A MANUAL FOR THE ESTABLISHMENT AND ASSESSMENT OF HORIZONTAL SURVEY NETWORKS IN THE MARITIME PROVINCES

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PREFACE

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A MANUAL FOR THE ESTABLISHMENT AND ASSESSMENT OF HORIZONTAL SURVEY NETWORKS IN THE MARITIME PROVINCES

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PREFACE

This "manual" is the third in the series designed to assist surveyors, in the maritime provinces, on the correct and practical use of the geodetic information of the redefined Maritime Geodetic Network. It has been written as a surveyors handbook for the design, computation, and assessment of horizontal geodetic networks. In this report, a geodetic network is considered to be any geometric configuration of three or more terrestrial survey points. The points may be connected via any combination of direction, angle, azimuth, and distance observations; furthermore, there may be redundant observations leading to overdetermined cases. The networks are treated in only one environment in this manual, the conformal mapping plane. There are two sound reasons for this: (1) this is the environment in which most practicing surveyors wish to do their network computations, (ii) derived quantities - coordinates, distances, azimuths and their associated covariance matrices - can be transformed, if required, to the 2-D ellipsoidal and 3-D environments using the methodologies outlined in "A Manual for Geodetic Coordinate Transformations in the Maritimes" [Krakiwsky et. al., 1977] and "A Manual for Geodetic Position Computations in the Maritime Provinces" [Thomson et. al., 1978] respectively. This approach (rigorous transformation of 2-D plane information to 2-D ellipsoidal or 3-D) is equivalent to carrying out the original computations in the environment itself (e.g. 2-D ellipsoidal, 3-D).

No extensive derivations or explantions of the mathematical formulae used are given. The equations required to solve certain problems are stated, the notation is explained, and numerical examples are presented. A reader desiring extensive background information is referred to the reference material.

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1. INTRODUCTION

It has been shown that the coordinate definition and associated accuracy estimates for any terrain point can be expressed equivalently in three dimensional cartesian coordinates ($X, Y, Z; C_{x,y,z}$), in ellipsoidal coordinates ($\sigma, \lambda; C_{\phi,\lambda}$), or in conformal mapping plane coordinates ($x, y; C_{x,y}$), since the rigorous transformations between these quantities are well known [e.g. Krakiwsky et. al., 1977]. When an unknown terrain point is observed (e.g. an azimuth and a distance) from a known terrain point, the determination of the unknown coordinates and associated covariance matrix can be done in three dimensional space, on the surface of a reference ellipsoid, or on a conformal mapping plane [e.g. Thomson et. al, 1978]. The equivalence of results (coordinates, covariance matrix) in the three environments is attained through the rigorous reduction (recall that no reductions are required for three dimensional computations) of the spatial measurements to the chosen computation surface. We can conclude from this that the choice of an environment in which to carry out position computations is, from a mathematical point of view, arbitrary. This fact is very important in the present context as it permits us to study the establishment and assessment of horizontal geodetic networks in the conformal mapping plane environment with the assurance that the procedures used yield results equivalent to those used in the three dimensional and ellipsoidal surface environments. Since the conformal mapping plane mathematical models involved with the establishment of horizontal geodetic networks are easy to understand, and since in practice a majority of surveyors prefer to use plane coordinates, the entire subject matter of this manual is treated in only one environment - the conformal mapping plane. For a treatment of this subject matter in the three-dimensional and ellipsoidal surface environments, the interested reader is referred to, for example, Vincenty [1973] and Krakiwsky and Thomson [1978] respectively. The establishment and assessment of one-dimensional vertical networks is beyond the scope of this present work. A knowledge of the treatment of vertical networks is vital for surveyors, and the reader is referred, for example, to Vanicek and Krakiwsky [in prep.].

In this manual, a horizontal geodetic network is considered to be any geometric configuration of three or more terrain points connected via any combination of azimuth, direction, angle, and distance observations. The horizontal network may be such that only a unique solution for the coordinates of unknown points is possible (no. observations n = no. unknowns u), or there may be redundant observations in which case we say the network is overdetermined (no. observations $n >$ no. unknowns u). For both cases, contemporary mathematical and statistical concepts and methodology are used.

The fundamental concept utilized is that of a mathematical model. A mathematical model is defined as a functional relationship between some unknown parameters x (coordinates of unknown points) and some observables ℓ (azimuths, directions, angles, distances). There are two mathematical models that are of interest to us: the direct (explicit) form

$$x = g(\ell) \tag{1-1}$$

in which g is an explicit, functional relationship, and the inverse (explicit) form

$$\ell = h(x) \qquad (1\text{-}2)$$

in which h is another explicit, functional relationship. Both forms of these models are well-known to surveyors. For example, the direct (explicit) form is used in position computations on a conformal mapping plane [e.g. Thomson et. al., 1978; p. 112, eqs. (4-39) and (4-40)]

$$x_j = x_i + \ell_{ij} \sin t_{ij} ,$$

$$y_j = y_i + \ell_{ij} \cos t_{ij} .$$

The solution for either of the coordinates of the new point (x_j, y_j) simply involves the evaluation of either of the two equations (x_i, y_i are known, ℓ_{ij} (chord length) and t_{ij} (grid azimuth of chord) are reduced measured quantities). This type of mathematical model (direct explicit) lends itself to geodetic position computations (only one unknown point to be considered). The inverse (explicit) form is also well known, for example, the expressions for a distance and azimuth respectively [e.g. Thomson et. al., 1978; p. 112, eqs. (4-41) and (4-42)]

$$\ell_{ij} = [(x_j - x_i)^2 + (y_j - y_i)^2]^{1/2} ,$$

$$t_{ij} = \tan^{-1} \frac{(x_j - x_i)}{(y_j - y_i)} .$$

Recall, however, that the objective is to solve for the coordinates of an unknown point (say x_j, y_j) using observed ℓ_{ij} and t_{ij}. In this instance, neither equation can be solved directly. They must be used together to get a solution for (x_j, y_j). This requires special techniques which become extremely important when there are redundant observations. The inverse (explicit) form of the mathematical model leads itself particularly to the computation of geodetic networks, and can also be easily used for geodetic position computations. In this manual, the inverse (explicit) model is used exclusively.

There are, of course, other forms of mathematical models. These are considered to be outside the scope of this work, but for the solution of certain special problems are important. For a complete coverage of the topic of mathematical models, the reader is referred, for example, to Vanicek and Krakiwsky [in prep].

This manual presents the step-by-step mathematical and computational procedures required for the establishment and assessment of a horizontal geodetic network on a conformal mapping plane. The procedures with respect to different conformal mapping planes only vary in the reduction of measured quantities; therefore, this distinction is only made in Chapter 2 which covers Observations and Their Reductions. In addition to a review of the reductions of measurements to a conformal mapping plane, Chapter 2 also includes (i) a brief review of the instrumentation used to observe azimuths, directions, angles, and distances and the accuracy estimates (variances) one should expect to be associated with the measured quantities, and (ii) an introduction to the concept of screening (assessing) observations prior to their reduction and use in network computations. Chapters 3 and 4 respectively cover the topics of Mathematical Models for Azimuth, Direction and Angle Observations and Mathematical Models for Distance Observations. For each observable, the inverse explicit model is presented, first in it's original non-linear form, then in it's linearized form. The linearized equation is often referred to as the observation equation. The relationships of the elements of the linearized model with the matrix expressions of the method of least-squares used for solving a set (2 or more) of these equations is given for each case. In Chapter 5, entitled the Solution of Unique Cases, the linearized mathematical models of Chapters 3 and 4 are taken in several practical combinations to yield unique (no. unknowns u = no. observations n) solutions of well-known surveying problems (e.g. direct problem, azimuth intersection, distance intersection, resection, special traverses). Numerical examples for each problem are presented. The Solution of Overdetermined Cases, which constitute the main body of work in the establishment of a surveying network, is the subject of Chapters 6 and 7. The advantages of using the method of least squares is given; the combination of the observation equations (Chapters 3 and 4) in several practical situations are shown.

In addition, the implications of differenct conditions imposed on a horizontal network (e.g. fixed or weighted coordinates, orientation, scale) are discussed. Numerical examples for several types of survey networks are given (e.g. traverse, triangulation, trilateration). Chapters 8 and 9 deal with the analyses of networks. The Preanalysis of a network, which is basically an optimization process, is important for surveyors when considering geometric design, economics, tolerances, etc. The Post-analysis, treated in the final chapter (9) is most important to a surveyor. It is here that a certain "confidence" in the work done can be ascertained. The manual is concluded by Three Appendices deemed to be necessary for a complete understanding of this work, namely Taylor Series (I), Least Squares Method (Parametric) (II), and Error Ellipses (III).

2. OBSERVATIONS AND THEIR REDUCTIONS

The planning, execution, and treatment of observed azimuths, directions, angles, and distances are important aspects of the establishment and analysis of a horizontal geodetic network. The execution (actual field measurement techniques) are not covered in this manual; the interested reader is referred to, for example, Faig [1972], Thomson [1978], Cooper [1971], Burnside [1971], and Saastamoinen [1967]. The planning of observations is treated in Chapter 8 (Preanalysis). The mathematical treatment of the observed quantities is given here in three sections, namely (i) the accuracies of observed azimuths, directions, angles, and distances, (ii) a review of the reductions of observations to a conformal mapping plane, and (iii) data screening.

2.1 Accuracies of Observed Azimuths, Directions, Angles and Distances

A knowledge of the accuracy of an observable (a proposed measurement) is an important aspect of the preanalysis of survey networks (Chapter 8), and a knowledge of the accuracy of an observation (a completed measurement) is important for network computations. The determination of these accuracies, expressed as variances (σ_ℓ^2), is the subject of this section. Note that the effects of systematic errors are assumed to have been removed by either observing or mathematical procedures or a combination of the two.

The variance σ_A^2 of an astronomic azimuth determination by observation of celestial bodies (e.g. stars or sun) is dependent on the method used. For astronomic azimuths determined by the hour angle method σ_A^2 is given as [e.g. Nickerson, 1978; Mueller, 1969]

$$\sigma_A^2 = \frac{1}{n} F \sigma_t^2 + \frac{1}{n} (\sigma_p^2 + \sigma_c^2) \quad , \qquad (2\text{-}1)$$

where n = number of pointings on the star,

σ_t^2 = variance of the observation of time in arcseconds (1" = 0.067s),

σ_p^2 = variance of a single pointing on a star (cf. eq. (2-5)),

σ_c^2 = combined variance of two readings of the horizontal circle and pointing on the reference mark (cf. eqs. (2-6) and (2-4)),

$$F = \cos^2 \Phi \ (\tan \Phi - \cos A \cot Z)^2 + m \ (2 \tan^2\Phi + \cot^2 Z - 2 \tan\Phi \cos A \cot Z) \ ,$$

Z = zenith angle of star,

Φ = astronomic latitude of station,

A = measured astronomic azimuth,

$m = (\sigma_p^2 + \sigma_v^2)/\sigma_t^2$,

σ_v^2 = variance of levelling the theodolite (cf. eq. (2-9)).

Some typical default values for σ_A assuming $\Phi = A = Z = 45°$ and different typical theodolites are shown in Table 2.1.

Astronomic azimuths determined by observing star altitude have the following expected variance [e.g. Nickerson, 1978; Mueller, 1969]:

$$\sigma_A^2 = \frac{1}{n} \{ (\sigma_p^2 + \sigma_v^2) \tan^2 a + (\tan\Phi - \cos A \tan a)^2 \ [(\sigma_{vc}^2 + \sigma_p^2) \operatorname{cosec}^2 A + \sigma_{tr}^2 \cos^2\Phi]$$

$$+ (\sigma_p^2 + \sigma_c^2) \} , \qquad (2\text{-}2)$$

where σ_{vc}^2 = combined variance of levelling the vertical circle bubble and reading the vertical circle,

σ_{tr}^2 = variance of tracking for simultaneous horizontal and vertical pointing on a star $\simeq$ 1" ,

a = altitude of the observed star corrected for refraction.

	Typical 20" Inst.				Typical 1" Inst.				Typical 0".5 inst.			
	M=20, d=20", v=30"				M = 30, d=1 , v=20"				M = 40, d=0".5, V=10"			
	yields				yields				yields			
	σ_p = 3".5, σ_c=8".78, σ_v=6"				σ_p = 2".33, σ_c=3".84, σ_v=4"				σ_p = 1".88, σ_c=2".095, σ_v=2"			
	n=2	n=4	n=8	n=16	n=2	n=4	n=8	n=16	n=2	n=4	n=8	n=16
σ_t = 0".5(0.03s)	9".11	6".44	4".55	3".22	5".20	3".68	2".60	1".84	3".15	2".23	1".58	1".11
σ_t = 1".5(0.10s)	9".11	6".44	4".55	3".22	5".21	3".68	2".60	1".84	3".16	2".23	1".58	1".12
σ_t = 15"(1.0s)	9".37	6".62	4".68	3".31	5".65	3".99	2".82	2".00	3".84	2".72	1".92	1".36
σ_t = 2'30"(10.0s)	23".78	16".81	11".89	8".41	22".57	15".96	11".29	7".98	22".19	15".69	11".10	7".85

Table 2.1. Expected Values of σ_A Using Hour Angle Method for $\Phi = A = Z = 45°$

Table 2.2 shows some typical values of σ_A (for the same types of theodolites considered in Table 2.1) assuming $A = \Phi = 45°$.

For azimuths observed with gyro-theodolites, the most reliable method for obtaining a variance is to compute the sample variance of the mean of a set of many observation of the azimuth. This yields

$$\sigma_A^2 = \frac{1}{n-1} \sum_{i=1}^{n} (A_i - \bar{A}) \; , \qquad (2\text{-}3)$$

where n = number of observed gyro-azimuths,

A_i = individual gyro-azimuths,

$\bar{A}$ = mean of the set of observed gyro-azimuths.

Some default values to be expected are σ_A = 20" to 30" for a single observed azimuth determined by a gyro attachment similar to the Wild GAK1 [Bomford, 1975], and σ_A = 3" for a single gyro-azimuth observed with gyro-theodolites such as the MOM Gi-B2 or GYMO-GI-B1/A which have electronic time registration [Halmos, 1977].

The expected variance of direction observations is [e.g Nickerson, 1978]

$$\sigma_d^2 = \frac{\sigma_p^2 + \sigma_r^2}{n} + \sigma_L^2 + \rho^2 \frac{2\sigma_c^2}{D^2} \; , \qquad (2\text{-}4)$$

where σ_p^2 = variance of pointing the telescope on the target (cf. eqs. (2-6) and (2-5)),

σ_r^2 = variance of reading the horizontal circle of the theodolite (cf. eqs. (2-7) and (2-8)),

σ_L^2 = effect of variance of levelling the theodolite (cf. eq. (2-9)),

σ_c^2 = variance of centering the instrument and target (see Table 2.3),

n = number of pointings and readings for the direction,

ρ = 206264.8 = number of arc seconds in one radian,

D = distance between instrument and target.

	Typical 20" Inst.				Typical 1" Inst.				Typical 0".5 Inst.			
	M = 20, d = 20", v = 30"				M = 30, d = 1", v = 20"				M = 40, d = 0".5, v = 10"			
	yields				yields				yields			
	$\sigma_p = 3''.5, \sigma_c = 8''.78, \sigma_v = 6'', \sigma_{vc} = 6''.03$				$\sigma_p = 2''.33, \sigma_c = 3''.84, \sigma_v = 4'', \sigma_{vc} = 2''.53$				$\sigma_p = 1''.88, \sigma_c = 2''.095, \sigma_v = 2'', \sigma_{vc} = 1''.27$			
a°	n=2	n=4	n=8	n=16	n=2	n=4	n=8	n=16	n=2	n=4	n=8	n=16
25	8.48	5.99	4.24	3.00	4.22	2.99	2.11	1.49	2.68	1.90	1.34	0.95
35	8.30	5.87	4.15	2.94	4.29	3.03	2.15	1.52	2.68	1.90	1.34	0.95
45	8.54	6.04	4.27	3.02	4.67	3.30	2.34	1.65	2.86	2.02	1.43	1.01
55	9.69	6.85	4.84	3.43	5.65	4.00	2.83	2.00	3.41	2.41	1.71	1.21
65	12.99	9.18	6.49	4.59	7.91	5.59	3.96	2.80	4.77	3.37	2.38	1.69
75	22.63	16.00	11.31	8.00	13.85	9.79	6.92	4.90	8.42	5.96	4.21	2.98
85	75.15	53.14	37.57	26.57	44.90	31.75	22.45	15.87	27.69	19.58	13.85	9.79

Table 2.2. Expected Values of σ_A Using Star Altitude Method for $\Phi = A = 45°$

The pointing error is dependent on the magnification M of the particular theodolite being used (see Table 2.5) and is given as

$$\sigma_p'' \simeq \frac{45''}{M} \;, \tag{2-5}$$

for stationary targets and good observing conditions. For moving targets (e.g. star), the pointing error is

$$\sigma_p'' = \frac{70''}{M} \;. \tag{2-6}$$

The reading error is a function of the least count of the theodolite and the readout system. For theodolites with a least count of d", and using coincidence micrometers (usually the case for 1" and 0".5 instruments (see Table 2.5)),

$$\sigma_r = 2.5\ d'' \;, \tag{2-7}$$

and for theodolites with a microscope or direct reading system (typically for d = 10" to 1'),

$$\sigma_r = 0.3\ d'' \;. \tag{2-8}$$

The effect σ_L^2 of the variance of levelling the instrument σ_v^2 is dependent on the vertical angle h to the target, and is given as

$$\sigma_L = \sigma_v'' \tan h \;, \tag{2-9}$$

where

$$\sigma_v = 0.2\ v'' \;, \tag{2-10}$$

and v" is the value of one division (2 mm) of the plate level (see Table 2.5). The centering error σ_c summarised in

Method of Centering	Expected Error σ_c
spring plumb-bob	1 mm/m
optical plummet	0.5 mm/m
plumbing rods	0.5 mm/m
forced or self-centering	0.1 mm

Table 2.3 Expected Centering Error

Table 2.3 is for normal conditions (i.e no wind, equipment in good adjustment). Table 2.4 lists typical default values for σ_d assuming σ_c = 0.5 mm/m (i.e. σ_c = 1 mm for instrument height = 2 m) and a vertical angle h = 5°. It should be noted that this table is representative of good observing conditions. If observing conditions were poor, then the pointing error σ_p would increase accordingly. From equation (2-4), it is obvious that centering error σ_c is more critical for short lines of sight. As well, σ_L (eq. 2-9) will contribute increasingly for steeper lines of sight.

Horizontal angles B can be considered as the difference of two direction observations. Propagation of errors through the formula

$$B = d_2 - d_1 \tag{2-11}$$

yields the expected variance for an observed angle as

$$\sigma_B^2 = 2\left\{\frac{\sigma_p^2 + \sigma_r^2}{n} + \sigma_L^2 + \rho^2 \frac{2\sigma_c^2}{D^2}\right\}, \tag{2-12}$$

or twice the variance of a single direction. Thus, typical values for observed angles can be obtained from Table 2.4 by multiplying the values by $\sqrt{2}$.

	Typical 20" Instrument M=20, d=20", v=30" ∴ σ_p=2".25, σ_r=6".0, σ_L=0".52				Typical 1" Instrument M=30, d=1", v=20" ∴ σ_p=1".5, σ_r=2".5, σ_L=0".35				Typical 0".5 Instrument M=40, d=0".5, v=10" ∴ σ_p=1".13, σ_r=1".25, σ_L=0".18			
D(m)	n=2	n=4	n=8	n=16	n=2	n=4	n=8	n=16	n=2	n=4	n=8	n=16
100	5".41	4".36	3".73	3".37	3".59	3".28	3".11	3".03	3".16	3".04	2".98	2".95
200	4".79	3".56	2".74	2".23	2".55	2".09	1".82	1".67	1".89	1".69	1".59	1".53
400	4".62	3".33	2".44	1".84	2".21	1".67	1".31	1".09	1".41	1".13	0".96	0".86
800	4".58	3".27	2".35	1".72	2".12	1".54	1".15	0".89	1".26	0".94	0".72	0".59
1600	4".56	3".25	2".33	1".69	2".10	1".51	1".10	0".83	1".22	0".88	0".65	0".49
3200	4".56	3".25	2".33	1".69	2".09	1".50	1".09	0".81	1".21	0".87	0".63	0".47

Table 2.4. Expected Values of σ_d for h = 5° and σ_c = 1 mm

INSTRUMENT	MANUFACTURER	COUNTRY	Telescope					H. Circle		V. Circle		Reading		Spirit Levels Value of 2 mm			Weight (kg)
			Magnification	Objective diam(mm)	Length (mm)	Shortest Focus (m)	Field of View (°)	Diam. (mm)	Graduation	Diam. (mm)	Graduation	Direct to	System	Plate (")	Altitude (")	Spherical (')	
FT1A	Fennel	W. Germany	30	40	175	1.2	1.6	90	1°	70	1°	1'	Opt. Scale	40	auto.	8	4.0
DKM-1	Kern	Switzerland	20	30	120	0.9	1.7	50	20'	50	20'	10"	Opt. micro	30	30	-	1.8
K1-A	Kern	Switzerland	28	45	155	1.8	1.5	89	1°	70	1°	20"	Opt. micro	40	auto.	-	4.2
Te-E6	Mom	Hungary	20	28	123	1.3	2.0	80	20'	40	20'	10"	Opt. micro	50	auto.	6	2.6
Microptic 1	Rank	U.K.	25	38	146	1.6	1.5	89	20'	64	20'	20"	Opt. micro	40	30	-	4.5
4149-A	Salmoiraghi	Italy	30	36	172	2.0	1.4	90	30"	90	30"	30"	Direct	30	auto.	10	4.7
V22	Vickers	U.K.	25	38	137	1.8	2.0	78	1°	63	1°	20"	Opt. scale	45	90	17	5.2
T16	Wild	Switzerland	28	40	150	1.4	1.6	79	1°	79	1°	1'	Opt. scale	30	30	8	4.5
T1A	Wild	Switzerland	28	40	150	1.4	1.6	73	1°	65	1°	20"	Opt. micro	30	auto.	8	5.0
Theo 020	Ziess(Jena)	E. Germany	25	35	195	2.1	1.6	96	1°	74	1°	1'	Opt. scale	30	auto.	8	4.3
Th 3	Zeiss(Ober.)	W. Germany	25	35	150	1.2	1.7	78	1°	70	1°	30"	Opt. micro	30	auto.	15	3.5
Th 4	Zeiss(Ober.)	W. Germany	25	35	150	1.2	1.7	98	1°	85	1°	1'	Opt. scale	30	auto.	10	4.5
Tu	Askania	W. Germany	30	45	165	1.5	1.6	90	20'	70	20'	1"	Coinc. micro	20	auto.	10	4.6
FT 2	Fennel	W. Germany	30	45	174	2.0	1.6	93	20'	60	20'	1"	Coinc. micro	20	20	6	5.5
DKM 2	Kern	Switzerland	30	45	170	1.7	1.3	75	10'	70	10'	1"	Coinc. micro	20	20	-	3.6
DKM 2-A	Kern	Switzerland	30	45	170	1.7	1.3	75	10'	70	10'	1"	Coinc. micro	20	auto.	-	6.8
TB -1	Mash-priboritorg	USSR	26	40	180	1.2	1.3	85	20'	75	20'	1"	Coinc. micro	20	25	12	5.1
Te-B3	Mom	Hungary	30	40	175	2.5	1.5	78	20'	66	20'	1"	Coinc. micro	20	auto.	6	5.5
Microptic 2	Rank	U.K.	28	41	165	1.8	1.5	98	10'	76	10'	1"	Coinc. micro	20	20	-	6.3
4200-A	Salmoiraghi	Italy	30	40	172	2.5	1.5	40	10'	90	10'	1"	Coinc. micro	20	auto.	10	6.1
Tavistock 2	Vickers	U.K.	25	38	159	1.8	2.0	85	20'	70	20'	1"	Coinc. micro	20	20	20	4.8
T2	Wild	Switzerland	28	40	150	1.5	1.6	90	20'	70	20'	1"	Coinc. micro	20	30	8	5.6
Theo 010	Zeiss(Jena)	E. Germany	31	53	135	2.0	1.2	84	20'	60	20'	1"	Coinc. micro	20	20	8	5.3
Th2	Zeiss (Ober.)	W. Germany	30	40	155	1.6	1.3	100	10'	85	10'	1"	Coinc. micro	20	auto.	10	5.2
DKM3	Kern	Switzerland	27,45	72	140	19	1.6	100	10'	100	10'	0".5	Coinc. micro	10	10	-	12.2
OT-02	Mash-priboritorg	USSR	24,30, 40	60	265	5.0	1.6	135	4'	90	8'	0".2	Coinc. micro	7	12	-	11.0
Microptic 3	Rank	U.K.	40	50	170	1.8	1.0	98	5'	76	5'	0".2	Coinc. micro	10	20	-	8.0
Geod. Tavi.	Vickers	U.K.	20,30	60	225	5.0	1.3	127	20'	70	20'	0".5&1"	Coinc. micro	20	10	-	9.8
T3	Wild	Switzerland	24,30 40	60	265	3.6	1.6	135	4'	90	8'	0".2	Coinc. micro	7	30	-	11.2
T4	Wild	Switzerland	70	60	-	100	-	240	2'	135	4'	0".1/0".2	Coinc. micro	1	2	8	60

Table 2.5 Major Features of Some Modern Theodolites

The covariance between angles derived from a set of three or more directions cannot be overlooked. Considering the situation illustrated in Figure 2.1, the angles are usually derived from the directions as

$$B_{ijk} = d_{ik} - d_{ij} \quad ,$$
$$B_{ik\ell} = d_{i\ell} - d_{ik} \quad . \qquad (2\text{-}13)$$

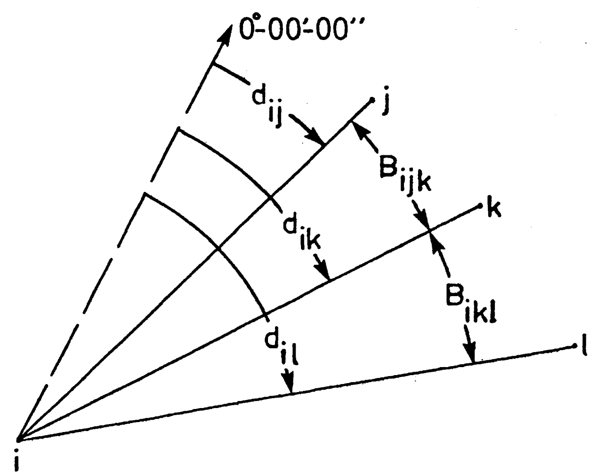


Figure 2.1 Angles and Directions

Use of the covariance law to propagate errors from equations (2-13) into the angles B_{ijk} and $B_{ik\ell}$ gives the variance covariance matrix C_B of the angles as

$$C_B = \begin{bmatrix} \sigma^2_{d_{ij}} + \sigma^2_{d_{ik}} & -\sigma^2_{d_{ik}} \\ -\sigma^2_{d_{ik}} & \sigma^2_{d_{ik}} + \sigma^2_{d_{i\ell}} \end{bmatrix} , \qquad (2\text{-}14)$$

i.e. covariances equal to minus the variance of the common direction between the angles will exist. For angles not derived from directions, but measured independently, C_B will be a diagonal matrix.

The variance σ_r^2 of spatial distances observed with EDM is characterized by [e.g. Nickerson, 1978]

$$\sigma_r^2 = \frac{\sigma_{ph}^2}{m_1} + \sigma_z^2 + \frac{r^2 \sigma_n^2}{m_2 n^2} , \tag{2-15}$$

where σ_{ph}^2 = variance of phase difference determination (cf. eq. (2-16)),

σ_z^2 = variance of the so-called zero error,

σ_n^2 = variance of determination of the index of refraction n (cf. eqs. (2-17) and (2-18)),

m_1 and m_2 = number of determinations of phase difference and meteorological readings, respectively.

The variance of phase determination is computed as

$$\sigma_{ph}^2 = \{\tfrac{1}{2}\lambda\}^2 \quad \sigma_\theta^2 \quad , \tag{2-16}$$

where λ = modulation wavelength used by the specific instrument (see Table 2.6),

σ_θ^2 = variance of determination of the phase difference for one distance measurement in fractions of a wavelength.

Most modern EDM equipment can easily achieve an accuracy of phase difference determination σ_θ = 0.001 [Burnside, 1971], but more accurate values for an individual instrument should be available from the manufacturer's specifications. The zero error σ_z results from inaccurate knowledge of the electrical center of the instrument with respect to the geometric center which is aligned over the point. This value is usually small (e.g. 5 mm) for instruments using light waves as the carrier frequency, but for

Model	Manufacturer	Radiation Source	Modulation Frequency		Modulator	Power Consumed (W)	Method of Phase Measurement	Range (Km)		Standard Deviation
			Base (MH_Z)	Total #				Day	Night	
Geodimeter Model 8	AGA Sweden	5mW He-Ne Laser	30	4	KDP Crystal	75	null meter	30	60	$\pm$ (5 mm + 1 . 10^{-6} s)
Geodolite 3 G	Spectra-Physics U.S.A.	5mW He-Ne Laser	49	5		400	digital	60	80	$\pm 1.10^{-6}$ s or 1 mm whichever greater
Geodimeter Model 6	AGA Sweden	30 W Mercury Lamp	30	3	Kerr Cell	70 300	resolver null meter	3 5	15 25	$\pm$(1 cm + 2.10^{-6} s)
Geodimeter 76	AGA Sweden	2mW Laser		2	Kerr Cell			3 (3 prisms)		$\pm$(1 cm + 1.10^{-6} s)
DM 1000	Kern	GaAs-Diode 900 nm	15	2	-	11	digital	2.5 (3 Refl.)		$\pm$ 1 cm
Mekometer ME 3000	Kern	Xenon-flash (100 Hz)	500	5	ADP Crystal	18	optomechanical null meter	3 (3 prisms)		$\pm$(0.2 mm + 1.10^{-6} s)
DM 500	Kern	GaAs Diode 875 nm	15	2	-	11	digital	0.5 (3 prisms)		$\pm$ 1 cm
SM 11	Zeiss Oberkochen	GaAs Diode 910 nm	15	2	-	12	automatic digital	2 (19 prisms)		$\pm$ 5 to 10 mm
ELDI 2	Zeiss Oberkochen					4		5		$\pm$ 5 mm
MA 100	Tellurometer	GaAs Diode 930 nm	75	4	-	14	digital	2		$\pm$(1.5 mm + 2.10^{-6} s)
CD 6	Tellurometer	GaAs Diode					digital	2		$\pm$(5 mm + 5.10^{-6} s)
SDM-3	Sokkisha Ltd, Tokyo	GaAs Diode 900 nm	15	2	-	10	digital	1 (3 prisms		$\pm$ 1 cm
DI 3	Wild Heerbrugg	GaAs Diode 875 nm	7.5	2	-	14	digital	0.6 (3 prisms)		$\pm$(5 mm + 5.10^{-6} s)
DM-60	Cubic Ind. Co., USA	GaAs Diode 900 nm	75	3	-	15	automatic digital	2		$\pm$(5 mm + 1.10^{-5} s)
Cubitape 3800 B	Hewlett-Packard, USA	GaAs Diode	15	4	-	12	digital null meter	3 (3 prisms)		$\pm$(5 mm + 1.10^{-5} s)
Ranger II	Laser Syst. & Electronics USA	3mW He-Ne Laser	15	4	KDP Crystal		automatic digital	6		$\pm$(5mm + 2.10^{-5} s)

Table 2.6 Characteristics of Modern EDM

Model	Manufacturer	Carrier Frequency (GH_z)	Measuring Frequency (MH_z)	Antenna Diameter (cm)	Antenna Divergence (°)	Power Consumed (w)	Readout	Measuring Range (Km)	Standard Deviation
MRA 101	Tellurometer Ltd.	10.05 to 10.45	7.5	33	6	38	digital	0.1 to 50	$\pm(1.5\ cm + 3.10^{-6}s)$
MRA 3	Tellurometer Ltd.	10.025 to 10.45	7.5	33	9		digital	0.1 to 50	$\pm(1.5\ cm + 3.10^{-6}s)$
MRA 4	Tellurometer Ltd.	34.5 to 35.1	75	33	2		digital	0.05 to 60	$\pm(3\ mm + 3.10^{-6}s)$
CA 1000	Tellurometer Ltd.	10.1 to 10.45	19 to 25				digital	0.05 to 30	$\pm(1.5\ cm + 5.10^{-6}s)$
Electrotape DM20	Cubic Corp. U.S.A.	10.5 to 10.5	7.5	33	6		digital	0.05 to 50	$\pm(1\ cm + 3.10^{-6}s)$
Distomat DI50	Wild Heerbrugg	10.2 to 10.5	15	36	6	50	digital	0.1 to 50	$\pm(2\ cm + 5.10^{-6}s)$
Distomat DI60	Siemens-Albiswerk	10.3	150	35	6	38	digital	0.02 to 150	$\pm(1\ cm + 3.10^{-6}s)$

Table 2.6 Continued.

microwave instruments the value may be up to 20 mm. This value is normally supplied with the instrument. The variance of the refractive index is different for lightwaves and microwaves. For lightwaves, the variance is [e.g. Nickerson, 1978; Laurila, 1976]

$$\sigma_n^2 = [\{\frac{1}{T^2}(\frac{-N_G P}{3.709} + 11.27e)\}^2 \sigma_T^2 + \{\frac{N_G}{3.709T}\}^2 \sigma_p^2 + \{\frac{11.27}{T}\}^2 \sigma_e^2] \cdot 10^{-12}, \tag{2-17}$$

where T = temperature in degrees Kelvin ($t°C + 273.15$),

$N_G = (287.604 + \frac{4.8864}{\lambda_c^2} + \frac{0.068}{\lambda_c^4})$ for λ_c = carrier wavelength (see Table 2.6),

p = air pressure in millibars (1 mbar ≃ 0.75 mm Hg @ 0°C),

e = water vapour pressure in mbar (for detailed computation see Bomford [1975], p. 54),

σ_T^2 = variance of temperature measurement in $°C^2$,

σ_p^2 = variance of pressure measurements in $mbar^2$,

σ_e^2 = variance of water vapour determination in $mbar^2$.

For microwaves the variance of the refractive index is [e.g. Nickerson, 1978; Laurila, 1976]

$$\sigma_n^2 = \left[\{\frac{-77.62P}{T^2} + (\frac{12.92}{T^2} - \frac{74.38 \cdot 10^4}{T^3})e\}^2 \sigma_T^2 + \{\frac{77.62}{T}\}^2 \sigma_p^2 + \{\frac{-12.92}{T} + \frac{37.19 \cdot 10^4}{T^2}\} \sigma_e^2\right] \cdot 10^{-12}, \tag{2-18}$$

where the elements in this equation are defined the same as those in equation (2-17). Table 2.7 summarizes the effect of errors in meteorological measurements on observed distances.

METEOROLOGICAL ERROR	EFFECT ON DISTANCE	
	Light waves	Microwaves
± 1 mbar in air pressure	0.22 ppm	0.22 ppm
± 1°C in temperature	1.0 ppm	1.6 ppm
± 1°C in the difference between dry and wet bulbs	0.05 ppm	8.0 ppm

Table 2.7 Effect of Meteorological Errors on Measured Distances

Table 2.8 below lists some expected values of σ_r for both lightwave and microwave instruments.

For treatment of distances observed by mechanical or optical means, one is referred to e.g. Nickerson [1978] or Smith [1970].

The above discussion has treated only the accuracy of observed azimuths, directions, angles and distances. However, the observations used in horizontal network computations are considered to be reduced to the plane. Any inaccuracies resulting from these reductions must also be accounted for. This propagation of errors through the reduction formulae (see section 2.2) has already been covered in section 3.2.8 of Thomson et al [1978].

2.2 Reduction of Observations to a Conformal Mapping Plane

The reduction of observed azimuths, directions, angles, and distances to a conformal mapping plane is essentially a two-phased process: terrain to reference ellipsoid, and reference ellipsoid to conformal mapping plane. Each phase, depending on the observed quantity, may contain one or more reduction steps. These procedures, for the 3° Transverse Mercator and

	Lightwaves; λ_c = 900 nm λ = 20 m, σ_z = 0.005 m, σ_θ = 0.001				Microwaves; λ_c = 3 cm λ=40 m, σ_z = 0.015 m, σ_θ= 0.001			
	σ_p = 1 mbar $\sigma_T = \sigma_{\Delta T}$=0.2°C		σ_p = 5 mbar $\sigma_T = \sigma_{\Delta T}$=1°C		σ_p = 1 mbar $\sigma_T{}^2 = \sigma_{\Delta T}$ = 0°2C		σ_p = 5 mbar $\sigma_T = \sigma_{\Delta T}$= 1°C	
S (m)	m_1=2,m_2=1	m_1=4,m_2=2	m_1=2,m_2=1	m_1=4,m_2=2	m_1=2,m_2=1	m_1=4,m_2=2	m_1=2,m_2=1	m_1=4,m_2=2
100	0.009	0.007	0.009	0.007	0.021	0.018	0.021	0.018
200	0.009	0.007	0.009	0.007	0.021	0.018	0.021	0.018
400	0.009	0.007	0.009	0.007	0.021	0.018	0.021	0.018
800	0.009	0.007	0.009	0.007	0.021	0.018	0.022	0.019
1600	0.009	0.007	0.009	0.007	0.021	0.018	0.024	0.020
3200	0.009	0.007	0.010	0.008	0.021	0.018	0.033	0.026
6400	0.009	0.007	0.013	0.010	0.023	0.020	0.057	0.041
12800	0.010	0.008	0.021	0.015	0.029	0.023	0.107	0.077
25600	0.012	0.009	0.040	0.028	0.045	0.035	0.212	0.150

Table 2.8 Expected Values for σ_r

Double Stereographic conformal map projections, are given, for example, in Thomson et. al. [1978]. The entire process is reviewed here in the context of horizontal geodetic networks. The primary reason for this is that for network computations the sequence of events is different than that used for position computations; in addition, the software used to generate the numerical examples given in this report follows the sequence given here.

The first problem to be solved is the determination of the approximate coordinates, (X^a, Y^a) and (ϕ^a, λ^a), for each unknown point in the network. This can be done in several ways, but the most often used are (i) to determine them graphically using a large scale map or a plan, or (ii) to compute them using observed quantities, well known geometric/ trigonometric solutions and coordinate transformation procedures. The coordinate transformations are given in, for example, Krakiwsky, et. al [1977]. The main point to bear in mind when determining approximate coordinates is that they must be sufficiently close to the final values so that the effects on the reduction of observations will be negligible. A conservative estimate of "sufficiently close" is 20 m. This can be achieved easily, in most instances, using observed quantities and unique geometric/trigonometric solutions. Well determined approximate coordinates are also important in the solution for final coordinates as this will minimize the number of required iterations [e.g. Steeves, 1978].

The reduction of an astronomic azimuth (A_{ij}), obtained from astronomic observations or a gyrotheodolite, to a conformal mapping plane (grid) azimuth (t_{ij}) is outlined in Figure 2.2. The observed, known, and computed quantities required are given in Table 2.9. The equation numbers listed in Figure 2.2 and Table 2.9 refer to those found in Thomson et. al [1978].

The reduction of a measured terrain normal section direction (d^t_{ij}) to a conformal mapping plane (grid) direction (d_{ij}) of the corresponding chord is outlined in Figure 2.3. The observed, known, and computed quantities required are given in Table 2.10. The equation numbers listed in Figure 2.3 and Table 2.10 refer to those found in Thomson et. al. [1978].

A measured angle (B_{jik}), since it is simply the difference of two terrain normal section directions ($d^t_{ik} - d^t_{ij}$) follows the same reduction procedure as the directions themselves. The procedure is outlined in Figure 2.4. The observed, known, and computed quantities involved are given in Table 2.11. The equation numbers listed in Figure 2.4 and Table 2.11 refer to those found in Thomson et. al. [1978].

The reduction of a terrain spatial distance (r_{ij}) (measured distance corrected for atmospheric and instrumental effects) to a conformal mapping plane (grid) distance (ℓ_{ij}) of the corresponding chord is outlined in Figure 2.5. The observed, known, and computed quantities involved are given in Table 2.12. The equation numbers given in Figure 2.5 and Table 2.12 refer to those found in Thomson et. al [1978].

An examination of Figures 2.2 to 2.5 and Tables 2.9 to 2.12 inclusive shows a significant overlap in observed (e.g. Z_{ij}), computed (e.g. approximate coordinates), and known (e.g. ellipsoidal and conformal mapping system constants) quantities involved in the reduction of measured azimuths, directions, angles, and distances. In practice, these quantities need only be specified once. For example, in the program GEOPAN [Steeves, 1978],

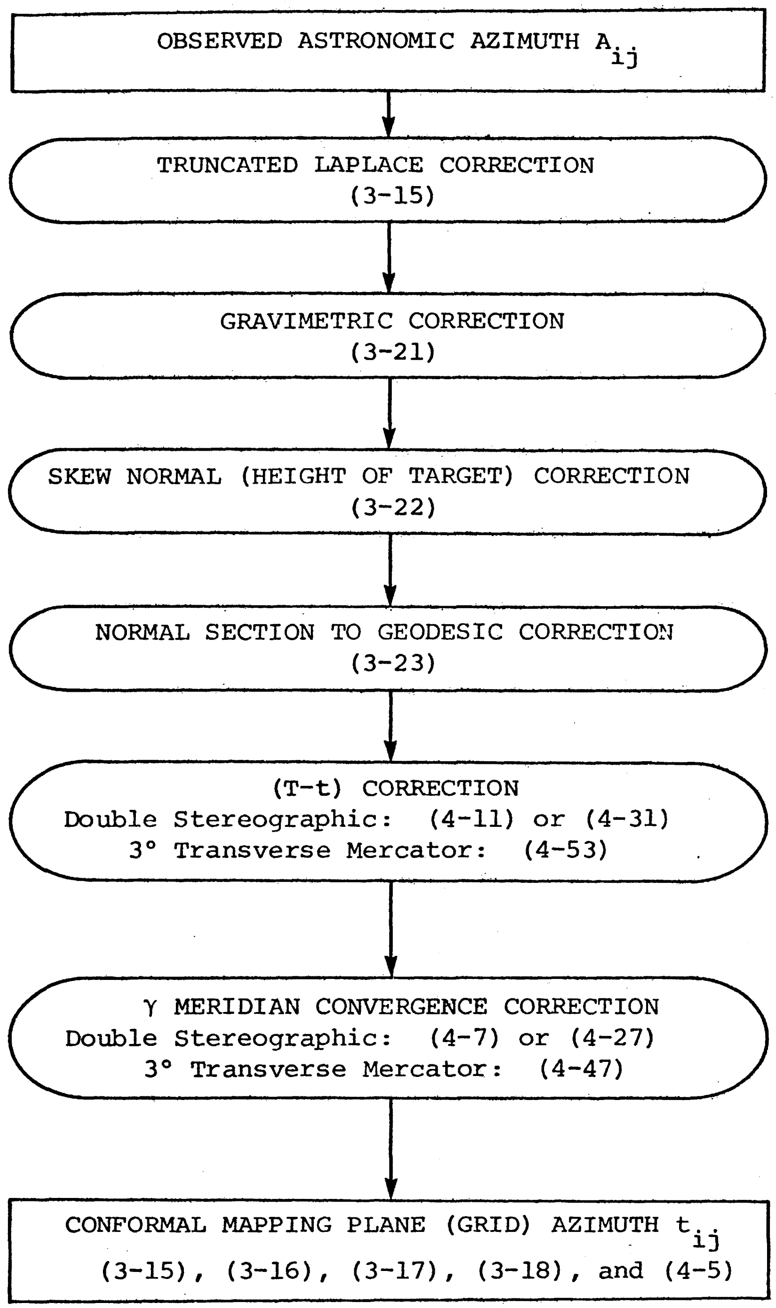


Figure 2.2

Reduction of Observed Astronomic Azimuth to a Conformal Mapping Plane

Status	Quantity	Remarks
Observed	A_{ij}: Astronomic Azimuth	obtained via stellar, solar, or gyrotheodolite observations.
	Z_{ij} (or h_{ij} or H_{ij}): zenith distance (or elliposidal or orthometric height difference)	needed for reduction purposes
Computed	(x_i^a, y_i^a), (x_j^a, y_j^a) (ϕ_i^a, λ_i^a), (ϕ_j^a, λ_j^a) (x_i^a, Λ_i^a)	see Krakiwsky et al. [1977] re coordinate transformations.
	Z_{ij}: reduced zenith distance	(3-14); use partially reduced azimuth, from (3-15), for this computation
	S_{ij}^a: approximate ellipsoidal distance	(3-39) or (3-62); use approximate coordinates (σ^a, λ^a) to compute all quantities, including auxilaries.
	$(T-t)_{ij}$	(4-11) or (4-31) for Double Stereographic, (4-53) for 3° Transverse Mercator; use approximate coordinates for all computations.
	γ_i: Meridian convergence	(4-7) or (4-27) for Double Stereographic, (4-47) for 3° Tranverse Mercator; use approximate coordinates for all computations.
	h_j: ellipsoidal height of target	
Known	a, b (or a,f)	parameters of reference ellipsoid.
	ψ_o, Λ_o, R; ϕ_o, λ_o; x_o, y_o; k_o	all defining parameters of the conformal mapping system.
	H_i (or h_i): orthometric (or ellipsoid) height	
	N_i^*, N_j^*	geoidal heights required to determine Δh_{ij}.
	ξ_i, η_i {deflection of vertical components at observed station	

Table 2.9

Reduction of Observed Astronomic Azimuth to a Conformal Mapping Plane

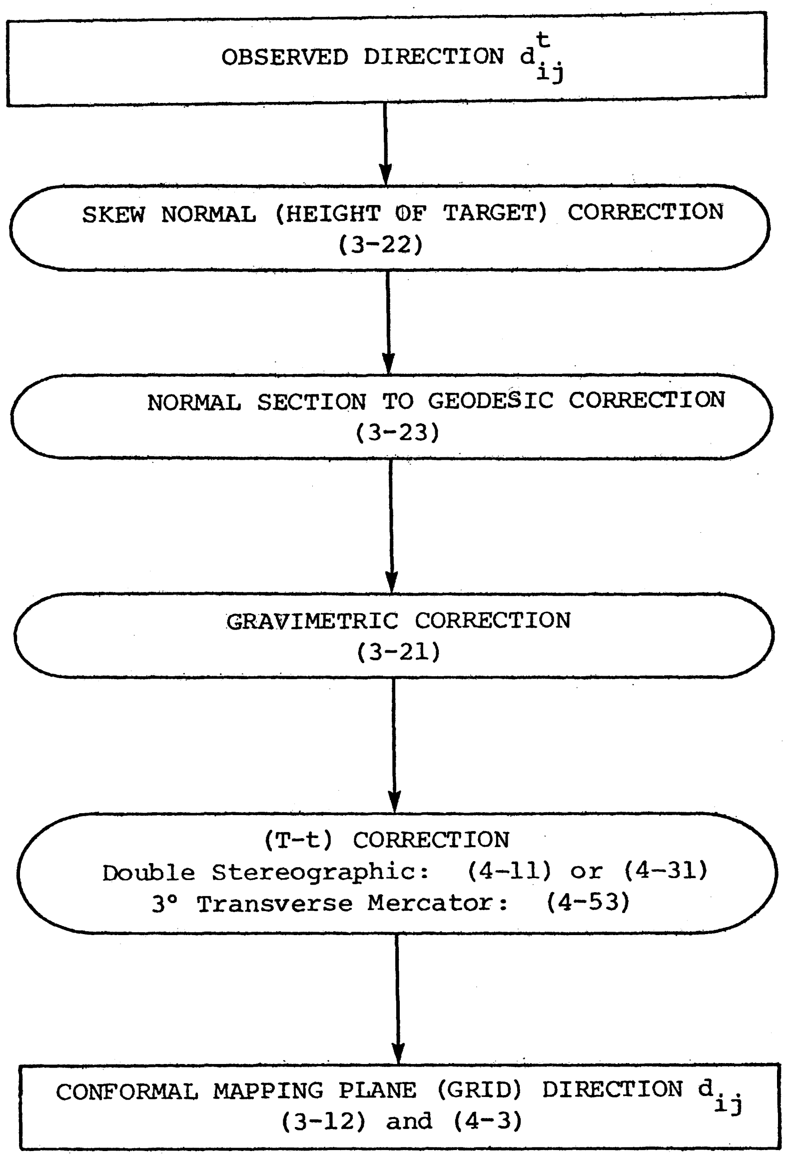


Figure 2.3

Reduction of an Observed Direction to a Conformal Mapping Plane

Status	Quantity	Remarks
Observed	d^t_{ij}: terrain normal section direction	
	Z_{ij} (or ΔH_{ij} or Δh_{ij})	needed for reduction purposes
Computed	(x^a_i, y^a_i), (x^a_j, y^a_j)	see Krakiwsky et al. [1977] re coordinate transformations.
	(ϕ^a_i, λ^a_i), (ϕ^a_j, λ^a_j)	
	S^a_{ij}: approximate ellipsoidal distance	(3-39) or (3-62) } use approximate coordinates for all computations
	α^a_{ij}: approximate geodetic azimuth	(3-37) or (3-59) } use approximate coordinates for all computations
	Z_{ij}: reduced zenith distance	(3-14): use α^a_{ij} for this computation
	$(T-t)_{ij}$	(4-11) or (4-31) for Double Stereographic, (4-53) for 3° Transverse Mercator; use approximate coordinates for all computations.
	h_j: ellipsoidal height of target	
Known	a,b (or a,f)	parameters of reference ellipsoid
	ψ_o, Λ_o, R;	all defining parameters of the particular conformal mapping system.
	ϕ_o, λ_o; x_o, y_o;	
	k_o	
	H_i (or h_i): orthometric (or ellipsoidal) height of instrument	
	N^*_i, N^*_j: geoidal heights	required to determine Δh_{ij}
	ξ_i, η_i: deflection of vertical components	

Table 2.10

Reduction of an Observed Direction to a Conformal Mapping Plane

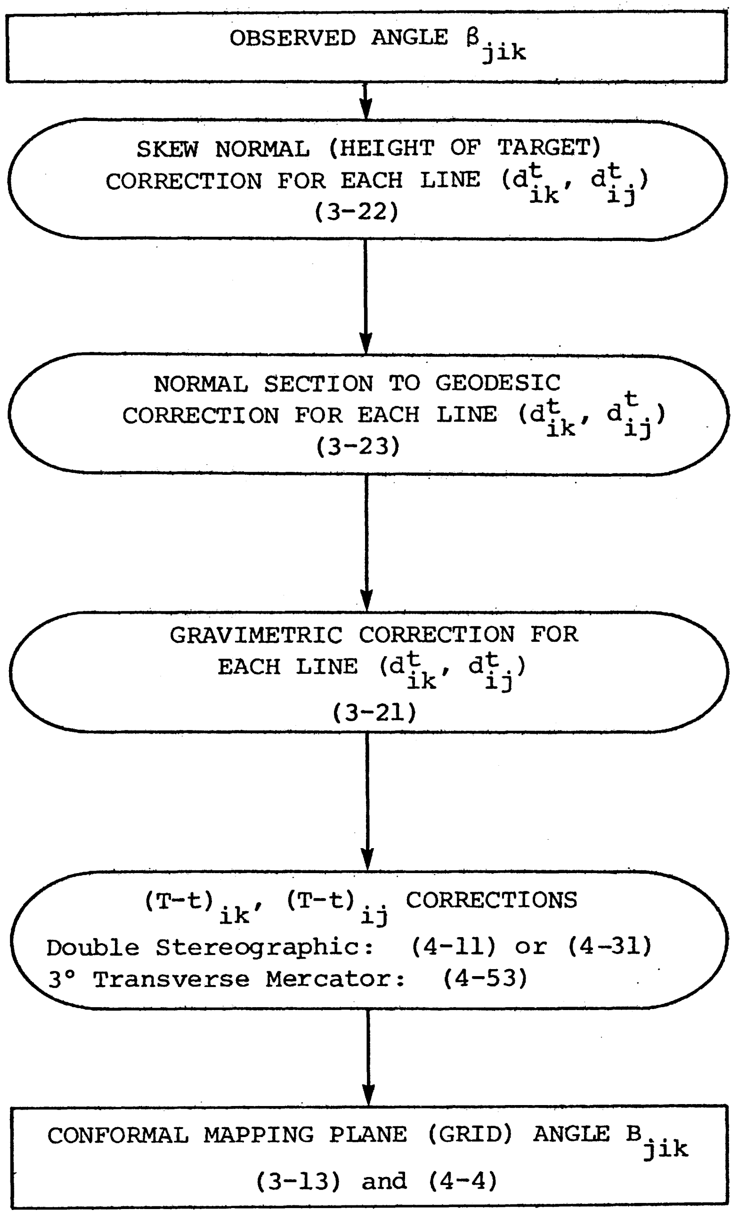


Figure 2.4

Reduction of an Observed Angle to a Conformal Mapping Plane

Status	Quantity	Remarks
Observed	β_{jik}: angle Z_{ij}, Z_{ik} (or ΔH_{ij}, ΔH_{ik}, or Δh_{ij}, Δh_{ik}	$\beta_{jik} = d^t_{ik} - d^t_{ij}$ needed for reduction purposes
Computed	(x^a_i, y^a_i), (x^a_j, y^a_j), (x^a_k, y^a_k) (ϕ^a_i, λ^a_i), (ϕ^a_j, λ^a_j), (ϕ^a_k, λ^a_k)	see Krakiwsky et al. [1977] re coordinate transformations.
	S^a_{ij}, S^a_{ik}: approx. ellipsoidal distances α^a_{ij}, α^a_{ik}: approx. geodetic azimuths	(3-39) or (3-62) (3-37) or (3-59) } use approximate coordinates for all computations
	Z_{ij}, Z_{ik}: reduced zenith distances	(3-14): use $\alpha^a_{ij}, \alpha^a_{ik}$ for these computations
	$(T-t)_{ij}$, $(T-t)_{ik}$	(4-11) or (4-31) for Double Stereographic (4-53) for 3° Transverse Mercator, use approximate coordinates for all computations
	h_j, h_k: ellipsoidal heights of targets	
Known	a,b (or a,f) ψ_o, Λ_o, R; ϕ_o, λ_o; $x_o, y_o; k_o$ H_i (or h_i): orthometric (or ellipsoidal) height of instrument	parameter of reference ellipsoid all defining parameters of the particular conformal mapping system.
	N^*_i, N^*_j, N^*_k: geoidal heights	required to determine Δh_{ij}, Δh_{ik}
	ξ_i, η_i: deflection of vertical components	

Table 2.11

Reduction of an Observed Angle to a Conformal Mapping Plane

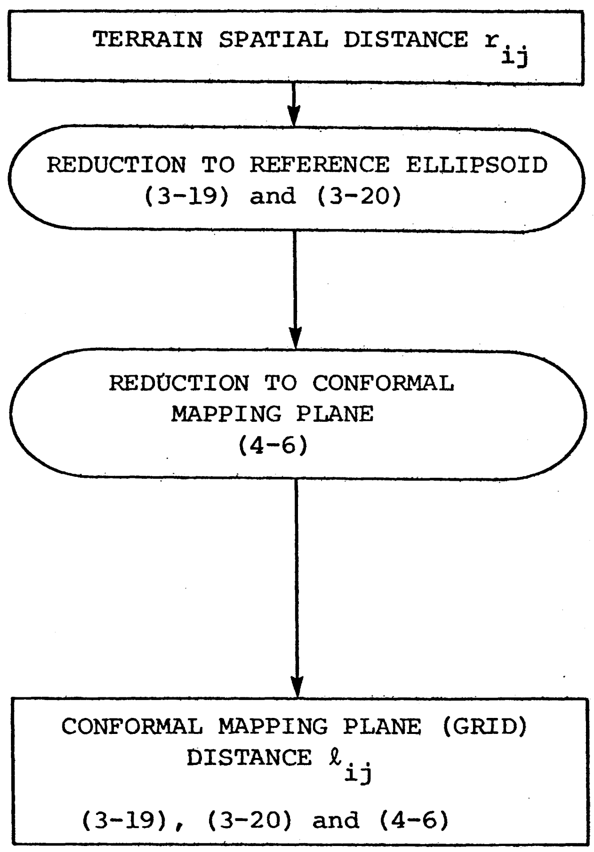


Figure 2.5

Reduction of a Terrain Spatial Distance to a Conformal Mapping Plane

Status	Quantity	Remarks
Observed	r_{ij}: terrain spatial distance Z_{ij} (or ΔH_{ij} or Δh_{ij})	Instrumental and atmospheric effects have been removed needed for reduction purposes.
Computed	$(x_i^a, y_i^a), (x_j^a, y_j^a)$ (ϕ_i^a, λ_i^a), (ϕ_j^a, λ_j^a)	see Krakiwsky et al. [1977] re coordinate transformations.
	α_{ij}^a, α_{ji}^a: approx. geodetic azimuths	(3-37), (3-35) and (3-36), or (3-59), (3-60) and (3-61); use approximate coordinates for all computations.
	h_j: ellipsoidal height of target	
	$\bar{k}_{ij}$: line scale factor	(4-13) or (4-33) for Double Stereographic, (4-57) for 3° Transverse Mercator; use approximate coordinates for all computations
Known	a,b (or a,f)	parameters of reference ellipsoid
	x_o, y_o; k_o	three parameters pertaining to the particular conformal mapping system
	H_i (or h_i): orthometric (or ellipsoidal) height of instrument	
	N_i^*, N_j^*: geoidal heights	required to determine Δh_{ij}

Table 2.12

Reduction of a Terrain Spatial Distance to a Conformal Mapping Plane used for the numerical computations in this report, one set of approximate coordinates are used for any one project.

2.3 Data Screening

Prior to being used in network computations, each piece of data (azimuth, direction, angle, distance) should be tested individually to ensure that it is self-consistent. These tests are accomplished by methods of univariate analysis, which means the examination of the repeated measurement of the same observable (e.g. a distance). These repeated measurements are represented by a data series ℓ_i, $i = 1, N$ where N is the sample size. The problem here is to discover which individual observations ℓ_i are statistically incompatible with the rest of the series. This subject is commonly known as the detection of outliers [e.g. Krakiwsky, 1978; Pope, 1976].

The specific test which is used to detect outliers depends on the underlying assumptions about the population mean μ and population variance σ^2. If μ and σ^2 are assumed unknown, they are estimated by the sample mean $\bar{\ell}$ and sample variance S^2. The following interpretations can be made: (a) 'μ known' corresponds to measuring a line of known length (e.g. a a calibration baseline); (b) 'σ^2 known' corresponds to measuring with an instrument of known accuracy; (c) 'μ unknown' corresponds to measuring a line of unknown length; (d) 'σ^2 unknown' corresponds to measuring with an instrument of unknown accuracy. The four possible combinations of the above cases are shown in Table 2.13.

The so-called null hypothesis H_o being tested is

$$H_o : \ell_i \text{ is a member of a sample with normal distribution.}$$

Name	Situation θ_1	Situation θ_2	H_o (null hypothesis)	Statistic y	pdf* $\phi(y)$	1-a** Confidence Interval for the Quantity Tested	Remarks
Normal Test of a Single Observation	μ known	σ^2 known	ℓ_i belongs to a sample having the pdf $\mathcal{N}(\ell;\ \mu,\ \sigma^2)$	$\frac{\ell - \mu}{\sigma}$	standard normal n(0,1)	$\mu - \sigma\ n_{\frac{a}{2}} < \ell_i < \mu + \sigma\ n_{1-\frac{a}{2}}$	σ known thus the normal distribution.
Student's t Test of a Single Observation	μ known	s^2	ℓ_i belongs to a sample having the pdf $\mathcal{N}(\ell;\ \mu,\ s^2)$	$\frac{\ell - \mu}{s}$	Student's t t_{N-1}	$\mu - s\ t_{N-1,\frac{a}{2}} < \ell_i < \mu + s\ t_{N-1,1-\frac{a}{2}}$	s is computed using $\overline{\ell}$ estimated from sample of size N thus t distribution.
Normal Test of a Single Observation	$\overline{\ell}$	σ^2 known	ℓ_i belongs to a sample having the pdf $\mathcal{N}(\ell;\ \overline{\ell},\ \sigma^2)$	$\frac{\ell - \overline{\ell}}{\left(\frac{N-1}{N}\right)^{1/2}\sigma}$	standard normal n(0,1)	$\overline{\ell}-\left(\frac{N-1}{N}\right)^{\frac{1}{2}}\sigma\ n_{\frac{a}{2}}<\ell_i<\overline{\ell}+\left(\frac{N-1}{N}\right)^{\frac{1}{2}}\sigma\ n_{1-\frac{a}{2}}$	σ known thus the normal distribution.
τ Test of a Single Observation	$\overline{\ell}$	s^2	ℓ_i belongs to a sample having the pdf $\mathcal{N}(\ell;\ \overline{\ell},\ s^2)$	$\frac{\ell - \overline{\ell}}{\left(\frac{N-1}{N}\right)^{1/2}s}$	Tau τ_{N-1}	$\overline{\ell}-\left(\frac{N-1}{N}\right)^{\frac{1}{2}}s\tau_{N-1,\frac{a}{2}}<\ell_i<\overline{\ell}+\left(\frac{N-1}{N}\right)^{\frac{1}{2}}s\tau_{N-1,1-\frac{a}{2}}$	$\overline{\ell}$ and s computed from the same sample thus the τ distribution.

* Pope [1976]. n(0,1) - standard normal distribution of 0 mean and variance 1.
t_{N-1} - student's t distribution with N-1 degrees of freedom.
τ_N - tau distribution with N-1 degrees of freedom.

** a = α/N, where N is the number of members in the series.

Table 2.13

Testing for Outliers

The test which is applicable most often is for σ^2 known and μ unknown. This corresponds to the third test in Table 2.13, i.e.

$$\bar{\ell} - (\frac{N-1}{N})^{1/2} \sigma \, n_{1-a/2} < \ell_i < \bar{\ell} + (\frac{N-1}{N})^{1/2} \sigma \, n_{1-a/2} \, , \qquad (2\text{-}19)$$

where $\bar{\ell}$ = sample mean,

N = sample size,

σ = known standard deviation,

n = standard normal distribution (see Table 2.14),

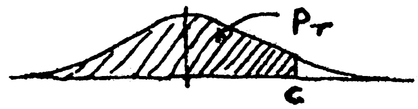
$a = \alpha/N$ for α = significance level (e.g. $\alpha = 0.5$).

If the observation ℓ_i being tested does not lie within the limits given by equation (2-19), then the null hypothesis H_o is rejected at the $(1-\alpha)\%$ confidence level.

For example, Table 2.15 lists 11 observed values of astronomic azimuth for the same line. The mean value of the series is $\bar{\ell} = 85° \, 36' \, 18.''71$, and the assumed known standard deviation $\sigma = 5.''77$. For a significance level $\alpha = 0.05$, $a/2 = \alpha/2N = 0.05/22 = 2.273 \cdot 10^{-3}$, and $1 - a/2 = 0.99773$. From Table 2.14, the value for $n_{1-a/2}$ is 2.83. Thus, the rejection limits for an individual observation ℓ_i are

$$85° \, 36' \, 03.''14 < \ell_i < 85° \, 36' \, 34.''28 \, . \qquad (2\text{-}20)$$

Performing the test for each astronomic azimuth in Table 2-15, it is seen that azimuth numbers 1, 2, 3 and 10 are rejected at the 95% confidence level, i.e. the hypothesis that they are members of a sample with normal distribution is rejected. Thus, only the seven remaining azimuths are taken as representative of the sample, and the mean value $\bar{\ell} = 85° \, 36' \, 25.''14$ computed from these seven remaining azimuths is used for further computations.



Values of Pr corresponding to c for the normal curve.
The value of Pr for (-c) equals one minus the value of Pr for (+c).

c	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Table 2.14

Cumulative Normal Distribution - Values of Pr

SET NO.	OBS'D AZIMUTH		
1	85	35	27.84
2	85	35	34.36
3	85	36	51.15
4	85	36	27.95
5	85	36	28.89
6	85	36	28.43
7	85	36	20.46
8	85	36	22.29
9	85	36	24.02
10	85	36	36.46
11	85	36	23.93

Table 2.15 Astronomic Azimuth Data Series

If the underlying assumptions for the univariate test are different (e.g. σ unknown, μ known), then one of the three other tests of Table 2.13 should be used. Essentially, this changes only the rejection limits (cf. eq. (2-20)). The necessary tables (i.e. Student's t and tau distributions) can be found in e.g. Rainsford [1957] and Pope [1976].

3. MATHEMATICAL MODELS FOR AZIMUTH, DIRECTION AND ANGLE OBSERVATIONS

In this chapter, the mathematical models relating angular observations and coordinates are given. Coordinates are the x(easting) and y(northing) coordinates referred to a conformal mapping plane (e.g. Krakiwsky et al., 1977; Thomson et al., 1978]. Both the linear and nonlinear forms of the mathematical model for azimuths, directions, and angles are given.

3.1 Azimuth Mathematical Model

The nonlinear form of the azimuth mathematical model is

$$F_{ij} = \arctan\left(\frac{x_j - x_i}{y_j - y_i}\right) - t_{ij} = 0 \quad , \tag{3-1}$$

where the first term is a nonlinear function of the coordinates of two points i and j (see Figure 3.1), and t_{ij} is the observed azimuth from point i to point j reduced to the mapping plane [e.g. Thomson et al., 1978, section 4.2.3]. A linear Taylor series (see Appendix I) is used to approximate this nonlinear model. The resulting equation is

$$F_{ij} = F^o_{ij} + dF_{ij} = \arctan\left(\frac{x^o_j - x^o_i}{y^o_j - y^o_i}\right) - t_{ij} + dt_{ij} - V_{t_{ij}} + \ldots = 0 \, , \tag{3-2}$$

where $\arctan\left(\frac{x^o_j - x^o_i}{y^o_j - y^o_i}\right)$ = computed value of the azimuth based on approximate values of the coordinates (x^o, y^o),

dt_{ij} = differential change in the computed azimuth resulting from differential changes in the approximate coordinates (see eq. (3-3)),

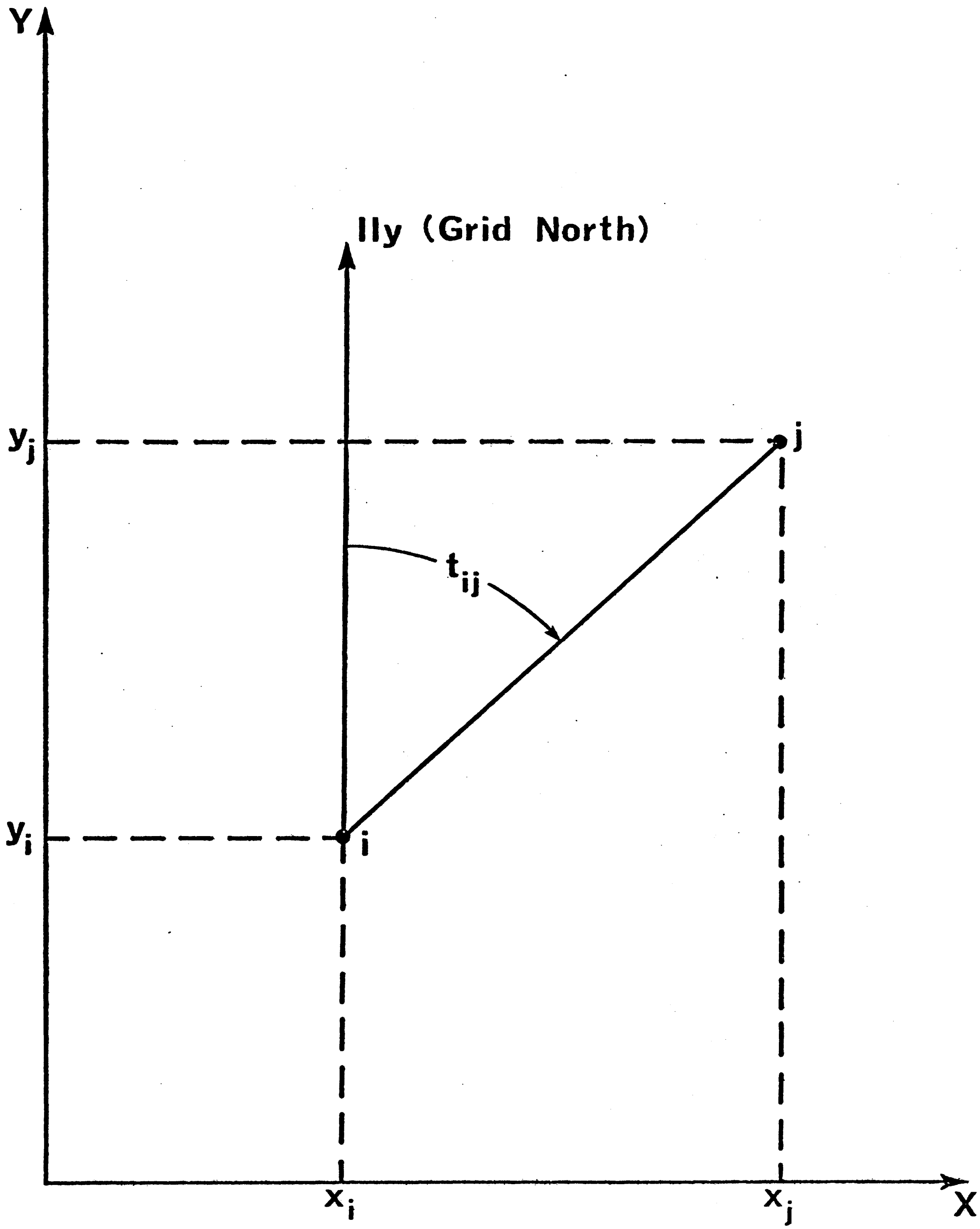


Figure 3.1 Mapping Plane Azimuth

$V_{t_{ij}}$ = correction to the observed mapping plane (grid) azimuth.

The differential change in azimuth dt_{ij} is given as

$$dt_{ij} = \frac{\partial t_{ij}}{\partial x_i} dx_i + \frac{\partial t_{ij}}{\partial y_i} dy_i + \frac{\partial t_{ij}}{\partial x_j} dx_j + \frac{\partial t_{ij}}{\partial y_j} dy_j \quad . \tag{3-3}$$

Evaluation of the partial derivatives in equation (3-3) yields

$$\frac{\partial t_{ij}}{\partial x_i} = \frac{-(y_j^o - y_i^o)}{(\ell_{ij}^o)^2} = a_{ij} \quad , \tag{3-4}$$

$$\frac{\partial t_{ij}}{\partial y_i} = \frac{(x_j^o - x_i^o)}{(\ell_{ij}^o)^2} = b_{ij} \quad , \tag{3-5}$$

$$\frac{\partial t_{ij}}{\partial x_j} = \frac{(y_j^o - y_i^o)}{(\ell_{ij})^2} = -a_{ij} \quad , \tag{3-6}$$

$$\frac{\partial t_{ij}}{\partial y_j} = \frac{-(x_j^o - x_i^o)}{(\ell_{ij}^o)^2} = -b_{ij} \quad , \tag{3-7}$$

where ℓ_{ij}^o is the mapping plane distance between points i and j computed using the approximate coordinates. Substituting equations (3-3) to (3-7) into equation (3-2) yields the so-called observation equation as

$$V''_{t_{ij}} = \left[\arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - t_{ij}\right]'' + \rho'' a_{ij}\delta x_i + \rho'' b_{ij}\delta y_i - \rho'' a_{ij}\delta x_j - \rho'' b_{ij}\delta y_j , \tag{3-8}$$

where $\rho'' = 206264''.8062$ is used to proportion the elements, and

$\delta x_i \rightarrow \delta y_j$ = differential changes in the coordinates.

Converting equation (3-8) to matrix notation yields the matrix form of the observation equation as

$$V''_{t_{ij}} = [\arctan (\frac{x_j^o - x_i^o}{y_j^o - y_i^o}) - t_{ij}]'' + \rho'' [a_{ij} \; b_{ij} \; -a_{ij} \; -b_{ij}] \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \end{bmatrix} . \qquad (3\text{-}9)$$

The matrix form of the evaluated partial derivatives (i.e. a_{ij}, b_{ij}, etc.) is called the design matrix A, the difference between computed and observed azimuths is called the misclosure vector W, the vector of differential changes in coordinates (δx, δy) is called the solution vector $\hat{X}$, and the correction to the observed azimuth is called the residual vector V. Rewriting equation (3-9) in this notation yields

$$\underset{(1,1)}{V_{t_{ij}}} = \underset{(1,1)}{W_{t_{ij}}} + \underset{(1,4)}{A_{t_{ij}}} \; \underset{(4,1)}{\hat{X}} . \qquad (3\text{-}10)$$

3.2 Direction Mathematical Model

Direction observations are relative to the 'zero' direction of the horizontal circle of a theodolite. The azimuth of this zero direction is called the orientation unknown Z (see Figure 3.2) and it must be solved for along with the unknown coordinates. Orientation unknowns are not desired quantities and thus are called nuisance parameters. The nonlinear form of the direction mathematical model is

$$F_{ij} = \arctan (\frac{x_j - x_i}{y_j - y_i}) - (d_{ij} + Z_i) = 0, \qquad (3\text{-}11)$$

where d_{ij} = observed direction from point i to point j reduced to the mapping plane [e.g. Thomson et al., 1978, section 4.2.1],

Z_i = orientation unknown at point i.

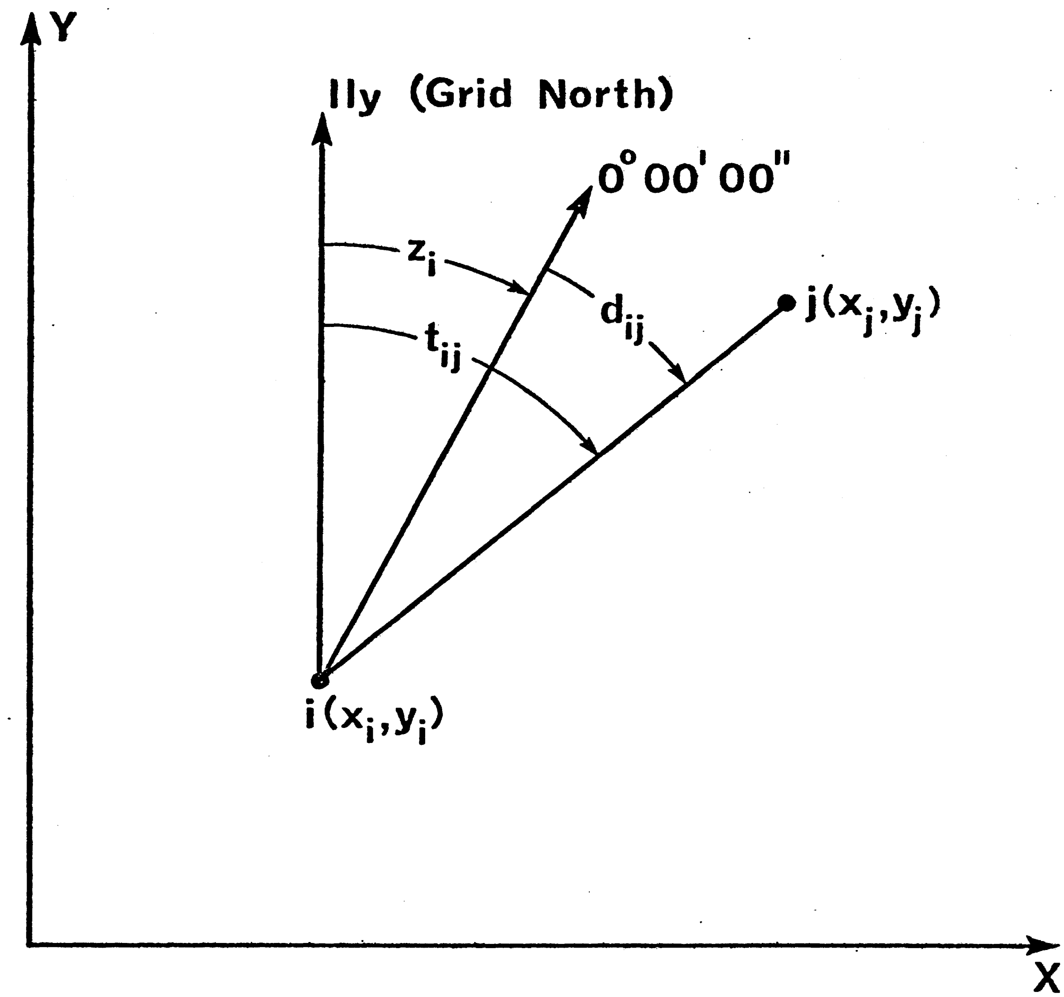


Figure 3.2 Direction on the Mapping Plane

The linear Taylor series expansion of equation (3-11) gives the linearized direction mathematical model as

$$F_{ij} = \arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - Z_i^o - d_{ij} + dt_{ij} - dZ_i - V_{d_{ij}} + \ldots = 0 , \quad (3\text{-}12)$$

where Z_i^o = approximate value of the orientation unknown,

dZ_i = differential change in the orientation unknown Z_i caused by an inaccurate approximate value Z_i^o ,

$V_{d_{ij}}$ = correction to the observed direction.

The approximate value of the orientation unknown Z_i^o is obtained by subtracting the observed direction to a station j from the azimuth to the same station computed from the approximate coordinates, i.e.

$$Z_i^o = \arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - d_{ij} \quad . \quad (3\text{-}13)$$

Realizing that dt_{ij} has already been evaluated for the linearized azimuth mathematical model (see eqs. (3-3) to (3-7)), the observation equation for a direction is

$$V''_{d_{ij}} = [\arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - Z_i^o - d_{ij}]'' + \rho'' a_{ij}\delta x_i + \rho'' b_{ij}\delta y_i - \rho'' a_{ij}\delta x_j - \rho'' b_{ij}\delta y_j - \delta Z_i \quad , \quad (3\text{-}14)$$

or, in matrix form

$$V''_{d_{ij}} = [\arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - Z_i^o - d_{ij}]'' + \rho'' \begin{bmatrix} a_{ij} & b_{ij} & -a_{ij} & -b_{ij} & -1 \end{bmatrix} \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \\ \delta Z_i \end{bmatrix} . \quad (3\text{-}15)$$

Using the symbolic matrix notation of section 3.1, equation (3-15) becomes

$$\underset{(1,1)}{V_{d_{ij}}} = \underset{(1,1)}{W_{d_{ij}}} + \underset{(1,5)}{A_{d_{ij}}} \underset{(5,1)}{\hat{X}} \quad .$$

3.3 Angle Mathematical Model

The nonlinear mathematical model for an angle is (see Figure 3.3)

$$F_{ijk} = \arctan\left(\frac{x_k - x_i}{y_k - y_i}\right) - \arctan\left(\frac{x_j - x_i}{y_j - y_i}\right) - B_{ijk} = 0 \, , \qquad (3\text{-}17)$$

where B_{ijk} = angle observed at point i from point j to point k reduced to the mapping plane [e.g. Thomson et al., 1978, section 4.2.2].

The difference between two direction mathematical models F_{ik} and F_{ij} has the same form as the angle mathematical model F_{ijk}. The linearized form of the angle mathematical model is

$$F_{ijk} = \arctan\left(\frac{x_k^o - x_i^o}{y_k^o - y_i^o}\right) - \arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - B_{ijk} + dB_{ijk} - V_{B_{ijk}} + \ldots = 0, \qquad (3\text{-}18)$$

where dB_{ijk} = differential change in the computed angle resulting from differential changes in the approximate coordinates,

$V_{B_{ijk}}$ = correction to the observed mapping plane angle.

The differential change in the angle dB_{ijk} is given as

$$dB_{ijk} = \frac{\partial B_{ijk}}{\partial x_i} dx_i + \frac{\partial B_{ijk}}{\partial y_i} dy_i + \frac{\partial B_{ijk}}{\partial x_j} dx_j + \frac{\partial B_{ijk}}{\partial y_j} dy_j + \frac{\partial B_{ijk}}{\partial x_k} dx_k + \frac{\partial B_{ijk}}{\partial y_k} dy_k \, . \qquad (3\text{-}19)$$

Evaluation of the partial derivatives in equation (3-19) yields

$$\frac{\partial B_{ijk}}{\partial x_i} = \frac{-(y_k^o - y_i^o)}{(\ell_{ik}^o)^2} + \frac{(y_j^o - y_i^o)}{(\ell_{ij}^o)^2} = c_{ijk} \, , \qquad (3\text{-}20)$$

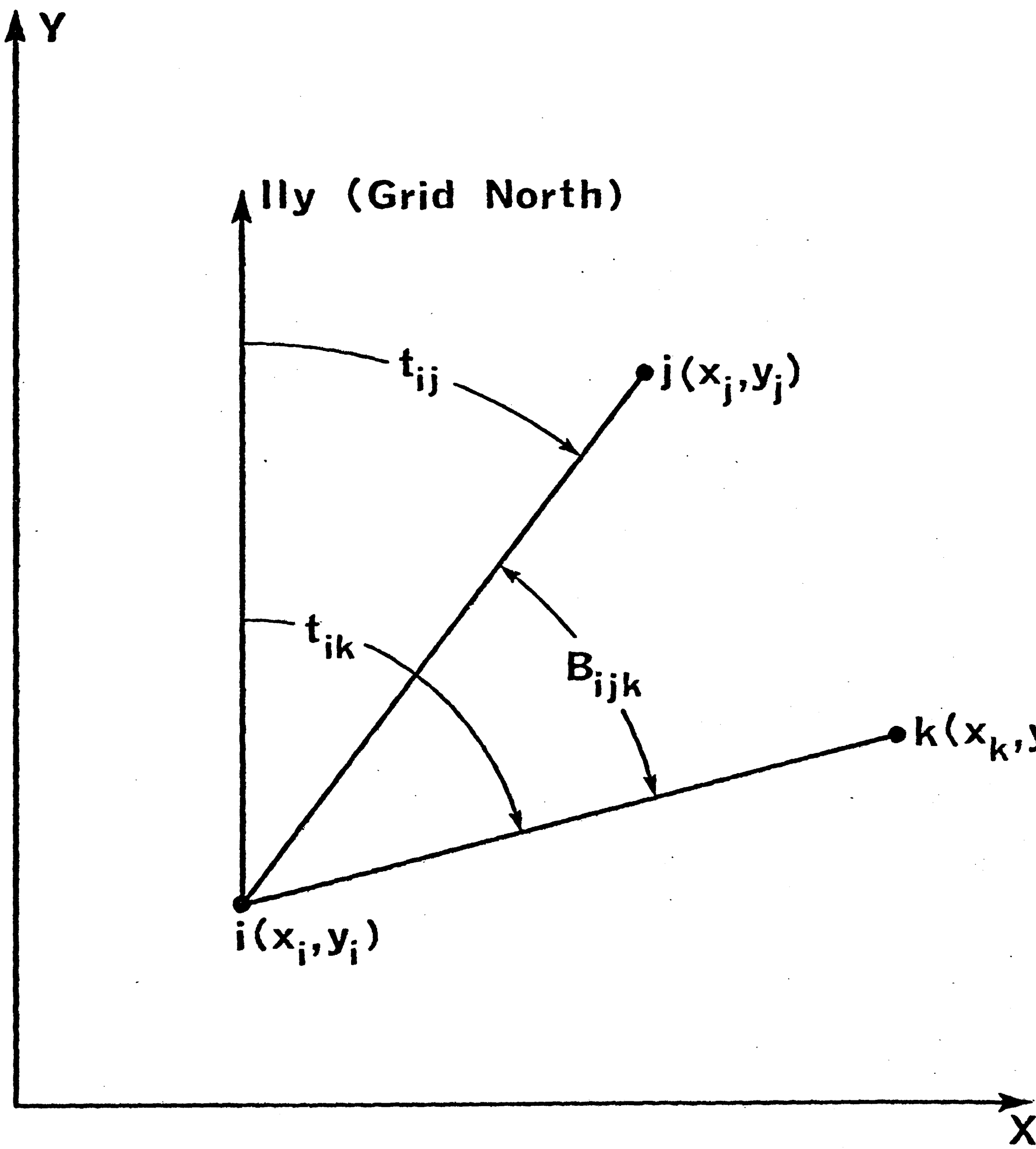


Figure 3.3 Angle on the Mapping Plane

$$\frac{\partial B_{ijk}}{\partial y_i} = \frac{(x_k^o - x_i^o)}{(\ell_{ik}^o)^2} - \frac{(x_j^o - x_i^o)}{(\ell_{ij}^o)^2} = d_{ijk} \quad , \qquad (3\text{-}21)$$

$$\frac{\partial B_{ijk}}{\partial x_j} = \frac{-(y_j^o - y_i^o)}{(\ell_{ij}^o)^2} = a_{ij} \quad , \qquad (3\text{-}22)$$

$$\frac{\partial B_{ijk}}{\partial y_j} = \frac{(x_j^o - x_i^o)}{(\ell_{ij}^o)^2} = b_{ij} \quad , \qquad (3\text{-}23)$$

$$\frac{\partial B_{ijk}}{\partial x_k} = \frac{(y_k^o - y_i^o)}{(\ell_{ik}^o)} = -a_{ik} \quad , \qquad (3\text{-}24)$$

$$\frac{\partial B_{ijk}}{\partial y_k} = \frac{-(x_k^o - x_i^o)}{(\ell_{ik}^o)^2} = -b_{ik} \quad , \qquad (3\text{-}25)$$

where ℓ_{ik}^o is the mapping plane distance between points i and k computed using the approximate coordinates. Substituting the above values into equations (3-19) and subsequently evaluating the linearized mathematical model (eq. (3-18)) gives the observation equation as

$$V''_{B_{ijk}} = [\arctan\left(\frac{x_k^o - x_i^o}{y_k^o - y_i^o}\right) - \arctan\left(\frac{x_j^o - x_i^o}{y_j^o - y_i^o}\right) - B_{ijk}]'' + \rho'' \; c_{ijk}\delta x_i + \rho'' d_{ijk}\,\delta y_i$$

$$+ \rho'' a_{ij}\delta x_j + \rho'' b_{ij}\delta y_j - \rho'' a_{ik}\delta x_k - \rho'' b_{ik}\delta y_k \quad , \qquad (3\text{-}26)$$

or, in matrix form

$$V''_{B_{ijk}} = [\arctan(\frac{x_k^o - x_i^o}{y_k^o - y_i^o}) - \arctan(\frac{x_j - x_i}{y_j^o - y_i^o}) - B_{ijk}] + \rho'' \; [c_{ijk} \; d_{ijk} \; a_{ij} \; b_{ij} \; -a_{ik} \; -b_{ik}] \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \\ \delta x_k \\ \delta y_k \end{bmatrix} . \quad (3\text{-}27)$$

Again, using the symbolic matrix notation, equation (3-27) becomes

$$\underset{(1,1)}{V_{B_{ijk}}} = \underset{(1,1)}{W_{B_{ijk}}} + \underset{(1,6)}{A_{B_{ijk}}} \; \underset{(6,1)}{\hat{X}} . \quad (3\text{-}28)$$

4. MATHEMATICAL MODELS FOR DISTANCE OBSERVATIONS

This chapter describes the nonlinear and linearized mathematical models relating plane coordinates (x,y) to observed distances ℓ_{ij} from point i to point j reduced to a conformal mapping plane [e.g. Thomson et al., 1978, section 4.2.4]. The nonlinear form of the distance mathematical model is (see Figure 4.1)

$$F_{ij} = ((x_j-x_i)^2 + (y_j-y_i)^2)^{1/2} - \ell_{ij} = 0 \; . \qquad (4\text{-}1) \; (4\text{-}1)$$

Linearization of equation (4-1) by a linear Taylor series (see Appendix I) expansion gives the linearized form of the distance mathematical model as

$$F_{ij} = F^o_{ij} + dF_{ij} = ((x^o_j-x^o_i)^2+(y_j-y_i)^2)^{1/2} - \ell_{ij} + d\ell_{ij} - V_{\ell_{ij}} + \ldots = 0 \; , \qquad (4\text{-}2)$$

where $((x^o_j-x^o_i)^2 + (y^o_j-y^o_i)^2)^{1/2}$ = computed value of the distance based on approximate values of the coordinates (x^o,y^o),

$d\ell_{ij}$ = differential change in the computed distance resulting from differential changes in the approximate coordinates (see eq. (4-3)),

$V_{\ell_{ij}}$ = correction to the observed mapping plane distance.

The differential change in distance $d\ell_{ij}$ is given as

$$d\ell_{ij} = \frac{\partial \ell_{ij}}{\partial x_i} dx_i + \frac{\partial \ell_{ij}}{\partial y_i} dy_i + \frac{\partial \ell_{ij}}{\partial x_j} dx_j + \frac{\partial \ell_{ij}}{\partial y_j} dy_j \; , \qquad (4\text{-}3)$$

where the partial derivatives are

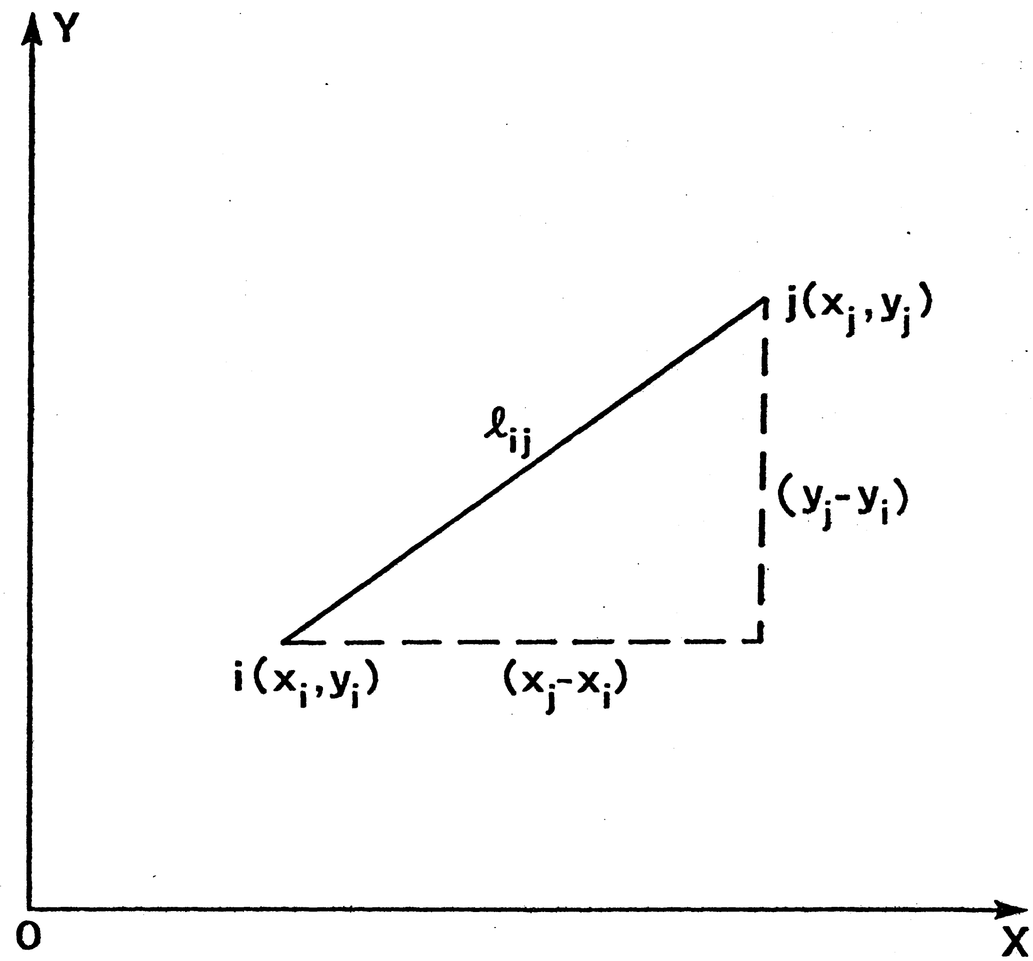


Figure 4.1 Distance on the Mapping Plane

$$\frac{\partial \ell_{ij}}{\partial x_i} = \frac{-(x_j^o - x_i^o)}{\ell_{ij}^o} = e_{ij} \quad , \tag{4-4}$$

$$\frac{\partial \ell_{ij}}{\partial y_i} = \frac{-(y_j - y_i)}{\ell_{ij}^o} = f_{ij} \quad , \tag{4-5}$$

$$\frac{\partial \ell_{ij}}{\partial x_j} = \frac{(x_j^o - x_i^o)}{\ell_{ij}^o} = -e_{ij} \quad , \tag{4-6}$$

$$\frac{\partial \ell_{ij}}{\partial y_j} = \frac{(y_j^o - y_i^o)}{\ell_{ij}^o} = -f_{ij} \quad , \tag{4-7}$$

where ℓ_{ij}^o is the mapping plane distance between points i and j computed using the approximate coordinates. Substituting these partial derivatives into equation (4-3) to obtain $d\ell_{ij}$, and, in turn, substituting $d\ell_{ij}$ back into equation (4-2) yields the distance observation equation as

$$V_{\ell_{ij}} = [((x_j^o - x_i^o)^2 + (y_j^o - y_i^o)^2)^{1/2} - \ell_{ij}] + e_{ij}\delta x_i + f_{ij}\delta y_i - e_{ij}\delta x_j - f_{ij}\delta y_j , \tag{4-8}$$

or in matrix form

$$V_{\ell_{ij}} = [((x_j^o - x_i^o)^2 + (y_j^o - y_i^o)^2)^{1/2} - \ell_{ij}] + [e_{ij} \; f_{ij} \; - e_{ij} \; - f_{ij}] \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \end{bmatrix} . \tag{4-9}$$

The units of both equations (4-8) and (4-9) are metres. Converting the observation equation into symbolic matrix notation as in section (3-1) (cf. eq. (3-10)) yields

$$\underset{(1,1)}{V_{\ell_{ij}}} = \underset{(1,1)}{W_{\ell_{ij}}} + \underset{(1,4)}{A_{\ell_{ij}}} \; \underset{(4,1)}{\hat{X}} \; . \tag{4-10}$$

5. SOLUTION OF UNIQUE CASES

This chapter covers the unique cases (i.e number of observations n = number of parameters u) of coordinate determination encountered in practice (e.g. direct problem, intersection, resection, traverse). The inverse (explicit) mathematical models developed in chapters 3 and 4 are combined using the method of least squares (see Appendix II) to solve these unique cases. This leads to a unified approach when the overdetermined (n>u) cases are considered in chapter 6. For treatment of these unique cases by the direct method (cf. eq. (1-1)), the reader is referred to e.g. Faig [1972], Thomson et al. [1978], Richardus [1974].

For all of the examples considered in this chapter, it is assumed that the observations have been reduced to a conformal mapping plane as explained in section 2.2. Thus, although the examples are not explicitly spelled out for each of the three existing Maritime conformal mapping planes [Krakiwsky et al., 1977], the methods used are equally applicable to all three provinces. The only differences are in the actual values of the initial approximate coordinates and the final adjusted coordinates.

All of the examples in this and following chapters have been performed by program GEOPAN [Steeves, 1978].

5.1 Direct Problem

The direct problem considered here (see Figure 5.1) is essentially the same as that used in section 4.8.1 of Thomson et al. [1978]. The only difference is that here point 1 is considered fixed (i.e. its covariance matrix is zero) whereas in Thomson et al. [1978], point 1 had a covariance matrix associated with it.

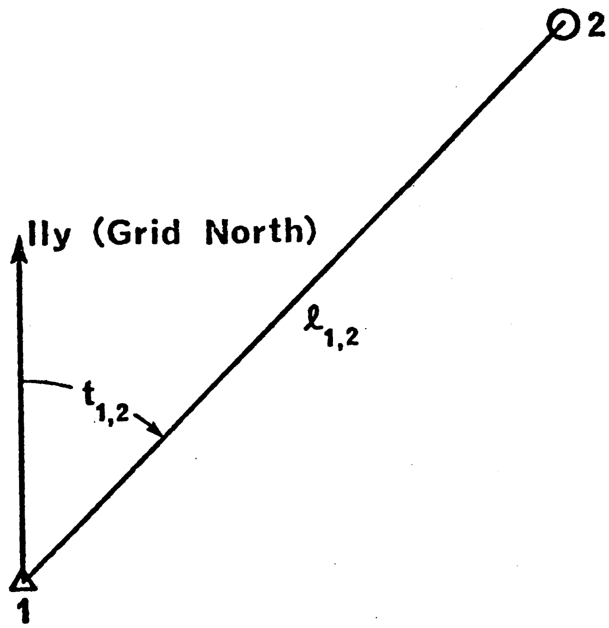


Figure 5.1 Direct Problem

The approximate coordinates of point 2, the fixed coordinates of point 1, the reduced observations and their standard deviations are given in Table 5.1. The standard deviations are derived through the formulae developed in section 2.1. For instance, the standard

Coordinates of Points			Observations on the Mapping Plane				
Station	X(m)	Y(m)	Type	From	To	Value	σ
1	377164.887	862395.774	Az.	1	2	44° 15'28.''97	5.''0
2	378907.0	864184.0	Dist.	1	2	2496.423 m	0.03 m

Table 5.1. Initial Data for Direct Problem

deviation of 5".0 for the observed azimuth could result from the azimuth observed three times by the hour angle method with a 1" theodolite and $\sigma_t = 1.0s$ (cf. Table 2.1). Similarly, the distance standard deviation of 0.03 m could result (assuming $\sigma_r = \sigma_\ell$) from the distance observed by a CA 1000 with $\sigma_p = 5$ mbar and $\sigma_T = \sigma_{\Delta T} = 1°C$ (cf. Table 2.8). The approximate coordinates are determined graphically or analytically as suggested in section 2.2.

The mathematical model used here is a combination of equations (3-10) and (4-10). The residual vector V is defined as

$$\underset{(2,1)}{V} = \underset{(2,1)}{W} + \underset{(2,2)}{A}\ \underset{(2,1)}{\hat{X}},$$

or explicitly

$$\underset{(2,1)}{V} = \begin{bmatrix} \arctan\left(\frac{x_2^o - x_1}{y_2^o - y_1}\right) - t_{1,2} \\ \\ ((x_2^o - x_1)^2 + (y_2^o - y_1)^2)^{1/2} - \ell_{1,2} \end{bmatrix} + \begin{bmatrix} \rho''\frac{(y_2^o - y_1)}{(\ell_{1,2}^o)^2} & \rho''\frac{-(x_2^o - x_1)}{(\ell_{1,2}^o)^2} \\ \frac{(x_2^o - x_1)}{\ell_{1,2}^o} & \frac{(y_2^o - y_1)}{\ell_{1,2}^o} \end{bmatrix} \begin{bmatrix} \delta x_2 \\ \delta y_2 \end{bmatrix}$$

(5-2)

where the units are

$$\underset{(2,1)}{V} = \begin{bmatrix} '' \\ m \end{bmatrix} + \begin{bmatrix} ''m^{-1} & ''m^{-1} \\ - & - \end{bmatrix} \begin{bmatrix} m \\ m \end{bmatrix}. \qquad (5\text{-}3)$$

The solution $\hat{X}$ is given from Appendix II, equation (AIII-11) as

$$\hat{X} = -[A^T P A]^{-1} A^T P W. \qquad (5\text{-}4)$$

Thus, using the above coordinates and the observations with their standard deviations, the A, P and W matrices are numerically evaluated yielding

$$A = \begin{bmatrix} 59.17941 & -57.65335 \\ 0.6978111 & 0.7162819 \end{bmatrix}, \qquad W = \begin{bmatrix} -23.0324 \\ 0.11655 \end{bmatrix},$$

and, assuming $\sigma_o^2 = 1$ (cf. eq. (AII-3)),

$$P = \begin{bmatrix} 0.04 & 0 \\ 0 & 1111.11 \end{bmatrix} \text{ in units of } \begin{bmatrix} ('')^{-2} & \\ & m^{-2} \end{bmatrix}.$$

Using these matrices, evaluation of equation (5-4) yields

$$\hat{X} = \begin{bmatrix} 0.11835 \text{ m} \\ -0.27802 \text{ m} \end{bmatrix}$$

for the first iteration, and the corresponding values of the parameters are (eq. (AII-12))

$$X = X^o + \hat{X} = \begin{bmatrix} 378907.0 \\ 864184.0 \end{bmatrix} + \begin{bmatrix} 0.11835 \\ -0.27802 \end{bmatrix} = \begin{bmatrix} 378907.118 \text{ m} \\ 864183.722 \text{ m} \end{bmatrix}. \tag{5-5}$$

These values for the parameters are now taken as new approximate coordinates, and the A and W matrices are reevaluated. They are

$$A = \begin{bmatrix} 59.17573 & -57.66265 \\ 0.6978911 & 0.7162039 \end{bmatrix}, \quad W = \begin{bmatrix} 0.001075 \\ 0.000016 \end{bmatrix}.$$

Evaluating equation (5-4) for the second time (iteration) yields

$$\hat{X} = \begin{bmatrix} -0.00002 \\ 0.00000 \end{bmatrix},$$

which is insignificant (i.e. less than 0.001 m), and thus the solution has converged. The final least squares estimate of the coordinates of point 2 are

$$X = \begin{bmatrix} x_2 \\ \\ y_2 \end{bmatrix} = \begin{bmatrix} 378907.118 \quad m \\ \\ 864183.722 \quad m \end{bmatrix}.$$

These are identical to the results obtained in Thomson et al. [1978], section 4.8.1.

The variance covariance matrix C_x of the parameters is given by equation (AII-16) as

$$C_x = [A^T C_L^{-1} A]^{-1} . \qquad (5\text{-}6)$$

If $\hat{X}$ has been computed according to formula (5-4), then this is a by-product of computing the solution vector $\hat{X}$. In this case it is

$$C_x = \begin{bmatrix} 0.2305 \cdot 10^{-2} & -0.13925 \cdot 10^{-2} \\ -0.13925 \cdot 10^{-2} & 0.2233 \cdot 10^{-2} \end{bmatrix} ,$$

in units of m^2. Computing the standard error ellipse according to formulae (AIII-8), (AIII-5), (AIII-6) and (AIII-14) gives

$$a_s = 0.061 \text{ m},$$

$$b_s = 0.030 \text{ m},$$

$$\theta = -45° \, 44' \, 32''.$$

Assuming the a priori variance factor known, the c factor to increase the confidence level to 95% is (see Table AIII.1)

$$c = (\chi^2_{2,0.95})^{1/2} = (5.99)^{1/2} = 2.45 .$$

Thus, the 95% confidence ellipse has a semi-major axis

$$a = 0.149 \text{ m} ,$$

and a semi-minor axis

$$b = 0.074 \text{ m} .$$

The orientation remains the same as the standard ellipse. The 95% confidence ellipse is depicted in Figure 5.2.

In this as in all of the unique cases, the residual vector V computed (eq. (AII-17)) after the final iteration is equal to zero. The observations can give only one value for the parameters, and thus there are no residual corrections for the observations. The a posteriori variance factor $\hat{\sigma}_o^2$ is zero in this case as well.

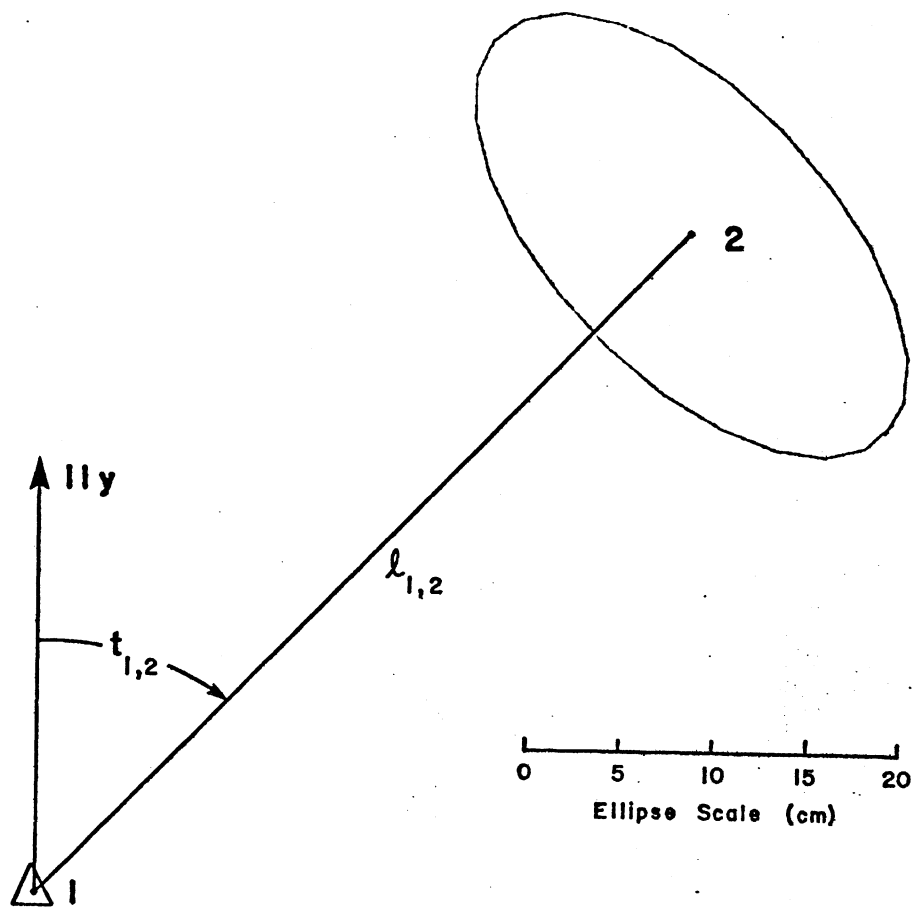


Figure 5.2 Confidence Ellipse for Direct Problem

5.2 Azimuth Intersection

Figure 5.3 shows the azimuth intersection example considered in this section. The mathematical model used here is that of

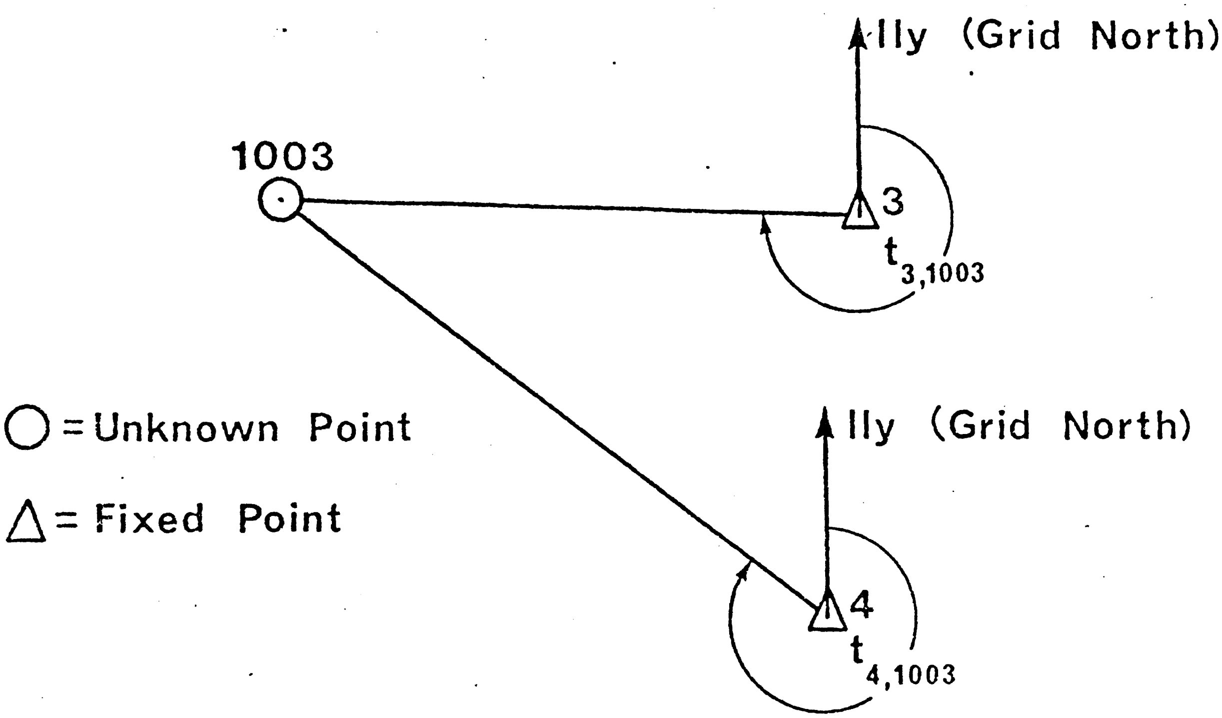


Figure 5.3 Azimuth Intersection on the Plane

section 3.1, specifically equations (3-8), (3-9) and (3-10). In this case, there are two azimuth observations (namely $t_{3,1003}$ and $t_{4,1003}$) and two unknowns (coordinates of station 1003). Thus, the matrix form of the observation equation is

$$\underset{(2,1)}{V} = \underset{(2,1)}{W} + \underset{(2,2)}{A}\ \underset{(2,1)}{\hat{X}} , \qquad (5\text{-}8)$$

or, explicitly

$$\underset{(2,1)}{V} = \begin{bmatrix} \arctan \dfrac{x^o_{1003}-x_3}{y^o_{1003}-y_3} - t_{3,1003} \\ \arctan \dfrac{x^o_{1003}-x_4}{y^o_{1003}-y_4} - t_{4,1003} \end{bmatrix} + \rho'' \begin{bmatrix} \dfrac{y^o_{1003}-y_3}{(\ell^o_{3,1003})^2} & \dfrac{(x^o_{1003}-x_3)}{(\ell^o_{3,1003})^2} \\ \dfrac{(y^o_{1003}-y_4)}{(\ell^o_{4,1003})^2} & \dfrac{-(x^o_{1004}-x_4)}{(\ell^o_{4,1003})^2} \end{bmatrix} \begin{bmatrix} \delta x_{1003} \\ \delta y_{1003} \end{bmatrix} . \qquad (5\text{-}9)$$

The approximate coordinates of points 1003 and the known coordinates of points 3 and 4 as well as the values and standard deviations of the observed azimuths reduced to the mapping plane are listed in Table 5.2. The standard deviation of 4".0 could result from 4 determinations using the hour angle method with a 1" theodolite and σ_t = 1.0s (see Table 2.1).

Coordinates of Points			Observed Azimuths Reduced to Mapping Plane			
Station	X(m)	Y(m)	From	To	Value	σ
1003	3265.0	645.0	3	1003	272° 10' 29".71	4".0
3	3660.0	630.0	4	1003	308° 15' 15".94	4".0
4	3635.0	355.0				

Table 5.2 Initial Data for Azimuth Intersection

The solution vector $\hat{X}$ is given as (cf. eq. (AII-11))

$$\hat{X} = - [A^T P A]^{-1} A^T P W \quad .$$

Thus, using the above approximate coordinates for point 1003 and the observed azimuths and their standard deviations, the A, P and W matrices and vector are evaluated. They are

$$A = \begin{bmatrix} 19.80142 & 521.4374 \\ 270.6642 & 345.3302 \end{bmatrix}, \qquad W = \begin{bmatrix} -0''.63035 \\ 0''.36394 \end{bmatrix},$$

and, assuming $\sigma_o^2 = 1$,

$$P = \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.0625 \end{bmatrix}.$$

The units for P are $('')^{-2}$ and for A are ('' /m). Evaluation of the solution vector $\hat{X}$ gives

$$\hat{X} = \begin{bmatrix} -0.01575 \text{ m} \\ 0.00181 \text{ m} \end{bmatrix} .$$

Thus, the least squares estimate of the parameters after this first iteration are

$$X = X^o + \hat{X} ,$$

$$X = \begin{bmatrix} 3265.0 \\ 645.0 \end{bmatrix} + \begin{bmatrix} -0.01575 \\ 0.00181 \end{bmatrix} = \begin{bmatrix} 3264.984 \text{ m} \\ 645.002 \text{ m} \end{bmatrix} . \qquad (5\text{-}10)$$

These values for the parameters are now taken as new approximate values X^o, and the A and W matrices are reevaluated. They are

$$A = \begin{bmatrix} 19.8022 & 521.4165 \\ 270.6504 & 345.3251 \end{bmatrix}, \qquad W = \begin{bmatrix} 0''.0001046 \\ -0''.00002522 \end{bmatrix}.$$

This results in a solution vector $\hat{X}$ of

$$\hat{X} = \begin{bmatrix} 0.00000 \\ 0.00000 \end{bmatrix} .$$

Thus, the parameters X are unchanged by the results of this second solution or iteration, and it has converged. The variance covariance matrix C_x of the parameters is

$$C_x = \begin{bmatrix} 0.347049.10^{-3} & -0.920918.10^{-4} \\ -0.920918.10^{-4} & 0.6534486.10^{-4} \end{bmatrix} .$$

The standard ellipse computed according to Appendix III is

$$a_s = 0.019 \text{ m} ,$$

$$b_s = 0.006 \text{ m} ,$$

$$\theta = -72° \; 24' \; 41'' .$$

Assuming σ_o^2 known, the 95% confidence ellipse is

$$a = 0.047 \text{ m} ,$$

$$b = 0.015 \text{ m} .$$

Figure 5.4 shows the 95% confidence ellipse.

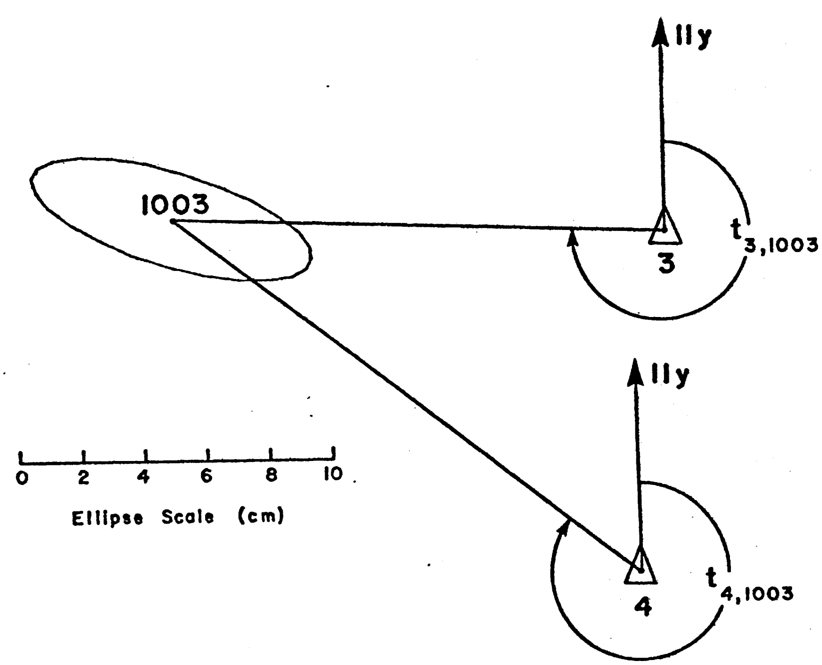


Figure 5.4 95% Confidence Ellipse for Azimuth Intersection

5.3 Distance Intersection

Figure 5.5 shows the distance intersection example used for

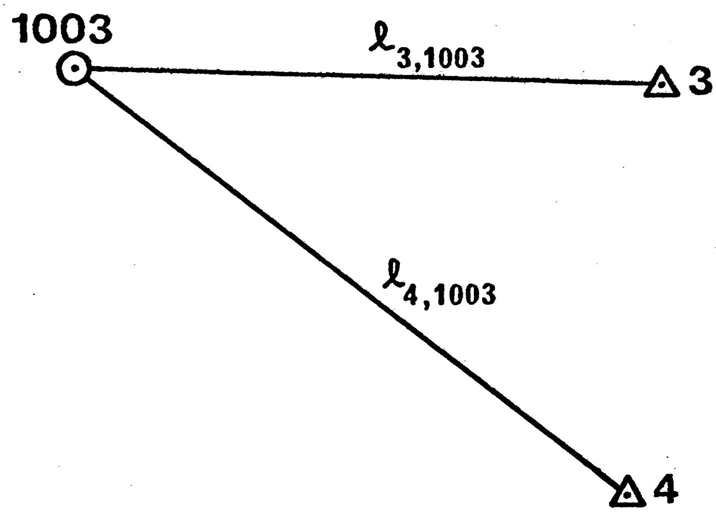


Figure 5.5 Distance Intersection

this section. Points 3 and 4 are fixed, and point 1003 is unknown. Thus, there are two unknown parameters and two observations giving the unique case again.

The approximate coordinates for point 1003, the fixed coordinates of

Coordinates of Points			Observations on the Mapping Plane				
Station	X(m)	Y(m)	Type	From	To	Value	σ
1003	3265.0	645.0	Dist.	3	1003	395.840 m	0.005 m
3	3660.0	630.0	Dist.	4	1003	464.103 m	0.006 m
4	3635.0	355.0					

Table 5.3 Initial Data for Distance Intersection

points 3 and 4, and the observed distances and their standard deviations are listed in Table 5.3. These standard deviations could be a result of six determinations of the distance with a lightwave instrument such as the Hewlett-Packard 3800 (assuming $\sigma_r = \sigma_\ell$).

The observation equation in the form of formulae (4-9) and (4-10) is

$$\underset{(2,1)}{V} = \begin{bmatrix} ((x^o_{1003}-x_3)^2 + (y^o_{1003}-y_3)^2)^{1/2} - \ell_{3,1003} \\ ((x^o_{1003}-x_4)^2 + (y^o_{1003}-y_4)^2)^{1/2} - \ell_{4,1003} \end{bmatrix} + \begin{bmatrix} \frac{(x^o_{1003}-x_3)}{\ell^o_{3,1003}} & \frac{(y^o_{1003}-y_3)}{\ell^o_{3,1003}} \\ \frac{(x^o_{1003}-x_4)}{\ell^o_{4,1003}} & \frac{(y^o_{1003}-y_4)}{\ell^o_{4,1003}} \end{bmatrix} \begin{bmatrix} \delta x_{1003} \\ \delta y_{1003} \end{bmatrix} . \qquad (5\text{-}11)$$

Using the data from Table 5.3, the A, P and W metrices are

$$A = \begin{bmatrix} -0.9992797 & 0.0379473 \\ -0.7870559 & 0.6168817 \end{bmatrix} , \qquad W = \begin{bmatrix} -0.55498 \text{ m} \\ 6.00325 \text{ m} \end{bmatrix} ,$$

and, assumed the a priori variance factor $\sigma_o^2 = 1$,

$$P = \begin{bmatrix} 40000 & 0 \\ 0 & 27777.777 \end{bmatrix} ,$$

where A is unitless and P has units of m^{-2}.

Using these matrices to evaluate $\hat{X}$ for the first iteration (again using equation (AII-11)) yields

$$\hat{X} = \begin{bmatrix} -0.97203 \text{ m} \\ \\ -10.97179 \text{ m} \end{bmatrix} ,$$

which results in the least squares estimate of the coordinates being

$$X = X^{o} + \hat{X} = \begin{bmatrix} 3265.0 \\ \\ 645.0 \end{bmatrix} + \begin{bmatrix} -0.97203 \\ \\ -10.97179 \end{bmatrix} = \begin{bmatrix} 3264.028 \text{ m} \\ \\ 634.028 \text{ m} \end{bmatrix} . \quad (5\text{-}12)$$

Using these parameter values as approximate coordinates now, and reevaluating the A and W matrices for the second iteration gives

$$A = \begin{bmatrix} -0.9999483 & 0.01017244 \\ \\ -0.7991728 & 0.6011013 \end{bmatrix} , \qquad W = \begin{bmatrix} 0.15283 \\ \\ 0.09187 \end{bmatrix} .$$

Using these matrices (as well as P) to compute $\hat{X}$ again using equation (AII-11) gives

$$\hat{X} = \begin{bmatrix} 0.15336 \text{ m} \\ \\ 0.05105 \text{ m} \end{bmatrix} , \quad (5\text{-}13)$$

which, when added to this iterations' approximate coordinates (eq. (5-12)) gives the parameters from the second iteration as

$$X = X^{o} + \hat{X} = \begin{bmatrix} 3264.028 \\ \\ 634.028 \end{bmatrix} + \begin{bmatrix} 0.15336 \\ \\ 0.05105 \end{bmatrix} = \begin{bmatrix} 3264.181 \text{ m} \\ \\ 634.079 \text{ m} \end{bmatrix} . \quad (5\text{-}14)$$

Because the correction or solution vector X was not insignificant (i.e. less than 0.001 m) on the second iteration (eq.(5-13)), a third iteration is required. Thus, the parameters of the second iteration (eq. (5-14)) now becomes the approximate coordinates X^{o} for the third iteration, and the A and W matrices are again computed. They are

$$A = \begin{bmatrix} -0.9999469 & 0.01030534 \\ -0.7990006 & 0.6013303 \end{bmatrix}, \quad W = \begin{bmatrix} 0.3496 \cdot 10^{-5} \text{ m} \\ 0.1905 \cdot 10^{-4} \text{ m} \end{bmatrix} .$$

The solution vector $\hat{X}$ from this third iteration is

$$\hat{X} = \begin{bmatrix} 0.00000 \\ -0.00003 \end{bmatrix},$$

which is less than 0.001 m, and thus insignificant. The solution has converged, and the final least squares estimates for the coordinates of point 1003 are given by equation (5-14).

The variance covariance matrix of the parameters computed according to equation (AII-16) is

$$C_x = \begin{bmatrix} 0.2972685 \cdot 10^{-4} & 0.40380781 \cdot 10^{-4} \\ 0.40380781 \cdot 10^{-4} & 0.13924377 \cdot 10^{-3} \end{bmatrix},$$

which results in a standard error ellipse of

$$a_s = 0.01235 \text{ m} ,$$

$$b_s = 0.00406 \text{ m} ,$$

$$\theta = 18° \; 12' \; 11" .$$

Increasing the confidence lebel to 95% as in the previous two examples yields

$$a = 0.030 \text{ m} ,$$

$$b = 0.010 \text{ m} .$$

Figure 5.6 illustrates the result.

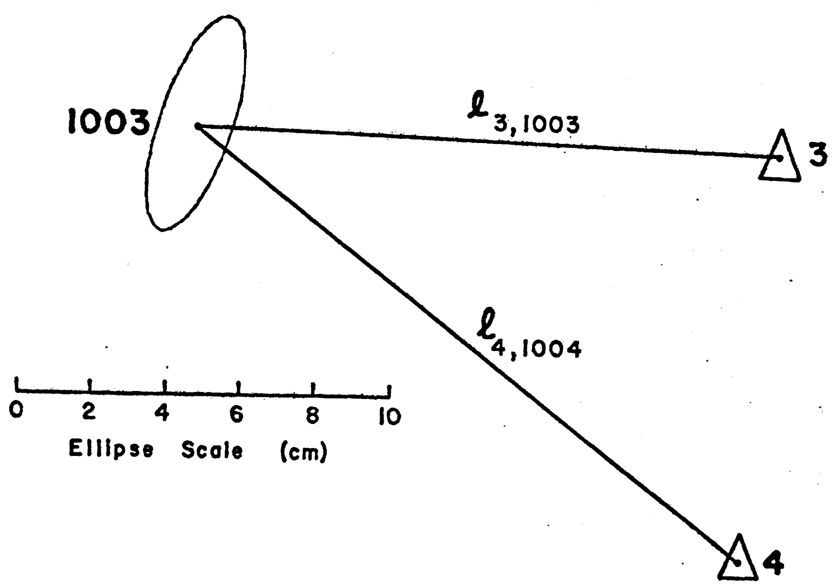


Figure 5.6 95% Confidence Ellipse for Distance Intersection

5.4 Angle Resection

The angle resection considered here is depicted in figure 5.7. There are two angle observations $B_{1007,2,1}$ and $B_{1007,1,3}$, where the subscripts stand for stations 'at, from, to'. Point 1007 is the unknown station and points 1,2 and 3 are fixed.

Table 5.4 lists the point coordinates as well as the observations

Coordinates of Points			Observations on the Mapping Plane				
Station	X	Y	At	From	To	Value	σ
1007	3160.0	865.0	1007	2	1	23°13'37".33	3".0
1	2640.0	1160.0	1007	1	3	175°36'21".70	3".5
2	2530.0	935.0					
3	3660.0	630.0					

Table 5.4 Initial Data for Angle Resection

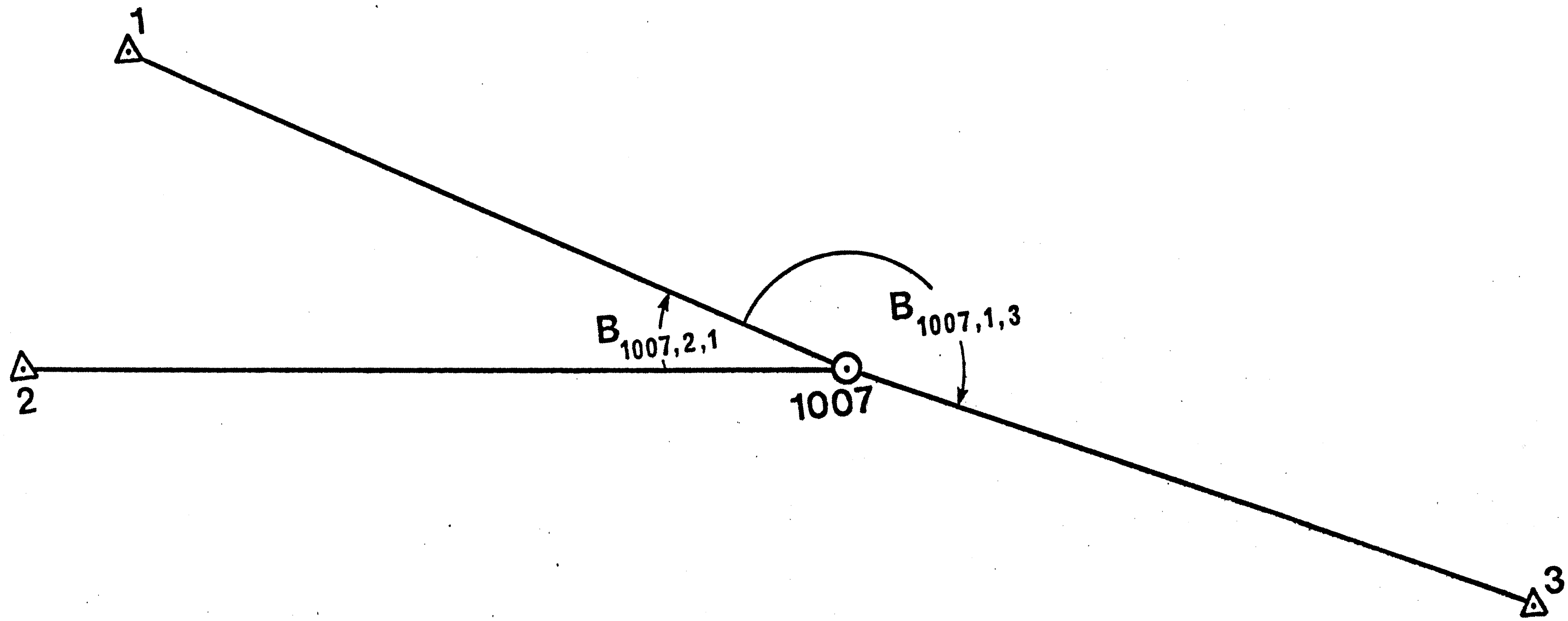


Figure 5.7 Angle Resection

and their standard deviations. These standard deviations could result from 2 sets of observations with a 1" theodolite or 8 to 10 sets with a 20" instrument (see Table 2.4).

The observation equation as developed in section 3.3 (eqs. (3-26) and (3-27)) is

$$\underset{(2,1)}{V} = \begin{bmatrix} \arctan\left(\dfrac{x_1-x^o_{1007}}{y_1-y^o_{1007}}\right) - \arctan\left(\dfrac{x_2-x^o_{1007}}{y_2-y^o_{1007}}\right) - B_{1007,2,1} \\ \arctan\left(\dfrac{x_3-x^o_{1007}}{y_3-y^o_{1007}}\right) - \arctan\left(\dfrac{x_1-x^o_{1007}}{y_1-y^o_{1007}}\right) - B_{1007,1,3} \end{bmatrix}''$$

$$+ \rho'' \begin{bmatrix} \dfrac{-(y_1-y^o_{1007})}{(\ell^o_{1007,1})^2} + \dfrac{(y_2-y^o_{1007})}{(\ell^o_{1007,2})^2} & \dfrac{(x_1-x^o_{1007})}{(\ell^o_{1007,2})^2} - \dfrac{(x_2-x^o_{1007})}{(\ell^o_{1007,2})^2} \\ \dfrac{-(y_3-y^o_{1007})}{(\ell^o_{1007,3})^2} + \dfrac{(y_1-y^o_{1007})}{(\ell^o_{1007,1})^2} & \dfrac{(x_3-x^o_{1007})}{(\ell^o_{1007,3})^2} - \dfrac{(x_1-x^o_{1007})}{(\ell^o_{1007,1})^2} \end{bmatrix} \begin{bmatrix} \delta x_{1007} \\ \delta y_{1007} \end{bmatrix} . \quad (5\text{-}15)$$

Substituting the initial data from Table 5.4 into the above expressions results in

$$A = \begin{bmatrix} -134.3056 & 23.32723 \\ 329.0484 & 637.9743 \end{bmatrix} , \quad W = \begin{bmatrix} -2''34886 \\ 3''317 \end{bmatrix} ,$$

and, assuming $\sigma_o^2 = 1$,

$$P = \begin{bmatrix} 0.11111 & 0 \\ 0 & 0.08163 \end{bmatrix} .$$

The units for P are $('')^{-2}$ and for A are $('').m^{-1}$.

The solution vector $\hat{X}$ resulting from this first iteration (computed by eq. (AII-11)) is

$$\hat{X} = \begin{bmatrix} -0.01688 \text{ m} \\ 0.00351 \text{ m} \end{bmatrix}, \tag{5-16}$$

which gives the parameter vector as

$$X = X^{o} + \hat{X} = \begin{bmatrix} 3161.0 \\ 865.0 \end{bmatrix} + \begin{bmatrix} -0.01688 \\ 0.00351 \end{bmatrix} = \begin{bmatrix} 3159.983 \text{ m} \\ 865.004 \text{ m} \end{bmatrix}. \tag{5-17}$$

These parameter values are now taken as new approximate coordinates X^{o}, the A and W matrices are recomputed to enable the second iteration value for $\hat{X}$ to be found. They are

$$A = \begin{bmatrix} -134.3128 & 23.32934 \\ 329.0485 & 637.9719 \end{bmatrix}, \quad W = \begin{bmatrix} 0''6428 \cdot 10^{-4} \\ -0''4607 \cdot 10^{-5} \end{bmatrix},$$

which results in a solution vector of

$$\hat{X} = \begin{bmatrix} 0.00000 \\ 0.00000 \end{bmatrix}.$$

Thus, the solution has converged, and the final least squares estimate of the coordinates of point 1007 is given by equation (5-17).

The variance covariance matrix of the parameters computed according to equation (AII-16) is

$$C_x = \begin{bmatrix} 0.42099214 \cdot 10^{-3} & -0.21233829 \cdot 10^{-3} \\ -0.21233829 \cdot 10^{-3} & 0.13714132 \cdot 10^{-3} \end{bmatrix},$$

which gives the standard error ellipse computed by the equations of Appendix III as

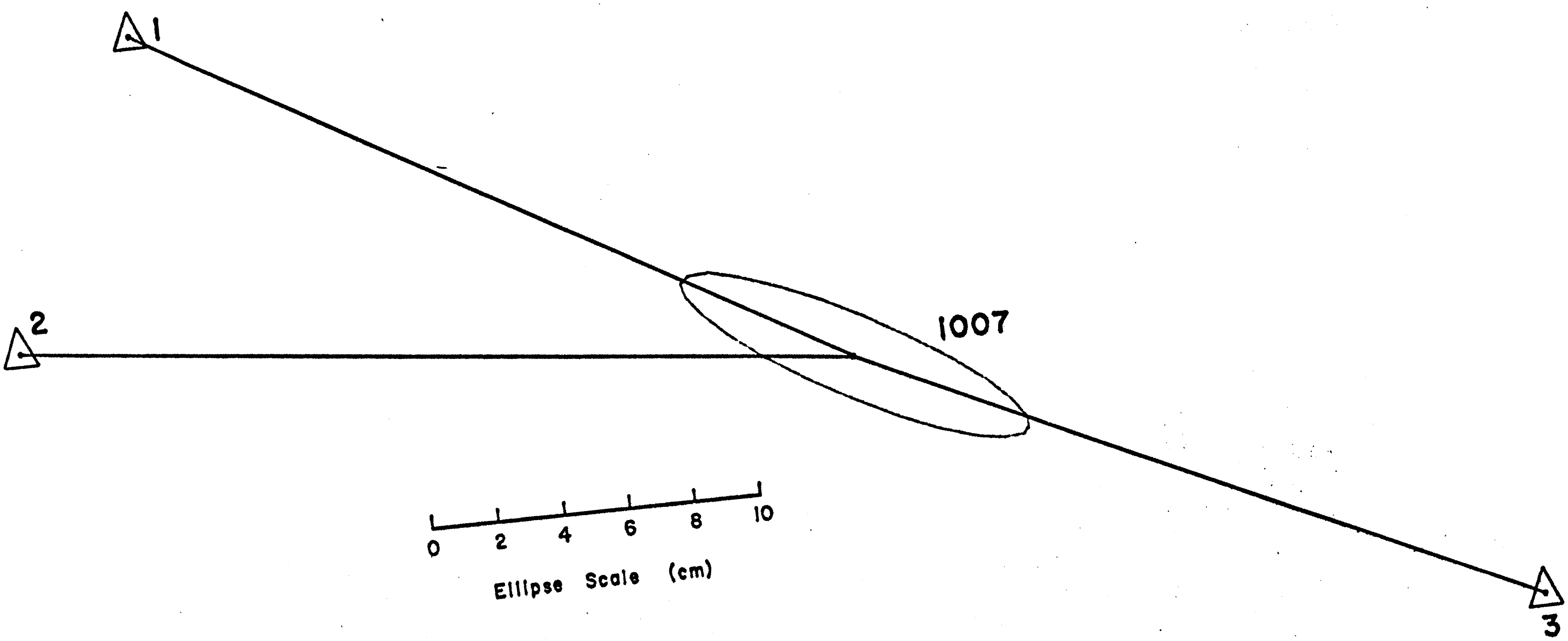


Figure 5.8 95% Confidence Ellipse from Angle Resection

$$a_s = 0.02312 \text{ m} ,$$

$$b_s = 0.00486 \text{ m} ,$$

$$\theta = - 61° \ 52' \ 46'' \quad .$$

Increasing the confidence level to 95% yields

$$a = 0.057 \text{ m} ,$$

$$b = 0.012 \text{ m} .$$

The resultant confidence ellipse is depicted in Figure 5.8.

The variance covariance matrix for the parameters C_x would be difficult to compute using the direct (explicit) mathematical model for an angle resection, mainly due to the complicated partial derivatives $\frac{\partial X}{\partial L}$ needed for the Jacobian of transformation in the covariance law (cf. Thomson et al. [1978], Appendix II). Thus, it is seen that the method of least squares offers a consistently convenient method of error propogation.

5.5 Open Traverse

Figure 5.9 shows the open ended traverse considered in this example. As in section 5.1, angle and distance observations are combined in the same model. There are three angles and three distances for a total of six observations, and three unknown points, or six unknown coordinates. Thus, n = u and this is a unique case.

The fixed coordinates for points 1 and 2, the approximate coordinates for points 1001, 1002 and 1003, and the observations and their standard deviations are listed below in Table 5.5. The observational standard deviations are computed according to section 2.1.

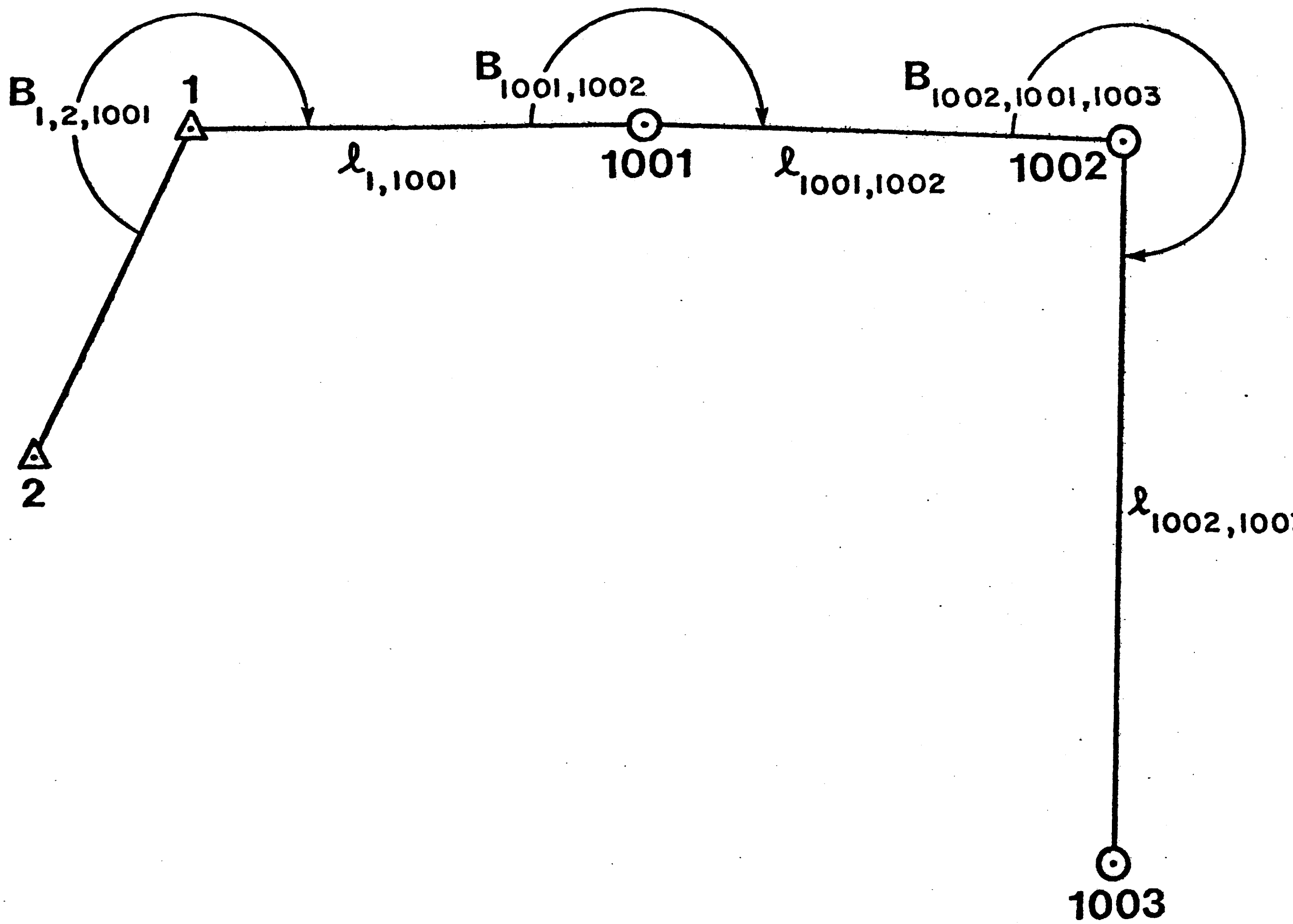


Figure 5.9 Open Traverse

Coordinates of Points			Observations on the Mapping Plane					
Station	X	Y	Type	At	From	To	Value	σ
1001	2950.0	1160.0	Dist.	1	1	1001	307.997 m	0.010 m
1002	3280.0	1145.0	Dist.	1001	1001	1002	330.355 m	0.012 m
1003	3265.0	645.0	Dist.	1002	1002	1003	500.243 m	0.011 m
1	2640.0	1160.0	Angle	1	2	1001	243°56'55".16	3".5
2	2530.0	935.0	Angle	1001	1	1002	182°36'5".44	4".0
			Angle	1002	1001	1003	269°6'51".39	3".0

Table 5.5 Initial Data for Open Traverse

Combining the mathematical models developed in section 3.3 and chapter 4, the observation equation for this example is

$$\underset{(6,1)}{V} = \underset{(6,1)}{W} + \underset{(6,6)}{A}\ \underset{(6,1)}{\hat{X}} ,$$

or evaluating the matrices individually,

$$\underset{(6,1)}{W} = \begin{bmatrix} ((x^o_{1001}-x_1)^2 + (y^o_{1001}-y_1)^2)^{1/2} - \ell_{1,1001} \\ ((x^o_{1002}-x^o_{1001})^2 + (y^o_{1002}-y^o_{1001})^2)^{1/2} - \ell_{1001,1002} \\ ((x^o_{1003}-x^o_{1002})^2 + (y^o_{1003}-y^o_{1002})^2) - \ell_{1002,1003} \\ \arctan(\frac{x^o_{1001}-x_1}{y^o_{1001}-y_1}) - \arctan(\frac{x_2-x_1}{y_2-y_1}) - B_{1,2,1001} \\ \arctan(\frac{x^o_{1002}-x^o_{1001}}{y^o_{1002}-y^o_{1001}}) - \arctan(\frac{x_1-x^o_{1001}}{y_1-y^o_{1001}}) - B_{1001,1,1002} \\ \arctan(\frac{x^o_{1003}-x^o_{1002}}{y^o_{1003}-y^o_{1002}}) - \arctan(\frac{x^o_{1001}-x^o_{1002}}{y^o_{1001}-y^o_{1002}}) - B_{1002,1001,1003} \end{bmatrix} , \quad (5\text{-}18)$$

$$\underset{(6,6)}{A} = \begin{bmatrix} \frac{(x^o_{1001}-x_1)}{\ell^o_{1,1001}} & \frac{(y^o_{1001}-y_1)}{\ell^o_{1,1001}} & 0 & 0 & 0 & 0 \\ \frac{-(x^o_{1002}-x^o_{1001})}{\ell^o_{1001,1002}} & \frac{-(y^o_{1002}-y^o_{1001})}{\ell^o_{1001,1002}} & \frac{(x^o_{1002}-x^o_{1001})}{\ell^o_{1001,1002}} & \frac{(y^o_{1002}-y^o_{1001})}{\ell^o_{1001,1002}} & 0 & 0 \\ 0 & 0 & \frac{-(x^o_{1003}-x^o_{1002})}{\ell^o_{1002,1003}} & \frac{-(y^o_{1003}-y^o_{1002})}{\ell^o_{1002,1003}} & \frac{(x^o_{1003}-x^o_{1002})}{\ell^o_{1002,1003}} & \frac{(y^o_{1003}-y^o_{1002})}{\ell^o_{1002,1003}} \\ \frac{(y^o_{1001}-y_1)}{(\ell^o_{1,1001})^2} & \frac{-(x^o_{1001}-x_1)}{(\ell^o_{1,1001})^2} & 0 & 0 & 0 & 0 \\ \frac{-(y^o_{1002}-y^o_{1001})}{(\ell^o_{1001,1002})^2} + \frac{(y_1-y^o_{1001})}{(\ell^o_{1,1001})^2} & \frac{(x^o_{1002}-x^o_{1001})}{(\ell^o_{1001,1002})^2} - \frac{(x^o_1 - x^o_{1001})}{(\ell^o_{1,1001})^2} & \frac{(y^o_{1002}-y^o_{1001})}{(\ell^o_{1001,1002})^2} & \frac{-(x^o_{1002}-x^o_{1001})}{(\ell^o_{1001,1002})^2} & 0 & 0 \\ \frac{-(y^o_{1001}-y^o_{1002})}{(\ell^o_{1002,1001})^2} & \frac{(x^o_{1001}-x^o_{1002})}{(\ell^o_{1002,1001})^2} & \frac{-(y^o_{1003}-y^o_{1002})}{(\ell^o_{1002,1003})^2} + \frac{(y^o_{1001}, y^o_{1002})}{(\ell^o_{1002,1003})^2} & \frac{(x^o_{1003}-x^o_{1002})}{(\ell^o_{1002,1003})^2} - \frac{(x^o_{1001}-x^o_{1002})}{(\ell^o_{1002,1001})^2} & \frac{(y^o_{1003}-y^o_{1002})}{(\ell^o_{1002,1003})^2} & \frac{-(x^o_{1003}-x^o_{1002})}{(\ell^o_{1002,1003})^2} \end{bmatrix} \quad (5\text{-}19)$$

Note that ρ'' has been omitted from the non-zero elements of the last three rows of A. The first three rows of A are unitless, whereas the last three have units of $('')m^{-1}$. The solution vector has the form

$$\hat{X} = \begin{bmatrix} \delta x_{1001} \\ \delta y_{1001} \\ \delta x_{1002} \\ \delta y_{1002} \\ \delta x_{1003} \\ \delta y_{1003} \end{bmatrix}. \qquad (5\text{-}20)$$

Using the initial values from Table 5.5 yields the following A, W and P matrices:

$$A = \begin{bmatrix} 100.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.9989685 & 0.04540766 & 0.9989685 & -0.04540766 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.02998651 & 0.9995503 & -0.02998651 & -0.9995503 \\ 0.0 & -665.3703 & 0.0 & 0.0 & 0.0 & 0.0 \\ 28.35255 & 1289.126 & -28.35255 & -623.7561 & 0.0 & 0.0 \\ -28.35255 & -623.7561 & 440.5112 & 611.3914 & -412.1587 & 12.36476 \end{bmatrix},$$

(note that ρ'' has been multiplied onto the appropriate elements for this A matrix)

$$W = \begin{bmatrix} 2.0033911 \text{ m} \\ -0.01411715 \text{ m} \\ -0.01848697 \text{ m} \\ -7''.7426476 \\ 3''.7866587 \\ 5''.4711834 \end{bmatrix}, \quad P = \begin{bmatrix} 10000.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6944.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8264.46 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.08163 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1111 \end{bmatrix}.$$

The solution vector $\hat{X}$ computed (using eq. (AII-11)) from these first iteration matrices is

$$\hat{X} = \begin{bmatrix} -2.00339 \text{ m} \\ -0.01164 \text{ m} \\ -1.98958 \text{ m} \\ -0.1861 \text{ m} \\ -1.98646 \text{ m} \\ -0.03720 \text{ m} \end{bmatrix},$$

which results in the parameters X being

$$X = X^{o} + \hat{X} = \begin{bmatrix} 2950.0 \\ 1160.0 \\ 3280.0 \\ 1145.0 \\ 3265.0 \\ 645.0 \end{bmatrix} + \begin{bmatrix} -2.00339 \\ -0.01164 \\ -1.98958 \\ -0\ 1861 \\ -1.98646 \\ -0.03720 \end{bmatrix} = \begin{bmatrix} 2947.997 \text{ m} \\ 1159.988 \text{ m} \\ 3278.010 \text{ m} \\ 1144.981 \text{ m} \\ 3263.014 \text{ m} \\ 644.963 \text{ m} \end{bmatrix} . \tag{5-21}$$

Using these parameters as new approximate coordinates and recomputing A and W to get a new solution vector $\hat{X}$ results in

$$\hat{X} = \begin{bmatrix} 0.00000 \\ 0.00008 \\ 0.00000 \\ 0.00007 \\ 0.00000 \\ 0.00007 \end{bmatrix} .$$

All elements of X are less than 0.001 m, and thus the solution has converged. Thus, the final least squares estimate of the coordinates of points 1001, 1002 and 1003 is given by equation (5-21).

The variance covariance matrix of the parameters assuming the a posteriori variance factor known (eq. (AII-16)) is

$$C_x = \begin{bmatrix} 0.10086490.10^{-3} & -0.27788873.10^{-8} & 0.10086495.10^{-3} & -0.16731703.10^{-8} & 0.10086663.10^{-3} & -0.17234176.10^{-8} \\ & 0.27313521.10^{-4} & 0.13280578.10^{-5} & -0.56579556.10^{-4} & 0.45670326.10^{-4} & -0.55249614.10^{-4} \\ & & 0.24569748.10^{-3} & -0.19616485.10^{-5} & 0.25068166.10^{-3} & -0.21111371.10^{-5} \\ & & & -0.15846045.10^{-3} & 0.15194931.10^{-3} & 0.15384425.10^{-3} \\ & & & & 0.47467789.10^{-3} & 0.14892772.10^{-3} \\ \text{Symmetric} & & & & & 0.27256622.10^{-3} \end{bmatrix}. \quad (5\text{-}22)$$

Table 5.6 shows the standard and 95% station error ellipses as well as the relative error ellipses between the unknown points. Again, it is assumed that σ_o^2 is known, and the c factor is 2.45.

Station Ellipses

	Standard			95%	
Station	a_s (m)	b_s (m)	θ	a (m)	b (m)
1001	0.010	0.005	-89°59'52"	0.025	0.013
1002	0.016	0.013	-88°42'45"	0.038	0.031
1003	0.024	0.014	62° 4'46"	0.058	0.034

Relative Ellipses

	Standard			95%	
Station	a_s (m)	b_s (m)	θ	a (m)	b (m)
1001 to 1002	0.012	0.009	-87°23'47"	0.030	0.021
1002 to 1003	0.015	0.011	-88°16'56"	0.036	0.027

Table 5.6 Station and Relative Ellipses for Open Traverse

The traverse with both station and relative 95% confidence ellipses is shown in Figure 5.10.

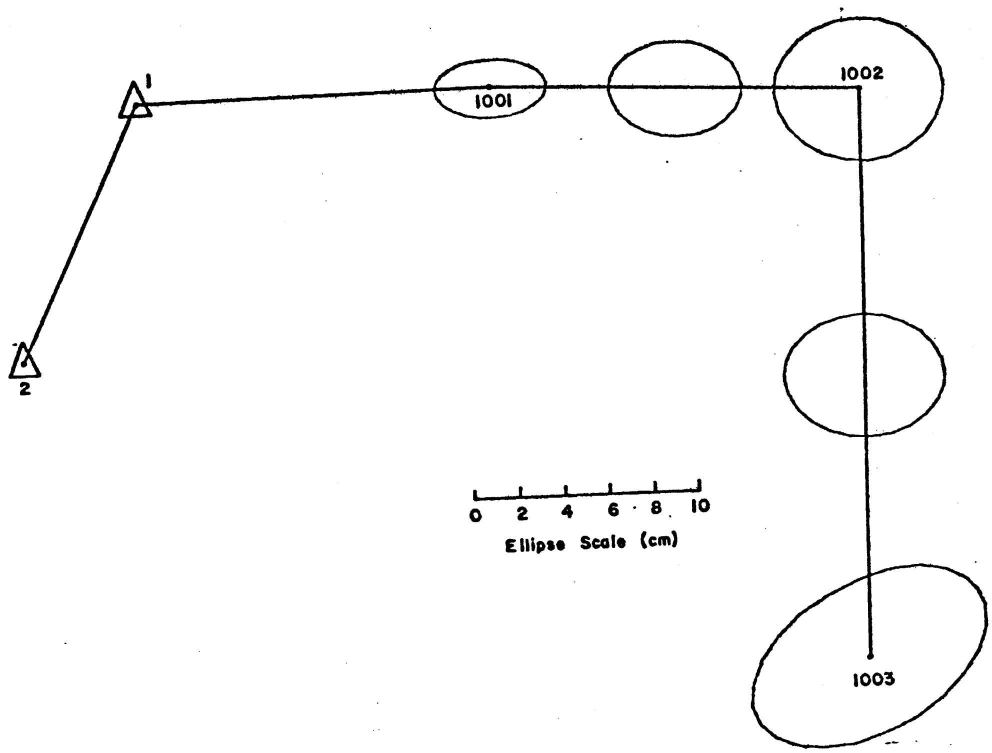


Figure 5.10 95% Confidence Ellipses for Open Traverse

6. SOLUTION OF OVERDETERMINED CASES

As already mentioned in the introduction, overdetermined cases include any network in which the number of observations n is greater than the number of unknowns u. Any overdetermined network has more than one unique solution for the coordinates of the unknown points. Thus, the best solution based on all of the available information must be found. This is accomplished by the method of least squares (see Appendix II) which gives the minimum weighted sum of squares of the residuals (corrections to the observations), i.e. V^TPV = minimum. A simple example is the least squares line fitting technique shown in Figure 6.1. The observations are the y coordinate (horizontal axis t is known) and the unknowns are the slope of the line a and the y intersect b (i.e. $y = at + b$). The least squares technique minimizes the sum square of the residuals, which in this case is the distance parallel to the y axis from the observation point to the line.

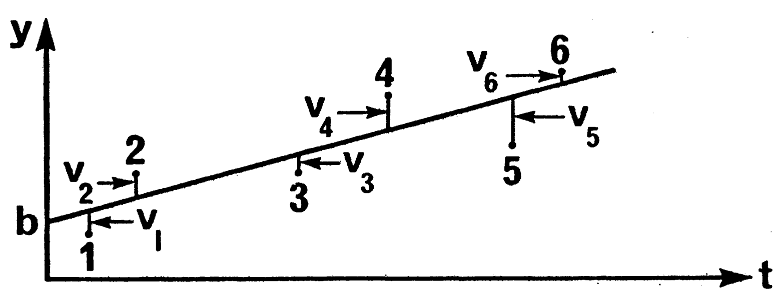


Figure 6.1 Least Squares Line Fitting

Besides having at least as many observations as unknowns, a horizontal network must also have certain other basic information before it can be solved. A network must have scale, orientation, and one known position. Orientation is introduced by observing an azimuth, or by "fixing" one point along with the x or y coordinate of another point. The scale is provided by measuring at least one distance and including it in the network, or by "fixing" at least two points. At least one point must be assumed known to provide the minimum position information for a horizontal network. If two or more points are assumed known, then the scale and orientation are inherent as well. If this minimum information is not provided, then usually the normal equations matrix $N = A^TPA$ is singular, and its inverse cannot be found.

Classical horizontal networks were usually measured by triangulation methods; i.e. having only angular observations between stations. The scale was introduced by baselines measured with invar wires or tapes. Since the introduction of EDM equipment, trilateration networks composed mainly of distance observations have been measured, with the orientation provided by azimuth observations. Modern day horizontal networks are usually composed of a mixture of both angular and distance observations, and the phrase triangulateration network has been coined to characterize them. A traverse is a simple example of a triangulateration network. The following examples show some of the various network types for over-determined cases.

6.1 Closed Traverse

Figure 6.2 depicts the closed traverse considered in this example.

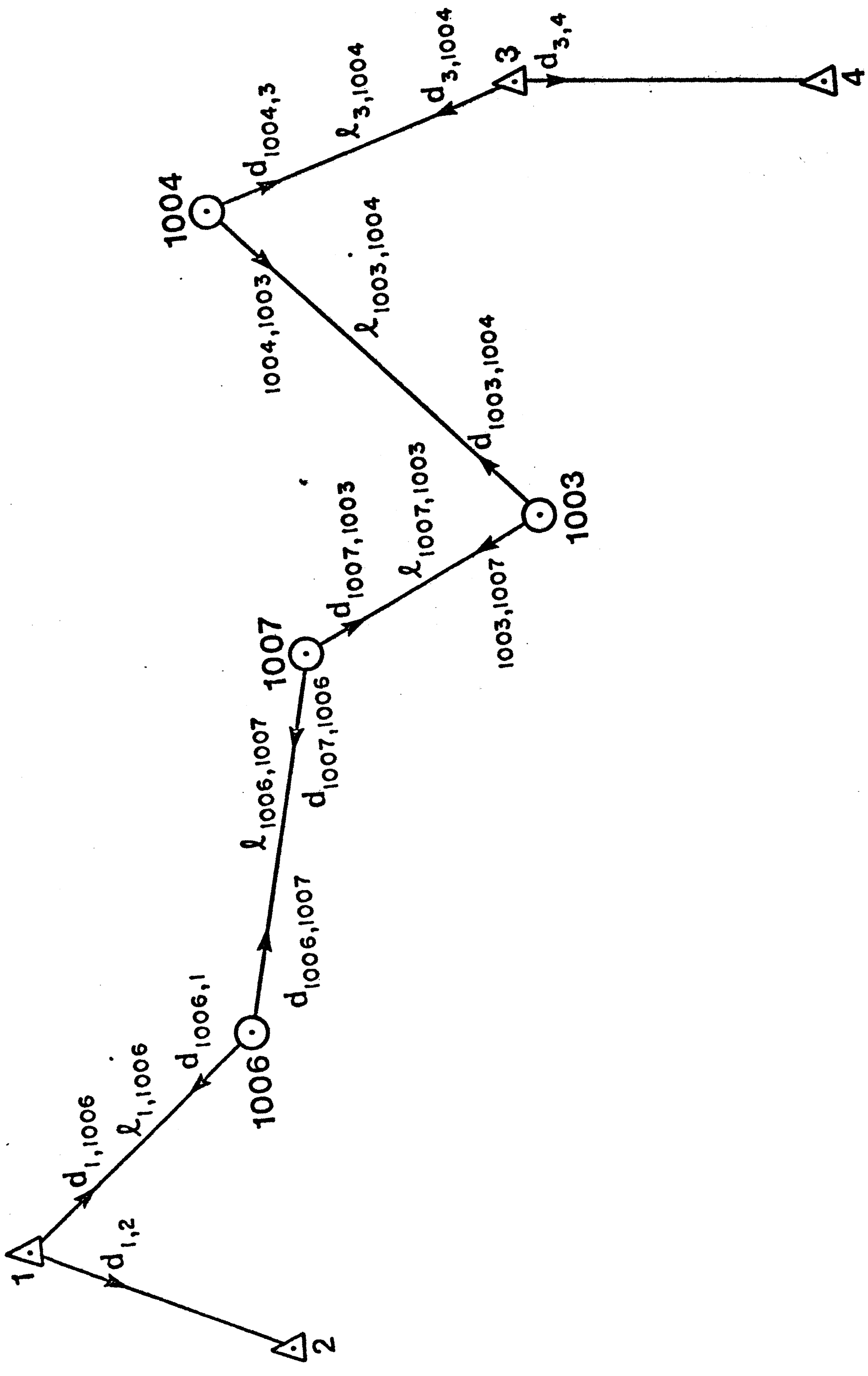


Figure 6.2 Closed Traverse

There are four unknown points and four fixed points. The x, y coordinates of the four unknown points along with the six orientation unknowns (see section 3.2) gives a total of fourteen unknown parameters. It is seen from Figure 6.2 and Table 6.1 that there are a total of seventeen observations, and the degrees of freedom is, therefore, three. Since the degrees of freedom is greater than zero, this problem is overdetermined.

Coordinates of Points			Observations on the Mapping Plane				
Station	X	Y	Type	From	To	Value	σ
1003	3265.0	645.0	Dir.	1	1006	0°00'00".0	2".0
1004	3570.0	915.0	Dir.	1	2	66°1' 1".0	2".0
1006	2820.0	945.0	Dir.	1006	1007	0°00'00".0	2".0
1007	3160.0	865.0	Dir.	1006	1	216°53'42".0	2".0
1	2640.0	1160.0	Dir.	1007	1003	0°00'00".0	2".0
2	2530.0	935.0	Dir.	1007	1006	128°41'52".0	2".0
3	3660.0	630.0	Dir.	1003	1004	0"00'00".0	2".0
4	3635.0	355.0	Dir.	1003	1007	286°1'57".0	2".0
			Dir.	1004	3	0"00'00".0	2".0
			Dir.	1004	1003	65°48'6".0	2".0
			Dir.	3	4	0"00'00".0	2".0
			Dir.	3	1004	157°25'5".0	2".0
			Dist.	3	1004	301.200 m	.01 m
			Dist.	1	1006	279.747 m	.01 m
			Dist.	1006	1007	348.982 m	.01 m
			Dist.	1007	1003	243.623 m	.01 m
			Dist.	1003	1004	408.310 m	.01 m

Table 6.1 Initial Data for Closed Traverse

Again, it is assumed that the observations have already been reduced to the mapping plane, and that the standard deviations have been computed according to section 2.1.

Combining the mathematical models developed in section 3.2 and chapter 4, the observation equation takes the form

$$\underset{(17,1)}{V} = \underset{(17,1)}{W} + \underset{(17,14)}{A} \ \underset{(14,1)}{\hat{X}} \quad .$$

The W, A and $\hat{X}$ matrices are evaluated exactly as for the unique case examples in chapter 5. The primary difference between the unique case and this overdetermined case is that the residuals V are no longer zero.

The W matrix is computed exactly as in equations (3-15) (directions) and (4-9) (distances). For a direction, the W matrix element is

$$W_{d_{ij}} = \arctan \left(\frac{x_j^\circ - x_i^\circ}{y_j^\circ - y_i^\circ}\right) - Z_i^\circ - d_{ij} \quad .$$

Taking the direction $d_{1,1006}$ as an example and using the initial approximate coordinates of Table 6.1 yields

$$W_{d_{1,1006}} = \arctan \left(\frac{2820.0-2640.0}{945.0-1160.0}\right) - \arctan \left(\frac{2820.0-2640.0}{945.0-1160.0}\right) - 0^\circ 00' 00''\!.0 \ ,$$

$$W_{d_{1,1006}} = 0''\!.0 \quad .$$

The approximate value for the orientation unknown Z_i° is always computed as the azimuth between the from and to stations of the first direction in the set of directions. This causes the W matrix element for the first direction of a set to be zero assuming that the directions are reduced such that the first direction of the set always has an observed value of 0°00'00"0. In this case, $d_{1,1006}$ is the first direction of the set. For the second direction of the set $d_{1,2}$, the W matrix element is

$$W_{d_{1,2}} = \arctan \left(\frac{2530.0-2640.0}{935.0-1160.0}\right) - \arctan \left(\frac{2820.0-2640.0}{945.0-1160.0}\right) - 66^\circ 1' 1''\!.0 \ ,$$

$$W_{d_{1,2}} = 206^\circ\ 03'\ 12''\!.58 - 140^\circ\ 03' 49''\!.02 - 66^\circ 1' 1''\!.0 \ ,$$

$$W_{d_{1,2}} = -97''\!.44 \ .$$

The distance W matrix elements are computed as in chapter 5, for example

$$W_{\ell_{3,1004}} = ((3570.0-3660.0)^2 + (915.0-630.0)^2)^{1/2} - 301.200 \quad ,$$

$$W_{\ell_{3,1004}} = 298.873 - 301.200 \quad ,$$

$$W_{\ell_{3,1004}} = -2.327 \text{ m.}$$

After computing all of the elements of W using the initial coordinates in Table 6.1, W is

$$\underset{(1,17)}{W^T} = (0''0, -97''44, 0''0, -258''85, 0''0, 203''80, 0''0, -106''71, 0''0, 745''82,$$

$$0''0, -496''99, -2.327 \text{ m}, 0.654 \text{ m}, 0.303 \text{ m}, 0.149 \text{ m}, -0.971 \text{ m}) \quad ,$$

where the transpose of W is given for ease of writing.

The A matrix is computed via formulae (3-15) and (4-9) as well. For example, the direction $d_{1,1006}$ has nonzero elements $-\rho a_{1,1006}$, $-\rho b_{1,1006}$, -1 in columns 5, 6 and 9 of row 1 of A. Distance $\ell_{3,1004}$ has nonzero elements $-e_{3,1004}$ and $f_{3,1004}$ in columns 3 and 4 of row 13 of A. The entire A matrix is given to five significant digits below.

$\underset{(17,14)}{A} =$

x_{1003}	y_{1003}	x_{1004}	y_{1004}	x_{1006}	y_{1006}	x_{1007}	y_{1007}	z_1	z_2	z_3	z_4	z_5	z_6
0	0	0	0	-564.03	-472.21	0	0	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0
0	0	0	0	135.26	574.84	--135.26	-574.84	0	-1	0	0	0	0
0	0	0	0	-564.03	-472.21	0	0	0	-1	0	0	0	0
-763.62	-364.46	0	0	0	0	763.62	364.46	0	0	-1	0	0	0
0	0	0	0	135.26	574.84	-135.26	-574.84	0	0	-1	0	0	0
-335.64	379.15	335.64	-379.15	0	0	0	0	0	0	0	-1	0	0
-763.62	-364.46	0	0	0	0	763.62	364.46	0	0	0	-1	0	0
0	0	658.11	207.82	0	0	0	0	0	0	0	0	-1	0
-335.64	379.15	335.64	-379.15	0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-1
0	0	658.11	207.82	0	0	0	0	0	0	0	0	0	-1
0	0	-0.30113	0.95358	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.64194	-0.76676	0	0	0	0	0	0	0	0
0	0	0	0	-0.97342	0.22904	0.97342	-0.22904	0	0	0	0	0	0
0.43073	-0.90248	0	0	0	0	-0.43073	0.90248	0	0	0	0	0	0
-0.74876	-0.66284	0.74876	0.66284	0	0	0	0	0	0	0	0	0	0

.

where Z_1, Z_2, Z_3, Z_4, Z_5 and Z_6 refer to the orientation unknowns at stations 1, 1006, 1007, 1003, 1004, and 3, respectively.

The P matrix is diagonal of size 17 x 17 with the inverse of the standard deviations (Table 6.1) squared on the diagonal. It is

$$\underset{(17,17)}{P} = \left[\begin{array}{ccccccccccccccccc}
0.25 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0.25 & 0 & & & & & & & & & & & & & & \\
 & 0 & 0.25 & 0 & & & & & & & & & & & & & \cdot \\
 & & 0 & 0.25 & 0 & & & & & & & & & & & & \\
\cdot & & & 0 & 0.25 & 0 & & & & & & & & & & & \cdot \\
 & & & & 0 & 0.25 & 0 & & & & & & & & & & \\
\cdot & & & & & 0 & 0.25 & 0 & & & & & & & & & \cdot \\
 & & & & & & 0 & 0.25 & 0 & & & & & & & & \\
\cdot & & & & & & & 0 & 0.25 & 0 & & & & & & & \cdot \\
 & & & & & & & & 0 & 0.25 & 0 & & & & & & \\
\cdot & & & & & & & & & 0 & 0.25 & 0 & & & & & \cdot \\
 & & & & & & & & & & 0 & 0.25 & 0 & & & & \\
 & & & & & & & & & & & 0 & 10000 & 0 & & & \cdot \\
\cdot & & & & & & & & & & & & 0 & 10000 & 0 & & \\
 & & & & & & & & & & & & & 0 & 10000 & 0 & \\
 & & & & & & & & & & & & & & 0 & 10000 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 10000
\end{array}\right].$$

Computing the solution vector $\hat{X}$ via equation (AII-11) and the above W, A and P matrices yields (in units of metres)

$$\hat{X}^T = (-0.40492,\ 1.4367,\ -0.01428,\ 2.43956,\ -0.32453,\ 0.58429,\ -0.49373,\ 1.23052)$$

for this first iteration. The orientation unknowns are not given as they are considered as nuisance parameters. This results in the parameters

$$X^T = X^{\circ T} + \hat{X}^T = (3264.595,\ 646.437,\ 3569.986,\ 917.440,\ 2819.675,\ 945.584,\ 3159.506,\ 866.231),$$

where X° are the approximate coordinates of Table 6.1.

These parameters X are taken as new approximate coordinates X°, and the A and W matrices are reevaluated using them. Equation (AII-11) is again used to compute the solution vector for the second iteration as

$\hat{X}^T$= (0.00451, -0.00163, 0.00500, 0.00123, -0.00103, 0.00361,

-0.00111) ,

which results in the parameter vector

$X^T = X^{\circ T} + \hat{X}^T$ = (3264.600, 646.435, 3569.991, 917.441, 2819.677, 945.583,

3159.510, 866.229). (6-1)

These parameters are taken as new approximate coordinates X° again, and the third solution vector is (after reevaluating A and W)

$\hat{X}^T$ = (0.00000, 0.00000, 0.00000, 0.00000, 0.00000, 0.00000, 0.00000,

0.00000) ,

and the solution has converged. The final parameters (x,y coordinates of the unknown points) are thus given by equation (6-1).

The variance covariance matrix of the parameters is computed via equation (AII-16) assuming the a priori variance factor is 1. To five significant digits, C_x is

$$C_x = \begin{bmatrix} 0.59682.10^{-4} & -0.22694.10^{-5} & 0.65123.10^{-5} & -0.24399.10^{-4} & 0.15629.10^{-4} & -0.70930.10^{-5} & 0.44888.10^{-4} & 0.76894.10^{-6} \\ & 0.72621.10^{-4} & -0.16742.10^{-4} & 0.56083.10^{-4} & -0.70447.10^{-5} & 0.24325.10^{-4} & 0.56478.10^{-5} & 0.39107.10^{-4} \\ & & 0.20302.10^{-4} & -0.22861.10^{-4} & 0.25438.10^{-5} & -0.50328.10^{-5} & 0.95914.10^{-5} & -0.11126.10^{-4} \\ & & & 0.69924.10^{-4} & -0.10576.10^{-4} & 0.19808.10^{-4} & -0.20030.10^{-4} & 0.32084.10^{-4} \\ & & & & 0.37112.10^{-4} & -0.32346.10^{-4} & 0.18196.10^{-4} & -0.20442.10^{-4} \\ & & & & & 0.46543.10^{-4} & -0.15496.10^{-4} & 0.43721.10^{-4} \\ & & \text{symmetric} & & & & 0.57933.10^{-4} & -0.20907.10^{-4} \\ & & & & & & & 0.65816.10^{-4} \end{bmatrix}$$

The 95% (c factor = 2.45) station and relative error ellipses computed by the equations of Appendix III from the above C_x matrix are listed in Table 6.2, and plotted in Figure 6.3.

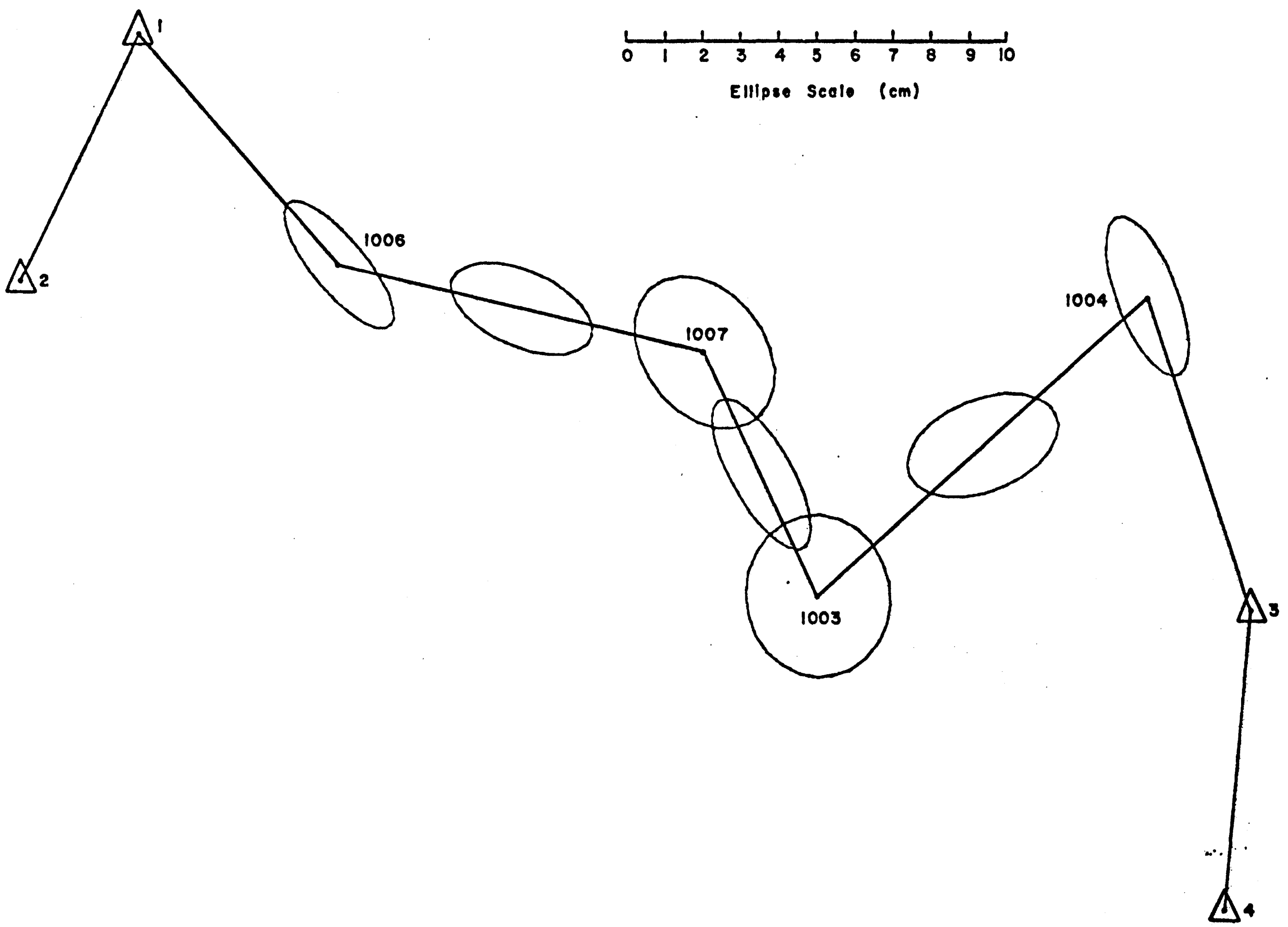


Figure 6.3 Plot of 95% Error Ellipses for Closed Traverse

Station Ellipses				Relative Ellipses				
Station	a(m)	b(m)	θ	Station	Station	a(m)	b(m)	θ
1003	.021	.019	-9°39'53"	1003	1004	.021	.012	69°24'07"
1004	.022	.008	-21°19'45"	1003	1007	.022	.008	-30°39'26"
1006	.021	.007	-40°51'12"	1007	1006	.020	.010	-67°07'34"
1007	.022	.016	-39°39'44"					

Table 6.2 95% Error Ellipses for Closed Traverse

The residuals V computed by equation (AII-17) are

$$\underset{(1,17)}{V^T} = (2.07, -2.07, 1.48, -1.48, 0.91, -0.91, 0.41, -0.41, 0.45, -0.45, -0.14, 0.14, 0.004, 0.000, -0.007, 0.003, -0.011), \qquad (6\text{-}2)$$

where the units are arcseconds for the first 12 residuals (i.e. for directions), and metres for the last five (distance residuals). Using these residuals and the P matrix computed earlier, the a posteriori variance factor $\hat{\sigma}_o^2$ is computed via equation (AII-18) as

$$\hat{\sigma}_o^2 = \frac{V^T PV}{df} = 1.9214 \quad . \qquad (6\text{-}3)$$

Both the residuals and a posteriori variance factor are used in chapter 9 for the post analysis procedures.

6.2 Network

Figure 6.4 shows the network considered in this example. It consists of 10 unknown stations and one fixed station with 38 directions, 17 distances, and 2 azimuths observed. Accounting for the 11 orientation unknowns, then, the degrees of freedom is 26. The initial point coordinates are listed in Table 6.3 and the observations and their standard deviations are given in Table 6.4.

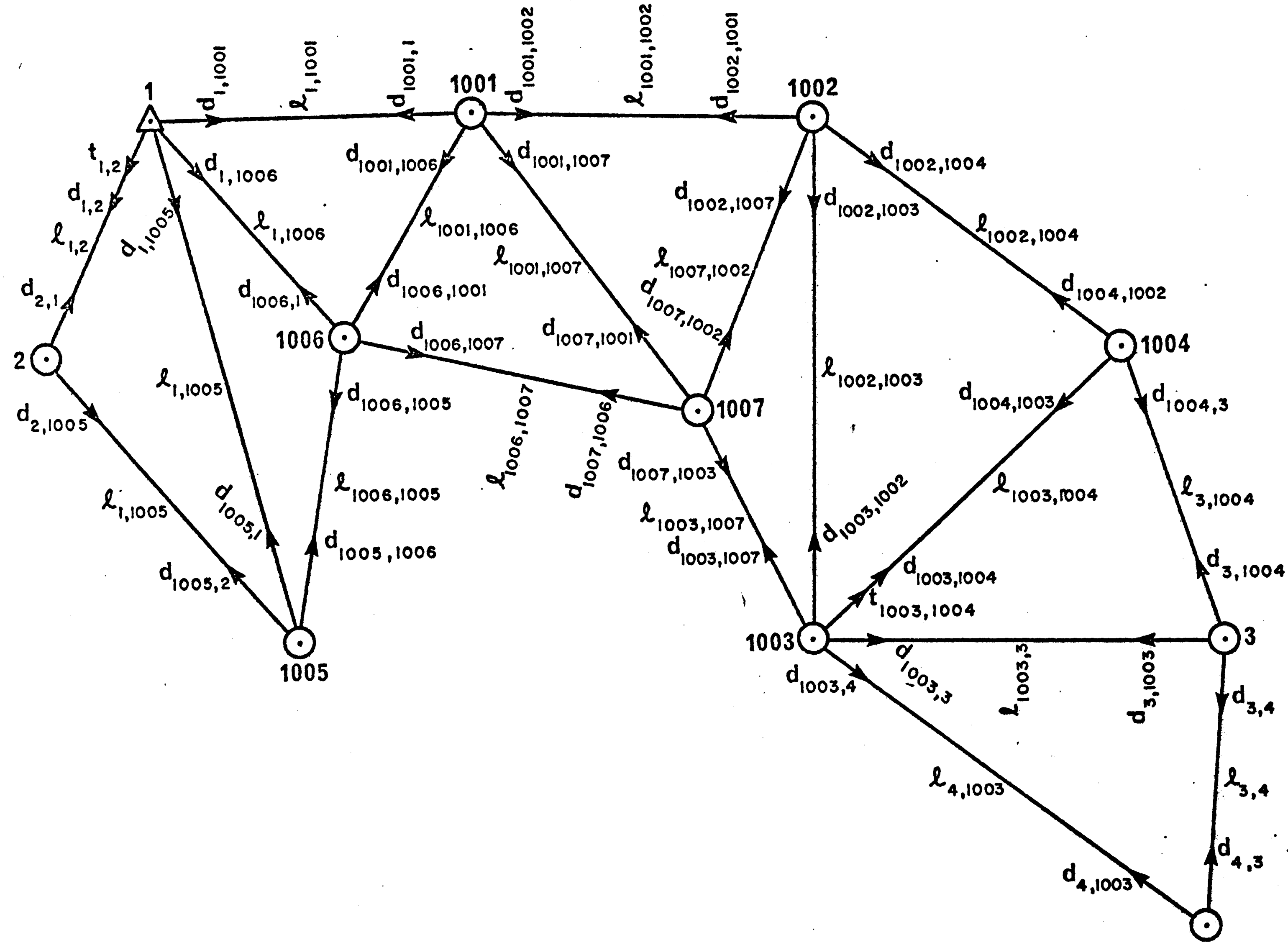


Figure 6.4 Network

Station	X	Y	Station	X	Y
1	2640.0	1160.0	1003	3265.0	645.0
2	2530.0	935.0	1004	3570.0	915.0
3	3660.0	630.0	1005	2770.0	655.0
4	3635.0	355.0	1006	2820.0	945.0
1001	2950.0	1160.0	1007	3160.0	865.0
1002	3280.0	1145.0			

Table 6.3 Initial Coordinates for Network Stations

Combining the mathematical models of sections 3.1 and 3.2 and chapter 4, the matrix form of the observation equations is

$$\underset{(57,1)}{V} = \underset{(57,1)}{W} + \underset{(57,31)}{A}\ \underset{(31,1)}{\hat{X}} .$$

Using the same techniques as in the previous example in section 6.1, the W matrix is computed as

$$\underset{(1,57)}{W^T} = (327''58,\ 928''78,\ 0.191\text{ m},\ 0.003\text{ m},\ 0.010\text{ m},\ -0.613\text{ m},\ -0.48\text{ m},$$
$$0.827\text{ m},\ -0.588\text{ m},\ 0.448\text{ m},\ 0.287\text{ m},\ 0.571\text{ m},\ 0.223\text{ m},\ 0.242\text{ m},$$
$$-1.011\text{ m},\ -0.198\text{ m},\ -0.112\text{ m},\ -0.603\text{ m},\ -0.774\text{ m},\ 0''0,\ -215''98,\ -388''51,$$
$$-337''42,\ 0''0,\ -344''78,\ -371''73,\ -527''22,\ 0''0,\ -59''62,\ -240''83,$$
$$-369''48,\ 0''0,\ 544''73,\ -304''18,\ 0''0,\ -497''86,\ -446''99,\ -367''48,$$
$$0''0,\ -161''91,\ 60''93,\ 41''22,\ 0''0,\ -67''31,\ -1361''01,\ -1193''51,$$
$$-276''02,\ 0''0,\ 236''22,\ -1184''82,\ 0''0,\ -204''24,\ 0''0,\ -688''87,$$
$$-582''99,\ 0''0,\ 522''36)\ ,$$

where the order of the elements of W is the same as the order of the observations in Table 6.4. The A matrix is too large to put conveniently on one page, so the nonzero elements of it are given on the following

Type	From	To	Value	σ	Type	From	To	Value	σ
Az	1	2	205°57'45".0	5".0	Dir	1006	1005	158°41'24".0	2".0
Az	1003	1004	48°13'31".0	5".0	Dir	1006	1	289°00'25".0	2".0
Dist	1003	3	395.094 m	.01 m	Dir	1005	1006	0°00'00".0	2".0
Dist	3	4	276.131 m	.01 m	Dir	1005	1003	81°13'25".0	2".0
Dist	1	2	250.440 m	.01 m	Dir	1005	1	335°51'58".0	2".0
Dist	4	1003	470.719 m	.01 m	Dir	1007	1002	0°00'00".0	2".0
Dist	2	1005	369.262 m	.01 m	Dir	1007	1003	131"25'33".0	2".0
Dist	1	1001	309.173 m	.01 m	Dir	1007	1006	260°08'58".0	2".0
Dist	1	1005	522.052 m	.01 m	Dir	1007	1001	301°27'28".0	2".0
Dist	1	1006	279.953 m	.01 m	Dir	1002	1004	0°00'00".0	2".0
Dist	1001	1006	250.960 m	.01 m	Dir	1002	1003	53°20'43".0	2".0
Dist	1001	1002	329.770 m	.01 m	Dir	1002	1007	74°45'49".0	2".0
Dist	1001	1007	361.889 m	.01 m	Dir	1002	1001	144°10'23".0	2".0
Dist	1006	1007	349.043 m	.01 m	Dir	1003	1002	0"00'00".0	2".0
Dist	1006	1005	295.290 m	.01 m	Dir	1003	1004	46°47'01".0	2".0
Dist	1007	1002	304.829 m	.01 m	Dir	1003	3	90°50'04".0	2".0
Dist	1007	1003	243.884 m	.01 m	Dir	1003	4	126°42'07".0	2".0
Dist	1002	1004	370.738 m	.01 m	Dir	1003	1007	332°50'40".0	2".0
Dist	1003	1004	408.113 m	.01 m	Dir	1004	1003	0°00'00".0	2".0
Dir	1	1001	0°0'00".0	2".0	Dir	1004	1002	79°52'09".0	2".0
Dir	1	1006	50°07'25".0	2".0	Dir	1004	3	294°19'13".0	2".0
Dir	1	1005	75°40'19".0	2".0	Dir	2	1	0°00'00".0	2".0
Dir	1	2	116°08'50".0	2".0	Dir	2	1005	113°24'07".0	2".0
Dir	1001	1002	0° 0'00".0	2".0	Dir	3	4	0°00'00".0	2".0
Dir	1001	1007	52°02'51".0	2".0	Dir	3	1003	87°10'18".0	2".0
Dir	1001	1006	118°39'36".0	2".0	Dir	3	1004	157°26'31".0	2".0
Dir	1001	1	177°32'38".0	2".0	Dir	4	1003	0°00'00".0	2".0
Dir	1006	1001	0° 0'00".0	2".0	Dir	4	3	56°57'38".0	2".0
Dir	1006	1007	72°05'52".0	2".0					

Table 6.4 Observations on the Mapping Plane for the Network

two pages. Note that the row number is given under the observation type, and the column numbers are given between dashes (e.g.-18-) over top of the actual number. Station 1 is fixed, and thus there are no columns appearing in A for it. All elements resulting from azimuth or direction observations are multiplied by ρ'' as was done in section 6.1.

The P matrix is diagonal of size 57 x 57 with the inverse of the standard deviations (see Table 6.4) squared on the diagonal. The first two diagonal elements are 0.04, the next 17 are 10000.0, and the last 38 elements are 0.25. P thus takes the shape

$$\underset{(57,57)}{P} = \begin{matrix} \\ \\ 3- \\ \\ \\ \\ 20- \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 0.04 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0.04 & 0 & & & & & \cdot \\ \cdot & 0 & 10000.0 & & & & & \cdot \\ \cdot & & & \ddots & & & & \cdot \\ \cdot & & & & 10000.0 & & & \cdot \\ \cdot & & & & & 0.25 & & \cdot \\ \cdot & & & & & & \ddots & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0.25 \end{bmatrix} .$$

Using the above A, P and W matrices in equation (AII-11) to compute the solution vector for the first iteration yields

$$\underset{(1,20)}{\hat{X}^T} = (0.3616, -0.1768, 0.8559, 1.6254, 1.2828, 1.5808, -0.8268, 1.0078, -1.3181, 2.9488, 1.0736, 2.3229, 0.4415, 4.2027, 0.8414, -0.3917, 0.1877, 0.7411, 0.2573, 2.0613) ,$$

in units of metres. The orientation unknowns are not given as they are not required. Adding this solution to the approximate coordinates of

Observation	Row	Station	X col	Y col	X coef.	Y coef.	Station	X col	Y col	X coef.	Y coef.	Orient. col	Orient. coef.
AZIMUTH	1	1	(X,Y) FIXED		0.7398897D+03	-0.3617239D+03	2	- 1-	- 2-	-0.7398897D+03	0.3617239D+03		
AZIMUTH	2	1003	-11-	-12-	-0.3356426D+03	0.3791518D+03	1004	-13-	-14-	0.3356426D+03	-0.3791518D+03		
DISTANCE	3	1003	-11-	-12-	-0.9992797D+00	0.3794733D-01	3	- 3-	- 4-	0.9992797D+00	-0.3794733D-01		
DISTANCE	4	3	- 3-	- 4-	0.9053575D-01	0.9958932D+00	4	- 5-	- 6-	-0.9053575D-01	-0.9958932D+00		
DISTANCE	5	1	(X,Y) FIXED		0.4392101D+00	0.8983844D+00	2	- 1-	- 2-	-0.4392101D+00	-0.8983844D+00		
DISTANCE	6	4	- 5-	- 6-	0.7870559D+00	-0.6168817D+00	1003	-11-	-12-	-0.7870559D+00	0.6168817D+00		
DISTANCE	7	2	- 1-	- 2-	-0.6507914D+00	0.7592566D+00	1005	-15-	-16-	0.6507914D+00	-0.7592566D+00		
DISTANCE	8	1	(X,Y) FIXED		-0.1000000D+01	0.0	1001	- 7-	- 8-	0.1000000D+01	0.0		
DISTANCE	9	1	(X,Y) FIXED		-0.2492980D+00	0.9684268D+00	1005	-15-	-16-	0.2492980D+00	-0.9684268D+00		
DISTANCE	10	1	(X,Y) FIXED		-0.6419367D+00	0.7667577D+00	1006	-17-	-18-	0.6419367D+00	-0.7667577D+00		
DISTANCE	11	1001	- 7-	- 8-	0.5174193D+00	0.8557320D+00	1006	-17-	-18-	-0.5174193D+00	-0.8557320D+00		
DISTANCE	12	1001	- 7-	- 8-	-0.9989685D+00	0.4540766D-01	1002	- 9-	-10-	0.9989685D+00	-0.4540766D-01		
DISTANCE	13	1001	- 7-	- 8-	-0.5799313D+00	0.8146654D+00	1007	-19-	-20-	0.5799313D+00	-0.8146654D+00		
DISTANCE	14	1006	-17-	-18-	-0.9734172D+00	0.2290393D+00	1007	-19-	-20-	0.9734172D+00	-0.2290393D+00		
DISTANCE	15	1006	-17-	-18-	0.1699069D+00	0.9854601D+00	1005	-15-	-16-	-0.1699069D+00	-0.9854601D+00		
DISTANCE	16	1007	-19-	-20-	-0.3939193D+00	-0.9191450D+00	1002	- 9-	-10-	0.3939193D+00	0.9191450D+00		
DISTANCE	17	1007	-19-	-20-	-0.4307296D+00	0.9024810D+00	1003	-11-	-12-	0.4307296D+00	-0.9024810D+00		
DISTANCE	18	1002	- 9-	-10-	-0.7834977D+00	0.6213947D+00	1004	-13-	-14-	0.7834977D+00	-0.6213947D+00		
DISTANCE	19	1003	-11-	-12-	-0.7487622D+00	-0.6628387D+00	1004	-13-	-14-	0.7487622D+00	0.6628387D+00		
DIRECTION 1	20	1	(X,Y) FIXED		0.0	0.6653703D+03	1001	- 7-	- 8-	0.0	-0.6653703D+03	-21-	-0.1D+01
DIRECTION 2	21	1	(X,Y) FIXED		0.5640309D+03	0.4722120D+03	1006	-17-	-18-	-0.5640309D+03	-0.4722120D+03	-21-	-0.1D+01
DIRECTION 3	22	1	(X,Y) FIXED		0.3830605D+03	0.9860963D+02	1005	-15-	-16-	-0.3830605D+03	-0.9860963D+02	-21-	-0.1D+01
DIRECTION 4	23	1	(X,Y) FIXED		0.7398897D+03	-0.3617239D+03	2	- 1-	- 2-	-0.7398897D+03	0.3617239D+03	-21-	-0.1D+01
DIRECTION 1	24	1001	- 7-	- 8-	0.2835255D+02	0.6237561D+03	1002	- 9-	-10-	-0.2835255D+02	-0.6237561D+03	-22-	-0.1D+01
DIRECTION 2	25	1001	- 7-	- 8-	0.4640467D+03	0.3303383D+03	1007	-19-	-20-	-0.4640467D+03	-0.3303383D+03	-22-	-0.1D+01
DIRECTION 3	26	1001	- 7-	- 8-	0.7025257D+03	-0.4247830D+03	1006	-17-	-18-	-0.7025257D+03	0.4247830D+03	-22-	-0.1D+01
DIRECTION 4	27	1001	- 7-	- 8-	0.0	-0.6653703D+03	1	(X,Y) FIXED		0.0	0.6653703D+03	-22-	-0.1D+01
DIRECTION 1	28	1006	-17-	-18-	-0.7025257D+03	0.4247830D+03	1001	- 7-	- 8-	0.7025257D+03	-0.4247830D+03	-23-	-0.1D+01
DIRECTION 2	29	1006	-17-	-18-	0.1352556D+03	0.5748363D+03	1007	-19-	-20-	-0.1352556D+03	-0.5743363D+03	-23-	-0.1D+01
DIRECTION 3	30	1006	-17-	-18-	0.6907251D+03	-0.1190905D+03	1005	-15-	-16-	-0.6907251D+03	0.1190905D+03	-23-	-0.1D+01

A matrix, rows 1-30

DIRECTION 31	4	1006	-17- (X,Y) -18-	-0.5640309D+03	-0.4722120D+03	1	(X,Y) FIXED	0.5640309D+03	0.4722120D+03	-23- -0.1D+01
DIRECTION 32	1	1005	-15- (X,Y) -16-	-0.6907251D+03	0.1190905D+03	1006	-17- (X,Y) -18-	0.6907251D+03	-0.1190905D+03	-24- -0.1D+01
DIRECTION 33	2	1005	-15- (X,Y) -16-	0.8414678D+01	0.4165266D+03	1003	-11- (X,Y) -12-	-0.8414678D+01	-0.4165266D+03	-24- -0.1D+01
DIRECTION 34	3	1005	-15- (X,Y) -16-	-0.3830605D+03	-0.9860963D+02	1	(X,Y) FIXED	0.3830605D+03	0.9860963D+02	-24- -0.1D+01
DIRECTION 35	1	1007	-19- (X,Y) -20-	-0.6223507D+03	0.2667217D+03	1002	- 9- (X,Y) -10-	0.6223507D+03	-0.2667217D+03	-25- -0.1D+01
DIRECTION 36	2	1007	-19- (X,Y) -20-	0.7636223D+03	0.3644561D+03	1003	-11- (X,Y) -12-	-0.7636223D+03	-0.3644561D+03	-25- -0.1D+01
DIRECTION 37	3	1007	-19- (X,Y) -20-	-0.1352556D+03	-0.5748363D+03	1006	-17- (X,Y) -18-	0.1352556D+03	0.5748363D+03	-25- -0.1D+01
DIRECTION 38	4	1007	-19- (X,Y) -20-	-0.4640467D+03	-0.3303383D+03	1001	- 7- (X,Y) - 8-	0.4640467D+03	0.3303383D+03	-25- -0.1D+01
DIRECTION 39	1	1002	- 9- (X,Y) -10-	0.3462840D+03	0.4366189D+03	1004	-13- (X,Y) -14-	-0.3462840D+03	-0.4366189D+03	-26- -0.1D+01
DIRECTION 40	2	1002	- 9- (X,Y) -10-	0.4121587D+03	-0.1236476D+02	1003	-11- (X,Y) -12-	-0.4121587D+03	0.1236476D+02	-26- -0.1D+01
DIRECTION 41	3	1002	- 9- (X,Y) -10-	0.6223507D+03	-0.2667217D+03	1007	-19- (X,Y) -20-	-0.6223507D+03	0.2667217D+03	-26- -0.1D+01
DIRECTION 42	4	1002	- 9- (X,Y) -10-	-0.2835255D+02	-0.6237561D+03	1001	- 7- (X,Y) - 8-	0.2835255D+02	0.6237561D+03	-26- -0.1D+01
DIRECTION 43	1	1003	-11- (X,Y) -12-	-0.4121587D+03	0.1236476D+02	1002	- 9- (X,Y) -10-	0.4121587D+03	-0.1236476D+02	-27- -0.1D+01
DIRECTION 44	2	1003	-11- (X,Y) -12-	-0.3356426D+03	0.3791518D+03	1004	-13- (X,Y) -14-	0.3356426D+03	-0.3791518D+03	-27- -0.1D+01
DIRECTION 45	3	1003	-11- (X,Y) -12-	0.1980142D+02	0.5214374D+03	3	- 3- (X,Y) - 4-	-0.1980142D+02	-0.5214374D+03	-27- -0.1D+01
DIRECTION 46	4	1003	-11- (X,Y) -12-	0.2706642D+03	0.3453302D+03	4	- 5- (X,Y) - 6-	-0.2706642D+03	-0.3453302D+03	-27- -0.1D+01
DIRECTION 47	5	1003	-11- (X,Y) -12-	-0.7636223D+03	-0.3644561D+03	1007	-19- (X,Y) -20-	0.7636223D+03	0.3644561D+03	-27- -0.1D+01
DIRECTION 48	1	1004	-13- (X,Y) -14-	0.3356426D+03	-0.3791518D+03	1003	-11- (X,Y) -12-	-0.3356426D+03	0.3791518D+03	-28- -0.1D+01
DIRECTION 49	2	1004	-13- (X,Y) -14-	-0.3462840D+03	-0.4366189D+03	1002	- 9- (X,Y) -10-	0.3462840D+03	0.4366189D+03	-28- -0.1D+01
DIRECTION 50	3	1004	-13- (X,Y) -14-	0.6581077D+03	0.2078235D+03	3	- 7- (X,Y) - 8-	-0.6581077D+03	-0.2078235D+03	-28- -0.1D+01
DIRECTION 51	1	2	- 1- (X,Y) - 2-	-0.7398897D+03	0.3617239D+03	1	(X,Y) FIXED	0.7398897D+03	-0.3617239D+03	-29- -0.1D+01
DIRECTION 52	2	2	- 1- (X,Y) - 2-	0.4246628D+03	0.3639967D+03	1005	-15- (X,Y) -16-	-0.4246628D+03	-0.3639967D+03	-29- -0.1D+01
DIRECTION 53	1	3	- 3- (X,Y) - 4-	0.7439059D+03	-0.6762781D+02	4	- 5- (X,Y) - 6-	-0.7439059D+03	0.6762781D+02	-30- -0.1D+01
DIRECTION 54	2	3	- 3- (X,Y) - 4-	-0.1980142D+02	-0.5214374D+03	1003	-11- (X,Y) -12-	0.1980142D+02	0.5214374D+03	-30- -0.1D+01
DIRECTION 55	3	3	- 3- (X,Y) - 4-	-0.6581077D+03	-0.2078235D+03	1004	-13- (X,Y) -14-	0.6581077D+03	0.2078235D+03	-30- -0.1D+01
DIRECTION 56	1	4	- 5- (X,Y) - 6-	-0.2706642D+03	-0.3453302D+03	1003	-11- (X,Y) -12-	0.2706642D+03	0.3453302D+03	-31- -0.1D+01
DIRECTION 57	2	4	- 5- (X,Y) - 6-	-0.7439059D+03	0.6762781D+02	3	- 3- (X,Y) - 4-	0.7439059D+03	-0.6762781D+02	-31- -0.1D+01

A matrix, rows 31-57

Table 6.3 results in the parameter vector

$X^T = X^{\circ T} + \hat{X}^T$ = (2530.362, 934.823, 3660.856, 631.625, 3636.283, 356.581, 2949.173, 1161.008, 3278.682, 1147.949, 3266.074, 647.323, 3570.442, 919.203, 2770.841, 654.608, 2820.188, 945.741, 3160.257, 867.061).

Taking these parameters as new approximate coordinates X°, reevaluating the A, P and W matrices, and computing a second iteration solution vector via equation (AII-11) results in

$\hat{X}^T$ = (-0.00006, -0.00004, -0.00898, -0.00012, -0.00821, 0.00147, -0.00136, -0.00253, -0.00649, -0.00518, -0.00375, -0.00082, -0.00707, 0.00096, 0.00073, -0.00062, -0.00155, -0.00031, -0.00369, -0.00124),

which yields a second iteration parameter vector of

$X^T = X^{\circ T} + X^T$ = (2530.362, 934.823, 3660.847, 631.625, 3636.275, 356.582, 2949.172, 1161.005, 3278.675, 1147.944, 3266.070, 647.322, 3570.434, 919.204, 2770.842, 654.608, 2820.186, 945.741, 3160.254, 867.060). (6-4)

Using these parameters as new approximate coordinates results in a zero vector for the third iteration solution vector, and thus the parameters in equation (6-4) are the final adjusted coordinates.

The variance covariance matrix C_x of the parameters is computed by equation (AII-16) (assuming a priori variance factor equals 1) and is given on the following page. Note that since the C_x matrix is too large to display in a normal fashion, it has been printed in rows of six columns at a time.

The 95% station and relative error ellipses are computed using the equations of Appendix III. They are listed below in Table 6.5 and plotted in Figure 6.5.

	X 2	Y	X 3	Y	X 4	Y
	(COL 1)	(COL 2)	(COL 3)	(COL 4)	(COL 5)	(COL 6)
1	0.23133042D-04	0.50139064D-05	0.33533155D-04	0.76588172D-04	0.51176153D-04	0.75715723D-04
2	0.50139064D-05	0.39027383D-04	-0.15854217D-04	-0.11625130D-04	-0.19519027D-04	-0.10484462D-04
3	0.33533155D-04	-0.15854217D-04	0.15827713D-03	0.14838047D-03	0.20431126D-03	0.13363231D-03
4	0.76588172D-04	-0.11625130D-04	0.14838047D-03	0.33620915D-03	0.24385731D-03	0.37526903D-03
5	0.51176153D-04	-0.19519027D-04	0.20431126D-03	0.24385731D-03	0.28553796D-03	0.22674168D-03
6	0.75715723D-04	-0.10484462D-04	0.13363231D-03	0.37526903D-03	0.22674168D-03	0.39056075D-03
7	0.10432330D-06	-0.32763930D-05	0.18077056D-04	-0.11528971D-04	0.15788683D-04	-0.11942956D-04
8	0.23908104D-04	-0.61470731D-05	0.52503122D-04	0.10745258D-03	0.77656787D-04	0.10513297D-03
9	0.11627271D-05	-0.56477673D-05	0.42166143D-04	-0.13500268D-04	0.38989935D-04	-0.15514193D-04
10	0.47773747D-04	-0.10890695D-04	0.10854620D-03	0.22286236D-03	0.16264025D-03	0.21704862D-03
11	0.33933038D-04	-0.13737982D-04	0.13132208D-03	0.15302481D-03	0.17621438D-03	0.14714397D-03
12	0.51452599D-04	-0.60363358D-05	0.81622584D-04	0.24370604D-03	0.13547169D-03	0.24305130D-03
13	0.16244278D-04	-0.11728576D-04	0.10258854D-03	0.59132437D-04	0.12107980D-03	0.50573273D-04
14	0.67410160D-04	-0.12015950D-04	0.14481827D-03	0.32651904D-03	0.22536122D-03	0.31837600D-03
15	0.39970625D-04	-0.11163481D-04	0.89750751D-04	0.16743677D-03	0.12987859D-03	0.16454358D-03
16	0.15419217D-04	0.86461748D-05	0.60772597D-05	0.60415475D-04	0.18080192D-04	0.61114659D-04
17	0.16522575D-04	-0.65523955D-05	0.45525936D-04	0.65518081D-04	0.61801215D-04	0.63952211D-04
18	0.14466803D-04	-0.20732048D-05	0.18873242D-04	0.68622795D-04	0.34142247D-04	0.67960424D-04
19	0.21077059D-04	-0.90867760D-05	0.85413401D-04	0.86395069D-04	0.10863831D-03	0.83092960D-04
20	0.40083471D-04	-0.77443964D-05	0.72880893D-04	0.13817722D-03	0.11755636D-03	0.18569338D-03

	X 1001	Y	X 1002	Y	X 1003	Y
	(COL 7)	(COL 8)	(COL 9)	(COL 10)	(COL 11)	(COL 12)
1	0.10432330D-06	0.23908104D-04	0.11627271D-05	0.47773747D-04	0.33933038D-04	0.51452599D-04
2	-0.32763930D-05	-0.61470731D-05	-0.56477673D-05	-0.10890695D-04	-0.13737982D-04	-0.60363358D-05
3	0.18077056D-04	0.52503122D-04	0.42166143D-04	0.10854620D-03	0.13132208D-03	0.81622584D-04
4	-0.11528971D-04	0.10745258D-03	-0.13500268D-04	0.22286236D-03	0.15302481D-03	0.24370604D-03
5	0.15788683D-04	0.77656787D-04	0.38989935D-04	0.16264025D-03	0.17621438D-03	0.13547169D-03
6	-0.11942956D-04	0.10513297D-03	-0.15514193D-04	0.21704862D-03	0.14714397D-03	0.24305130D-03
7	0.18679580D-04	-0.15035713D-05	0.21810387D-04	-0.58391609D-05	0.16759839D-04	-0.84933246D-05
8	-0.15035713D-05	0.37871576D-04	0.26960182D-05	0.71928502D-04	0.50618532D-04	0.71203980D-04
9	0.21810387D-04	0.26960182D-05	0.45985773D-04	-0.86318238D-06	0.36966390D-04	-0.98669309D-05
10	-0.58391609D-05	0.71928502D-04	-0.86318238D-06	0.14898700D-03	0.10276375D-03	0.14466694D-03
11	0.16759839D-04	0.50618532D-04	0.36966390D-04	0.10276375D-03	0.12342306D-03	0.87457363D-04
12	-0.84933246D-05	0.71203980D-04	-0.98669309D-05	0.14466694D-03	0.87457368D-04	0.16776205D-03
13	0.20115962D-04	0.27253733D-04	0.44370538D-04	0.53677624D-04	0.85752681D-04	0.29965617D-04
14	-0.91574834D-05	0.96907336D-04	-0.70625029D-05	0.20192228D-03	0.13886118D-03	0.21087157D-03
15	0.81462727D-05	0.53219069D-04	0.13081241D-04	0.10681412D-03	0.88554252D-04	0.10991036D-03
16	-0.88522772D-05	0.16815244D-04	-0.11491520D-04	0.33794055D-04	0.93338924D-05	0.43818934D-04
17	0.98999305D-05	0.21432761D-04	0.13354679D-04	0.42840319D-04	0.44128937D-04	0.42031007D-04
18	-0.87294432D-05	0.22483273D-04	-0.10077109D-04	0.44044240D-04	0.19476094D-04	0.46922453D-04
19	0.18579378D-04	0.30337487D-04	0.37658607D-04	0.53494800D-04	0.80902041D-04	0.52441866D-04
20	-0.96201704D-05	0.58769272D-04	-0.13144562D-04	0.11951536D-03	0.73282790D-04	0.12443133D-03

	X 1004	Y	X 1005	Y	X 1006	Y
	(COL 13)	(COL 14)	(COL 15)	(COL 16)	(COL 17)	(COL 18)
1	0.16244278D-04	0.67410160D-04	0.39970625D-04	0.15419217D-04	0.16522575D-04	0.14466803D-04
2	-0.11728576D-04	-0.12015950D-04	-0.11163481D-04	0.86461748D-05	-0.65523955D-05	-0.20732048D-05
3	0.10258854D-03	0.14481827D-03	0.89750751D-04	0.60772597D-05	0.45525936D-04	0.18873242D-04
4	0.59132437D-04	0.32651904D-03	0.16743677D-03	0.60415475D-04	0.65518081D-04	0.68622795D-04
5	0.12107980D-03	0.22536122D-03	0.12987859D-03	0.18080192D-04	0.61801215D-04	0.34142247D-04
6	0.50573273D-04	0.31837600D-03	0.16454358D-03	0.61114659D-04	0.63952211D-04	0.67960424D-04
7	0.20115962D-04	-0.91574834D-05	0.81462727D-05	-0.88522772D-05	0.98999305D-05	-0.87294432D-05
8	0.27253733D-04	0.96907336D-04	0.53219069D-04	0.16815244D-04	0.21432761D-04	0.22483273D-04
9	0.44370538D-04	-0.70625029D-05	0.13081241D-04	-0.11491520D-04	0.13354679D-04	-0.10077109D-04
10	0.53677624D-04	0.20192228D-03	0.10681412D-03	0.33794055D-04	0.42840319D-04	0.44044240D-04
11	0.85752681D-04	0.13886118D-03	0.88554252D-04	0.93338924D-05	0.44128937D-04	0.19476094D-04
12	0.29965617D-04	0.21087157D-03	0.10991036D-03	0.43818934D-04	0.42031007D-04	0.46922453D-04
13	0.82059826D-04	0.60966534D-04	0.49948079D-04	-0.49891397D-05	0.29184408D-04	0.38878236D-05
14	0.60966534D-04	0.29655347D-03	0.14873723D-03	0.51077042D-04	0.58852218D-04	0.60535116D-04
15	0.49948079D-04	0.14873723D-03	0.10035699D-03	0.17003850D-04	0.44576263D-04	0.27314809D-04
16	-0.49891397D-05	0.51077042D-04	0.17003850D-04	0.32883711D-04	0.32366147D-05	0.15523093D-04
17	0.29184408D-04	0.58852218D-04	0.44576263D-04	0.32366147D-05	0.23948026D-04	0.83038718D-05
18	0.38878236D-05	0.60535116D-04	0.27314809D-04	0.15523093D-04	0.83038718D-05	0.20975137D-04
19	0.61618083D-04	0.79315743D-04	0.57623321D-04	0.30820896D-05	0.31078955D-04	0.74300498D-05
20	0.28840419D-04	0.16587234D-03	0.87034098D-04	0.31421840D-04	0.33249303D-04	0.40942867D-04

	X 1007	Y
	(COL 19)	(COL 20)
1	0.21077059D-04	0.40083471D-04
2	-0.90867760D-05	-0.77443964D-05
3	0.85413401D-04	0.72880893D-04
4	0.86395069D-04	0.13817722D-03
5	0.10863831D-03	0.11755636D-03
6	0.83092960D-04	0.18569338D-03
7	0.18579378D-04	-0.96201704D-05
8	0.30337487D-04	0.58769272D-04
9	0.37658607D-04	-0.13144562D-04
10	0.53494800D-04	0.11951536D-03
11	0.80902041D-04	0.73282790D-04
12	0.52441866D-04	0.12443133D-03
13	0.61618083D-04	0.28840419D-04
14	0.79315743D-04	0.16587234D-03
15	0.57623321D-04	0.87034098D-04
16	0.30820896D-05	0.31421840D-04
17	0.31078955D-04	0.33249303D-04
18	0.74300498D-05	0.40942867D-04
19	0.62069715D-04	0.37195858D-04
20	0.37195858D-04	0.10701254D-03

C_v matrix for Network.

STATION ELLIPSES

Station	a(m)	b(m)	
2	.016	.011	16°07'25"
3	.052	.023	26°14'12"
4	.058	.025	38°28'49"
1001	.015	.011	-4°27'10"
1002	.030	.017	-0°28'49"
1003	.038	.018	37°53'16"
1004	.043	.020	14°48'29"
1005	.025	.013	76°37'28"
1006	.014	.009	50°14'28"
1007	.028	.016	29°25'52"

RELATIVE ELLIPSES

Station	Station	a(m)	b(m)	
2	1005	.020	.014	36°20'31"
1005	1006	.015	.011	-69°55'41"
1001	1006	.012	.008	-63°44'52"
1001	1002	.016	.011	2°01'21"
1001	1007	.018	.011	59°14'16"
1006	1007	.017	.012	11°45'51"
1002	1007	.015	.009	-65°34'59"
1002	1003	.024	.013	-84°45'14"
1002	1004	.018	.013	42°27'42"
1003	1004	.019	.010	-39°02'59"
1003	1007	.013	.012	-21°58'22"
3	1004	.015	.013	58°21'17"
3	1003	.020	.011	1°26'04"
3	4	.015	.012	-76°04'26"
4	1003	.024	.014	38°01'35"

Table 6.5. 95% Error Ellipses for Network

The residuals V computed using equation (AII-17) are

$$\underset{(1,57)}{V^T} = (-3.89,\ 3.89,\ -0.005,\ 0.007,\ 0.010,\ 0.005,\ -0.004,\ 0.000,\ 0.003,\ 0.001,\ -0.010,\ -0.008,\ -0.006,\ 0.008,\ -0.005,\ -0.002,\ 0.005,\ -0.002,\ 0.002,\ 0.13,\ -0.72,\ -1.31,\ 1.90,\ -0.01,\ 1.34,\ -0.55,\ -0.78,\ 0.45,\ -1.29,\ -0.51,\ 1.36,\ 1.02,\ -0.33,\ -0.69,\ -0.37,\ 0.45,\ 1.68,\ -1.76,\ -1.20,\ 0.84,\ 0.05,\ 0.32,\ -0.08,\ 1.17,\ 0.31,\ -0.35,\ -1.05,\ -1.98,\ 1.72,\ 0.26,\ -0.88,\ 0.88,\ -0.16,\ 0.03,\ 0.13,\ -0.73,\ 0.73), \quad (6\text{-}5)$$

where the units are arcseconds for the first two residuals, metres for the next 17, and arcseconds for the last 38 residuals. The a posteriori variance factor $\hat{\sigma}_o^2$ is computed using the above residuals, the P matrix given above, and equation (AII-18) as

$$\hat{\sigma}_o^2 = \frac{V^T P V}{df} = 0.58488 \ . \qquad (6\text{-}6)$$

The residuals and a posterior variance factor are considered again in Chapter 9 for postanalysis of the network.

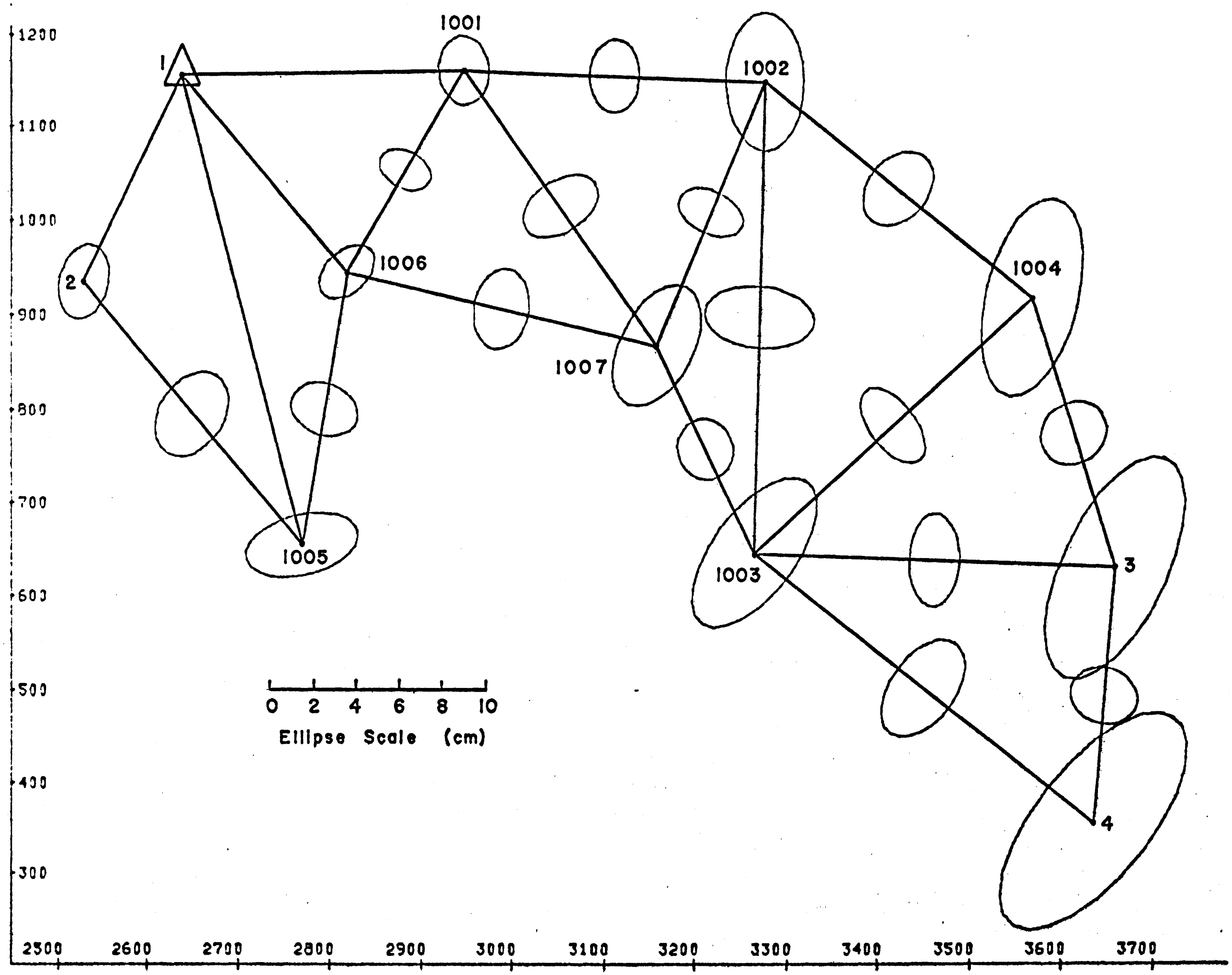


Figure 6.5 Plot of 95% Error Ellipses for the Network

7. A PRIORI KNOWLEDGE OF PARAMETERS

This chapter considers points which have some independent a priori estimate of their position. The coordinates of these points are treated as observables, and have the following simple observation equation:

$$L_x = X \quad , \tag{7-1}$$

with associated variance covariance matrix C_{L_x}, the accuracy estimate of these so-called weighted parameters. By expanding the matrices of Appendix II to include these new observables, the least squares estimate of the solution vector (cf. eq. (AII-11)) becomes

$$X = -\left[[A^T \; I] \begin{bmatrix} C_L & 0 \\ 0 & C_{L_x} \end{bmatrix}^{-1} \begin{bmatrix} A \\ I \end{bmatrix} \right]^{-1} [A^T \; I] \begin{bmatrix} C_L & 0 \\ 0 & C_{L_x} \end{bmatrix}^{-1} \begin{bmatrix} W \\ W_{L_x} \end{bmatrix}, \tag{7-2}$$

or, multiplying the matrices together,

$$X = -[A^T PA + P_x]^{-1} [A^T PW + P_x W_{L_x}] \; , \tag{7-3}$$

where $P_x = C_{L_x}^{-1}$,

and $W_{L_x} = X^\circ - L_x$.

Note that $W_{L_x} = 0$ for the first iteration if X° is taken equal to L_x. This is not the case for the second and subsequent iterations.

Although not done here, it can be shown (e.g. Krakiwsky [1975]), that the corresponding variance covariance matrix of the parameters X (cf. eq. (AII-16)) is

$$C_x = [A^T PA + P_x]^{-1} \; . \tag{7-4}$$

Thus, the only difference between a priori knowledge and no a priori knowledge of the parameters for the accuracy estimate of the parameters is the addition of the P_x matrix to the normal equations.

Since the weighted parameters are treated as observables, the degrees of freedom for the adjustment change from df = n-u to

$$df = n - u + u_x \; , \tag{7-5}$$

where u_x = number of weighted parameters.

Another consequence of this is that the a posteriori variance factor (cf. eq. (AII-18)) is now computed as

$$\hat{\sigma}_o^2 = \frac{V^T P V + V_{L_x} P_x V_{L_x}}{df} \; , \tag{7-6}$$

where $V_{L_x} = A\hat{X} + W_{L_x}$ (cf. eq. (AII-17)) are the residual corrections to the weighted coordinates. Noting that the A matrix for L_x is equal to I, then V_{L_x} is simply

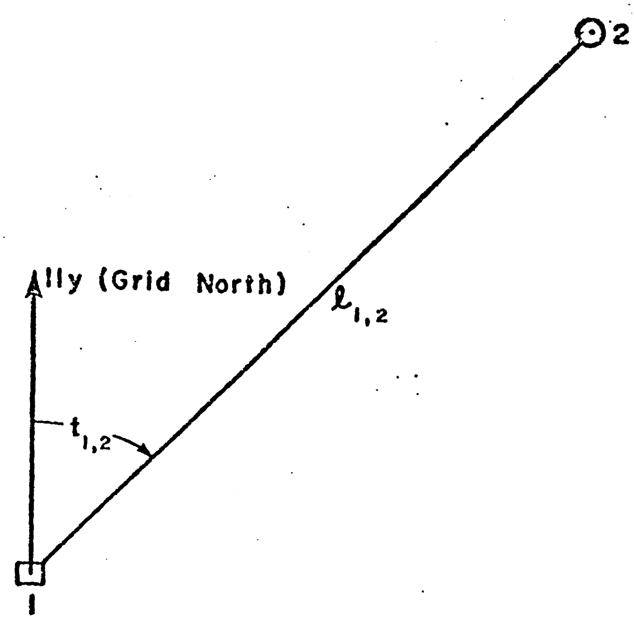
$$V_{L_x} = \hat{X} + W_{L_x} = \sum_{i=1}^{m} \hat{X}_i \; , \tag{7-7}$$

where m = number of iterations in the adjustment.

The example shown in Figure 7.1 is exactly the same as that of section 5.1 except that station 1 is now weighted with an a priori variance covariance matrix of

$$C_{L_{x_1}} = \begin{bmatrix} 0.4455.10^{-1} & -0.709.10^{-3} \\ -0.709.10^{-3} & 0.9535.10^{-1} \end{bmatrix} \tag{7-8}$$

in units of m^2.



This corresponds to a standard error ellipse a = 0.309 m, b = 0.211 m and $\theta = -0° \, 47' \, 58''$. The initial data (i.e. approximate point coordinates, observations and their standard deviations) are given in Table 5.1. The observation equation is

$$\underset{(2,1)}{V} = \underset{(2,1)}{W} + \underset{(2,4)}{A} \; \underset{(4,1)}{\hat{X}} , \tag{7-9}$$

or, explicitly

$$\underset{(2,1)}{V} = \begin{bmatrix} \arctan\left(\dfrac{x_2^\circ - x_1^\circ}{y_2^\circ - y_1^\circ}\right) - t_{1,2} \\ \left((x_2^\circ - x_1^\circ)^2 + (y_2^\circ - y_1^\circ)^2\right)^{1/2} - \ell_{1,2} \end{bmatrix} + \begin{bmatrix} \dfrac{-\rho''(y_2^\circ - y_1^\circ)}{(\ell_{1,2}^\circ)^2} & \dfrac{\rho''(x_2^\circ - x_1^\circ)}{(\ell_{1,2}^\circ)^2} & \dfrac{\rho''(y_2^\circ - y_1^\circ)}{(\ell_{1,2}^\circ)^2} & \dfrac{-\rho''(x_2^\circ - x_1^\circ)}{(\ell_{1,2}^\circ)^2} \\ \dfrac{-(x_2^\circ - x_1^\circ)}{\ell_{1,2}^\circ} & \dfrac{-(y_2^\circ - y_1^\circ)}{\ell_{1,2}^\circ} & \dfrac{(x_2^\circ - x_1^\circ)}{\ell_{1,2}^\circ} & \dfrac{(y_2^\circ - y_1^\circ)}{\ell_{1,2}^\circ} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta y_1 \\ \delta x_2 \\ \delta y_2 \end{bmatrix} , \tag{7-10}$$

where the units are

$$\underset{(2,1)}{V} = \begin{bmatrix} '' \\ m \end{bmatrix} + \begin{bmatrix} ('').m^{-1} & ('').m^{-1} & ('').m^{-1} & ('').m^{-1} \\ - & - & - & - \end{bmatrix} \begin{bmatrix} m \\ m \\ m \\ m \end{bmatrix} . \tag{7-11}$$

Evaluating A and W using the coordinates and observations of Table 5.1 yields

$$A = \begin{bmatrix} -59.17941 & 57.65335 & 59.17941 & -57.65335 \\ -0.6978111 & -0.7162819 & 0.6978111 & 0.7162819 \end{bmatrix} , \quad W = \begin{bmatrix} -23.0324 \text{ m} \\ 0''.11655 \end{bmatrix} .$$

The weight matrix P of the observations is identical to that of section 5.1. Employing formula (7-3) to compute the solution vector $\hat{X}$ gives $\hat{X}$ for the first iteration as

$$\hat{X} = \begin{bmatrix} 0.00000 \text{ m} \\ 0.00000 \text{ m} \\ 0.11835 \text{ m} \\ -0.27802 \text{ m} \end{bmatrix} .$$

which yield parameters X of

$$X = X^\circ + \hat{X} = \begin{bmatrix} 377164.887 \\ 862395.774 \\ 378907.0 \\ 864184.0 \end{bmatrix} + \begin{bmatrix} 0.00000 \\ 0.00000 \\ 0.11835 \\ -0.27802 \end{bmatrix} = \begin{bmatrix} 377164.877 \text{ m} \\ 862395.774 \text{ m} \\ 378907.118 \text{ m} \\ 864183.722 \text{ m} \end{bmatrix}. \quad (7\text{-}12)$$

Using these parameter values as new approximate coordinates, and recomputing A and W gives the second iteration solution vector as

$$\hat{X} = \begin{bmatrix} 0.00000 \\ 0.00000 \\ -0.00002 \\ 0.00000 \end{bmatrix},$$

and the solution has converged. Thus, the final least squares estimate of the parameters is given by equation (7-12), which is identical to the solution obtained in section 5.1.

The variance covariance matrix of the parameters computed according to equation (7-4) is

$$C_x = \begin{bmatrix} 0.4455.10^{-1} & -0.709.10^{-3} & 0.4455.10^{-1} & -0.709.10^{-3} \\ & 0.9535.10^{-1} & -0.709.10^{-3} & 0.9535.10^{-1} \\ & & 0.46855185.10^{-1} & -0.21014856.10^{-2} \\ \text{Symmetric} & & & 0.97583041.10^{-1} \end{bmatrix}, \quad (7\text{-}13)$$

which gives the following standard error ellipses:

Point #1:	a = 0.309 m	b = 0.211 m	θ = -0° 47' 58"
Point #2:	a = 0.313 m	b = 0.216 m	θ = -2° 22' 05"
Relative 1-2:	a = 0.061 m	b = 0.030 m	θ = -45° 44' 31" .

From these error ellipses, it is seen that the relative error ellipses give the precision of the actual surveying being done, whereas the station ellipses reflect the fact that point 2 cannot be established more accurately than the accuracy of the starting point #1. Increasing the confidence level to 95% (c factor = 2.45) gives

Point # 1: a = 0.756 m b = 0.517 m

Point # 2: a = 0.766 m b = 0.530 m

Relative 1-2: a = 0.148 m b = 0.072 m .

These error ellipses are plotted in Figure 7.2.

The residuals V are still zero because this is a unique case. These results compare identically with the example of section 4.8.1 in Thomson et al. [1978] which uses the direct formulae, and propagation of errors to arrive at the result. Thus, the equivalence of the least squares method and the direct approach for the unique case is seen.

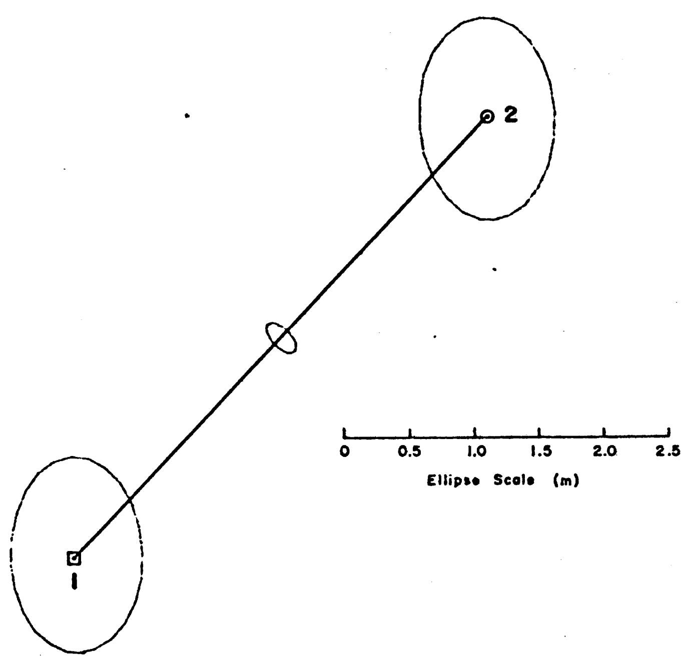


Figure 7.2 Plot of 95% Error Ellipses for Direct Case with Weighted Parameters

The following example depicted in Figure 7.3 is almost the same as the closed traverse in section 6.1. The only difference is that the points which were considered fixed (i.e pts. 1, 2, 3 and 4) in section 6.1 are now weighted. The initial data in Table 6.1 is the same for this case with weighted parameters. Additional initial data includes the weight matrix P_x (cf. eq. (7-3)) which is (for points 1, 2, 3 and 4, respectively)

$$P_x = \begin{bmatrix} 0.2501.10^{-4} & -0.1232.10^{-5} & 0.5201.10^{-6} & 0.1222.10^{-6} & 0.0 & 0.0 & 0.0 & 0.0 \\ & 0.2169.10^{-4} & 0.5941.10^{-6} & -0.9860.10^{-6} & 0.0 & 0.0 & 0.0 & 0.0 \\ & & 0.2460.10^{-4} & -0.1006.10^{-5} & 0.0 & 0.0 & 0.0 & 0.0 \\ & & & 0.2300.10^{-4} & 0.0 & 0.0 & 0.0 & 0.0 \\ & \text{symmetric} & & & 0.2399.10^{-4} & -0.1663.10^{-5} & 0.6120.10^{-6} & 0.1001.10^{-6} \\ & & & & & 0.2243.10^{-4} & 0.8045.10^{-6} & -0.8000.10^{-6} \\ & & & & & & 0.2379.10^{-4} & -0.1611.10^{-5} \\ & & & & & & & 0.2607.10^{-4} \end{bmatrix} .$$

Note that points 1 and 2 are considered uncorrelated to points 3 and 4. Figure 7.3 shows the 95% error ellipses represented by P_x above for the four weighted points.

Strictly speaking, the general matrix form of the observation equations for this example is

$$\underset{(25,1)}{V} = \underset{(25,22)}{A} \; \underset{(22,1)}{\hat{X}} + \underset{(25,1)}{W} ,$$

where W includes W_{L_x}. Realizing that the rows of A corresponding to the L_x observations reduce to the unity matrix (cf. eq. (7-2)), and that W_{L_x} reduces to zero for the first iteration (i.e. $L_x = X°$), then the observation equations are written as

$$\underset{(17,1)}{V} = \underset{(17,22)}{A} \; \underset{(22,1)}{\hat{X}} + \underset{(17,1)}{W} . \qquad (7\text{-}14)$$

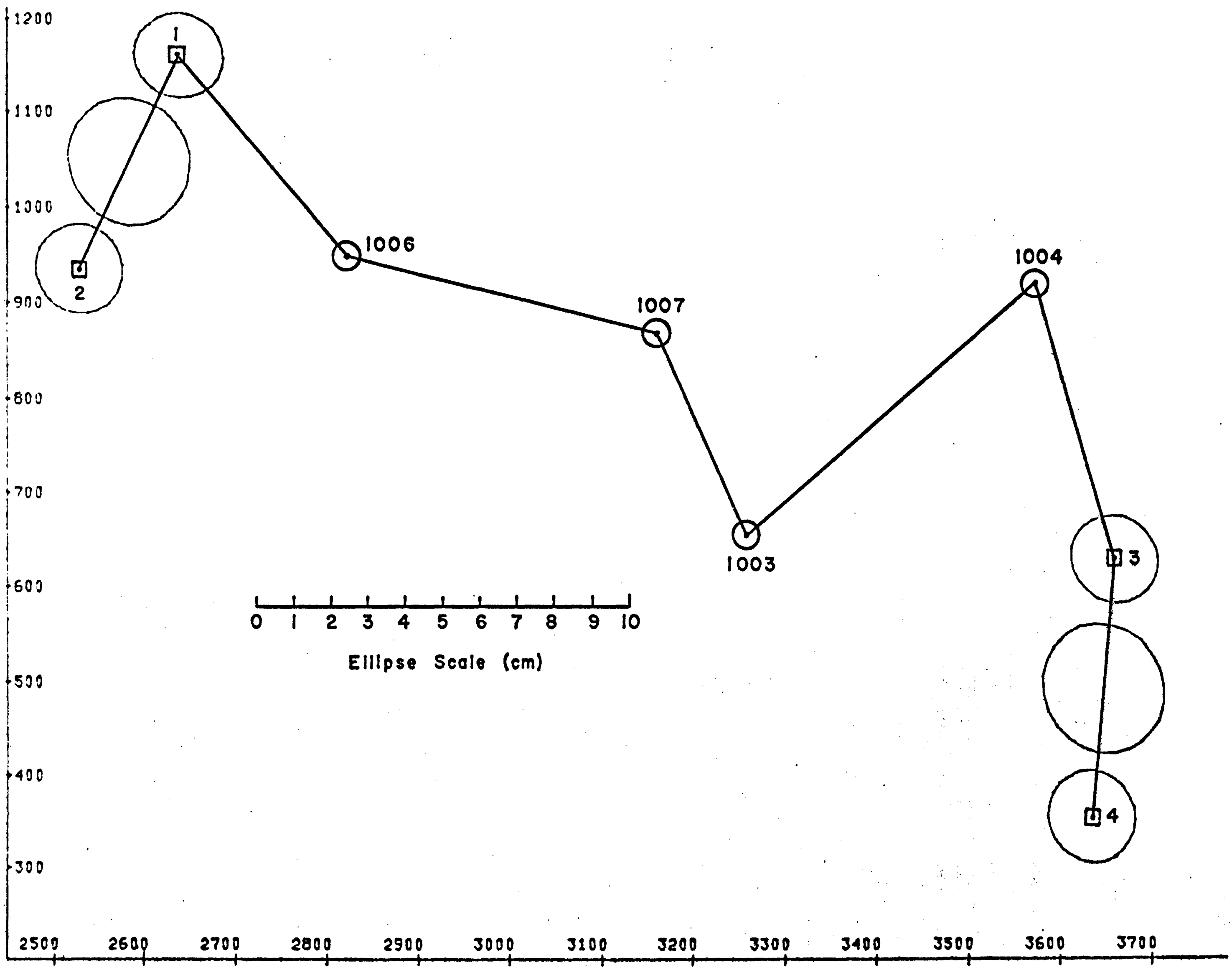


Figure 7.3 Initial Configuration for Closed Traverse
with Weighted Points

The A matrix of section 6.1 is part of the A matrix for this example. Here, however, there are eight more columns for stations 1, 2, 3 and 4. The A matrix for this example is

$$\underset{(17,22)}{A} = \begin{array}{c} \begin{array}{cccccccccc} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 & y_4 & & \end{array} \\ \left[\begin{array}{cccccccccc}
564.03 & 472.21 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
739.89 & -361.72 & -739.89 & 361.72 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
564.03 & 472.21 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{Next 14 columns} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{same as A in} & \\
0 & 0 & 0 & 0 & -658.11 & -207.82 & 0 & 0 & \text{Section 6.1} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \\
0 & 0 & 0 & 0 & 743.90 & -676.28 & -743.90 & 676.28 & & \\
0 & 0 & 0 & 0 & -658.11 & -207.82 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0.30113 & -0.95358 & 0 & 0 & & \\
-0.64194 & 0.76676 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & &
\end{array}\right] \end{array}$$

where the numbers have been rounded to five significant figures and only the first eight columns are given (the final 14 are identical to A of section 6.1). The W matrix in equation (7-14) as well as the P matrix are both identical to those in section 6.1.

Noting that W_{L_x} is zero for this first iteration, equation (7-3) is employed to compute the first solution vector as

$$\hat{X}^T = (0.00271, -0.00239, -0.00436, 0.00215, 0.00170, 0.00018, -0.00007,$$
$$0.00002, -0.40785, 1.42952, -0.01027, 2.43608, -0.32709, 0.57935,$$
$$-0.49390, 1.22106),$$

for stations 1, 2, 3, 4, 1003, 1004, 1006, 1007, respectively. Thus, the updated parameter vector is

$$X^T = X^{\circ T} + \hat{X}^T = (2640.003,\ 1159.998,\ 2529.996,\ 935.002,\ 3660.002,\ 630.000,\ 3635.000,\ 355.000,\ 3264.592,\ 646.430,\ 3569.990,\ 917.436,\ 2819.673,\ 945.579,\ 3159.506,\ 866.221).$$

Using this parameter vector as new approximate coordinates X°, the A and W matrices are reevaluated (note that W_{L_x} is no longer zero) and equation (7-3) is again employed to give a second iteration solution vector of

$$\hat{X}^T = (0.00003,\ 0.00014,\ 0.00048,\ -0.00023,\ -0.00069,\ 0.00014,\ 0.00019,\ -0.00004,\ 0.00441,\ -0.00096,\ 0.00361,\ 0.00156,\ 0.00150,\ -0.00063,\ 0.00322,\ -0.00030)\ , \quad (7\text{-}15)$$

which yields the second iteration parameter vector as

$$X^T = X^{\circ T} + \hat{X}^T = (2640.003,\ 1159.998,\ 2529.996,\ 935.002,\ 3660.001,\ 630.000,\ 3635.000,\ 355.000,\ 3264.597,\ 646.429,\ 3569.993,\ 917.438,\ 2819.674,\ 945.579,\ 3159.509,\ 866.221)\ . \quad (7\text{-}16)$$

Evaluating A, W and W_{L_x} a third time to compute the third iteration solution vector $\hat{X}$ yields a zero solution vector. The final parameter vector of adjusted coordinates is thus given by equation (7-16).

The variance covariance matrix C_x of the parameters is computed using equation (7-4) to yield

C_x =

	X 1 (COL 1)	Y 1 (COL 2)	X 2 (COL 3)	Y 2 (COL 4)	X 3 (COL 5)	Y 3 (COL 6)
1	0.19652029D-04	0.23958093D-05	0.55410165D-05	-0.23799186D-05	0.10803111D-05	-0.13109755D-05
2	0.23958093D-05	0.18765149D-04	-0.33319641D-05	0.18483291D-05	0.11778042D-05	0.82335640D-06
3	0.55410165D-05	-0.33319641D-05	0.18158363D-04	0.21624219D-05	0.11243383D-05	0.76766771D-06
4	-0.23799186D-05	0.18483291D-05	0.21624219D-05	0.21421597D-04	-0.52133802D-06	-0.33758310D-06
5	0.10803111D-05	0.11778042D-05	0.11243383D-05	-0.52133802D-06	0.15991498D-04	-0.11396669D-05
6	-0.13109755D-05	0.82335640D-06	0.76766771D-06	-0.33758310D-06	-0.11396669D-05	0.21968524D-04
7	-0.79459153D-06	-0.65240955D-06	0.30613918D-06	-0.17026784D-06	0.64470222D-05	0.60075892D-06
8	0.17666737D-06	0.12439638D-06	-0.65945965D-07	0.36187247D-07	-0.93126575D-06	-0.78254713D-06
9	0.17302967D-06	0.32207817D-05	0.71759510D-05	-0.34735467D-05	0.11774514D-04	-0.21643472D-06
10	-0.40391751D-05	0.11224189D-04	0.66999466D-05	-0.32284658D-05	-0.19281765D-05	0.13827776D-04
11	0.52285445D-05	0.25948374D-05	0.12177935D-05	-0.48881682D-06	0.22329675D-04	-0.45526854D-05
12	-0.55454989D-05	0.54642598D-05	0.26282010D-05	-0.13176297D-05	0.95229809D-07	0.18297477D-04
13	0.98821736D-05	0.42689205D-05	0.96155974D-05	-0.45050544D-05	0.41738142D-05	-0.72668092D-06
14	0.24432063D-06	0.16929750D-04	0.14386755D-05	-0.53478662D-06	-0.17557044D-06	0.50030457D-05
15	0.59868620D-05	0.34742726D-05	0.56114738D-05	-0.26160882D-05	0.12179738D-04	-0.21289755D-05
16	-0.69139802D-05	0.15267440D-04	0.67739490D-05	-0.32699634D-05	-0.10102625D-05	0.99876188D-05

	X 4 (COL 7)	Y 4 (COL 8)	X 1003 (COL 9)	Y 1003 (COL 10)	X 1004 (COL 11)	Y 1004 (COL 12)
1	-0.79459153D-06	0.17666737D-06	0.17302967D-06	-0.40391751D-05	0.52285445D-05	-0.55454939D-05
2	-0.65240955D-06	0.12439638D-06	0.32207817D-05	0.11224139D-04	0.25948374D-05	0.54642598D-05
3	0.30613918D-06	-0.65945965D-07	0.71759510D-05	0.66999486D-05	0.12177935D-05	0.26282010D-05
4	-0.17026784D-06	0.36187247D-07	-0.34735467D-05	-0.32284658D-05	-0.48881682D-06	-0.13176297D-05
5	0.64470222D-05	-0.93126575D-06	0.11774514D-04	-0.19281765D-05	0.22329675D-04	0.95229809D-07
6	0.60075892D-06	-0.78254713D-06	-0.21643472D-06	0.13827776D-04	-0.45526854D-05	0.18297477D-04
7	0.16442448D-04	-0.68923722D-06	0.56361025D-05	-0.14634679D-05	-0.33985119D-05	0.18671034D-05
8	-0.68923722D-06	0.25910460D-04	-0.85098421D-06	-0.17483192D-06	0.92016095D-06	-0.84364762D-06
9	0.56361025D-05	-0.85098421D-06	0.73176925D-04	0.42770475D-05	0.21289436D-04	-0.20186520D-04
10	-0.14634679D-05	-0.17483192D-06	0.42770476D-05	0.97156414D-04	-0.21783386D-04	0.75172961D-04
11	-0.36985119D-05	0.92016095D-06	0.21289435D-04	-0.21783386D-04	0.62765348D-04	-0.27015393D-04
12	0.18671084D-05	-0.84364762D-06	-0.20186520D-04	0.75172961D-04	-0.27015393D-04	0.89533775D-04
13	0.15341162D-05	-0.20592782D-06	0.23371052D-04	-0.13969825D-05	0.89699759D-05	-0.94526825D-05
14	-0.21155949D-05	0.20241693D-06	-0.19591565D-05	0.41781902D-04	-0.48449816D-05	0.30441760D-04
15	0.11954289D-05	-0.57925073D-07	0.56100058D-04	0.72550638D-05	0.29950500D-04	-0.20319026D-04
16	0.34983817D-06	-0.35655266D-06	0.99839646D-05	0.66114453D-04	-0.16455629D-04	0.50951052D-04

	X 1006 (COL 13)	Y 1006 (COL 14)	X 1007 (COL 15)	Y 1007 (COL 16)
1	0.98821736D-05	0.24432063D-06	0.59868620D-05	-0.69139802D-05
2	0.42689205D-05	0.16929750D-04	0.34742726D-05	0.15267440D-04
3	0.96155974D-05	0.14386755D-05	0.56114738D-05	0.67739490D-05
4	-0.45050544D-05	-0.53478662D-06	-0.26160882D-05	-0.32699634D-05
5	0.41738142D-05	-0.17557044D-06	0.12179738D-04	-0.10102625D-05
6	-0.72668092D-06	0.50030457D-05	-0.21289755D-05	0.99876188D-05
7	0.15341162D-05	-0.21155949D-05	0.11954289D-05	0.34983817D-06
8	-0.20592782D-06	0.20241693D-06	-0.57925073D-07	-0.35655266D-06
9	0.23371052D-04	-0.19591565D-05	0.56100058D-04	0.99839646D-05
10	-0.13969825D-05	0.41781902D-04	0.72550633D-05	0.66114453D-04
11	0.89699759D-05	-0.48449816D-05	0.29950500D-04	-0.16455629D-04
12	-0.94526825D-05	0.30441760D-04	-0.20319026D-04	0.50951052D-04
13	0.48079778D-04	-0.26581723D-04	0.26528812D-04	-0.13914582D-04
14	-0.26581723D-04	0.65014449D-04	-0.12074354D-04	0.65080222D-04
15	0.26528812D-04	-0.12074354D-04	0.70775065D-04	-0.18379388D-04
16	-0.13914582D-04	0.65080222D-04	-0.18379388D-04	0.98208227D-04

where C_x has been printed in rows of six columns at a time. The equations of Appendix III (C factor = 2.45) are used to compute the 95% station and relative error ellipses listed in Table 7.1 and plotted in Figure 7.4.

Station Ellipses

Station	a(m)	b(m)	θ
1	.011	.010	50°14'35"
2	.012	.010	26°36'31"
3	.012	.010	-10°26'14"
4	.012	.010	-5°13'45"
1003	.024	.021	9°48'58"
1004	.025	.017	-31°49'52"
1006	.022	.013	-36°09'57"
1007	.025	.019	-26°37'59"

Relative Ellipses

Station	Station	a(m)	b(m)	θ
1	2	.016	.011	32°18'42"
1	1006	.022	.011	-44°01'33"
1006	1007	.021	.012	-65°23'33"
1003	1007	.022	.009	-31°42'07"
1003	1004	.024	.014	73°00'12"
3	1004	.023	.012	-24°38'31"
3	4	.017	.011	-3°28'13"

Table 7.1 95% Error Ellipses for Closed Traverse with Weighted Parameters

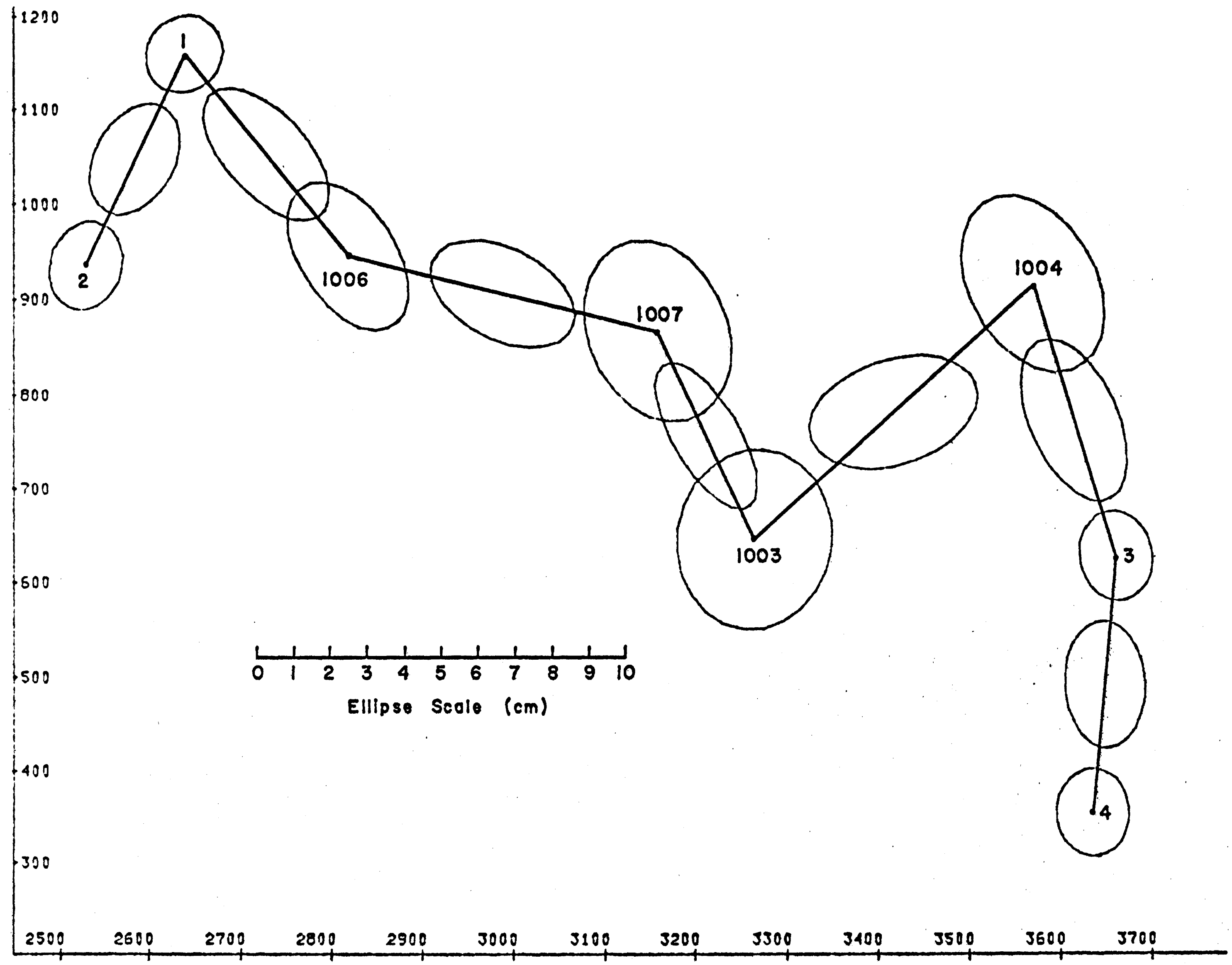


Figure 7.4 Plot of 95% Error Ellipses for Closed Traverse with Weighted Points

the residuals V are computed using equation (AII-17) and the solution vector of equation (7-15) to be

$$\underset{(1,17)}{V^T} = (0.84, -0.84, 0.58, -0.58, 0.40, -0.40, 0.16, -0.16, 0.31, -0.31, 0.02, -0.02, 0.000, -0.002, -0.004, -0.000, -0.005), \quad (7\text{-}17)$$

where the units of the first 12 elements are arcseconds and the final five elements are in metres. Using these residuals and the summation of $\hat{X}$'s for all of the iterations (cf. eq. (7-7)) the a posteriori variance factor $\hat{\sigma}_o^{\,2}$ is computed via equation (7-6) as

$$\hat{\sigma}_o^{\,2} = \frac{V^T P V + V_{L_X}^T P_X V_{L_X}}{df} = 0.79712 \quad .$$

The results (i.e. adjusted coordinates and accuracy estimates) for this example and that of section 6.1 where points 1, 2, 3 and 4 were considered fixed are significantly different. With weighted points, the observations are allowed to affect the final coordinates of the weighted points to a degree dictated by the observation as well as coordinate weight matrices (P and P_x). This gives a more realistic least squares solution than does the fixed point approach.

8. PREANALYSIS

Preanalysis is the study of the design of a network. The design is carried out prior to the establishment of the network in the field, and thus no observations are necessary for a preanalysis. By optimizing the accuracy and distribution of the observables before entering the field, the required accuracy of the network points is achieved most expediently.

Preanalysis is based on equation (AII-16), i.e.

$$C_x = [A^T C_L^{-1} A]^{-1} \quad .$$

Since the variance covariance matrix of the parameters C_x does not require knowledge of the actual observations (the only place where the observations are necessary is for computation of W), it can be computed knowing the approximate coordinates of the unknown points along with some proposed observations (and their standard deviations) amongst them. If some parameters are weighted, then equation (7-4) applies, i.e.

$$C_x = [A^T PA + P_x]^{-1} \quad .$$

Computing the station and relative error ellipses from C_x, the results of a network design are readily apparent. The design can be altered by proposing different observations and standard deviations and/or changing the position or number of unknown points, and recomputing C_x.

Better use of the already existing design is made when the sequential design approach [e.g. Nickerson et al., 1978] is used. This method is characterized by the following equation:

$$C_{x_i} = C_{x_{i-1}} - C_{x_{i-1}} A_i^T (\pm C_{L_i} + A_i C_{x_{i-1}} A_i^T)^{-1} A_i C_{x_{i-1}} \quad , \qquad (8\text{-}1)$$

where C_{x_i} = covariance matrix of the parameters utilizing all observables,

$C_{x_{i-1}}$ = previous covariance matrix which is being altered,

A_i = design matrix for the observables to be added or deleted,

C_{L_i} = variance covariance matrix of the observables being added or deleted.

The plus and minus signs preceding C_{L_i} refer to addition and deletion of observables, respectively. The size of inverse to be computed (usually the most time consuming task) is equal to the number of observables being added or deleted, not the number of parameters as in the nonsequential equations. The standard deviations of specific observables are changed by subtracting the old observable with its standard deviation, and adding it back with the new standard deviation. The following examples illustrate the preanalysis process.

8.1 Traverse Design

Figure 8.1 depicts the initial information (see Table 8.1) for the traverse design. There are four fixed points and three unknown

UNKNOWN POINTS				KNOWN POINTS		
Station	x(m)	y(m)	Des.Acc.(m)	Station	x(m)	y(m)
1	293682	225293	0.05	1102	293054.171	225214.674
2	293976	225607	0.05	1116	293571.011	225598.373
3	294421	225284	0.05	1105	295267.293	225419.706
				1106	295004.038	225951.144

Table 8.1 Initial Data for Traverse Design

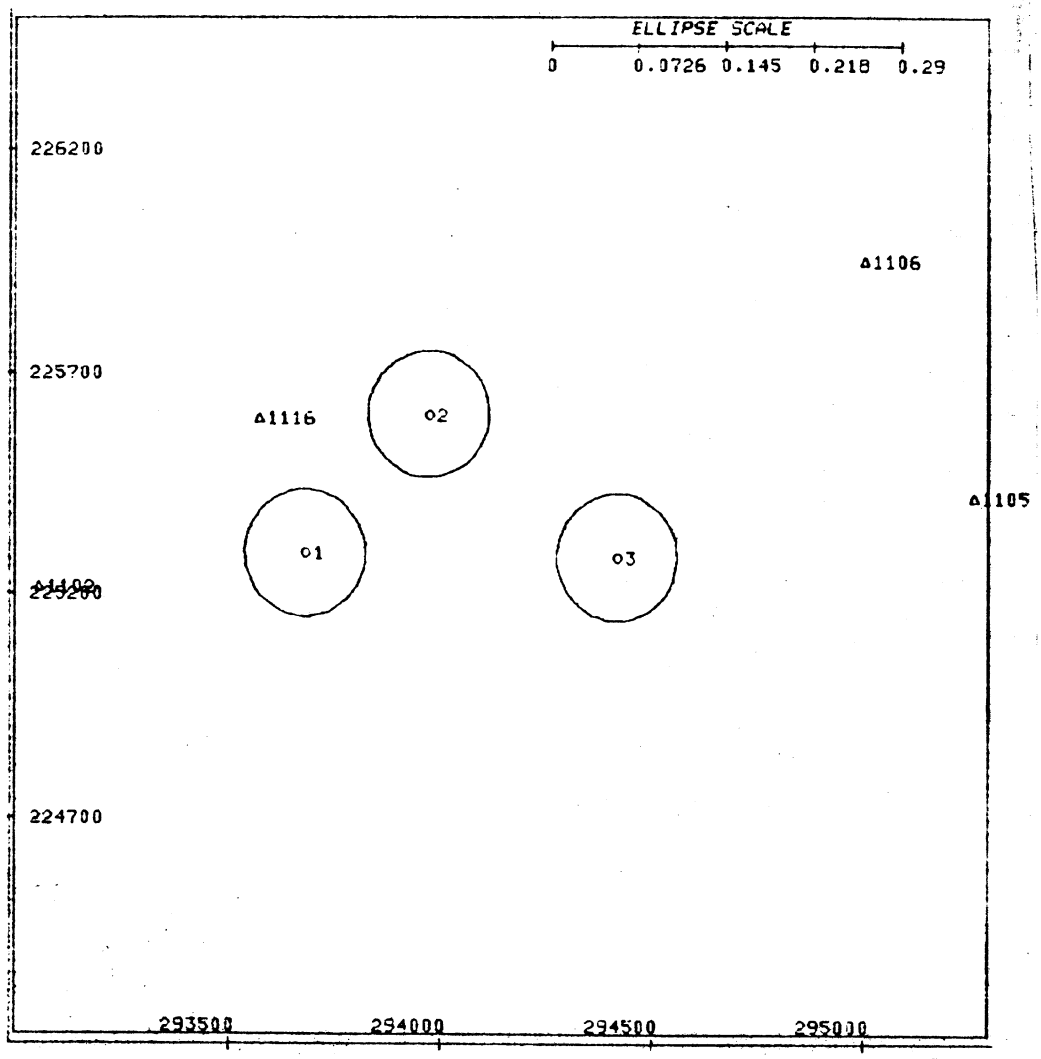


Figure 8.1 Initial Data Plot

OBSERVABLE TYPE	STANDARD DEVIATION	FROM STATION	TO STATION
direction	3"0	1116	1102
direction	3"0	1116	1
direction	2"0	1	1116
direction	2"0	1	2
direction	2"0	2	1
direction	2"0	2	3
direction	0.02 m	1116	1
direction	0.02 m	2	1
direction	0.02 m	2	3

Table 8.2 Initial Observables

points. The required accuracy is represented by the design circles of 5 cm. radius around the unknown points.

The proposed initial observables between these points are listed in Table 8.2 and plotted in Figure 8.2. The design is first treated as an open ended traverse similar to the example in section 5.5. The main difference is that directions are used here instead of angles. Thus, the A matrix is of size 9 x 9 (i.e. n = 9, u = 9 (6 unknown coordinates, 3 orientation unknowns)), and is given as

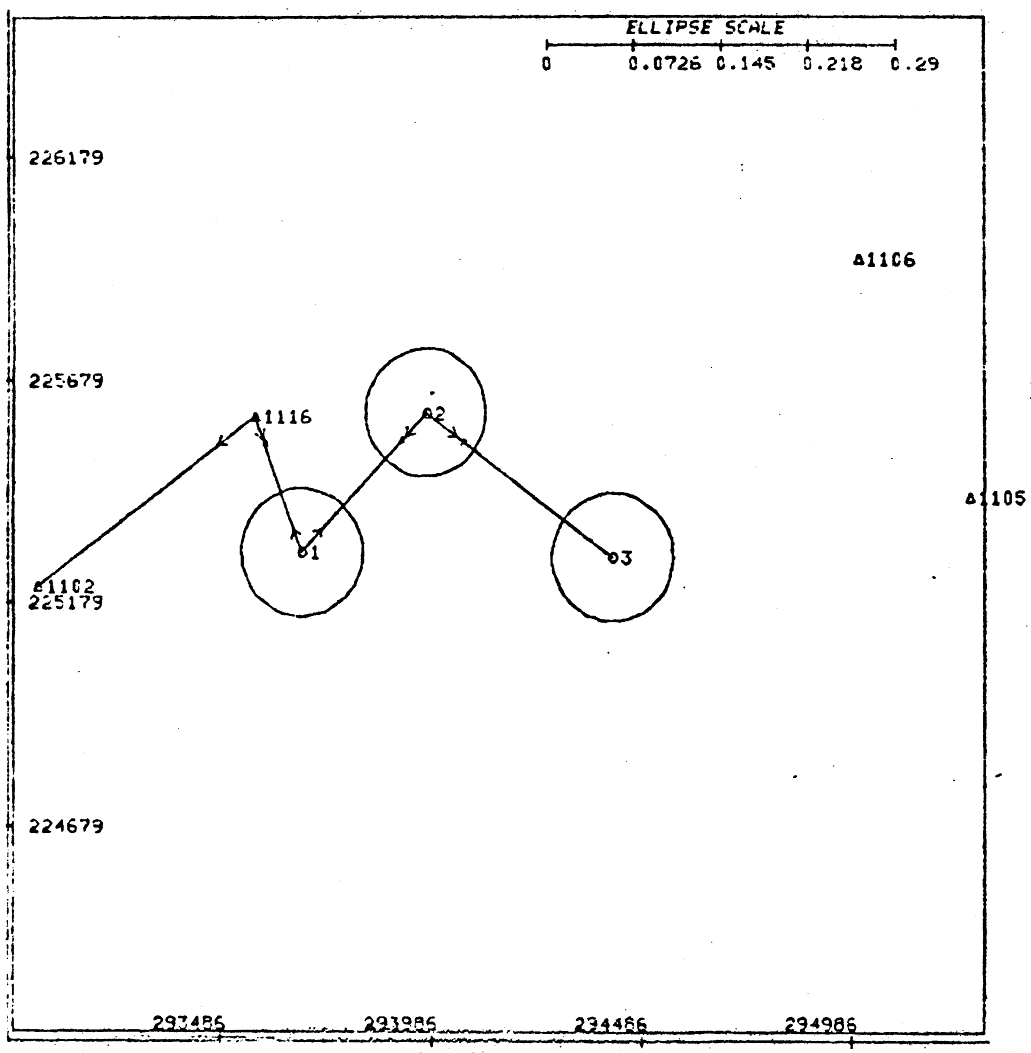


Figure 8.2 Initial observables Plot

$$
\underset{(9,9)}{A} = \begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 \\
-596.637 & -216.850 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 \\
-596.637 & -216.850 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 \\
-350.0322 & 327.7371 & 350.0322 & -327.7371 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 \\
-350.0322 & 327.7371 & 350.0322 & -327.7371 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 \\
0.0 & 0.0 & 220.3494 & 303.5774 & -220.3494 & -303.5774 & 0.0 & 0.0 & -1.0 \\
0.3415916 & -0.9398485 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.683477 & -0.729972 & 0.683477 & 0.729972 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -0.8092862 & 0.5874145 & 0.8092862 & -0.5874145 & 0.0 & 0.0 & 0.0
\end{bmatrix} .
$$

This results from a combination of the direction mathematical model developed in section 3.2 (for the first six rows of A) and the distance mathematical model of chapter 4 (last three rows of A above). The last three columns of A are for the orientation unknowns Z_{1116}, Z_1, and Z_2. The P matrix is (assuming $\sigma_o^2 = 1$)

$$\underset{(9,9)}{P} = \begin{bmatrix} 0.111 & 0.0 & 0.0 & \cdots & & & & & 0.0 \\ 0.0 & 0.111 & 0.0 & & & & & & \vdots \\ \vdots & & 0.25 & & & & & & \vdots \\ & & & 0.25 & & & & & \vdots \\ & & & & 0.25 & & & & \vdots \\ & & & & & 0.25 & & & \vdots \\ & & & & & & 2500.0 & & \vdots \\ \vdots & & & & & & & 2500.0 & 0.0 \\ 0.0 & \cdots & & & & & & 0.0 & 2500.0 \end{bmatrix} .$$

Computing $(A^T PA)$ and taking its inverse yields

$$C_x = \begin{bmatrix} 8.61272.10^{-5} & -1.14078.10^{-4} & 4.55593.10^{-5} & -7.60943.10^{-5} & 8.72900.10^{-5} & -1.86016.10^{-5} & -1.33244.10^{-2} & -2.66488.10^{-2} & -2.66488.10^{-2} \\ & 3.58538.10^{-4} & -1.28823.10^{-4} & 3.72343.10^{-4} & -1.13656.10^{-4} & 3.93239.10^{-4} & -4.84281.10^{-3} & -9.68562.10^{-3} & -9.68562.10^{-3} \\ & & 2.52101.10^{-4} & 5.23131.10^{-5} & 2.31851.10^{-4} & 2.44147.10^{-5} & 3.76422.10^{-4} & 6.84210.10^{-3} & 1.29314.10^{-2} \\ & & & 6.52115.10^{-4} & 1.25513.10^{-4} & 7.52963.10^{-4} & -1.76710.10^{-2} & -4.10434.10^{-2} & -4.67448.10^{-2} \\ & & & & 5.56954.10^{-4} & 2.23259.10^{-5} & -1.37171.10^{-2} & -2.76088.10^{-2} & -3.49471.10^{-2} \\ & & & & & 1.15009.10^{-3} & -3.70878.10^{-2} & -8.85066.10^{-2} & -1.11467.10^{-1} \\ & & & & & & 9.0 & 9.0 & 9.0 \\ & \text{symmetric} & & & & & & 22.0 & 22.0 \\ & & & & & & & & 30.0 \end{bmatrix} . \quad (8\text{-}2)$$

The first six rows and columns are the variance covariance matrix for the coordinates of points 1, 2 and 3. The last three rows and columns represent the variance and covariance of the orientation unknowns, and are of no practical concern. Since the orientation unknowns are nuisance parameters, only the first (6 x 6) submatrix of C_x will be considered as representative of the traverse being designed. The 99% error ellipses assuming σ_o^2 known and nonsimultaneous ellipses (c factor = 3.035, see Appendix II) are listed in Table 8.3. From Figure 8.3

STATION ELLIPSES (99%) RELATIVE ELLIPSES (99%)

Point	Semimajor(m)	Semiminor(m)	θ	Points	Semimajor(m)	Semiminor(m)	θ
1	0.061	0.021	-19°58'26"	1-2	0.061	0.033	43°06'57"
2	0.079	0.049	7°19'43"	2-3	0.061	0.049	-54°01'35"
3	0.103	0.073	2°09'09"				

Table 8.3 Confidence Ellipses from Initial Observables

and the above table it is seen that all of the station ellipses lie outside the required accuracy circle of 5 cm radius. New confidence ellipses

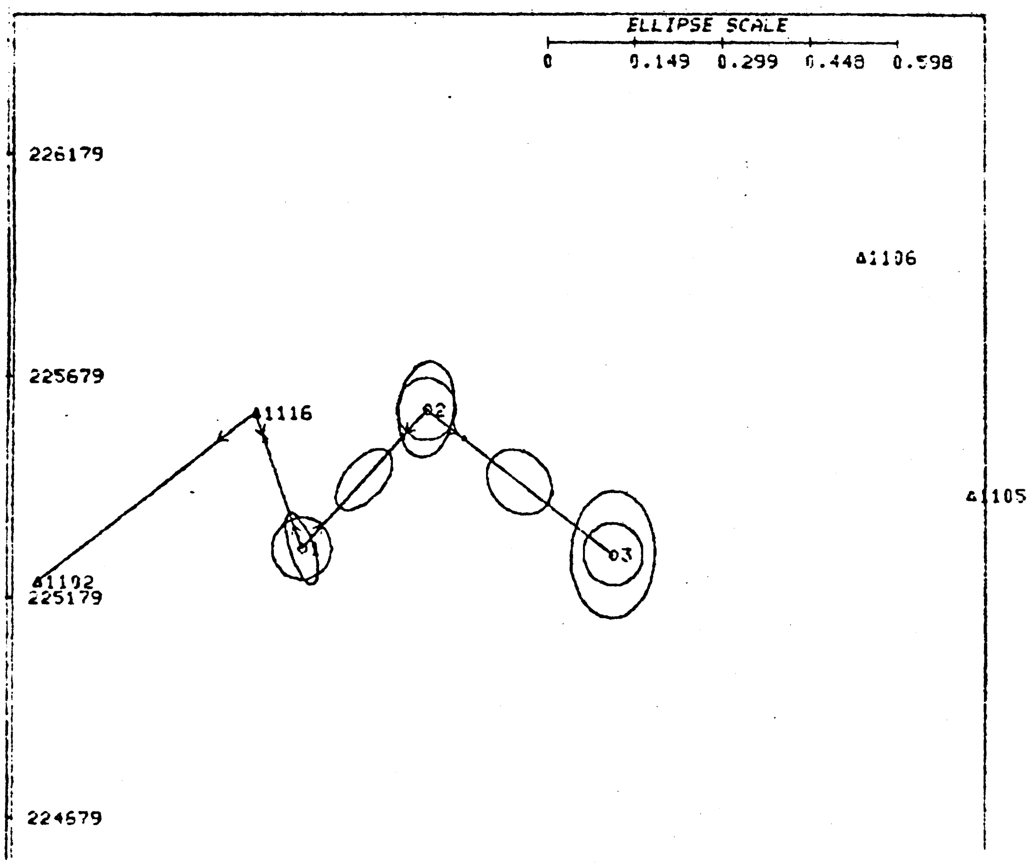


Figure 8.3 Plot of 99% Confidence Ellipses from Initial Observables.

are computed using equation (8-1) to update the design. Table 8.4 shows the update observables which are depicted in Figure 8.4. Referring to

OBSERVATION TYPE	STANDARD DEVIATION	FROM STATION	TO STATION
Direction	1\".5	3	2
Direction	1\".5	3	1105
Direction	1\".5	1105	3
Direction	1\".5	1105	1106
Distance	0.02 m	3	2

Table 8.4 Update Observables

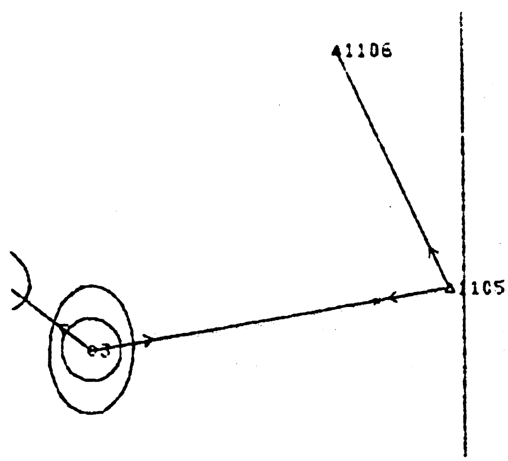


Figure 8.4 Plot of Update Observables

equation (8-1), these new observables are added by first computing A_i and C_{L_i}, and in turn C_{x_i} taking equation (8-2) as $C_{x_{i-1}}$. After performing these operations, the new C_{x_i} corresponding to the 3 unknown points is (to 5 significant digits)

$$C_x = \begin{bmatrix} 5.9200\text{E}^{-}5 & {}^{-}8.2518\text{E}^{-}5 & 4.2258\text{E}^{-}5 & {}^{-}5.8677\text{E}^{-}5 & 2.3325\text{E}^{-}5 & {}^{-}1.7375\text{E}^{-}5 \\ {}^{-}8.2518\text{E}^{-}5 & 2.0463\text{E}^{-}4 & {}^{-}1.1573\text{E}^{-}4 & 1.3496\text{E}^{-}4 & {}^{-}3.1692\text{E}^{-}5 & 3.4932\text{E}^{-}5 \\ 4.2258\text{E}^{-}5 & {}^{-}1.1573\text{E}^{-}4 & 1.7800\text{E}^{-}4 & 4.3074\text{E}^{-}6 & 1.0786\text{E}^{-}4 & 3.2542\text{E}^{-}5 \\ {}^{-}5.8677\text{E}^{-}5 & 1.3496\text{E}^{-}4 & 4.3074\text{E}^{-}6 & 1.7620\text{E}^{-}4 & 7.3305\text{E}^{-}5 & 7.1295\text{E}^{-}5 \\ 2.3325\text{E}^{-}5 & {}^{-}3.1692\text{E}^{-}5 & 1.0786\text{E}^{-}4 & 7.3305\text{E}^{-}5 & 2.2146\text{E}^{-}4 & 1.4604\text{E}^{-}5 \\ {}^{-}1.7375\text{E}^{-}5 & 3.4932\text{E}^{-}5 & 3.2542\text{E}^{-}5 & 7.1295\text{E}^{-}5 & 1.4604\text{E}^{-}5 & 5.5169\text{E}^{-}5 \end{bmatrix}.$$

The 99% error ellipses (again assuming σ_o^2 known and nonsimultaneous ellipses for a c factor of 3.035) computed from the above C_x matrix are listed in

STATION ELLIPSES (99%)

Point	Semimajor(m)	Semiminor(m)	θ
1	0.047	0.014	-24°18'25"
2	0.041	0.040	50°54'24"
3	0.045	0.022	85°01'08"

RELATIVE ELLIPSES (99%)

Points	Semimajor(m)	Semiminor(m)	θ
1-2	0.046	0.018	51°07'30'
2-3	0.047	0.019	-59°19'18'

Table 8.5 99% Error Ellipses After Update

Table 8.5 and plotted in Figure 8.5. Obviously, all of the error ellipses

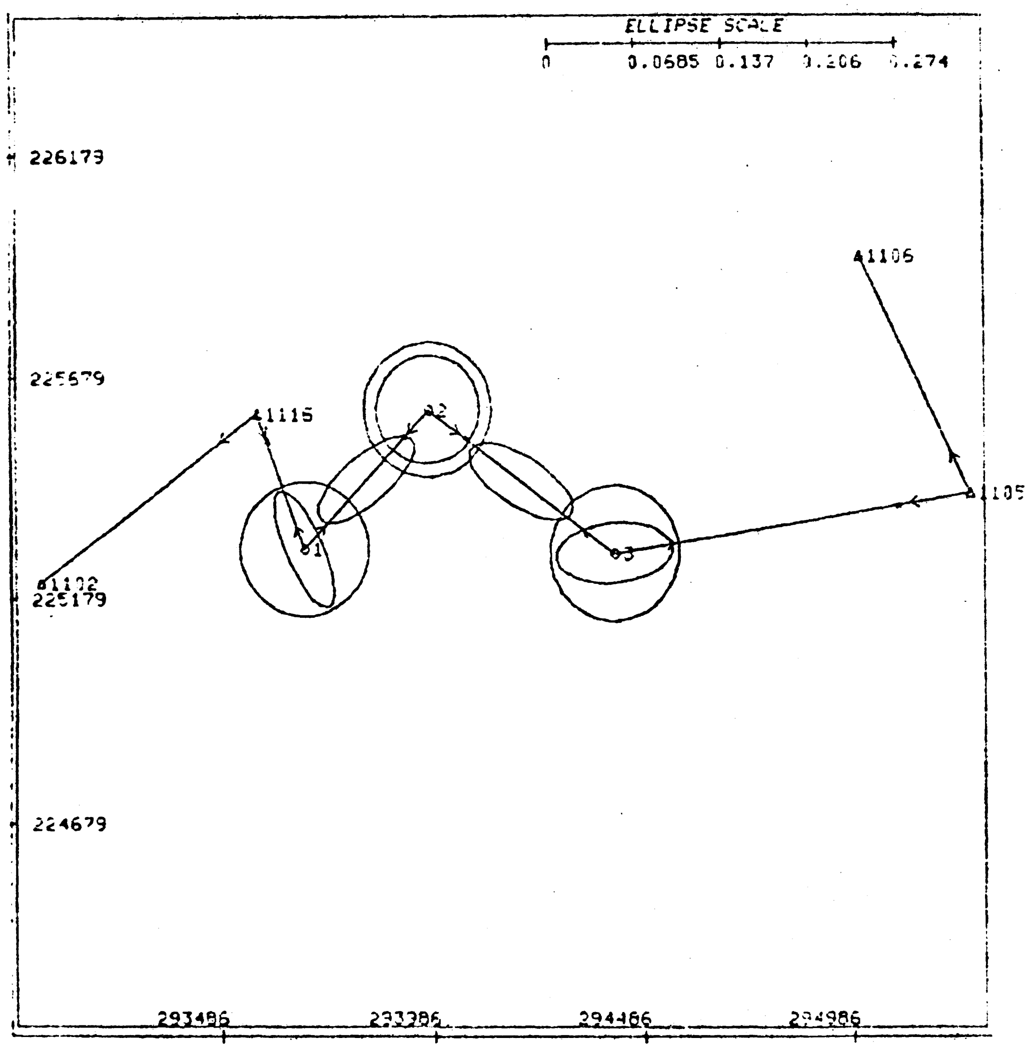


Figure 8.5 Plot of Updated 99% Error Ellipses

now meet the required accuracy, and the design is finished. A summary of the proposed observations is given in Table 8.6.

FROM	TO	TYPE	σ
1116	1102	1	3.0000
1116	1	1	3.0000
1116	1	2	.0200
1105	3	1	1.5000
1105	1106	1	1.5000
1105	3	2	.0200
1	1106	1	2.0000
1	2	1	2.0000
2	1	1	2.0000
2	3	1	2.0000
2	1	2	.0200
2	3	2	.0200
3	2	1	1.5000
3	1105	1	1.5000

Note: Type 1 = Direction

Type 2 = Distance

Table 8.6 Observable Summary

8.2 Property Survey Design

The initial data of this design (see Table 8.7 and Figure 8.6) is characteristic of a simple lot layout often encountered in practice.

UNKNOWN POINTS				KNOWN POINTS		
Station	x(m)	y(m)	Des.Acc.	Station	x(m)	y(m)
1	155721.0	119687.0	0.05	1004	155221.688	119515.558
2	156019.0	119595.0	0.05	1005	155493.110	119604.892
3	156027.0	119386.0	0.05			
4	156204.0	119596.0	0.05			
5	156213.0	119388.0	0.05			

Table 8.7 Initial Data for Property Survey Design

Fixed points 1004 and 1005 represent two second order monuments, point 1 an intermediate point , and points 2, 3, 4 and 5 the four lot corners to be established. Again, the required accuracy circle is of 5 cm. radius, but this time at a confidence level of 95%.

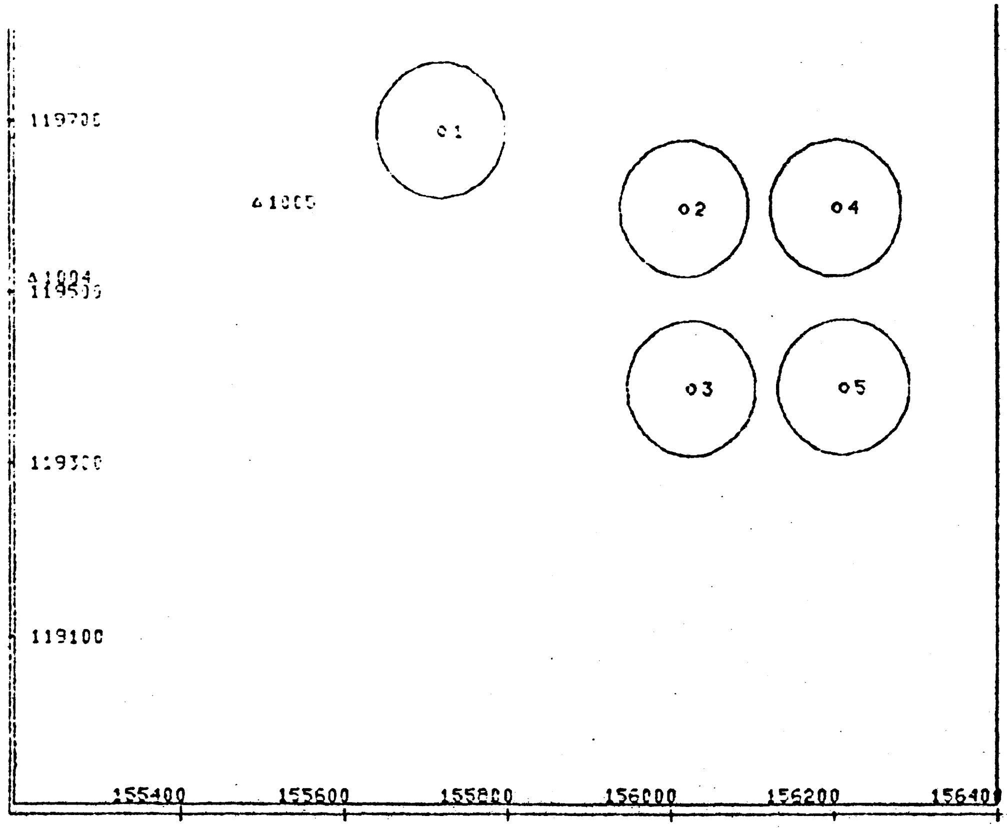


Figure 8.6 Initial Data Plot for Property Survey Design

The initial observables considered for the design are listed in Table 8.8 and plotted in Figure 8.7. These initial accuracies

OBSERVABLE TYPE	STANDARD DEVIATION	FROM STATION	TO STATION
Direction	2″0	1005	1004
Direction	2″0	1005	1
Direction	2″0	1	1005
Direction	2″0	1	2
Distance	0.02 m	1	1005
Distance	0.02 m	1	2
Direction	3″0	2	1
Direction	3″0	2	4
Direction	3″0	2	3
Distance	0.02 m	2	4
Direction	3″0	4	2
Direction	3″0	4	5
Direction	3″0	5	4
Direction	3″0	5	3
Distance	0.02 m	5	4
Distance	0.02 m	5	3

Table 8.8. Initial Observables for Property Survey Design

could be achieved using 4 sets of direction observables at station 1005 and station 1, and 2 sets at stations 2, 4 and 5 with a 1" theodolite (see Table 2.4). The distance's accuracy of 0.02 m is easily achieved using either lightwave or microwave EDM with normal meteorological readings (see Table 2.8). The A matrix for the initial C_x is (16 x 15) since there are 16 observables

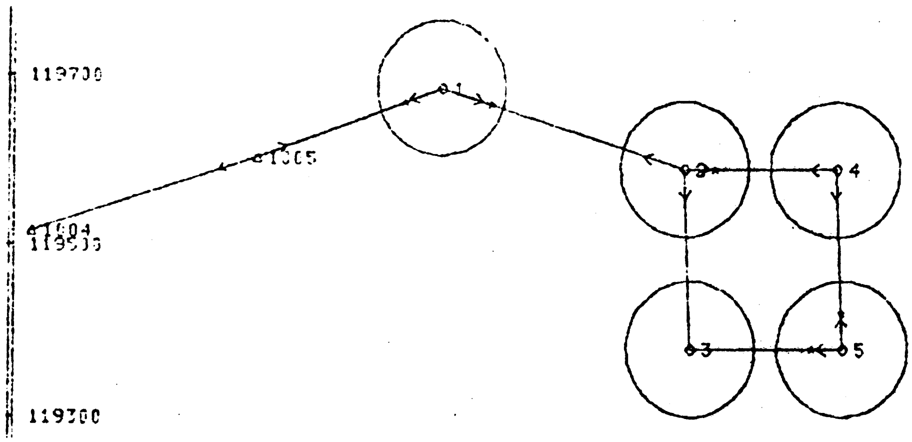


Figure 8.7 Plot of Initial Observables for Property Survey Design

(see Table 8.8) and 15 unknowns (10 coordinates and 5 orientation unknowns). The A matrix is formed using the equations developed in section 3.2 and chapter 4. After computing A and P, equation (AII-16) is used to compute C_x as

	X 1	Y	X 2	Y	X 3	Y
	(COL 1)	(COL 2)	(COL 3)	(COL 4)	(COL 5)	(COL 6)
1	0.35530836D-03	0.12404122D-03	0.35388796D-03	0.11944034D-03	0.35066116D-03	0.11931582D-03
2	0.12404122D-03	0.55724723D-04	0.12798355D-03	0.68494440D-04	0.13693949D-03	0.63837251D-04
3	0.35388796D-03	0.12793355D-03	0.72084369D-03	0.20948855D-04	0.72484798D-03	0.21102129D-04
4	0.11944034D-03	0.68494440D-04	0.20948855D-04	0.14946768D-03	0.53327244D-04	0.15070704D-03
5	0.35066116D-03	0.13693949D-03	0.72484798D-03	0.53327244D-04	0.76431864D-03	0.38004149D-04
6	0.11931682D-03	0.63837251D-04	0.21102129D-04	0.15070704D-03	0.38004149D-04	0.57958023D-03
7	0.35390340D-03	0.12794079D-03	0.72082453D-03	0.20793934D-04	0.72887243D-03	0.27883232D-04
8	0.11658408D-03	0.76421948D-04	0.24493327D-04	0.17812798D-03	0.79689233D-04	0.18069542D-03
9	0.35069204D-03	0.13685379D-03	0.72480966D-03	0.53017403D-04	0.75993483D-03	0.28935628D-04
10	0.11644513D-03	0.76807611D-04	0.24665761D-04	0.17952226D-03	0.65701068D-04	0.58070559D-03

	X 4	Y	X 5	Y
	(COL 7)	(COL 8)	(COL 9)	(COL 10)
1	0.35390340D-03	0.11653408D-03	0.35069204D-03	0.11644513D-03
2	0.12794079D-03	0.76421948D-04	0.13685379D-03	0.76807611D-04
3	0.72082453D-03	0.24493327D-04	0.72480966D-03	0.24665761D-04
4	0.20793934D-04	0.17812798D-03	0.53017403D-04	0.17952226D-03
5	0.72887243D-03	0.79689233D-04	0.75993483D-03	0.65701068D-04
6	0.27883232D-04	0.18069542D-03	0.28935628D-04	0.58070559D-03
7	0.92948792D-03	0.21401159D-04	0.92030858D-03	0.21958859D-04
8	0.21401159D-04	0.23402273D-03	0.84037515D-04	0.23675776D-03
9	0.92030858D-03	0.84037515D-04	0.96387214D-03	0.69636548D-04
10	0.21958859D-04	0.23675776D-03	0.69636548D-04	0.63884349D-03

Figure 8.8 depicts the 95% (assuming σ_o^2 known, nonsimultaneous ellipses, c factor = 2.45) error ellipses resulting from this initial design.

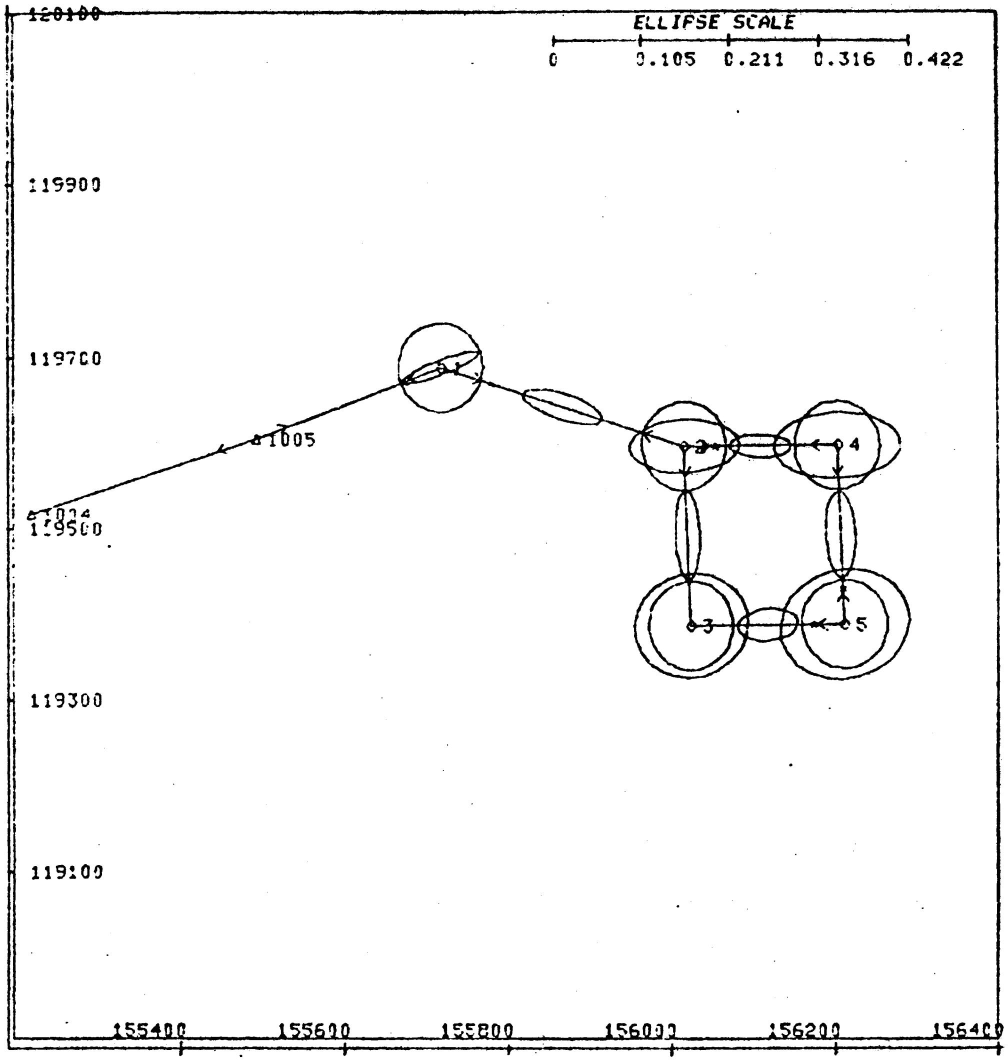


Figure 8.8 Initial Design Results

Point 1 is the only point which meets the required accuracy. From the shape of the relative error ellipses (i.e. long and skinny along the line of sight), it is obvious that the distances are less accurate relative to the direction observables. Thus, the design is altered by changing the standard deviation of most of the distance observables from 0.02 m to 0.01 m. This would correspond to using a more accurate instrument for observing distances (e.g. lightwave EDM with standard meteorological readings (cf. table 2.8)). These changes are effected by first subtracting (use - sign in eq. (8-1)) the existing distances with 0.02m for a standard deviation, and then adding them back again (+ sign in eq. (8-1)) with a standard deviation of 0.01 m. After performing this operation for each distance observable except that from point 5 to point 3, the C_x matrix is

	X 1 (COL 1)	Y 1 (COL 2)	X 2 (COL 3)	Y 2 (COL 4)	X 3 (COL 5)	Y 3 (COL 6)
1	0.89777853D-04	0.23371474D-04	0.88357446D-04	0.23770588D-04	0.85130650D-04	0.23647074D-04
2	0.23371474D-04	0.21255235D-04	0.32313803D-04	0.34024951D-04	0.41269745D-04	0.34367762D-04
3	0.88357446D-04	0.32313803D-04	0.12141837D-03	0.98372335D-05	0.18542267D-03	0.99905079D-05
4	0.23770588D-04	0.34024951D-04	0.98372335D-05	0.68929968D-04	0.42215623D-04	0.90132362D-04
5	0.85130650D-04	0.41269745D-04	0.18542267D-03	0.42215623D-04	0.22440193D-03	0.38293266D-04
6	0.23647074D-04	0.34367762D-04	0.99905079D-05	0.90132362D-04	0.38293266D-04	0.21893587D-03
7	0.89372885D-04	0.32270951D-04	0.18159921D-03	0.96823130D-05	0.18691531D-03	0.12255452D-04
8	0.20914333D-04	0.41952459D-04	0.13381705D-04	0.11755329D-03	0.68612540D-04	0.12014303D-03
9	0.85161528D-04	0.41134042D-04	0.16553435D-03	0.41905782D-04	0.21764206D-03	0.20425218D-04
10	0.20775381D-04	0.42338122D-04	0.13554139D-04	0.11894758D-03	0.66106046D-04	0.22074673D-03

	X 4 (COL 7)	Y 4 (COL 8)	X 5 (COL 9)	Y 5 (COL 10)
1	0.89372885D-04	0.20914333D-04	0.85161528D-04	0.20775381D-04
2	0.32270951D-04	0.41952459D-04	0.41134042D-04	0.42338122D-04
3	0.18159921D-03	0.13381705D-04	0.16553435D-03	0.13554139D-04
4	0.96823130D-05	0.11755329D-03	0.41905782D-04	0.11894758D-03
5	0.18691531D-03	0.68612540D-04	0.21764206D-03	0.66106046D-04
6	0.12255452D-04	0.12014303D-03	0.20425218D-04	0.22074673D-03
7	0.26274303D-03	0.11991372D-04	0.26150775D-03	0.12031467D-04
8	0.11991372D-04	0.17342529D-03	0.74525217D-04	0.17613918D-03
9	0.26150775D-03	0.74525217D-04	0.31196712D-03	0.72499913D-04
10	0.12031467D-04	0.17613918D-03	0.72499913D-04	0.27876436D-03

As can be seen in Figure 8.9, all of the station ellipses now fall within the required accuracy circle, and the design is finished. A list of the final 95% station and relative error ellipse (using the same assumptions as for the initial design) is made in Table 8.9.

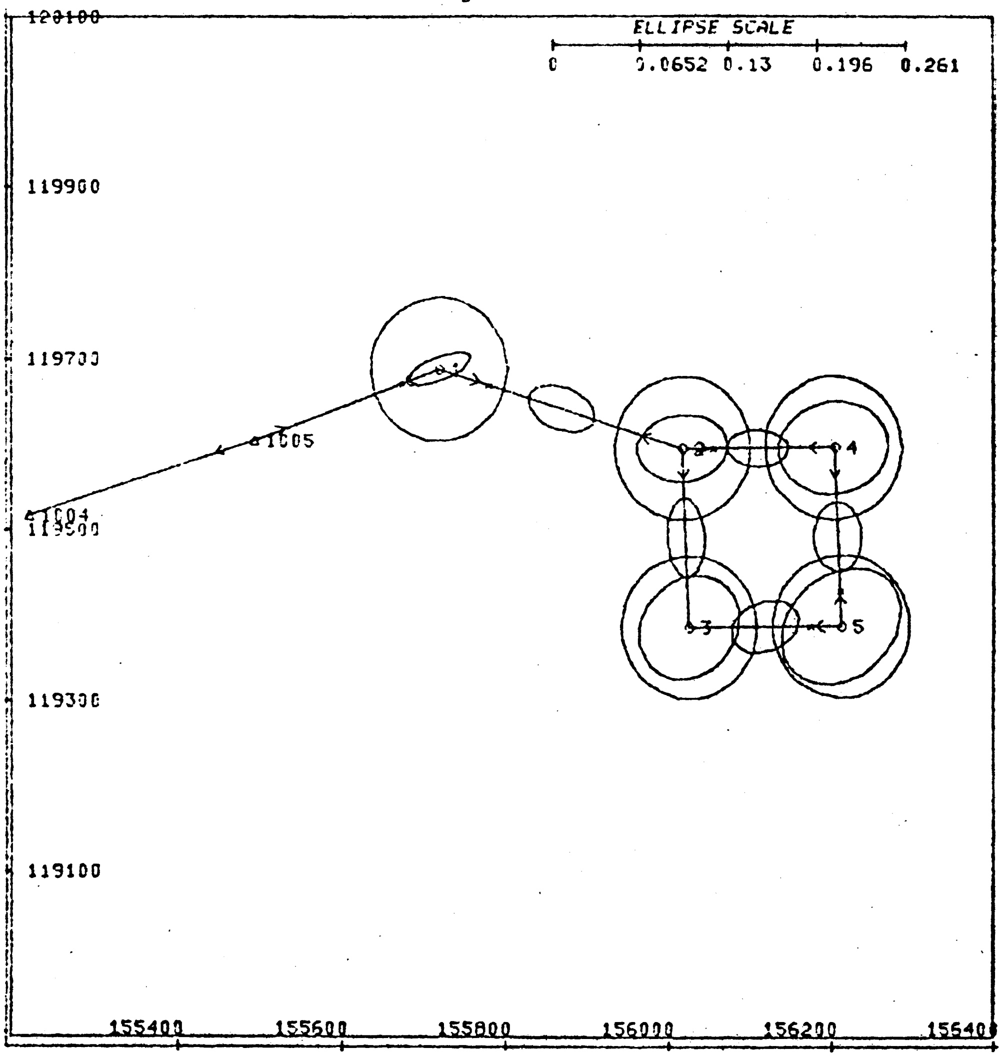


Figure 8.9 Final Plot of Property Survey Design

STATION ELLIPSES (95%)

Point	Semimajor(m)	Semiminor(m)	θ
1	0.025	0.008	70°11'09"
2	0.033	0.023	87°54'11"
3	0.040	0.034	78°50'42"
4	0.040	0.032	88°14'20"
5	0.047	0.037	78°24'09"

RELATIVE ELLIPSES (95%)

Points	Semimajor(m)	Semiminor(m)	θ
1-2	0.025	0.015	-72°50'36"
2-3	0.028	0.014	- 2°15'34"
2-4	0.022	0.013	-89°04'23"
3-4	0.025	0.017	-12°02'02"
4-5	0.025	0.017	- 2°28'02"

Table 8.9 Final 95% Error Ellipses for Property Survey Design

This chapter has shown how the preanalysis or design of a horizontal network can be done. This technique is used wherever a priori knowledge of the expected accuracies of the points in a network is desired. The next chapter treats the equally important matter of testing the results of an adjustment of a network to see if they are reasonable. This process is known as postanalysis.

9. POSTANALYSIS

Postanalysis of a horizontal network tests whether the results of an adjustment are reliable, and is based on multivariate analysis. One important test which is performed is the chi-square test of the variance factor. This test takes the form

$$\frac{df\ \hat{\sigma}_o^2}{\chi^2_{df,1-\frac{\alpha}{2}}} < \sigma_o^2 < \frac{df\ \hat{\sigma}_o^2}{\chi_{df,\frac{\alpha}{2}}} , \qquad (9\text{-}1)$$

where df = degrees of freedom of the adjustment,

$\hat{\sigma}_o^2$ = a posteriori variance factor computed by equation (AII-18),

α = significance level (e.g. 0.05),

σ_o^2 = a priori variance factor (usually assumed to be 1),

χ^2 = chi-square distribution value from Table AIII-1 (replace u by df).

Equation (9-1) tests the null hypothesis H_o

$$H_o : \sigma_o^2 = \sigma_o^2 \text{ hypothesized} ,$$

i.e. is the actual value of σ_o^2 equal to what it was assumed to be (e.g. 1)? If the test fails, then this hypothesis is rejected at the $(1-\alpha)$% confidence level. Two possible reasons for its failure are

1) Incorrect a priori covariance matrix C_L of the observations (i.e. wrong weights for the observations),

2) Observations are not normally distributed.

The first reason given is usually the first to be investigated. If it is found that the weights are chosen correctly and the test still fails, then the second reason is examined. The observations are examined by testing the residuals for outliers similar to the data screening process explained in section 2.3. Each residual v_i from the adjustment is tested as follows:

$$n_{\frac{\alpha}{2}} \sigma_{v_i} < v_i < n_{1-\frac{\alpha}{2}} \sigma_{v_i} , \tag{9-2}$$

where $n_{\frac{\alpha}{2}}$ = values of the normal distribution for probability $\frac{\alpha}{2}$ (see Table 2.14),

α = significance level (e.g. 0.05),

σ_{v_i} = known a priori standard deviation of the observation whose residual is being tested,

v_i = residual being tested (computed by eq. (AII-17)).

Assuming that the a priori standard deviation σ_{v_i} is known, then if this test fails the observation does not come from a normal distribution. This usually implies that some systematic bias has affected the observation, and it should be reobserved.

Another test which is useful is one for comparing two determinations of the same set of parameters to see if they are significantly different. This test assumes that the difference vector $(X_2 - X_1)$ between the two determinations is a random variable which is normally distributed. The test is

$$(X_2 - X_1)^T C_{x_2}^{-1} (X_2 - X_1) < \chi^2_{u,1-\alpha} , \tag{9-3}$$

where X_2 = vector of parameters being tested,

X_1 = originally determined parameters,

C_{x_2} = variance covariance matrix of the parameters being compared,

$\chi^2_{u,1-\alpha}$ = chi-square distribution with u degrees of freedom at probability level $1-\alpha$ (see Table AIII.1) , u = number of parameters being compared (i.e. dimension of vector $(X_2 - X_1)$) ,

α = significance level (e.g. 0.05).

If the test fails, then the two parameter determinations are considered different at the $(1-\alpha)\%$ confidence level. If it passes, then the two sets of parameters cannot be considered significantly different (again at $(1-\alpha)\%$ confidence level). One precautionary note when using this test is that the two sets of parameters should be determined using approximately the same level of accuracy; i.e. C_{x_1} and C_{x_2} should not be greatly different. If a network is being designed or adjusted specifically for the purpose of comparing to a previous adjustment of the same network, then the simultaneous error ellipses (Appendix III, eqs. (AIII-12) and (AIII-13)) should be computed since all of the points are required to be inside the $(1 - \alpha)\%$ confidence ellipse simultaneously.

The following examples illustrate some of the postanalysis concepts described above.

Using the a posteriori variance factor from section 6.1 and equation (9-1), the chi-square test of the variance factor (for $\alpha = 0.05$) yields

$$\frac{3 \cdot 1.9214}{\chi^2_{3,0.975}} < 1 < \frac{3 \cdot 1.9214}{\chi^2_{3,0.025}} ,$$

$$\frac{5.76}{9.35} < 1 < \frac{5.76}{0.216} ,$$

$$0.62 < 1 < 26.69 ,$$

and the test passes. The hypothesis that the a priori variance factor is 1 cannot be rejected at the 95% confidence level.

The chi-square test on the variance factor should always be performed after an adjustment as an overall check on the validity of the results. If the test fails, then there is a good chance that something is wrong in the adjustment.

As an example of testing of the residual for outliers, the first residual of equation (6-2) is tested. From equation (9-2), the test is

$$n_{0.025}\ 2''\!.0 < 2''\!.07 < n_{0.975}\ 2''\!.0 \quad ,$$

where α is assumed to be 0.05, and the a priori standard deviation for this direction (from 1 to 1006) is 2".0. From Table 2.14, the value of $n_{0.025} = -1.96$, and $n_{0.975} = 1.96$. The test becomes

$$-3''\!.92 < 2''\!.07 < 3''\!.92 \quad ,$$

which is true, and the test passes. This test should be performed on all residuals of an adjustment even though the chi-square test on the variance factor passes. It is easily verified that all of the residuals for the examples in sections 6.1 and 6.2 pass the outlier test. It is said, then, that we are 95% confident that the residuals come from a normal distribution.

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APPENDIX I

Taylor's Series

The Taylor series is used to linearize non-linear mathematical models. Given a single function f(x) of a single variable x (unidimensional case), and a known value of this function f(a) at x = a, then the function f(x) is given by Taylor's series as

$$f(x) = f(a) + \left.\frac{\partial f}{\partial x}\right|_a (x-a) + \left.\frac{\partial^2 f}{\partial x^2}\right|_a \frac{(x-a)^2}{2!} + \ldots + \left.\frac{\partial^n f}{\partial x^n}\right|_a \frac{(x-a)^n}{n!} + \ldots. \qquad \text{(AI-1)}$$

For values of x close to a, the linear approximation is used. This is

$$f(x) = f(a) + \left.\frac{\partial f}{\partial x}\right|_a (x-a)\,. \qquad \text{(AI-2)}$$

From Figure AI.1, the geometric meaning of this linear approximation is clear. The function f(x) is approximated by a straight tangent to f(x) at a

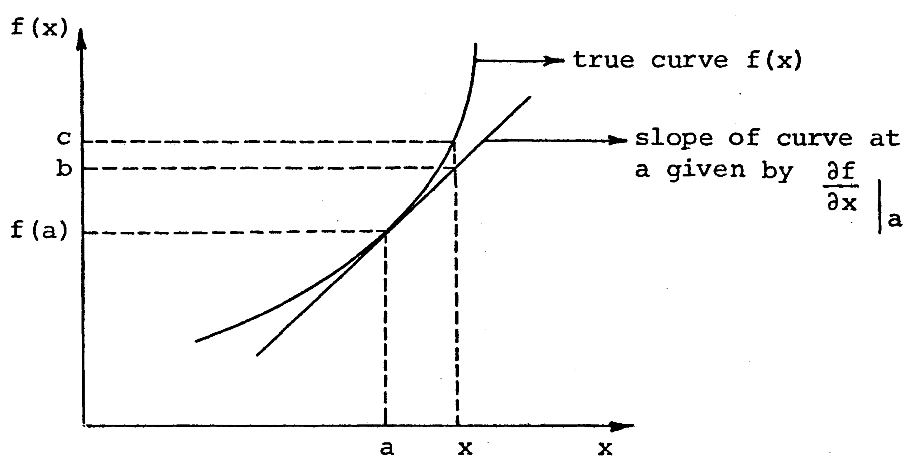

Figure AI.1. Geometric Interpretation of Linear Taylor's Series with slope $\left.\frac{\partial f}{\partial x}\right|_a$. When (x - a) is given (or evaluated), the value of the function f(x) is approximated by b, and the exact value is c. Thus, the error arising from using the linear approximation is c-b.

If f is a function of more than one variable, say $f(x_1, x_2)$ and its value is known at $x_1 = a_1$ and $x_2 = a_2$,
then for values of (a_1, a_2) close to (x_1, x_2) the linear approximation is

$$f(x_1, x_2) = f(a_1, a_2) + \left.\frac{\partial f}{\partial x_1}\right|_{a_1,a_2} (x_1-a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{a_1,a_2} (x_2-a_2) \ . \qquad \text{(AI-3)}$$

Setting

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \Delta X = \begin{bmatrix} x_1-a_1 \\ x_2-a_2 \end{bmatrix}, \quad X° = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix},$$

then

$$f(X) = f(X°) + \left.\frac{\partial f}{\partial X}\right|_{X°} \Delta X \ . \qquad \text{(AI-4)}$$

If there is more than one function of X (e.g. f_1, f_2) then the following set of equations exists:

$$f_1(X) = f_1\ (X°) + \left.\frac{\partial f_1}{\partial X}\right|_{X°} \Delta X \ ,$$

$$f_2(X) = f_2\ (X°) + \left.\frac{\partial f_2}{\partial X}\right|_{X°} \Delta X \ .$$

Setting

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix},$$

then

$$f(X) = F(X^{\circ}) + \left.\frac{\partial F}{\partial X}\right|_{X^{\circ}} \Delta X . \qquad \text{(AI-5)}$$

Equation (AI-5) is the matrix form of the Taylor's series linear approximation. Equations (AI-2) and (AI-4) can be thought of as special cases of this matrix form.

APPENDIX II

Least Squares Method

The least squares method is usually used to give a unique solution for an overdetermined case (i.e. number of observations n greater than number of parameters u). Only the inverse (explicit) form of mathematical model (cf. eq. (1-2)), which is sometimes called the parametric case of adjustment, is considered here. In matrix form, this model is expressed as

$$L = F(X) \; ,$$

$$\text{or} \qquad F(X) - L = 0 \; , \qquad \text{(AII-1)}$$

where L = vector of observations,

F(X) = non-linear functions of the parameters X.

The linearized form of this inverse (explicit) model is (see also Appendix I and chapters 3 and 4)

$$\underset{(n,u)}{A} \; \underset{(u,1)}{\hat{X}} + \underset{(n,1)}{W} - \underset{(n,1)}{V} = 0 \; , \qquad \text{(AII-2)}$$

where V = vector of residuals or corrections to the observations,

$A = \left.\frac{\partial F}{\partial X}\right|_{X^o}$ = design matrix or Jacobian of transformation from observation space to parameter space,

$\hat{X}$ = solution vector of corrections, which, when added to the approximate values X^o gives the parameters X (see eq. AII-12),

$W = F(X^o) - L$ = misclosure vector.

The least squares estimate for X is obtained subject to the condition

$$V^T PV = \text{minimum}, \tag{AII-3}$$

where $\underset{(n,n)}{P} = \sigma_o^2 \, C_L^{-1}$ is called the weight matrix of the observables,

σ_o^2 = a priori variance factor,

$\underset{(n,n)}{C_L}$ = variance covariance matrix of the observables .

The variation function ϕ relating the unknown quantities $\hat{X}$ and V to the known quantities A, W, and P is

$$\phi = V^T PV + 2K^T \; (A\hat{X} + W - V) \; , \tag{AII-4}$$

where K = unknown vector of Lagrange correlates.

To find the minimum of the variation function, the derivatives with respect to $\hat{X}$ and V are found and set to zero. Thus

$$\frac{1}{2} \frac{\partial \phi}{\partial v} = V^T P - K^T = 0 \quad , \tag{AII-5}$$

$$\frac{1}{2} \frac{\partial \phi}{\partial \hat{x}} = K^T A = 0 \quad . \tag{AII-6}$$

The transpose of the above two equations and the linearized mathematical model (eq. (AII-2)) make up the following least squares normal equations system:

$$PV - K = 0 \; ,$$

$$A^T K = 0 \; ,$$

$$A\hat{X} + W - V = 0 \quad .$$

Writing these equations in hypermatrix form yields the most expanded matrix form of the normal equations system as

$$\begin{bmatrix} P & -I & 0 \\ -I & 0 & A \\ 0 & A^T & 0 \end{bmatrix} \begin{bmatrix} V \\ K \\ \hat{X} \end{bmatrix} + \begin{bmatrix} 0 \\ W \\ 0 \end{bmatrix} = 0 . \qquad \text{(AII-7)}$$

The solution vector $\hat{X}$ is obtained by using a matrix elimination technique [e.g Thompson, 1969]. Given the matrix equation system

$$\left[\begin{array}{cc} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} X \\ \hline Y \end{array}\right] + \left[\begin{array}{c} U \\ \hline V \end{array}\right] = 0 ,$$

X is eliminated by forming a modified coefficient matrix and known vector as follows:

$$[D - CA^{-1} B]\, Y + [V - CA^{-1} U] = 0 . \qquad \text{(AII-9)}$$

Applying this method to equation (AIII-7) to first eliminate V gives

$$\left\{ \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} - \begin{bmatrix} -I \\ 0 \end{bmatrix} P^{-1} [-I \quad 0] \right\} \begin{bmatrix} K \\ \hat{X} \end{bmatrix} + \left\{ \begin{bmatrix} W \\ 0 \end{bmatrix} - \begin{bmatrix} -I \\ 0 \end{bmatrix} P^{-1} [0] \right\} = 0 ,$$

or

$$\begin{bmatrix} -P^{-1} & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} K \\ \hat{X} \end{bmatrix} + \begin{bmatrix} W \\ 0 \end{bmatrix} = 0 . \qquad \text{(AII-10)}$$

Using the same technique to eliminate K from equation (AII-10) yields

$$[A^T PA]\, \hat{X} + A^T PW = 0 \quad ,$$

or, solving for $\hat{X}$

$$\hat{X} = - [A^T PA]^{-1} \; A^T PW \quad . \qquad \text{(AII-11)}$$

This is the least squares estimate for the solution vector $\hat{X}$. The parameters X are now computed as

$$X = X^{\circ} + \hat{X} \quad . \qquad \text{(AII-12)}$$

Usually, the solution vector $\hat{X}$ is iterated (i.e. now $X^\circ \leftarrow X$, and a new $\hat{X}$ is computed) until it is very small (e.g. < 1 mm). This is necessary because the use of a linear approximation is not exact (see Appendix I). The final expression for X is

$$X = X^\circ - [A^T PA]^{-1} A^T PW . \qquad \text{(AII-13)}$$

To find the variance covariance matrix C_x of the parameters, the covariance law [e.g. Thomson et al., 1978] is used to propogate errors through equation (AII-13). Since the only independent random variable in equation (AII-13) is L (because $W = F(X^\circ) - L$), then

$$C_x = (\frac{\partial X}{\partial L}) \quad C_L \, (\frac{\partial X}{\partial L})^T . \qquad \text{(AII-14)}$$

Realizing that

$$\frac{\partial X}{\partial L} = [A^T PA]^{-1} \quad A^T P , \qquad \text{(AII-15)}$$

since

$$\frac{\partial W}{\partial L} = \frac{\partial (F(X^\circ) - L)}{\partial L} = -I \quad ,$$

then

$$C_x = [A^T PA]^{-1} A^T PC_L PA \; [A^T PA]^{-1} .$$

Noting that $P = \sigma_o^2 \; C_L^{-1}$, then

$$C_x = \sigma_o^2 \; [A^T PA]^{-1} A^T PA \; [A^T PA]^{-1} ,$$

or

$$C_x = [A^T \, C_L^{-1} \, A]^{-1}. \qquad \text{(AII-16)}$$

Thus, C_x is simply the normal equations inverse of the solution vector (see eq. (AII-11)).

Once the final solution vector is found, then the residual vector V is computed as

$$V = A\hat{X} + W \quad . \qquad \text{(AII-17)}$$

From the residuals V, the a posteriori variance factor $\hat{\sigma}_o^2$ is evaluated as

$$\hat{\sigma}_o^2 = \frac{V^T P V}{df} \qquad \text{(AII-18)}$$

where df = degrees of freedom = n - u .

This a posteriori variance factor is useful when performing a post analysis of adjustment results (see chapter 9).

APPENDIX III

Error Ellipses

Error ellipses (see Figure AIII-1), are characterized by the length of their semimajor and semiminor axes a and b, respectively and the azimuth θ of the semimajor axis a. These ellipses are representative of the error of a point in a network (sometimes called station ellipses) or of the error in the difference of coordinates between two points (relative error ellipses). These error ellipses are computed knowing the variance covariance matrix C_x of the parameters, and the so-called c factor. The c factor is used to increase the confidence level of the ellipse from standard(~39%) to a desired (e.g. 95%) confidence level in the following way:

$$a = ca_s \ ,$$
$$b = cb_s \ , \qquad \text{(AII-1)}$$

where a_s and b_s are the semimajor and semiminor axes of the standard error ellipse.

The basis of error ellipse computation lies in multivariate statistics [e.g. Wells and Krakiwsky, 1971; Hogg and Craig, 1970]. The quadratic form of the parameters for the a priori variance factor σ_o^2 known is distributed as

$$X^T C_x^{-1} X = \chi^2_{u,1-\alpha} \ , \qquad \text{(AII-2)}$$

where X = difference between the least squares estimate of the parameters and the true value of the parameters,

$\chi^2_{u,1-\alpha}$ = random variable with a chi-square distribution and degrees of freedom u (see Table AIII-1),

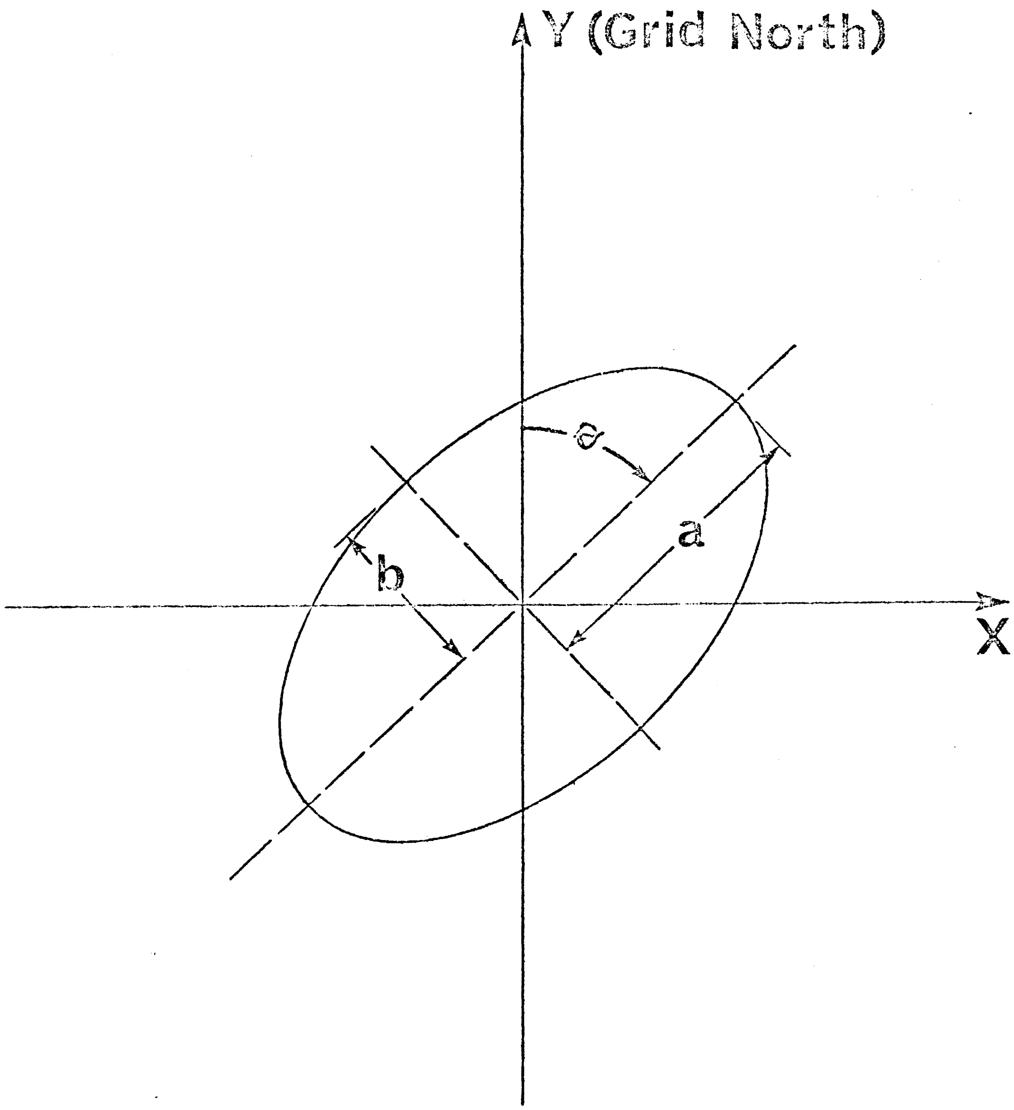

Figure AIII.1 Error Ellipse

u \ 1-α	.005	.010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.900	.995
1	.0000393	.000157	.000982	.00393	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	.4.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	10.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

Table AIII.1 χ^2 Distribution

$1-\alpha$ = desired confidence level (e.g. 0.95) ,

u = dimensionality of the problem.

For the case of horizontal geodetic networks, $u = 2$, and equation (AIII-2) is written as

$$[x \quad y] \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \chi^2_{2,1-\alpha} . \qquad \text{(AIII-3)}$$

The C_x element in equations (AIII-2) and (AIII-3) is the submatrix for a single point of the full C_x matrix for the whole network. An eigenvalue problem [e.g. Kreyszig,1972; Mikhail, 1976] is performed an equation (AIII-3) to transform it to an equation without cross product terms as follows;

$$[x' \quad y'] \begin{bmatrix} \sigma^2_{max} & 0 \\ 0 & \sigma^2_{min} \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \chi^2_{2,1-\alpha} ,$$

where x', y' = transformed coordinates with respect to the rotated coordinate axes resulting from the eigenvalue problem,

σ^2_{max} = largest eigenvalue of C_x (see eq. (AIII-5)),

σ^2_{min} = smallest eigenvalue of C_x (see eq. (AIII-6)).

Specfically,the eigenvalues are

$$\sigma^2_{max} = \frac{1}{2} [(\sigma_x^2 + \sigma_y^2) + \{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma^2_{xy}\}^{1/2}] \qquad \text{(AIII-5)}$$

$$\sigma^2_{min} = \frac{1}{2} [(\sigma_x^2 + \sigma_y^2) - \{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma^2_{xy}\}^{1/2}] . \qquad \text{(AIII-6)}$$

Writing out equation (AIII-4) explicitly results in

$$\frac{(x')^2}{\sigma^2_{max}\ \chi^2_{2,1-\alpha}} + \frac{(y')^2}{\sigma^2_{min}\ \chi^2_{2,1-\alpha}} = 1 \ , \qquad \text{(AIII-7)}$$

which is the familiar equation of an ellipse with axes $(\sigma^2_{max}\ \chi^2_{2,1-\alpha})^{1/2}$ and $(\sigma^2_{min}\ \chi^2_{2,1-\alpha})^{1/2}$. The standard error ellipse is found when $\chi^2_{2,1-\alpha}$ is equal to 1, which corresponds to $(1-\alpha) = 0.3935$, or a 39.35% confidence level. Thus

$$a_s = (\sigma^2_{max})^{1/2} \ ,$$

$$b_s = (\sigma^2_{min})^{1/2} \qquad \text{(AIII-8)}$$

are the axes of the standard error ellipse.

It is obvious from equation (AIII-7) that the required c factor to compute a and b is

$$c = (\chi^2_{2,1-\alpha})^{1/2} \ , \qquad \text{(AIII-9)}$$

for the case of σ_o^2 assumed known. If the a priori variance factor is assumed unknown, however, then the a posterior variance factor $\hat{\sigma}_o^2$ (see Appendix II, eq. (AII-18)) is used to estimate the variance covariance matrix as $\hat{C}_x$, where the ^ stands for an estimated quantity. In this case, the quadratic form of the parameters is distributed as

$$X^T\ \hat{C}_x^{-1}\ X = u\ F_{u,df,1-\alpha} \ , \qquad \text{(AIII-10)}$$

where F = random variable with a Fischer distribution (see table AIII.2) and degrees of freedom u and df,

df = degrees of freedom of the adjustment.

Using the same development as for σ_o^2 known, the c factor when using the estimated variance factor $\hat{\sigma}_o^2$ to estimate $\hat{C}_x$ is

df \ u	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table AIII.2 F Distribution for (1-α) = 0.95

$$c = (2F_{2,df,1-\alpha})^{1/2} . \qquad \text{(AIII-11)}$$

Equations (AIII-9) and (AIII-11) above give the c factor for computation of a single error ellipse without regard to other stations in the network. If, however, it is required that N station ellipses all have the desired confidence level (1-α) simultaneously, then the c factor is computed as

$$c = (\chi^2_{2,1-\alpha/N})^{1/2} , \qquad \text{(AIII-12)}$$

or

$$c = (2F_{2,df,1-\alpha/N})^{1/2} , \qquad \text{(AIII-13)}$$

where α has been replaced by α/N. This is a direct result of Bonferroni's inequality [Vanicek and Krakiwsky, in prep.] which states that the given confidence level is at least 1-α for the simultaneous case.

The orientation of the error ellipse is given by the normalized eigenvector corresponding to the eigenvalue σ^2_{max} of the eigenvalue problem performed on equation (AIII-3). The azimuth of the semimajor axis is thus

$$\theta = \text{sign}(\sigma_{xy}) \cdot \arccos \left[\frac{(\sigma^2_{max} - \sigma_x^2)}{(\sigma^2_{xy} + (\sigma^2_{max} - \sigma_x^2)^2)^{1/2}}\right] , \qquad \text{(AIII-14)}$$

where σ^2_{max} is given by equation (AIII-5).

The computation of relative error ellipses is facilitated by applying the covariance law to the following expressions:

$$\Delta x_{ij} = x_j - x_i ,$$
$$\Delta y_{ij} = y_j - y_i . \qquad \text{(AIII-15)}$$

This gives

$$C_{\Delta x,\Delta y} = B\, C_{x,y}\, B^T \quad ,$$

$$\text{where } B = \begin{bmatrix} \frac{\partial \Delta x_{ij}}{\partial x_i} & \frac{\partial \Delta x_{ij}}{\partial y_i} & \frac{\partial \Delta x_{ij}}{\partial x_j} & \frac{\partial \Delta x_{ij}}{\partial y_j} \\ \frac{\partial \Delta y_{ij}}{\partial x_i} & \frac{\partial \Delta y_{ij}}{\partial y_i} & \frac{\partial \Delta y_{ij}}{\partial x_j} & \frac{\partial \Delta y_{ij}}{\partial y_j} \end{bmatrix}$$

or carrying out the partial derivatives and writing $C_{x,y}$ in full

$$C_{\Delta x,\Delta y} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^2_{x_i} & \sigma_{x_i y_i} & \sigma_{x_i x_j} & \sigma_{x_i y_j} \\ & \sigma^2_{y_i} & \sigma_{y_i x_j} & \sigma_{y_i y_j} \\ & & \sigma^2_{x_j} & \sigma_{x_j y_j} \\ \text{symmetric} & & & \sigma^2_{y_j} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_{\Delta x,\Delta y} = \begin{bmatrix} \sigma^2_{x_i} - 2\sigma_{x_i x_j} + \sigma^2_{x_j} & \sigma_{x_i y_i} - \sigma_{y_i x_j} - \sigma_{x_i y_j} + \sigma_{x_j y_j} \\ \sigma_{x_i y_i} - \sigma_{y_i x_j} - \sigma_{x_i y_j} + \sigma_{x_j y_j} & \sigma^2_{y_i} - 2\sigma_{y_i y_j} + \sigma^2_{y_j} \end{bmatrix}$$

(AIII-16)

Thus, to compute the standard relative error ellipse between points i and j, the equations for station ellipses (i.e. eqs. (AIII-5), (AIII-6), (AIII-14)) are employed, but making the substitutions

$$\sigma_x^2 = \sigma_{x_i}^2 - 2\sigma_{x_i x_j} + \sigma_{x_j}^2 ,$$

$$\sigma_{xy} = \sigma_{x_i y_i} - \sigma_{y_i x_j} - \sigma_{x_i y_j} + \sigma_{x_j y_j} , \qquad \text{(AIII-17)}$$

$$\sigma_y^2 = \sigma_{y_i}^2 - 2\sigma_{y_i y_j} + \sigma_{y_j}^2 .$$

The Surveys and Mapping Branch of the Dept. of Energy, Mines and Resources uses relative error ellipses to classify different order surveys [Energy, Mines and Resources, 1973]. A survey station of a network is classified according to whether the semimajor axis of the 95% confidence ellipse with respect to other stations of the network is less than or equal to

$$r = k\ d , \qquad \text{(AIII-18)}$$

where r = radius of an error circle in cm (see Figure AIII.2),

d = distance in km to any station,

k = factor assigned according to the order of survey (see Table AIII.3).

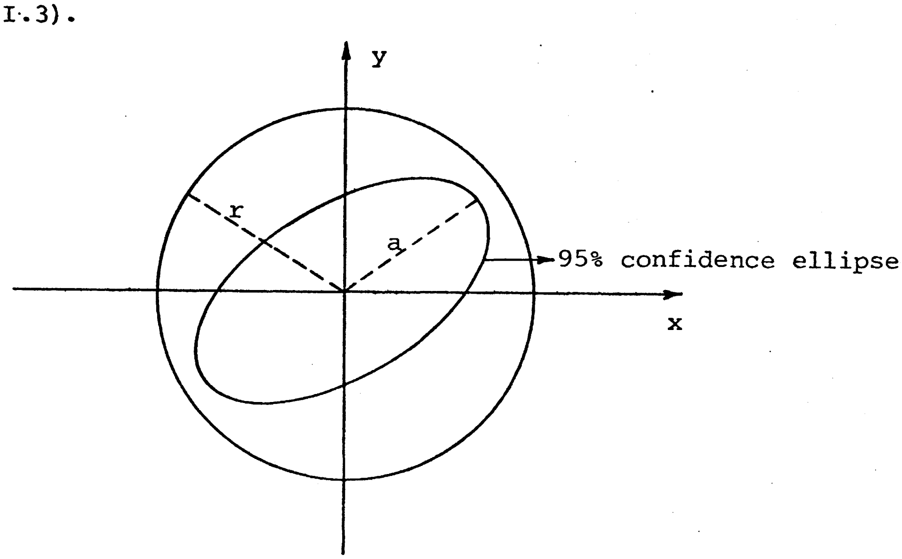

Figure AIII.2. Error Circle and Confidence Ellipse

Thus, if $a < r$ for all the relative error ellipses (at 95% confidence level) between station i and the rest of the network stations, then station i is classified in that specific order of survey. For example, a

order	$\bar{k}$	r in ppm
1st	2	20
2nd	5	50
3rd	12	120
4th	30	300

Table AIII.3. Horizontal Survey Classification.

second order survey station must have the semimajor axis a of the 95% relative error ellipse less than 25 cm for stations 5 km apart.