INVESTIGATION ON THE ANALYTICAL FORM OF THE TRANSITION MATRIX IN INERTIAL GEODESY

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PREFACE

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ABSTRACT

The error behaviour of inertial survey systems can best be described by a system of differential equations. Its solution in analytical form, by way of a transition matrix, is discussed in this report.

After a review of the methods available to solve systems of differential equations, the dynamics matrix of the local-level system operating in three dimensions is treated in detail. Two methods are used to derive the analytical form of the transition matrix: the inverse Laplace transform technique and the series expansion of the matrix exponential. Analytical and numerical comparisons show that the two derived solutions are not completely equivalent but agree very well for time intervals up to 1000 seconds. For large time spans the inverse Laplace transform solution is more accurate. The report concludes with a brief discussion of the effects which the variation of certain parameters has on the error behaviour.

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1. INTRODUCTION

During the last four years inertial survey systems have been routinely used to establish second-order control and a recent analysis of extensive test data indicates that first order accuracy may be achievable (Schwarz, 1979a). Considering the amount of information generated by these systems, the integration of this new data type into existing control is a task of growing importance. Its solution requires error propagation models which take into account the time dependence of all quantities involved. Thus, the basic mathematical model is a system of differential equations.

The solution of such systems is usually done today by numerical integration procedures. The methods are well developed mathematically and excellent program packages have become available in recent years. A review of the state of the art is given in Lambert (1977).
The advantage of this approach is that it is almost universally applicable, i.e. inhomogeneous and nonlinear systems can be treated as well as the homogeneous case. However, it is very time consuming computerwise and it does not allow general statements about the stability of the system and its characteristical features. Thus, analytical solutions, besides requiring less computer time, can be of great value to study the general behaviour of a specific system. Generally, such

solutions can only be obtained for a more restricted class of problems and even then the amount of formal manipulations can be exasperating. It was felt, however, that this effort would be well spent for the study of the error propagation in inertial survey systems. The following report documents how far this attempt has been successful.

It should be noted at this point that the accuracy of the final coordinates produced by an inertial survey system is a mixture of continuous error propagation through a system of differential equations and update procedures at discrete instants in time which basically are discontinuous. Here, only the first part is treated. Its solution sets the scene for the filtering and smoothing procedures which are used when discrete control measurements become available. The application of such methods to the error propagation in inertial positioning has been treated in Schwarz (1979b).

The report has been organized in such a way that a brief review on the solution of systems of differential equations is given first. Then, the system of error equations used in inertial geodesy is described. Its solution is obtained in two ways, by using the inverse Laplace transform and by a series expansion of the matrix exponential. Comparison of results and some numerical studies conclude the report.

2. SOLUTION OF SYSTEMS OF DIFFERENTIAL EQUATIONS

The inhomogeneous linear system of vector differential equations

$$\dot{\vec{x}}(t) = F(t)\,\vec{x}(t) + G(t)\,\vec{u}(t) \qquad (2\text{-}1)$$

with initial conditions

$$\vec{x}(o) = \vec{c}$$

is used as a model for the errors in an inertial survey system. In this equation vectors are denoted by an arrow and matrices by capital letters. Let us first discuss the homogeneous case $\vec{u}(t) = 0$

$$\dot{\vec{x}}(t) = F(t)\,\vec{x}(t) \qquad (2\text{-}2)$$

with

$$\vec{x}(0) = \vec{c} \quad .$$

It describes a physical system whose state at any time t is completely defined by the N functions $x_i(t)$ contained in $\vec{x}^T = \{x_1, x_2 \ldots x_N\}^T$ and whose rate of change at a given time t_k depends only on the values of the functions $x_k(t)$ at t_k. This means that in a formal way the solution can be written as

$$\vec{x}(t) = \Phi(t, t_o) \cdot \vec{c} \qquad (2\text{-}3)$$

where $\Phi(t, t_o)$ is as yet undetermined. We will call $\vec{x}(t)$ the state vector, $F(t)$ the dynamics matrix, and $\Phi(t, t_o)$ the transition matrix.

These names are often used in optimal estimation literature. $\Phi(t, t_o)$ is a square matrix which in general is nonsymmetric. The vector $\vec{c}$ represents the initial state of the system. If the elements of $\vec{c}$ are such that equation (2-2) becomes zero, we speak of an equilibrium state. The behaviour of the system in the neighbourhood of such equilibrium states is of special interest because it determines the stability of the system. A system is called stable if after perturbations it returns to the equilibrium state; otherwise, we speak of an unstable system.

In our case the functions $x_i(t)$ describes the error characteristics of an inertial survey system. It is obvious that in such a case equation (2-2) can only be an approximation. It is dependent on the current knowledge of the error sources and on the requirement that they can be modelled in the form (2-2). Errors of this type are e.g. the position, velocity and misalignment errors at the start of the measurement, gyro drifts and accelerometer biases. If any of these error sources produce an unbounded error growth, we have an unstable system. It is well-known (see e.g. Britting, 1971) that this is the case for the general inertial navigation problem in three dimensions when no outside information is provided for the altitude channel. The formulation of the dynamics matrix has therefore been done in such a way as to exclude this instability.

So far only the homogeneous system has been discussed. The solution of the more general system (2-1) can be written in the form

$$\vec{x}(t) = \Phi(t, t_o)\, \vec{c} + \int_{t_o}^{t} \Phi(t, t_o)\, \Phi^{-1}(s, t_o)\, G(s)\, \vec{u}(s)\, ds \qquad (2\text{-}4)$$

where s is a time variable. A comparison with equation (2-3) shows that the solution obtained from the homogeneous system is changed by adding an integral containing the transition matrix and the functions $G(t) \cdot \vec{u}(t)$. Since the homogeneous equation describes the unforced motion of the system, the functions $u_i(t)$ are often called the forcing functions. They can be either deterministic or random. In the first case, equation (2-1) is said to define a control problem, in the second an estimation problem. A combination of both types of problems is obviously possible. In the applications discussed here the gravity vector can be considered as a control function while accelerometer noise represents a random forcing function.

In order to obtain a solution of type (2-3) or (2-4) the transition matrix must be determined. The remainder of this section will therefore discuss methods to obtain $\Phi(t, t_o)$ in analytical form. The inhomogeneous part of the solution (2-4) will not be treated in detail. In the problems discussed in this report, the forcing functions are always considered as random functions with mean zero, and are therefore handled as part of the optimal estimation procedure.

Equation (2-2) has a unique solution if F(t) is continuous for $t \geq 0$. This solution can be written in the form (2-3) where $\Phi(t, t_o)$ is the unique matrix satisfying the matrix differential equation

$$\dot{\Phi}(t, t_o) = F(t)\, \Phi(t, t_o) \qquad (2\text{-}5)$$

with initial conditions

$$\Phi(t_o, t_o) = I \quad .$$

Among the continuous matrices those with constant coefficients represent the simplest case. For such a matrix the solution of (2-2) can be formally written as a matrix exponential

$$\vec{x}(t) = e^{F(t-t_o)} \vec{c} \quad . \tag{2-6}$$

Comparison with equation (2-3) results in

$$\Phi(t, t_o) = e^{F(t-t_o)} . \tag{2-7}$$

The matrix exponential can be expanded into an infinite series

$$e^{F(t-t_o)} = \sum_{i=0}^{\infty} \frac{(t-t_o)^i F^i}{i!} \tag{2-8}$$

which in conjunction with equation (2-7) gives the first method to compute an analytical form of the transition matrix. Using the Cayley-Hamilton theorem the infinite series expansion can be replaced by a finite series of order N where N is the rank of F. A discussion of this case and its usefulness in applications is given in Bierman (1971).

The meaning of the formal solution (2-6) becomes clearer when considering the special case $F = \lambda I$

$$\vec{x}(t) = e^{\lambda I(t-t_o)} \vec{d} \tag{2-9}$$

where λ is an unknown scalar and $\vec{d}$ an unknown vector. When this solution is introduced into equation (2-2), we obtain for $t_o = 0$

$$\lambda e^{\lambda t} \vec{d} = e^{\lambda t} F \vec{d} \tag{2-10}$$

and thus

$$(F - \lambda I) \vec{d} = 0 \quad . \tag{2-11}$$

If $\vec{d}$ is not a zero vector, equation (2-2) will only have a solution if

$$|F - \lambda I| = 0 \quad . \tag{2-12}$$

This equation represents a standard eigenvalue problem where λ is the eigenvalue and $\vec{d}$ the eigenvector. If all λ_i are distinct then we can transform equation (2-9) into

$$\vec{x}(t) = \sum_{i=1}^{N} e^{\lambda_i t} \vec{d}_i \qquad (2\text{-}12)$$

where N is again the rank of F. This equation shows that a second method to obtain an analytical solution is by way of the eigenvalue problem. If instead of the special case leading to equation (2-9), a full matrix F with distinct eigenvalues has to be treated, the canonical matrix Λ can always be obtained by the transformation

$$T^{-1} F T = \Lambda \qquad (2\text{-}13)$$

where Λ is the diagonal matrix of eigenvalues and the columns of T are the eigenvectors. The transition matrix is then of the simple form

$$\Phi(t, t_o) = T e^{\Lambda t} T^{-1} \quad . \qquad (2\text{-}14)$$

where $e^{\Lambda t}$ can be written as a diagonal matrix of the form

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & e^{\lambda_2 t} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & e^{\lambda_N t} \end{bmatrix} . \qquad (2\text{-}15)$$

In case of multiple eigenvalues the situation is more complicated and reference is made to textbooks as Gantmacher (1959) and Hochstadt (1975).

A third way to arrive at an analytical form for $\Phi(t,t_o)$ is by use of the inverse Laplace transform L^{-1}. Again it is only applicable to differential equations with constant coefficients. The Laplace transform L of a function x(t) is defined as

$$y(s) = L\,x(t)$$

$$= \int_o^\infty e^{-st}\,x(t)\,dt \tag{2-16}$$

and equivalently for the vector $\vec{x}(t)$ of functions $x_i(t)$

$$\vec{y}(s) = L\,\vec{x}(t)$$

$$= \int_o^\infty c^{-st}\,\vec{x}(t)\,dt \tag{2-17}$$

where the operation (2-16) is performed on each element of $\vec{x}(t)$. The Laplace transform of $\vec{x}(t)$ can be found by integrating by parts

$$\int_o^\infty e^{-st}\,\vec{x}(t)\,dt = s\,\vec{y}(s) - \vec{x}(o) \tag{2-18}$$

Using equations (2-17) and (2-18) the homogeneous equation (2-2) can be transformed to

$$s\vec{y}(s) - \vec{c} = F\,\vec{y}(s) \tag{2-19}$$

or

$$(sI - F)\,\vec{y}(s) = \vec{c} \tag{2-20}$$

Thus, the system of differential equations has been converted into a system of algebraic equations. If the inverse of (sI - F) exists we have

$$\vec{y}(s) = (sI - F)^{g}\,\vec{c} \tag{2-21}$$

where the index g denotes some suitably defined generalized inverse. For the Cayley inverse, which is the only one considered hereafter, equation (2-21) reads

$$\vec{y}(s) = (sI - F)^{-1}\,\vec{c} \tag{2-22}$$

Once $\vec{y}(s)$ has been found the inverse Laplace transform L^{-1} can be applied to obtain $\vec{x}(t)$

$$\begin{aligned}\vec{x}(t) &= L^{-1}\,\vec{y}(s) \\ &= \frac{1}{2\pi j}\int_{\sigma_o - i\omega}^{\sigma_o + i\omega} \vec{y}(s)\, e^{st}\, ds\end{aligned} \tag{2-23}$$

where $\sigma_o > \sigma_1$ and σ_1 is some allowable region of convergence. Thus, an $\vec{x}(t)$ must be found whose transform is $\vec{y}(s)$. Usually the individual function $y_i(s)$ can be brought into a form which is listed in one of the extensive tables of Laplace transforms. By using equation (2-3), (2-22) and (2-23) we obtain

$$\Phi(t, t_o) = L^{-1}\,(sI - F)^{-1} \tag{2-24}$$

which is the desired expression for the transition matrix. Applications of this technique will be given in chapter 4.

So far only a matrix F with constant coefficients has been treated. Homogeneous equations with variable coefficients are not so tractable. In general, it is not possible to derive closed form solutions. In certain cases, series solutions can be obtained. This applies for instance to systems where F can be expanded into a series with analytic coefficients. But usually the increase in complexity is considerable. In many cases the solution from a constant coefficient matrix represents a good first approximation and is sufficient for applications. When modelling errors of an inertial system, a constant F-matrix corresponds to a system without vehicle accelerations. Such a system reflects all the long term error frequencies. If necessary, the neglected accelerations can be modelled in a Taylor expansion about the first approximation as discussed by Lyon (1977) for the two-dimensional inertial navigation problem. This report treats only the case of constant coefficients.

3. DYNAMICS MATRIX

3.1 Notation

A number of symbols have been introduced in the last chapter. To simplify the reference they are listed below together with some notations used later on.

$(\dot{\ })$ - first derivative with respect to time

$(\ )^{-1}$ - inverse

F - dynamics matrix

Φ - transition matrix

R - distance from earth centre to platform

ϕ - geodetic latitude

λ - geodetic longitude

h - height above the reference ellipsoid

g - gravity

μ - Schuler frequence ($\sqrt{g/R}$)

ℓ - celestial longitude rate ($\ell = \dot{\lambda} + \omega_{ie}$)

ω_{ie} - earth rate (7.202115 x 10^{-5} rad/s)

ε - attitude errors

δ - coordinate and velocity errors

K_1, K_2 - damping loop gains

t - time

3.2 Description of the Dynamics Matrix

It has been mentioned in section 2 that the dynamics matrix F should model all time dependent errors of the inertial survey system. As equation (2-3) shows they can be presented as modulations of the error vector $\vec{c}$ at the initial point. Britting [1971] has proposed to formulate a unified error theory by using a nine state vector of three initial errors for position, velocity and attitude. This approach is followed here. The much larger state vectors used in the Kalman filters of inertial survey systems are obtained by splitting the above errors into physically meaningful components. Thus, the attitude errors may consists of a misalignment and drift part, the velocity errors of an accelerometer bias and a scale factor. The separation of different components is in general done by a priori weighting and belongs therefore to the estimation part of the problem. A thorough treatment of this matter would have to include the observability conditions of the dynamic system as discussed for instance in Fossard [1977] and Kortüm [1974].

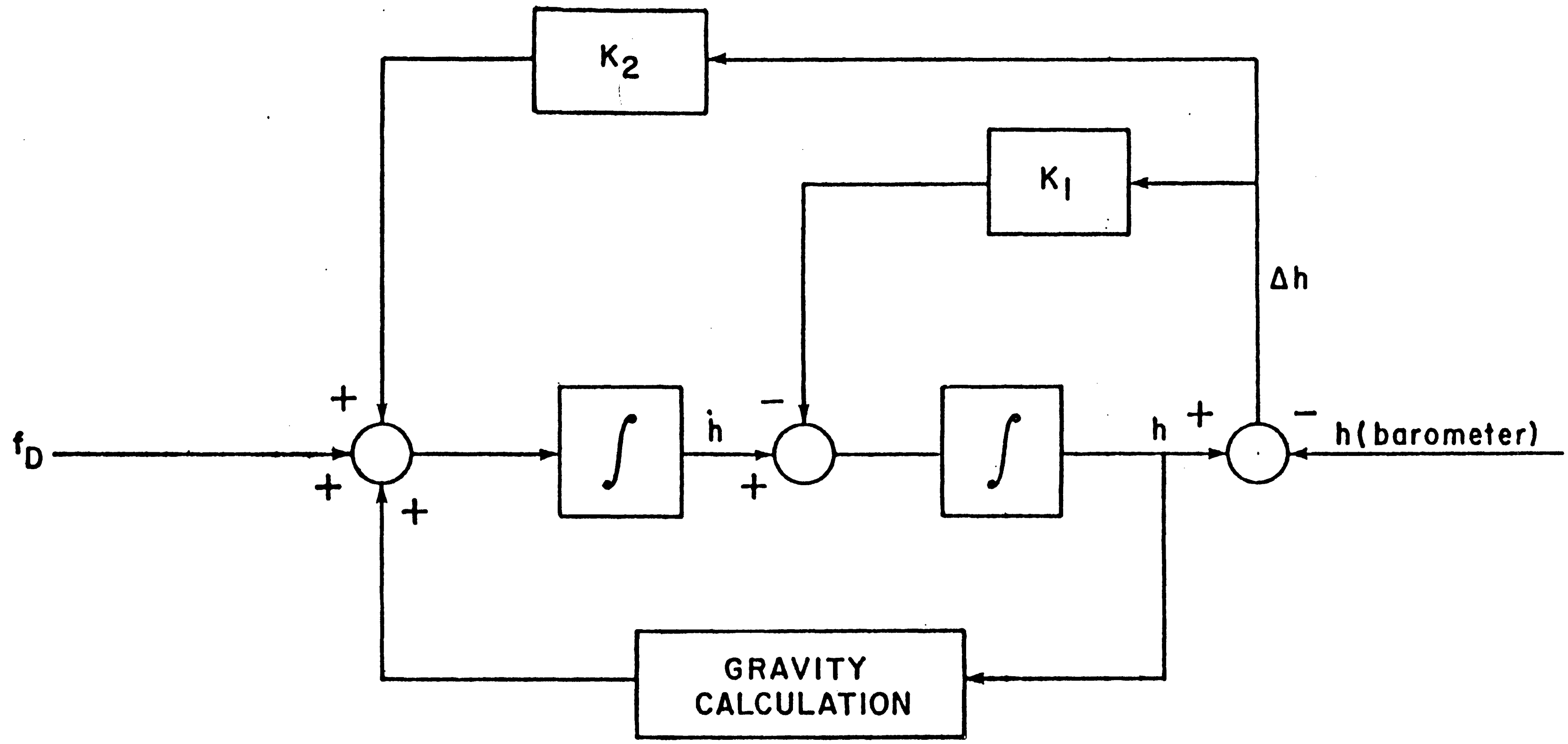
The nine state error vector considered in our investigation is given by

$$x^T = [\varepsilon_N, \varepsilon_E, \varepsilon_D, \delta\phi, \delta\lambda, \delta\dot{\phi}, \delta\dot{\lambda}, \delta h, \delta\dot{h}] \qquad (3\text{-}1)$$

where ε_N, ε_E, ε_D are the attitude errors, $\delta\phi$, $\delta\lambda$, and δh are the latitude, longitude, and height errors, and $\delta\phi$, $\delta\lambda$, and δh are velocity errors in latitude, longitude and height.

The dynamics matrix F associated with the state vector may be derived from the specific force equations and the misorientation error equation given e.g. in Britting [1971] and Adam [1979]. The F-matrix for a local level system that employs a barometric damping second-order loop (as shown in Figure 3-1) can be found in Schmidt [1978]. By neglecting the time dependent elements and the products of velocities, the dynamics matrix can be reduced to the form shown in Figure 3-2. In the aircraft mode, such a system can utilze the height difference information from the altimeter to dampen the exponential growth of errors in the height channel and its effect on the horizontal channels. In the terrestial mode no altimeter data are available, but the errors are controlled by the information obtained at every zero velocity update (ZUPT). Thus, the gain factors K_1 and K_2, although derived from a different set of measurements, can formally be used in exactly the same way in the F-matrix.

The final expression for the system of differential equation is given by the matrix in Figure (3-2) and the state vector (3-1).



where

f_D = specific force in "DOWN" direction

K_1, K_2 = damping loop gains

Figure 3-1 Barometric damping loop

$$F=\begin{bmatrix} 0 & -\ell \sin\phi & 0 & -\sin\phi & 0 & 0 & \cos\phi & 0 & 0 \\ \ell\sin\phi & 0 & \ell\cos\phi & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\ell\cos\phi & 0 & -\cos\phi & 0 & 0 & \sin\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \mu^2 & 0 & 0 & 0 & 0 & -\ell\sin 2\phi & 0 & \frac{-2\dot{\phi}}{R} \\ -\mu^2\sec\phi & 0 & 0 & 0 & 0 & 2\ell\tan\phi & 0 & 0 & \frac{-2\ell}{R} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2R\dot{\phi} & 2R\ell\cos^2\phi & 2\mu^2-K_2 & 0 \end{bmatrix} \quad (3\text{-}2)$$

Figure 3-2 Dynamics Matrix

4. INVERSE LAPLACE TRANSFORM SOLUTION

The inverse Laplace transform is one of the techniques used in solving for the transition matrix $\Phi(t)$ of the system of differential equations described in the last chapter. Here as in the following $t_o = 0$ has been assumed. The purpose of this chapter is to show how the matrix is derived and to list its final form.

By substituting the dynamics matrix F into equation (2-24), the system of differential equation may be rewritten in the form shown in Figure 4-1. The most difficult task in deriving the transition matrix using the inverse Laplace transform is the inversion of the matrix.

$$Q = (sI - F) \tag{4-1}$$

4.1 Matrix Inversion by Partitioning Method

The matrix partitioning method for inverting a matrix described by Faddeev and Faddeeva [1963] can be used for solving this problem.

To apply the method, the matrix Q in Figure 4-1 may be partitioned as follows:

$$\left[\begin{array}{c|c} \overbrace{\begin{array}{c:c} A_1 & B_1 \\ \hdashline C_1 & D_1 \end{array}}^{A_2} & B_2 \\ \hline C_2 & D_2 \end{array}\right] \tag{4-2}$$

and the inverse is given as

$$Q^{-1} = \left[\begin{array}{c|c} \overbrace{\begin{array}{c:c} O_1 & L_1 \\ \hdashline M_1 & N_1 \end{array}}^{O_2} & L_2 \\ \hline M_2 & N_2 \end{array}\right] \qquad (4\text{-}3)$$

where

$$N_i = (D_i - C_iA_i^{-1}B_i) \ , \qquad (4\text{-}4)$$

$$L_i = -A_i^{-1}B_iN_i \ , \qquad (4\text{-}5)$$

$$M_i = -N_iC_iA_i^{-1} \ , \qquad (4\text{-}6)$$

and

$$O_i = A_i^{-1} + A_i^{-1}B_iN_iC_iA^{-1} \ . \qquad (4\text{-}7)$$

In this case, the method has to be applied twice for inverting the whole matrix in Figure 4-1. First, the 7 x 7 portion on the top left hand corner is inverted and the partitioning is done as indicated by the dotted lines. The whole matrix is then partitioned and inverted according to the solid lines using the inverted 7 x 7 portion as the new A_2^{-1}. The inverted top left hand 7 x 7 portion of Figure 4-1 is shown in Appendix I and the N_2-matrix which was derived from standard co-factor techniques is

$$N_2 = \begin{bmatrix} \dfrac{s}{(s-a)(s-b)} & \dfrac{1}{(s-a)(s-b)} \\[2ex] \dfrac{(2\mu^2-K_2)}{(s-a)(s-b)} & \dfrac{s+K_1}{(s-a)(s-b)} \end{bmatrix} , \qquad (4\text{-}8)$$

where $a = -K_1 + \sqrt{[K_1^2 + 4(2\mu^2 - K_2)]}$

$b = -K_1 - \sqrt{[K_1^2 + 4(2\mu^2 - K_2)]}$

and s is the Laplace operator. More details on the derivation of N_2 are shown in Appendix I.

4.2 Inverse Laplace Transform Technique

After completing the inversion of the Q-matrix, the inverse Laplace transform of each individual element in Q^{-1} is taken to yield the final form of the transition matrix shown in section 4.3. The Q^{-1} matrix contains expressions which are divisions of high degree polynomials of s. Taking the inverse Laplace transform of these expression can be a very lengthy process. The convolution theorem

$$L^{-1}(f_1(s),\ f_2(s)) = \int_0^t F_1(t - x)\, F_2(x)\, dx \qquad (4\text{-}9)$$

is used to simplify the computations. The expressions are reduced to the product of two or more lower degree ploynomials whose inverse Laplace transform F_i can be found in tables [as e.g. in Spiegel, 1968; McCullum/Brown, 1965]. Integration of the right-hand side of equation (4-9) then results in the individual elements of the transition matrix $\Phi(t)$ given in section 4.3.

Two elements of the transition matrix are derived in detail to demonstrate the basic technique. In the first case it is shown how in certain cases approximations can be made which, without appreciable loss in accuracy, result in much simpler expressions. In the second case an example for the use of the convolution theorem is given. It should be noted that the arguments of Φ, e.g. Φ (6.1), now denote a matrix position rather than a time interval.

The element $\Phi(6, 1)$ of the transition matrix is given as

$$\Phi(6, 1) = L^{-1}\left(\frac{s\mu^2 \ell \sin \phi}{(s^2+\ell^2)(s^2+\mu^2)} + \frac{2s\mu^2 \ell \sin \phi}{(s^2 + \mu^2)^2}\right) ,$$

$$= L^{-1} \left(\frac{s\mu^2 \ell \sin \phi}{(s^2+\ell^2)(s^2+\mu^2)}\right) + L^{-1} \left(\frac{2s\mu^2 \ell \sin \phi}{(s^2 + \mu^2)^2}\right) \qquad (4\text{-}10)$$

Taking the inverse Laplace transform of (4-10), we get

$$\Phi(6, 1) = \frac{\mu^2 \ell \sin \phi(\cos \ell t - \cos \mu t)}{\mu^2 + \ell^2} + \frac{t\mu^2 \ell \sin \phi}{\mu} , \qquad (4\text{-}11)$$

and using the assumption that

$$\mu^2 >> \ell^2 ,$$

(4-11) may be reduced to

$$\Phi(6, 1) = \ell \sin \phi(\cos \ell t - \cos \mu t + \mu t \sin \mu t) \qquad (4\text{-}12)$$

Another more involved element $\Phi(6, 8)$ is given as

$$\Phi(6, 8) = L^{-1}\left\{-2(2\mu^2-K_2)\left(\frac{s\dot{\phi}}{(s^2+\mu^2)(s-a)(s-b)} - \frac{s^2 \ell^2 \sin^2\phi}{(s^2+\mu^2)(s-a)(s-b)}\right)\right\} . \qquad (4\text{-}13)$$

Using the convolution theorem, equation (4-13) may be reduced to

$$\Phi(6, 8) = -2(2\mu-K_2)\Big[\dot{\phi} \int_0^t \frac{(e^{a(t-\Gamma)}-e^{b(t-\Gamma)}) \cos \mu\Gamma}{(a - b)} d\Gamma$$

$$- \ell^2 \sin 2\phi \int_0^t \frac{\Gamma(ae^{a(t-\Gamma)}-be^{a(t-\Gamma)}) \sin \mu\Gamma}{2\mu (a - b)} d\Gamma\Big] . \qquad (4\text{-}14)$$

which when integrated yields the final form

$$\Phi(6,\ 8) = -2(2\mu^2 - K_2)\ \ [\dot{\Phi}\ (\frac{u \sin \mu t - a(\cos \mu t - e^{at})}{(a^2+\mu^2)\ (a-b)} - \frac{u \sin \mu t - b(\cos \mu t - e^{bt})}{(b^2+\mu^2)\ (a-b)}$$

$$+ \frac{\ell^2 \sin \phi}{2\mu(a-b)}\ \ \frac{a\ (t(a \sin \mu t - \mu \cos \mu t)}{(a^2 + \mu^2)} - \frac{b\ (t\ (b \sin \mu t - \mu \cos \mu t)}{(b^2 + \mu^2)}$$

$$+ \frac{((a^2 - \mu^2) \sin \mu t + 2a\mu\ (\cos \mu t - e^{at})}{(a^2 + \mu^2)^2} - \frac{((b^2 - \mu^2) \sin \mu t + 2b\mu\ (\cos \mu t - e^{bt})}{(b^2 + \mu^2)})]$$

$$(4\text{-}15)$$

In the first example, the final form of the element (6, 1) can be simplified considerably if the assumption $\mu^2 >> \ell^2$ holds. The accuracy lost, numerically, is less than 1% of the value computed from equations (4-21). The last example shows how the convolution theorem can be applied to find the inverse Laplace transform of an expression.

4.3 Final Expressions

Figure 4-1 shows the matrix Q as defined by equation (4-1). The elements of the transition matrix obtained by forming

$$\Phi(t) = L^{-1}\ \{Q^{-1}\ (s)\}$$

are listed below.

$$
Q = \left[\begin{array}{ccccc|cc|cc}
s & \ell \sin\phi & 0 & \ell \sin\phi & 0 & 0 & -\cos\phi & 0 & 0 \\
-\ell \sin\phi & s & -\ell \cos\phi & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \ell \cos\phi & s & \ell \cos\phi & 0 & 0 & \sin\phi & 0 & 0 \\
0 & 0 & 0 & s & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & 0 & -1 & 0 & 0 \\
\hline
0 & -\mu^2 & 0 & 0 & 0 & s & -\ell \sin 2\phi & 0 & \frac{2\dot{\phi}}{R} \\
\mu^2 \sec\phi & 0 & 0 & 0 & 0 & 2\ell \tan\phi & s & 0 & \frac{2\ell}{R} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & s + K_1 & -1 \\
0 & 0 & 0 & 0 & 0 & -2R\dot{\phi} & -2R\ell \cos^2\phi & -2\mu^2 + K_2 & s
\end{array}\right]
$$

Figure 4-1 Q matrix before inversion

$$\Phi(1, 1) = \cos \mu t$$

$$\Phi(1, 2) = -\ell t \sin \phi \cos \mu t - 4\phi\ell\mu^2(SS_3 + K_1 SS_2)$$

$$\Phi(1, 3) = -\frac{1}{2} \sin 2\phi \left(\frac{\ell^2}{\mu^2} (\cos \mu t - \cos \ell t) + \frac{\ell^2 t \sin \mu t}{u}\right)$$

$$\Phi(1, 4) = \frac{\ell}{\mu} \left(\frac{\ell}{\mu} \sin \ell t - \sin \mu t\right) \sin \phi$$

$$\Phi(1, 5) = 0$$

$$\Phi(1, 6) = \frac{\ell t \sin \phi \sin \mu t}{\mu} - 4\dot{\phi}\ell \cos \phi(SU_2 + K_1 SU_1)$$

$$\Phi(1, 7) = \frac{\cos \phi \sin ut}{\mu}$$

$$\Phi(1, 8) = \frac{0 - (2\mu^2 - K_2)(4\dot{\phi}\ell \sin\phi Su_1 + 2\ell \cos \phi EU_0)}{R}$$

$$\Phi(1, 9) = \frac{0 - (4\dot{\phi}\ell \sin \phi (SU_2 + K_1 SU_1) + 2\ell \cos\phi (EU_1 + K_1 EU_0))}{R}$$

$$\Phi(2, 1) = \ell t \sin \phi \cos \mu t - 4\dot{\phi}\ell\mu^2 (SU_1 + K_1 SU_o)$$

$$\Phi(2, 2) = \cos \mu t$$

$$\Phi(2, 3) = \frac{\ell}{\mu} \cos \phi(\sin \mu t - \frac{\ell}{\mu} \sin \ell t)$$

$$\Phi(2, 4) = \frac{\ell^2}{\mu^2} \cos 2\phi(\cos \mu t - \cos \ell t) - \frac{\ell^2 t \sin^2\phi \sin \mu t}{\mu}$$

$$\Phi(2, 5) = 0$$

$$\Phi(2, 6) = \frac{-\sin \mu t}{\mu}$$

$$\Phi(2,\ 7) = \frac{\ell t \text{ Sin } 2\phi \text{ Sin } \mu t}{2\mu} + 4\dot{\phi}\ell \cos^2\phi\ (SU_2 + K_1 SU_1)$$

$$\Phi(2,\ 8) = \frac{2(2\mu^2 - K_2)(\dot{\phi}\ EU_0 - \ell^2 \sin 2\phi\ SU_1)}{R}$$

$$\Phi(2,\ 9) = \frac{2(\dot{\phi}(EU_1 + K_1 EU_0) - \ell^2 \sin 2\phi\ (SU_2 + K_1 SU_1)}{R}$$

$$\Phi(3,\ 1) = \tan \phi(\cos \ell t - \cos \mu t)$$

$$\Phi(3,\ 2) = 0 - \sec \phi(\sin \ell t - \ell t \sin^2\phi \cos \mu t)$$

$$\Phi(3,\ 3) = \cos \phi t$$

$$\Phi(3,\ 4) = \sec \phi(\frac{\ell}{\mu} \sin^2\phi \sin \mu t - \sin \ell t)$$

$$\Phi(3,\ 5) = 0$$

$$\Phi(3,\ 6) = 4\dot{\phi}\ell \sin \phi(SU_2 + K_1 SU_1) - \frac{\ell t \sin^2\phi \sin\mu t}{\mu \cos \phi}$$

$$\Phi(3,\ 7) = \frac{-\sin \phi \sin \mu t}{\mu}$$

$$\Phi(3,\ 8) = \frac{(2\mu^2 - K_2) \tan \phi\ (4\dot{\phi}\ell\ \sin \phi\ SU_1 + 2\ell \cos\phi\ EU_0)}{R}$$

$$\Phi(3,\ 9) = \frac{(2\mu^2 - K_2)\tan \phi(4\dot{\phi}\ell \sin \phi(SU_2 + K_1 SU_1) + 2\ell \cos\phi(EU_1 + K_1 EU_0))}{R}$$

$$\Phi(4,\ 1) = \sin \phi(\sin \ell t - \ell t \cos \mu t)$$

$$\Phi(4,\ 2) = \cos \mu t - \cos \mu t$$

$$\Phi(4,\ 3) = \cos \phi(\sin \ell t - \frac{\ell}{\mu} \sin \mu t)$$

$$\Phi(4,\ 4) = 1 - \frac{\ell^2}{\mu^2} \cos 2\phi\ (\cos \mu t - \cos \ell t) + \frac{\ell^2 t \sin^2\phi \sin \mu t}{\mu}$$

$$\Phi(4, 5) = 0$$

$$\Phi(4, 6) = \frac{\sin \mu t}{\mu}$$

$$\Phi(4, 7) = 0 - \frac{\ell t \sin^2\phi \sin \mu t}{2\mu} - 4\dot{\phi}\ell \cos^2\phi \ (SU_2 + K_1 SU_1)$$

$$\Phi(4, 8) = 0 - \frac{2(2\mu^2 - K_2)(\dot{\phi} EU_o - \ell^2 \sin 2\phi \ SU_1)}{R}$$

$$\Phi(4, 9) = 0 - \frac{2[\dot{\phi}(EU_1 + K_1 EU_0) - \ell^2 \sin 2\phi \ (SU_2 + K_1 SU_1)]}{R}$$

$$\Phi(5, 1) = \sec \phi(\cos \mu t - \cos^2\phi - \sin^2\phi \cos \ell t)$$

$$\Phi(5, 2) = \tan \phi(\sin \ell t - \ell t \cos \mu t)$$

$$\Phi(5, 3) = \sin \phi(1 - \cos \ell t - \frac{\ell^2}{\mu^2} (\cos \mu t - 1) - \frac{\ell^2 t \sin \mu t}{\mu})$$

$$\Phi(5, 4) = \tan \phi \ (\sin \ell t - \frac{\ell}{\mu} \sin \mu t)$$

$$\Phi(5, 5) = 1$$

$$\Phi(5, 6) = \frac{\ell t \tan \phi \sin \mu t}{\mu} - 4\dot{\phi}\ell \ (SU_2 + K_1 SU_1)$$

$$\Phi(5, 7) = \frac{\sin \mu t}{\mu}$$

$$\Phi(5, 8) = \frac{(4\dot{\phi}\ell \tan \phi \ SU_1 + 2\ell\phi \ EU_0)(K_2 - 2\mu^2)}{R}$$

$$\Phi(5, 9) = \frac{-[4\dot{\phi}\ell \tan \phi(SU_2 + K_1 SU_1) + 2\ell \ (EU_1 + K_1 EU_0)]}{R}$$

$$\Phi(6, 1) = \ell \sin \phi(\cos \ell t - \cos \mu t + \mu t \sin \mu t)$$

$$\Phi(6, 2) = \mu \sin \mu t + \ell \sin^2 \phi(\sin \ell t - \frac{\ell}{\mu} \sin \mu t)$$

$$\Phi(6, 3) = \ell \cos \phi \, (\cos \ell t - \cos \mu t) - 2\mu^2 \ell^2 \sin^2 \phi \cos \phi \, UL_2$$

$$\Phi(6,4) = -(\ell(\sin \ell t - \frac{\ell}{\mu} \sin \mu t) + 2\mu^2 \ell \ \sin^2\phi \, UL_2))$$

$$\Phi(6, 5) = 0$$

$$\Phi(6, 6) = \cos (\sqrt{} (\mu^2 + 4\ell^2 \sin^2\phi \ ))$$

$$\Phi(6, 7) = - [\frac{\ell \sin 2\phi \, (\sin \mu t + \mu t \cos \mu t)}{2\mu} + 4\dot{\phi}\ell \cos^2\phi \, (SU_3 + K_1 SU_2)]$$

$$\Phi(6, 8) = \frac{2(2\mu^2 - K_2)(\ell^2 \sin 2\phi \, (SU_2 - \dot{\phi} EU_1)}{R}$$

$$\Phi(6, 9) = \frac{2[\ell^2 \sin 2\phi \, (SU_3 + K_1 SU_2) - \phi \, (EU_2 + K_1 EU_1)]}{R}$$

$$\Phi(7,1) = -\sec \phi [\mu \sin \mu t + \ell \sin^2\phi \, (\sin \ell t - \frac{\ell}{\mu} \sin \mu t)]$$

$$\Phi(7, 2) = \ell \tan \phi (\cos \ell t - \cos \mu t + \mu t \sin \mu t)$$

$$\Phi(7, 3) = \ell \sin \phi (\sin \ell t - \ell t \cos \mu t)$$

$$\Phi(7, 4) = \ell \tan \phi (\cos \ell t - \cos \mu t - \frac{\ell^2 t \ \sin \mu t}{2\mu})$$

$$\Phi(7, 5) = 0$$

$$\Phi(7, 6) = \frac{\ell \tan \phi (\sin \mu t + \mu t \cos \mu t)}{\mu} - 4\dot{\phi}\ell \, (SU_3 + K_1 SU_2)$$

$$\Phi(7, 7) = \cos [\sqrt{}(\mu^2 + 4\ell^2 \sin^2 \phi)] - 4\ell^2 \cos^2\phi (SU_3 + K_1 SU_2)$$

$$\Phi(7, 8) = \frac{2 \, (K_2 - 2\mu^2) \ \ \ell(EU_0 + 2\dot{\phi} \ \ \tan \phi \, SU_2)}{R}$$

$$\Phi(7, 9) = \frac{-2[\ell(EU + K_2 EU_D) + 2\dot{\phi}\ell \tan \phi (SU_3 + K_1 SU_2)]}{R}$$

$$\Phi(8, 1) = 2R\mu^2\ell[\dot{\phi} \sin \phi(EL_1+2SU_1) + (2\ell^2 \cos^2\phi \sin^2\phi SS_2 - \cos\phi EU_0)]$$

$$\Phi(8, 2) = R\mu^2 [2\dot{\phi}(EL_2 - \ell^2 \sin^2\phi SU_0) + \ell^2 \sin 2\phi (EL_1 + 2SU_1)]$$

$$\Phi(8, 3) = 2R\mu^2\ell \cos \phi[\dot{\phi} EL_1 + \ell^2 \sin\phi \cos\phi (EL_0 + 2 SS_2)]$$

$$\Phi(8, 4) = 2R\mu^2\ell^2 [\sin \phi \cos \phi EL_1 - \dot{\phi}(EL_0 + 2 \sin^2\phi SS_2)]$$

$$\Phi(8, 5) = 0$$

$$\Phi(8, 6) = 2R (\dot{\phi} EU_1 + \ell^2 \sin 2\phi SU_2)$$

$$\Phi(8, 7) = 2R\ell (\cos^2\phi EU_1 - \dot{\phi} \sin 2\phi SU_2)$$

$$\Phi(8, 8) = \frac{ae^{at} - be^{bt}}{a - b}$$

$$\Phi(8, 9) = \frac{e^{at} - e^{bt}}{a - b}$$

$$\Phi(9, 1) = 2\mu^2\ell R\{\dot{\phi} \sin [EL_2 + K_1EL_1 + 2(SU_2 + K_1SU_1)]$$

$$+ \cos \phi[\ell^2 \tan \phi \sin \phi(SS_3 + K_1SS_2) - EU_1 - K_1EU_0]\}$$

$$\Phi(9, 2) = R\mu^2 \{2\dot{\phi}[EL_3 + K_1EL_2 - \ell^2 \sin^2 \phi(SU_1 + K_1SU_0)]$$

$$+\ell^2 \sin 2\phi[EL_2 + K_1EL_1 + 2(SU_2 + K_1SU_1)]\}$$

$$\Phi(9, 3) = 2R\mu^2\ell \cos\phi\{\dot{\phi}(EL_2 + K_1EL_1) + \ell^2 \sin \phi \cos \phi [EL_1 + K_1EL_0$$

$$+ 2(SS_3 + K_1SS_2) - 2\dot{\phi} \tan \phi(SS_2 + K_1SS_1)]\}$$

$$\Phi(9,\ 4) = 2R\mu^2\ell^2\ \{\sin\phi\cos\phi(EL_2 + K_1EL_1) - \dot{\phi}[EL_1 + K_1EL_0$$

$$+ 2\sin^2\phi(SS_3 + K_1SS_2)] - \ell^2\sin 2\phi(SS_2 + K_1SS_1)\}$$

$$\Phi(9,\ 5) = 0$$

$$\Phi(9,\ 6) = 2R\ [\dot{\phi}(EU_2+K_1EU_1) + \ell^2\sin 2\phi(SU_3 + K_1SU_2)]$$

$$\Phi(9,\ 7) = 2\ell[\cos^2\phi(EU_2 + K_1EU_1) - \dot{\phi}\sin 2\phi\ (SU_3 + K_1SU_2)]$$

$$\Phi(9,\ 8) = \frac{(2\mu - K_2)(e^{at}-e^{bt})}{a - b}$$

$$\Phi(9,\ 9) = \frac{[e^{at}(a+K_1) - e^{bt}(b+K_1)]}{a - b} \qquad (4\text{-}21)$$

where

$$UL_2 = \frac{1}{4}\left(\frac{\sin\ \mu t - \sin\ \mu t}{\mu(\mu + \ell)^2} - \frac{2\mu t\ \cos\mu t}{\mu^3} + \frac{\sin\ \mu t - \sin\ \ell t}{\mu(\mu - \ell)^2}\right), \qquad (4\text{-}22)$$

$$EU_0 = \frac{1}{\mu(a-b)}\left[\frac{\mu(\cos\ \mu t-e^{bt}) + b\sin\ \mu t}{(b^2 + \mu^2)} - \frac{\mu(\cos\ \mu t-e^{at}) + a\sin\ \mu t}{(a^2 + \mu^2)}\right], \qquad (4\text{-}23)$$

$$EU_1 = \frac{1}{a-b}\left[\frac{\mu\sin\ \mu t - a(\cos\mu t-e^{at})}{a^2 + \mu^2} - \frac{\mu\sin\mu t - b(\cos\mu t-e^{bt})}{b^2 + \mu^2}\right], \qquad (4\text{-}24)$$

$$EU_2 = \frac{1}{a-b}\left[\frac{a\mu\sin\mu t - a^2(\cos\mu t-e^{at})}{a^2 + \mu^2} - \frac{b\mu\sin\mu t - b^2(\cos\mu t-e^{bt})}{b^2 + \mu^2}\right], \qquad (4\text{-}25)$$

$$EL_0 = \frac{1}{\mu^2(a-b)}\left(\frac{e^{bt}-\cos\mu t - \frac{b}{\mu}\sin\mu t}{b^2 + \mu^2} - \frac{e^{bt}-\cos\ell t - \frac{b}{\mu}\sin\ \ell t}{b^2 + \ell^2}\right.$$

$$\left. - \frac{e^{at}-\cos\mu t - \frac{a}{\mu}\sin\mu t}{a^2 + \mu^2} + \frac{e^{at}-\cos\ell t - \frac{a}{\mu}\sin\ell t}{a^2 + \ell^2}\right), \qquad (4\text{-}26)$$

$$EL_1 = \frac{1}{\mu^2(a-b)} \left[\frac{\mu\sin t - b(\cos\mu t-e^{bt})}{b^2 + \mu^2} - \frac{\ell \sin\ell t - b(\cos\ell t-e^{bt})}{b^2 + \ell^2}\right.$$

$$\left.- \frac{\mu \sin \mu t - a(\cos\mu t-e^{at})}{a^2 + \mu^2} + \frac{\ell\sin \ell t - a(\cos\ell t-e^{bt})}{a^2 + \ell^2}\right], \qquad (4\text{-}27)$$

$$EL_2 = \frac{1}{\mu^2(a-b)} \left[\frac{\mu b \sin\mu t + \mu^2(\cos\mu t-e^{bt})}{b^2 + \mu^2} - \frac{\ell b \sin\ell t + \ell^2(\cos\ell t-e^{bt})}{b^2 + \ell^2}\right.$$

$$\left.- \frac{\mu a \sin \mu t + \mu^2(\cos\mu t - e^{at})}{a^2 + \mu^2} + \frac{\ell a \sin\ell t + \ell^2(\cos \ell t - e^{at})}{a^2 + \ell^2}\right], \qquad (4\text{-}28)$$

$$EL_3 = \frac{1}{\mu^2(a-b)} \{b[\frac{\mu b \sin\mu t + {}^2(\cos\mu t-e^{bt})}{b^2 + \mu^2} - \frac{\ell b \sin\ell t + \ell^2(\cos\ell t-e^{bt})}{b^2 + \ell^2}]$$

$$-a \left[\frac{\mu a \sin\mu t + \mu^2(\cos\mu t-\ell^{at})}{a^2 + \mu^2} - \frac{\ell a \sin\ell t + \ell^2(\cos\ell t-e^{at})}{}\right]\} \qquad (4\text{-}29)$$

$$SU_0 = \frac{-1}{2\mu^2(a-b)}\{[\frac{t(b\cos\mu t + \mu \sin\mu t)}{b^2 + \mu^2} + \frac{(b^2-\mu^2)(\cos\mu t - e^{bt}) - 2b \mu\sin\mu t}{(b^2 + \mu^2)^2}$$

$$- \frac{t(a \cos\mu t - \mu \sin \mu t)}{a^2 + \mu^2} - \frac{(a^2-\mu^2)(\cos\mu t-e^{at})-2a\mu \sin\mu t}{(a^2 + \mu^2)^2}] \qquad (4\text{-}30)$$

$$- \frac{-b \sin\mu t + \mu(\cos \mu t - e^{bt})}{\mu(b^2 + \mu^2)} + \frac{a \sin \mu t + \mu(\cos \mu t - e^{at})}{\mu(a^2 + \mu^2)}\}$$

$$SU_1 = \frac{1}{2\mu(a-b)} [\frac{t(b\sin\mu t + \mu\cos\mu t)}{b^2 + \mu^2} + \frac{(b^2-\mu^2) \sin\mu t + 2b\mu(\cos\mu t - e^{bt})}{(b^2 + \mu^2)^2}$$

$$- \frac{t(a \sin\mu t + \mu \cos \mu t}{a^2 + \mu^2} - \frac{(a^2-\mu^2) \sin\mu t + 2a\mu(\cos\mu t-e^{at})}{(a^2 + \mu^2)^2} \qquad (4\text{-}31)$$

$$SU_2 = \frac{1}{2\mu(a-b)} \{b[\frac{t(b\ \sin\mu t-\mu\cos\mu t}{b^2+\mu^2} + \frac{(b^2-\mu^2)\ \sin\mu t\ +\ 2b\mu(\cos\mu t-e^{bt})}{(b^2\ +\ \mu^2)^2}]$$

$$-a[\frac{t(a\ \sin\mu t\ -\ \mu\cos\ \mu t)}{a^2\ +\ \mu^2} + \frac{(a^2-\mu^2)\ \sin\mu t\ +\ 2a\mu(\cos\mu t-e^{at})}{(a^2\ +\ \mu^2)^2}\ ]\ \}, \tag{4-32}$$

$$SU_3 = \frac{1}{2\ (a-b)}\{[\frac{\mu t\ (\mu\ \sin\mu t\ +\ a\ \cos\ \mu t)\ -\ a\ \sin\mu t\ -\ \mu\ \cos\ \mu t}{a^2\ +\ \mu^2}$$

$$-\ \frac{\mu(a^2-\mu^2)\ \cos\ \mu t\ +\ 2a\mu^2\ \sin\ \mu t\ -\ 2a^2\mu e^{-at}}{(a^2\ +\ \mu^2)^2}]a$$

$$+[\frac{\mu t(b\ \cos\ \mu t\ -\ \mu\sin\ \mu t)\ +\ b\ \sin\ \mu t\ +\ \mu\cos\ \mu t}{(b^2\ +\ \mu^2)}$$

$$+\ \frac{\mu(b^2-\mu^2)\ \cos\ \mu t\ +\ 2b\mu^2\ \sin\mu t\ -\ 2b^2\ \mu e^{-bt}}{9b^2\ +\ \mu^2)^2}\ ]\ b\}, \tag{4-33}$$

$$SS_1 = \frac{1}{2\mu(a-b)}\ [\frac{t(a\ \sin\mu t\ +\ \ \cos\mu t}{(a^2+\ell^2)(a^2+\mu^2)} + \frac{(a^2-\mu^2)\ \sin\mu t\ +\ 2\mu a\ (\cos\mu t-e^{-at})}{(a^2+\ell^2)(a^2+\mu^2)^2}$$

$$\frac{-t(b\ \sin\mu t\ +\ \mu\cos\ \mu t)}{(b^2+\ell^2)(b^2+\mu^2)} - \frac{(b^2-\mu^2)\ \sin\ \mu t\ +\ 2b\mu(\cos\mu t-e^{-bt})}{(b^2+\ell^2)(b^2+\mu^2)^2}\ ]$$

$$+\ \frac{1}{4\mu\ell\ \sqrt{[(2\mu^2-K_2+\ell^2)^2\ +\ \ell^2 K_1{}^2]}}\ [\frac{t\ \sin\ (\mu t-\theta)}{u\ +\ \ell} - \frac{t\ \sin\ (\mu t\ +\ \theta)}{u\ -\ \ell}$$

$$+\ \frac{\cos\ (\mu t\ -\ \theta)\ -\ \cos(\ell t\ +\theta)}{(u\ +\ \ell)^2} - \frac{\cos(\mu t\ +\ \theta)\ -\ \cos\ (\ell t\ +\ \theta)}{(\mu\ -\ \ell)^2} \tag{4-34}$$

$$SS_2 = \frac{1}{2\mu(a-b)} \{a[\frac{t(a \sin\mu t + \mu \cos \mu t)}{(a^2+\ell^2)(a^2+\mu^2)} + \frac{(a^2-\mu^2)\sin\mu t + 2a\mu(\cos\mu t - e^{-at})}{(a^2+\ell^2)(a^2+\mu^2)^2}$$

$$-b\ [\frac{t(b \sin\mu t + \mu\cos\mu t)}{(b^2+\ell^2)(a^2+\mu^2)} + \frac{(b^2-\mu^2)\ \sin\mu t + 2b\mu(\cos\mu t-e^{-bt})}{(b^2+\ell^2)(b^2+\mu^2)}\ ]\}$$

$$+ \frac{1}{4\mu\ \sqrt{[(2\mu^2-K_2+\ell^2)^2 + \ell^2 K_1{}^2]}} [\frac{t\ \sin(\mu t - \beta)}{\mu + \ell} - \frac{t\ \sin\ (\mu t + \beta)}{\mu - \ell}$$

$$+ \frac{\cos\ (\mu t - \beta) - \cos\ (\ell t + \beta)}{(\mu - \ell)^2} - \frac{\cos\ (\mu t + \beta) - \cos\ (\ell t + \beta)}{(\mu - \ell)^2}], \quad (4\text{-}35)$$

$$SS_3 = \frac{1}{2\mu(a-b)} \{\ a^2\ [\frac{t(a \sin\mu t + \mu \cos \mu t)}{(a^2+\ell^2)(a^2+\mu^2)} + \frac{(a^2-\mu^2)\ \sin\mu t + 2a\mu\ (\cos\mu t-e^{-at})}{(a^2+\ \ell^2)(a^2+\mu^2)^2}$$

$$-b^2\ [\frac{t(b \sin \mu t + \mu\cos\mu t)}{(b^2+\ell^2)(b^2+\mu^2)} + \frac{(b^2-\mu^2)\ \sin \mu t\ +2b\mu(\cos\mu t-e^{-bt})}{(b^2+\ell^2)(b^2+\mu^2)^2}]\}$$

$$- \frac{\ell}{2\mu\ \sqrt{[(\ell^2+a^2)(\ell^2+b^2]}}\ [\frac{t\ \sin\ (\mu t - \theta')}{u + \ell} - \frac{t\ \sin\ (\mu t + \theta)}{u - \ell}$$

$$+ \frac{\cos\ (\mu t - \beta) - \cos\ (\ell t + \theta)}{(\mu + \ell)^2} - \frac{\cos\ (\mu t + \beta) - \cos\ (\ell t + \theta)}{(\mu - \ell)^2}, \quad (4\text{-}36)$$

$$\theta = - \tan^{-1}(\frac{\ell}{-b}) - \tan^{-1}(\frac{\ell}{-a}) \quad , \qquad (4\text{-}37)$$

and

$$\beta = \tan^{-1}\ (\frac{K_2-2\mu^2-\ell^2}{\ell K_1}). \qquad (4\text{-}38)$$

4.4 Discussion of the Solution

Difficulties in finding the inverse Laplace transform of an expression are usually related to the degree and complexity of the polynomial in its denominator. In our case the denominator is basically given by the analytical expression for the determinant of the Q-matrix. Therefore the order of the dynamics matrix F and the complexity of its elements will to a large extent determine if the use of this technique is advisable.

When using the expressions in section 4.3 it should be realized that:

a) some of the elements in $\Phi(t)$ become undefined if

$$K_2 > (2\mu^2 + \frac{K_1^2}{4}) , \tag{4-16}$$

b) the elements in column 9 of the transition matrix, except $\Phi(8, 9)$, approach zero as

$$K_2 \to 2\mu^2 \tag{4-17}$$

c) if an element in column 8 is f_c^8, then

$$f_c^9 = \frac{f_c'^8 + K_1 f_c^8}{2\mu^2 - K_2} \tag{4-18}$$

except $\Phi(8, 8)$,

d) if an element in row 8 is f_r^8, then

$$f_r^9 = f_r'^8 + K_1 f_r^8 \tag{4-19}$$

except $\Phi(8, 8)$,

e) the elements in row 4 and row 5, except the diagonal elements, are the integrals of row 6 and 7 evaluated between intervals of zero and t.

In an undamped system the values of K_1 and K_2 are zero. The inverse Laplace transform of the matrix N_2 (4-8) becomes

$$L^{-1}N_2 = \begin{bmatrix} \cosh\sqrt{8\mu} & \dfrac{\sinh\sqrt{8\mu}}{c} \\ \dfrac{2\mu^2 \sinh\sqrt{8\mu}}{c} & \cosh\sqrt{8\mu} \end{bmatrix} \qquad (4\text{-}20)$$

The values of the elements in $\Phi(t)$ which are associated with the height channel increase exponentially as t increases. The increases in these elements are propagated to each element in the transition matrix, as indicated by equation (4-5), (4-6) and (4-7) and contaminate the sinusoidal behaviour of the elements associated with the horizontal channels when the value of t becomes very large.

5. SERIES SOLUTION

The series expansion approach is another way to find the solution of the transition matrix of equation (2-2). The dynamics matrix F can be substituted into equation (2-8) to form an infinite series representation for the transition matrix $\Phi(t)$ which converges for all values of t. Each element in the matrix is expressed in form of a Taylor series expanding around the point where time t is zero. By examining and regrouping the corresponding variables in each series, the elements in the transition matrix may be expressed as sums of several less complicated Taylor series of common functions. The analytical expressions for the elements in the transition matrix may then be derived from these series. However, this approach for the solution of the transition matrix can be a very lengthy and difficult process. The matrix $\Phi(t)$ has to be expanded analytically to the sixth, or perhaps the seventh term of the series to include all the essential components required to form the simpler series of common functions. Regrouping the series is another difficult task. Some of the series may not be easily rearranged to become sums of common functions. The final analytical expression for the matrix $\Phi(t)$ derived in the series expansion approach are listed in section 5.2.

5.1 Basic Technique

Two of the series approximations of elements of $\Phi(t\ )$ are derived here to show how the series were rearranged to get the final expressions.

The series expansion of element $\Phi(1, 1)$ is

$$\Phi(1, 1) = 1 - \frac{t^2}{2} (\ell^2 \sin^2\phi + \mu^2) + \frac{t^4}{24} (\ell^4 \sin^2\phi + \mu^4 + \mu^2\ell^2) \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\mu t)^n}{n!} - \sin^2\phi \sum_{n=0}^{\infty} \frac{(-1)^n(\ell t)^{n+2}}{(n+2)!} + \frac{\ell^2}{\mu^2} \sum_{n=0}^{\infty} \frac{(-1)^n(\mu t)^{n+4}}{(n+4)!} \qquad (5\text{-}1)$$

Using the assumption that

$$\mu^2 >> \ell^2 \quad , \qquad (5\text{-}2)$$

(5.1) may be reduced to

$$\Phi(1, 1) = \cos \mu t + \sin^2\phi\ (1 - \cos \ell t)\ . \qquad (5\text{-}3)$$

Another element $\Phi(2, 9)$ can be represented by the series

$$\Phi(2, 9) = \frac{t\dot{\phi}}{R} - \frac{t^3\ \ell^2 \sin 2\phi}{3R} + \frac{t^4\ (K_2-\mu^2)\dot{\phi}}{12R}$$

$$+ \frac{t^5\ (K_2-4\mu^2)\ \ell^2 \sin 2\phi}{R} + \frac{t^6\ (K_2-\mu^2)^2\dot{\phi}}{360R}$$

$$- \frac{t^7[(K_2-\mu^2)K_1 - 5\mu^2)\ \ell^2 \sin 2\phi}{5040R} + \ .\ .\ .\ . \qquad (5\text{-}4)$$

It can be rearranged and yields

$$(2,\ 9) \simeq \frac{2\dot{\phi}}{R} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}[\surd(K_2-\mu^2)t]^{n+2}}{(n+2)\,!} + \ell^2 \sin 2\phi \sum_{n=0}^{\infty}$$

$$[\frac{(-1)^{n+1}\ t\ \ (\mu t)^{n+2}}{(n+2)!} + \frac{(-1)^{n}\ (\mu t)^{n+3}}{^3(n+3)!}]$$

$$+ \ell^2 \sin 2\phi \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ t^5\ (K_2-\mu^2)(K_1\ t)^n}{(n+5)!}$$

$$= \ell^2 \sin 2\phi\ [\frac{t\ (\cos \mu t-1)}{\mu^2} - \frac{(\sin \mu t - \mu t)}{\mu^3} + (K_2-\mu^2)\ .\ EK5]$$

$$- 2\dot{\phi}\ (\frac{\cos\ [\surd(K_2-\mu^2)t]\ -1}{R(K_2 - \mu^2)}) \qquad (5\text{-}5)$$

where $$EK5 = \frac{e^{-kt}}{K_1^{\ 5}} - \sum_{n=0}^{4} \frac{(-1)^n\ (K_1 t)^n}{K_1^{\ 5}\ n!}\ )\quad , \qquad (5\text{-}6)$$

for

$$K_2 - \mu^2 > 0 \quad , \qquad (5\text{-}7)$$

and

$$\Phi(2,\ 9) \simeq \ell^2 \sin 2\phi\ [\frac{t\ (\cos \mu t-1)}{\mu^2} - \frac{(\sin \mu t - \mu t)}{\mu^3} + (K_2 - \mu^2)\ EK5]$$

$$-2\dot{\phi}\ (\frac{\cosh\ [\surd\ |(K_2-\mu^2)|\ t]\ -1}{R(K_2 - \mu^2)}) \qquad (5\text{-}8)$$

if $\quad K_2 - \mu^2 < 0.$

5.2 Final Expressions

The elements in the transition matrix derived from their series representations are:

$\Phi(1,\ 1) = \cos \mu t + \sin^2\phi\ (1 - \cos \ell t)\quad ,$

$$\Phi(1,\ 2) = -\ell t \sin\phi \cos\mu t - 4\dot{\phi}\ell\mu^2 \cos\phi\ CU_4 \quad ,$$

$$\Phi(1,\ 3) = \frac{-\sin 2\phi}{2} \left[\frac{\ell^2}{\mu^2} (\cos\mu t - \cos\ell t) + \frac{t\ \ell^2 \sin\mu t}{\mu}\right] \quad ,$$

$$\Phi(1,\ 4) = -\ \Phi(1,\ 2) \quad ,$$

$$\Phi(1,\ 5) = 0$$

$$\Phi(1,\ 6) = \frac{\ell t \sin\phi \sin\mu t}{\mu} + 4\dot{\phi}\ell \cos\phi\ SUE \quad ,$$

$$\Phi(1,\ 7) = \frac{\cos\phi \sin\mu t}{\mu}$$

$$\Phi(1,\ 8) = \frac{2}{R}\ \zeta\ [\ell \cos\phi\ (K_1\ EK_3 + SUE) - 2\ell\dot{\phi} \sin\phi (CU_4 + K_1\ EK_4)] \quad ,$$

$$\Phi(1,\ 9) = \frac{2\ell}{R} [\cos\phi\ (CUE - K_1\ \zeta\ EK_4) + 2\ell\dot{\phi}\ \sin\phi\ \ SUE] \quad ,$$

$$\Phi(2,\ 1) = t \sin\phi \cos\mu t - 4\dot{\phi}\ell\mu^2 \cos\phi\ CU_4 \quad ,$$

$$\Phi(2,\ 3) = \ell \cos\phi \left[\frac{\sin\mu t}{\mu} + \frac{\ell^2}{\mu^2} (\sin\mu t - \mu t)\right] \quad ,$$

$$\Phi(2,\ 4) = \frac{\ell^2}{\mu^2} \cos 2\phi\ (\cos\mu t - \cos\ell t) - \frac{t\ \ell^2 \sin^2\phi \sin\mu t}{\mu} \quad ,$$

$$\Phi(2,\ 5) = 0 \quad ,$$

$$\Phi(2,\ 6) = \frac{-\sin\mu t}{\mu} \quad ,$$

$$\Phi(2,\ 7) = \frac{\ell t \sin 2\phi\ \ \sin\mu t}{2\mu} - 4\dot{\phi}\ell \cos^2\phi\ \ CUE \quad ,$$

$$\Phi(2,\ 8) = \frac{2}{R}\ (2\mu^2 - K_2)\ [\dot{\phi}(SUE + K_1\ EK_3) + \ell^2 \sin 2\phi\ (K_1 EK_4 - CS_2 + \zeta\ EK_5)] \quad ,$$

$$\Phi(2,\ 9) = \frac{1}{R}\ [\ell^2 \sin 2\phi\ (CS_1 + 2\zeta\ EK_4) - 2\dot{\phi}\ CUE] \quad ,$$

$$\Phi(3,\ 1) = \tan\phi(1 - \cos\mu t) - \sin\phi\cos\phi\ (\cos \ell t - 1)\ ,$$

$$\Phi(3,\ 2) = \sec\phi(\ell t \sin^2\phi \cos\mu t - \sin\ell t)\ ,$$

$$\Phi(3,\ 3) = \cos^2\phi\ \ \sin^2\phi\cos\ell t\ ,$$

$$\Phi(3,\ 4) = \sec\phi\left(\frac{\ell\sin^2\phi\sin\mu t}{\mu} - \sin\ell t\right)\ ,$$

$$\Phi(3,\ 5) = 0\ ,$$

$$\Phi(3,\ 6) = -\tan\phi\ \ \Phi(1,\ 6)\ ,$$

$$\Phi(3,\ 7) = -\tan\phi\ \Phi(1,\ 7)\ ,$$

$$\Phi(3,\ 8) = -\tan\phi\ \ \Phi(1,\ 8)\ ,$$

$$\Phi(3,\ 9) = -\tan\phi\ \ \Phi\ (1,\ 9)\ ,$$

$$\Phi(4,\ 1) = \sin\phi\ (\sin\ell t - \ell t \cos\mu t)\ ,$$

$$\Phi(4,\ 2) = 1 - \cos\mu t\ ,$$

$$\Phi(4,\ 3) = \frac{\ell\cos\phi(ut - \sin\mu t)}{\mu}\ ,$$

$$\Phi(4,\ 4) = 1\ ,$$

$$\Phi(4,\ 5) = 0\ ,$$

$$\Phi(4,\ 6) = \Phi(2,\ 6)\ ,$$

$$\Phi(4,\ 7) = -\Phi(2,\ 7)$$

$$\Phi(4,\ 8) = -\Phi(2,\ 8)$$

$$\Phi(4,\ 9) = -\ \Phi(2,\ 9) \quad ,$$

$$\Phi(5,\ 1) = \sin\phi(\cos\mu t - 1) \quad ,$$

$$\Phi(5,\ 2) = \sec\phi\ \Phi(4,\ 1) \quad ,$$

$$\Phi(5,\ 3) = \sin\phi\ [1 - \cos \ell t + \frac{\ell^2}{\mu^2}(1 - \cos\mu t) - \frac{t\ \ell^2 \sin\mu t}{\mu}\ ] \ ,$$

$$\Phi(5,\ 4) = \frac{\ell \tan\phi(\mu t - \sin\mu t)}{\mu} \quad ,$$

$$\Phi(5,\ 5) = 1$$

$$\Phi(5,\ 6) = \sec\phi\ \Phi(1,\ 6) \ ,$$

$$\Phi(5,\ 7) = \sec\phi\ \Phi(1,\ 7) \quad ,$$

$$\Phi(5,\ 8) = \sec\phi\ \Phi(1,\ 8) \ ,$$

$$\Phi(5,\ 9) = \sec\phi\ \Phi(1,\ 8) \ ,$$

$$\Phi(6,\ 1) = \ell \sin\phi(\cos\ell t - \cos\mu t + \mu t \sin\mu t) \ ,$$

$$\Phi(6,\ 2) = \mu \sin\mu t + \frac{\ell^2}{\mu}(\sin\mu t - \mu t) \quad ,$$

$$\Phi(6,\ 3) = \ell \cos\phi\ (1 - \cos\mu t) \quad ,$$

$$\Phi(6,\ 4) = \frac{\ell^2}{\mu}(\sin\mu t - \mu t) + \ell^2 \sin^2\phi\ (t \cos\mu t - \frac{\sin\mu t}{\mu}\ ) \ ,$$

$$\Phi(6,\ 5) = 0 \quad ,$$

$$\Phi(6,\ 6) = \cos\ [\surd\ (\mu^2 + 4\ell^2 \sin^2\phi)\ t\ ] \ ,$$

$$\Phi(6,\ 7) = -4\dot{\phi}\ell \cos^2\phi[\frac{t \sin\mu t}{2\mu} + \zeta\ EK_3) - \frac{\ell \sin 2\phi\ (\sin\mu t + \mu t \cos\mu t)}{2\mu}] \ ,$$

$$\Phi(6,\ 8) = -\frac{2}{R}\ \zeta[\ell^2 \sin 2\phi\ (\frac{CS_1}{2} - 2K_1\mu^2\ EK_5 + \zeta EK_4 + K_1EK_3 - K_1EK_5)$$

$$+ \dot\phi(K_1EK_2 - CUE - K_1(K_2-\mu^2)\ EK_4)]$$

$$\Phi(6,\ 9) = \frac{2}{R}\ [\ell^2 \sin 2\phi(\frac{t \sin \mu t}{2\mu} + \zeta EK_3) - \dot\phi(SN_1 - K_1EK_3)] \quad ,$$

$$\Phi(7,\ 1) = -\sec\phi\ (\mu \sin \mu t + \ell^2 \sin^2\phi\ \frac{\sin \mu t - \mu t}{\mu}) \quad ,$$

$$\Phi(7,\ 2) = \ell \tan\phi(\cos \ell t - \cos \mu t + \mu t \sin \mu t) \quad ,$$

$$\Phi(7,\ 3) = \ell \sin\phi(\sin \mu t - \ell t \cos \mu t) \quad ,$$

$$\Phi(7,\ 4) = \ell \tan\phi(\cos \ell t - \cos \mu t + \frac{\ell^2 \sin \mu t}{2\mu}) \quad ,$$

$$\Phi(7,\ 5) = 0 \ ,$$

$$\Phi(7,\ 6) = \frac{\ell}{\mu}\ \tan\phi(\sin \mu t + \mu t \cos \mu t) - 4\dot\phi\ell\ (0.5\ SH_2 - K_1\zeta\ EK_4) \quad ,$$

$$\Phi(7,\ 7) = \cos\ [\surd\ (\mu^2 + 4\ell^2 \sin^2\phi\ )\ t] + 4\ \ell^2 \cos^2\phi\ CUE \ ,$$

$$\Phi(7,\ 8) = \frac{2}{R}\ \ell\zeta\ [CUE - K_1EK_2 + K_1(K_2-\mu^2)\ EK_4] + \frac{2}{R}\ \dot\phi\ell\tan\phi(CS_1+2\zeta EK_4+2K_1EK_3) \quad ,$$

$$\Phi(7,\ 9) = -\frac{2}{R}\ \ell\ (SN_1 - K_1\zeta EK_3 + \dot\phi \tan\phi\ SH_2) \ ,$$

$$\Phi(8,\ 1) = 2R\ell\mu^2\ [\cos\phi(SUE + K_1EK_3) + 2\dot\phi \sin\phi(CS_2+1.5K_1EK_4+1.5\zeta EK_5)] \ ,$$

$$\Phi(8,\ 2) = R\ell^2 \sin^2\phi\ [3\mu^2(K_1EK_4 + \zeta EK_5) + 2CS_2] - 2R\dot\phi\mu^2(SUE + K_1EK_3) \ ,$$

$$\Phi(8,\ 3) = R\ell[2\dot\phi\mu^2 \cos\phi(K_1EK_4 + CUE) + \ell^2\sin^2\phi \cos\phi(CS_3-3K_1\mu^2EK_5$$

$$+ \frac{\mu^2\zeta t^7}{1863})] \quad ,$$

$$\Phi(8,\ 4) = R\ell^2\mu^2\ [\sin 2\phi\ (CUE + K_1EK_4) + 2\dot\phi(1 + 2\sin^2\phi)\ EK_4] \ ,$$

$\Phi(8, 5) = 0$

$$\Phi(8, 6) = 2R\,\dot{\phi}\,[K_1 EK_2 - CUE - K_1(K_2-\mu^2)\, EK_4)] - 2R\ell^2 \sin^2 \phi(CS_1 + EK_4 + K_1 EK_3 - 2\mu^2 K_1 EK_5)$$

$$\Phi(8, 7) = R\ell\, \dot{\phi}\sin 2\phi(CS_1 + 2\zeta EK_4 + 2K_1 EK_3) + 2R\ell \cos^2\phi\ [CUE - K_1 EK_2 - K_1 EK_4\ (K_2-\mu^2)]\ .$$

$$\phi(8, 8) = e^{-K_1 t} + CHE - 1 - K_1\zeta[t\ (EK_1 + \zeta EK_3) + (EK_2 + 2\zeta EK_4)]$$

$$\Phi(8, 9) = \frac{SHE}{\sqrt{|2\mu^2-K_2|}} - K_1\ [EK_1 - \zeta(tEK_2 + 2EK_3)]\ ,$$

$$\Phi(9, 1) = 2R\ell\mu^2\ \{\cos\phi CUE + \dot{\phi}\sin\phi(\frac{t(1-\cos\mu t)}{\mu^2} - 3\zeta\ EK_4]\}$$

$$\Phi(9, 2) = 2R\mu^2\ [\dot{\phi}\ (\frac{1-\cos\mu t}{\mu^2} + \zeta EK_3) + \ell^2 \sin 2\phi(\frac{t - t\cos\mu t}{2\mu^2} - 1.5\ \zeta EK_4)]\ ,$$

$$\Phi(9, 3) = 2R\ell\mu^2[\dot{\phi}\cos\phi(\frac{\mu t - \sin\mu t}{3} - \zeta EK_4) + 3\ell^2 \sin\phi\cos^2\phi\ (\frac{\mu t^2 - t\sin\mu t}{4\mu^3} + \zeta EK_5)]\ ,$$

$$\Phi(9, 4) = 2R\dot{\phi}\mu^2\ell^2\ (2\sin^2\phi\ CS_2 - CUE) - \mu^2\ell^2 R\sin 2\phi\ SUE,$$

$\Phi(9, 5) = 0$,

$$\Phi(9, 6) = 2R\ [\dot{\phi}(SN_1 - K_1\zeta EK_3) + \ell^2\sin^2\phi\ (\frac{t\sin\mu t}{2\mu} + \zeta EK_3)]\ ,$$

$$\Phi(9, 7) = 2R\ell\ [\cos^2\phi\ (SN_1 - K_1\ \zeta EK_3) - \dot{\phi}\sin 2\phi(\frac{t\sin\mu t}{2\mu} + \zeta EK_3)]$$

$$\Phi(9,\ 8) = \zeta\ \Phi(8,\ 9) \quad ,$$

$$\Phi(9,\ 9) = CHE + K_1\ \zeta EK_2 - K_1\zeta^2\ (tEK_3 + 3EK_4) \tag{5-10}$$

where

$$\zeta = 2\mu^2 - K_2 \quad , \tag{5-11}$$

$$EK_1 = \frac{e^{-K_1 t}}{K_1^{2}} - \sum_{n=0}^{1} \frac{(-K_1 t)^n}{K_1^{2}\ n!} \quad , \tag{5-12}$$

$$EK_2 = \frac{e^{-K_1 t}}{K_1^{3}} - \sum_{n=0}^{2} \frac{(-K_1 t)^n}{K_1^{3} n!} \quad , \tag{5-13}$$

$$EK_3 = \frac{e^{-K_1 t}}{K_1^{4}} - \sum_{n=0}^{3} \frac{(-K_1 t)^n}{K_1^{4} n!} \quad , \tag{5-15}$$

$$EK_4 = \frac{e^{-K_1 t}}{K_1^{5}} - \sum_{n=0}^{4} \frac{(-K_1 t)^n}{K_1^{5} n!} \quad , \tag{5-16}$$

$$EK_5 = \frac{e^{-K_1 t}}{K_1^{6}} - \sum_{n=0}^{5} \frac{(-K_1 t)^n}{K_1^{6} n!} \quad , \tag{5-17}$$

$$SN_1 = 0.5\ \left(\frac{\sin \mu t - \mu t}{\mu} + \frac{SHE}{\sqrt{|2\mu^2 - K_2|}} + SU_1\right) \quad , \tag{5-17}$$

$$SUE = \begin{cases} \dfrac{\sin pt - pt}{p^3} & \text{when } K_2 > \mu^2, \quad (5\text{-}18) \\[2ex] \dfrac{-t^3}{6} & \text{when } K_2 = \mu^2\ , \quad (5\text{-}19) \\[2ex] \dfrac{pt - \sinh pt}{p^3} & \text{when } K_2 < \mu^2\ , \quad (5\text{-}20) \end{cases}$$

$$SU_1 = \begin{cases} \dfrac{\sin pt}{p} & \text{when } K_2 > \mu^2 \quad , & (5\text{-}21) \\ t & \text{when } K_2 = \mu^2 \quad , & (5\text{-}22) \\ \dfrac{\sinh pt}{p} & \text{when } K_2 < \mu^2 \quad , & (5\text{-}23) \end{cases}$$

$$p = \sqrt{|K_2 - \mu^2|}$$

$$CUE = \begin{cases} \dfrac{\cos pt - 1}{K_2 - \mu^2} & \text{when } K_2 > \mu^2 \quad , & (5\text{-}24) \\ \dfrac{-t^2}{2} & \text{when } K_2 = \mu^2 \quad , & (5\text{-}25) \\ \dfrac{\cosh pt - 1}{K_2 - \mu^2} & \text{when } K_2 < \mu^2 & (5\text{-}26) \end{cases}$$

$$CU_4 = \frac{CUE + 0.5\, t^2}{K_2 - \mu^2} \quad , \tag{5-27}$$

$$SHE = \begin{cases} \sin qt & \text{when } K_2 \geq 2\mu^2 \ , & (5\text{-}28) \\ \sinh qt & \text{when } K_2 < 2\mu^2 \ , & (5\text{-}29) \end{cases}$$

$$CHE = \begin{cases} \cos qt & \text{when } K_2 \geq 2\mu^2 \ , & (5\text{-}30) \\ \cosh qt & \text{when } K_2 < 2\mu^2 \ , & (5\text{-}31) \end{cases}$$

$$q = \sqrt{|2\mu^2 - K_2|} \quad , \tag{5-32}$$

$$CS_1 = \frac{t\,(\cos \mu t - 1)}{\mu^2} - \frac{\sin \mu t - \mu t}{\mu^3} \tag{5-33}$$

$$CS_2 = 0.5\left[\frac{t(\sin \mu t - \mu t)}{\mu^3} + \frac{2(\cos \mu t - 1 + 0.5\,\mu^2 t^2}{\mu^4}\right] , \qquad (5\text{-}34)$$

$$CS_3 = \frac{t(\cos \mu t - 1 + 0.5\,\mu^2 t^2}{\mu^2} - \frac{2(\sin \mu t - \mu t)}{\mu^3} - \frac{t^3}{3} , \qquad (5\text{-}35)$$

and

$$SH_2 = \frac{t \sin \mu t}{\mu} - \frac{1 + 0.5\zeta t^2 - CHE}{\zeta} . \qquad (5\text{-}36)$$

5.3 Discussion of the Solution

The elements of the matrix given in section 5.2 were derived from the first six terms of the series expansion of the transition matrix $\Phi(t)$. It was assumed that inferences on the high order terms could be made from the first six terms. The other assumptions made in deriving these expressions were the same as those made in the inverse Laplace Transform approach. The expressions in the last two rows and columns of the matrix are different from those derived in chapter 4 but they are numerically close.

The equations (5-17) to (5-32) indicate that for

$$K_2 < \mu^2 \qquad (5\text{-}39)$$

the expressions for elements associated with the height channel are functions of hyperbolic functions which grow exponentially as t becomes larger. Such growth can affect the other elements, especially those associated with the velocity corrections, when t becomes very large.

Generally, the analytical expressions for the matrix $\Phi(t)$ derived in this chapter are shorter and less complicated than those derived in chapter 4. However, it can be shown that many of the expressions associated with the elements in the horizontal channels are identical for both solutions. In most cases a series expansion of the matrix elements given in section 4.3 will show this. The expressions associated with the height channel are more complicated and their agreement can only be shown by numerical comparison. The series representation of the matrix exponential is less accurate because the summation was done on the basis of the first six terms only. Apparently, the effects of some high order terms have been omitted by this procedure.

6. TESTS AND NUMERICAL RESULTS

Two transition matrices have been derived in the last two chapters. As has been pointed out in section 5.3 the equivalence of the two solutions could not be shown analytically for each individual element. Numerical tests were therefore performed for varying values of t, ϕ, K_1, and K_2 in order to find out how closely the solutions agreed numerically and how close they came to the exact solution. This 'exact' solution was computed by a numerical method given in section 6.1 and it will be treated as the most accurate approximation to the actual transition matrix. The differences between this matrix and the other two matrices computed by using the analytical expressions of sections 4.3 and 5.2 will be regarded as the approximate errors of these expressions.

6.1 The Numerical Method

The numerical method is basically the series expansion solution. It generates the transition matrix $\Phi(t_n)$ by premultiplying a transition matrix for a small time interval Δt by itself for n times

$$t_n = n \cdot \Delta t, \tag{6-1}$$

where n is greater than zero. The matrix $\Phi(\Delta t)$ can be computed by using equation (2-8) truncated at a finite number of terms. The truncation error is negligible if Δt is very small. In our case Δt was equal to 1 second and the matrix was truncated after the sixth term. The solution of the equation (2-2) for time t_1 may be written as

$$\vec{x}(t_1) = \Phi(\Delta t, 0) \cdot x(\vec{0}) \tag{6-2}$$

where

$$t_1 = \Delta t , \tag{6-3}$$

and the solution for t_n is given by

$$\begin{aligned} \vec{x}(t_n) &= \Phi(t_n, t_{n-1}) \cdot \vec{x}(t_{n-1}) \\ &= \Phi(t_n , t_{n-1}) \cdot \Phi(t_{n-1}, t_{n-2}) \cdot \vec{x}(t_{n-2}) \\ &= \Phi(t_1, 0)^n \cdot \vec{x}(0) \end{aligned} \tag{6-4}$$

for

$$t_o = 0 . \tag{6-5}$$

Generating the solution with equation (6-4) is usually very time consuming when the value of t_n becomes large. If the value of n is, however, a power of 2

$$n = 2^k, \tag{6-6}$$

then the number of multiplications reduces from n to k. Formula (6-4) is then replaced by

step 1 $\qquad \Phi(t_2) = \Phi(\Delta t, 0)^2$

step 2 $\qquad \Phi(t_4) = \Phi(t_2)^2 = \Phi(\Delta t, 0)^4$

..... $\qquad$ ....

step k $\qquad \Phi(t_n) = \Phi(t_{\frac{n}{2}})^2 = \Phi(\Delta t, 0)^n \qquad (6-7)$

Since we are usually free to choose Δt the condition (6-6) can always be satisfied. The major error in applying this method is the truncation error occuring during the matrix multiplications.

6.2 Testing the Inverse Laplace Transform Solution

The solution for the transition matrix computed from the analytical expressions given in chapter 4 was compared to the numerical solution of the last section, for

$$25^\circ \leq \Phi \leq 85^\circ \quad , \tag{6-8}$$

$$0 \leq K_1 < \mu \quad , \tag{6-9}$$

$$0 \leq K_2 < 2\mu^2 \, , \tag{6-10}$$

$$0 \leq t \leq 1000 \text{ seconds}, \tag{6-11}$$

$$0 \leq \dot{\phi} \leq 1.31 \times 10^{-5} \text{ rad/second} \tag{6-12}$$

and

$$0 \leq \dot{\lambda} \leq 1.31 \times 10^{-5} \text{ rad/second} \quad . \tag{6-13}$$

The results are given in the table below:

time (sec)	max. percentage error
128	0.48
256	0.49
512	0.54
1024	1.13

Table 6-2 Error of Inverse Laplace Transform Solution

The inverse Laplace transform solution of the transition matrix for the values of t between 1000 and 3000 seconds was also computed to show the effect of the damping loop gains K_1 and K_2. The change of the matrix elements $\phi(8, 8)$ and $\phi(9, 9)$ with respect to changes of K_1 and K_2 are tabulated in Table 6-2.

The results indicate that the values of $\phi(8, 8)$ and $\phi(9, 9)$ increase rapidly for an undamped system.

The value of $\phi(8, 8)$ approaches $e^{-K_1 t}$ and the value of $\phi(9, 9)$ becomes 1.0 as

$$K_2 \to 2\mu^2 \tag{6-14}$$

and $\phi(8, 9)$ approaches t as

$$K_1 \to 0 \tag{6-15}$$

time(sec.)		10000		1500		2000		2500		3000	
K_1	K_2	$\Phi(8,8)$	$\Phi(9,9)$	$\Phi(8,8)$	$\Phi(9,9)$	$\Phi(8,8)$	$\Phi(9,9)$	$\Phi(8,8)$	$\Phi(9,9)$	$\Phi(8,8)$	$\Phi(9,9)$
0	0	2.98	2.98	6.99	6.99	16.73	16.73	40.21	40.21	96.69	96.69
0.2μ	$0.4\mu^2$	2.06	2.38	4.10	4.79	8.40	9.82	17.31	20.27	35.73	41.84
0.4μ	$0.8\mu^2$	1.39	1.91	2.29	3.26	3.98	5.70	6.98	10.04	12.30	17.60
0.6μ	$1.2\mu^2$	0.90	1.53	1.19	2.21	1.71	3.27	2.52	4.86	3.74	7.23
0.8μ	$1.6\mu^2$	0.54	1.23	0.54	1.49	0.60	1.83	0.71	2.27	0.86	2.81
μ	$2\mu^2$	0.29	1.00	0.155	1.00	0.08	1.00	0.05	1.00	0.02	1.00
1E-6	$2\mu^2$	0.99	1.00	0.99	1.00	0.99	1.00	0.99	1.000	0.99	1.00

Table 6-2 Effect of Damping

The values of other elements associated with the height channel also decrease under the conditions (6-14) and (6-15). The numerical method is generally a more time consuming way for generating a transition matrix. In our case, the method required 10 times as much computation time as the inverse Laplace transform solution to obtain the matrix $\Phi(t)$ when t is larger than 128 seconds.

6.3 Testing the Series Solution

The series solution described in chapter 5 was tested in a similar manner and under the same conditions mentioned in the last section. The series solution was compared to the numerical solution and the results are summarized in Table 6-3.

Time (sec.)	Max. percentage error
128	0.80
256	0.81
512	0.88
1024	1.45

Table 6-3 Error of the Series Solution

The results also indicate that the analytical expressions can generate the transition matrix $\Phi(t)$ 14 to 15 times faster than the numerical method for t larger than 128 seconds. However, the accuracy of the series solution decreases very rapidly as t increases beyond 1000 seconds. This is mainly because the solution was derived from the truncated series expansion. The analytical expression are not be accurate enough to give a close approximation to the actual solution when t becomes very large.

The series solution was also compared with the inverse Laplace transform solution for different values of t. The maximum differences between them are expressed as percentage of the values of their corresponding elements in the inverse Laplace transform solution in Table 6-4.

t(sec.)	maximum percentage differences	Average CPU time L (sec.)	S(sec.)
128	0.88	0.0050	0.0034
200	0.88	0.0050	0.0034
400	0.89	0.0051	0.0036
600	0.92	0.0051	0.0036
800	0.95	0.0049	0.0035
1000	1.40	0.0050	0.0037

Table 6-4 Differences between Series and Inverse Laplace Transform Solutions

The quantities listed in Table 6-4 show that the two solutions are in good agreement for time intervals up to 1000 seconds. The time needed to compute the transition matrix by way of the inverse Laplace transform solution is about 1.4 times larger than that required by the series solution.

7. CONCLUSIONS

The analytical form of the transition matrix for the local-level case of inertial navigation has been derived in two different ways: by using the inverse Laplace transform technique and by expanding the matrix exponential into a series. The equivalence of the two solutions can be shown for most matrix elements. Where it is not possible, numerical comparisons have been made. The agreement is always better than 1.5% of the respective value for time intervals up to 1000 seconds. Comparisons with an accurate numerical solution show agreement on the same level of accuracy. For time intervals larger than 1000 seconds the inverse Laplace transform solution is more accurate than the series solution. Both analytical solutions are superior to the numerical solution with respect to computer time by a factor of 10 to 15.

The analytical solutions can be used to discuss the behaviour of individual errors in a very general manner and to spot instabilities of the system. The best example is the instability of the height channel. The expressions show that, in an undamped system, the elements associated with the height channel contain hyperbolic functions of the Schuler frequency and grow in exponential manner when t becomes large. The damping loop gains, if properly chosen, can reduce these errors.

The analytical expressions were developed from a constant dynamics matrix, i.e. by neglecting the time dependent components. They can therefore only be treated as a first approximation to the elements of the actual transition matrix of the system of error equations. For further investigations, these time dependent components may be included in the dynamics matrix. A better approximation can then be obtained by using the inverse Laplace transform solution as a first approximation and by deriving the time dependent terms by a series expansion.

APPENDIX I

The purpose of this appendix is to show the elements of E_2^{-1} in Chapter 4 and the detailed derivation of the matrix N_2 described in the same chapter.

Let us label E_2^{-1} as $U(S)$ then the element in it are:

$$U(1,\ 1) = \frac{S^3 + S\ell^2 \cos\phi}{(S^2+\ell^2)(S^2+\mu^2)} \quad ,$$

$$U(1,\ 2) = \frac{\ell \sin\phi}{(S^2+\ell^2)} - \frac{u^2 \ell \sin\phi}{(S^2+\mu^2)}\left(\frac{1}{(S^2+\ell^2)} + \frac{2}{(S^2+\mu^2)}\right) ,$$

$$U(1,3) = \ell^2 \sin\phi \cos\phi\left[\frac{1}{S(S^2+\ell^2)} + \frac{\mu^2}{(S^2+\ell^2)(S^2+\ell^2)}\left(\frac{1}{S} + \frac{2S}{(S^2+\mu^2)}\right)\right]$$

$$U(1,\ 4) = \frac{-S^2 \ell \sin\phi}{(S^2+\ell^2)(S^2+\mu^2)} \quad ,$$

$$U(1,\ 5) = 0 \quad ,$$

$$U(1,\ 6) = \frac{2S\ell \sin\phi}{(S^2 + \mu^2)^2} \quad ,$$

$$U(1,\ 7) = \frac{\cos\phi}{(S^2 + \mu^2)} \quad ,$$

$$U(2,\ 1) = \frac{-\ell \sin\phi}{(S^2 + \ell^2)} + \frac{\mu^2 \ell \sin\phi}{(S^2 + \mu^2)}\left(\frac{1}{(S^2 + \ell^2)} + \frac{1}{(S^2 + \mu^2)}\right) \quad ,$$

$$U(2,\ 2) = \frac{S^3}{(S^2+\ell^2)(S^2+\mu^2)} \quad ,$$

$$U(2,\ 3) = \frac{S^2\ell \cos \phi}{(S^2+\ell^2)(S^2+\mu^2)} \quad ,$$

$$U(2,\ 4) = \frac{-\ell^2}{S(S^2+\ell^2)} + \frac{\mu^2\ell^2}{(S^2+\ell^2)(S^2+\ell^2)} \left(\frac{1}{S} + \frac{2S \sin^2\phi}{(S^2+\mu^2)}\right) \quad ,$$

$$U(2,\ 5) = 0 \quad ,$$

$$U(2,\ 6) = \frac{-1}{S^2 + \mu^2} \quad ,$$

$$U(2,\ 7) = \frac{S\ell \sin \phi}{(S^2 + \mu^2)^2} \quad ,$$

$$U(3,\ 1) = \tan \phi \left(\frac{S}{(S^2+\ell^2)} - \frac{S}{(S^2+\mu^2)}\right) \quad ,$$

$$U(3,\ 2) = \sec \phi \left[\frac{-\ell \cos^2\phi}{S^2+\ell^2} - \frac{\mu^2\ell \sin^2\phi}{S^2+\mu^2} \left(\frac{1}{S^2+\ell^2} + \frac{2}{S^2+\mu^2}\right)\right] \quad ,$$

$$U(3,\ 3) = \frac{S}{S^2+\ell^2} - \frac{\mu^2\ell^2 \sin^2\phi}{(S^2+\ell^2)(S^2+\mu^2)} \left(\frac{1}{S} + \frac{2S}{S^2+\mu^2}\right) \quad ,$$

$$U(3,\ 4) = \frac{\ell \cos \phi}{S^2 + \ell^2} + \frac{\mu^2\ell \sin \phi \tan \phi}{(S^2+\ell^2)(S^2+\mu^2)} \quad ,$$

$$U(3,\ 5) = 0$$

$$U(3,\ 6) = \frac{-2S\ell \sin^2\phi \ \sec\phi}{(S^2+\mu^2)^2} \quad ,$$

$$U(3,\ 7) = \frac{-\sin \phi}{S^2 + \mu^2} \quad ,$$

$$U(4,\ 1) = \frac{\mu^2 \ell \sin\phi}{(S^2 + \mu^2)} \left(\frac{1}{S^2 + \ell^2} + \frac{2}{S^2 + \mu^2} \right) \quad ,$$

$$U(4,\ 2) = \frac{\mu^2}{(S^2 + \mu^2)} \left(\frac{S}{S^2 + \ell^2} - \frac{\ell^2 \sin^2\phi}{S(S^2 + \mu^2)} \right) \ ,$$

$$U(4,\ 3) = \frac{\mu\ell^2 \ \cos\phi}{(S^2+\ell^2)(S^2+\mu^2)} \left(1 - \frac{2\ell^2 \sin^2\phi}{S^2 + \mu^2} \right) \quad ,$$

$$U(4,\ 4) = \frac{-\mu^2\ell^2}{(S^2+\ell^2)(S^2+\mu^2)} \left(\frac{1}{S} + \frac{2S \sin^2\phi}{S^2 + \mu^2} \right) \quad ,$$

$$U(4,\ 5) = 0$$

$$U(4,\ 6) = \frac{1}{S^2 + \mu^2} \quad ,$$

$$U(4,\ 7) = \frac{S\ell \sin 2\phi}{(S^2 + \mu^2)^2} \quad ,$$

$$U(5,\ 1) = \frac{-\mu^2}{(S^2+\ell^2)(S^2+\mu^2)} \left(\frac{2S\ell^2 \sin^2\phi}{S^2 + \mu^2} - S \sec\phi \right) \quad ,$$

$$U(5,\ 2) = \frac{\mu^2\ell \tan\phi}{(S^2+\mu^2)} \qquad \left(\frac{1}{S^2+\ell^2} + \frac{2}{S^2+\mu^2} \right) \ ,$$

$$U(5,\ 3) = \frac{\mu^2\ell^2 \sin\phi}{(s^2+\ell^2)(S^2+\mu^2)} \left(\frac{1}{S} + \frac{2S}{S^2+\mu^2} \right) \ ,$$

$$U(5,\ 4) = \frac{\mu^2\ell \tan\phi}{(S^2+\ell^2)(S^2+\mu^2)} \left(1 - \frac{2\ell^2}{S^2+\mu^2} \right) \quad ,$$

$$U(5,\ 5) = S \quad ,$$

$$U(5,\ 6) = \frac{2S\ell \tan\phi}{(S^2+ \mu^2)^2} \quad ,$$

$$U(5,\ 7) = \frac{1}{S^2 + \mu^2} \quad ,$$

$$U(6,\ 1) = \frac{\mu^2 \ell \sin\phi}{S^2 + \mu^2} \left(\frac{S}{S^2 + \ell^2} + \frac{2S}{S^2 + \mu^2}\right) \quad ,$$

$$U(6,\ 2) = \frac{\mu}{S^2 + \mu^2} \left(\frac{S^2}{S^2 + \ell^2} - \frac{\ell^2 \sin^2\phi}{S^2 + \mu^2}\right) \quad ,$$

$$U(6,\ 3) = \frac{\mu^2 \ell \cos\phi}{(S^2+\ell^2)(S^2+\mu^2)} \left(S - \frac{2S\ell^2 \sin^2\phi}{S^2 + \mu^2}\right) \quad ,$$

$$U(6,\ 4) = \frac{\mu^2 \ell^2}{(S^2+\ell^2)(S^2+\mu^2)} \left(1 + \frac{2S^2 \sin^2\phi}{S^2 + \mu^2}\right) \quad ,$$

$$U(6,\ 5) = 0 \quad ,$$

$$U(6,\ 6) = \frac{S}{S^2 + \mu^2 + 4\ell^2 \sin^2\phi} \quad ,$$

$$U(6,\ 7) = \frac{-S^2 \ell \sin^2\phi}{(S^2 + \mu^2)^2} \quad ,$$

$$U(7,\ 1) = \frac{\mu^2}{(S^2+\ell^2)(S^2+\mu^2)} \left(\frac{2S^2\ell^2 \sin^2\phi}{S^2 + \mu^2} - S^2 \sec\phi\right) \quad ,$$

$$U(7,\ 2) = \frac{\mu^2 \ell \tan\phi}{S^2 + \mu^2} \left(\frac{S}{S^2+\ell^2} + \frac{2S}{S^2 + \mu^2}\right) \quad ,$$

$$U(7,\ 3) = \frac{\mu^2 \ell^2 \sin}{(S^2+\ell^2)(S^2+\mu^2)} \left(1 + \frac{2S^2}{S^2+\mu^2}\right) \quad ,$$

$$U(7,\ 5) = 0 \quad ,$$

$$U(7,\ 6) = \frac{2S\ell \tan\phi}{(S^2 + \mu^2)^2} \quad ,$$

and $$U\ (7,\ 7) = \frac{S}{S^2 + \mu^2 + 4\ell^2 \sin^2\phi} \qquad \text{(AI-1)}$$

Substituting the above matrices, A_2 and C_2, into equation (4-4) results in

$$(D_2-C_2A_2^{-1}B_2) = \begin{bmatrix} S+K_1 & -1 \\ K_2-2\mu^2 & \frac{S + 4S(\dot{\phi}^2+\ell^2 \cos^2\phi)}{(S^2 + \mu^2)} \end{bmatrix} \qquad \text{(AI-2)}$$

Using the assumption that

$$\mu^2 >> \ell^2 \qquad \text{(AI-3)}$$

and

$$\mu^2 >> \dot{\phi}^2 \quad , \qquad \text{(AI-4)}$$

(AI-2) may be reduced to

$$(D_2-C_2A_2^{-1}B_2) = \begin{bmatrix} S+K_1 & -1 \\ K_2-2\mu^2 & S \end{bmatrix} \quad , \qquad \text{(AI-5)}$$

and its determinant is given by

$$|(D_2-C_2A_2^{-1}B_2)| = (S^2+K_1S - (2\mu^2-K_2))$$

$$= (S - a)(S - b) \quad , \qquad \text{(AI-6)}$$

where $a = -K_1 + \sqrt{(K_1^2 - 4K_2 + 8\mu^2)}$ (AI-7)

and $b = -K_1 - \sqrt{(K_1^2 - 4K_2 + 8\mu^2)}$. (AI-8)

The matrix N_2 may be derived by using standard co-factor techniques[5]

$$(D_2-C_2A_2^{-1}B_2)^{-1} = \begin{bmatrix} \frac{S}{(S-a)(S-b)} & \frac{1}{(S-a)(S-b)} \\ \frac{2\mu^2-K_2}{(S-a)(S-b)} & \frac{S + K_1}{(S-a)(S-b)} \end{bmatrix} \qquad \text{(AI-9)}$$

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