0069	-39.128

Table AI.2. 95% Confidence Ellipses from Initial Design of Distance Addition Example





Again, the 95% confidence ellipses are extracted from  $C_x$  according to section 3.4 to give the results displayed in Table AI.3. Figure AI.3 shows a plot of these error ellipses. The a priori variance factor was again assumed known and nonsimultaneous ellipses were computed. It is obvious from Figure AI.4 that station 60 has benefitted the most from the additional distances, which is to be expected as stations 50 and 70 have a fairly large weight matrix on them which contributes to their stability.

## STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	.0081	.0057	81.519
60	.0128	.0063	87.487
70	.0105	.0040	-85.809
80	.0118	.0091	-74.026

## RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	60	.0116	.0068	64.798
50	80	.0110	.0078	-18.602
50	70	.0077	.0054	73.205
60	70	.0057	.0046	63.919
70	80	.0105	.0066	-46.032

Table AI.3. 95% Confidence Ellipses After Addition of Distance Observables

As a final summary, Table AI.4 (which is of the same format as the example in Chapter 4) lists all of the observables that were considered in this design. Although this example has been very simple, it

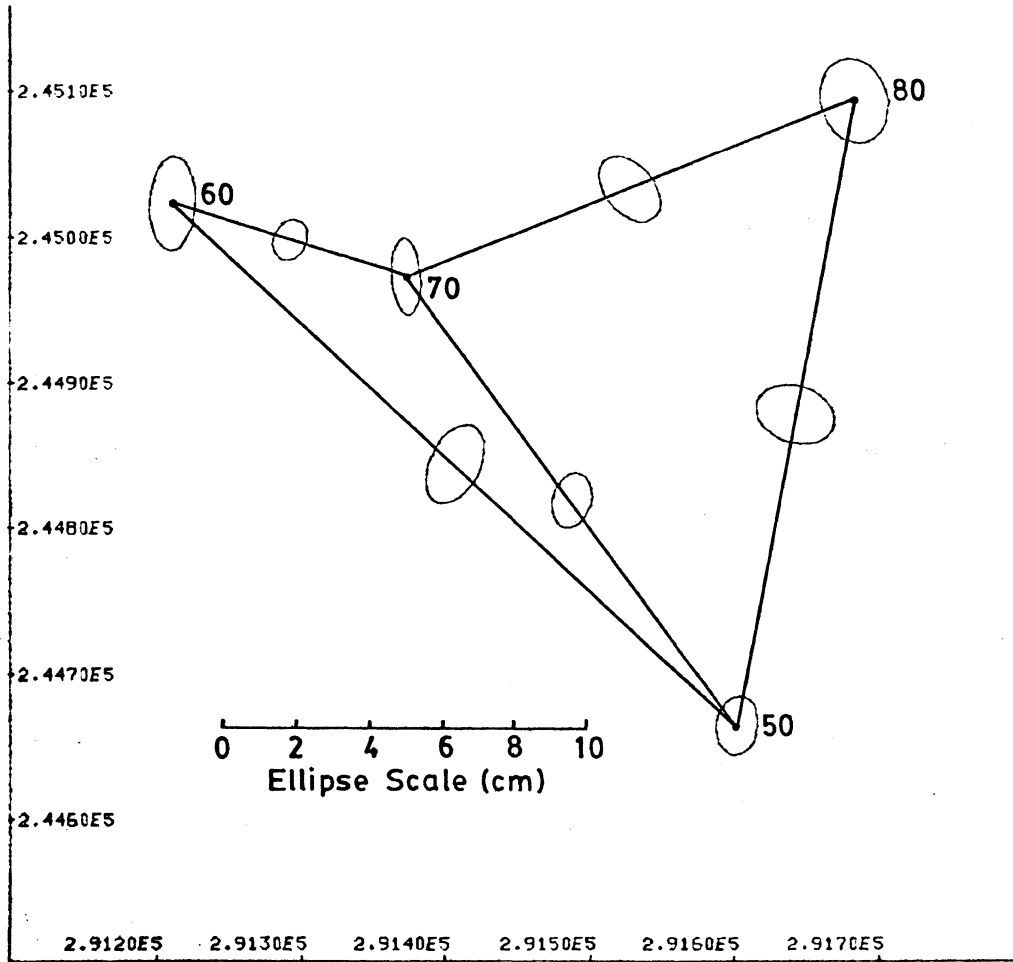


Figure AI.3. Plot of 95% Confidence Ellipses After Distance Addition

serves as a good check for someone considering the application of the models developed in this thesis.

OBSERVATIONS MADE:

1	50	60	2.000
1	50	80	1.000
1	60	70	3.000
1	60	50	3.000
2	60	50	.006
2	60	70	.002
1	70	50	2.000
1	70	60	2.000
1	70	80	2.000
2	70	80	.004
1	80	50	1.500
1	80	70	1.500
2	80	50	.005

Table AI.4. Summary of Observables Used in Distance Addition Example

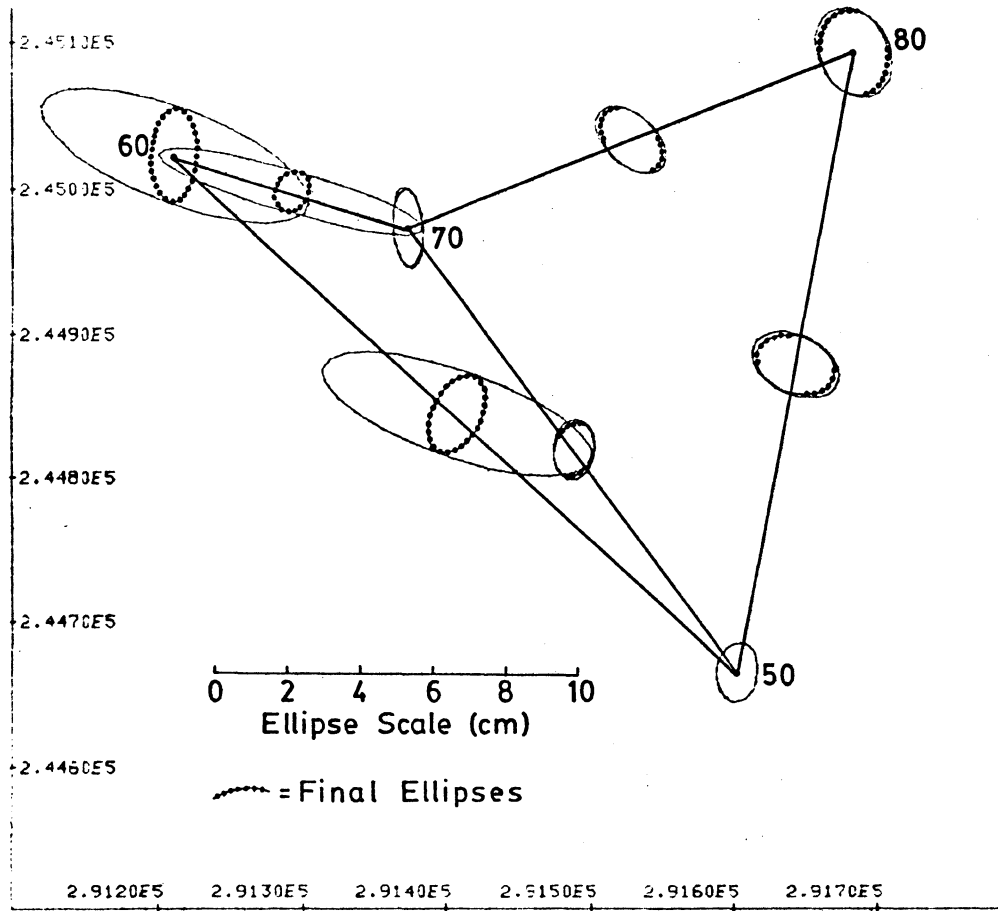


Figure AI.4. Comparison Plot of Initial and Final 95% Confidence Ellipses for Distance Addition Example

APPENDIX II. Simple Addition of Parameters

This appendix illustrates the procedure for the addition of new unknowns (i.e. point coordinates) when there are no nuisance parameters involved (cf. section 2.2). The network considered is the same as in Appendix I, except that for the initial case, only three points, as shown in Figure AII.1, are considered. Points 50, 70 and 80 are first considered

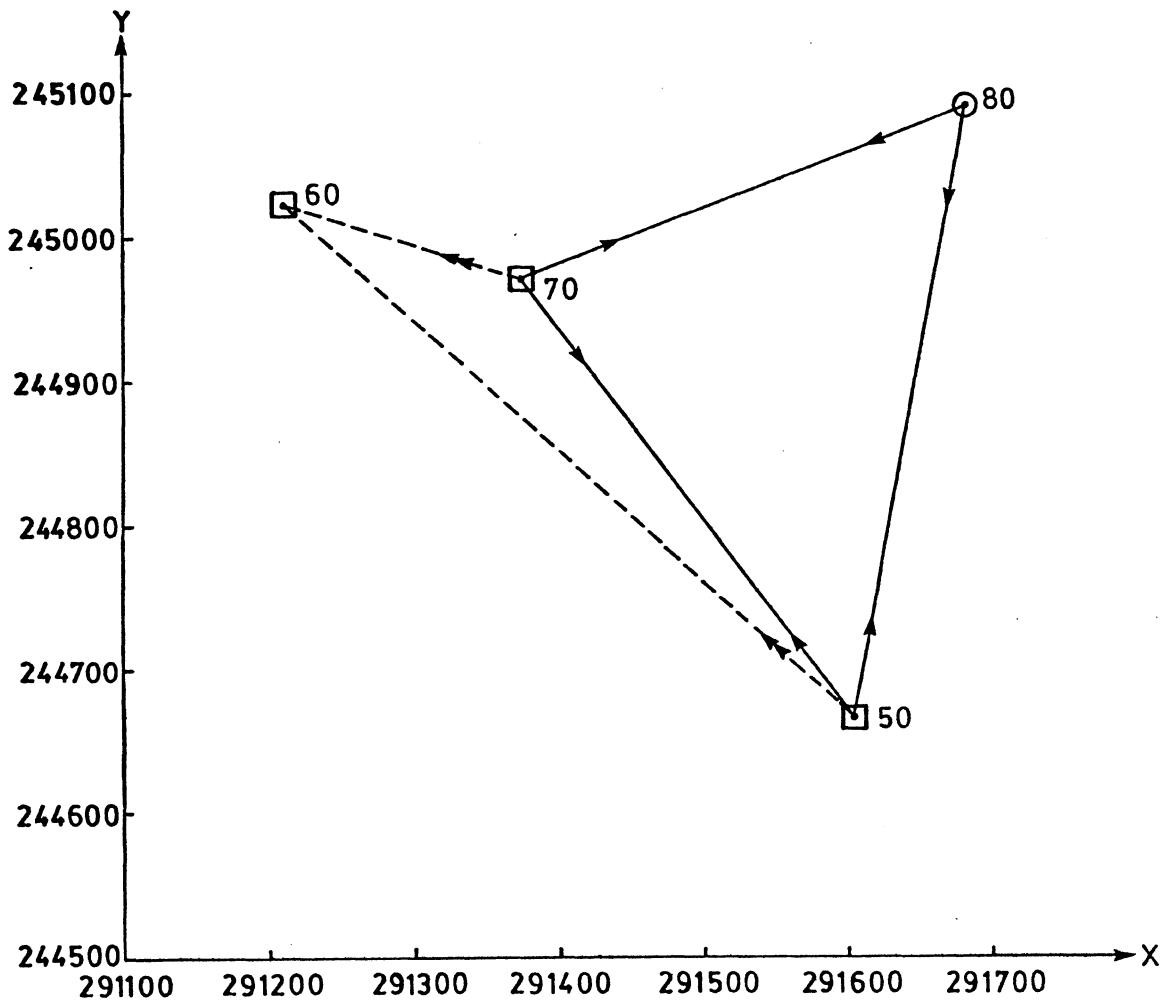


Figure AII.1. Example Network for Addition of Parameters by Azimuth Observables

with points 50 and 70 having the same weights as in Appendix I. The initial set of observables is as shown in Figure AII.1 and their standard deviations are listed in Table AII.1. The normal equations inverse or  $C_x$  matrix

OBSERVATIONS MADE				
Type	From	To	$\sigma$	
1	50	70	2.0	
1	50	80	1.0	
1	70	50	2.0	
1	70	80	2.0	
2	70	80	0.004	
1	80	50	1.5	
1	80	70	1.5	
2	80	50	0.005	

Table AII.1. Initial Observables Used for Simple Addition of Parameters  
Example

resulting from this initial setup is

$$C_x = \begin{bmatrix} 5.4599E-6 & 8.2476E-7 & 1.4184E-6 & -1.3454E-6 & 2.8716E-7 & 2.2932E-6 \\ 8.2476E-7 & 1.0886E-5 & -2.1003E-7 & 9.8071E-6 & -7.6618E-7 & 1.1011E-5 \\ 1.4184E-6 & -2.1003E-7 & 2.8816E-6 & -1.5570E-6 & 2.2042E-6 & -1.7620E-6 \\ -1.3454E-6 & 9.8071E-6 & -1.5570E-6 & 2.0116E-5 & 3.6958E-6 & 1.4870E-5 \\ 2.8716E-7 & -7.6618E-7 & 2.2042E-6 & 3.6958E-6 & 1.3367E-5 & -1.7054E-6 \\ 2.2932E-6 & 1.1011E-5 & -1.7620E-6 & 1.4870E-5 & -1.7054E-6 & 2.2720E-5 \end{bmatrix}$$

and the resulting 95% nonsimultaneous confidence ellipses, assuming the a priori variance factor is unknown, are listed in Table AII.2 and plotted in Figure AII.2. The C factor for extracting the 95% confidence ellipses in this case is

$$C = \sqrt{2F_{2,n-u,1-\alpha}}, \text{ or } C = \sqrt{2F_{2,3,0.95}} = 4.37 .$$

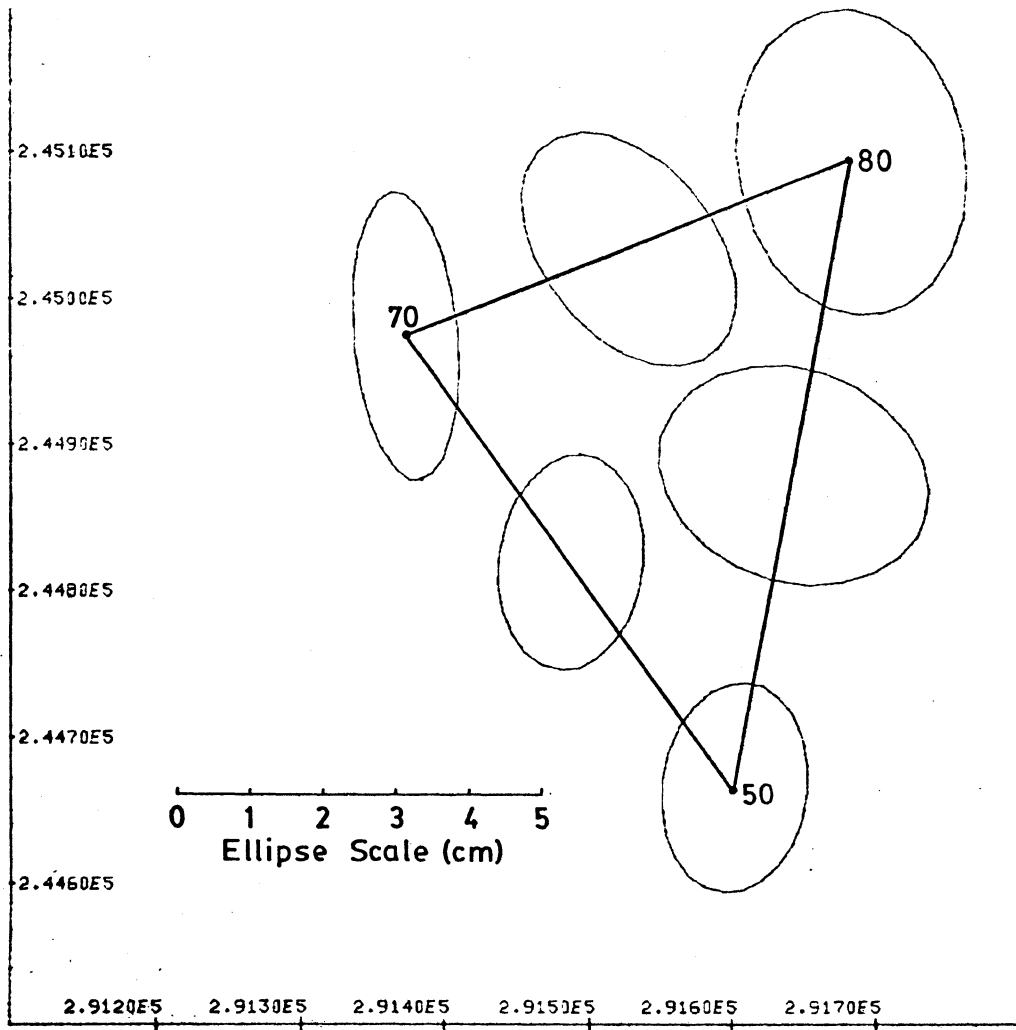


Figure AII.2. Plot of 95% Confidence Ellipses Resulting from the Initial Design of Simple Addition of Parameters Example

## STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	.0145	.0101	81.545
70	.0197	.0072	-84.879
80	.0210	.0158	-79.982

## RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	70	.0148	.0101	82.183
50	80	.0191	.0144	-17.912
70	80	.0184	.0118	48.441

Table AII.2. 95% Confidence Ellipses Resulting from Initial Design of Simple Addition of Parameters Example

The next step is to add the station number 60 to the existing  $C_x$  by means of two azimuth observations. For this case, the models developed in sections 2.2 and 3.2 are directly applicable. Considering that the first observable to be added is an azimuth from 50 to 60 with a standard deviation of 10", the  $A_{21}$  matrix, or  $\frac{\partial F_2}{\partial X_1}$  is

$$A_{21} = \rho \left[ \begin{array}{cc} \frac{-(y_{60}-y_{50})}{S_{50,60}^2} & \frac{(x_{60}-x_{50})}{S_{50,60}^2} \end{array} \right] = [-263.04 \quad -287.57].$$

The  $A_2$  matrix is similarly given as

$$A_2 = \frac{\partial F_2}{\partial X_2} = \rho \left[ \begin{array}{cc} \frac{(y_{60}-y_{50})}{S_{50,60}^2} & \frac{-(x_{60}-x_{50})}{S_{50,60}^2} \end{array} \right] = [263.04 \quad 287.57],$$

where  $\rho = 206264.8062$ . The constant  $\rho$  is used to make the units of  $A$  correspond to the units of the weight matrix for the observables  $C_L$  which are in units of ( $"$ )<sup>2</sup>. In this case,  $C_L = (1 \times 1) = 100''^2$ , and therefore  $M_2 = 100''^2$ .



Following the procedure outlined in section 2.2.1,  $\tilde{N}_1^{-1}$  is first formed by equation (2-47), which is

$$\tilde{N}_1^{-1} = N_1^{-1} - N_1^{-1} A_{21}^T [M_2 + A_{21} N_1^{-1} A_{21}^T]^{-1} A_{21} N_1^{-1} .$$

Considering first the expression inside the square brackets yields

$$M_2 + A_{21} N_1^{-1} A_{21}^T = 100 + \begin{bmatrix} -263.04 & -287.57 \end{bmatrix} \begin{bmatrix} 5.4599E-6 & 8.2476E-7 \\ 8.2476E-7 & 1.0886E-5 \end{bmatrix} \begin{bmatrix} -263.04 \\ -287.57 \end{bmatrix} ,$$

where  $N_1^{-1}$  is taken to be the portion of the previous  $C_x$  matrix corresponding to point 50. Evaluation of this expression leads to

$$[M_2 + A_{21} N_1^{-1} A_{21}^T]^{-1} = 0.0098617 .$$

Evaluation of  $A_{21} N_1^{-1}$  yields a matrix of size (1x6) which is

$$A_{21} N_1^{-1} = [-0.0016734 - 0.0033473 - 0.0003127 - 0.0024664 \quad 0.0001448 - 0.0037698] .$$

Computing  $N_1^{-1} A_{21}^T [M_2 + A_{21} N_1^{-1} A_{21}^T]^{-1} A_{21} N_1^{-1}$  and subtracting it from  $N_1^{-1}$  yields  $\tilde{N}_1^{-1}$  as

$$\tilde{N}_1^{-1} = \begin{bmatrix} 5.4323E-6 & 7.6952E-7 & 1.4132E-6 & -1.3861E-6 & 2.8955E-7 & 2.2310E-6 \\ 7.6952E-7 & 1.0775E-5 & -2.2036E-7 & 9.7257E-6 & -7.6140E-7 & 1.0887E-5 \\ 1.4132E-6 & -2.2036E-7 & 2.8806E-6 & -1.5646E-6 & 2.2046E-6 & -1.7736E-6 \\ -1.3861E-6 & 9.7257E-6 & -1.5646E-6 & 2.0056E-5 & 3.6993E-6 & 1.4779E-5 \\ 2.8955E-7 & -7.6140E-7 & 2.2046E-6 & 3.6993E-6 & 1.3367E-5 & -1.7000E-6 \\ 2.2310E-6 & 1.0887E-5 & -1.7736E-6 & 1.4779E-5 & -1.7000E-6 & 2.2579E-5 \end{bmatrix} .$$

Following the procedure of section 2.21,  $N_{12}$  and  $N_2$  are next computed. Firstly,

$$N_{12} = A_{21}^T M_2^{-1} A_2 = \begin{bmatrix} -263.04 \\ -287.57 \end{bmatrix} \begin{bmatrix} 0.01 & 263.04 & 287.57 \end{bmatrix} = \begin{bmatrix} -691.91 & -756.44 \\ -756.44 & -826.98 \end{bmatrix} .$$

Now,  $N_2$  is computed as

$$N_2 = P_{x_2} + A_2^T M_2^{-1} A_2 = \begin{bmatrix} 0.29513E6 & 0.28418E5 \\ 0.28418E5 & 0.37081E6 \end{bmatrix} + \begin{bmatrix} 263.04 \\ 287.57 \end{bmatrix} [0.01] [263.04 \ 287.57] ,$$

$$N_2 = \begin{bmatrix} 2.9582E5 & 2.9174E4 \\ 2.9174E4 & 3.7164E5 \end{bmatrix} .$$

This now enables us to compute  $Q_2$  which, according to equation (2-43), is

$$Q_2 = [N_2 - N_{12}^T \tilde{N}_1^{-1} N_{12}]^{-1} .$$

Carrying out the necessary computations,  $Q_2$  is found to be

$$Q_2 = \begin{bmatrix} 3.4069E-6 & -2.6736E-7 \\ -2.6736E-7 & 2.7119E-6 \end{bmatrix} .$$

Equation (2-42) is now employed to compute  $Q_{12}$  as

$$Q_{12} = \tilde{N}_1^{-1} N_{12} Q_2 .$$

At this point, it should be mentioned that  $\tilde{N}_1^{-1}$  and  $N_{12}$  do not seem to be conformable for multiplication. The main reason for this is that  $A_{21} = \frac{\partial F_2}{\partial X_1}$  should really be of dimension (1x6) instead of (1x2).  $A_{21}$  was given as (1x2) because the other four elements (i.e.  $\frac{\partial F_2}{\partial x_{70}}$ ,  $\frac{\partial F_2}{\partial y_{70}}$ ,  $\frac{\partial F_2}{\partial x_{80}}$ ,  $\frac{\partial F_2}{\partial y_{80}}$ ) are all zero. It now becomes necessary to insert these zeroes in their proper positions to compute  $Q_{12}$ . Carrying out the matrix multiplications yields

$$Q_{12} = \begin{bmatrix} 1.3520E-8 & 1.1709E-8 \\ 2.7044E-8 & 2.3422E-8 \\ 2.5264E-9 & 2.1880E-9 \\ 1.9927E-8 & 1.7257E-8 \\ -1.1699E-9 & -1.0132E-9 \\ 3.0457E-8 & 2.6378E-8 \end{bmatrix} .$$

The next step is to compute  $Q_1$  according to equation (2-45)

as

$$Q_1 = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} N_{12} Q_2 N_{12}^T \tilde{N}_1^{-1},$$

and upon computing this

$$Q_1 = \begin{bmatrix} 5.4324E-6 & 7.6975E-7 & 1.4133E-6 & -1.3859E-6 & 2.8954E-7 & 2.2312E-6 \\ 7.6975E-7 & 1.0776E-5 & -2.2031E-7 & 9.7260E-6 & -7.6142E-7 & 1.0888E-5 \\ 1.4133E-6 & -2.2031E-7 & 2.8806E-6 & -1.5646E-6 & 2.2046E-6 & -1.7736E-6 \\ -1.3859E-6 & 9.7260E-6 & -1.5646E-6 & 2.0056E-5 & 3.6993E-6 & 1.4779E-5 \\ 2.8954E-7 & -7.6142E-7 & 2.2046E-6 & 3.6993E-6 & 1.3367E-5 & -1.7000E-6 \\ 2.2312E-6 & 1.0888E-5 & -1.7736E-6 & 1.4779E-5 & -1.7000E-6 & 2.2580E-5 \end{bmatrix}$$

This matrix  $Q_1$  is now the  $C_x$  matrix for the "old" or previously considered points in the network, which, in this case, are points 50, 70 and 80. Appending the  $Q_{12}$  and  $Q_2$  matrices onto  $Q_1$  results in the final  $C_x$  matrix for all of the points, which is of size (8x8). It is given as

$$C_x = \begin{bmatrix} Q_1 & Q_{12} \\ \hline Q_{12}^T & Q_2 \end{bmatrix},$$

(6x6)                      (6x2)  
(2x6)                      (2x2)

$$C_x = \begin{bmatrix} 5.4324E-6 & 7.6975E-7 & 1.4133E-6 & -1.3859E-6 & 2.8954E-7 & 2.2312E-6 & 1.3520E-8 & 1.1709E-8 \\ & 1.0776E-5 & -2.2031E-7 & 9.7260E-6 & -7.6142E-7 & 1.0888E-6 & 2.7044E-8 & 2.3422E-8 \\ & & 2.8806E-6 & -1.5646E-6 & 2.2046E-6 & -1.7736E-6 & 2.5264E-9 & 2.1880E-9 \\ & & & 2.0056E-5 & 3.6993E-6 & 1.4779E-5 & 1.9927E-8 & 1.7257E-8 \\ & & & & 1.3367E-5 & -1.7000E-6 & -1.1699E-9 & -1.0132E-9 \\ symmetric & & & & & 2.2580E-5 & 3.0457E-8 & 2.6378E-8 \\ & & & & & & 3.4069E-6 & -2.6736E-7 \\ & & & & & & & 2.7119E-6 \end{bmatrix}$$

The final step in this example is to add the azimuth from 70 to 60 which also has a standard deviation of 10". This observable is between two stations which are now already included in the  $C_x$  matrix, and therefore its addition is identical to that of Appendix I, except for the elements of  $A_2$ , which are

$$A_2 = \frac{\partial F_2}{\partial X} = \rho \left[ \frac{-(y_{70}-y_{60})}{S_{60,70}^2} \quad \frac{(x_{70}-x_{60})}{S_{60,70}^2} \quad \frac{(y_{70}-y_{60})}{S_{60,70}^2} \quad \frac{-(x_{70}-x_{60})}{S_{60,70}^2} \right],$$

$$A_2 = [-358.84 \quad -1164.8 \quad 358.84 \quad 1164.8].$$

After adding the contribution of this azimuth to the above  $C_x$  matrix, the 95% nonsimultaneous ellipses assuming  $\sigma_0^2$  unknown are again extracted, and are tabulated in Table AII.3 below as well as plotted in Figure AII.3. The C

RESULTS ACHIEVED AT 95 PERCENT CONFIDENCE LEVEL.

STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	.0107	.0078	79.218
70	.0137	.0056	-83.902
80	.0157	.0120	-73.641
60	.0064	.0054	-18.582

RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	70	.0110	.0079	81.095
50	80	.0148	.0110	-19.494
50	60	.0118	.0101	81.144
70	80	.0141	.0091	-48.507
70	60	.0144	.0083	-80.852

Table AII.3. 95% Confidence Ellipses After Addition of Parameters

factor used in this case is

$$C = \sqrt{2F_{2,5,0.95}} = \sqrt{2 \times 5.79} = 3.40 .$$

Figure AII.4 shows the decrease in confidence ellipse size resulting from the addition of the two azimuth observables along with the new station.

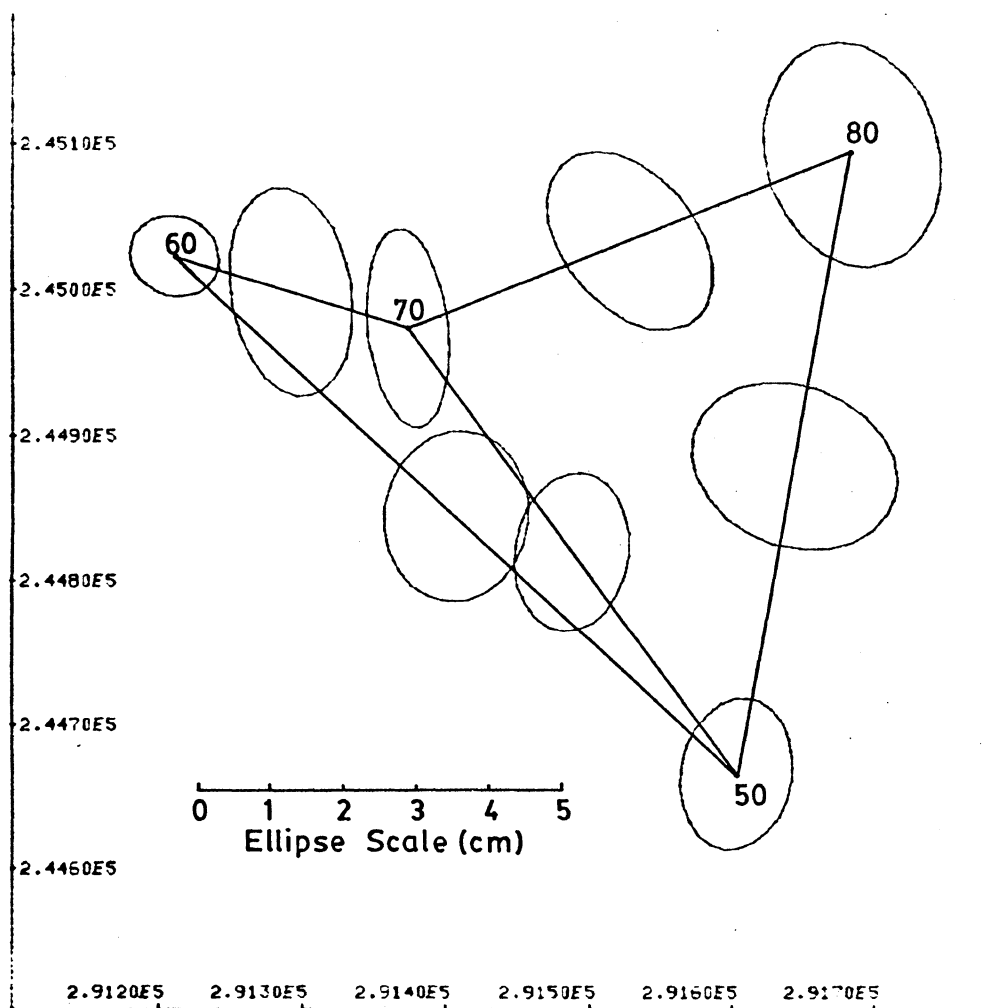


Figure AII.3. Plot of 95% Confidence Ellipses After Addition of a Station

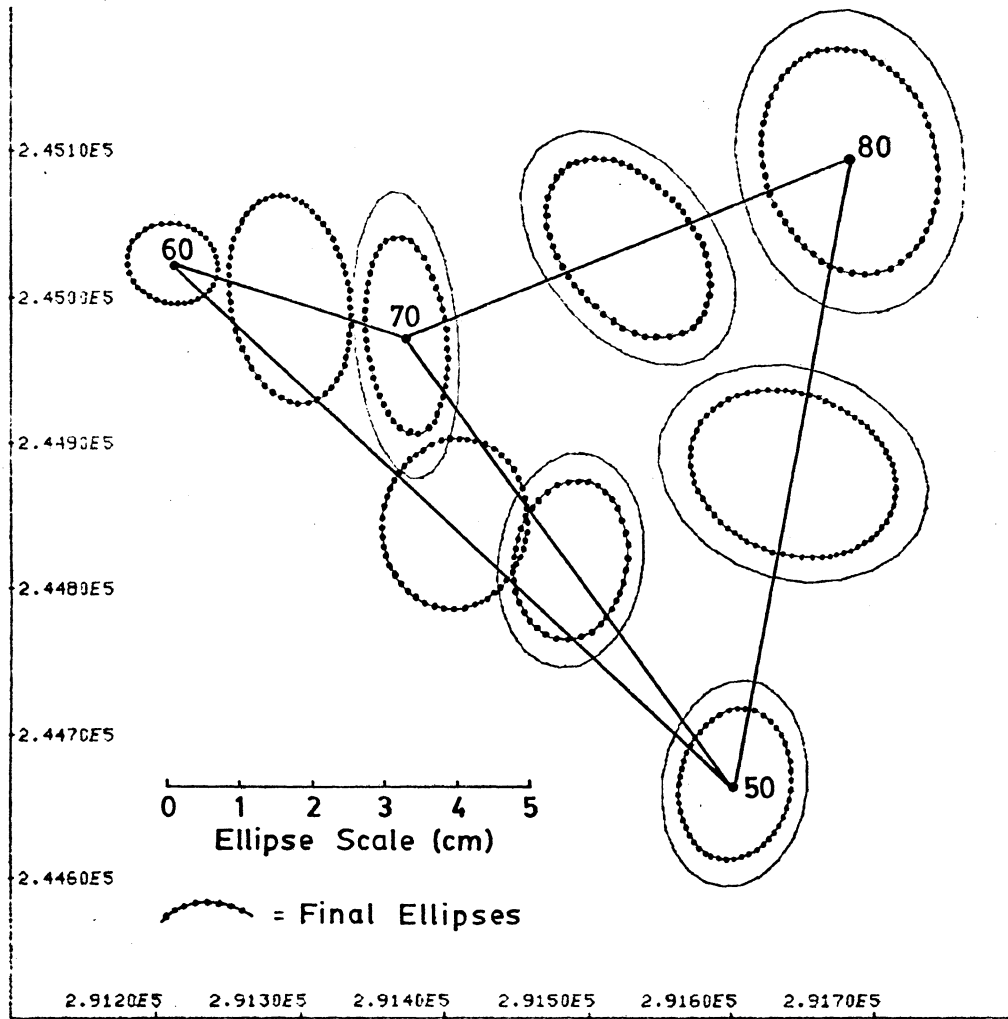


Figure AII.4. Comparison Plot of 95% Confidence Ellipses Before and After Parameter Addition

APPENDIX III. Addition of Parameters by Directions

Appendix III illustrates the models developed in Section 2.4 for the addition of parameters with common nuisance parameters (i.e. orientation unknowns) between steps. The example network this time includes the same stations as in Appendices I and II, but they are configured as shown in Figure AIII.1 below. Again, the dashed lines stand for observables added after the initial observable stage.

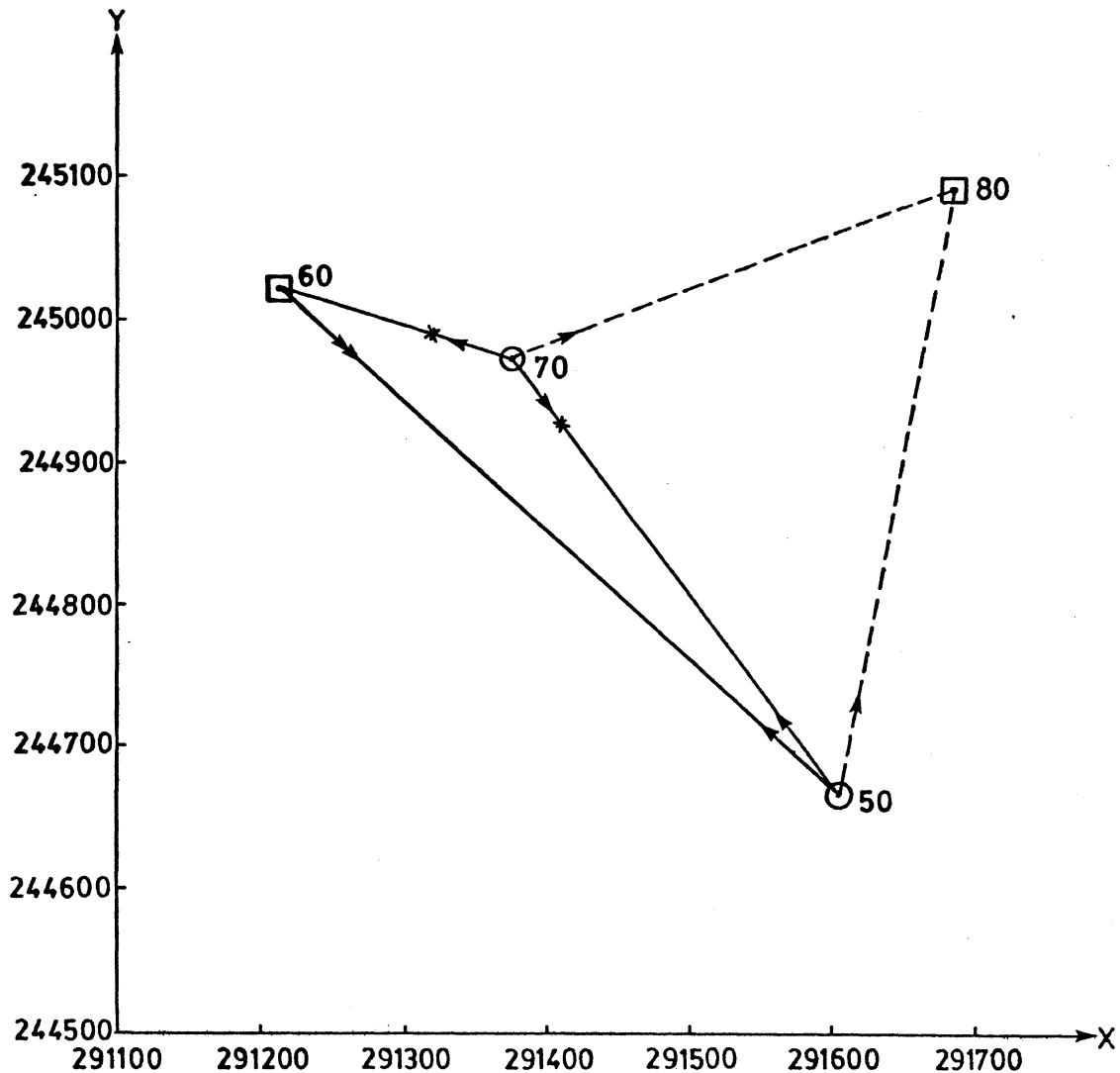


Figure AIII.1. Example Network for Addition of Parameters by Directions

Initially, stations 50, 60 and 70 are considered, with station 60 having a weight matrix of

$$P_{\mathbf{x}} = \begin{bmatrix} 0.29513E6 & 0.28418E5 \\ 0.28418E5 & 0.37081E6 \end{bmatrix} .$$

The  $C_{\mathbf{x}}$  matrix for stations 50, 60 and 70 is formed considering the initial observables depicted by the solid lines in Figure AIII.1. These observables are added in the manner outlined in Appendix I, and result in the following  $C_{\mathbf{x}}$  matrix:

$$C_{\mathbf{x}} = \begin{bmatrix} 1.5983E-4 & 7.2146E-6 & 3.4135E-6 & -2.6160E-7 & 5.3314E-5 & 1.8774E-5 \\ 7.2146E-6 & 1.6047E-4 & -2.6160E-7 & 2.7168E-6 & -2.4906E-5 & 5.3470E-5 \\ 3.4135E-6 & -2.6160E-7 & 3.4135E-6 & -2.6160E-7 & 3.4135E-6 & -2.6160E-7 \\ -2.6160E-7 & 2.7163E-6 & -2.6160E-7 & 2.7168E-6 & -2.6160E-7 & 2.7168E-6 \\ 5.3314E-5 & -2.4906E-5 & 3.4135E-6 & -2.6160E-7 & 5.3973E-5 & -1.7810E-5 \\ 1.8774E-5 & 5.3470E-5 & -2.6160E-7 & 2.7168E-6 & -1.7810E-5 & 3.1832E-5 \end{bmatrix} .$$

The observables used to obtain this initial solution are outlined in Table AIII.1 below.

<u>Type</u>	<u>From</u>	<u>To</u>	<u><math>\sigma</math></u>
1	50	60	3.0
1	50	70	3.0
1	70	50	2.0
1	70	60	2.0
2	70	60	0.008
2	70	50	0.010
3	60	50	5.0

Table AIII.1. Initial Observables Used for Addition of Parameters by Directions Example



Assuming that simultaneous error ellipses are required at the standard (~ 39%) confidence level, and that the a priori variance factor is known, the C factor is

$$C = \sqrt{\chi^2_{2,1-\frac{0.6065}{3}}} = \sqrt{\chi^2_{2,0.7978}} = \sqrt{3.197} = 1.79 .$$

Note that only the station error ellipses are multiplied by this factor, as simultaneous relative error ellipses are not required in this case.

Using the methods of section 3.4, the error ellipses are computed as shown in Table AIII.2.

RESULTS ACHIEVED AT 39.35 PERCENT CONFIDENCE LEVEL

STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	.0231	.0222	46.266
60	.0034	.0029	-18.453
70	.0143	.0084	-29.067

RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	60	.0128	.0122	47.551
50	70	.0104	.0092	-11.127
60	70	.0078	.0044	-29.287

Table AIII.2. Initial Confidence Ellipses for Addition of Parameters by Direction Example  
These ellipses are plotted in Figure AIII.2.

The next step is to add station 80 by means of the two directions depicted by dashed lines in Figure AIII.1. This involves the use of the models developed in section 2.4. Considering first that the direction from station 70 to station 80 is to be added, the model for

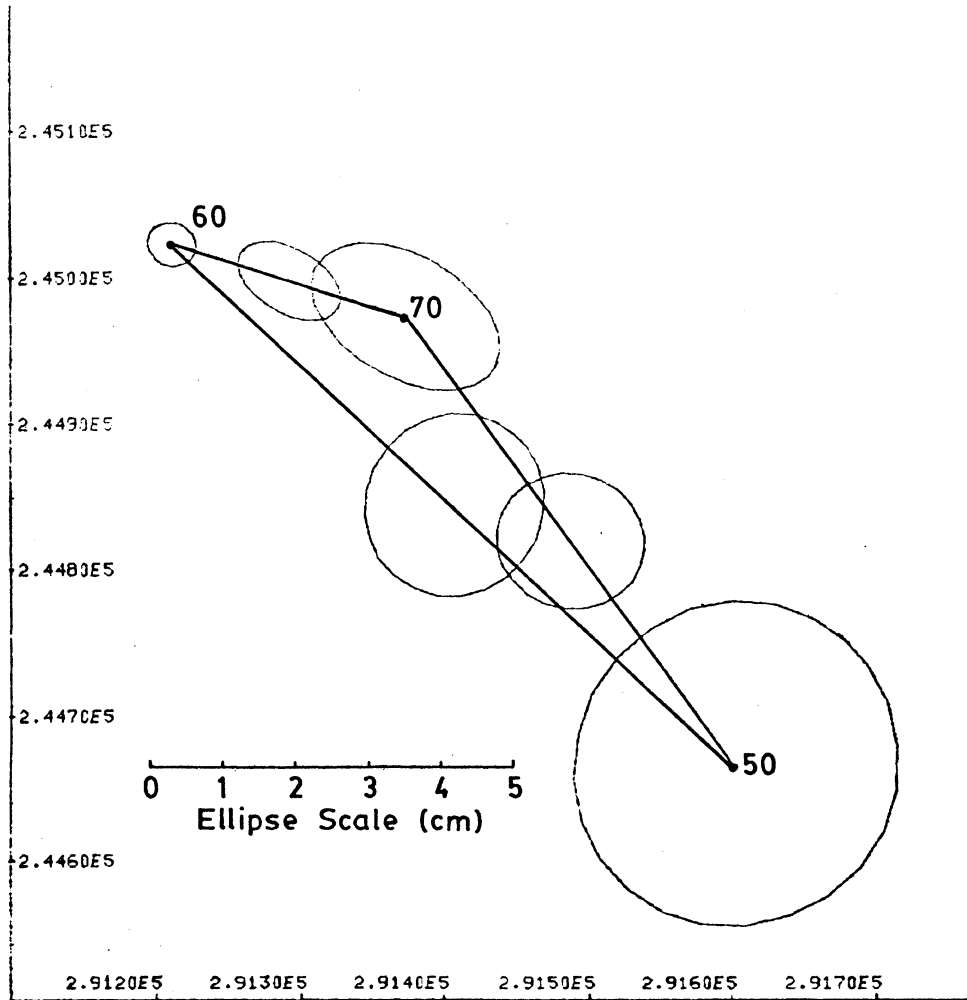


Figure AIII.2. Plot of Standard Error Ellipses Resulting from the Initial Design of the Addition of Parameters by Direction Example

sequential addition of this direction to the set of directions already existing at station 70 unfolds as follows.

First, the formulae for computation of the new  $C_x$  for all stations including station 80 are reviewed. The basic form of the expanded  $C_x$  is given by equation (2-102) as

$$C_x = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix},$$

where

$$Q_2 = [N_{x_2} - N_{12}^T N_{x_1}^{-1} N_{12}]^{-1},$$

$$Q_{12} = -N_{x_1}^{-1} N_{12} Q_2,$$

$$Q_1 = N_{x_1}^{-1} + N_{x_1}^{-1} N_{12} Q_2 N_{12}^T N_{x_1}^{-1}.$$

Equation (2-106) gives the equation for  $N_{x_1}^{-1}$  as

$$N_{x_1}^{-1} = \tilde{N}_{1x_1}^{-1} - \tilde{N}_{1x_1}^{-1} \tilde{A}_{2x_1}^T [\tilde{M}_2 + \tilde{A}_{2x_1} \tilde{N}_{1x_1}^{-1} \tilde{A}_{2x_1}^T]^{-1} \tilde{A}_{2x_1} \tilde{N}_{1x_1}^{-1},$$

where

$$\tilde{N}_{1x_1}^{-1} = \text{existing or old } C_x \text{ matrix,}$$

$$\tilde{A}_{2x_1} = A_{2x_1} - A_{2y} N_{1y}^{-1} N_{yx_1},$$

$$\tilde{M}_2 = M_2 + A_{2y} N_{1y}^{-1} A_{2y}^T.$$

As can be seen from the above formulae, the procedure for computing the new  $C_x$  is not a simple one. The steps involved in accomplishing this are outlined in point form below, and are then applied to the particular problem at hand.

1. Compute  $\tilde{A}_{2x_1}$ .

$$\tilde{A}_{2x_1} = A_{2x_1} - A_{2y} N_{ly}^{-1} N_{yx_1},$$

where

$$N_{ly} = A_{ly}^T M_1^{-1} A_{ly},$$

and

$$N_{yx_1} = A_{ly}^T M_1^{-1} A_{lx_1}.$$

Matrix  $A_{ly}$  is given as

$$A_{ly} = \frac{\partial F_1}{\partial Y} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix},$$

considering that there are 3 directions previously measured in this set. As well,  $M_1^{-1} = P_1$  is the weight matrix for the previous set of directions. Thus, this results in

$$N_{ly} = [-1 \ -1 \ -1] \begin{bmatrix} \sigma_{d_{ij}}^{-2} & 0 & 0 \\ 0 & \sigma_{d_{ik}}^{-2} & 0 \\ 0 & 0 & \sigma_{d_{il}}^{-2} \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \sum_{t=1}^n \sigma_{d_{it}}^{-2},$$

and therefore

$$N_{ly}^{-1} = \frac{1}{\sum_{t=1}^n \sigma_{d_{it}}^{-2}},$$

where  $n$  = number of previous directions in the set,

$i$  = from station for the previous set of directions, and

$j, k$  and  $\ell$  = to stations for the previous set of directions.

Turning now to the computation of  $N_{yx_1}$ ,  $A_{1x_1}$  is given as

$$A_{1x_1} = \frac{\partial F_1}{\partial X_1} = \begin{bmatrix} a & b & -a & -b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & -c & -d & 0 & 0 \\ e & f & 0 & 0 & 0 & 0 & -e & -f \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{-(y_j - y_i)}{S_{ij}^2}, & b &= \frac{(x_j - x_i)}{S_{ij}^2} \\ c &= \frac{-(y_k - y_i)}{S_{ik}^2}, & d &= \frac{(x_j - x_i)}{S_{ij}^2} \\ e &= \frac{-(y_l - y_i)}{S_{il}^2}, & f &= \frac{(x_l - x_i)}{S_{il}^2}. \end{aligned}$$

It is obvious that  $A_{1x_1}$  is the A design matrix for the set of directions as they were in the  $F_1$  model. Consequently,  $N_{yx_1}$  is given as

$$\begin{aligned} N_{yx_1}(1,1) &= \sum_{t=1}^n \frac{\sigma_{dit}^{-2} (y_t - y_i)}{S_{it}^2}, & N_{yx_1}(1,2) &= -\sum_{t=1}^n \frac{\sigma_{dit}^{-2} (x_t - x_i)}{S_{it}^2}, \\ N_{yx_1}(1,u) &= \frac{-\sigma_{dit}^{-2} (y_t - y_i)}{S_{it}^2}, & N_{yx_1}(1,v) &= \frac{\sigma_{dit}^{-2} (x_t - x_i)}{S_{it}^2}. \end{aligned}$$

where  $u = (2x(t+1)) - 1$  and  $v = 2x(t+1)$  for  $t = 1, n$ .

Assuming that only 1 direction is being added,

$$A_{2y} = \frac{\partial F_2}{\partial Y} = -1,$$

and, letting  $B = -A_{2y}^{-1} N_{ly}^{-1} N_{yx_1}$ , this results in

$$B(1, 1) = \frac{1}{\sum_{t=1}^n \sigma_{dit}^{-2}} \frac{\sum_{t=1}^n \sigma_{dit}^{-2} (y_t - y_i)}{S_{it}^2},$$

$$B(1, 2) = \frac{-1}{\sum_{t=1}^n \sigma_{dit}^{-2}} \frac{\sum_{t=1}^n \sigma_{dit}^{-2} (x_t - x_i)}{S_{it}^2},$$

$$B(1, u) = \frac{-\sigma_{dit}^{-2} (y_t - y_i)}{S_{it}^2 \sum_{t=1}^n \sigma_{dit}^{-2}}, \quad B(1, v) = \frac{\sigma_{dit}^{-2} (x_t - x_i)}{S_{it}^2 \sum_{t=1}^n \sigma_{dit}^{-2}},$$

where  $u, v$  and  $t$  are as before.

Finally,  $A_{2x_1}$  is given as

$$A_{2x_1} = \frac{\partial F_2}{\partial X_1} = \left[ \frac{-(y_m - y_i)}{S_{im}^2} \quad \frac{(x_m - x_i)}{S_{im}^2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right],$$

for  $m$  being the new to station, and  $\tilde{A}_{2x_1}$  is given as

$$\tilde{A}_{2x_1}(1, 1) = \frac{-(y_m - y_i)}{S_{im}^2} + \frac{1}{\sum_{t=1}^n \sigma_{dit}^{-2}} \frac{\sum_{t=1}^n \sigma_{dit}^{-2} (y_t - y_i)}{S_{it}^2},$$

$$\tilde{A}_{2x_1}(1, 2) = \frac{(x_m - x_i)}{S_{im}^2} - \frac{1}{\sum_{t=1}^n \sigma_{dit}^{-2}} \frac{\sum_{t=1}^n \sigma_{dit}^{-2} (x_t - x_i)}{S_{it}^2}, \quad (\text{AIII-1})$$

$$\tilde{A}_{2x_1}(1, u) = \frac{-\sigma_{dit}^{-2} (y_t - y_i)}{S_{it}^2 \sum_{t=1}^n \sigma_{dit}^{-2}}, \quad \tilde{A}_{2x_1}(1, v) = \frac{\sigma_{dit}^{-2} (x_t - x_i)}{S_{it}^2 \sum_{t=1}^n \sigma_{dit}^{-2}}.$$

2. Compute  $\tilde{M}_2$ .

From equation (2-90), the expression for  $\tilde{M}_2$  is

$$\tilde{M}_2 = M_2 + A_{2y} N_{ly}^{-1} A_{2y}^T .$$

From the computation of  $\tilde{A}_{2x_1}$ ,  $A_{2y}$  and  $N_{ly}^{-1}$  are known, and thus

$$\tilde{M}_2 = \sigma_{d_{im}}^2 + \frac{1}{\sum_{t=1}^n \sigma_{d_{it}}^{-2}} \quad (\text{AIII-2})$$

is the required expression for  $\tilde{M}_2$ .

3. Compute  $N_{12}$ .

From equation (2-47),  $N_{12}$  is given as

$$N_{12} = \tilde{A}_{2x_1}^T \tilde{M}_2^{-1} A_{2x_2} .$$

$\tilde{A}_{2x_1}$  and  $\tilde{M}_2$  are both known from the previous computations, and  $A_{2x_2}$  is given as

$$A_{2x_2} = \frac{\partial F_2}{\partial X_2} = \left[ \frac{(y_m - y_i)}{S_{im}} \quad \frac{-(x_m - x_i)}{S_{im}} \right] . \quad (\text{AIII-3})$$

Thus,  $N_{12}$  is of dimension  $((2xn)+2)$  by 2.

4. Compute  $N_{x_2}$  which, according to equation (2-98), is given as

$$N_{x_2} = P_{x_2} + A_{2x_2}^T \tilde{M}_2^{-1} A_{2x_2} .$$

$N_{x_2}$  is of size  $(2x2)$  for one station being added.

5. Compute  $N_{x_1}^{-1}$  which has already been given as

$$N_{x_1}^{-1} = \tilde{N}_{1x_1}^{-1} - \tilde{N}_{1x_1}^{-1} \tilde{A}_{2x_1}^T [\tilde{M}_2 + \tilde{A}_{2x_1} \tilde{N}_{1x_1}^{-1} \tilde{A}_{2x_1}^T]^{-1} \tilde{A}_{2x_1} \tilde{N}_{1x_1}^{-1},$$

where  $\tilde{N}_{1x_1}^{-1}$  is the already existing  $C_x$  matrix and  $\tilde{A}_{2x_1}$  and  $\tilde{M}_2$  are given above.

6. Compute  $Q_2$ ,  $Q_{12}$  and  $Q_1$  according to formulae (2-103), (2-104) and (2-105), and form the new  $C_x$  matrix for all of the points as shown in equation (2-102).

To add the direction from station 70 to station 80 ( $\sigma = 2''0$ ) requires the use of all 6 of the steps outlined above. First, formulae (AIII-1) are utilized in order to compute  $\tilde{A}_{2x_1}$ . The previous set of directions at station 70 includes one to station 50 and one to station 60, both with a standard deviation of 2 arcsec. Thus,

$$\tilde{A}_{2x_1}(1, 1) = \rho'' \left[ \frac{-(y_{80} - y_{70})}{S_{70,80}^2} + \frac{1}{(0.25+0.25)} \left\{ \frac{0.25(y_{50} - y_{70})}{S_{50,70}^2} + \frac{0.25(y_{60} - y_{70})}{S_{60,70}^2} \right\} \right] = -258.46,$$

$$\tilde{A}_{2x_1}(1, 2) = \rho'' \left[ \frac{(x_{70} - x_{70})}{S_{70,80}^2} - \frac{1}{0.5} \left\{ \frac{0.25(x_{50} - x_{70})}{S_{50,70}^2} + \frac{0.25(x_{60} - x_{70})}{S_{60,70}^2} \right\} \right] = 1002.2,$$

$$\tilde{A}_{2x_1}(1, 3) = \rho'' \left[ \frac{-0.25 (y_{50} - y_{70})}{0.5 S_{50,70}^2} \right] = 215.9,$$

$$\tilde{A}_{2x_1}(1, 4) = \rho'' \left[ \frac{0.25(x_{50} - x_{70})}{0.5 S_{50,70}^2} \right] = 160.7,$$



$$\tilde{A}_{2 \times 1} (1, 5) = \rho'' \left[ \frac{-0.25(y_{60} - y_{70})}{0.5 S_{60,70}^2} \right] = -179.42 ,$$

$$\tilde{A}_{2 \times 1} (1, 6) = \rho'' \left[ \frac{0.25(x_{60} - x_{70})}{S_{60,70}^2} \right] = -582.4 .$$

Having computed  $\tilde{A}_{2 \times 1}$ ,  $\tilde{M}_2$  is now found to be

$$\tilde{M}_2 = \sigma_{d_{im}}^2 + \frac{1}{\sum_{t=1}^n \sigma_{d_{it}}^{-2}} = 4 + \frac{1}{0.5} = 6 .$$

In order to compute  $N_{12}$ ,  $A_{2 \times 2}$  is first needed and is computed

as

$$A_{2 \times 2} = \rho'' \left[ \begin{array}{cc} \frac{(y_{80} - y_{70})}{S_{70,80}^2} & \frac{-(x_{80} - x_{70})}{S_{70,80}^2} \\ \frac{-(x_{80} - x_{70})}{S_{70,80}^2} & \frac{(y_{80} - y_{70})}{S_{70,80}^2} \end{array} \right] = [221.97 \quad -580.54] .$$

Therefore

$$N_{12} = (6 \times 2) = \tilde{A}_{2 \times 1}^T \tilde{M}_2^{-1} A_{2 \times 2} = \begin{bmatrix} -9561.7 & 25007 \\ 37078 & -96972 \\ 7987.5 & -20890 \\ 5945.1 & -15549 \\ -6637.7 & 17360 \\ -21546 & 56351 \end{bmatrix} .$$

$N_{x_2}$  is the next matrix to be computed, and, considering the weight for station 80 as

$$P_{x_{80}} = \begin{bmatrix} 0.15031E6 & 0.19647E5 \\ 0.19647E5 & 0.15906E6 \end{bmatrix} ,$$

$N_{x_2}$  is

$$N_{x_2} = P_{x_{80}} + A_{2x_2} \tilde{M}_2^{-1} A_{2x_2} = \begin{bmatrix} 1.5852E5 & -1.8302E3 \\ -1.8302E3 & 2.1523E5 \end{bmatrix}.$$

Equation (2-106) is now used to compute  $N_{x_1}^{-1}$  from the old  $C_x$ ,  $\tilde{A}_{2x_1}$  and  $\tilde{M}_2$ , and it is

$$N_{x_1}^{-1} = \begin{bmatrix} 1.4004E-4 & -3.5009E-5 & 3.8610E-6 & -1.0660E-6 & 6.5489E-5 & -4.6489E-6 \\ -3.5009E-5 & 7.0415E-5 & 6.9270E-7 & 1.0011E-6 & 1.0624E-6 & 3.5122E-6 \\ 3.8610E-6 & 6.9270E-7 & 3.4034E-6 & -2.4342E-7 & 3.1384E-6 & 2.6778E-7 \\ -1.0660E-6 & 1.0011E-6 & -2.4342E-7 & 2.6842E-6 & 2.3312E-7 & 1.7651E-6 \\ 6.5489E-5 & 1.0624E-6 & 3.1384E-6 & 2.3312E-7 & 4.6486E-5 & -3.4043E-6 \\ -4.6489E-6 & 3.5122E-6 & 2.6778E-7 & 1.7651E-6 & -3.4043E-6 & 4.1183E-6 \end{bmatrix}.$$

Finally,  $Q_2$ ,  $Q_{12}$  and  $Q_1$  are calculated to give the new  $C_x$  for all four stations. The final  $C_x$  after adding the direction from 70 to 80 to the already existing set of directions at 70 is

$$C_x = \begin{bmatrix} 1.4067E-4 & -3.3660E-5 & 3.8467E-6 & -1.0403E-6 & 6.5100E-5 & -3.9001E-6 & -9.4565E-7 & 1.8548E-6 \\ & 7.3294E-5 & 6.6219E-7 & 1.0560E-6 & 2.3222E-7 & 5.1094E-6 & -2.0169E-6 & 3.9560E-6 \\ & & 3.4037E-6 & -2.4401E-7 & 3.1472E-6 & 2.5086E-7 & 2.1373E-8 & -4.1920E-8 \\ & & & 2.6852E-6 & 2.1730E-7 & 1.7955E-6 & -3.8425E-8 & 7.5366E-8 \\ & & & & 4.6725E-5 & -3.8648E-6 & 5.8158E-7 & -1.1407E-6 \\ & symmetric & & & & 5.0044E-6 & -1.1189E-6 & 2.1945E-6 \\ & & & & & & 6.7154E-6 & -7.4373E-7 \\ & & & & & & & 6.2106E-6 \end{bmatrix}$$

The next observable to be added is a direction from station 50 to station 80 with a standard deviation of 3.0 arcsec. The procedure for adding this direction to the already existing set at station 50 is similar to that already discussed, except that station 80 is now no longer a new station, and the  $C_x$  matrix remains the same size. When computing  $\tilde{A}_{2x_1}$ ,  $A_{1x_1}$  contains station 80 as well as stations 50, 60 and 70, and thus  $\tilde{A}_{2x_1}$  is of size (1x8) instead of (1x6) as before.  $\tilde{A}_{2x_1}$  is given as



Assuming again that simultaneous confidence ellipses are required at the standard confidence level and that the a priori variance factor is known, the C factor for the confidence ellipses is

$$C = \sqrt{\chi^2_{2, 1 - \frac{0.6065}{4}}} = \sqrt{\chi^2_{2, 0.8484}} = \sqrt{3.773} = 1.94 .$$

Thus the standard point confidence ellipses derived directly from  $C_x$  must be multiplied by 1.94 so that the probability of all station ellipses occurring simultaneously at 39.35% confidence is given by these ellipses. The relative error ellipses are again assumed to be nonsimultaneous. The final confidence ellipses are given in Table AIII.3.

## STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	0.0181	0.0132	6.454
60	0.0037	0.0031	-18.377
70	0.0120	0.0043	-5.318
80	0.0052	0.0047	-39.089

## RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	60	0.0092	0.0069	5.509
50	70	0.0073	0.0053	-54.595
50	80	0.0094	0.0067	7.555
60	70	0.0061	0.0019	-6.592
70	80	0.0064	0.0026	-2.808

Table AIII.3. Final Standard Confidence Ellipses for Addition of Parameters by Direction Example.

These standard confidence ellipses are plotted in Figure AIII.3.

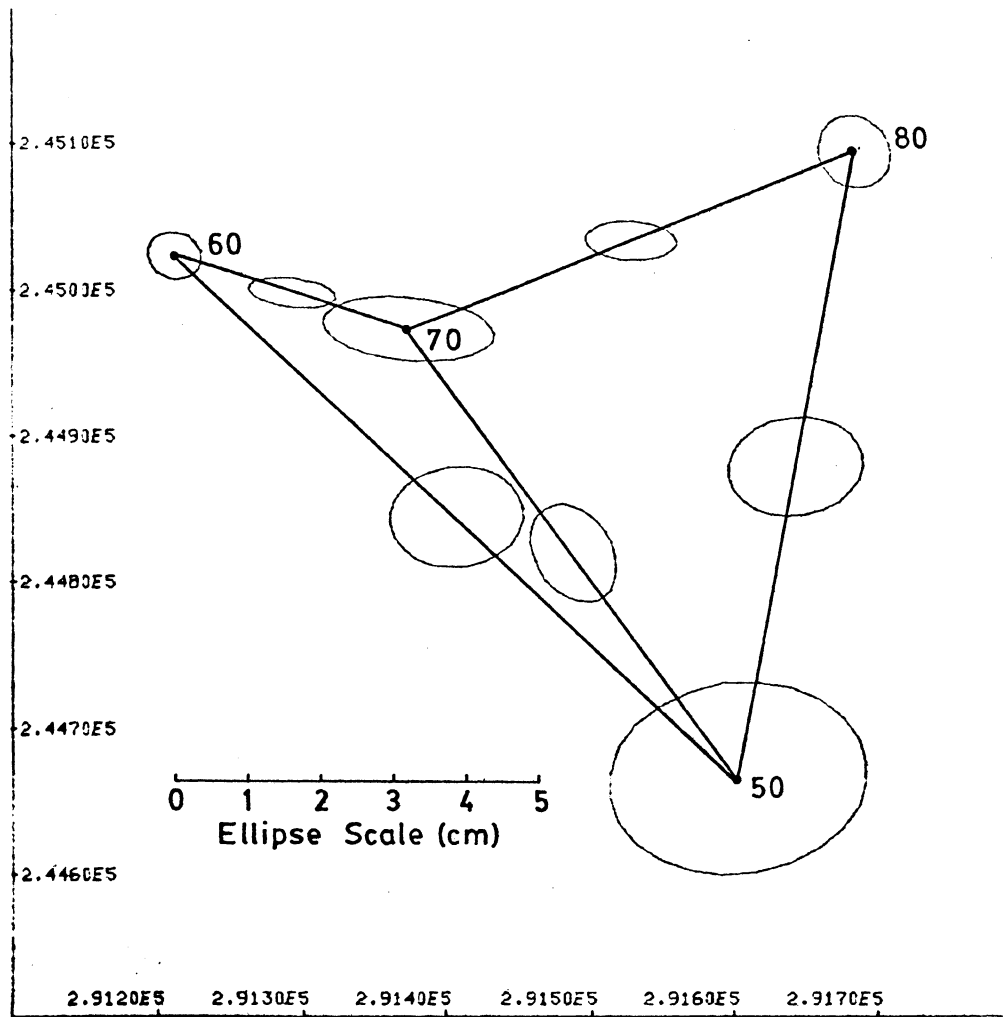


Figure AIII.3. Plot of Standard Confidence Ellipses After Addition of Parameters by Directions

The comparison plot in Figure AII.4 shows that even though the C-factor for the point ellipses is larger, the point confidence ellipses are smaller because of the addition of the new station and direction observables.

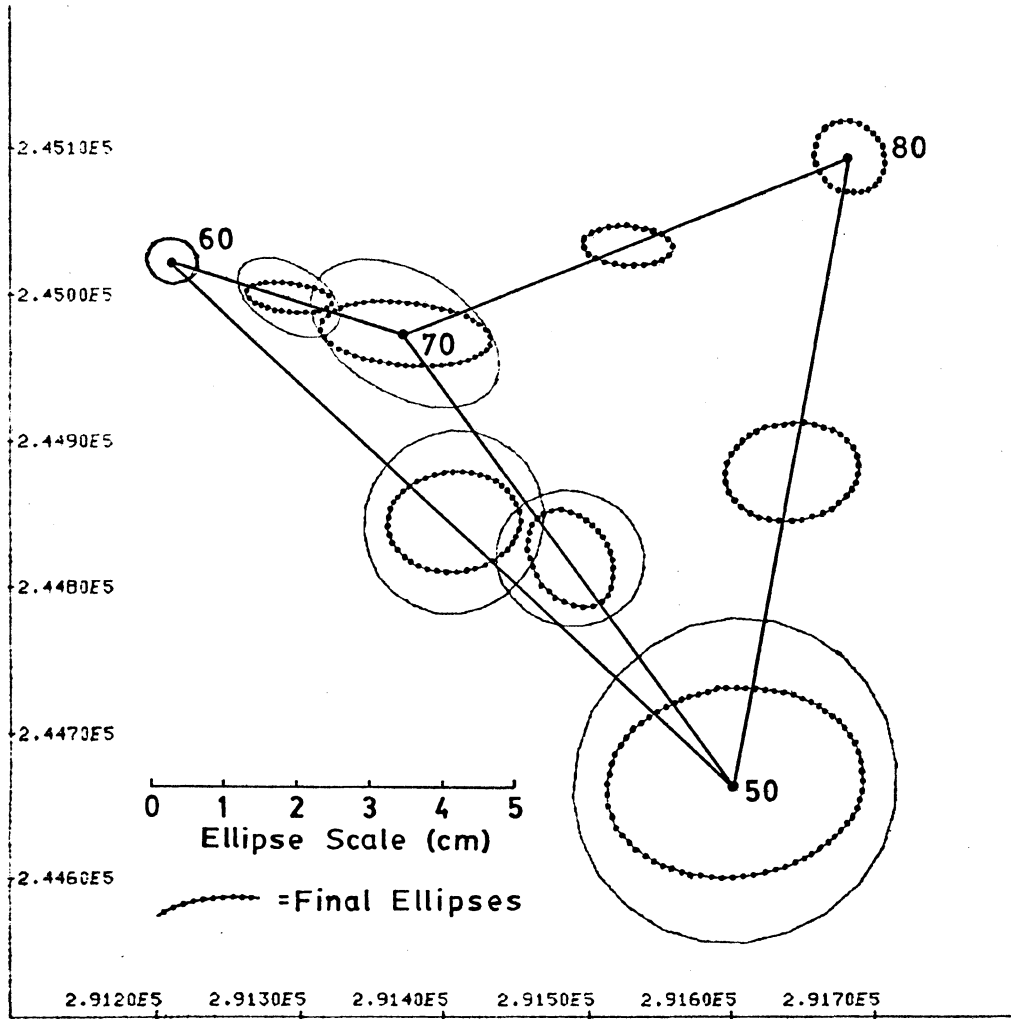


Figure AIII.4. Comparison Plot of Initial and Final Designs for Addition of Parameters by Directions Example

APPENDIX IV. Subtraction of Parameters

This appendix illustrates the model developed in section 2.2 for the deletion of unknown parameters. The network for this illustration is depicted in Figure AIV.1 below. The stations have the same coordinates as they do in the first appendix, but this time stations 60 and 80 are weighted. The dotted lines in this case represent the angle observable at point 60 from 80 to 50. In the initial case,

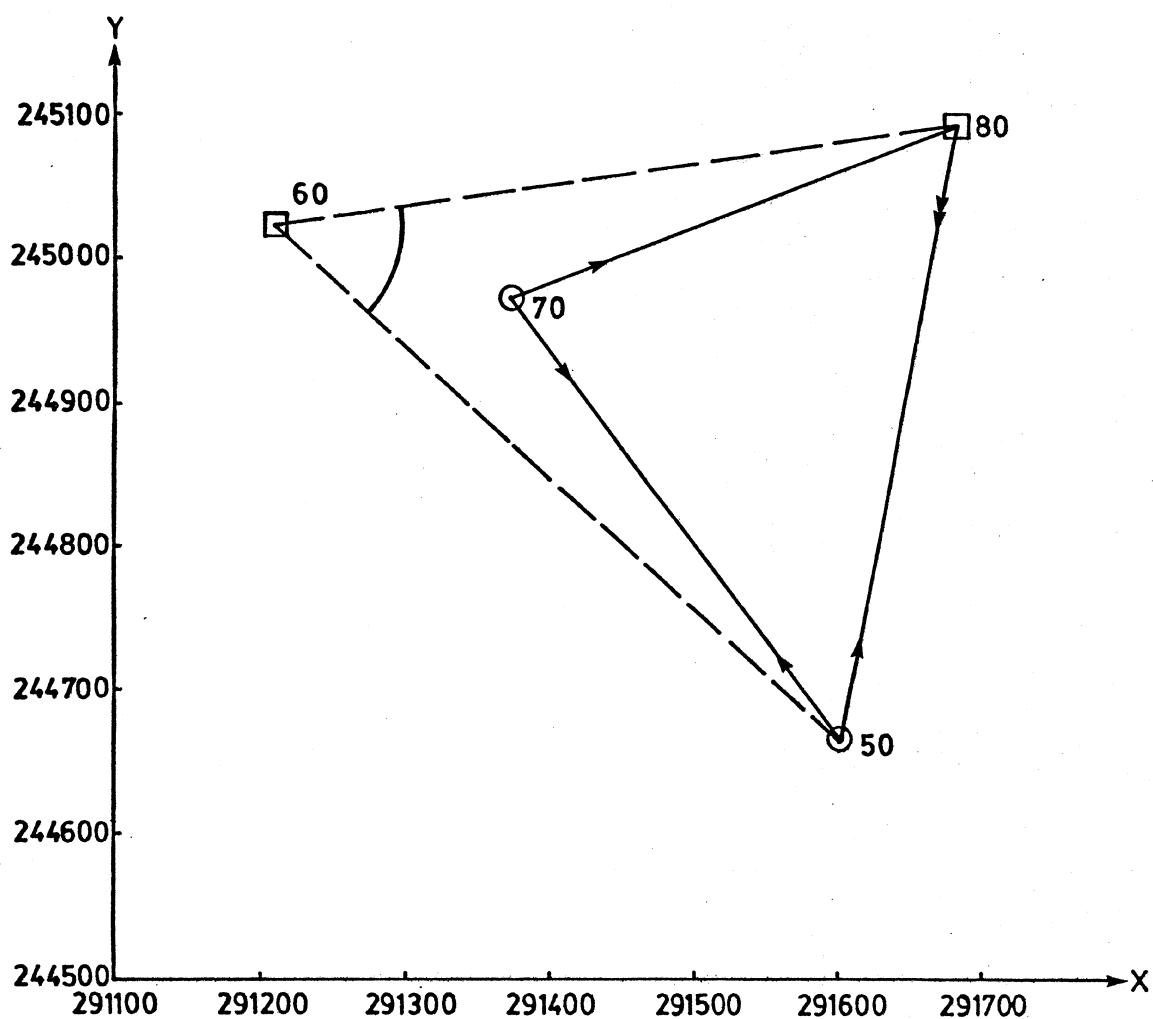


Figure AIV.1. Example Network for Subtraction of Parameters by an Angle Observable

it is included, but for the update or new situation, it is deleted, and, it being the only observable connecting station 60 with the rest of the network, causes station 60 to be eliminated.

To begin with, then, the initial observables are shown in Table AIV.1. In this case, the simultaneous ellipses are computed

<u>Type</u>	<u>At</u>	<u>From</u>	<u>To</u>	<u><math>\sigma</math></u>
1	-	70	80	2.5
1	-	70	50	2.5
1	-	50	70	2.0
1	-	50	80	2.0
2	-	70	80	0.010
2	-	70	50	0.012
3	-	80	50	5.0
4	60	80	50	3.0

Table AIV.1. Initial Observables Used for Subtraction of Parameters Example at the standard confidence level, and the a priori variance factor is assumed unknown, which leads to the C factor being computed according to equation (3-48) as

$$C = \sqrt{2F_{2, n-u, 1-\frac{\alpha}{N}}} = \sqrt{2F_{2, 2, 0.8484}} = \sqrt{11.19} = 3.35.$$

This factor applies to the station ellipses; however, the relative ellipses are not required to be simultaneous, and thus their C factor is

$$C = \sqrt{2F_{2, 2, 1-\alpha}} = \sqrt{2F_{2, 2, 0.3935}} = \sqrt{1.30} = 1.14.$$

With this in mind, the confidence ellipses shown in Table AIV.2 and plotted in Figure AIV.2 result from the initial observables in Table AIV.1. It is interesting to note that the weighted points have



## STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	0.0361	0.0224	-45.240
60	0.0064	0.0054	-18.195
70	0.0368	0.0191	88.701
80	0.0090	0.0080	-40.682

## RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	70	0.0108	0.0090	29.539
50	80	0.0120	0.0074	-42.100
50	60	0.0125	0.0077	-46.415
60	80	0.0036	0.0033	-33.918
70	80	0.0122	0.0063	89.034

Table AIV.2. Standard Confidence Ellipses Resulting from Initial Design for Subtraction of Parameters Example

significantly smaller confidence ellipses than do the fully unknown points. This indicates that the error resulting from the actual observables is quite a bit larger than that of the weighted points. The weights used for station 60 and 80 are

$$P_{x_{60,80}} = \begin{bmatrix} 0.29513E6 & 0.28418E5 & 0 & 0 \\ & 0.37081E6 & 0 & 0 \\ & & 0.15031E6 & 0.19647E5 \\ \text{symmetric} & & & 0.15906E6 \end{bmatrix} .$$

The next step is to subtract the angle observed at station 60, and in so doing, delete station 60 from the design as well. Following the development of section 2.2.2, we wish to compute the  $C_x$  matrix existing before the addition of station 60 and its angle observable. Equation (2-48) expresses this previous  $C_x$  matrix  $N_1^{-1}$  as

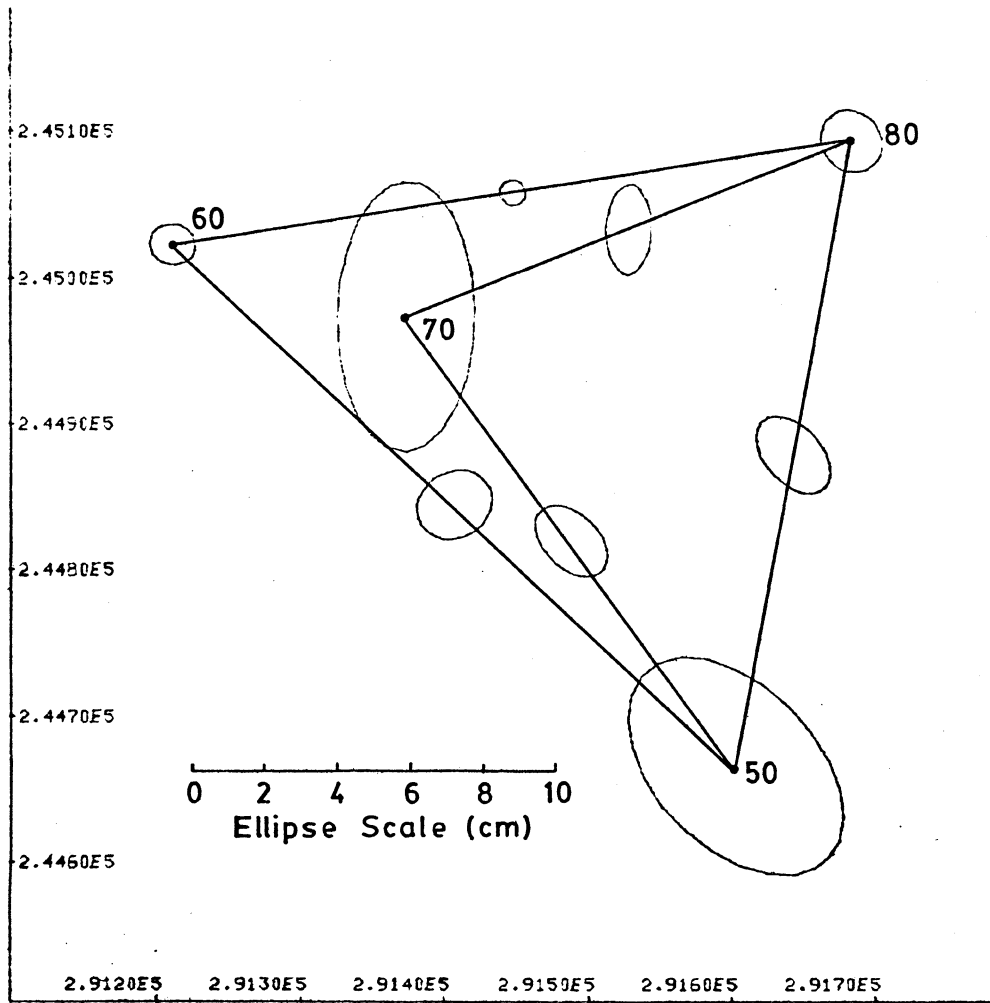


Figure AIV.2. Plot of Standard Confidence Ellipses for Initial Design of Subtraction of Parameters Example

$$\tilde{N}_1^{-1} = \tilde{N}_1^{-1} + \tilde{N}_1^{-1} A_{21}^T [M_2 - A_{21} \tilde{N}_1^{-1} A_{21}^T]^{-1} A_{21} \tilde{N}_1^{-1},$$

where

$$\tilde{N}_1^{-1} = Q_1 - Q_1 N_{12} [N_2 + N_{12}^T Q_1 N_{12}]^{-1} N_{12}^T Q_1,$$

$$N_{12} = A_{21}^T M_2^{-1} A_2,$$

and

$$N_2 = P_{x_2} + A_2^T M_2^{-1} A_2.$$

The matrix  $Q_1$  is the portion of the nonexisting  $C_x$  matrix which corresponds to stations 50, 70 and 80; i.e. rows and columns (1, 2, 5, 6, 7, 8) at the present  $C_x$ . Following the development of the angle model in section 3.3.2, the matrices  $A_{21}$  and  $A_2$  are given as

$$A_{21} = \frac{\partial F_2}{\partial X_1} = \left[ \frac{\partial F_2}{\partial x_{80}} \quad \frac{\partial F_2}{\partial y_{80}} \quad \frac{\partial F_2}{\partial x_{50}} \quad \frac{\partial F_2}{\partial y_{50}} \right] = \rho'' \left[ \frac{-(y_{80} - y_{60})}{s_{60,80}^2} \quad \frac{(x_{80} - x_{60})}{s_{60,80}^2} \quad \frac{(y_{50} - y_{60}) - (x_{50} - x_{60})}{s_{50,60}^2} \quad \frac{-(x_{50} - x_{60})}{s_{50,60}^2} \right],$$

$$A_{21} = [-62.36 \quad 428.18 \quad -263.04 \quad -287.57],$$

$$A_2 = \frac{\partial F_2}{\partial X_2} = \left[ \frac{\partial F_2}{\partial x_{60}} \quad \frac{\partial F_2}{\partial y_{60}} \right] = \rho'' \left[ \frac{-(y_{50} - y_{60})}{s_{50,60}^2} + \frac{(y_{80} - y_{60})(x_{50} - x_{60})}{s_{60,80}^2} - \frac{(x_{80} - x_{60})}{s_{60,80}^2} \right] =$$

$$= [325.4 \quad -140.61].$$

The matrix  $P_{x_2}$  is the weight matrix for point 60, which has already been defined, and  $M_2$  is the inverse of the weight of the angle observable, which in this case is

$$M_2 = P_2^{-1} = C_{L_2} = 9''^2.$$

Thus,  $N_{12}$  and  $N_2$  are now computed as

$$N_{12} = \begin{bmatrix} -9510.5 & 4109.7 \\ -10397 & 4492.9 \\ -2254.7 & 974.28 \\ 15481 & -6689.8 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 3.0690E5 & 2.3334E4 \\ 2.3334E4 & 3.7301E5 \end{bmatrix}.$$

To now compute  $\tilde{N}_1^{-1}$ , first the expression under the inverse sign in equation (2-49) is found. To do this, considering that  $Q_1$  is the old or previous inverse,  $N_{12}$  is multiplied onto rows and columns (7, 8, 1, 2) of  $Q_1$  (i.e. those rows and columns corresponding to stations 80 and 50). When computing  $Q_1 N_{12}$ , rows (1, 2, 5, 6, 7, 8) and columns (7, 8, 1, 2) are chosen from the old  $C_x$  matrix  $Q_1$  for multiplication onto  $N_{12}$ . This results in

$$\tilde{N}_1^{-1} = \begin{bmatrix} 8.0256E-5 & -3.6496E-5 & 1.3335E-6 & -5.4284E-7 & 1.4205E-5 & -6.6583E-5 & 4.0240E-6 & 5.4729E-7 \\ & 8.0910E-5 & 1.2732E-6 & -5.1828E-7 & 2.0143E-5 & 6.6377E-5 & -3.4495E-6 & 7.7101E-6 \\ & & 3.3645E-6 & -2.4166E-7 & 1.2064E-6 & -1.4210E-7 & 9.8983E-8 & -4.9978E-8 \\ & & & 2.7087E-6 & -4.9109E-7 & 5.7848E-8 & -4.0294E-8 & 2.0345E-8 \\ & & & & 3.2325E-5 & 2.0564E-6 & 4.2850E-6 & 4.1548E-7 \\ & symmetric & & & & 1.2089E-4 & -5.4346E-7 & 6.2428E-6 \\ & & & & & & 6.5588E-6 & -7.3263E-7 \\ & & & & & & & 6.3383E-6 \end{bmatrix}.$$

Again when computing  $N_1^{-1}$ , rows and columns (7, 8, 1, 2) of  $\tilde{N}_1^{-1}$  are used to compute  $A_{21} \tilde{N}_1^{-1} A_{21}^T$ , and rows (7, 8, 1, 2) and columns (1, 2, 5, 6, 7, 8) of  $\tilde{N}_1^{-1}$  are required to compute  $A_{21} \tilde{N}_1^{-1}$ . After carrying this out, and deleting the rows and columns for station 60, the final  $N_1^{-1}$  is

$$N_1^{-1} = \begin{bmatrix} 1.1714E-4 & -1.2776E-6 & 4.7576E-5 & -7.0514E-5 & 6.7621E-6 & -8.3525E-7 \\ -1.2776E-6 & 1.1454E-4 & 5.2004E-5 & 6.2623E-5 & -8.3525E-7 & 6.3901E-6 \\ 4.7576E-5 & 5.2004E-5 & 6.2515E-5 & -1.4997E-6 & 6.7621E-6 & -8.3525E-7 \\ -7.0514E-5 & 6.2623E-5 & -1.4997E-6 & 1.2131E-4 & -8.3525E-7 & 6.3901E-6 \\ 6.7621E-6 & -8.3525E-7 & 6.7621E-6 & -8.3525E-7 & 6.7621E-6 & -8.3525E-7 \\ -8.3525E-7 & 6.3901E-6 & -8.3525E-7 & 6.3901E-6 & -8.3525E-7 & 6.3901E-6 \end{bmatrix}$$

For the computation of confidence ellipses, the simultaneous standard station ellipses are again required assuming the a priori variance factor unknown. Because there are now only 3 points instead of 4, and the degrees of freedom is reduced to 1, the C factor is

$$C = \sqrt{2F_{2,1,1-\frac{0.6065}{3}}} = \sqrt{2F_{2,1,0.7978}} = \sqrt{23.46} = 4.84.$$

The C factor for the relative ellipses also changes as the degrees of freedom change, and results in

$$C = \sqrt{2F_{2,1,0.3935}} = \sqrt{1.719} = 1.311 .$$

Taking these into account, the final resulting standard confidence ellipses are given in Table AIV.3 and plotted in Figure AIV.3. The comparison plot in Figure AIV.4 clearly shows the difference before and after subtraction of station 60.

## STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	0.0523	0.0518	-22.207
70	0.0533	0.0383	-88.540
80	0.0131	0.0116	-38.723

## RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
50	70	0.0143	0.0115	64.834
50	80	0.0138	0.0136	-10.793
70	80	0.0140	0.0098	-89.357

Table AIV.3. Standard Confidence Ellipses After Subtraction of  
Parameters.

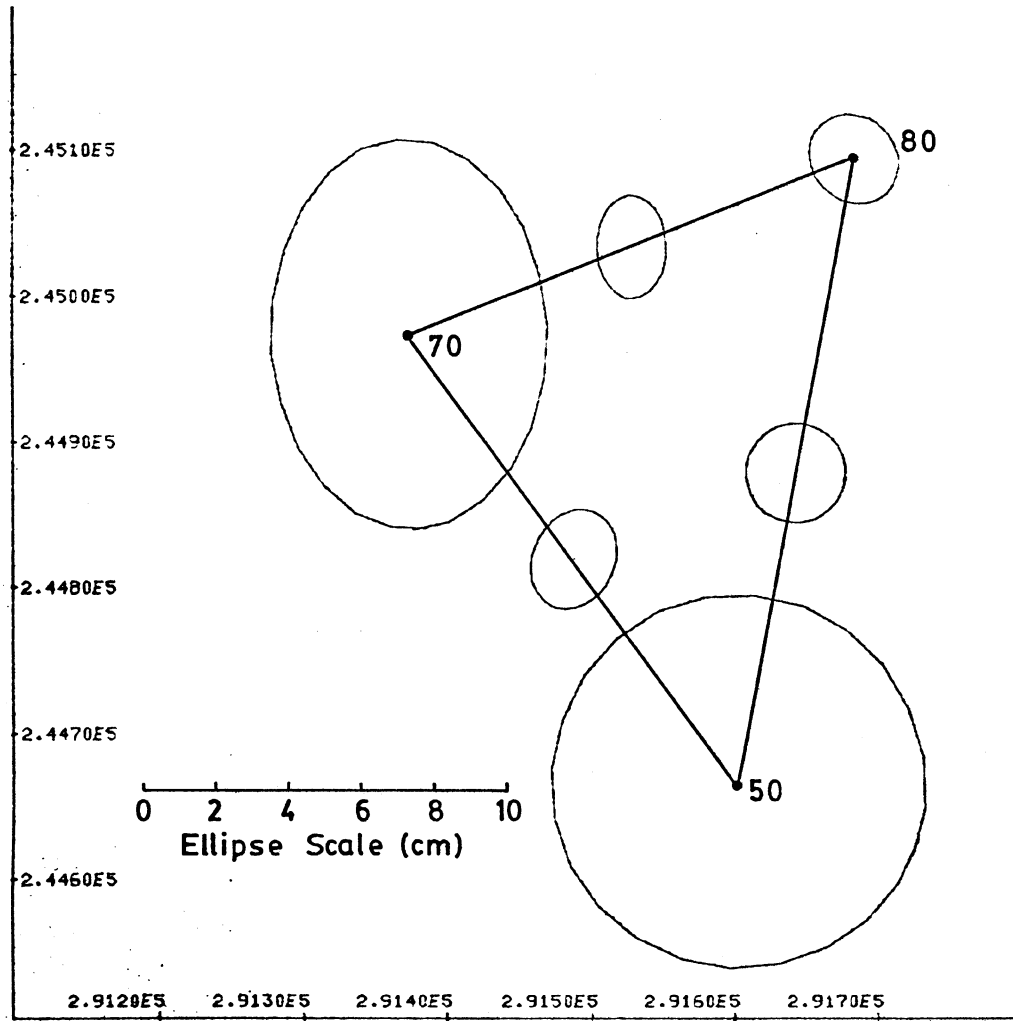


Figure AIV.3. Plot of Standard Confidence Ellipses After Subtraction of a Station

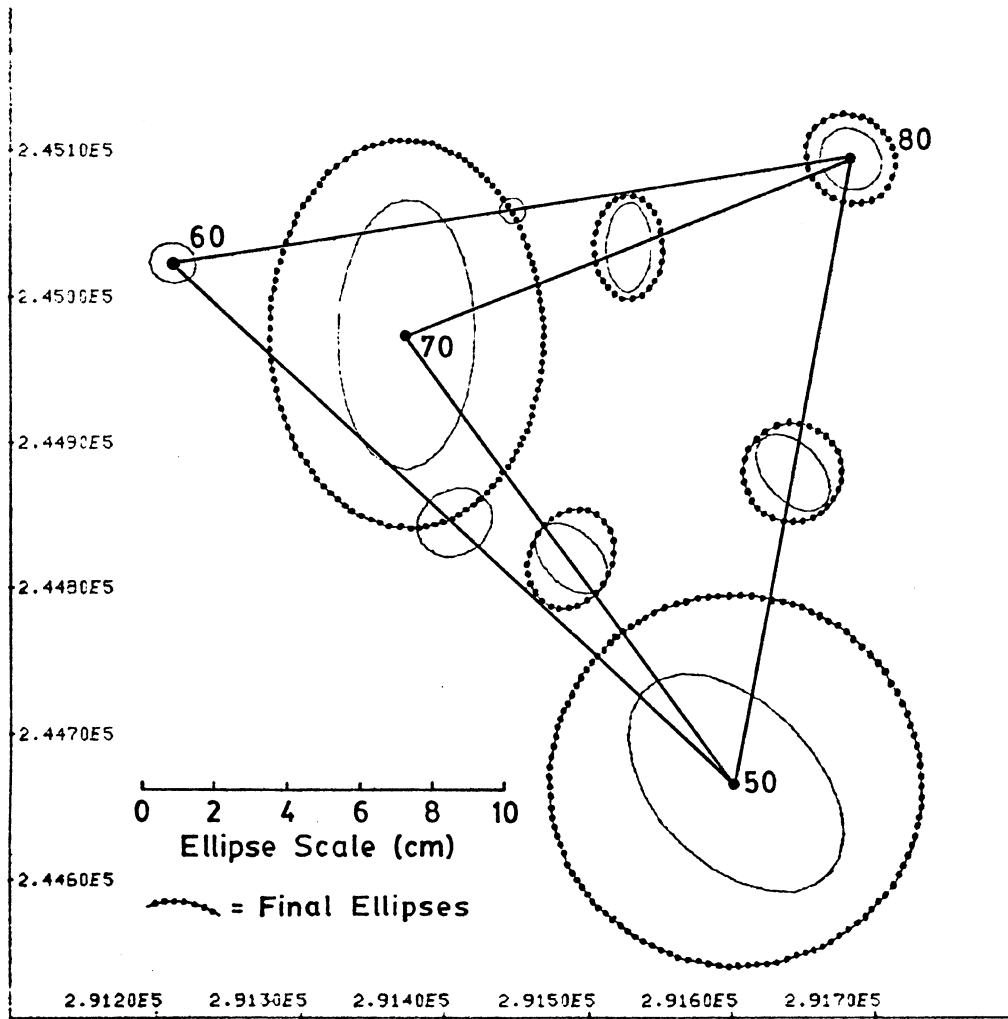


Figure AIV.4. Comparison Plot of Standard Error Ellipses Before and After Subtraction of Station 60



APPENDIX V. Illustration of Angle and Direction Comparison

In this appendix the five point network depicted in Figure AV.1 below has been used to perform three different preanalyses. This network is similar to one used in a paper by Allman and Bennett [1966] in their

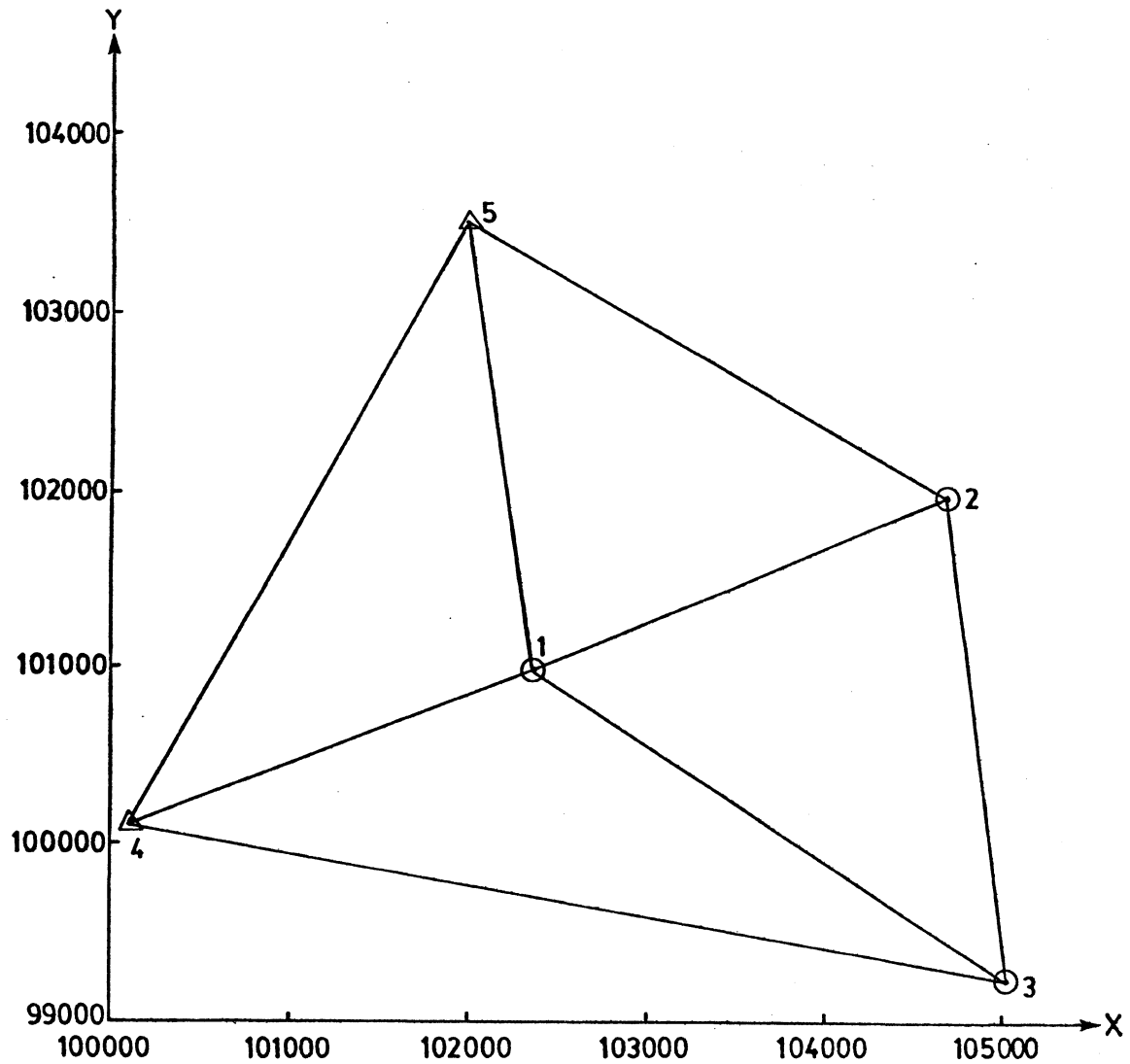


Figure AV.1. Example Network for Comparison of Angles and Directions

comparison of directions and angles by the condition adjustment.

Points 4 and 5 are considered as fixed points and points 1, 2 and 3 are unknown. The coordinates of the points are listed in Table AV.1.

FIXED STATIONS:			UNKNOWN STATIONS:		
STATION	X COORD.	Y COORD.	STATION	X COORD.	Y COORD.
4	100100.000	100100.000	1	102368.134	100987.503
5	102000.000	103500.000	2	104679.915	101954.045
			3	105016.967	99244.124

Table AV.1. Point Coordinates for Angle and Direction Comparison  
Examples

The three separate preanalyses include one using directions only, one using correlated angles and one using independent angles.

These three preanalyses are presented below.

#### 1. Directions

In this case, all of the possible directions are measured in the network, which brings the total to 16 measured directions. Each direction was assumed to have a standard deviation of 1.9997", and thus the P or weight matrix for this case is a (16x16) diagonal matrix with 0.2501 on the diagonal.

There are 3 unknown points and 5 sets of directions which lead to 6 unknown coordinates and 5 orientation unknowns for a total of 11 parameters. This produces an A design matrix as follows:

$$\begin{array}{l}
 A = \\
 (16,11)
 \end{array}
 \begin{array}{c}
 \begin{array}{cccccccccccc}
 X_1 & Y_1 & X_2 & Y_2 & X_3 & Y_3 & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\
 d_{4,5} & \left[ \begin{array}{cccccccccccc}
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
 a_{4,1} & -b_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & a_{4,3} & -b_{4,3} & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & a_{5,2} & -b_{5,2} & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 a_{5,1} & -b_{5,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 -a_{1,5} & b_{1,5} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 -a_{1,2} & b_{1,2} & a_{1,2} & -b_{1,2} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 -a_{1,3} & b_{1,3} & 0 & 0 & a_{1,3} & -b_{1,3} & -1 & 0 & 0 & 0 & 0 \\
 -a_{1,4} & b_{1,4} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & -a_{2,3} & b_{2,3} & a_{2,3} & -b_{2,3} & 0 & -1 & 0 & 0 & 0 \\
 a_{2,1} & -b_{2,1} & -a_{2,1} & b_{2,1} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & -a_{2,5} & b_{2,5} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -a_{3,4} & b_{3,4} & 0 & 0 & -1 & 0 & 0 \\
 a_{3,1} & -b_{3,1} & 0 & 0 & -a_{3,1} & b_{3,1} & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & a_{3,2} & -b_{3,2} & -a_{3,2} & b_{3,2} & 0 & 0 & -1 & 0 & 0
 \end{array} \right.
 \end{array}
 \end{array}$$

where

$$a_{i,j} = \frac{(y_j - y_i)}{S_{ij}^2} \quad \text{and} \quad b_{i,j} = \frac{(x_j - x_i)}{S_{ij}^2}$$

The A matrix appears here as it would if formed for the whole network at once. In the program which actually performs the preanalysis, the A design matrix is formed only in parts, generally being able to avoid the many unnecessary zero elements. As well, the method

discussed in Appendix I (cf. eq. (AI-1) is applied in order to eliminate the orientation unknowns. The results obtained after the preanalysis with directions are listed in Table AV.2 and plotted in Figure AV.2 assuming nonsimultaneous ellipses with the a priori variance factor known.

RESULTS ACHIEVED AT 39.35 PERCENT CONFIDENCE LEVEL

STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
1	.0281	.0173	56.238
2	.0413	.0338	-45.628
3	.0590	.0459	-67.498

RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
1	2	.0396	.0242	-1.488
2	3	.0456	.0338	-78.162
1	3	.0508	.0314	-41.586

Table AV.2. Standard Confidence Ellipses Using Directions

2. Correlated Angles.

The preanalysis was repeated using angles which were correlated in the manner described in section 3.3.3 Although it looks as if there are 12 possible angles in Figure AV.1, in fact there are only 3 angles at station 1 resulting from the 4 measured directions. Thus, there are 11 measured angles and 6 unknown coordinates involved.

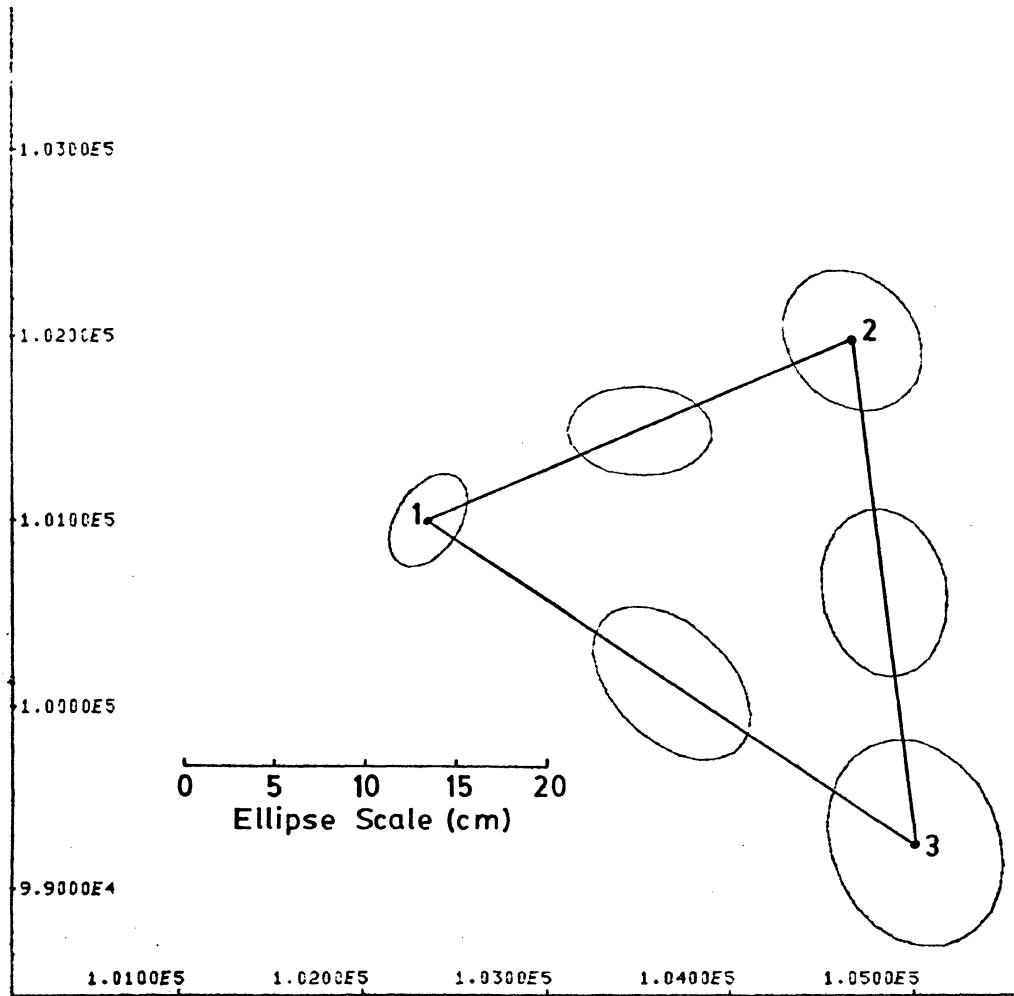


Figure AV.2. Plot of Standard Error Ellipses Resulting from  
Design by Directions or Correlated Angles

The P or weight matrix in this case is (11x11) and block diagonal. Assuming that the angles have been derived from the directions in the previous example, the standard deviation of one measured angle is 2.828". Thus, the  $C_a$  matrix takes the form

$$C_a = (11 \times 11) = \begin{bmatrix} a_{4,5,1} & 7.9976 & -3.9988 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{4,1,3} & -3.9988 & 7.9976 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{5,2,1} & 0 & 0 & 7.9976 & -3.9988 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{5,1,4} & 0 & 0 & -3.9988 & 7.9976 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{1,5,2} & 0 & 0 & 0 & 0 & 7.9976 & -3.9988 & 0 & 0 & 0 & 0 & 0 \\ a_{1,2,3} & 0 & 0 & 0 & 0 & -3.9988 & 7.9988 & -3.9988 & 0 & 0 & 0 & 0 \\ a_{1,3,4} & 0 & 0 & 0 & 0 & 0 & -3.9988 & 7.9976 & 0 & 0 & 0 & 0 \\ a_{2,3,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7.9976 & -3.9988 & 0 & 0 \\ a_{2,1,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.9988 & 7.9976 & 0 & 0 \\ a_{3,4,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7.9976 & -3.9988 \\ a_{3,1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.9988 & 7.9976 \end{bmatrix}$$

and the weight matrix P is the inverse of  $C_a$ .

The A design matrix in this case is

$$A(11 \times 6) = \begin{bmatrix} a_{4,5,1} & X_1 & Y_1 & X_2 & Y_2 & X_3 & Y_3 \\ a_{4,1,3} & -c_{4,1} & b_{4,1} & 0 & 0 & c_{4,3} & -b_{4,3} \\ a_{5,2,1} & c_{5,1} & -b_{5,1} & -c_{5,2} & b_{5,2} & 0 & 0 \\ a_{5,1,4} & -c_{5,1} & b_{5,1} & 0 & 0 & 0 & 0 \\ a_{1,5,2} & e_{1,5,2} & f_{1,5,2} & c_{1,2} & -b_{1,2} & 0 & 0 \\ a_{1,2,3} & e_{1,2,3} & f_{1,2,3} & -c_{1,2} & b_{1,2} & c_{1,2} & -b_{1,2} \\ a_{1,3,4} & e_{1,3,4} & f_{1,3,4} & 0 & 0 & -c_{1,3} & b_{1,3} \\ a_{2,3,1} & c_{2,1} & -b_{2,1} & e_{2,3,1} & f_{2,3,1} & -c_{2,3} & b_{2,3} \\ a_{2,1,5} & -c_{2,1} & b_{2,1} & e_{2,1,5} & f_{2,1,5} & 0 & 0 \\ a_{3,4,1} & c_{3,1} & -b_{3,1} & 0 & 0 & e_{3,4,1} & f_{3,4,1} \\ a_{3,1,2} & -c_{3,1} & b_{3,1} & c_{3,2} & -b_{3,2} & e_{3,1,2} & f_{3,1,2} \end{bmatrix}$$

where

$$c_{i,j} = \frac{(y_j - y_i)}{s_{ij}^2}, \quad b_{i,j} = \frac{(x_j - x_i)}{s_{ij}^2}, \quad e_{i,j,k} = \frac{-(y_k - y_i)}{s_{ik}^2} + \frac{(y_j - y_i)}{s_{ij}^2},$$

and

$$f_{i,j,k} = \frac{(x_k - x_i)}{s_{ik}^2} - \frac{(x_j - x_i)}{s_{ij}^2}.$$

Again, this matrix is never formed entirely by the program which performs the preanalysis, but it produces the same normal equations matrix in the end.

By looking at Table AV.3, one can see that the resulting confidence ellipses are identical to those of the preanalysis with directions. One can conclude from this, then, that both the method of correlated angles and that of directions give identical variance covariance matrices for the unknown stations.

RESULTS ACHIEVED AT 39.35 PERCENT CONFIDENCE LEVEL

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RELATIVE ELLIPSES

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2	3	.0456	.0338	-78.162
1	3	.0508	.0314	-41.586

Table AV.3. Standard Confidence Ellipses Using Correlated Angles.

### 3. Uncorrelated Angles.

The uncorrelated angles method of preanalysis differs from the correlated angles method only in the formation of the P matrix. It is now diagonal with 0.1250 ( $\sigma_a = 2''828$ ) on the diagonal instead of being the inverse of the nondiagonal  $C_a$  as in the previous preanalysis. The A design matrix is identical to the correlated angles example.

The results using this method are given in Table AV.4 and plotted in Figure AV.3 below. It can be seen from the comparison plot in Figure AV.4 that this method gives significantly different results. The size of all of the ellipses is larger and the orientation is slightly different. Thus, it can be concluded that at least in this case, the method of directions or correlated angles is preferable for network design and adjustment.

#### RESULTS ACHIEVED AT 39.35 PERCENT CONFIDENCE LEVEL

##### STATION ELLIPSES

STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
1	.0337	.0218	57.757
2	.0484	.0449	-72.163
3	.0794	.0562	-71.221

##### RELATIVE ELLIPSES

STATION NUMBER	STATION NUMBER	SEMI-MAJOR AXIS	SEMI-MINOR AXIS	THETA
1	2	.0436	.0297	3.028
2	3	.0593	.0424	-81.413
1	3	.0623	.0364	-49.798

Table AV.4. Standard Confidence Ellipses Using Uncorrelated Angles



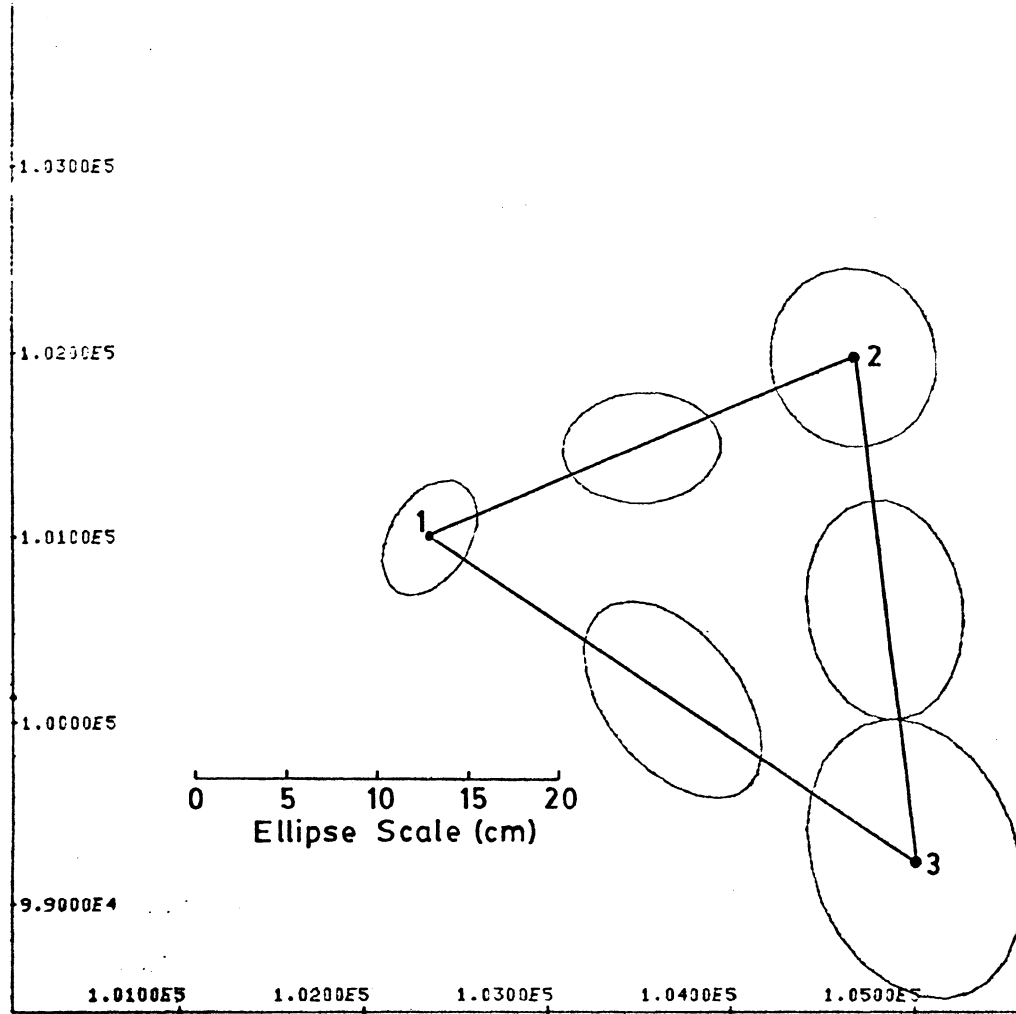


Figure AV.3. Plot of Standard Error Ellipses Resulting From Design by Uncorrelated Angles

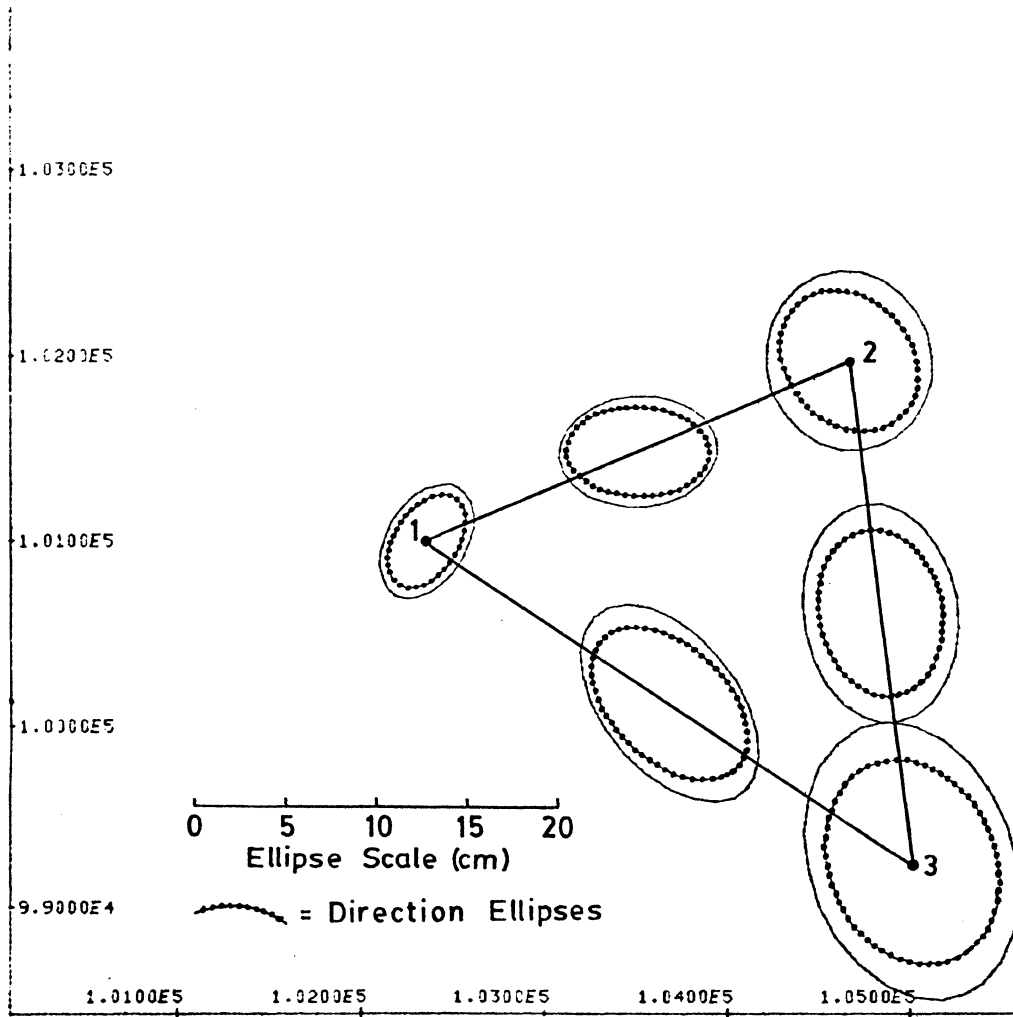


Figure AV.4. Comparison Plot of Standard Error Ellipses Using Directions and Uncorrelated Angles

VITA

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Thomson and E.J. Krakiwsky). Dept. of Surveying Engineering,  
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