A MANUAL FOR GEODETIC POSITION COMPUTATIONS IN THE MARITIME PROVINCES

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PREFACE

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PREFACE

This "manual" is the second of three being written to cover the correct and practical use of the geodetic information of the redefined Maritime Geodetic Network. While the first manual dealt with a single terrain point, this involves two points and the observations between them. The third manual will centre on terrestrial networks (many terrain points and observations amongst them).

This manual was written as a guide to the use and interpretation of geodetic information for two terrain points. It is to serve mainly as a surveyors handbook for Geodetic Position Computations in the three-dimensional, ellipsoidal, and conformal mapping plane environments in the maritime provinces. No derivations or extensive explanations of the mathematical formulae are given. The equations required to solve the position and associated error transformation problems are stated, the notation used is explained, and a numerical example is presented. A reader desiring extensive background information as to the relevance of this manual, and a detailed explanation of the origins of the mathematical formulae, is referred to the reference material. It should be noted that the material presented in this manual has been rigorously developed. Approximations made, and their affects are clearly indicated. Further approximations, for whatever reasons, are left to the professional judgement of the surveyor.

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1. INTRODUCTION

In "A Manual for Geodetic Coordinate Transformations in the Maritime Provinces" [Krakiwsky et al, 1977], it was shown that a terrain point i could be described mathematically by any one of three different sets of coordinates (three-dimensional (X_i, Y_i, Z_i), ellipsoidal (ϕ_i, λ_i), conformal mapping plane (x_i, y_i)) and their associated accuracies (variance-covariance matrices). Furthermore, it was shown, by rigorous coordinate and variance-covariance matrix transformations, that the coordinates and associated accuracies in all three systems were equivalent. In this second handbook, we introduce a second point j, and treat two different problems involving i and j simultaneously in each of the three-dimensional, ellipsoidal, and conformal mapping plane environments.

One of the problems - the so-called <u>inverse problem</u> - involves the computation of the azimuths, distance , and associated accuracies between the two points. A rigorous procedure for each of the three environments is given, and it is shown, via appropriate "reductions", that the solutions are equivalent.

The other problem - the so-called <u>direct problem</u> - involves the computation of the coordinates and associated variance-covariance matrix of the second point j using observations made from i to j. Again, solutions are given for the three-dimensional, ellipsoidal, and conformal mapping plane environments and, using appropriate "reductions" of observed and computed data and coordinate transformations, it is shown that the solutions are equivalent.

Since this is the first time we introduce observed quantities - directions, angles, azimuths, and distances, some special attention is given to the reduction of observations. We begin with terrain angular and distance measurements as a result of our terrestrial observing procedures. After correcting for atmospheric and instrumental effects, we are left with measurements that can be used directly in three-dimensional position computations (Chapter 2). To express the computed coordinates of the new point in other than a topocentric coordinate system, say geodetic Cartesian coordinates (X_j, Y_j, Z_j), certain coordinate transformations are required. If geodetic curvilinear (ellipsoidal) or conformal mapping plane coordinates are desired, the coordinate transformations outlined in 'A Manual for Geodetic Coordinates Transformations in the Maritime Provinces' [Krakiwsky et al, 1977] are used. In most instances, however, the practicing surveyor finds it desirable to carry out position computations in the environment in which the coordinates of point j must be expressed, usually the surface of a reference ellipsoid or a conformal mapping plane. In this case, observations must be "reduced" to the appropriate surface prior to position computation. For ellipsoidal computations, the corrected terrain measurements must be reduced to the surface of the reference ellipsoid (Chapter 3), while for conformal mapping plane computations, one must first "reduce" measurements to the reference ellipsoid, thence make further "reductions" to express the measurements correctly on the conformal mapping plane (Chapter 4). This entire process - terrain to ellipsoid to conformal mapping plane - is depicted, for a distance measurement, in Figure 1-1. This manual treats all of the reduction

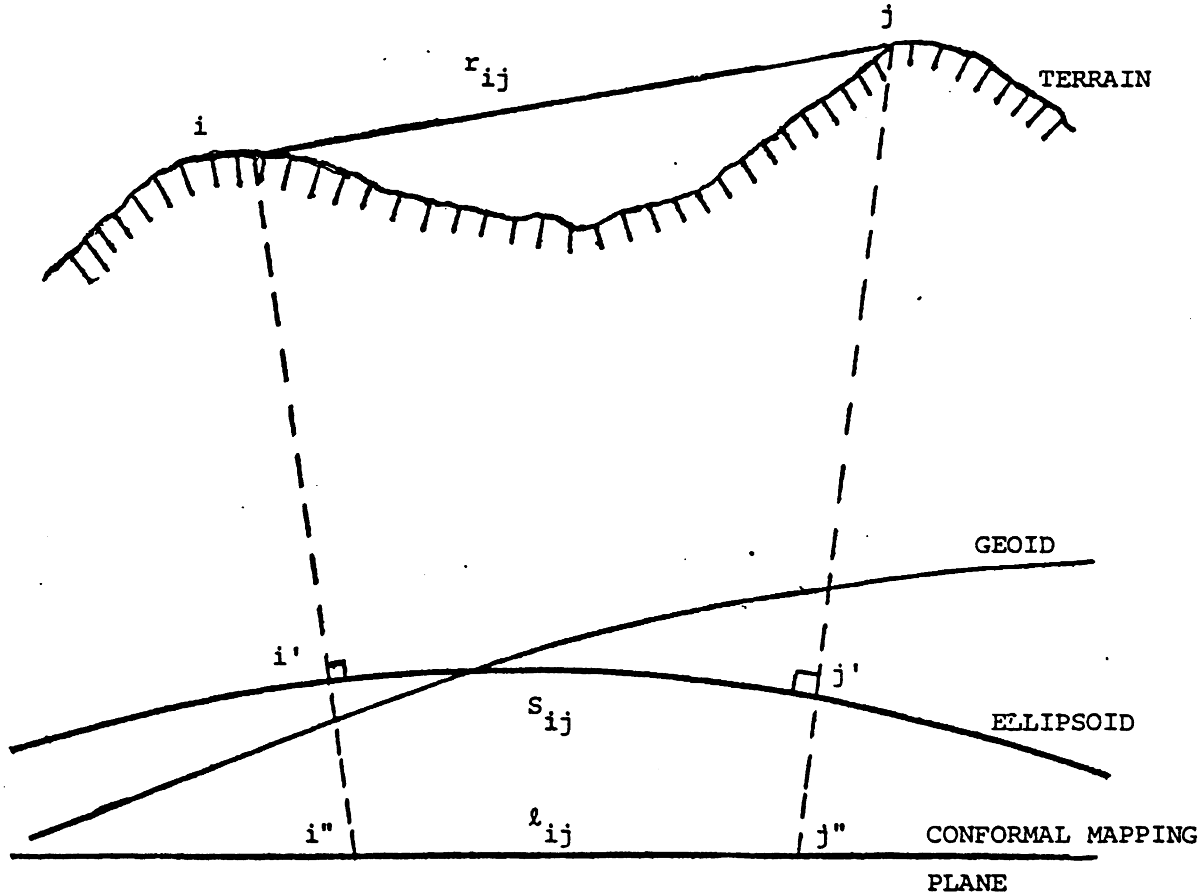


Figure 1-1

Distance Reduction

processes required for position computations in the maritime provinces.

In closing, the reader should take special note of the fact that the measurement reduction processes are reversible; that is, one may compute a distance on a conformal mapping plane and "reduce" it up to the terrain. This is an important point for surveyors who are often faced with the need for terrain values for computed distances and azimuths. This inverse reduction process is covered in Chapters 3 and 4.

2. COMPUTATIONS IN THREE DIMENSIONS

2.1 Notation

Before giving the general concepts and various formulae, the notation used in this chapter is listed.

$\eta_i \equiv$ prime vertical deflection component at point i

$\xi_i \equiv$ meridian deflection component at point i

$\Delta az_i \equiv$ difference between astronomic and geodetic azimuth from point i to point j

$r_{ij} \equiv$ spatial distance from point i to point j ($r_{ij} = |r_{ij}|$)

$\phi_i, \lambda_i \equiv$ geodetic latitude and longitude of a point i

$\Phi_i, \Lambda_i \equiv$ astronomic latitude and longitude of a point i

$Z_{ij} \equiv$ zenith angle in the local astronomic system (measured from Z axis)

$A_{ij} \equiv$ astronomic azimuth in the astronomic system of line ij

$(\vec{r}_{ij})_G \equiv$ position vector in Geodetic coordinate system

$(\vec{r}_{ij})_{LG} \equiv$ position vector in Local Geodetic coordinate system

$h_i \equiv$ ellipsoidal height of a point i

$(\vec{r}_{ij})_{LA} \equiv$ position vector in Local Astronomic coordinates

$\rho \equiv 6.48 \times 10^5/\pi = 206264.8....$

2.2 General Concepts

In chapter 5 of "Geodetic Coordinate Transformations in the Maritimes",we saw that geodetic positions may be defined by a triplet of Cartesian coordinates $(X,Y,Z)_G$ or by the triplet $(\phi, \lambda, h)_G$

referred to the reference ellipsoid. Computations of geodetic positions in three dimensions, for which formulae are given in this chapter, are based on three dimensional Euclidean geometry and employ vector and matrix algebra. Since distances, zenith angles and azimuths of lines are actually observed in three dimensional space, they require no "reduction" to some surface and need only be corrected for refraction effects and instrumental corrections such as heights of instrument above the actual terrain point or zero error for electromagnetic distance measurements.

Readers not familiar with the Average Terrestrial, Geodetic, Local Geodetic and Local Astronomic coordinate systems are referred to, for example, Krakiwsky and Wells [1971].

It should be mentioned here that we present only one method for solving the direct and inverse problems in the 3-D environment. There are other methods and the interested reader is referred to, for example, Krakiwsky and Thomson [1974]. If the reader is unfamiliar with rotation matrices please review Appendix I before continuing.

2.2.1 The Direct Problem

The direct problem may be stated as; given the coordinates $(X_i, Y_i, Z_i)_G$ of point i, the terrestrial spatial distance r_{ij}, astronomic azimuth A_{ij}, and zenith angle Z_{ij} from i to a second point j, compute the coordinates $(X_j, Y_j, Z_j)_G$ of point j. We note here that if we are given $(\phi_i, \lambda_i, h_i)_G$ of point i a coordinate transformation [Krakiwsky et al., 1977] yields the Cartesian coordinates $(X_i, Y_i, Z_i)_G$.

To solve the direct problem we must know the relationship between the Local Geodetic coordinate system and Geodetic system, and between the Local Astronomic system, where we observe, and the Local Geodetic system in which we compute. If we know these relationships the observed quantities of azimuth, zenith angle and distance can be used to determine the coordinates of a second point.

The relationship between the Local Geodetic and Geodetic system is examined first. From Figure 2-1 we can rotate the vector $(\vec{r}_{ij})_{LG}$ from the Local Geodetic to the Geodetic system using [Krakiwsky and Wells, 1971].

$$(\vec{r}_{ij})_G = R_3\ (180 - \lambda_i)\ R_2\ (90 - \phi_i)\ P_2 \cdot (\vec{r}_{ij})_{LG} \tag{2-1}$$

We can then obtain the $(\vec{r}_j)_G$ using

$$(\vec{r}_j)_G = (\vec{r}_i)_G + (\vec{r}_{ij})_G \tag{2-2}$$

Expanding (2-1) and substituting into (2-2) yields

$$(X_j)_G = (X_i)_G - [\sin \phi_i \cos \lambda_i\ (X_i)_{LG} + \sin \lambda_i\ (Y_i)_{LG} - \cos \phi_i \cos \lambda_i\ (Z_i)_{LG}] \quad , \tag{2-3}$$

$$(Y_j)_G = (Y_i)_G - [\sin \phi_i \sin \lambda_i\ (X_i)_{LG} - \cos \lambda_i\ (Y_i)_{LG} - \cos \phi_i \sin \lambda_i\ (Z_i)_{LG}] \quad , \tag{2-4}$$

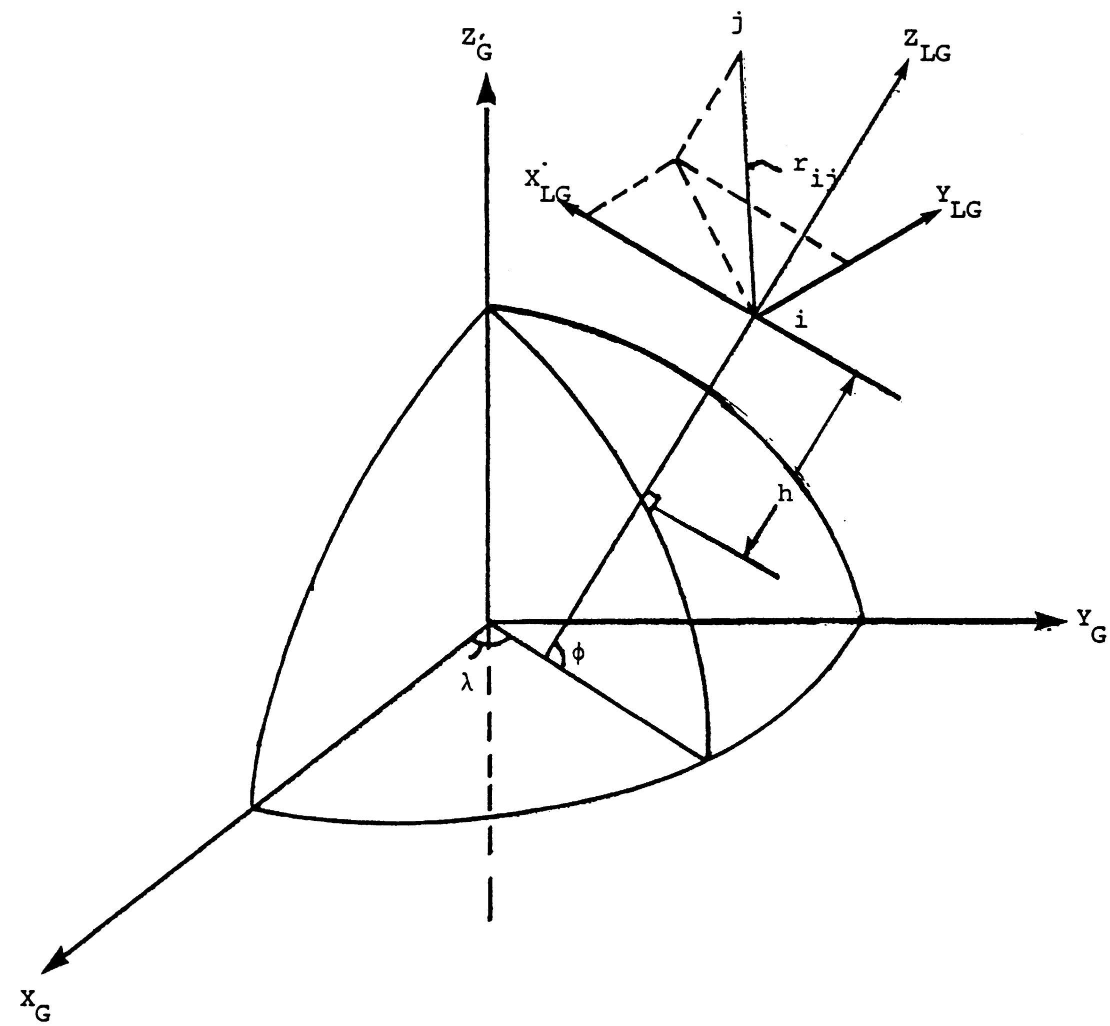


Figure 2-1

Local Geodetic System Position Vector

and $(Z_j)_G = (Z_i)_G + [\cos\phi_i \ (X_i)_{LG} + \sin\phi_i \ (Z_i)_{LG}]$. (2-5)

Continuing and referring to Figure 2.2, the position vector $(\vec{r}_{ij})_{LA}$ can be written as a function of the observables, namely

$$(\vec{r}_{ij})_{LA} = \begin{bmatrix} (X_i) \\ (Y_i) \\ (Z_i) \end{bmatrix}_{LA} = \begin{bmatrix} \vec{r}_{ij} \sin Z_{ij} \cos A_{ij} \\ \vec{r}_{ij} \sin Z_{ij} \sin A_{ij} \\ \vec{r}_{ij} \cos Z_{ij} \end{bmatrix} . \qquad (2\text{-}6)$$

The relationship between the vector $(\vec{r}_{ij})_{LA}$ and $(\vec{r}_{ij})_{LG}$ is given by [Krakiwsky and Wells, 1971].

$$(\vec{r}_{ij})_{LG} = R_3 \ (\Delta az_{ij}) R_2 \ (-\xi_i) \ R_1 \ (+\eta_i) \ (\vec{r}_{ij})_{LA} \ . \qquad (2\text{-}7)$$

ξ_i and η_i are the meridian and prime vertical components of the deflection of the vertical at the terrain point i. Δaz_{ij} is the difference between the geodetic and astronomic azimuths of the terrain point i and is given by [Krakiwsky and Wells, 1971]

$$\Delta az_{ij} = \eta_i \tan\phi_i - (\xi_i \sin A_{ij} - \eta_i \cos A_{ij}) \cot Z_{ij} . \qquad (2\text{-}8)$$

Now (2-7) is expanded with (2-6) substituted in it. We are going to assume that the deflection components and Δaz are all less than 30"0 of arc in the maritimes which allows us to write with better than .01 m accuracy that

$$(\vec{r}_{ij})_{LG} = \begin{bmatrix} r_{ij} \ (\sin Z_{ij} \cos A_{ij} + \Delta az_{ij} \sin Z_{ij} \sin A_{ij} + \xi_i \cos Z_{ij}) \\ r_{ij} \ (-\Delta az_{ij} \sin Z_{ij} \cos A_{ij} + \sin Z_{ij} \sin A_{ij} + \eta_i \cos Z_{ij}) \\ r_{ij} \ (-\xi_i \sin Z_{ij} \cos A_{ij} - \eta_i \sin Z_{ij} \sin A_{ij} + \cos Z_{ij}) \end{bmatrix} .$$

(2-9)

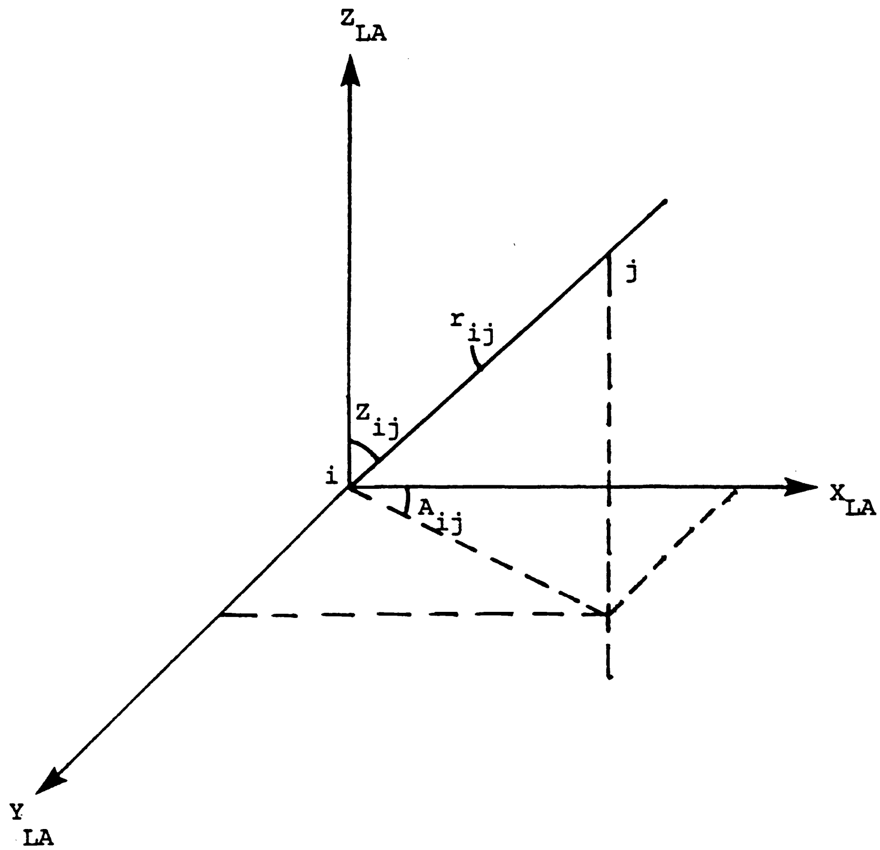


Figure 2-2

Local Astronomic Observations

The quantities $\Delta az_{ij}, \eta_i, \xi_i$ are all expressed in radians units.

Substituting (2-9) into (2-3),(2-4),and (2-5) the final solution for the geodetic coordinates of point j are

$$\begin{aligned}(X_j)_G = (X_i)_G - r_{ij} \; [&\sin\phi_i \cos\lambda_i \; (\sin Z_{ij} \cos A_{ij} + \\ &\Delta az_{ij} \sin Z_{ij} \sin A_{ij} + \xi_i \cos Z_{ij}) \\ &+ \sin\lambda_i \; (-\Delta az_{ij} \sin Z_{ij} \cos A_{ij} + \\ &\sin Z_{ij} \sin A_{ij} + \eta_i \cos Z_{ij}) \\ &- \cos\phi_i \cos\lambda_i \; (-\xi_i \sin Z_{ij} \cos A_{ij} - \\ &\eta_i \sin Z_{ij} \sin A_{ij} + \cos Z_{ij})] \;, \end{aligned} \qquad (2\text{-}10)$$

$$\begin{aligned}(Y_j)_G = (Y_i)_G - r_{ij} \; [&\sin\phi_i \sin\lambda_i \; (\sin Z_{ij} \cos A_{ij} + \Delta az_{ij} \sin Z_{ij} \\ &\sin A_{ij} + \xi_i \cos Z_{ij}) \\ &- \cos\lambda_i (-\Delta az_{ij} \sin Z_{ij} \cos A_{ij} + \sin Z_{ij} \sin A_{ij} \\ &+ \eta_i \cos Z_{ij}) \\ &- \cos\phi_i \sin\lambda_i \; (-\xi_i \sin Z_{ij} \cos A_{ij} - \eta_i \sin Z_{ij} \\ &\sin A_{ij} + \cos Z_{ij})] \;, \end{aligned} \qquad (2\text{-}11)$$

and

$$\begin{aligned}(Z_j)_G = (Z_i)_G + r_{ij} \; [&\cos\phi_i \; (\sin Z_{ij} \cos A_{ij} + \Delta az_{ij} \sin Z_{ij} \sin A_{ij} \\ &+ \xi_i \cos Z_{ij}) \\ &+ \sin\phi_i \; (-\xi_i \sin Z_{ij} \cos A_{ij} - \eta_i \sin Z_{ij} \\ &\sin A_{ij} + \cos Z_{ij})] \;. \end{aligned} \qquad (2\text{-}12)$$

This completes the direct problem.

2.2.2 The Inverse Problem

The inverse problem may be stated as: given the coordinates $(X_i, Y_i, Z_i)_G$ of a point i and $(X_j, Y_j, Z_j)_G$ of a point j, compute the spatial distance r_{ij}, the direct astronomic azimuth A_{ij}, and the direct zenith angle Z_{ij}.

We begin by computing

$$(\vec{r}_{ij})_G = \begin{bmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{bmatrix}_G = \begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix}_G - \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_G . \tag{2-13}$$

Taking the inverse of (2-1) yields [Krakiwsky and Wells, 1971].

$$(\vec{r}_{ij})_{LG} = P_2 \, R_2 \, (\phi_i - 90°) \, R_3 \, (\lambda_i - 180°)(\vec{r}_{ij})_G . \tag{2-14}$$

Substituting (2-13) into (2-14) and expanding gives

$$(\Delta X_{ij})_{LG} = - (\Delta X_{ij})_G \sin \phi_i \cos \lambda_i - (\Delta Y_{ij})_G \sin \phi_i \sin \lambda_i + (\Delta Z_{ij})_G \cos \phi_i , \tag{2-15}$$

$$(\Delta Y_{ij})_{LG} = - (\Delta X_{ij})_G \sin \lambda_i + (\Delta Y_{ij})_G \cos \lambda_i , \tag{2-16}$$

and

$$(\Delta Z_{ij})_{LG} = (\Delta X_{ij})_G \cos \phi_i \cos \lambda_i + (\Delta Y_{ij})_G \cos \phi_i \sin \lambda_i + (\Delta Z_{ij})_G \sin \phi_i . \tag{2-17}$$

We must now rotate the Local Geodetic vector into the Local Astronomic system. This is accomplished by taking the inverse of (2-7) which is

$$(\vec{r}_{ij})_{LA} = R_1\ (-\eta_i)\ R_2\ (\xi_i)\ R_3\ (-\Delta az_{ij})(\vec{r}_{ij})_{LG}\ . \tag{2-18}$$

Making the same assumptions for small angles as mentioned in deriving (2-9) we can write

$$(\Delta X_{ij})_{LA} = (\Delta X_{ij})_{LG} - \Delta az_{ij}(\Delta Y_{ij})_{LG} - \xi_i(\Delta Z_{ij})_{LG}\ , \tag{2-19}$$

$$(\Delta Y_{ij})_{LA} = \Delta az_{ij}(\Delta X_{ij})_{LG} + (\Delta Y_{ij})_{LG} - \eta_i\ (\Delta Z_{ij})_{LG}\ , \tag{2-20}$$

and

$$(\Delta Z_{ij})_{LA} = \xi_i\ (\Delta X_{ij})_{LG} + \eta_i\ (\Delta Y_{ij})_{LG} + (\Delta Z_{ij})_{LG}\ , \tag{2-21}$$

where $(\Delta X_{ij})_{LG}$, $(\Delta Y_{ij})_{LG}$, $(\Delta Z_{ij})_{LG}$ come from equations (2-15), (2-16) and (2-17) respectively and Δaz_{ij}, η_i, ξ_i are expressed in radians.

Having obtained the Local Astronomic vector, the equations for the distance, azimuth, and zenith angle are given as

$$r_{ij} = [(\Delta X_{ij})^2{}_G + (\Delta Y_{ij})^2{}_G + (\Delta Z_{ij})^2{}_G]^{1/2}\ , \tag{2-22}$$

$$A_{ij} = \tan^{-1}\{\frac{(\Delta Y_{ij})_{LA}}{(\Delta X_{ij})_{LA}}\}\ , \tag{2-23}$$

and

$$z_{ij} = \cos^{-1}\{\frac{(\Delta Z_{ij})_{LA}}{r_{ij}}\}\ . \tag{2-24}$$

This completes the inverse problem.

2.3 Error Propagation

It should be noted that the error propagation given here does not include any propagation through the various rotation matrices. That is η_i, ξ_i, az_{ij}, ϕ_i and λ_i rotations are assumed errorless in equations (2-10) to (2-12) and (2-19) to (2-21). If the user is measuring azimuths with a standard deviation of less than 5 arc seconds then a more rigorous error propagation is advisable.

2.3.1 Error Propagation in the Direct Problem

Given the covariance matrix for the initial point i and the variances of the spatial distance, astronomic azimuth, and zenith distance, the covariance matrix is computed as follows for the second point j.

The covariance matrix of the initial point i and the observations is given by

$$C_1 = \left[\begin{array}{ccc|ccc} \sigma_{x_i}^2 & \sigma_{x_i y_i} & \sigma_{x_i y_i} & 0 & 0 & 0 \\ \sigma_{x_i y_i} & \sigma_{y_i}^2 & \sigma_{y_i z_i} & 0 & 0 & 0 \\ \sigma_{x_i z_i} & \sigma_{y_i z_i} & \sigma_{z_i}^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \sigma_{r_{ij}}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{A_{ij}}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{Z_{ij}}^2 \end{array}\right], \qquad (2\text{-}25)$$

in units of

$$\left[\begin{array}{ccc|ccc} m^2 & m^2 & m^2 & & & \\ m^2 & m^2 & m^2 & & & \\ m^2 & m^2 & m^2 & & & \\ \hline & & & m^2 & & \\ & & & & rad^2 & \\ & & & & & rad^2 \end{array}\right]$$

The variances of the astronomic azimuth and zenith angle are in radians squared. To convert the variance from arc sec^2 to rad^2 the variance is multiplied by $\frac{1}{\rho^2}$.

The output of the direct error propagation must include the initial covariance information for the point i. To do this we simply supplement equations 2-10, 2-11, and 2-12 with three more equations of the form

$$X_i = X_i \quad , \tag{2-26}$$

$$Y_i = Y_i \quad , \tag{2-27}$$

and

$$Z_i = Z_i \quad . \tag{2-28}$$

The Jacobian of transformation is(taken in the order of equations (2-30) to (2-32) and (2-10) to (2-12));

e.g. $B_1(1,1) = \dfrac{\partial x_i}{\partial x_i} = 1,\ B_1\ (4,4) = \dfrac{\partial x_j}{\partial r_{ij}}$:

$$B_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & B_1(4,4) & B_1(4,5) & B_1(4,6) \\ 0 & 1 & 0 & B_1(5,4) & B_1(5,5) & B_1(5,6) \\ 0 & 0 & 1 & B_1(6,4) & B_1(6,5) & B_1(6,6) \end{array}\right] , \tag{2,29}$$

where

$$\begin{aligned} B_1(4,4) = -\ [&\sin\phi_i \cos\lambda_i\ (\sin Z_{ij} \cos A_{ij} + \Delta az_{ij} \sin Z_{ij}\ \sin A_{ij} \\ &+ \xi_i \cos Z_{ij}) \\ &+ \sin\lambda_i (-\Delta az_{ij} \sin Z_{ij} \cos A_{ij} + \sin Z_{ij} \sin A_{ij} \\ &+ \eta_i \cos Z_{ij}) \\ &- \cos\phi_i \cos\lambda_i\ (-\xi_i \sin Z_{ij} \cos A_{ij} - \eta_i \sin Z_{ij} \sin A_{ij} \\ &+ \cos Z_{ij})], \end{aligned} \tag{2-30}$$

$$B_1(4,5) = - r_{ij} [\sin \phi_i \cos \lambda_i (- \sin Z_{ij} \sin A_{ij} + \Delta az_{ij} \sin Z_{ij} \cos A_{ij})$$

$$+ \sin \lambda_i (\Delta az_{ij} \sin Z_{ij} \sin A_{ij} + \sin Z_{ij} \cos A_{ij}) \tag{2-31}$$

$$- \cos \phi_i \cos \lambda_i (\xi_i \sin Z_{ij} \sin A_{ij} - \eta_i \sin Z_{ij} \cos A_{ij})],$$

$$B_1(4,6) = - r_{ij} [\sin \phi_i \cos \lambda_i (\cos Z_{ij} \cos A_{ij} + \Delta az_{ij} \cos Z_{ij} \sin A_{ij} - \xi_i \sin Z_{ij})$$

$$+ \sin \lambda_i (- \Delta az_{ij} \cos Z_{ij} \cos A_{ij} + \cos Z_{ij} \sin A_{ij} - \eta_i \sin Z_{ij}) \tag{2-32}$$

$$- \cos \phi_i \cos \lambda_i (- \xi_i \cos Z_{ij} \cos A_{ij} - \eta_i \cos Z_{ij} \sin A_{ij} - \sin Z_{ij})],$$

$$B_1(5,4) = - [\sin \phi_i \sin \lambda_i (\sin Z_{ij} \cos A_{ij} + \Delta az_{ij} \sin Z_{ij} \sin A_{ij} + \xi_i \cos Z_{ij})$$

$$- \cos \lambda_i (- \Delta az_{ij} \sin Z_{ij} \cos A_{ij} + \sin Z_{ij} \sin A_{ij} + + \eta_i \cos Z_{ij}) \tag{2-33}$$

$$- \cos \phi_i \sin \lambda_i (- \xi_i \sin Z_{ij} \cos A_{ij} - \eta_i \sin Z_{ij} \sin A_{ij} + \cos Z_{ij})],$$

$$
\begin{aligned}
B_1(5,5) = - r_{ij} \, [\sin\phi_i \sin\lambda_i \, & (-\sin Z_{ij} \sin A_{ij} + \Delta az_{ij} \sin Z_{ij} \\
& \cos A_{ij}) \\
& - \cos\lambda_i \, (\Delta az_{ij} \sin Z_{ij} \sin A_{ij} + \sin Z_{ij} \cos A_{ij}) \\
& - \cos\phi_i \sin\lambda_i \, (\xi_i \sin Z_{ij} \sin A_{ij} - \eta_i \sin Z_{ij} \\
& \cos A_{ij})] ,
\end{aligned} \tag{2-34}
$$

$$
\begin{aligned}
B_1(5,6) = - r_{ij} \, [\sin\phi_i \sin\lambda_i \, & (\cos Z_{ij} \cos A_{ij} + \Delta az_{ij} \cos Z_{ij} \\
& \sin A_{ij} - \xi_i \sin Z_{ij}) \\
& - \cos\lambda_i (-\Delta az_{ij} \cos Z_{ij} \cos A_{ij} + \cos Z_{ij} \sin A_{ij} \\
& - \eta_i \sin Z_{ij}) \\
& - \cos\phi_i \sin\lambda_i \, (- \xi_i \cos Z_{ij} \cos A_{ij} - \eta_i \cos Z_{ij} \\
& \sin A_{ij} - \sin Z_{ij})] ,
\end{aligned} \tag{2-35}
$$

$$
\begin{aligned}
B_1(6,4) = \cos\phi_i \, & (\sin Z_{ij} \cos A_{ij} + \Delta az_{ij} \sin Z_{ij} \sin A_{ij} + \xi_i \cos Z_{ij}) \\
& + \sin\phi_i \, (-\xi_i \sin Z_{ij} \cos A_{ij} - \eta_i \sin Z_{ij} \sin A_{ij} \\
& + \cos Z_{ij}) ,
\end{aligned} \tag{2-36}
$$

$$
\begin{aligned}
B_1(6,5) = r_{ij} \, [\cos\phi_i \, & (-\sin Z_{ij} \sin A_{ij} + \Delta az_{ij} \sin Z_{ij} \cos A_{ij}) \\
& + \sin\phi_i \, (\xi_i \sin Z_{ij} \sin A_{ij} - \eta_i \sin Z_{ij} \cos A_{ij})] ,
\end{aligned} \tag{2-37}
$$

and

$$B_1(6,6) = r_{ij}\,[\cos\phi_i\,(\cos Z_{ij}\cos A_{ij} + \Delta az_{ij}\cos Z_{ij}\sin A_{ij}$$

$$-\xi_i \sin Z_{ij}) \tag{2-38}$$

$$+\sin\phi_i\,(-\xi_i\cos Z_{ij}\cos A_{ij} - \eta_i\cos Z_{ij}$$

$$\sin A_{ij} - \sin Z_{ij})]\ .$$

Now with B_1^T equal to the transposed B_1 matrix, we may write, using the covariance law [Vanicek, 1974]

$$C_2 = B_1\, C_1\, B_1^T\ , \tag{2-39}$$

where C_2 is the full variance covariance matrix of the two points i and j and has the form

$$C_2 = \left[\begin{array}{ccc|ccc}
\sigma_{X_i}^2 & \sigma_{X_iY_i} & \sigma_{X_iZ_i} & \sigma_{X_iX_j} & \sigma_{X_iY_j} & \sigma_{X_iZ_j} \\
\sigma_{X_iY_i} & \sigma_{Y_i}^2 & \sigma_{Y_iZ_i} & \sigma_{Y_iX_j} & \sigma_{Y_iY_j} & \sigma_{Y_iZ_j} \\
\sigma_{X_iZ_i} & \sigma_{Y_iZ_i} & \sigma_{Z_i}^2 & \sigma_{Z_iX_j} & \sigma_{Z_iY_j} & \sigma_{Z_iZ_j} \\
\hline
\sigma_{X_iX_j} & \sigma_{Y_iX_j} & \sigma_{Z_iX_j} & \sigma_{X_j}^2 & \sigma_{X_jY_j} & \sigma_{X_jZ_j} \\
\sigma_{X_iY_j} & \sigma_{Y_iY_j} & \sigma_{Z_iY_j} & \sigma_{X_jY_j} & \sigma_{Y_j}^2 & \sigma_{Y_jZ_j} \\
\sigma_{X_iZ_j} & \sigma_{Y_iZ_j} & \sigma_{Z_iZ_j} & \sigma_{X_jZ_j} & \sigma_{Y_jZ_j} & \sigma_{Z_j}^2
\end{array}\right]\ . \tag{2-40}$$

All the elements of C_2 are in units of m^2.

This completes the error propagation in the direct case.

2.3.2 Error Propagation in the Inverse Problem

In the inverse problem we are given the covariance matrix of points i and j. This is in the form of the matrix C_2 described in section 2.3.1. This matrix is then used to derive the covariance matrix for the spatial distance, astronimic azimuth, and zenith angle. The procedure is as follows.

The Jacobian of transformation is (from equations (2-22), (2-23) and (2-24))

$$B_2 = \begin{bmatrix} B_2(1,1) & B_2(1,2) & B_2(1,3) & B_2(1,4) & B_2(1,5) & B_2(1,6) \\ B_2(2,1) & B_2(2,2) & B_2(2,3) & B_2(2,4) & B_2(2,5) & B_2(2,6) \\ B_2(3,1) & B_2(3,2) & B_2(3,3) & B_2(3,4) & B_2(3,5) & B_2(3,6) \end{bmatrix} . \tag{2-41}$$

The elements of B_2 are

$$B_2(1,1) = \frac{-(\Delta X_{ij})_G}{r_{ij}} ,$$

$$B_2(1,2) = \frac{-(\Delta Y_{ij})_G}{r_{ij}} ,$$

$$B_2(1,3) = \frac{-(\Delta Z_{ij})_G}{r_{ij}} ,$$

$$B_2(1,4) = - B_2(1,1) ,$$

$$B_2(1,5) = - B_2(1,2) ,$$

$$B_2(1,6) = - B_2(1,3) ,$$

$$B_2(2,1) = \frac{(\Delta X_{ij})_{LA}}{(\Delta X_{ij})^2_{LA} + (\Delta Y_{ij})^2_{LA}} [\Delta az_{ij} \sin \phi_i \cos \lambda_i + \sin \lambda_i +$$

$$\eta_i \cos \phi_i \cos \lambda_i - \tan A_{ij} (\sin \phi_i \cos \lambda_i - \Delta az_{ij} \sin \lambda_i$$

$$+ \xi_i \cos \phi_i \cos \lambda_i)] ,$$

$$B_2(2,2) = \frac{(\Delta X_{ij})_{LA}}{(\Delta X_{ij})^2_{LA} + (\Delta Y_{ij})^2_{LA}} [\Delta az_{ij} \sin \phi_i \sin \lambda_i - \cos \lambda_i +$$

$$\eta_i \cos \phi_i \sin \lambda_i - \tan A_{ij} (\sin \phi_i \sin \lambda_i$$

$$+ \Delta az_{ij} \cos \lambda_i + \xi_i \cos \phi_i \sin \lambda_i)] \quad ,$$

$$B_2(2,3) = \frac{(\Delta X_{ij})_{LA}}{(\Delta X_{ij})^2_{LA} + (\Delta Y_{ij})^2_{LA}} \quad [-\Delta az_{ij} \cos \phi_i + \eta_i \sin \phi_i$$

$$+ \tan A_{ij} (\cos \phi_i - \xi_i \sin \phi_i)] \quad ,$$

$$B_2(2,4) = - B_2(2,1) \ ,$$

$$B_2(2,5) = - B_2(2,2) \ ,$$

$$B_2(2,6) = - B_2(2,3) \ ,$$

$$B_2(3,1) = \frac{-1}{((\Delta X_{ij})^2_{LA} + (\Delta Y_{ij})^2_{LA})^{1/2}} \ [\ \xi_i \sin \phi_i \cos \lambda_i +$$

$$\eta_i \sin \lambda_i - \cos \phi_i \cos \lambda_i + \cos Z_{ij} \frac{(\Delta X_{ij})_G}{r_{ij}}] \quad ,$$

$$B_2(3,2) = \frac{-1}{((\Delta X_{ij})^2_{LA} + (\Delta Y_{ij})^2_{LA})^{1/2}} \ [\xi_i \sin \phi_i \sin \lambda_i - \eta_i \cos \lambda_i$$

$$- \cos \phi_i \sin \lambda_i + \cos Z_{ij} \frac{(\Delta Y_{ij})_G}{r_{ij}}] \ ,$$

$$B_2(3,3) = \frac{-1}{((\Delta X_{ij})^2_{LA} + (\Delta Y_{ij})^2_{LA})^{1/2}} [-\xi_i \cos \phi_i - \sin \phi_i$$

$$+ \cos Z_{ij} \frac{(\Delta Z_{ij})_G}{r_{ij}}] \quad ,$$

$B_2\ (3,4) = -\ B_2\ (3,1)$,

$B_2\ (3,5) = -\ B_2(3,2)$,

$B_2\ (3,6) = -\ B_2(3,3)$.

With B_2^T equal to the transpose of B_2 it follows that

$$C_3 = B_2\ C_2\ B_2^{\ T} \qquad (2\text{-}42)$$

where C_3 has the form

$$C_3 = \begin{bmatrix} \sigma_{r_{ij}}^{\ 2} & \sigma_{r_{ij}A_{ij}} & \sigma_{r_{ij}Z_{ij}} \\ \sigma_{r_{ij}A_{ij}} & \sigma_{A_{ij}}^{\ 2} & \sigma_{A_{ij}Z_{ij}} \\ \sigma_{r_{ij}Z_{ij}} & \sigma_{A_{ij}Z_{ij}} & \sigma_{Z_{ij}}^{\ 2} \end{bmatrix} \qquad (2\text{-}43)$$

with the units

$$\begin{bmatrix} m^2 & m\cdot rad & m\ .\ rad \\ m.rad & rad^2 & rad^2 \\ m.rad & rad^2 & rad^2 \end{bmatrix} .$$

To convert the rad^2 to $arcsec^2$ the term is multiplied by ρ^2. To convert the off diagonal m.rad to arcsec, the terms are multiplied by ρ.

2.4 New Brunswick Numerical Example

2.4.1 Direct Problem

The following information is given for the solution of the direct problem and its associated error propagation.

The coordinates of point 1 are

$$(\phi_1)_G = 47° \ 03' \ 24''.644 \quad ,$$

$$(\lambda_1)_G = 65° \ 29' \ 3''.453 \ W \quad ,$$

and

$$(h_1)_G = 100.0 \text{ metres} \quad .$$

The components of the deflection of the vertical of point 1 are

$$\xi_1 = 4''.0 \quad ,$$

and

$$\eta_1 = 6''.0 \quad .$$

The observations are

$$r_{12} = 2\ 500.0 \text{ m} \quad ,$$

$$A_{12} = 45° \ 00' \ 0''.00 \quad ,$$

and

$$Z_{12} = 87° \ 00' \ 0''.00 \quad .$$

Equation (2-8) gives

$$\Delta az_{12} = 3.1615 \times 10^{-5} \text{ rad} \quad .$$

Also given is the associated covariance matrix.

$$C_1 = \left[\begin{array}{ccc|ccc} 1.0000 \times 10^{-4} & -8.0000 \times 10^{-8} & 0 & 0 & 0 & 0 \\ -8.0000 \times 10^{-8} & 1.0000 \times 10^{-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 7.840 \times 10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & 225.00 \end{array}\right]$$

in units of

$$\left[\begin{array}{ccc|ccc} \text{arcsec}^2 & \text{arcsec}^2 & & & & \\ \text{arcsec}^2 & \text{arcsec}^2 & & & & \\ & & m^2 & & & \\ \hline & & & m^2 & & \\ & & & & \text{arcsec}^2 & \\ & & & & & \text{arcsec}^2 \end{array}\right] ,$$

The curvilinear coordinates $(\phi_1)_G$, $(\lambda_1)_G$, and $(h_1)_G$ and their associated covariance matrix (top left (3,3) quadrant of C_1) must be converted to $(X_1)_G$, $(Y_1)_G$, $(Z_1)_G$ with its associated covariance matrix and the variances of the azimuth and zenith angle must be converted from arcsec^2 to rad^2.

The coordinates become (using equations from, for example, Krakiwsky et al. [1977])

$$(X_1)_G = 1\ 806\ 355.970 \text{ m} ,$$

$$(Y_1)_G = -3\ 960\ 808.539 \text{ m} ,$$

and

$$(Z_1)_G = 4\ 645\ 941.572 \text{ m} .$$

The associated covariance matrix becomes, using equations from, for example, Krakiwsky et al. [1977] and multiplying the variances of the astronomic azimuth and zenith angle by $\frac{1}{\rho^2}$,

$$C_1 = \left[\begin{array}{ccc|ccc} .365 & -.703 & .808 & 0 & 0 & 0 \\ -.703 & 1.587 & -1.772 & 0 & 0 & 0 \\ .808 & -1.772 & 2.188 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 7.84 \times 10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.876 \times 10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.288 \times 10^{-9} \end{array}\right],$$

in units of

$$\left[\begin{array}{ccc|ccc} m^2 & m^2 & m^2 & & & \\ m^2 & m^2 & m^2 & & & \\ m^2 & m^2 & m^2 & & & \\ \hline & & & m^2 & & \\ & & & & rad^2 & \\ & & & & & rad^2 \end{array}\right]$$

The coordinates $(X_2)_G$, $(Y_2)_G$, $(Z_2)_G$ using equations (2-10), (2-11) and (2-12) are

$$(X_2)_G = 1\ 807\ 462.838 \text{ m},$$

$$(Y_2)_G = -3\ 958\ 981.272 \text{ m},$$

and

$$(Z_2)_G = 4\ 647\ 240.008 \text{ m}.$$

Using formulae from, for example, Krakiwsky et al, [1977], the Cartesian coordinates are converted to curvilinear coordinates yielding

$(\phi_2)_G = 47° \ 4' \ 21''.801$,

$(\lambda_2)_G = 65° \ 27' \ 39''.788$ W ,

and

$(h_2)_G = 231.243$ m .

Beginning the direct problem error propagation, the Jacobian of transformation, B_1, (equation (2-29)) is

$$B_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & .44275 & 2.1424 \times 10^3 & -6.4974 \times 10^2 \\ 0 & 1 & 0 & .73091 & -4.4319 \times 10^2 & 1.6475 \times 10^3 \\ 0 & 0 & 1 & .51937 & -1.2027 \times 10^3 & -1.7646 \times 10^3 \end{array}\right]$$

Using equation (2-39) the resultant covariance matrix C_2 for points 1 and 2 is

$$C_2 = \left[\begin{array}{ccc|ccc} .365 & -.703 & .808 & .365 & -.703 & .808 \\ -.703 & 1.587 & -1.772 & -.703 & 1.587 & -1.772 \\ .808 & -1.772 & 2.188 & .808 & -1.772 & 2.188 \\ \hline .365 & -.703 & .808 & .370 & -.709 & .813 \\ -.703 & 1.587 & -1.772 & -.709 & 1.602 & -1.787 \\ .808 & -1.772 & 2.188 & .813 & -1.787 & 2.205 \end{array}\right] .$$

where the units of all the elements are m^2 .

The lower right hand 3 x 3 sub matrix is converted to a covariance matrix of the curvilinear coordinates (using formulae from, for example, Krakiwsky et al. [1977]) yielding

$$C_{\phi_2,\lambda_2,h_2} = \begin{bmatrix} 1.024 \times 10^{-4} & -2.196 \times 10^{-6} & -7.431 \times 10^{-5} \\ -2.196 \times 10^{-6} & 1.052 \times 10^{-4} & -1.093 \times 10^{-4} \\ -7.431 \times 10^{-5} & -1.093 \times 10^{-4} & 4.033 \end{bmatrix},$$

in units of

$$\begin{bmatrix} \text{arcsec}^2 & \text{arcsec}^2 & \text{arcsec.m} \\ \text{arcsec}^2 & \text{arcsec}^2 & \text{arcsec.m} \\ \text{arcsec.m} & \text{arcsec.m} & \text{m}^2 \end{bmatrix}.$$

2.4.2 Inverse Problem

In the inverse problem the coordinates of points 1 and 2 and their covariance matrix C_2 are known (in this example as the results of the direct problem). Using equations (2-22), (2-23), and (2-24) the distance, astronomic azimuth and zenith distance are

$$r_{12} = 2\,500.000 \text{ metres},$$

$$A_{12} = 45°00'\,0''.00\ ,$$

and

$$Z_{12} = 87°\,0'\,0''.00\ .$$

The inverse problem Jacobian of transformation matrix B_2 (equation (2-41)) is

$$B_2 = \begin{bmatrix} -.44275 & -.73091 & -.51937 & .44275 & .73091 & .51937 \\ -3.4373\times10^{-4} & 7.1106\times10^{-5} & 1.9295\times10^{-4} & 3.4373\times10^{-4} & -7.1106\times10^{-5} & -1.9295\times10^{-} \\ 1.0396\times10^{-4} & -2.6359\times10^{-4} & 2.8233\times10^{-4} & -1.0396\times10^{-4} & 2.6359\times10^{-4} & -2.8233\times10^{-} \end{bmatrix}$$

Using equation (2-42) the resultant covariance matrix C_3 for the distance, astronomic azimuth and zenith distance is

$$C_3 = \begin{bmatrix} 7.840 \times 10^{-4} & 8.82 \times 10^{-11} & 7.239 \times 10^{-10} \\ 8.82 \times 10^{-11} & 25.00 & -2.712 \times 10^{-7} \\ 7.239 \times 10^{-10} & -2.712 \times 10^{-7} & 225.00 \end{bmatrix}$$

in units of

$$\begin{bmatrix} m^2 & m.arcsec & m.arcsec \\ m.arcsec & arcsec^2 & arcsec^2 \\ m.arcsec & arcsec^2 & arcsec^2 \end{bmatrix}$$

where m.rad have been converted to arc.sec by multiplication by ρ and rad^2 have been converted to $arcsec^2$ by multiplication by ρ^2. Note that the off diagonal terms are negligible because of the nature of our example (see page 25) but the terms could be significant.

2.5 Prince Edward Island Numerical Example

2.5.1 Direct Problem

The following information is given for the solution of the direct problem and its associated error propagation.

The coordinates of point 1 are

$$(\phi_1)_G = 46° \; 42' \; 28''.147 \; ,$$

$$(\lambda_1)_G = 64° \; 29' \; 34''.014 \; W,$$

and

$$(h_1)_G = 100.0 \; m \quad .$$

The components of the deflection of the vertical of point 1 are

$$\xi_1 = 4''.0 \quad ,$$

and

$$\eta_1 = 6''.0 \quad .$$

The observations are

$$r_{12} = 2\,500.00 \; m$$

$$A_{12} = 135° \; 00' \; 0''.00 \quad ,$$

and

$$z_{12} = 87° \; 00' \; 0''.00 \quad .$$

Equation (2-8) gives

$$\Delta az_{12} = 2.9080 \times 10^{-5} \; rad \quad .$$

Also given is the associated covariance matrix

$$C_1 = \left[\begin{array}{ccc|ccc} 1.0000 \times 10^{-4} & -8.0000 \times 10^{-8} & 0 & 0 & 0 & 0 \\ -8.0000 \times 10^{-8} & 1.0000 \times 10^{-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 7.840 \times 10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & 225.00 \end{array}\right]$$

in units of

$$\left[\begin{array}{ccc|ccc}
\text{arcsec}^2 & \text{arcsec}^2 & & & & \\
\text{arcsec}^2 & \text{arcsec}^2 & & & & \\
 & & m^2 & & & \\
\hline
 & & & m^2 & & \\
 & & & & \text{arcsec}^2 & \\
 & & & & & \text{arcsec}^2
\end{array}\right].$$

The curvilinear coordinates $(\phi_1)_G$, $(\lambda_1)_G$ and $(h_1)_G$ and their associated covariance matrix (top left (3,3) quadrant of C_1) must be converted to $(X_1)_G$, $(Y_1)_G$, $(Z_1)_G$ with its associated covariance matrix and the variances of the azimuth and zenith angle must be converted from arc sec ^{2} to rad^2.

The coordinates become (using equations from, for example, Krakiwsky et al. [1977]),

$$(X_1)_G = 1\ 886\ 820.969 \text{ m} \quad ,$$

$$(Y_1)_G = -3\ 954\ 520.208 \text{ m} \quad ,$$

and

$$(Z_1)_G = 4\ 619\ 420.996 \text{ m}.$$

The associated covariance matrix becomes, using equations from, for example, Krakiwsky et al. [1977] and multiplying the variances on the astronomic azimuth and zenith angle by $\frac{1}{\rho^2}$,

$$
C_1 = \left[\begin{array}{ccc|ccc}
.395 & -.733 & .839 & 0 & 0 & 0 \\
-.733 & 1.581 & -1.759 & 0 & 0 & 0 \\
.839 & -1.759 & 2.164 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 7.84 \times 10^{-4} & 0 & 0 \\
0 & 0 & 0 & 0 & 5.876 \times 10^{-10} & 0 \\
0 & 0 & 0 & 0 & 0 & 5.288 \times 10^{-5}
\end{array}\right]
$$

in units of

$$
\left[\begin{array}{ccc|ccc}
m^2 & m^2 & m^2 & & & \\
m^2 & m^2 & m^2 & & & \\
m^2 & m^2 & m^2 & & & \\
\hline
 & & & m^2 & & \\
 & & & & rad^2 & \\
 & & & & & rad^2
\end{array}\right].
$$

The coordinates $(X_2)_G$, $(Y_2)_G$, $(Z_2)_G$, using equations (2-10), (2-11) and (2-12), are

$$(X_2)_G = 1\ 889\ 006.235 \text{ m} \quad ,$$

$$(Y_2)_G = -3\ 955\ 000.606 \text{ m} \quad ,$$

and

$$(Z_2)_G = 4\ 618\ 305.724 \text{ m}.$$

Using formulae, from for example Krakiwsky et al. [1977], the above Cartesian coordinates are converted to curvilinear coordinates yielding

$$(\phi_2)_G = 46° \; 41' \; 30''\!.973 \quad ,$$

$$(\lambda_2)_G = 64° \; 28' \; 10''\!.933 \text{ W},$$

and

$$(h_2)_G = 231.311 \text{ m} \quad .$$

Beginning the direct problem error propagation, the Jacobian of transformation, B_1, (equation (2-29)) is

$$B_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & .87411 & -1.0399 \times 10^3 & -6.2476 \times 10^2 \\ 0 & 1 & 0 & -.19216 & -1.9200 \times 10^3 & 1.5241 \times 10^3 \\ 0 & 0 & 1 & -.44611 & -1.2105 \times 10^3 & -1.8806 \times 10^3 \end{array}\right]$$

Using the equation (2-39) the resultant covariance matrix C_2 for points 1 and 2 is

$$C_2 = \left[\begin{array}{ccc|ccc} .395 & -.733 & .839 & .395 & -.733 & .839 \\ -.733 & 1.582 & -1.759 & -.733 & 1.582 & -1.759 \\ .839 & -1.759 & 2.164 & .839 & -1.759 & 2.164 \\ \hline .395 & -.733 & .839 & .398 & -.737 & .846 \\ -.733 & 1.582 & -1.759 & -.737 & 1.596 & -1.773 \\ .839 & -1.759 & 2.164 & .846 & -1.773 & 2.184 \end{array}\right] ,$$

where the units of all the elements are m^2.

The lower right hand 3 x 3 sub matrix is converted to a covariance matrix of the curvilinear coordinates (using formulae from, for example, Krakiwsky et al. [1977]) yielding

$$C_{\phi_2,\lambda_2,h_2} = \begin{bmatrix} 1.024 \times 10^{-4} & 2.067 \times 10^{-6} & 7.359 \times 10^{-5} \\ 2.067 \times 10^{-6} & 1.050 \times 10^{-4} & -1.085 \times 10^{-4} \\ 7.359 \times 10^{-5} & -1.085 \times 10^{-4} & 4.030 \end{bmatrix},$$

in units of

$$\begin{bmatrix} \text{arcsec}^2 & \text{arcsec}^2 & \text{arcsec.m} \\ \text{arcsec}^2 & \text{arcsec}^2 & \text{arcsec.m} \\ \text{arcsec.m} & \text{arcsec.m} & \text{m}^2 \end{bmatrix}.$$

2.5.2 Inverse Problem

In the inverse problem the coordinates of points 1 and 2 and their covariance matrix C_2 are known (in this example the results of the direct problem). Using equations (2-22), (2-23), and (2-24) the distance, astronomic azimuth and zenith distance are

$$r_{12} = 2\ 500.000 \text{ m} ,$$

$$A_{12} = 135°00'\ 0''.00 ,$$

and

$$Z_{12} = 87°00'\ 0''.00 .$$

The inverse problem Jacobian of transformation matrix B_2 (equation (2-41)) is

$$B_2 = \begin{bmatrix} -.87411 & .19216 & .44611 & .87411 & -.19216 & -.44611 \\ 1.6684\text{x}10^{-4} & 3.0804\text{x}10^{-4} & 1.9421\text{x}10^{-4} & -1.6684\text{x}10^{-4} & -3.0804\text{x}10^{-4} & -1.9421\text{x}10^{-4} \\ 9.9961\text{x}10^{-5} & -2.4385\text{x}10^{-4} & 3.0090\text{x}10^{-4} & -9.9961\text{x}10^{-5} & 2.4385\text{x}10^{-4} & -3.0090\text{x}10^{-4} \end{bmatrix}$$

Using equation (2-42) the resultant covariance matrix C_3 for the distance, astronomic azimuth, and zenith distance is

$$C_3 = \begin{bmatrix} 7.840 \times 10^{-4} & 6.79 \times 10^{-11} & -2.751 \times 10^{-9} \\ 6.79 \times 10^{-11} & 25.00 & -5.026 \times 10^{-8} \\ -2.751 \times 10^{-9} & -5.026 \times 10^{-8} & 225.00 \end{bmatrix},$$

in units of

$$\begin{bmatrix} m^2 & \text{arcsec.m} & \text{arcsec.m} \\ \text{arcsec.m} & \text{arcsec}^2 & \text{arcsec}^2 \\ \text{arcsec.m} & \text{arcsec}^2 & \text{arcsec}^2 \end{bmatrix}.$$

where m.rad have been converted to arcsec.m by multiplication by ρ and rad^2 have been converted to acrsec^2 by multiplication by ρ^2. Note that the off diagonal terms are negligible because of the nature of our example (see page 31) but the terms could be significant.

2.6 Nova Scotia Numerical Example

2.6.1 Direct Problem

The following information is given for the solution of the direct problem and its associated error propagation.

The coordinates of point 1 are

$$(\phi_1)_G = 44° \ 39' \ 3''.123$$

$$(\lambda_1)_G = 63° 00' \ 0''.000 \ W$$

and

$$(h_1)_G = 100.0 \text{ metres} .$$

The components of the deflection of the vertical of point 1 are

$$\xi_1 = 4''.0 \quad ,$$

and

$$\eta_1 = 6''.0 \quad .$$

The observations are

$$r_{12} = 2\ 500.0 \text{ metres}$$

$$A_{12} = 225° \ 00' \ 0''.00 \quad ,$$

and

$$Z_{12} = 87° 00' \ 0''.00 \quad .$$

Equation (2-8) gives

$$\Delta az_{12} = 2.8377 \times 10^{-5} \text{ rad} \quad .$$

Also given is the associated covariance matrix C_1

$$C_1 = \left[\begin{array}{ccc|ccc} 1.0000\text{x}10^{-4} & -8.0000\text{x}10^{-8} & 0 & 0 & 0 & 0 \\ -8.0000\text{x}10^{-8} & 1.0000\text{x}10^{-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 7.840\text{x}10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & 225.00 \end{array}\right]$$

in units of

$$\left[\begin{array}{ccc|ccc} \text{arcsec}^2 & \text{arcsec}^2 & & & & \\ \text{arcsec}^2 & \text{arcsec}^2 & & & & \\ & & \text{m}^2 & & & \\ \hline & & & \text{m}^2 & & \\ & & & & \text{arcsec}^2 & \\ & & & & & \text{arcsec}^2 \end{array}\right]$$

The curvilinear coordinates $(\phi_1)_G$, $(\lambda_1)_G$, and $(h_1)_G$ and their associated covariance matrix (top left (3,3) quadrant of C_1) must be converted from arcsec^2 to rad^2.

The coordinates become (using equations from, for example, Krakiwsky et al. [1977]),

$$(X_1)_G = 2\ 063\ 453.133 \text{ m} \quad ,$$

$$(Y_1)_G = -4\ 049\ 754.797 \text{ m} \quad ,$$

and

$$(Z_1)_G = 4\ 459\ 697.671 \text{ m} \quad .$$

The associated covariance matrix becomes, using equations from, for example, Krakiwsky et al. [1977], and multiplying the variances on the astronomic azimuth and zenith angle by $\frac{1}{\rho^2}$,

$$C_1 = \left[\begin{array}{ccc|ccc} .465 & -.818 & .886 & 0 & 0 & 0 \\ -.818 & 1.654 & -1.739 & 0 & 0 & 0 \\ .886 & -1.739 & 2.024 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 7.84\text{x}10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.876\text{x}10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.288\text{x}10^{-9} \end{array}\right]$$

in units of

$$\left[\begin{array}{ccc|ccc} m^2 & m^2 & m^2 & & & \\ m^2 & m^2 & m^2 & & & \\ m^2 & m^2 & m^2 & & & \\ \hline & & & m^2 & & \\ & & & & rad^2 & \\ & & & & & rad^2 \end{array}\right]$$

The coordinates $(X_2)_G$, $(Y_2)_G$, $(Z_2)_G$ using equations (2-10) (2-11) and (2-12) are

$$(X_2)_G = 2\ 062\ 485.795 \text{ m} \quad ,$$

$$(Y_2)_G = -4\ 051\ 744.675 \text{ m} \quad ,$$

and

$$(Z_2)_G = 4\ 458\ \ 533.780 \text{ m} \ .$$

Using formulae from, for example, Krakiwsky et al. [1977] the above Cartesian coordinates are converted to curvilinear coordinates yielding

$$(\phi_2)_G = 44°\ 38'\ 5''.925 \quad ,$$

$$(\lambda_2)_G = 63°01'\ 20''.088 \ \text{ W} \quad ,$$

and

$$(h_2)_G = 231.414 \text{ m} \quad .$$

Beginning the direct problem error propagation, the Jacobian of transformation, B_1, (equation (2-29)) is

$$B_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -.38694 & -2.1362\text{x}10^3 & -8.5928\text{x}10^2 \\ 0 & 1 & 0 & -.79595 & 3.0392\text{x}10^2 & 1.4825\text{x}10^3 \\ 0 & 0 & 1 & -.46556 & 1.2558\text{x}10^3 & -1.8204\text{x}10^3 \end{array}\right].$$

Using equation (2-43) the resultant covariance matrix C_2 for points 1 and 2 is

$$C_2 = \left[\begin{array}{ccc|ccc} .465 & -.818 & .886 & .465 & -.818 & .886 \\ -.818 & 1.654 & -1.739 & -.818 & 1.654 & -1.739 \\ .886 & -1.739 & 2.024 & .886 & -1.739 & 2.024 \\ \hline .465 & -.818 & .886 & .472 & -.825 & .893 \\ -.818 & 1.654 & -1.739 & -.825 & 1.667 & -1.753 \\ .886 & -1.739 & 2.024 & .893 & -1.753 & 2.042 \end{array}\right],$$

where the units of all the elements are m^2.

The lower right hand 3 x 3 sub matrix is converted to a covariance matrix of the curvilinear coordinates (using formulae from, for example, Krakiwsky et al. [1977]) yielding

$$C_{\phi_2,\lambda_2,h_2} = \begin{bmatrix} 1.024 \times 10^{-4} & -2.148 \times 10^{-6} & 7.364 \times 10^{-5} \\ -2.148 \times 10^{-6} & 1.046 \times 10^{-4} & 1.035 \times 10^{-4} \\ 7.364 \times 10^{-5} & 1.035 \times 10^{-4} & 4.033 \end{bmatrix},$$

in units of

$$\begin{bmatrix} \text{arcsec}^2 & \text{arcsec}^2 & \text{arcsec.m} \\ \text{arcsec}^2 & \text{arcsec}^2 & \text{arcsec.m} \\ \text{arcsec.m} & \text{arcsec.m} & \text{m}^2 \end{bmatrix} .$$

2.6.2 Inverse Problem

In the inverse problem the coordinates of points 1 and 2 and their covariance matrix C_2 are known (in this example the results of the direct problem). Using equations (2-22), (2-23), and (2-24) the distance, astronomic azimuth and zenith distance are

$$r_{12} = 2\ 500.00 \text{ m} \quad ,$$

$$A_{12} = 225°\,00'\,0''.00 \quad ,$$

and

$$Z_{12} = 87°\,00'\,0''.00 \quad .$$

The inverse problem Jacobian of transformation matrix B_2 (equation (2-41)) is

$$B_2 = \begin{bmatrix} .38694 & .79595 & .46556 & -.38694 & -.79595 & -.46556 \\ 3.4273\text{x}10^{-4} & -4.8761\text{x}10^{-5} & -2.0149\text{x}10^{-4} & -3.4273\text{x}10^{-4} & 4.8761\text{x}10^{-5} & 2.0149\text{x}10^{-4} \\ 1.3748\text{x}10^{-4} & -2.3720\text{x}10^{-4} & 2.9126\text{x}10^{-4} & -1.3748\text{x}10^{-4} & 2.3720\text{x}10^{-4} & -2.9126\text{x}10^{-4} \end{bmatrix}$$

Using equation (2-46) the resultant covariance matrix C_3 for the distance, astronomic azimuth and zenith distance is

$$C_3 = \begin{bmatrix} 7.840 \times 10^{-4} & 5.56 \times 10^{-11} & -4.194 \times 10^{-10} \\ 5.56 \times 10^{-11} & 25.00 & 2.440 \times 10^{-7} \\ -4.194 \times 10^{-10} & 2.440 \times 10^{-7} & 225.00 \end{bmatrix},$$

in units of

$$\begin{bmatrix} m^2 & m.\text{arcsec} & m.\text{arcsec} \\ m.\text{arcsec} & \text{arcsec}^2 & \text{arcsec}^2 \\ m.\text{arcsec} & \text{arcsec}^2 & \text{arcsec}^2 \end{bmatrix}$$

where m.rad have been converted to m.arcsec by multiplication by ρ and rad^2 have been converted to $arcsec^2$ by multiplication by ρ^2. Note that the off diagonal terms are negligible because of the nature of our numerical example (see page 36) but the terms could be significant.

3. Computations on the Ellipsoid

In this chapter equations are given for the reduction of observed directions, angles, azimuths, distances and zenith distances, from the terrain to the reference ellipsoid (and conversely), after which equations are given for computing the direct and inverse problems on the ellipsoid.

3.1 Notation

The notation used in this chapter is listed here for convenience.

a, b ≡ semi-major and semi-minor axes respectively of the Clarke 1866 reference ellipsoid,

$a = 6\ 378\ 206.4$ m

$b = 6\ 356\ 583.8$ m

e ≡ first eccentricity of the reference ellipsoid ,

$$e^2 = (a^2 - b^2) / a^2 \tag{3-1}$$

ϕ_i, λ_i ≡ ellipsoidal coordinates of a point i

ϕ_m, λ_m ≡ mean ellipsoidal coordinates of two points i and j

$$\phi_m = \frac{\phi_i + \phi_j}{2} \tag{3-2}$$

and

$$\lambda_m = \frac{\lambda_i + \lambda_j}{2} \tag{3-3}$$

r_{ij} ≡ observed spatial distance between points i and j, corrected for refraction and instrumental corrections

S_{ij} ≡ distance between points i and j on the surface of the reference ellipsoid

d_{ij} ≡ observed horizontal direction on the terrain from terrain point i to point j

α_{ij} ≡ geodetic azimuth on the ellipsoid from point i to point j

A_{ij} ≡ terrain astronomic azimuth from point i to point j

R_α ≡ Euler radius of curvature in the azimuth α_{ij}

$$R_{\alpha_{ij}} = \frac{M_i N_i}{M_i \sin^2 \alpha_{ij} + N_i \cos^2 \alpha_{ij}} \tag{3-4}$$

M_i ≡ radius of curvature of the ellipsoid in the meridian plane at point i

$$M_i = a(1-e^2)/(1 - e^2 \sin^2 \phi_i)^{3/2} \tag{3-5}$$

N_i ≡ radius of curvature of the ellipsoid in the prime vertical plane at point i

$$N_i = a/(1 - e^2 \sin^2 \phi_i)^{1/2} \tag{3-6}$$

M_m ≡ mean meridian radius of curvature, $M_m = (M_i + M_j)/2$ (3-7)

N_m ≡ mean prime vertical radius of curvature; $N_m = (N_i + N_j)/2$ (3-8)

Z_{ij} ≡ observed zenith distance on the terrain from terrain point i to point j, corrected for refraction and instrumental corrections

h_i ≡ height of terrain point i above the reference ellipsoid measured along the ellipsoid normal

ξ_i ≡ deflection of the vertical component in the meridian plane at point i

η_i ≡ deflection of the vertical component in the prime vertical plane at point i

β_{ijk} ≡ terrain horizontal angle at point j from point i to point k

z_{ij} ≡ zenith distance corrected for terrain deflection of the vertical

$(\ )^e$ ≡ denotes an ellipsoidal quantity

$(\ )^a$ ≡ an approximate quantity

H_j ≡ orthometric height of a terrain point j (height of point above the geoid)

N^*_j ≡ geoidal height of point j (geoid reference ellipsoid separation)

3.2 Reduction Formulae

3.2.1 Introduction

Upon examination of the various reduction formulae, it will be seen that the corrective terms are sometimes functions of the position to be solved for or the quantity being corrected. If the position of the point being solved for is required, then the coordinates may be computed using the formulae,

$$\phi_j^a = \phi_i + \frac{r_{ij} \cos (A_{ij})}{M_i} \tag{3-9}$$

$$\phi_m^a = \frac{\phi_i + \phi_j^a}{2}$$

$$\lambda_j^a = \lambda_i + \frac{r_{ij} \sin (A_{ij})}{N_m \cos \phi_m} \tag{3-10}$$

Deflection components for horizontal control points will be given along with the published redefined coordinates. The means for computing ξ_i and η_i, for any new points in the Maritimes, will be available through the Surveys and Mapping Division of L.R.I.S. Heights of points above the ellipsoid must be as accurate as possible and can be obtained by adding the orthometric height to the geoidal height,

$$h_j = H_j + N_j^*. \tag{3-11}$$

Although we are now working in the two-dimensional domain of the ellipsoidal surface, the heights of the points are needed for the reduction of various observed quantities to the ellipsoidal surface. The height H_j is the orthometric height. The geoid height, N_j^*, for known control points will be given along with the published redefined coordinates. As with ξ_j and η_j methods for computing N_j^* for any Maritime points will be available.

Having reduced the observations, the direct and inverse position computations may be done on the ellipsoid surface using the Puissant's formulae or the Gauss Mid Latitude formulae (or any of many other equivalent formulae). Upon completion of the direct problem new coordinates for point 2 are available. These should now be used in the reduction formulae to obtain more precise corrections. This is most essential when the ellipsoidal height difference of the two points is very large. The error propagation through the reduction formula are formulated assuming that the estimates of the second point are with 1" of their final value or approximately 30 metres. The coordinates obtained from the solution of the direct problem should therefore be tested against the estimates used in the reduction formulae.

3.2.2 Reduction of Horizontal Directions

A horizontal direction is reduced from the terrain to the ellipsoid by [Krakiwsky, and Thomson, 1974].

$$d^e_{ij} = d_{ij} + (\frac{h_j}{M_m} e^2 \sin\alpha_{ij} \cos\alpha_{ij} \cos^2\phi_j)$$

$$- (\frac{e^2 S^2_{ij} \cos^2\phi_m \sin 2\alpha_{ij}}{12\, N^2_m})$$

$$- ([\xi_i \sin\alpha_{ij} - \eta_i \cos\alpha_{ij}] \cot z^e_{ij}) \ , \quad (3\text{-}12)$$

where,

$$N_m = \frac{N_i + N_j^a}{2} \ ,$$

$$M_m = \frac{M_i + M_j^a}{2} \ ,$$

$$\phi_m = \frac{\phi_i + \phi_j^a}{2} \ ,$$

and $N_j{}^a$ and $M_j{}^a$ are evaluated at $\phi_j{}^a$ and α_{ij}=A_{ij} in a first approximation.

3.2.3 Reduction of Horizontal Angles

Since a horizontal angle is actually composed of two directions, we reduce it from the terrain to the ellipsoid by applying equation (3-12) twice. This yields

$$\beta^e_{jik} = \beta_{jik} + (\frac{h_j}{M_{m_{ij}}} e^2 \sin\alpha_{ij} \cos\alpha_{ij} \cos^2\phi_j)$$

$$+ (\frac{h_k}{M_{m_{ik}}} e^2 \sin\alpha_{ik} \cos\alpha_{ik} \cos^2\phi_k)$$

$$+ (\frac{e^2 S^2_{ij} \cos^2\phi_{m_{ij}} \sin 2\alpha_{ij}}{12 N^2_{m_{ij}}})$$

$$- (\frac{e^2 S^2_{ik} \cos^2\phi_{m_{ik}} \sin 2\alpha_{ik}}{12 N^2_{m_{ik}}})$$

$$+ ([\xi_i \sin \alpha_{ij} - \eta_i \cos \alpha_{ij}] \cot z_{ij})$$

$$- ([\xi_i \sin \alpha_{ik} - \eta_i \cos \alpha_{ik}] \cot z_{ik}) \ . \qquad (3\text{-}13)$$

and all quantities are the same as those in section 3.2.2.

3.2.4 Reduction of Zenith Distances

A terrain zenith distance is reduced from the terrain to the ellipsoid [Krakiwsky and Thomson, 1974]

$$z_{ij} = Z_{ij} + (\xi_i \cos \alpha_{ij} + \eta_i \sin \alpha_{ij}) . \qquad (3\text{-}14)$$

3.2.5 Reduction of Astronomic Azimuths

The observed astronomic azimuth is best reduced from the terrain to the ellipsoid in a series of steps as follows. First

$$\alpha'_{ij} = A_{ij} - \eta_i \tan \phi_i \ , \qquad (3\text{-}15)$$

then

$$\alpha''_{ij} = \alpha'_{ij} - ((\xi_i \sin \alpha'_{ij} - \eta_i \cos \alpha'_{ij}) \cot z_{ij}) \ , \qquad (3\text{-}16)$$

where Z_{ij} has been corrected as described in section 3.2.4. The next reduction is

$$\alpha'''_{ij} = \alpha''_{ij} + (\frac{h_j}{M_m} e^2 \sin \alpha''_{ij} \cos \alpha''_{ij} \cos^2 \phi_j) , \qquad (3\text{-}17)$$

and finally, using S_{ij} as computed by equation (3-20)

$$\alpha_{ij} = \alpha'''_{ij} - (\frac{e^2 S_{ij}^2 \cos^2 \phi_m \sin 2 \alpha'''_{ij}}{12 N_m^2}) \ , \qquad (3\text{-}18)$$

where α_{ij} is the desired geodetic azimuth.

3.2.6 Reduction of Spatial Distances

A terrain spatial distance, r_{ij}, between two points i and j (see Figure 1 below) is reduced from the terrain to the ellipsoid as follows [Krakiwsky and Thomson, 1974].

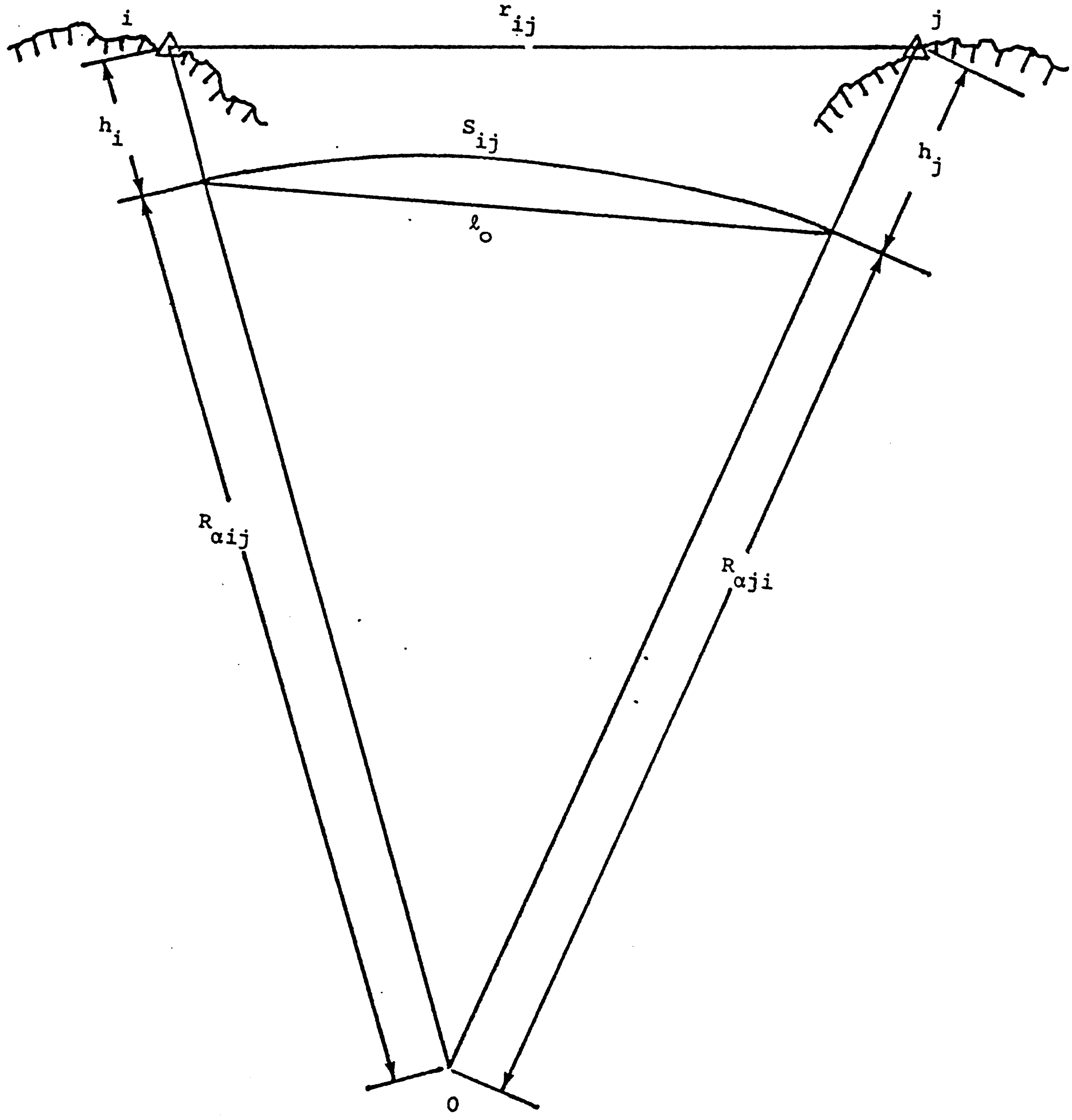


Figure 3-1

Spatial Distance Reduction

Compute,

$$\ell_o = \left[\frac{(r_{ij})^2 - \Delta h^2}{(1 + \frac{h_i}{R})\ (1 + \frac{h_j}{R})} \right]^{1/2} . \tag{3-19}$$

Then the ellipsoid distance is given by

$$S_{ij} = 2R \sin^{-1} \left(\frac{\ell_o}{2R}\right) , \tag{3-20}$$

where $\Delta h = h_j - h_i$,

and

$$R = \frac{R_{\alpha_{ij}} + R_{\alpha_{ji}}}{2} ,$$

in which

$$R_{\alpha_{ij}} = \frac{M_i N_i}{M_i \sin^2 \alpha_{ij} + N_i \cos^2 \alpha_{ij}} ,$$

and

$$R_{\alpha_{ji}} = \frac{M_j N_j}{M_j \sin^2 \alpha_{ji} + N_j \cos^2 \alpha_{ji}} .$$

3.2.7 Magnitude of Corrections

To give the user an idea of the magnitude of the various corrective terms several graphical illustrations are given. It must be noted here that the graphs are used solely for illustration and should not be used to obtain the corrective terms.

The first term illustrated is the so called gravimetric corrective term or deflection of the vertical term, C_1'' (which appears in the direction, angle, and astronomic azimuth reductions), and is given as

$$C_1'' = \rho((-\xi_i \sin \alpha_{ij} + \eta_i \cos \alpha_{ij}) \cot z_{ij}) . \tag{3-21}$$

An examination of Figure (3-2) shows that the corrective term, C_1'', can be significant and should be taken into account.

The second corrective term to be examined is the skew normal correction (applied to directions, angles, and astronomic azimuths), which is a geometrical correction resulting from the height of the target above the reference ellipsoid. This takes the form

$$C_2'' = \rho \frac{h_j}{M_m} (e^2 \sin \alpha_{ij} \cos \alpha_{ij} \cos^2 \phi_j) . \tag{3-22}$$

From Figure (3-3) it can be seen that the corrective term can be signigicant and should be taken into account for control surveys.

The next term to be examined is the normal section to geodesic term (applied to directions, angles, and astronomic azimuths), which is the result of the normal section-geodesic separation, and is given by

$$C_3'' = \rho \left(\frac{-e^2 S_{ij}^2 \cos^2 \phi_m \sin 2 \alpha_{ij}}{12 N_m^2}\right) . \tag{3-23}$$

Examining Figure (3-4) we see that the corrective terms are a magnitude smaller than those of the skew normal and only become critical on longer lines.

Distance reductions (3-21) and (3-22) are significant and should always be considered. Very often, however, the geoid-ellipsoid separation, N*, is neglected. It is well known that this leads to a scale error of 1 ppm for every 6 m of N* that is neglected.

Table 1 illustrates the errors introduced when H (orthometric height) is used in place of h (ellipsoidal height); that is, N is neglected.

CONDITIONS

$\xi = 10''0$

$\eta = 10''0$

$Z_{ij} = 80° \ 0' \ 0''00$

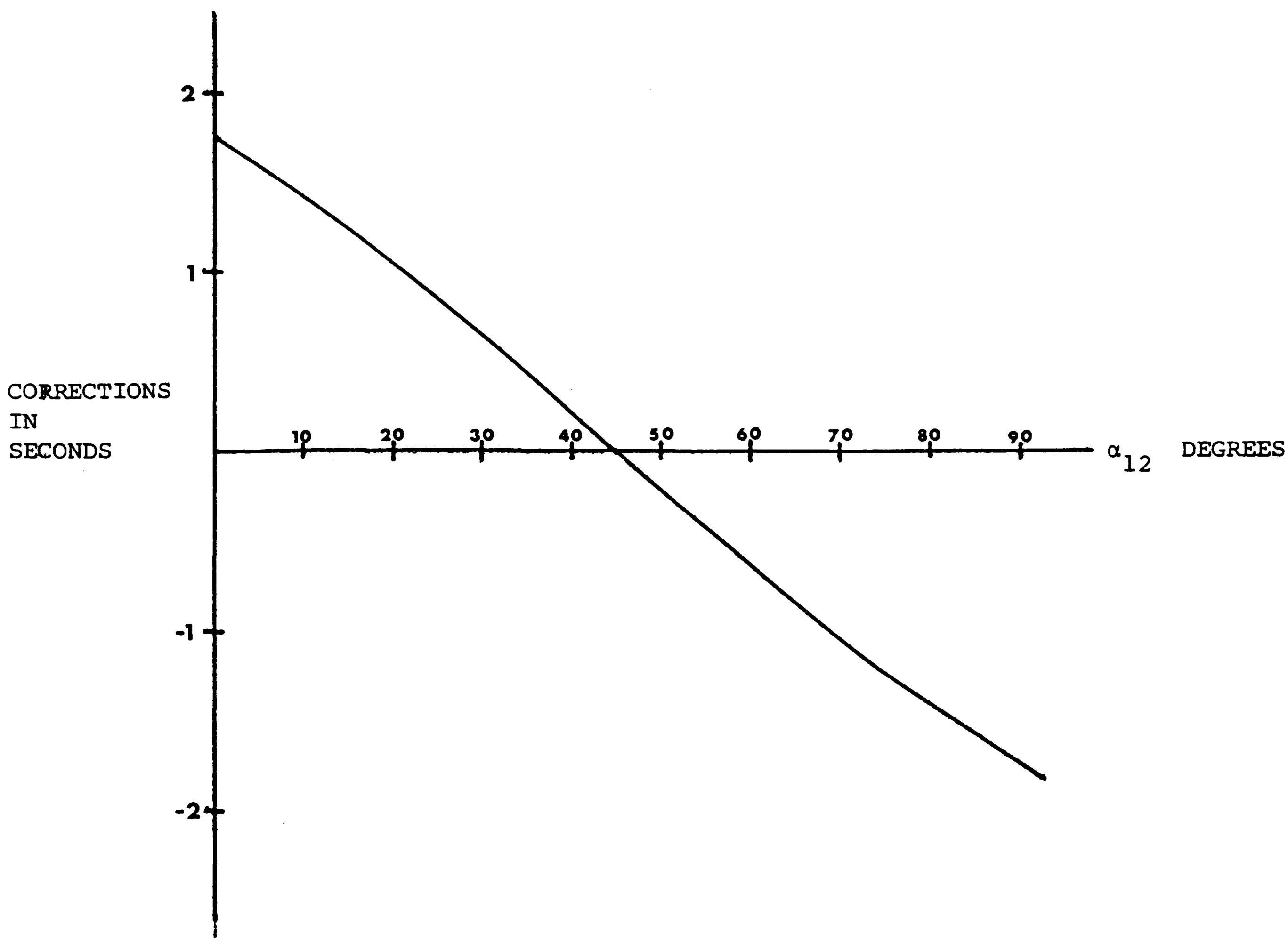


Figure 3-2

GRAVIMETRIC CORRECTION

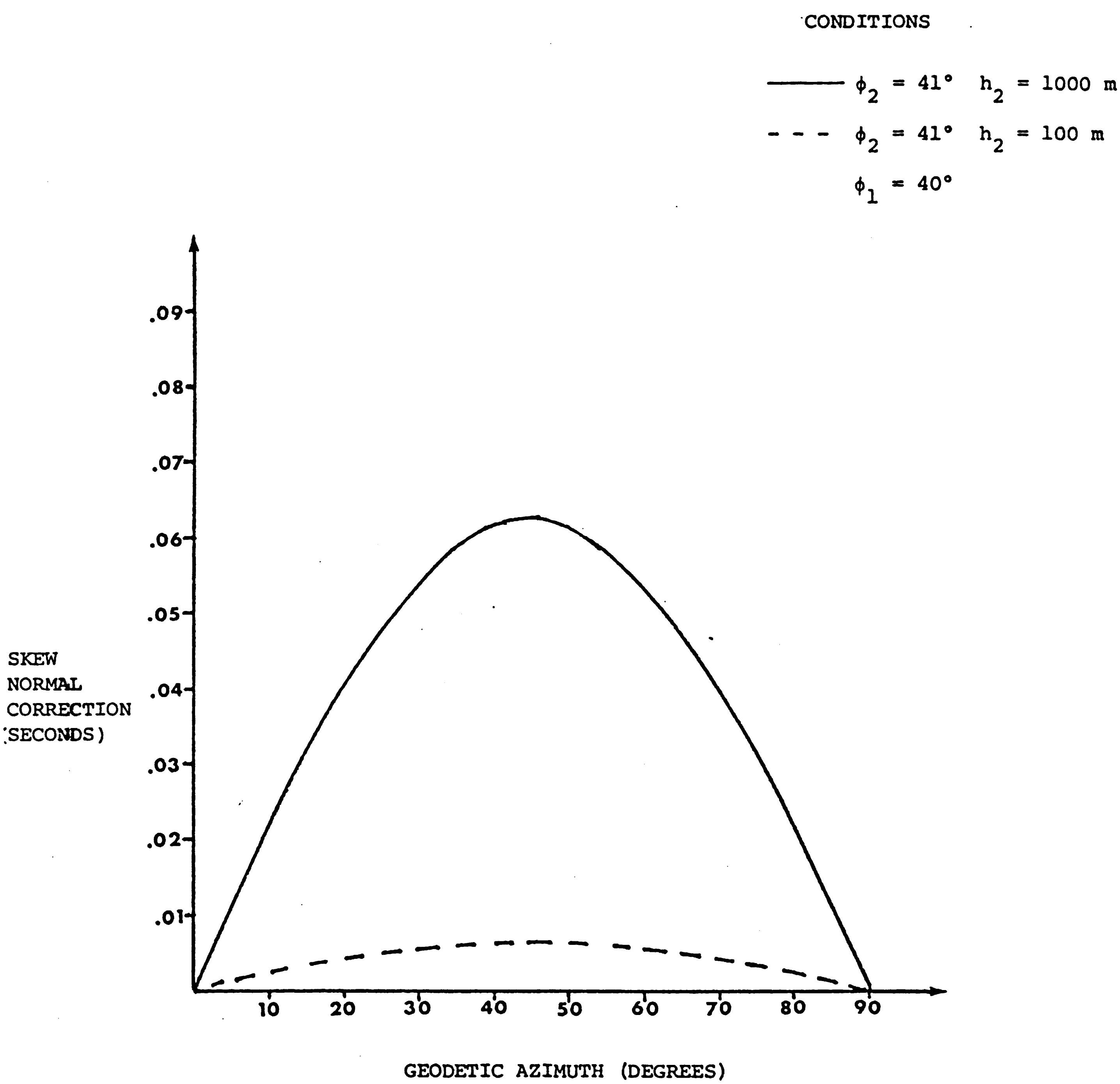


Figure 3-3

SKEW NORMAL CORRECTION

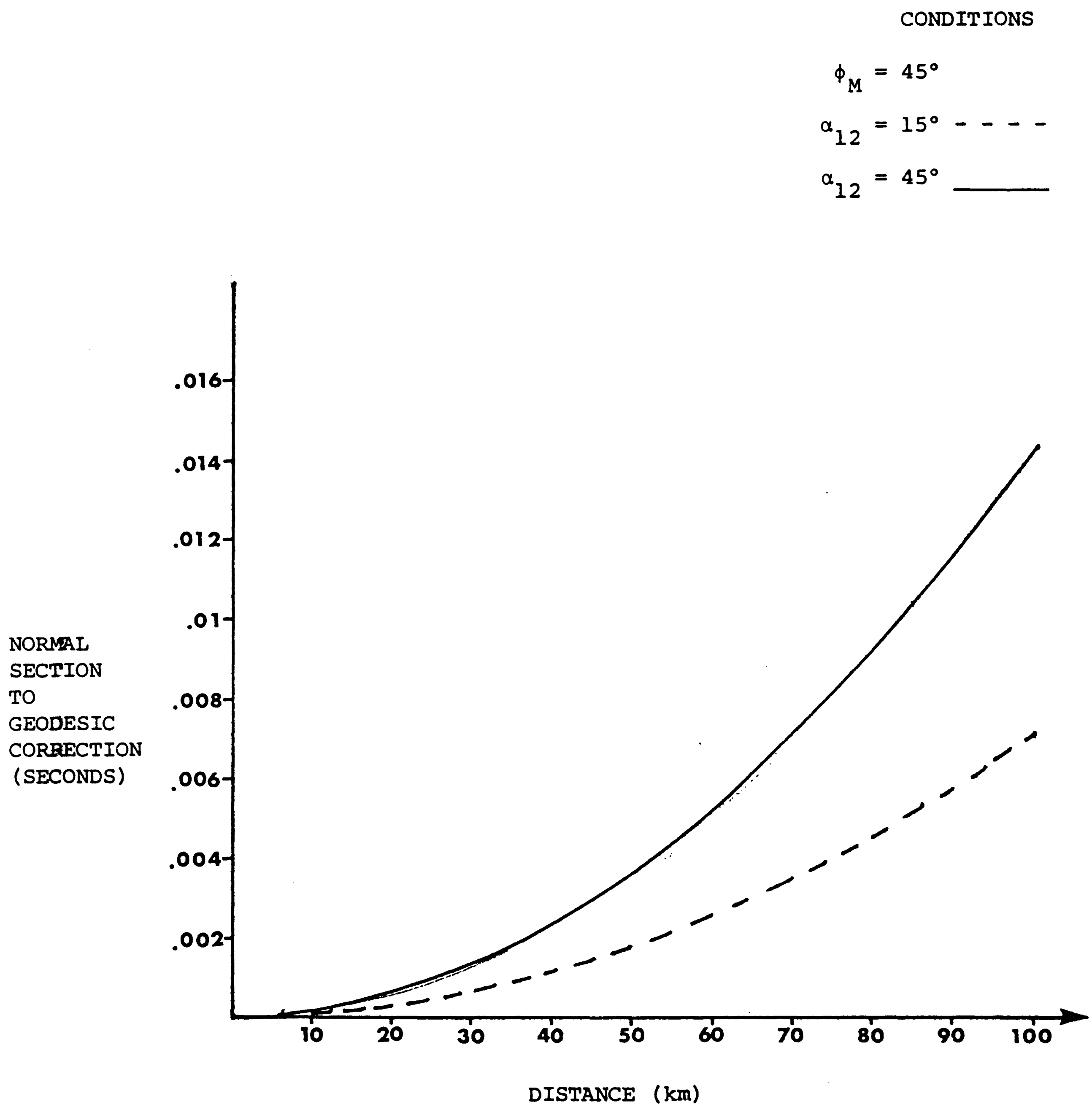


Figure 3-4

NORMAL SECTION TO GEODESIC CORRECTION

As a final note the surveyor should be aware that errors introduced by improper reduction of observed quantities are systematic and propagate through a network as such.

Geoid Distance $H_1 = 75$ m $H_2 = 50$ m	Ellipsoid Distance $h_1 = 81$ m $h_2 = 56$ m	Difference m	ppm
8 027.95	8 027.94	.01	1.2×10^{-6}
16 053.19	16 053.17	.02	1.2×10^{-6}
24 075.71	24 075.69	.02	$.9 \times 10^{-6}$
32 095.52	32 095.49	.03	$.9 \times 10^{-6}$
40 112.60	40 112.56	.04	1.0×10^{-6}
48 126.95	48 126.90	.05	1.0×10^{-6}
56 138.57	56 138.52	.05	0.9×10^{-6}
64 147.46	64 147.40	.06	0.9×10^{-6}
72 153.60	72 153.53	.07	1.0×10^{-6}
80 157.00	80 156.93	.07	0.9×10^{-6}
88 157.65	88 157.57	.08	0.9×10^{-6}
96 155.55	96 155.46	.09	0.9×10^{-6}

Table 3-1

Effect of Geoidal Height on Distance Reduction

3.2.8 <u>Error Propagation Through Reduction Formula</u>

The variance for the reduced quantities is taken to be the variance of the observation itself except for distances. This is not entirely rigorous but is practical for most surveying applications.

For precise work such as first order geodetic work the contributions could be significant. For example if one assumed the conditions

$$\phi_i = 45° ,$$

$$\eta_i = 20''0 \quad (\sigma_{\eta_i} = 2''0),$$

$$\xi_i = 20''0 \quad (\sigma_{\xi_i} = 2''0),$$

and

$$z_{ij} = 75°,$$

$$\alpha_{ij} = 45°,$$

the contribution of σ_{η_i} and σ_{ξ_i} to the standard deviation of a direction would be approximately 1"0. Assuming the same conditions the contributio to the standard deviation of an azimuth would be 2"4 .

The same conditions would add approximately 2"0 to the standard deviation of the observed zenith distance.

The propagation of errors through the distance reduction formulas concerns only the error in the ellipsoid height of the end points of the measured distance. The covariance matrix of the heights of the end points is needed and it has the form

$$C_l = \begin{bmatrix} \sigma_{r_{ij}}^2 & 0 & 0 \\ 0 & \sigma_{h_i}^2 & \sigma_{h_i h_j} \\ 0 & \sigma_{h_i h_j} & \sigma_{h_j}^2 \end{bmatrix} , \tag{3-24}$$

where all units are in m^2.

The Jacobian of transformation matrix B_l is (from equation (3-19))

$$B_l = [B_l(1,\ 1) \quad B_l\ (1,\ 2) \quad B_l\ (1,3)] , \tag{3-25}$$

where

$$B_1\ (1,\ 1) = \frac{r_{ij}}{\ell_o(1 + \frac{h_i}{R})\ (1 + \frac{h_j}{R})},$$

$$B_1\ (1,2) = \frac{1}{2\ell_o}\ [\frac{(\Delta h^2 - r_{ij}^2)}{R\ (1 + \frac{h_i}{R})^2\ (1 + \frac{h_j}{R})} + \frac{2\Delta h}{(1 + \frac{h_i}{R})\ (1 + \frac{h_j}{R})}],$$

and

$$B_1\ (1,3) = \frac{1}{2\ell_o}\ [\frac{(\Delta h^2 - r_{ij}^{\ 2})}{R(1 + \frac{h_i}{R})\ (1 + \frac{h_j}{R})^2} - \frac{2\Delta h}{(1 + \frac{h_i}{R})\ (1 + \frac{h_j}{R})}].$$

With B_1^T equal to the transpose of B_1, the variance for the ellipsoid distance, S_{ij}, is

$$C_2 = B_1\ C_1\ B_1^T, \tag{3-26}$$

where C_2 is given by

$$C_2 = \sigma_{S_{ij}}^2,$$

and is in units of m^2.

3.3 "Reduction" of Computed Geodetic Quantities to the Terrain

It is sometimes desirable to compare observed geodetic quantities (directions, azimuths, distances) with computed geodetic quantities. If the latter are given on the ellipsoid, they may be "reduced" to the terrain so that they may be compared with the observed quantities.

In order to "reduce" the directions, horizontal angles, zenith distances, and azimuths, we simply re-arrange terms in equations (3-12), (3-13), (3-14) and (3-15) to (3-18) respectively. For example, to "reduce" a direction from the ellipsoid to an observed direction on the terrain, we get

$$d_{ij} = d^e_{ij} + (\xi_i \sin \alpha_{ij} - \eta_i \cos \alpha_{ij}) \cot z_{ij}$$
$$- (\frac{h_i}{M_m} e^2 \sin \alpha_{ij} \cos \alpha_{ij} \cos^2 \phi_j)$$
$$+ (\frac{e^2 S^2_{ij} \cos^2\phi_m \sin 2\alpha_{ij}}{12 N^2_m}) \quad . \qquad (3\text{-}27)$$

To reduce distances from the ellipsoid to the terrain we use a similar procedure. Re-arrangment of terms in equations (3-19) and (3-20) yield

$$\ell_o = 2R \sin (\frac{S_{ij}}{2R}) \quad , \qquad (3\text{-}28)$$

and

$$r_{ij} = [\ell^2_o (1 + \frac{h_i}{R}) (1 + \frac{h_j}{R}) + \Delta h^2]^{1/2} \quad . \qquad (3\text{-}29)$$

Note that in all these "reductions" to the terrain we should not expect to have complete agreement between the computed quantity and the newly observed quantity since both of these quantities have some statistical fluctuation.

3.4 Puissant's Formula

It should be noted here at the outset that the derivation of Puissant's formulae is based on a spherical approximation, thus they are correct to 1 ppm (part per million) at 100 km, beyond which they break down rapidly (40 ppm at 250 km when $\phi = 60°$)[Bomford, 1971, p. 134].

3.4.1 Direct Problem

The direct problem is: given the geodetic quantities ϕ_i, λ_i, S_{ij} α_{ij}, compute the geodetic coordinates ϕ_j, λ_j. The solution for ϕ_j is iterative and proceeds as [Krakiwsky and Thomson, 1974]

$$\Delta\phi_{k} = [\frac{S_{ij}}{N_i} \cos \alpha_{ij} - \frac{S_{ij}^2}{2N_i^2} \tan \phi_i \sin^2 \alpha_{ij}$$

$$- \frac{S_{ij}^3}{6N_i^3} \cos \alpha_{ij} \sin^2\alpha_{ij} (1 + 3 \tan^2 \phi_i)] , \tag{3-30}$$

then,

$$\Delta\phi_{k+1} = [\frac{S_{ij} \cos \alpha_{ij}}{M_i} - \frac{S_{ij}^2 \tan \phi_i \sin^2 \alpha_{ij}}{2M_iN_i}$$

$$- \frac{S_{ij}^3 \cos \alpha_{ij} \sin^2 \alpha_{ij}(1 + 3 \tan^2 \phi_i)}{6M_iN_i^2}] [1 - \frac{3e^2 \sin\phi_i \cos \phi_i}{2(1-e^2 \sin^2 \phi_i)} \cdot \Delta\phi_k] .$$

where the letter k is a iteration counter. (3-31)

Finally

$$\phi_j = \phi_i + \Delta\phi_{k+1} \tag{3-32}$$

Examining equation (3-31) it can be seen that $\Delta\phi$ is a function of $\Delta\phi$ and therefore iteration is necessary. To accomplish this the solution $\Delta\phi_{k+1}$ is substituted for $\Delta\phi_k$ and $\Delta\phi_{k+2}$ is obtained. This process is repeated until the difference between successive $\Delta\phi$ values is less than 1×10^{-9} radians. This procedure is shown numerically in the example given in section 3.8, 3.9, and 3.10.

Now

$$\Delta\lambda = \frac{S_{ij}}{N_j} \sin \alpha_{ij} \sec \phi_j (1 - \frac{S_{ij}^2}{6N_j^2} (1 - \sin^2 \alpha_{ij} \sec^2 \phi_j)), \tag{3-33}$$

and

$$\lambda_j = \lambda_i + \Delta\lambda . \tag{3-34}$$

As a further step, one may compute the inverse azimuth.

First compute

$$\Delta\alpha = \Delta\lambda \sin \phi_m \sec (\frac{\Delta\phi}{2}) + \frac{\Delta\lambda^3}{12} (\sin \phi_m \sec \frac{\Delta\phi}{2}$$

$$-\sin^3 \phi_m \sec^3 (\frac{\Delta\phi}{2})) \quad , \tag{3-35}$$

then

$$\alpha_{ji} = \alpha_{ij} + \Delta\alpha + 180° \quad . \tag{3-36}$$

3.4.2 Inverse Problem

Puissant's inverse problem is: given ϕ_i, λ_i of point i and ϕ_j, λ_j of point j, compute the quantities S_{ij}, α_{ij} and α_{ji}. The solution proceeds as follows [Krakiwsky and Thomson 1974]

First compute,

$$\alpha_{ij_k} = \tan^{-1} \left[\frac{\dfrac{\Delta\lambda\ N_j}{\sec \phi_j}}{\dfrac{\Delta\phi\ \ M_i}{(1 - \dfrac{3e^2 \sin \phi_i \cos \phi_i \Delta\phi}{2(1 - e^2 \sin^2 \phi_i)})}} \right] , \tag{3-37}$$

and

$$S_{ij_k} = \frac{\Delta\lambda\ N_j}{\sec \phi_j \sin \alpha_{ij_k}} \quad , \tag{3-38}$$

or

$$S_{ij_k} = \frac{\Delta\phi}{\cos \alpha_{ij_k}} \left[\frac{M_i}{(1 - \dfrac{3e^2 \sin \phi_i \cos \phi_i \Delta\phi}{2(1-e^2 \sin^2 \phi_i)})} \right] , \tag{3-39}$$

where $\Delta\phi = \phi_j - \phi_i$,

and

$$\Delta\lambda = \lambda_j - \lambda_i .$$

Next, new values of α_{ij} and S_{ij} are computed as follows.

Compute

$$T_1 = \frac{\Delta\lambda\ N_j}{\sec\ \phi_j} + \frac{(S_{ij_k})^3}{6N_j^2} \sin\ \alpha_{ij_k} - \frac{(S_{ij_k})}{6N_j^2} \sin^3\ \alpha_{ij_k}\ \sec^2\ \phi_j , \tag{3-40}$$

and

$$T_2 = \Delta\phi \left[\frac{M_i}{(1 - \frac{3e^2\ \sin\ \phi_i\ \cos\ \phi_i}{2(1 - e^2\ \sin^2\ \phi_i)} \cdot \Delta\phi)} \right] + \frac{(S_{ij_k})^2 \tan\ \phi_i\ \sin^2\ \alpha_{ij_k}}{2N_i} \tag{3-41}$$

$$+ \frac{(S_{ij_k})^3 \cos\alpha_{ij_k}\ \sin^2\alpha_{ij_k}\ (1 + 3\ \tan^2\ \phi_i)}{6\ N_i^2} .$$

Now

$$\alpha_{ij_{k+1}} = \tan^{-1}\ (\frac{T_1}{T_2}) , \tag{3-42}$$

and

$$S_{ij_{k+1}} = \frac{T_1}{\sin\ \alpha_{ij_k}} , \tag{3-43}$$

or

$$S_{ij_{k+1}} = \frac{T_2}{\cos\ \alpha_{ij_k}} . \tag{3-44}$$

Note that the new $\alpha_{ij_{k+1}}$ (equation (3-43)), is used in (3-43) or (3-44). Now using the new values of $S_{ij_{k+1}}$ and $\alpha_{ij_{k+1}}$ we may again compute updated values by returning to equations (3-40), (3-41), (3-42) and (3-43) or (3-44). This iteration process continues until changes in α_{ij} and S_{ij} are negligible ($\Delta\alpha_{ij} \leq 1 \times 10^{-9}$ radians).

Once we have obtained a final value for α_{ij}, α_{ji} is computed using equations (3-35) and (3-36). This completes the inverse problem using Puissant's formulae.

3.5 The Gauss Mid-Latitude Formulae

These formulae are also based on a spherical approximation of the earth and because of the degree of approximation should only be used for points separated by less than 40 km at latitudes less than 80° [Allen et al, 1968] and are accurate to 2 ppm within these bounds.

3.5.1 Direct Problem

The direct problem is: given the quantities ϕ_i, λ_i, S_{ij} and α_{ij}, compute the geodetic coordinates ϕ_j, λ_j. The solution is iterative. The first iteration is

$$\Delta\phi_k = [\frac{S_{ij} \cos \alpha_{ij}}{M_i}] \tag{3-45}$$

$$\phi_{j_k} = \phi_i + \Delta\phi_k \quad , \tag{3-46}$$

$$\Delta\lambda_k = [\frac{S_{ij} \sin\alpha_{ij}}{N_m \cos \phi_m}] \tag{3-47}$$

and

$$\lambda_{j_k} = \lambda_i + \Delta\lambda_k . \tag{3-48}$$

Then

$$\Delta\alpha_k = \Delta\lambda_k \sin \phi_m . \tag{3-49}$$

The second iteration proceeds as

$$\Delta\phi_{k+1} = \frac{S_{ij} \cos (\alpha_{ij} + \frac{\Delta\alpha_k}{2})}{M_m} \tag{3-50}$$

$$\phi_j = \phi_i + \Delta\phi_{k+1}$$

$$\Delta\lambda_{k+1} = \frac{S_{ij} \sin (\alpha_{ij} + \frac{\Delta\alpha_k}{2})}{N_m \cos \phi_m} \tag{3-51}$$

and

$$\Delta\alpha_{k+1} = \Delta\lambda_{k+1} \sin \phi_m . \tag{3-52}$$

Having obtained a new $\Delta\alpha_{k+1}$ we can return to (3-50) and repeat the procedure solving for $\Delta\phi_{k+2}$, a new $\phi_{j_{k+2}}$, $\Delta\lambda_{k+2}$ and $\Delta\alpha_{k+2}$. This procedure is repeated until the difference between successive cycles is less than 1×10^{-9} radians for all quantities. Upon completion of the iterations we compute

$$\phi_j = \phi_i + \Delta\phi \quad , \tag{3-53}$$

$$\lambda_j = \lambda_i + \Delta\lambda \quad , \tag{3-54}$$

and as a further step we compute the inverse azimuth

$$\alpha_{ji} = \alpha_{ij} + \Delta\alpha + 180 \ . \tag{3-55}$$

3.5.2 Inverse Problem

The inverse problem is: given ϕ_i, λ_i of point i and ϕ_j, λ_j of point j, compute the direct and inverse geodetic azimuths α_{ij} and α_{ji} and the ellipsoid distance S_{ij}. The procedure is as follows. First

$$\Delta\phi = \phi_j - \phi_i \ , \tag{3-56}$$

$$\Delta\lambda = \lambda_j - \lambda_i \ , \tag{3-57}$$

and

$$\Delta\alpha = \Delta\lambda \sin \phi_m \ . \tag{3-58}$$

Next compute

$$\alpha_m = (\alpha_{ij} + \frac{\Delta\alpha}{2}) = \tan^{-1} [\frac{\Delta\lambda}{\Delta\phi} \ \frac{N_m \cos \phi_m}{M_m}] \ . \tag{3-59}$$

Then

$$\alpha_{ij} = \alpha_m - \frac{\Delta\alpha}{2} \ , \tag{3-60}$$

and

$$\alpha_{ji} = \alpha_{ij} + \Delta\alpha + 180° \ . \tag{3-61}$$

Finally, S_{ij} (= S_{ji}) is computed either from

$$S_{ij} = \frac{\Delta\lambda \; N_m \; \cos \phi_m}{\sin \left(\alpha_{ij} + \frac{\Delta\alpha}{2}\right)} , \tag{3-62}$$

or

$$S_{ij} = \frac{M_m \; \Delta\phi}{\cos \left(\alpha_{ij} + \frac{\Delta\alpha}{2}\right)} . \tag{3-63}$$

3.6 Error Propagation Through Position Computations

The notation used in this section is identical with that used in Section 3.5. This is because the Gauss Mid-Latitude formulae have been used for the generation of the necessary Jacobian matrix elements. This approximation amounts to errors well within the accuracy of the formulae themselves, that is less than 1 ppm at 100 km.

3.6.1 Direct Problem Error Propagation

The direct problem error propagation proceeds in the following manner. The covariance matrix for point i is combined with the variance of the geodetic azimuth and the variance of the ellipsoidal distance to produce the covariance matrix C_3. The form of C_3 is

$$C_3 = \left[\begin{array}{cc|cc} \sigma_{\phi_i}^2 & \sigma_{\phi_i\lambda_i} & 0 & 0 \\ \sigma_{\phi_i\lambda_i} & \sigma_{\lambda_i}^2 & 0 & 0 \\ \hline 0 & 0 & \sigma_{\alpha_{ij}}^2 & \\ 0 & 0 & & \sigma_{s_{ij}}^2 \end{array}\right] , \qquad (3\text{-}64)$$

in units of

$$\begin{bmatrix} rad^2 & rad^2 & & \\ rad^2 & rad^2 & & \\ & & rad^2 & \\ & & & m^2 \end{bmatrix}$$

Arcsec^2 may be converted to rad^2 for use in the above covariance matrix by multiplying by $\frac{1}{\rho^2}$.

To include the covariance information of point i in the output covariance matrix the equations

$$\phi_i = \phi_i , \qquad (3\text{-}65)$$

and

$$\lambda_i = \lambda_i , \qquad (3\text{-}66)$$

are required.

The Jacobian of transformation matrix B_2 is (from equations (3-66), (3-67), (3-53) and (3-54))

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_2(3,1) & B_2(3,2) & B_2(3,3) & B_2(3,4) \\ B_2(4,1) & B_2(4,2) & B_2(4,3) & B_2(4,4) \end{bmatrix} \qquad (3\text{-}67)$$

where,

$$B_2(3,1) = 1 - \frac{3\, S_{ij} \cos \alpha_m\, M_i N_i^{\,2}\, e^2 \sin \phi_i \cos \phi_i}{2M_m^{\,2}\, a^2}$$

$$- \frac{S_{ij} \sin \alpha_m\, \Delta\lambda \cos \phi_m}{4M_m} ,$$

$$B_2(3,2) = \frac{S_{ij} \sin \alpha_m \sin \phi_m}{2M_m} ,$$

$$B_2(3,3) = \frac{- S_{ij} \sin \alpha_m}{M_m} ,$$

$$B_2(3,4) = \frac{\cos \alpha_m}{M_m} ,$$

$$B_2(4,1) = S_{ij} \sin \alpha_m \left(N_m \sin \phi_m - \frac{M_i e^2 \sin \phi_i \cos \phi_i \cos \phi_m}{(1 - e^2)}\right)$$

$$\left(\frac{1}{2\, N_m^{\,2} \cos^2 \phi_m}\right) + \frac{S_{ij} \cos \alpha_m\, \Delta\lambda\, .}{4N_m} ,$$

$$B_2(4,2) = 1 - \frac{S_{ij} \cos \alpha_m \sin \phi_m}{2\, N_m \cos \phi_m}$$

$$B_2(4,3) = \frac{S_{ij} \cos \alpha_m}{N_m \cos \phi_m} ,$$

and

$$B_2(4,4) = \frac{\sin \alpha_m}{N_m \cos \phi_m} .$$

With $B_2^{\,T}$ equal to the transpose of B_2, the covariance matrix for points i and j will then be

$$C_4 = B_2\, C_3\, B_2^{\,T}, \tag{3-68}$$

where C_4 has the form

$$C_4 = \left[\begin{array}{cc|cc} \sigma_{\phi_i}^{\;2} & \sigma_{\phi_i\lambda_i} & \sigma_{\phi_i\phi_j} & \sigma_{\phi_i\lambda_j} \\ \sigma_{\phi_i\lambda_i} & \sigma_{\lambda_i}^{\;2} & \sigma_{\lambda_i\phi_j} & \sigma_{\lambda_i\lambda_j} \\ \hline \sigma_{\phi_i\phi_j} & \sigma_{\lambda_i\phi_j} & \sigma_{\phi_j}^{\;2} & \sigma_{\phi_j\lambda_j} \\ \sigma_{\phi_i\lambda_j} & \sigma_{\lambda_i\lambda_j} & \sigma_{\phi_j\lambda_j} & \sigma_{\lambda_j}^{\;2} \end{array}\right]$$

All elements in C_4 are in rad.^{2} which can be converted to arcsec^2 by simply multiplying each element of C_4 by ρ^2.

3.6.2 Inverse Problem Error Propagation

The inverse problem error propagation proceeds as follows. First, the covariance matrix for the points i and j is written as

$$C_4 = \begin{bmatrix} \sigma_{\phi_i}^{\;2} & \sigma_{\phi_i\lambda_i} & \sigma_{\phi_i\phi_j} & \sigma_{\phi_i\lambda_j} \\ \sigma_{\phi_i\lambda_i} & \sigma_{\lambda_i}^{2} & \sigma_{\lambda_i\phi_j} & \sigma_{\lambda_i\lambda_j} \\ \sigma_{\phi_i\phi_j} & \sigma_{\lambda_i\phi_j} & \sigma_{\phi_j}^{\;2} & \sigma_{\phi_j\lambda_j} \\ \sigma_{\phi_i\lambda_j} & \sigma_{\lambda_i\lambda_j} & \sigma_{\phi_j\lambda_j} & \sigma_{\lambda_j}^{\;2} \end{bmatrix} \tag{3-69}$$

where all units, in C_4 are in rad^2 which is obtained from a covariance matrix in arcsec^2 by multiplying each element by $\frac{1}{\rho^2}$.

Then using equations (3-65), (3-66), (3-60), (3-61) along with

$$\phi_j = \phi_j \tag{3-70}$$

$$\lambda_i = \lambda_i \tag{3-71}$$

the Jacobian of transformation B_3 is

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ B_3(5,1) & B_3(5,2) & B_3(5,3) & B_3(5,4) \\ B_3(6,1) & B_3(6,2) & B_3(6,3) & B_3(6,4) \end{bmatrix} \tag{3-72}$$

where

$$B_3(5,\ 1) = B_3\ (6,\ 1) - \frac{\Delta\lambda\ \cos\phi_m}{2}\ ,$$

$$B_3(5,\ 2) = B_3\ (6,\ 2) + \sin\phi_m\ ,$$

$$B_3(5,\ 3) = B_3(6,\ 3) - \frac{\Delta\lambda\ \cos\phi_m}{2}\ ,$$

$$B_3(5,\ 4) = B_3(6,\ 4) - \frac{\sin\phi_m}{2}\ ,$$

$$B_3\ (6,1) = \frac{\Delta\lambda}{\left(1 + \left(\frac{\Delta\lambda N_m \cos\phi_m}{\Delta\phi\ M_m}\right)^2\right)\Delta\phi M_m} \left(\frac{N_m\ \cos\phi_m}{\Delta\phi} - + \right.$$

$$\cdot\ \frac{M_i\ e^2 \sin\phi_i\ \cos\phi_i\ \cos\phi_m}{2(1 - e^2)} - \frac{3}{2}\,\frac{N_m\ \cos\phi_m\ M_i\ e^2\ \sin\phi_i\ \cos\phi_i}{M_m\ (1 - e^2\ \sin^2\phi_i)}$$

$$\left. - \frac{1}{2} N_m\ \sin\phi_m \right) + \frac{\Delta\lambda\ \cos\phi_m}{4}\ ,$$

$$B_3\,(6,2) = \left[\frac{1}{1 + \left(\dfrac{\Delta\lambda\, N_m}{\Delta\phi\ \ M_m}\cos\phi_m\right)^2}\right]\left(\frac{-N_m\cos\phi_m}{\Delta\phi\ M_m}\right) - \frac{\sin\phi_m}{2}\ ,$$

$$B_3\,(6,3) = \left[\frac{1}{1 + \left(\dfrac{\Delta\lambda\ N_m\ \cos\phi_m}{\Delta\phi\ M_m}\right)^2}\right]\frac{\Delta\lambda}{\Delta\phi\ M_m}\ \left(\frac{-N_m\ \cos\phi_m}{\Delta\phi}\right.$$

$$+ \frac{M_j\ e^2\sin\phi_j\ \cos\phi_j\ \cos\phi_m}{2(1-e)^2} - \frac{3}{2}\,\frac{N_m\ \cos\phi_m\ M_j\ e^2\sin\phi_j\ \cos\phi_j}{M_m\ \ (1-e^2\sin^2\phi_j)}$$

$$\left. - \frac{1}{2}N_m\ \sin\phi_m\ \right) + \frac{\Delta\lambda\ \cos\phi_m}{4}\ ,$$

and

$$B_3(6,4) = \left[\frac{1}{1 + \left(\dfrac{\Delta\lambda\ N_m\ \cos\ \phi_m}{\Delta\phi\ M_m}\right)^2}\right]\left(\frac{N_m\ \cos\phi_m}{\Delta\phi\ M_m}\right) + \frac{\sin\phi_m}{2}\ .$$

With B_4^T equal to the transpose of B_4 the covariance matrix for the points i and j and the direct (α_{ij}) and inverse (α_{ji}) azimuths is given by

$$C_5 = B_3 \, C_4 \, B_3^T \,, \tag{3-73}$$

where C_5 has the form

$$C_5 = \left[\begin{array}{cccc|cc} & & & & \sigma_{\phi_i \alpha_{ij}} & \sigma_{\phi_i \alpha_{ji}} \\ & & & & \sigma_{\lambda_i \alpha_{ij}} & \sigma_{\lambda_i \alpha_{ji}} \\ & C_4 & & & \sigma_{\phi_j \alpha_{ij}} & \sigma_{\phi_j \alpha_{ji}} \\ & & & & \sigma_{\lambda_j \alpha_{ij}} & \sigma_{\lambda_j \alpha_{ji}} \\ \hline \sigma_{\phi_i \alpha_{ij}} & \sigma_{\lambda_i \alpha_{ij}} & \sigma_{\phi_j \alpha_{ij}} & \sigma_{\lambda_j \alpha_{ij}} & \sigma^2_{\alpha_{ij}} & \sigma_{\alpha_{ij}\alpha_{ji}} \\ \sigma_{\phi_i \alpha_{ji}} & \sigma_{\lambda_i \alpha_{ji}} & \sigma_{\phi_j \alpha_{ji}} & \sigma_{\lambda_j \alpha_{ji}} & \sigma_{\alpha_{ij}\alpha_{ji}} & \sigma^2_{\alpha_{ji}} \end{array}\right] \tag{3-74}$$

with all units in rad^2.

If the accuracy of the distance and its relationship with the coordinates, and azimuths α_{ij} and α_{ji} is required, then using equations (3-64), (3-66) (3-70) and (3-71), plus

$$\alpha_{ij} = \alpha_{ij} \quad , \tag{3-75}$$

and

$$\alpha_{ji} = \alpha_{ji} \quad , \tag{3-76}$$

the Jacobian of transformation B_4, is

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ B\,(3,1) & 0 & B\,(3,3) & 0 & B\,(3,5) & B\,(3,6) \end{bmatrix}, \quad (3\text{-}77)$$

where

$$B_4(3,1) = \frac{-M_m}{\cos\left(\frac{\alpha_{ij} + \alpha_{ji} - 180}{2}\right)} + \frac{3\,\Delta\phi\,M_i\,e^2 \sin\phi_i \cos\phi_i}{2\,(1 - e^2 \sin^2\phi_i)\cos\left(\frac{\alpha_{ij} + \alpha_{ji} - 180}{2}\right)}$$

$$B_4(3,3) = \frac{M_m}{\cos\left(\frac{\alpha_{ij} + \alpha_{ji} - 180}{2}\right)} + \frac{3\,\Delta\phi\,M_j\,e^2 \sin\phi_j \cos\phi_j}{2\,(1 - e^2 \sin^2\phi_j)\cos\left(\frac{\alpha_{ij} + \alpha_{ji} - 180}{2}\right)}$$

$$B_4(3,5) = \frac{\Delta\phi\,M_m \sin\left(\frac{\alpha_{ij} + \alpha_{ji} - 180}{2}\right)}{2\cos^2\left(\frac{\alpha_{ij} + \alpha_{ji} - 180}{2}\right)},$$

and

$$B_4(3,6) = B_4(7,5).$$

With B_4^T equal to the transpose of B_4 the covariance matrix is

$$C_6 = B_4\,C_5\,B_4^T, \quad (3\text{-}78)$$

where C_6 has the form

$$C_6 = \begin{bmatrix} \sigma_{\alpha_{ij}}^2 & \sigma_{\alpha_{ij}\alpha_{ji}} & \sigma_{\alpha_{ij}S_{ij}} \\ \sigma_{\alpha_{ij}\alpha_{ji}} & \sigma_{\alpha_{ji}}^2 & \sigma_{\alpha_{ji}S_{ij}} \\ \sigma_{\alpha_{ij}S_{ij}} & \sigma_{\alpha_{ji}S_{ij}} & \sigma_{S_{ij}}^2 \end{bmatrix}$$

in units of

$$\begin{bmatrix} \mathrm{rad}^2 & \mathrm{rad}^2 & \mathrm{rad.m} \\ \mathrm{rad}^2 & \mathrm{rad}^2 & \mathrm{rad.m} \\ \mathrm{rad.m} & \mathrm{rad.m} & \mathrm{m}^2 \end{bmatrix},$$

where the rad.m may be converted to arcsec.m by multiplying by ρ and rad^2 may be converted to arcsec^2 by multiplying by ρ^2.

3.7 Introduction to Numerical Examples

3.7.1 Use of Computed Geodetic Azimuth

Before commencing with the numerical examples for direct and inverse problems on the reference ellipsoid, let us examine the determination of the geodetic azimuth of a line by means other than the reduction of a terrain astronomic azimuth. A common situation is to know the geodetic coordinates of the instrument station i and those of the reference station j, along with the covariance matrix (C_2) for those points. The geodetic azimuth α_{ij} for the line ij can be computed using the Puissant's (section 3.4) or Gauss Mid Latitude (section 3.5) inverse formulae. The covariance matrix involving the points and the azimuth can be derived using the inverse problem error propagation (section 3.6.2).

The terrain angle β_{jik} (k is the unknown point) can be measured and then using the reduction formulae outlined in Section 3.2.3 the angle is reduced to the ellipsoid giving β^e_{jik}. This angle is then added to α_{ij} yielding

$$\alpha_{ik} = \beta^e_{jik} + \alpha_{ij} \quad . \tag{3-79}$$

The variance of α_{ik} is computed as

$$\sigma^2_{\alpha_{ik}} = \sigma^2_{\beta^e_{jik}} + \sigma^2_{\alpha_{ij}} \quad . \tag{3-80}$$

The C_3 matrix (equation 3-65) would take the form

$$C_3 = \left[\begin{array}{cc|cc} \sigma^2_{\phi_i} & \sigma_{\phi_i\lambda_i} & \sigma_{\phi_i\alpha_{ij}} & 0 \\ \sigma_{\phi_i\lambda_i} & \sigma^2_{\lambda_i} & \sigma_{\lambda_i\alpha_{ij}} & 0 \\ \hline \sigma_{\phi_i\alpha_{ij}} & \sigma_{\lambda_i\alpha_{ij}} & \sigma^2_{\alpha_{ik}} & 0 \\ 0 & 0 & 0 & \sigma^2_{S_{ik}} \end{array}\right] \tag{3-81}$$

in units of

$$\left[\begin{array}{cc|cc} rad^2 & rad^2 & rad^2 & \\ rad^2 & rad^2 & rad^2 & \\ \hline rad^2 & rad^2 & rad^2 & \\ & & & m^2 \end{array}\right]$$

The terms $\sigma_{\phi_i \alpha_{ij}}$ and $\sigma_{\lambda_i \alpha_{ij}}$ result from the fact that α_{ij}, which was used in equation (3-79) to form α_{ik}, is derived from the coordinates of points i and j and therefore is correlated with point i. The $\sigma_{\phi_i \alpha_{ij}}$ and $\sigma_{\lambda_i \alpha_{ij}}$ terms can be taken from the appropriate location in equation (3-74).

Having obtained the above information (equations (3-79) and (3-81)) the direct problem can be solved using the Puissant or Gauss Mid-Latitude solution for the direct problem as outlined in Sections 3.4.1 and 3.5.1.

The numerical examples that follow are done assuming that an astronomic azimuth has been determined by observation.

3.7.2 Ellipsoid Direct Problem Flow Chart

Figure (3-5) contains a flow chart which depicts the steps to be executed in doing the direct problem. The flow chart begins with the observed astronomic azimuth, zenith distance and spatial distance, followed by the reduction of these observations. These reduced observations are then used in either the Puissant or Guass Mid Latitude direct problem solution.

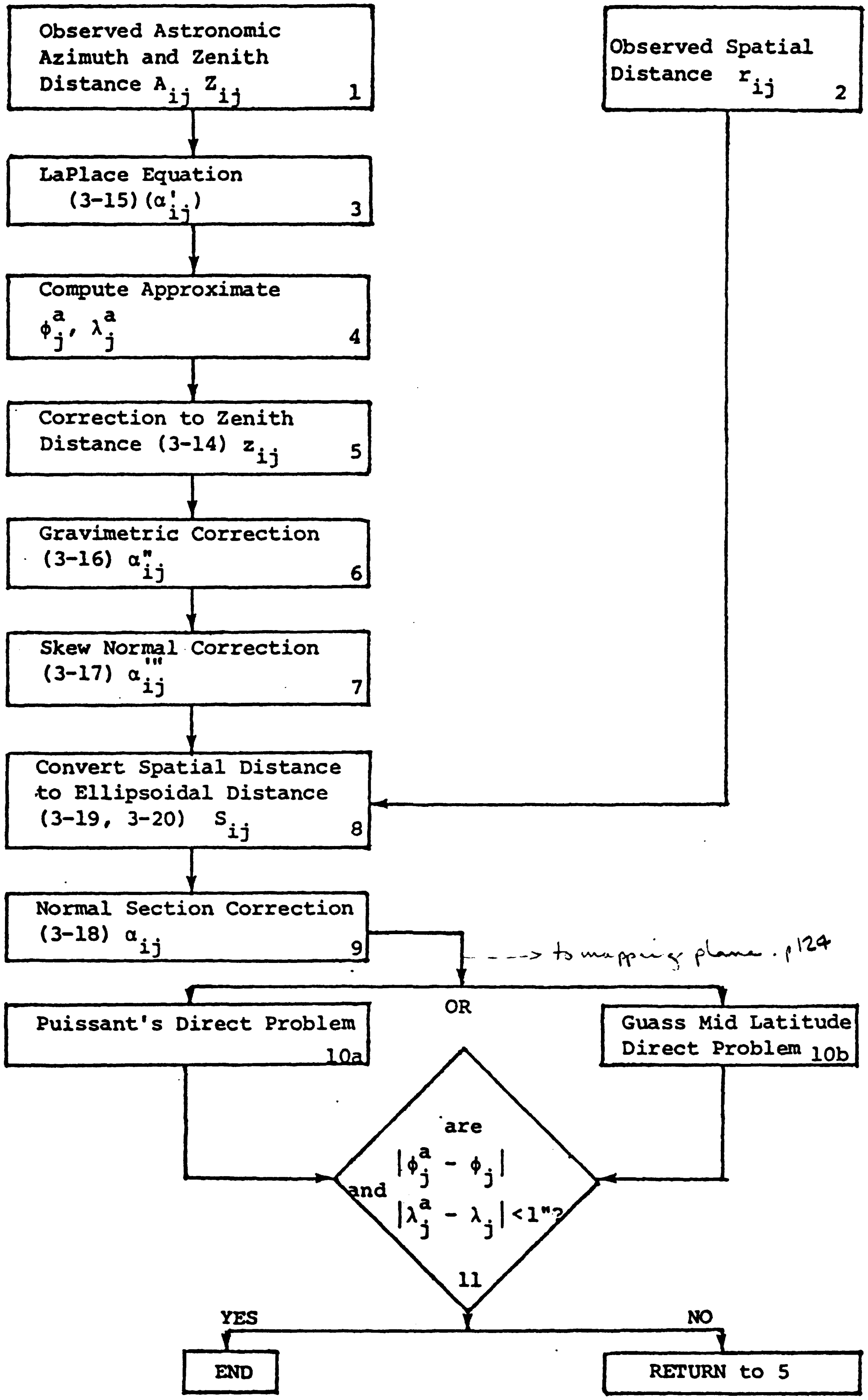


Figure 3-5

Ellipsoid Direct Problem Flow Chart

3.8 New Brunswick Numerical Example

3.8.1 Direct Problem

The information given here for the solution of the direct problem on the reference ellipsoid and its associated error propagation, is identical to that used for the numerical example for the three dimensional case in Section 2.4.

The coordinates of point 1 are

$$\phi_1 = 47° \ 03' \ 24''.644 \quad ,$$

$$\lambda_1 = 65° \ 29' \ 3''.453 \ \ \mathrm{W}$$

$$= -65° \ 29' \ 3''.453 \quad ,$$

and the associated covariance matrix is

$$C_{\phi_i \lambda_i} = \begin{bmatrix} 2.3504 \times 10^{-15} & -1.8804 \times 10^{-18} \\ -1.8804 \times 10^{-18} & 2.3504 \times 10^{-15} \end{bmatrix} ,$$

in units of

$$\begin{bmatrix} \mathrm{rad}^2 & \mathrm{rad}^2 \\ \mathrm{rad}^2 & \mathrm{rad}^2 \end{bmatrix} .$$

The deflection of the vertical components for point 1 are

$$\xi_1 = 4''.0 \quad ,$$

and

$$\eta_1 = 6''.0 \quad .$$

The observations are

$$r_{12} = 2\,500.00 \ \mathrm{m} \ ,$$

$$A_{12} = 45° \ 00' 00''.00 \quad ,$$

and

$$z_{12} = 87° \ 00' 00''.00 \quad .$$

The variance of the astronomic azimuth is

$$\sigma_{A_{ij}}^{\ 2} = 25''.00 \ \mathrm{arcsec}^2$$

$$= 5.876 \times 10^{-10} \ \mathrm{rad}^2$$

This value is taken to be the variance of the geodetic azimuth

$$\sigma_{\alpha}{}^{2}{}_{ij} = \sigma_{A}{}^{2}{}_{ij} \quad .$$

The ellipsoid heights of the two points are

$$h_1 = 100.00 \text{ m.}$$

$$h_2 = 231.243 \text{ m.}$$

The covariance matrix C_1 of the spatial distance and heights is given by

$$C_1 = \begin{bmatrix} 7.840 \times 10^{-4} & 0 & 0 \\ 0 & 4.0 & 4.0 \\ 0 & 4.0 & 4.033 \end{bmatrix} ,$$

in units of

$$\begin{bmatrix} m^2 & & \\ & m^2 & m^2 \\ & m^2 & m^2 \end{bmatrix} .$$

The approximate coordinates for point 2 (equations (3-9) and (3-10)) are

$$\phi^{a}{}_{2} = 47°04'\ 21''.889,$$

and

$$\lambda^{a}{}_{2} = 65°\ 27'\ 39''.680 \quad W$$

$$= -65°\ 27'\ 39''.680.$$

The corrected zenith distance (equation (3-14)) is

$$z_{ij} = 87°00'\ 7''.07$$

The first steps through the azimuth reduction give (equations(3-15), (3-16) and (3-17))

$$\alpha_{12}''' = 44°\ 59'\ 53''.64 \quad .$$

The distance is now reduced using equations (3-19), (3-20), and assuming

$$\alpha_{12} \simeq \alpha_{12}''' \quad ,$$

and

$$\alpha_{21} \simeq \alpha_{12}''' + 180° \quad .$$

The reduced distance is

$$S_{12} = 2\,496.488 \text{ m} \quad .$$

The variance of the distance is (using equation (3-26))

$$\sigma_{S_{12}}^2 = 8.760 \times 10^{-4} \text{ m}^2 .$$

The final correction to the azimuth (equation (3-18)) yields the geodetic azimuth

$$\alpha_{12} = 44° \ 59' \ 53''.64 \quad .$$

The covariance matrix C_3 (equation (3-64)) is given by

$$C_3 = \begin{bmatrix} 2.3504 \times 10^{-15} & -1.8804 \times 10^{-18} & 0 & 0 \\ -1.8804 \times 10^{-18} & 2.3504 \times 10^{-15} & 0. & 0 \\ 0 & 0 & 5.876 \times 10^{-10} & 0 \\ 0 & 0 & 0 & 8.760 \times 10^{-4} \end{bmatrix}$$

in units of

$$\begin{bmatrix} \text{rad}^2 & \text{rad}^2 & & \\ \text{rad}^2 & \text{rad}^2 & & \\ & & \text{rad}^2 & \\ & & & \text{m}^2 \end{bmatrix} .$$

The direct problem solution using the reduced quantities (S_{12} and α_{12}) is done using Puissant's formulae. After the first iteration the difference in the $\Delta\phi$ terms is

$$|\Delta\phi_1 - \Delta\phi_2| = 0''.18014$$
$$= 8.733 \times 10^{-7} \text{ rad}$$

After the second iteration

$$|\Delta\phi_2 - \Delta\phi_3| < 2 \times 10^{-4} \text{ arcsec}$$
$$< 1 \times 10^{-9} \text{ rad}$$

This indicates that the stopping criteria has been met and from the final iteration of equation (3-31)

$$\Delta\phi = 57''.157 \ .$$

Now (from equation (3-32))

$$\phi_2 = 47° \ 04' \ 21''.801 \ ,$$

then (from (3-33) and (3-34))

$$\lambda_2 = 65° \ 27' \ 39''.787 \quad W$$

$$= -65° \ 27' \ 39''.787 \ ,$$

and finally (from (3-35) and (3-36))

$$\alpha_{21} = 225°00' \ 54''.89 \ .$$

At this point the approximate values ϕ^a, λ^a are tested against the values above to see if they are within 1 arc second of the final coordinates as determined by Puissant's formulae. In this example

$$\phi_2{}^a - \phi_2 = 0''.0872 \ ,$$

and

$$\lambda_2{}^a - \lambda_2 = 0''.1079 \ .$$

This indicates that there is no need to repeat the observation reduction process and the error propagation assumptions will be valid (see section 3.2.8). If the magnitude of either coordinate difference had been greater than 1".0, the ϕ_2, λ_2 solved for using Puissant's formula would have become $\phi_2{}^a$, $\lambda_2{}^a$ and the whole process from where $\phi_2{}^a$, $\lambda_2{}^a$ were first computed would have to be repeated.

Turning to the direct problem error propagation (section (3.6.1)) the Jacobian matrix, B_2 (equation (3-67)), of the direct error propagation is

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1.0146 \times 10^{-4} & -2.7717 \times 10^{-4} & 1.1100 \times 10^{-7} \\ 2.1732 \times 10^{-4} & 9.9985 \times 10^{-1} & 4.0553 \times 10^{-4} & 1.6248 \times 10^{-7} \end{bmatrix} .$$

Using equation (3-68) the covariance matrix, C_4, of points 1 and 2 (converted to arcsec^2) is

$$C_4 = \begin{bmatrix} 1.000 \times 10^{-4} & -8.000 \times 10^{-8} & 1.000 \times 10^{-4} & -5.826 \times 10^{-8} \\ -8.000 \times 10^{-8} & 1.000 \times 10^{-4} & -6.985 \times 10^{-8} & 9.999 \times 10^{-5} \\ 1.000 \times 10^{-4} & -6.985 \times 10^{-8} & 1.024 \times 10^{-4} & -2.186 \times 10^{-6} \\ -5.826 \times 10^{-8} & 9.999 \times 10^{-5} & -2.186 \times 10^{-6} & 1.051 \times 10^{-4} \end{bmatrix} .$$

3.8.2 Inverse Problem

In the inverse problem the coordinates of the two points 1 and 2 are provided along with the corresponding variance covariance matrix. In this example, the coordinates are those determined in the direct problem (section 3.8.1)

$$\phi_1 = 47° \; 03' \; 24\overset{''}{.}644 ,$$

$$\lambda_1 = 65° \; 29' \; 3\overset{''}{.}453 \; W$$

$$= -65° \; 29' \; 3\overset{''}{.}453 ,$$

$$\phi_2 = 47° \; 04' \; 21\overset{''}{.}801 ,$$

$$\lambda_2 = 65° \; 27' \; 39\overset{''}{.}787 \; W$$

$$= -65° \; 27' \; 39\overset{''}{.}787 ,$$

and the covariance matrix, C_4, (section 3.8.1) is given in units of rad^2.

Using Puissant's inverse formulae the solution of the inverse problem on the reference ellipsoid is executed. After the first iteration the azimuth difference is

$$\left|(\alpha_{12})_1 - (\alpha_{12})_2\right| = 7.42 \times 10^{-5} \text{ rad}$$

$$= 15''\!.313 \quad .$$

Upon completeion of the second iteration

$$\left|(\alpha_{12})_2 - (\alpha_{12})_3\right| < 1 \times 10^{-9} \text{ rad}$$

$$< 2 \times 10^{-4} \text{ arcsec} \quad .$$

This indicates that the stopping criteria has been met and from the final iteration (equations (3-42) and (3-43)) the geodetic azimuth and distance are

$$\alpha_{12} = 44° \ 59' \ 53''\!.64$$

and

$$S_{12} = 2\ 496.488 \ \text{ m} \quad .$$

Finally (from equation (3-35) and (3-37))

$$\alpha_{21} = 225°00' \ 54''\!.892 \quad .$$

The Jacobian of transformation, B_3, is (using equation (3-72))

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1804.090 & -1232.308 & -1804.631 & 1232.308 \\ 1804.090 & -1233.041 & -1804.631 & 1233.041 \end{bmatrix} .$$

Using equation (3-73) the covariance matrix C_5 (converted to arcsec^2) is

$$
C_5 = \left[\begin{array}{cccc|cc}
 & & & & 2.703\times10^{-5} & -2.698\times10^{-5} \\
 & & C_4 & & -3.658\times10^{-5} & -3.659\times10^{-5} \\
 & & & & -6.956\times10^{-3} & -6.958\times10^{-3} \\
 & & & & 1.010\times10^{-2} & 1.010\times10^{-2} \\
\hline
2.703\times10^{-5} & -3.658\times10^{-5} & -6.956\times10^{-3} & 1.010\times10^{-2} & 25.00 & 25.00 \\
-2.698\times10^{-5} & -3.659\times10^{-5} & -6.958\times10^{-3} & 1.010\times10^{-2} & 25.00 & 25.01
\end{array}\right]
$$

If error propagation for the computed distance is required, the Jacobian of transformation B_4 is (using equation (3-77))

$$
B_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-9.009\times10^{6} & 0 & 9.009\times10^{6} & 0 & 1248.538 & 1248.538
\end{bmatrix} .
$$

The covariance matrix C_6 is (using equation (3-78) and converting elements to arcsec.m and arcsec^2)

$$
C_6 = \begin{bmatrix}
25.00 & 25.00 & 6.175\times10^{-6} \\
25.00 & 25.01 & 2.768\times10^{-5} \\
6.175\times10^{-6} & 2.768\times10^{-5} & 8.759\times10^{-4}
\end{bmatrix} .
$$

3.9 Prince Edward Island Numerical Example

3.9.1 Direct Problem

The information given here for the solution of the direct problem on the reference ellipsoid and its associated error propagation, is identical to that used for the numerical example for the three dimensional case in section 2.5.

The coordinates of point 1 are

$$\phi_1 = 46° \; 42' \; 28''\!.147 \; ,$$

$$\lambda_1 = 64° \; 29' \; 34''\!.014 \quad W$$

$$= -64° \; 29' \; 34''\!.014 \; ,$$

and its covariance matrix is

$$C_{\phi_i \lambda_i} = \begin{bmatrix} 2.3504 \times 10^{-15} & -1.8804 \times 10^{-18} \\ -1.8804 \times 10^{-18} & 2.3504 \times 10^{-15} \end{bmatrix} ,$$

in units of

$$\begin{bmatrix} rad^2 & rad^2 \\ rad^2 & rad^2 \end{bmatrix} .$$

The deflection of the vertical components for point 1 are

$$\xi_1 = 4''.0 \quad ,$$

and

$$\eta_1 = 6''.0 \quad .$$

The observations are

$$r_{12} = 2\ 500.00 \text{ m} \quad ,$$

$$A_{12} = 135°\ 00'\ 0''.00 \quad ,$$

and

$$Z_{12} = 87°\ 00'\ 0''.00 \quad .$$

The variance of the astronomic azimuth is

$$\sigma_{A_{ij}}^{2} = 25''.0 \text{ arcsec}^2$$
$$= 5.876 \times 10^{-10} \text{ rad}^2 \quad .$$

This value is taken to be the variance of the geodetic azimuth

$$\sigma_{\alpha_{ij}}^{2} = \sigma_{A_{ij}}^{2}$$

The ellipsoid heights of the two points are

$$h_1 = 100.00 \text{ m} \quad ,$$

$$h_2 = 231.31 \text{ m} \quad .$$

The covariance matrix C_1 of the spatial distance and heights is given by

$$C_1 = \begin{bmatrix} 7.840 \times 10^{-10} & 0 & 0 \\ 0 & 4.0 & 4.0 \\ 0 & 4.0 & 4.032 \end{bmatrix} ,$$

in units of

$$\begin{bmatrix} m^2 & & \\ & m^2 & m^2 \\ & m^2 & m^2 \end{bmatrix} .$$

The approximate coordinates for point 2 (equations (3-9) and (3-10)) are

$$\phi_2^a = 46° \;\; 41' \;\; 30''.899 \quad ,$$

$$\lambda_2^a = 64° \; 28' \; 10''.807 \quad W$$

$$= -64° \; 28' \; 10''.807 \; .$$

The corrected zenith distance (equation (3-14)) is

$$z_{ij}^e = 87°00'01''.41 \quad .$$

The first steps through the azimuth reduction give (equations(3-15), (3-16) and (3-17))

$$\alpha''_{12} = 134° \;\; 59' \;\; 53''.25 \quad .$$

The distance is now reduced using equations (3-19), (3-20) and assuming

$$\alpha_{12} \doteq \alpha_{12}''' \quad ,$$

and

$$\alpha_{21} \doteq \alpha_{12}''' + 180° \quad .$$

The reduced distance is

$$S_{12} = 2\,496.484 \text{ m.}$$

The variance of the distance is (equation (3-26))

$$\sigma_{S_{12}}^2 = 8.761 \times 10^{-4} \text{ m}^2 \; .$$

The final correction to the azimuth (equation (3-18)) yields the geodetic azimuth

$$\alpha_{12} = 134° \; 59' \; 53''.25 \quad .$$

The covariance matrix C_3 (equation (3-64)) is given by

$$C_3 = \begin{bmatrix} 2.3504 \times 10^{-15} & -1.8804 \times 10^{-18} & 0 & 0 \\ -1.8804 \times 10^{-18} & 2.3504 \times 10^{-15} & 0 & 0 \\ 0 & 0 & 5.876 \times 10^{-10} & 0 \\ 0 & 0 & 0 & 8.761 \times 10^{-4} \end{bmatrix},$$

in units of

$$\begin{bmatrix} rad^2 & rad^2 & & \\ rad^2 & rad^2 & & \\ & & rad^2 & \\ & & & m^2 \end{bmatrix}.$$

The direct problem solution using the reduced quantities (S_{12} and α_{12}) is done using Puissant's formulae. After the first iteration the difference in the $\Delta\phi$ term is

$$|\Delta\phi_1 - \Delta\phi_2| = 0\overset{''}{.}18270$$

$$= 8.86 \times 10^{-7} \text{ rad.}$$

After the second iteration

$$|\Delta\phi_2 - \Delta\phi_3| < 2 \times 10^{-4} \text{ arcsec}$$

$$< 1 \times 10^{-9} \text{ rad.}$$

This indicates that the stopping criteria has been met and from the final iteration of equation (3-31)

$$\Delta\phi = -57\overset{''}{.}174$$

Now (from equation (3-32))

$$\phi_2 = 46° \; 41' \; 30\overset{''}{.}973$$

and (from (3-33) and (3-34))

$$\lambda_2 = 64° \ 28' \ 10''.933 \ \ W$$

$$= -64° \ 28' \ 10''.933 \ ,$$

and finally (from (3-35) and (3-36))

$$\alpha_{21} = 315°00' \ 53''.71 \ .$$

At this point the approximate coordinate values ϕ^a, λ^a, are tested against the values above to see if they are within 1 arc second of the final coordinates as determined by Puissant's formulae. In this example

$$\phi_a^2 - \phi_2 = -0''.074 \quad ,$$

$$\lambda_2^a - \lambda_2 = 0''.126 \quad .$$

This indicates that there is no need to repeat the observation reduction process and the error propagation assumptions will be valid. (see section 3.2.8). If the magnitude of either coordinate difference had been greater than 1''.0, the $\phi_2{}^a$, $\lambda_2{}^a$ solved for using Puissant's formula would have become $\phi_2{}^a$, $\lambda_2{}^a$, and the whole process from where $\phi_2{}^a$,$\lambda_2{}^a$ were first computed would have to be repeated.

Turning to the direct problem error propagation (Section (3.6.1)) the Jacobian matrix, B_2 (equation (3-67)), of the direct error propagation is

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1.0084 \times 10^{-4} & -2.7713 \times 10^{-4} & -1.1103 \times 10^{-7} \\ 2.1300 \times 10^{-4} & 1.0002 & -4.0288 \times 10^{-4} & 1.6134 \times 10^{-7} \end{bmatrix}$$

Using equation (3-68) the covariance matrix C_4, of points 1 and 2 (converted to arcsec^2) is

$$C_4 = \begin{bmatrix} 1.000 \times 10^{-4} & -8.000 \times 10^{-8} & 1.000 \times 10^{-4} & -5.871 \times 10^{-8} \\ -8.000 \times 10^{-8} & 1.000 \times 10^{-4} & -6.992 \times 10^{-8} & 1.000 \times 10^{-4} \\ 1.000 \times 10^{-4} & -6.992 \times 10^{-8} & 1.024 \times 10^{-4} & 2.075 \times 10^{-6} \\ -5.871 \times 10^{-8} & 1.000 \times 10^{-4} & 2.075 \times 10^{-6} & 1.051 \times 10^{-4} \end{bmatrix}$$

3.9.2 Inverse Problem

In the inverse problem the coordinates of the two points 1 and 2 are provided along with the corresponding variance covariance matrix. In this example, the coordinates are those determined in the direct problem (section 3.8.1)

$$\phi_1 = 46° \; 42' \; 28''.147 \; ,$$

$$\lambda_1 = 64° \; 29' \; 34''.014 \; \text{W}$$
$$-64° \; 29' \; 34''.014 \; ,$$
$$\phi_2 = 46° \; 41' \; 30''.973 \; ,$$
$$\lambda_2 = 64° \; 28' \; 10''.933 \; \text{W}$$
$$= -64° \; 28' \; 10''.933 \; ,$$

and the covariance matrix, C_4 (section 3.8.1) is given in units of rad^2 .

Using Puissant's inverse formulae the solution of the inverse problem on the reference ellipsoid is executed. After the first iteration the azimuth difference is

$$|(\alpha_{12})_1 - (\alpha_{12})_2| = 7.33 \times 10^{-5} \text{ rad}$$
$$= 15''.116 \; .$$

Upon completion of the second iteration

$$|(\alpha_{12})_2 - (\alpha_{12})_3 < 1 \times 10^{-9} \text{ rad}$$
$$< 2 \times 10^{-4} \text{ arc sec} \; .$$

This indicates that the stopping criteria has been met and from the final iteration (equations (3-42) and (3-43)) the geodetic azimuth and distance are

$$\alpha_{12} = 134° \; 59' \; 53''.25 \quad ,$$

and

$$S_{12} = 2\,496.484 \text{ m} \quad .$$

Finally (from equation (3-35) and (3-37))

$$\alpha_{21} = 315°00' \; 53''.71 \quad .$$

The Jacobian of transformation, B_3, is (using equation (3-72))

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1804.096 & 1241.714 & -1803.562 & -1241.714 \\ 1804.096 & 1240.986 & -1803.562 & -1240.986 \end{bmatrix} .$$

Using equation (3-73) the covariance matrix C_5 (converted to arcsec^2) is

$$C_5 = \left[\begin{array}{cccc|cc} & & & & 2.671\times10^{-5} & 2.676\times10^{-5} \\ & C_4 & & & -3.641\times10^{-5} & -3.640\times10^{-5} \\ & & & & -6.902\times10^{-3} & -6.900\times10^{-3} \\ & & & & -1.011\times10^{-2} & -1.011\times10^{-2} \\ \hline 2.671\times10^{-5} & -3.641\times10^{-5} & -6.902\times10^{-3} & -1.011\times10^{-2} & 25.00 & 25.00 \\ 2.676\times10^{-5} & -3.640\times10^{-5} & -6.900\times10^{-3} & -1.011\times10^{-2} & 25.00 & 24.99 \end{array}\right]$$

If error propagation for the computed distance is required, the Jacobian of transformation B_4 is (using equation (3-77))

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 9.007 \times 10^{6} & 0 & -9.006 \times 10^{6} & 0 & -1247.958 & -1247.958 \end{bmatrix} .$$

The covariance matrix C_6 is (using equation (3-78) and converting elements to arcsec.m and arcsec^2)

$$C_6 = \begin{bmatrix} 25.00 & 25.00 & 5.904 \times 10^{-6} \\ 25.00 & 24.99 & 2.712 \times 10^{-5} \\ 5.904 \times 10^{-6} & 2.712 \times 10^{-5} & 8.762 \times 10^{-4} \end{bmatrix} .$$

3.10 Nova Scotia Numerical Example

3.10.1 Direct Problem

The information given here for the solution of the direct problem on the reference ellipsoid and its associated error propagation, is identical to that used for the numerical example for the three dimensional case Section 2.6.

The coordinates of point 1 are

$$\phi_1 = 44° \ 39' \ 3''.123 \quad ,$$

$$\lambda_1 = 63° \ 00' \ 0''.000 \ \ W$$

$$= -63° 00' \ 0''.000 \ ,$$

and its covariance matrix is

$$C_{\phi_i\lambda_i} = \begin{bmatrix} 2.3404 \times 10^{-15} & -1.8804 \times 10^{-18} \\ -1.8804 \times 10^{-18} & 2.3504 \times 10^{-15} \end{bmatrix} ,$$

in units of

$$\begin{bmatrix} \text{rad}^2 & \text{rad}^2 \\ \text{rad}^2 & \text{rad}^2 \end{bmatrix} .$$

The deflection of the vertical components for point 1 are

$$\xi_1 = 4''0 \quad ,$$

and

$$\eta_1 = 6''0 \quad .$$

The observations are

$$r_{12} = 2\ 500.00 \text{ m} \; ,$$

$$A_{12} = 225° \ 0' \ 0''00 \; ,$$

and

$$Z_{12} = 87° \ 0' \ 0''00 \; .$$

The variance of the astronomic azimuth is

$$\sigma^2_{A_{ij}} = 25''0 \text{ arcsec}^2$$

$$= 5.876 \times 10^{-10} \ \text{rad}^2 \; .$$

This value is taken to be the variance of the geodetic azimuth

$$\sigma^2_{\alpha_{ij}} = \sigma^2_{A_{ij}} \; .$$

The ellipsoid heights of the two points are

$$h_1 = 100.000 \text{ m} \quad ,$$

$$h_2 = 231.414 \text{ m} \quad .$$

The covariance matrix C_1 is given by (spatial distance and heights)

$$C_1 = \begin{bmatrix} 7.840 \times 10^{-4} & 0 & 0 \\ 0 & 4.0 & 4.0 \\ 0 & 4.0 & 4.033 \end{bmatrix} \; ,$$

in units of

$$\begin{bmatrix} m^2 & & \\ & m^2 & m^2 \\ & m^2 & m^2 \end{bmatrix} \; .$$

The approximate coordinates for point 2 (from equations (3-9) and (3-10)) are

$$\phi_2^a = 44° \ 38' 05''854$$

and

$$\lambda_2^a = 63°01'\ 20''.214\ \ W$$

$$= -63°\ 01'\ 20''.214\ .$$

The corrected zenith distance (equation (3-14)) is

$$z_{ij} = 86°\ 59'\ 52''.93\ .$$

The first steps through the azimuth reduction give (equations (3-15), (3-16) and (3-17))

$$\alpha_{12}''' = 224°\ 59'\ 54''.00\ .$$

The distance is now reduced using equations (3-19), (3-20), and assuming

$$\alpha_{12} \cong \alpha_{12}''' \ ,$$

and

$$\alpha_{21} = \alpha_{12}''' + 180°\ .$$

The reduced distance is

$$S_{12} = 2\ 496.479\ \text{m}\ .$$

The variance of the distance is (using equation (3-26))

$$\sigma_{S_{12}}^2 = 8.762 \times 10^{-4}\ \text{m}^2\ .$$

The final correction to the azimuth (equation (3-18)) yields the geodetic azimuth

$$\alpha_{12} = 224°\ 59'\ 54''.011\ .$$

The covariance matrix C_3 (equation (3-64)) is given by

$$C_3 = \begin{bmatrix} 2.3504 \times 10^{-15} & -1.8804 \times 10^{-18} & & \\ -1.8804 \times 10^{-18} & 2.3504 \times 10^{-15} & & \\ & & 5.876 \times 10^{-10} & \\ & & & 8.762 \times 10^{-4} \end{bmatrix}$$

in units of

$$\begin{bmatrix} rad^2 & rad^2 & & \\ rad^2 & rad^2 & & \\ & & rad^2 & \\ & & & m^2 \end{bmatrix} .$$

The direct problem solution using the reduced quantities (S_{12} and α_{12}) is done using Puissant's formulae. After the first iteration the difference in the $\Delta\phi$ terms is

$$|\Delta\phi_1 - \Delta\phi_2| = 0''.19667$$
$$= 9.53 \times 10^{-7} \text{ rad} .$$

After the second iteration

$$|\Delta\phi_2 - \Delta\phi_3| < 2 \times 10^{-4} \text{ arcsec}$$
$$< 1 \times 10^{-9} \text{ rad} .$$

This indicates that the stopping criteria has been met and from the final iteration of equation (3-31)

$$\Delta\phi = 57''.198 .$$

Now (from equation (3-32))

$$\phi_2 = 44° \; 38' \; 5''.925 ,$$

then (from (3-33) and (3-34))

$$\lambda_2 = 63° \; 01' \; 20''.088 \text{ W}$$
$$= -63° \; 01' \; 20''.088$$

and finally (from (3-35) and (3-36))

$$\alpha_{21} = 44° \; 58' \; 57''.73 .$$

At this point the approximate coordinate values ϕ^a, λ^a, are tested against the values above to see if they are within 1 arc second of the final coordinates as determined by Puissant's formulae. In this example

$$\phi_2{}^a - \phi_2 = -0''.071 \quad ,$$

and

$$\lambda_2{}^a - \lambda_2 = -0''.126 \quad .$$

This indicates that there is no need to repeat the observation reduction process and the error propagation assumptions will be valid (see section 3.2.8). If the magnitude of either coordinate difference had been greater than 1".0, the ϕ_2, λ_2 solved for using Puissant's formula would have become $\phi_2{}^a$, $\lambda_2{}^a$ and the whole process from where $\phi_2{}^a$, $\lambda_2{}^a$ were first computed would have to be repeated.

Turning to the direct problem error propagation (section (3.6.1)) the Jacobian matrix, B_2 (equation (3-67)), of the direct error propagation is

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -9.740 \times 10^{-5} & 2.772 \times 10^{-4} & -1.111 \times 10^{-7} \\ 1.911 \times 10^{-4} & 1.000 & -3.884 \times 10^{-4} & -1.555 \times 10^{-7} \end{bmatrix}$$

Using equation (3-68) the covariance matrix, C_4, of points 1 and 2 (converted to arcsec^2) is

$$C_4 = \begin{bmatrix} 1.000 \times 10^{-4} & -8.000 \times 10^{-8} & 1.000 \times 10^{-4} & -9.912 \times 10^{-8} \\ -8.000 \times 10^{-8} & 1.000 \times 10^{-4} & -8.974 \times 10^{-8} & 1.000 \times 10^{-4} \\ 1.000 \times 10^{-4} & -8.974 \times 10^{-8} & 1.024 \times 10^{-4} & -2.157 \times 10^{-6} \\ -9.912 \times 10^{-8} & 1.000 \times 10^{-4} & -2.157 \times 10^{-6} & 1.047 \times 10^{-4} \end{bmatrix} .$$

3.10.2 Inverse Problem

In the inverse problem the coordinates of the two points 1 and 2 are provided along with the corresponding variance covariance matrix. In this example, the coordinates are those determined in the direct problem (section 3.8.1)

$$\phi_1 = 44°\ 39'\ 3''.123\ ,$$

$$\lambda_1 = 63°\ 0'\ 0''.000\ \ W$$
$$= -63°00'\ 0''.000\ ,$$

$$\phi_2 = 44°\ 38'\ 5''.925\ ,$$

$$\lambda_2 = 63°01'\ 20''.088$$
$$= -63°01'\ 20''.088\ ,$$

and the covariance matrix, C_4, (section 3.8.1) is given in units of rad^2 .

Using Puissant's inverse formulae the solution of the inverse problem on the reference ellipsoid is executed. After the first iteration the azimuth difference is

$$|(\alpha_{12})_1 - (\alpha_{12})_2| = 6.82 \times 10^{-5}\ \mathrm{rad}\ .$$
$$= 14''.068\ .$$

Upon completion of the second iteration

$$|(\alpha_{12})_2 - (\alpha_{12})_3| < 1 \times 10^{-9}\ \mathrm{rad}$$
$$< 2 \times 10^{-4}\ \mathrm{arcsec}\ .$$

This indicates that the stepping criteria has been met and from the final iteration (from equations (3-42) and (3-43)) the geodetic azimuth and distance are

$$\alpha_{12} = 224° \; 59' \; 54''.01 \;,$$

and

$$S_{12} = 2\,496.479 \text{ m}.$$

Finally (from equation (3-35) and (3-37))

$$\alpha_{21} = 44° \; 58' \; 57''.73 \;.$$

The Jacobian of transformation, B_3, is (using equation (3-72))

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1803.329 & 1288.095 & 1802.832 & -1288.095 \\ -1803.329 & 1287.392 & 1802.832 & -1287.392 \end{bmatrix} .$$

Using equation (3-73) the covariance matrix C_5 (converted to arcsec^2) is

$$C_6 = \left[\begin{array}{cccc|cc} & & & & -2.482\times10^{-5} & -2.486\times10^{-5} \\ & & & & -3.512\times10^{-5} & -3.511\times10^{-5} \\ & & C_4 & & 6.906\times10^{-3} & 6.904\times10^{-3} \\ & & & & -9.746\times10^{-3} & -9.743\times10^{-3} \\ \hline -2.482\times10^{-5} & -3.512\times10^{-5} & 6.906\times10^{-3} & -9.746\times10^{-3} & 25.00 & 25.00 \\ -2.486\times10^{-5} & -3.511\times10^{-5} & 6.904\times10^{-3} & -9.743\times10^{-3} & 25.00 & 24.99 \end{array}\right]$$

If error propagation for the computed distance is required, the Jacobian of transformation B_4 is (using equation (3-77))

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -9.003 \times 10^6 & 0 & 9.003 \times 10^6 & 0 & -1247.826 & -1247.826 \end{bmatrix}$$

The covariance matrix C_6 is (using equation (3-78) and converting elements to arcsec.m and arcsec^2)

$$C_6 = \begin{bmatrix} 25.00 & 25.00 & 5.477 \times 10^{-6} \\ 25.00 & 24.99 & 2.522 \times 10^{-5} \\ 5.477 \times 10^{-6} & 2.522 \times 10^{-5} & 8.764 \times 10^{-4} \end{bmatrix}.$$

4. COMPUTATIONS ON A CONFORMAL MAPPING PLANE

In Chapter 2, the solution of the direct and inverse problems in the 3-D environment were given. In Chapter 3 the reduction of observed quantities to the reference ellipsoid were treated for the solution of the direct and inverse problems on that surface. This chapter presents the completion of the process with the further reduction of the ellipsoidal quantities to the conformal mapping plane and the solution of the direct and inverse problems on that surface.

4.1 Notation

d_{ij} ≡ direction from point i to point j

$(T-t)_{ij}$ ≡ arc to chord correction for line from point i to point j

X_i, Y_i ≡ Mapped coordinates of point i

χ_i, Λ_i ≡ spherical coordinates of point i

$\Delta\Lambda_i$ ≡ spherical longitude of point to be mapped minus the spherical longitude of the origin

$$\Delta\Lambda_i = \Lambda_i - \Lambda_o \tag{4-1}$$

a,b ≡ semi-major and semi minor axes respectively of the Clarke 1866 reference ellipsoid,

$$a = 6\ 378\ 206.4 \text{ m} \quad ,$$

$$b = 6\ 356\ 583.8 \text{ m} \quad .$$

α_{ij} ≡ geodetic azimuth from point i to point j

B_{ijk} ≡ horizontal angle from point i to point k with instrument at point j

γ_i ≡ meridian convergence at point i

S_{ij} ≡ ellipsoid distance from point i to point j

ℓ_{ij} ≡ distance from point i to point j

S^*_{ij} ≡ projected geodesic length

$\phi_o, \lambda_o, \chi_o, \Lambda_o, X_o, Y_o$ ≡ coordinates of the origin of the projections

k_o ≡ scale factor at the origin of the double stereographic projection or scale factor at the central meridian of the 3° Transverse Mercator

λ_{CM} ≡ geodetic longitude of central meridian

R ≡ radius of the conformal sphere ($R = (MN)^{1/2}$ evaluated at ϕ_o)

X_i^a, Y_i^a ≡ approximate mapped coordinates of point i

$(\ )^e$ ≡ ellipsoidal quantity

$(\ )^a$ ≡ approximate quantity

N_i ≡ Prime vertical radius of curvature at point i

M_i ≡ Meridian radius of curvature at point i

$\Delta\lambda_i$ ≡ geodetic longitude at point minus geodetic longitude at central meridian

$$\Delta\lambda_i = \lambda_i - \lambda_{CM} \tag{4-2}$$

t_{ij} ≡ grid azimuth from point i to point j.

T_{ij} ≡ grid azimuth of the projected geodesic from point i to point j

$\bar{k}_{ij}$ ≡ the line scale factor

4.2 Reduction of Observations

The quantities needed in the reduction of observations from the reference ellipsoid to the mapping plane are the (T-t) correction (sometimes called the "arc to chord" correction), the meridian convergence, and the line scale factor [Bomford, 1971; Krakiwsky, 1973]. Specific formulae for these quantities must be derived for each projection and are given in this chapter.

4.2.1 Reduction of Horizontal Directions (Ellipsoid to the Mapping Plane).

$$d_{ij} = d^e_{ij} - (T-t)_{ij} . \tag{4-3}$$

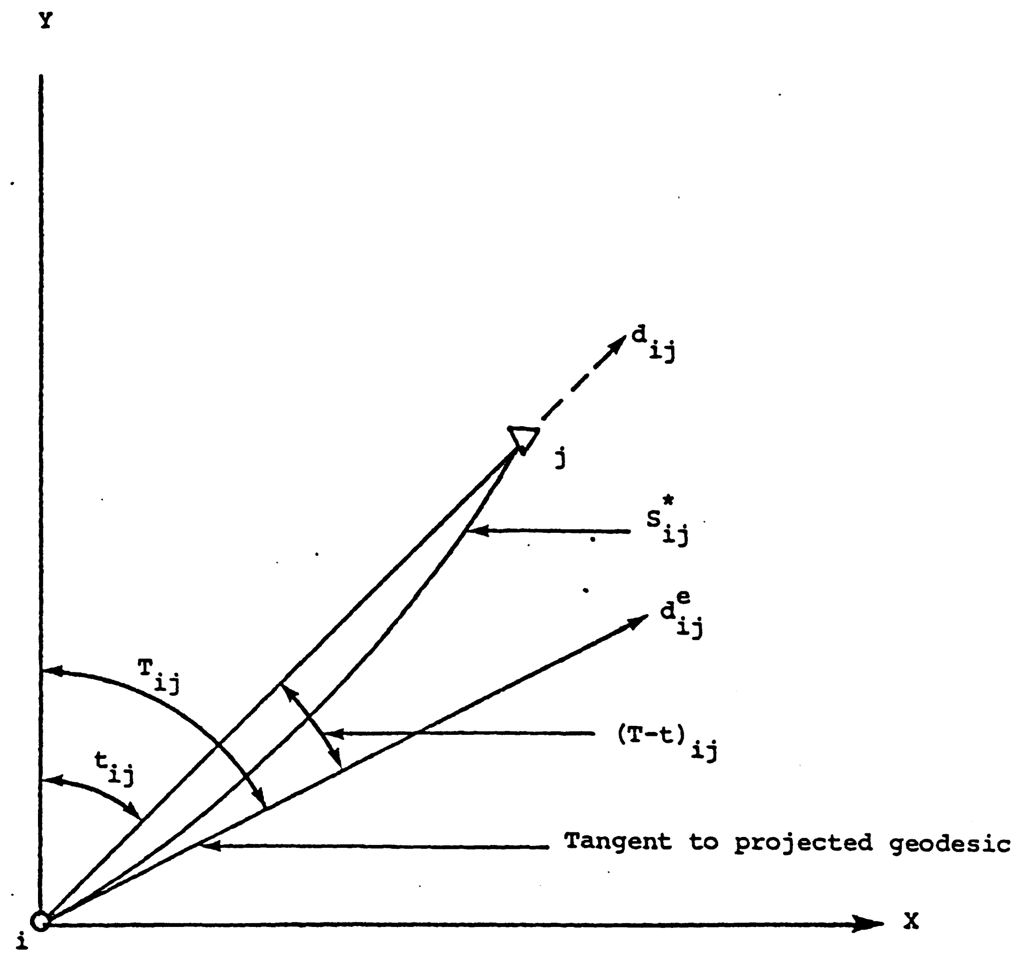


Figure 4-1

REDUCTION OF HORIZONTAL DIRECTIONS

4.2.2 Reduction of Horizontal Angles (Ellipsoid to the Mapping Plane)

From Figure 4-2

$$B_{ijk} = \beta^{e}_{ijk} + (T-t)_{ij} - (T-t)_{ik} . \tag{4-4}$$

4.2.3 Reduction of Azimuth (Ellipsoid to the Mapping Plane)

From Figure 4-3

$$t_{ij} = \alpha_{ij} - \gamma_i - (T-t)_{ij} . \tag{4-5}$$

4.2.4 Reduction of Distances (Ellipsoid to the Mapping Plane)

The distance on the plane from a point i to a point j is computed by;

$$\ell_{ij} = \bar{k}_{ij} S_{ij}, \tag{4-6}$$

where $\bar{k}_{ij}$ is the line scale factor of the line i to j.

4.3 New Brunswick Stereographic Double Projection

4.3.1 Direct Problem

The direct problem on the N.B. Stereographic projection plane is stated as: given the grid coordinates X_i, Y_i of point i and the astronomic azimuth A_{ij} and spatial distance r_{ij}, to a point j, compute the grid coordinates X_j, Y_j of the point j. The solution is as follows.

First the azimuth, A_{ij}, and the distance, r_{ij}, must be reduced to the geodetic azimuth, α_{ij}, and elliposid distance, S_{ij}, as described in Chapter 3. These quantities must then be reduced to the conformal mapping plane. Beginning with the azimuth, the meridian convergence and (T-t) corrections must be applied. Meridian convergence is given by [Thomson et al., 1977]

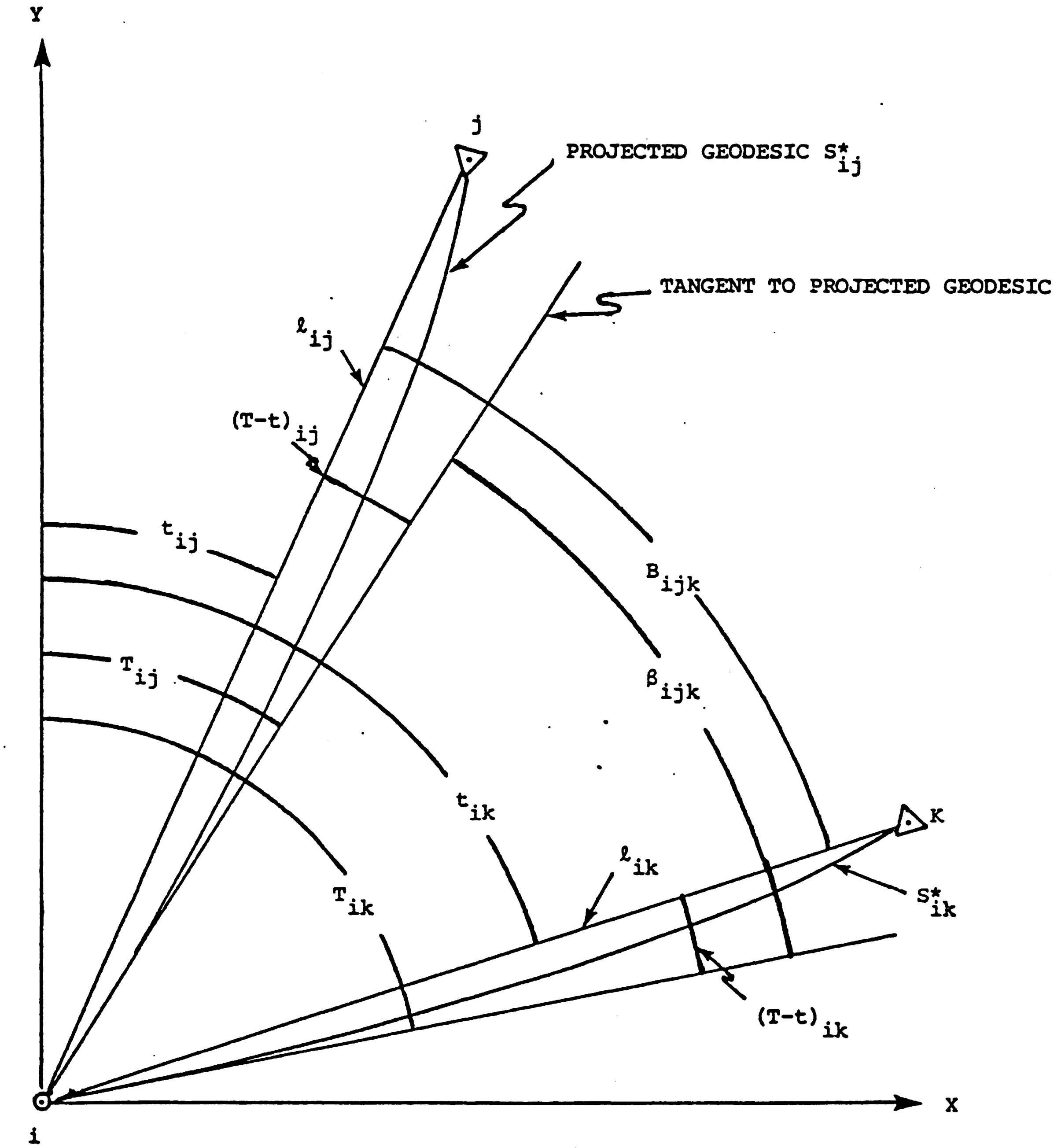


Figure 4-2

REDUCTION OF HORIZONTAL ANGLES

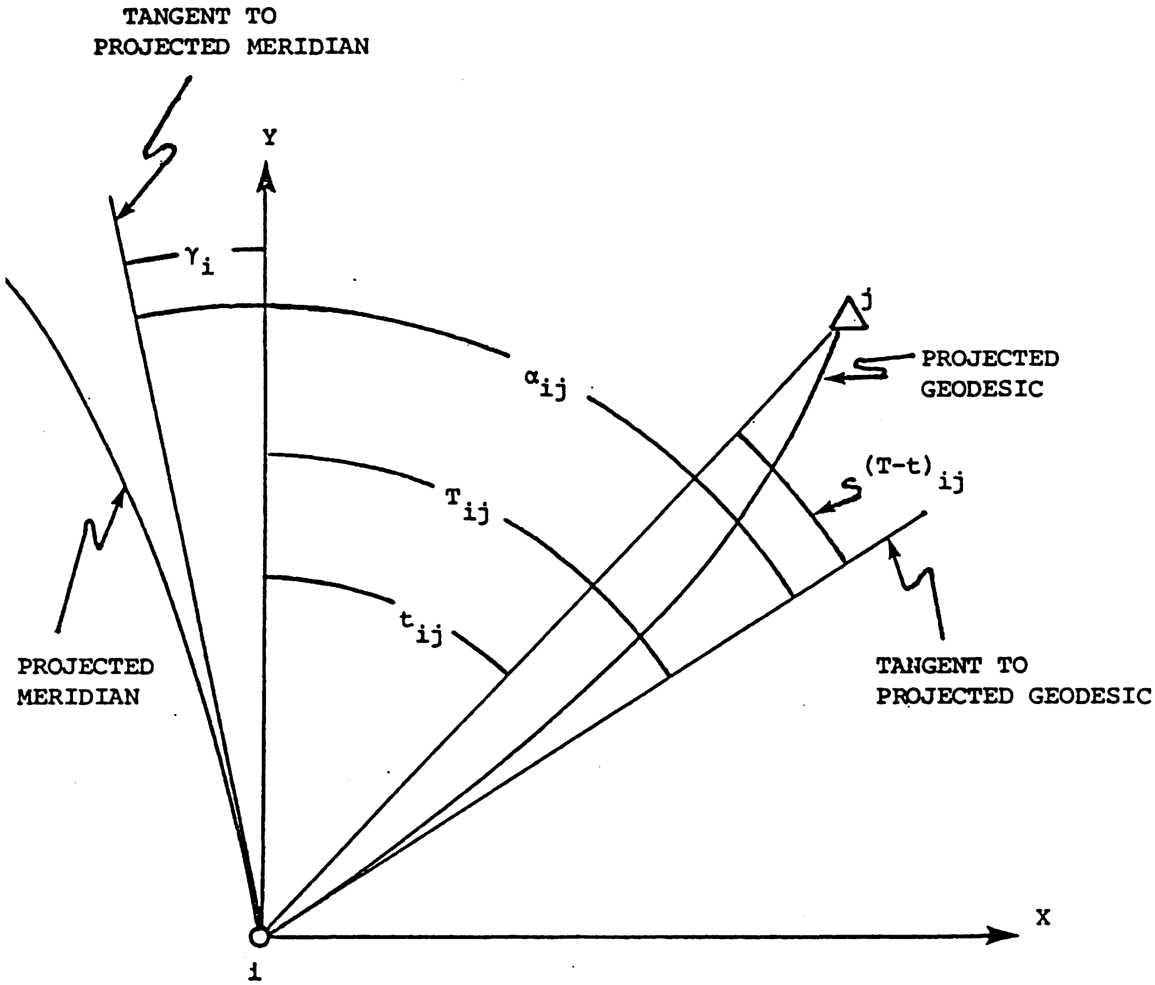


Figure 4-3

REDUCTION OF AZIMUTH

$$\gamma_i = \tan^{-1} \left[\frac{\sin \Delta\Lambda_i \; (\sin \chi_i + \sin \chi_o)}{\cos \chi_i \, \cos \chi_o + (1 + \sin \chi_i \, \sin \chi_o) \cos \Delta\Lambda_i} \right] . \tag{4-7}$$

The meridian convergence obtained from (4-7) is then applied to the geodetic azimuth yielding the grid azimuth of the projected geodesic T_{ij}, namely

$$T_{ij} = \alpha_{ij} - \gamma_i \; . \tag{4-8}$$

T_{ij} is used in the computation of the approximate coordinates X_j^a, Y_j^a for the second point, yielding

$$X_j^{\,a} = X_i + S_{ij} \sin T_{ij} \quad , \tag{4-9}$$

and

$$Y_j^{\,a} = Y_i + S_{ij} \cos T_{ij} \quad . \tag{4-10}$$

The $(T-t)_{ij}$ correction is now written as [Thomson et al., 1977]

$$(T-t)_{ij} = \tan^{-1} \left[\frac{\Delta X_j^a \, \Delta Y_i - \Delta X_i \, \Delta Y_j^a}{\Delta X_i \, \Delta X_j^{\,a} + \Delta Y_i \, \Delta Y_j^{\,a} + (2k_o R)^2} \right] , \tag{4-11}$$

in which

$$\begin{aligned} \Delta X_i &= X_i - X_o \quad , \\ \Delta Y_i &= Y_i - Y_o \quad , \\ \Delta X_j^a &= X_j^a - X_o \quad , \end{aligned} \tag{4-12}$$

and

$$\Delta Y_j^a = Y_j^a - Y_o \; .$$

The sign on the (T-t) correction must now be determined. Referring to Figure 4-3 it can be seen that the sign on the correction varies from quadrant to quadrant. To determine the sign the following tests must be performed. If the grid azimuth t_{ij} is between the grid azimuth from point i to the origin and the grid azimuth from the origin to the point i then the sign on the correction (used in equations (4-3) and (4-5)) is positive. If the grid azimuth t_{ij} is between the grid azimuth from the origin to the point i and the grid azimuth from the point i to the origin then the sign on the correction is negative.

If the grid azimuth t_{ij} is equal to either the grid azimuth from the point i to the origin or from the origin to the point i, $(T-t) = 0$ [Thomson et al., 1977].

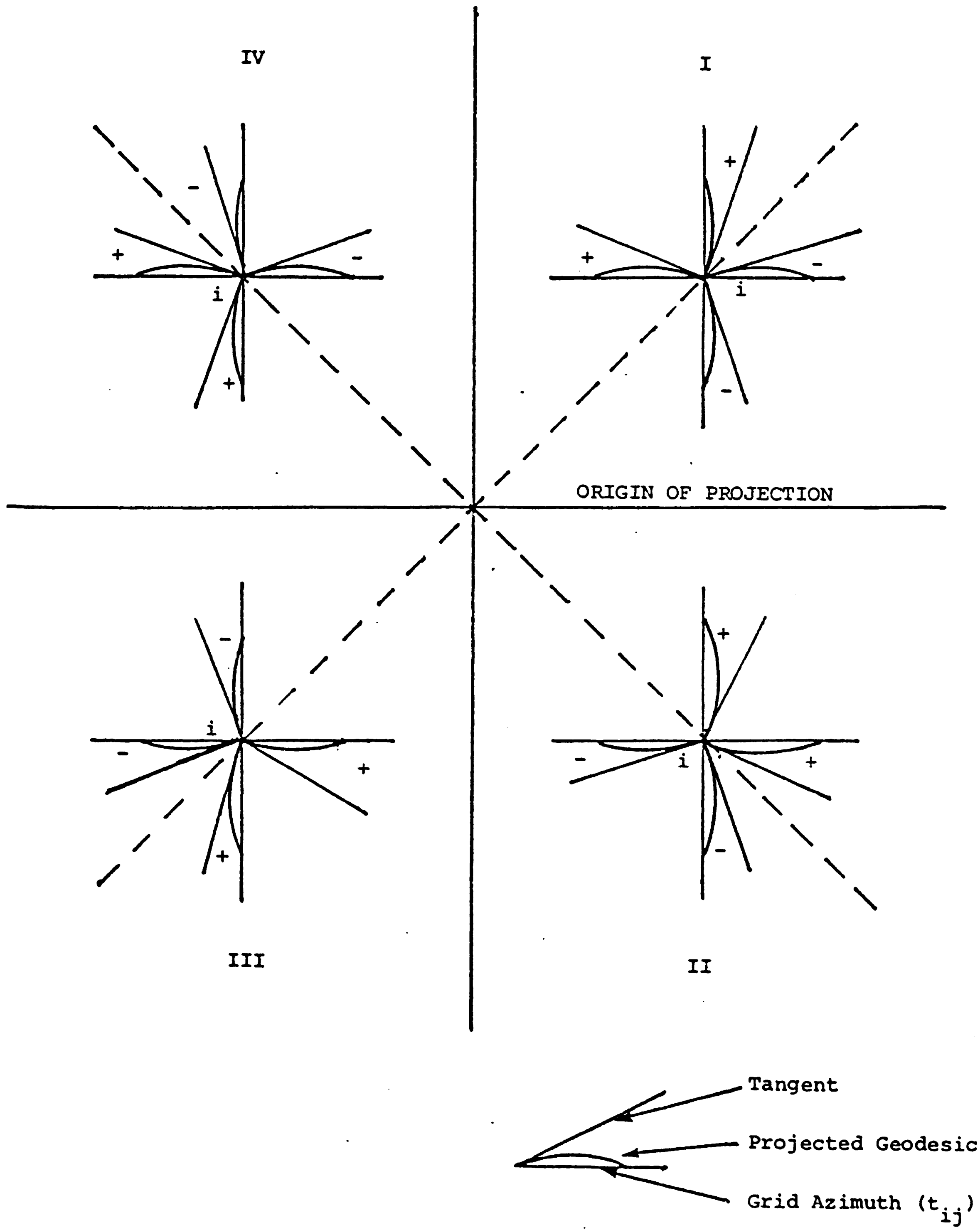


Figure 4-4

SIGN OF (T-t) CORRECTION STEREOGRAPHIC

It should be noted that (4-11) gives the $(T-t)_{ij}$ correction for the mapping of the line from the conformal sphere to the plane and is missing an ellipsoidal term. This term has been proven insignificant [Thomson et al., 1977] and can be safely neglected. The azimuth is now written as (equation (4-5)),

$$t_{ij} = \alpha_{ij} - \gamma_i - (T-t)_{ij} \quad .$$

Turning to the distance, and using the previously obtained approximate coordinates of point j the line scale factor k_{ij} is (following Simpson's rule)

$$\bar{k}_{ij} = [\frac{1}{6} (\frac{1}{k_i} + \frac{4}{k_m^a} + \frac{1}{k_j^a})]^{-1} \quad , \tag{4-13}$$

where k_i is the point scale factor at point i and is computed from [Thomson et al.,1977]

$$k_i = k_o + \frac{(X_i - X_o)^2 + (Y_i - Y_o)^2}{4\ k_o\ R^2} \quad , \tag{4-14}$$

and k_j^a is the approximate point scale factor at point j and is given by [Thomson et al.,1977],

$$k_j^a = k_o + \frac{(X_j^a - X_o)^2 + (Y_j^a - Y_o)^2}{4\ k_o\ R^2} \quad . \tag{4-15}$$

k_m is the point scale factor at the midpoint of the line ij and is given by [Thomson et al.,1977]

$$k_m^a = k_o + \frac{(X_m - X_o)^2 + (Y_m^a - Y_o)^2}{4\ k_o\ R^2} \quad , \tag{4-16}$$

where

$$X_m^a = \frac{X_i + X_j^a}{2} \quad , \tag{4-17}$$

and

$$Y_m^a = \frac{Y_i + Y_j^a}{2} \quad . \tag{4-18}$$

The point scale factor given here (equation (4-14)) accounts only for the mapping of the conformal sphere on the plane. It is accurate to 1×10^{-7} [Thomson et al., 1977] if it is used as an approximation to the scale factor for the mapping of ellipsoidal information on the conformal plane. The reduced distance is now written as(equation (4-6)),

$$\ell_{ij} = \bar{k}_{ij} \, S_{ij} \quad .$$

To complete the direct problem we have

$$X_j = X_i + \ell_{ij} \sin t_{ij} \ , \tag{4-19}$$

and

$$Y_j = Y_i + \ell_{ij} \cos t_{ij} \ . \tag{4-20}$$

4.3.2 Inverse Problem

The inverse problem on the N.B. stereographic plane is stated as: given the grid coordinates X_i, Y_i of point i and the grid coordinates X_j, Y_j of point j compute the grid and geodetic azimuths t_{ij}, t_{ji}, α_{ij} and α_{ji}, and the grid and geodetic distance ℓ_{ij} and S_{ij}.

If further reduction from the ellipsoid to the terrain is required refer to Section 3.3. The grid distance and azimuths are given respectively by

$$\ell_{ij} = [(X_j - X_i)^2 + (Y_j - Y_i)^2]^{1/2} \ , \tag{4-21}$$

and

$$t_{ij} = \tan^{-1} [\frac{X_j - X_i}{Y_j - Y_i}] \ . \tag{4-22}$$

and

$$t_{ji} = t_{ij} + 180° \quad . \qquad (4-23)$$

From equations (4-5), (4-7), and (4-11) we have that

$$\alpha_{ij} = t_{ij} + \gamma_i + (T-t)_{ij} \quad , \qquad (4-24)$$

and from equations (4-6) and (4-13)

$$S_{ij} = \frac{\ell_{ij}}{\bar{k}_{ij}} \quad . \qquad (4-25)$$

Since $(T-t)_{ji} = -(T-t)_{ij}$, then

$$\alpha_{ji} = t_{ji} + \gamma_j - (T-t)_{ij} \quad , \qquad (4-26)$$

where γ_j is computed from equation (4-7).

This completes the inverse problem.

4.4 Prince Edward Island Stereographic Double Projection

4.4.1 Direct Problem

The direct problem on the P.E.I. Stereographic projection plane is given the grid coordinates X_i, Y_i of point i and the astronomic azimuth, A_{ij} and spatial distance, r_{ij}, to a point j, compute the grid coordinates X_j, Y_j of the point j. The solution is as follows.

First the azimuth, A_{ij}, and the distance r_{ij} must be reduced to the geodetic azimuth, α_{ij}, and the ellipsoid distance, S_{ij}, as described in Chapter 3. These quantities must then be reduced to the conformal mapping plane. Beginning with the azimuth, the meridian convergence and (T-t) corrections must be applied. Meridian convergence is given by [Thomson et al. 1977]

$$\gamma_i = \tan^{-1} \left[\frac{\sin \Delta\Lambda_i \; (\sin \chi_i + \sin \chi_o)}{\cos \chi_i \cos \chi_o + (1 + \sin \chi_i \sin \chi_o) \cos \Delta\Lambda_i} \right] . \tag{4-27}$$

The meridian convergence obtained from (4-27) is then applied to the geodetic azimuth yielding the grid azimuth of the projected geodesic, T_{ij}, namely

$$T_{ij} = \alpha_{ij} - \gamma_i . \tag{4-28}$$

T_{ij} is used for the computation of approximate coordinates (X_j^a, Y_j^a) for the second point, yielding

$$X_j^a = X_i + S_{ij} \sin T_{ij} , \tag{4-29}$$

and

$$Y_j^a = Y_i + S_{ij} \cos T_{ij} . \tag{4-30}$$

The (T-t) correction is now written as [Thomson et al., 1977]

$$(T-t)_{ij} = \tan^{-1} \left[\frac{\Delta X_j^a \, \Delta Y_i - \Delta X_i \, \Delta Y_j^a}{\Delta X_i \, \Delta X_j^a + \Delta Y_i \, \Delta Y_j^a + (2k_o \, R)^2} \right] , \tag{4-31}$$

in which

$$\begin{aligned} \Delta X_i &= X_i - X_o , \\ \Delta Y_i &= Y_i - Y_o , \\ \Delta X_j^a &= X_j^a - X_o , \\ \Delta Y_j^a &= Y_j^a - Y_o . \end{aligned} \tag{4-32}$$

The sign on the (T-t) correction must now be determined. Referring to Figure 4-3 it can be seen that the sign on the correction varies from quadrant to quadrant. To determine the sign the following tests must be performed. If the grid azimuth t_{ij} is between the grid azimuth from point i to the origin and the grid azimuth from the origin to the point i then the sign on the correction (used in equations (4-3) and (4-5)) is positive. If the grid azimuth t_{ij} is between the grid azimuth from the origin to the point i and the grid azimuth from the point i to the origin then the sign on the correction is negative.

If the grid azimuth t_{ij} is equal to either the grid azimuth from the point i to the origin or from the origin to the point i, (T-t) = 0 [Thomson et al., 1977].

It should be noted that (4-31) gives the $(T-t)_{ij}$ correction for the mapping of the line from the conformal sphere to the plane and is missing an ellipsoidal term. This term has been proven insignificant [Thomson et al.,1977] and can be safely neglected. The azimuth is now written as (equation (4-5)),

$$t_{ij} = \alpha_{ij} - \gamma - (T-t)_{ij} \quad .$$

Turning to the distance and using the previously obtained approximate coordinates of point j, the line scale factor is (following Simpson's rule) [Thomson et al., 1977]

$$\bar{k}_{ij} = [\frac{1}{6} (\frac{1}{k_i} + \frac{4}{k_m^a} + \frac{1}{k_j^a})]^{-1} , \tag{4-33}$$

where k_i is the point scale factor at point i and is computed from [Thomson et al.,1977].

$$k_i = k_o + \frac{(X_i - X_o)^2 + (Y_i - Y_o)^2}{4\ k_o\ R^2} . \tag{4-34}$$

k_j^a is the approximate point scale factor at point j and is given by [Thomson et al., 1977]

$$k_j = k_o + \frac{(X_j^a - X_o)^2 + (Y_j^a - Y_o)^2}{4\ k_o\ R^2} . \tag{4-35}$$

k_m is the point scale factor computed at the midpoint of the line ij and is given by [Thomson et al., 1977]

$$k_m^a = k_o + \frac{(X_m^a - X_o)^2 + (Y_m^a - Y_o)^2}{4\ k_o\ R^2} , \tag{4-36}$$

where

$$X_m^a = \frac{X_i + X_j^a}{2} \; , \qquad (4\text{-}37)$$

and

$$Y_m^a = \frac{Y_i + Y_j^a}{2} \; . \qquad (4\text{-}38)$$

The point scale factor given here (equation (4-14)) accounts only for the mapping of the conformal sphere on the plane. It is accurate to 1×10^{-7} [Thomson et al.,1977] if it is used as an approximation to the scale factor for the mapping of the ellipsoidal information on the conformal plane. The reduced distance is now written as (equation (4-6)),

$$\ell_{ij} = \bar{k}_{ij} \; S_{ij} \; .$$

To complete the direct problem we have

$$X_j = X_i + \ell_{ij} \sin t_{ij} \; , \qquad (4\text{-}39)$$

and

$$Y_j = Y_i + \ell_{ij} \cos t_{ij} \; . \qquad (4\text{-}40)$$

4.4.2 Inverse Problem

The inverse problem on the P.E.I. stereographic projection plane is stated as: given the grid coordinates X_i, Y_i of point i and and the grid coordinates X_j, Y_j of point j compute the grid and geodetic azimuths t_{ij}, t_{ji}, α_{ij}, α_{ji}, and the grid and geodetic distances ℓ_{ij} and S_{ij}. If further reduction from the ellipsoid to terrain is required, refer to Section 3.3. The grid distance and azimuths are given respectively by

$$\ell_{ij} = [(X_j - X_i)^2 + (Y_j - Y_i)^2]^{1/2} \; , \qquad (4\text{-}41)$$

$$t_{ij} = \tan^{-1} [\frac{X_j - X_i}{Y_j - Y_i}] \quad , \tag{4-42}$$

and

$$t_{ji} = t_{ij} + 180° \quad . \tag{4-43}$$

From equations (4-5), (4-27) and (4-31) we have that

$$\alpha_{ij} = t_{ij} + \gamma_i + (T\text{-}t)_{ij} \quad , \tag{4-44}$$

and from equations (4-6) and (4-33)

$$S_{ij} = \frac{\ell_{ij}}{\bar{k}_{ij}} \quad . \tag{4-45}$$

Since $(T\text{-}t)_{ji} = - (T\text{-}t)_{ij}$

$$\alpha_{ji} = t_{ji} + \gamma_j - (T\text{-}t)_{ij} \tag{4-46}$$

where γ_j is computed from equation (4-27). This completes the inverse problem.

4.5 Nova Scotia 3° Transverse Mercator

4.5.1 Direct Problem

The direct problem on the N.S. 3° Transverse Mercator plane is stated as: given the grid coordinates X_i, Y_i of point i, the astronomic azimuth A_{ij}, and spatial distance r_{ij} to a point j, compute the grid coordinates X_j, Y_j of the point j.

As with all the map projections the first step is to reduce the observations A_{ij} and r_{ij} from the terrain to the ellipsoidal quantities of geodetic azimuth α_{ij} and ellipsoid distance S_{ij} as described in Chapter 3. These reduced quantities must be reduced once more to obtain the grid azimuth t_{ij} and the chord distance ℓ_{ij}. Beginning with the azimuth the meridian convergence and (T-t) corrections must be applied.

The meridian convergence accurate to 0".01 is computed using [Krakiwsky, 1973]

$$\gamma_i = \Delta\lambda_i \sin\phi_i [1 + \frac{\Delta\lambda_i^2 \cos^2\phi_i \,(1 + 3G_1 + 2G_1^2)}{3} + \frac{\Delta\lambda_i^4 \cos^4\phi_i \,(2 - G_2^{\,2})}{15}] \,. \qquad (4\text{-}47)$$

The values for G_1 and G_2 are given by

$$G_1 = (\frac{a^2 - b^2}{b^2}) \cos^2\phi_i \,, \qquad (4\text{-}48)$$

and

$$G_2 = \tan\phi_i \,. \qquad (4\text{-}49)$$

To evaluate (T-t) correction the approximate coordinates of point j are required. These can be computed by first computing the grid azimuth of the projected geodesic

$$T_{ij} = \alpha_{ij} - \gamma_i \,. \qquad (4\text{-}50)$$

Using this azimuth, T_{ij}, the approximate coordinates are

$$X_j^{\,a} = X_i + S_{ij} \sin T_{ij} \,, \qquad (4\text{-}51)$$

and

$$Y_j^{\,a} = Y_i + S_{ij} \cos T_{ij} \,. \qquad (4\text{-}52)$$

The $(T-t)_{ij}$ correction, accurate to 0".02 for a 100 km line is given by [Krakiwsky, 1973]

$$(T-t)_{ij} = (\frac{(Y_j^a - Y_i)\,(\Delta X_j^a + 2\,\Delta X_i)}{6\,R_m^{\,2}})(1 - \frac{(2\Delta X_i + \Delta X_j^a)^2}{27\,R_m^{\,2}}) \,, \qquad (4\text{-}53)$$

N_1

where R_m is the Gaussian mean radius evaluated at the mean latitude ϕ_m and is given by

$$R_m = \sqrt{M_m N_m} \quad . \tag{4-54}$$

The mapping plane coordinate differences are computed by

$$\Delta X_j^a = X_j^a - X_o , \tag{4-55}$$

and

$$\Delta X_i = X_i - X_o \quad . \tag{4-56}$$

The azimuth is now written as (equation (4-4))

$$t_{ij} = \alpha_{ij} - \gamma - (T-t)_{ij} \quad .$$

Turning to the distance, and using the previously obtained approximate coordinates we can write the line scale factor as [Krakiwsky, 1973]

$$\bar{k}_{ij} = \bar{k}_o \left(1 + \frac{\Delta X_u^{\,2}}{6R_m^{\,2}} \left(1 + \frac{\Delta X_u^{\,2}}{36R_m^{\,2}} \right) \right) \quad , \tag{4-57}$$

where

$$X_u^{\,2} = (\Delta X_i)^2 + \Delta X_i \; \Delta X_j + (\Delta X_j^a)^2 , \tag{4-58}$$

and R_m is computed using (4-51).

The distance ℓ_{ij} then computed using (4-6), namely

$$\ell_{ij} = \bar{k}_{ij} \; S_{ij} \quad .$$

The above formula is accurate to 1×10^{-7} for lines up to 150 km in length [Krakiwsky, 1973].

Finally, we write that

$$X_j = X_i + \ell_{ij} \sin t_{ij} , \qquad (4\text{-}59)$$

and

$$Y_j = X_i + \ell_{ij} \cos t_{ij} , \qquad (4\text{-}60)$$

which completes the direct problem.

4.5.2 Inverse Problem

The inverse problem on the N.S. 3° Transverse Mercator projection plane is: given the grid coordinates X_i, Y_i of point i and X_j, Y_j of point j compute the grid and geodetic azimuths, t_{ij}, t_{ji}, α_{ij}, and α_{ji}, and the grid and geodetic distance ℓ_{ij} and S_{ij} . If further reduction from the ellipsoid to the terrain is required refer to Section 3.3.

The grid distance and azimuths are given respectively by

$$\ell_{ij} = [(X_j - X_i)^2 + (Y_j - Y_i)^2]^{1/2} , \qquad (4\text{-}61)$$

$$t_{ij} = \tan^{-1} [\frac{X_j - X_i}{Y_j - Y_i}] , \qquad (4\text{-}62)$$

and

$$t_{ji} = t_{ij} + 180° . \qquad (4\text{-}63)$$

From equations (4-4), (4-47) and (4-50) the geodetic azimuth is

$$\alpha_{ij} = t_{ij} + \gamma_i + (T-t)_{ij} , \qquad (4\text{-}64)$$

and from equation (4-6)

$$S_{ij} = \frac{\ell_{ij}}{\bar{k}_{ij}} , \qquad (4\text{-}65)$$

where $\bar{k}_{ij}$ is computed from (4-57).

Since $(T-t)_{ji} = - (T-t)_{ij}$

$$\alpha_{ji} = t_{ji} + \gamma_j - (T-t)_{ij} , \qquad (4\text{-}66)$$

where γ_j is computed using equation (4-47).

This completes the inverse problem on the N.S. 3° Transverse Mercator projection.

4.6 Error Propagation

4.6.1 Direct Problem Error Propagation

The covariance matrix for the point i is combined with the variance of the plane azimuth and the variance of the plane distance to form the covariance matrix C_1. The variance on the plane azimuth is taken to be equal to the variance on the observed astronomic azimuth and the variance on the plane distance is taken to be equal to the variance on the ellipsoidal distance. The matrix C_1 has the form

$$C_1 = \left[\begin{array}{cc|cc} \sigma_{X_i}^2 & \sigma_{X_iY_i} & 0 & 0 \\ \sigma_{X_iY_i} & \sigma_{Y_i}^2 & 0 & 0 \\ \hline 0 & 0 & \sigma_{\ell_{ij}}^2 & 0 \\ 0 & 0 & 0 & \sigma_{t_{ij}}^2 \end{array}\right], \qquad (4\text{-}66)$$

in units of

$$\begin{bmatrix} m^2 & m^2 & & \\ m^2 & m^2 & & \\ & & m^2 & \\ & & & rad^2 \end{bmatrix},$$

where rad^2 can be obtained from $arcsec^2$ by multiplying by $(\frac{1}{\rho^2})$.

Using the equations

$$X_i = X_i \ , \tag{4-67}$$

$$Y_i = Y_i \ , \tag{4-68}$$

$$X_j = X_i + \ell_{ij} \sin t_{ij} \ , \tag{4-69}$$

and

$$Y_j = Y_i + \ell_{ij} \cos t_{ij} \ , \tag{4-70}$$

the Jacobian of transformation matrix B_1 is given as

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & B_1(3,3) & B_1(3,4) \\ 0 & 1 & B_1(4,3) & B_1(4,4) \end{bmatrix} , \tag{4-71}$$

in which

$$B_1(3,3) = \sin t_{ij} \ ,$$

$$B_1(3,4) = \ell_{ij} \cos t_{ij}$$

$$B_1(4,3) = \cos t_{ij} \ ,$$

and

$$B_1(4,4) = - \ell_{ij} \sin t_{ij} \ .$$

With B_1^T equal to the transpose of B_1, the covariance matrix for the points i and j will then be

$$C_2 = B_1 \, C_1 \, B_1^T \tag{4-72}$$

where C_2 has the form

$$C_2 = \left[\begin{array}{cc:cc} \sigma_{X_i}^2 & \sigma_{X_iY_i} & \sigma_{X_iY_j} & \sigma_{X_iY_j} \\ \sigma_{X_iY_i} & \sigma_{Y_i}^2 & \sigma_{Y_iX_j} & \sigma_{Y_iY_j} \\ \hdashline \sigma_{X_iX_j} & \sigma_{Y_iX_j} & \sigma_{X_j}^2 & \sigma_{X_jY_j} \\ \sigma_{X_iY_j} & \sigma_{Y_iY_j} & \sigma_{X_jY_j} & \sigma_{Y_j}^2 \end{array}\right] , \tag{4-73}$$

in which all units are in m^2 .

4.6.2 Inverse Problem Error Propagation

The covariance matrix C_2 (equation (4-73)) is known. Using equations

$$\ell_{ij}=[(X_j - X_i)^2 + (Y_j - Y_i)^2]^{1/2} , \qquad (4\text{-}74)$$

and

$$t_{ij} = \tan^{-1} \left[\frac{(X_j - X_i)}{(Y_j - Y_i)}\right] , \qquad (4\text{-}75)$$

the Jacobian of transformation, B_2, is

$$B_2 = \begin{bmatrix} B_2(1,1) & B_2(1,2) & B_2(1,3) & B_2(1,4) \\ B_2(2,1) & B_2(2,2) & B_2(2,3) & B_2(2,4) \end{bmatrix}, \qquad (4\text{-}76)$$

where

$$B_2(1,1) = \frac{(Y_i - Y_j)}{\ell_{ij}^2} ,$$

$$B_2(1,2) = \frac{(X_j - X_i)}{\ell_{ij}^2} ,$$

$$B_2(1,3) = \frac{(Y_j - Y_i)}{\ell_{ij}^2} ,$$

$$B_2(1,4) = \frac{(X_i - X_j)}{\ell_{ij}^2} ,$$

$$B_2(2,1) = \frac{(X_i - X_j)}{\ell_{ij}} ,$$

$$B_2(2,2) = \frac{(Y_i - Y_j)}{\ell_{ij}} ,$$

$$B_2(2,3) = \frac{(X_j - X_i)}{\ell_{ij}} ,$$

and

$$B_2(2,4) = \frac{(Y_j - Y_i)}{\ell_{ij}} .$$

With B_2^T equal to the transposed B_2, the covariance matrix for the derived plane azimuth and distance is given by

$$C_3 = B_2 \, C_2 \, B_2^T \tag{4-77}$$

where C_3 has the form

$$C_3 = \begin{bmatrix} \sigma_{t_{ij}}^{\;2} & \sigma_{t_{ij}\ell_{ij}} \\ \sigma_{t_{ij}\ell_{ij}} & \sigma_{\ell_{ij}}^{\;2} \end{bmatrix} , \tag{4-78}$$

in units of

$$\begin{bmatrix} \text{rad}^2 & \text{rad}\cdot\text{m} \\ \text{rad.m} & \text{m}^2 \end{bmatrix} .$$

The rad^2 can be converted to arcsec^2 by multiplying by ρ^2. The m.rad can be converted to m.arcsec by multiplying by ρ.

The variance on the plane azimuth is taken to be equivalent to the variance of the geodetic azimuth and the variance of the plane distance is taken to be equivalent to that of the ellipsoid distance. These are valid assumptions since the error propagation through the reduction equations proves insignificant.

4.7 Introduction to Numerical Examples

4.7.1 Use of Computed Grid Azimuths

Before commencing with the numerical examples for direct and inverse problems on the mapping plane, let us examine the determination of the grid azimuth of a line by means other than the reduction of a terrain astronomic azimuth. A common situation is to know the grid coordinates of the instrument station i and those of the reference station j, along with the covariance matrix (C_2) for those points. The grid azimuth t_{ij} for the line ij can be computed using equation (4-75). The covariance matrix involving the points and the azimuth can be derived using the inverse problem error propagation (section 4.6.2).

The terrain angle β_{jik} (k is the unknown point) can be measured and then using the reduction formulae outlined in Section 3.2.3 the angle is reduced to the ellipsoid giving β^e_{jik}. This angle is then reduced to the mapping plane angle B_{ijk}, using the reduction formulae outlined in Section 4.2.2. This angle is then added to t_{ij} yielding

$$t_{ik} = B_{jik} + t_{ij} \quad . \tag{4-79}$$

The variance t_{ik} is computed as

$$\sigma^2_{t_{ik}} = \sigma^2_{B_{jik}} + \sigma^2_{t_{ij}} \quad . \tag{4-80}$$

Equation (4-79) indicates that the grid azimuth t_{ik} is correlated to the grid azimuth t_{ij} which has been computed from the coordinates of point i and j. This implies that t_{ik} must be correlated to the coordinates of points i and j. This is completely analagous to the development on the ellipsoid, Section 3.7. To obtain the necessary covariance information between point i and the azimuth t_{ik} we first expand the Jacobian of transformation, B_2, to account for the following equations namely

$$X_i = X_i, \tag{4-81}$$

and

$$Y_i = Y_i \ . \tag{4-82}$$

The resulting Jacobian, B_3, is

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_2(1,1) & B_2(1,2) & B_2(1,3) & B_2(1,4) \end{bmatrix} . \tag{4-83}$$

With B_3^T equal to the transposed B_3, the covariance matrix for the derived plane azimuth and point i coordinates is given by

$$C_4 = B_3 \, C_2 \, B_3^T \tag{4-84}$$

where C_4 has the form

$$C_4 = \begin{bmatrix} \sigma_{X_i}^2 & \sigma_{X_iY_i} & \sigma_{X_it_{ij}} \\ \sigma_{X_iY_i} & \sigma_{Y_i}^2 & \sigma_{Y_it_{ij}} \\ \sigma_{X_it_{ij}} & \sigma_{Y_it_{ij}} & \sigma_{t_{ij}}^2 \end{bmatrix} , \tag{4-85}$$

in units of

$$\begin{bmatrix} m^2 & m^2 & m.rad \\ m^2 & m^2 & m.rad \\ m.rad & m.rad & rad^2 \end{bmatrix} ,$$

where m.rad can be converted to m.arcsec by multiplication by ρ and rad^2 can be converted to $arcsec^2$ by multiplication by ρ^2.

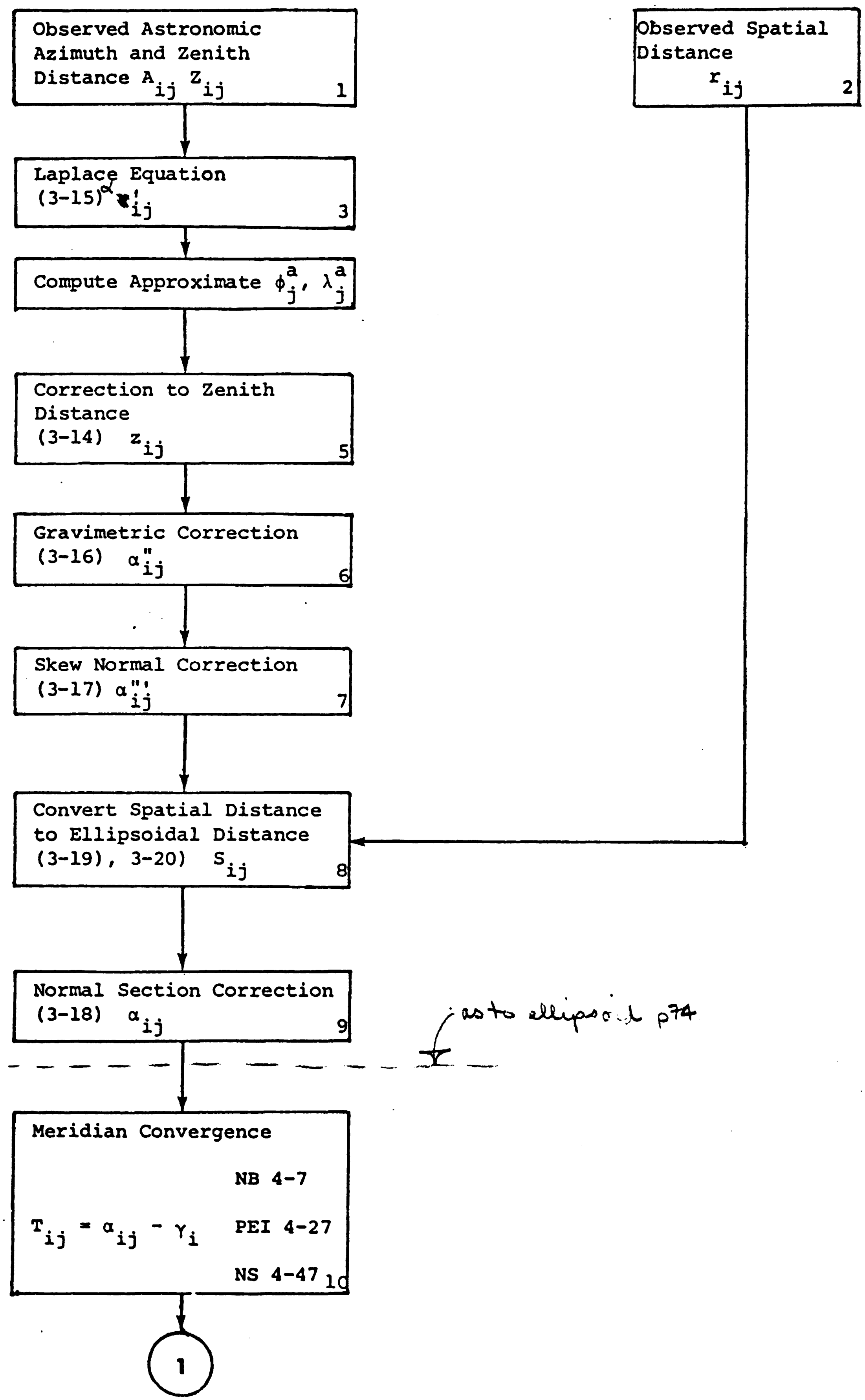
Substituting $\sigma^2_{t_{ik}}$ (equation (4-80)) for $\sigma^2_{t_{ij}}$ in equation (4-85) and including the observed distance variance $\sigma_{\ell_{ik}}^2$, between the points i and k (k is the unknown point), we obtain the alternate expression for the covariance matrix C_1 namely

$$C_1 = \begin{bmatrix} \sigma_{X_i}^2 & \sigma_{X_iY_i} & \sigma_{X_it_{ij}} & 0 \\ \sigma_{X_iY_i} & \sigma_{Y_i}^2 & \sigma_{Y_it_{ij}} & 0 \\ \sigma_{X_it_{ij}} & \sigma_{Y_it_{ij}} & \sigma_{t_{ik}}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\ell_{ik}}^2 \end{bmatrix} \quad (4\text{-}86)$$

where the new variance $\sigma_{\ell_{ik}}^2$ is in units of m^2 .

4.7.2 Mapping Plane Direct Problem Flow Chart

Figure 4-5 contains the flow chart for the mapping plane direct problem. The purpose of this flow chart is to indicate the steps required to reduce the observations from the terrain to the mapping plane environment and then perform the direct problem on the mapping plane. The last decision box, 15, checks to see if the final



Observed Astronomic Azimuth and Zenith Distance A_{ij} Z_{ij} 1
Observed Spatial Distance r_{ij} 2
Laplace Equation (3-15) α'_{ij} 3
Compute Approximate ϕ^a_j, λ^a_j
Correction to Zenith Distance (3-14) z_{ij} 5
Gravimetric Correction (3-16) α''_{ij} 6
Skew Normal Correction (3-17) α'''_{ij} 7
Convert Spatial Distance to Ellipsoidal Distance (3-19), 3-20) S_{ij} 8
Normal Section Correction (3-18) α_{ij} 9
as to ellipsoid p74
Meridian Convergence
NB 4-7
$T_{ij} = \alpha_{ij} - \gamma_i$
PEI 4-27
NS 4-47 10
1

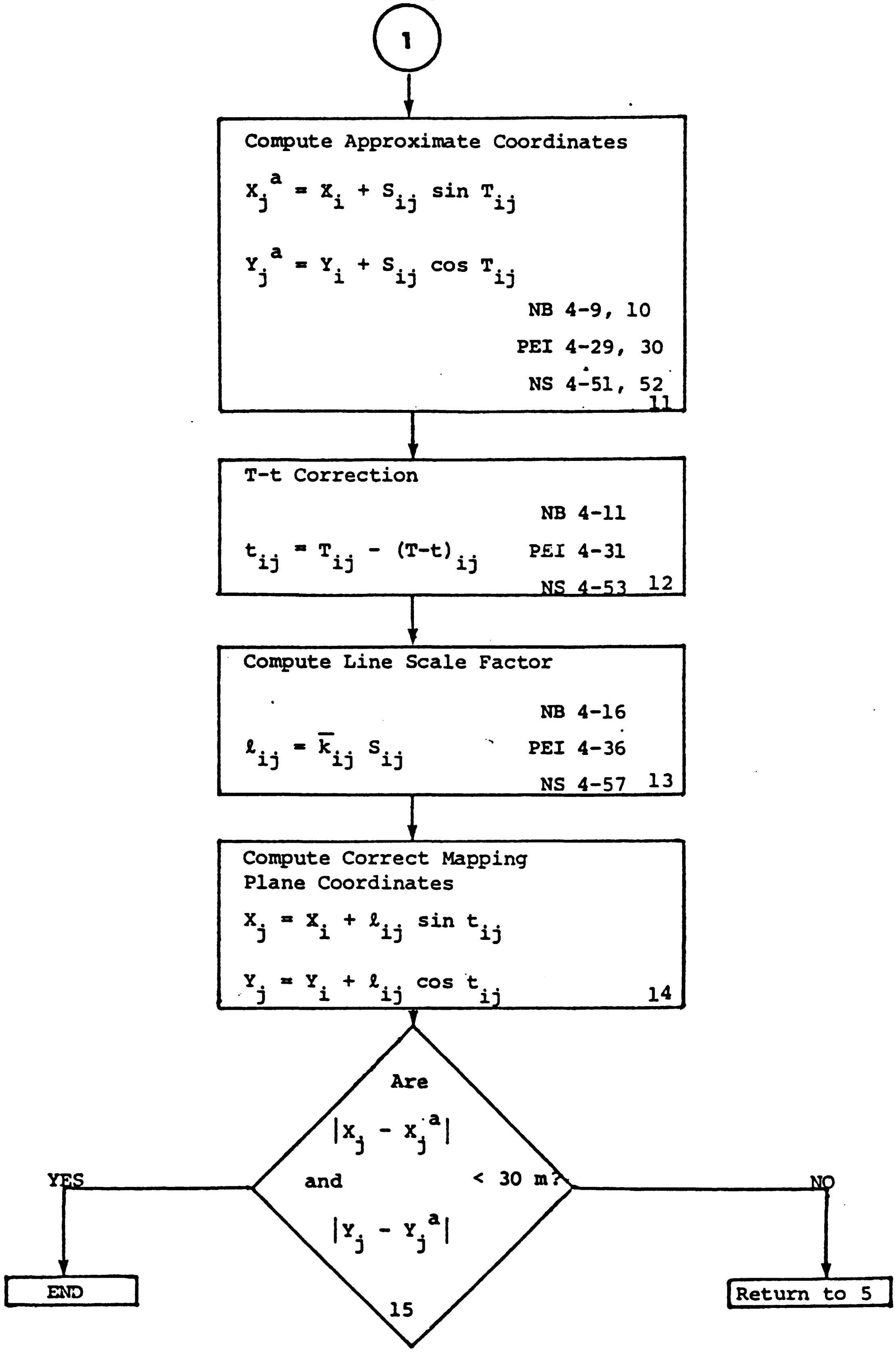


Figure 4-5

Mapping Plane Direct Problem Flow Chart

coordinates of the point j are within 30 m of the approximate values. This is in keeping with the error propagation assumptions mentioned in Section 3.2.1 concerning the reduction formulae.

4.8 New Brunswick Numerical Example

4.8.1 Direct Problem

The following information is given for the solution of the direct problem and its associated error propagation.

The coordinates of point 1 are

$$X_1 = 377\ 164.887 \text{ m} \quad ,$$

$$Y_1 = 862\ 395.774 \text{ m} \quad .$$

The geodetic azimuth is (see Section 3.8.1)

$$\alpha_{12} = 44°\ 59'\ 53''\!.64 \quad ,$$

and has a variance $\sigma^2_{\alpha_{12}} = 25.00 \text{ arcsec}^2$

The ellipsoid distance is (see Section 3.8.1)

$$S_{12} = 2\ 496.488 \text{ m} \quad ,$$

and has a variance $\sigma^2_{s_{12}} = 8.762 \times 10^{-4} \text{ m}^2$.

The covariance matrix, C_1, for the above information is

$$C_1 = \begin{bmatrix} 4.455 \times 10^{-2} & -7.09 \times 10^{-4} & 0 & 0 \\ -7.09 \times 10^{-4} & 9.535 \times 10^{-2} & 0 & 0 \\ 0 & 0 & 8.762 \times 10^{-4} & 0 \\ 0 & 0 & 0 & 5.876 \times 10^{-10} \end{bmatrix}$$

in units of

$$C_1 = \begin{bmatrix} m^2 & m^2 & & \\ m^2 & m^2 & & \\ & & m^2 & \\ & & & rad^2 \end{bmatrix}$$

The covariance matrix for the coordinate values has been obtained from the numerical examples used in Chapters 2 and 3 and has been converted to the mapping plane covariance matrix by formulae from, for example, Krakiwsky et al., [1977]. As mentioned in section 4.6.1 the variance of the plane azimuth is taken to be equal to the variance of the observed astronomic azimuth and the variance of the plane distance is taken to be equal to the variance of the ellipsoid distance.

Using equation (4-7) the meridian convergence is

$$\gamma_1 = 0° \ 44' \ 24''.63 \quad ,$$

and from equation (4-8) the grid azimuth of the projected geodesic is

$$T_{12} = 44° \ 15' \ 29''.01 \ .$$

The approximate coordinates (from equations (4-9) and (4-10)) are

$$X_2^a = 378\ 907.164 \ \ m \quad ,$$

$$Y_2^a = 864\ 183.768 \ \ m \quad .$$

The $(T-t)_{12}$ correction (using equation (4-11)) is

$$(T-t)_{12} = +\ 0°00' \ 0''.04 \quad ,$$

and the grid azimuth, t_{12}, (from equation (4-5)) is

$$t_{12} = 44° \ 15' \ 28''.97 \quad .$$

The line scale factor, $\bar{k}_{12}$, (from equation (4-13)) is

$$\bar{k}_{12} = .999974 \quad ,$$

and using equation (4-6) the chord distance is

$$\ell_{12} = 2\ 496.423 \ \ m \quad .$$

Using equations (4-19) and (4-20) the coordinates of point 2 are

$$X_2 = 378\ 907.118 \ \text{m},$$

$$Y_2 = 864\ 183.722 \ \text{m}.$$

The X_2, Y_2 values are converted to the geodetic coordinates ϕ_2, λ_2 (using formulae from, for example, Krakiwsky et al., [1977]) yielding

$$\phi_2 = 47°04'\ 21''.801 \ ,$$

$$\lambda_2 = 65°\ 27'\ 39''.787 \ W \ ,$$

which are identical to the solutions obtained in the Three Dimensional and Ellipsoid examples (Sections 2.4.1 and 3.8.1 respectively).

Turning to the error propagation (Section 4.6.1) the Jacobian of transformation, B_1, is (from equation (4-71))

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & .69789 & 1787.948 \\ 0 & 1 & .71620 & -1742.231 \end{bmatrix} .$$

The covariance matrix for points 1 and 2 is (from equation (4-72))

$$C_2 = \begin{bmatrix} 4.455 \times 10^{-2} & -7.09 \times 10^{-4} & 4.455 \times 10^{-2} & -7.09 \times 10^{-4} \\ -7.09 \times 10^{-4} & 9.535 \times 10^{-2} & -7.09 \times 10^{-4} & 9.535 \times 10^{-2} \\ 4.455 \times 10^{-2} & -7.09 \times 10^{-4} & 4.685 \times 10^{-2} & -2.101 \times 10^{-3} \\ -7.09 \times 10^{-4} & 9.535 \times 10^{-2} & -2.101 \times 10^{-3} & 9.758 \times 10^{-2} \end{bmatrix}$$

in which all units are in m^2.

The lower right hand (2 x 2) sub matrix is converted to the covariance matrix of the geodetic coordinates ϕ_2, λ_2 (using formula from, for example, Krakiwsky et al., [1977]) yielding

$$C_{\phi_2\lambda_2} = \begin{bmatrix} 1.024 \times 10^{-4} & -2.19 \times 10^{-6} \\ -2.19 \times 10^{-6} & 1.051 \times 10^{-4} \end{bmatrix} ,$$

in units of arcsec^2. It can be seen that this is equivalent to the covariance matrices derived in the Three Dimensional and Ellipsoidal examples (Sections 2.4.1 and 3.8.1 respectively).

4.8.2 Inverse Problem

In the inverse problem the coordinates of points 1 and 2 along with the associated covariance matrix, C_2, are known (in this example the results of the direct problem).

The grid distance and the direct and inverse grid azimuths are (from equations (4-21), (4-22) and (4-23))

$$\ell_{12} = 2\ 496.423 \ \text{m} \ ,$$

$$t_{12} = 44° \ 15' \ 28''.97 \ ,$$

and

$$t_{21} = 224° \ 15' \ 28''.97 \ .$$

The ellipsoidal distance S_{12} is (from equation (4-25))

$$S_{12} = 2\ 496.488 \ \text{m} \ .$$

The direct geodetic azimuth is (from equations (4-24))

$$\alpha_{12} = 44° \ 59' \ 53''.64 \ ,$$

and then (from equation (4-26)) the inverse geodetic azimuth is

$$\alpha_{21} = 225°00'\ 54''.89$$

The inverse problem error propagation begins with the Jacobian of transformation, B_2, (given by equation (4-76))

$$B_2 = \begin{bmatrix} -2.8689 \times 10^{-4} & 2.7956 \times 10^{-4} & 2.8689 \times 10^{-4} & -2.7956 \times 10^{-4} \\ -.69789 & -.71620 & .69789 & .71620 \end{bmatrix}.$$

The covariance matrix, C_3, (from equation (4-77)) is

$$C_3 = \begin{bmatrix} 25.00 & 0 \\ 0 & 8.762 \times 10^{-4} \end{bmatrix},$$

in units of

$$C_3 = \begin{bmatrix} \text{arcsec}^2 & \text{m.arcsec} \\ \text{m.arcsec} & \text{m}^2. \end{bmatrix},$$

where the arcsec^2 have been obtained from the rad.^2 by multiplication by ρ^2 and arcsec.m have been obtained from the rad.m by multiplication by ρ.

The variance of the plane azimuth is taken to be equivalent to that of the geodetic azimuth and the variance of the plane distance is taken to be equivalent to that of the ellipsoid distance (as described in Section 4.6.1).

4.9 Prince Edward Island Numerical Example

4.9.1 Direct Problem

The following information is given for the solution of the direct problem and its associated error propagation.

The coordinates of point 1 are

$$X_1 = 585\ 855.446 \text{ m},$$

$$Y_1 = 340\ 817.760 \text{ m}.$$

The geodetic azimuth is (see Section 3.9.1)

$$\alpha_{12} = 134° \ 59' \ 53''.25 \ ,$$

and has a variance $\sigma^2_{\alpha_{12}} = 25.00 \text{ arcsec}^2$.

The ellipsoid distance is (see Section 3.9.2).

$$S_{12} = 2\ 496.484 \text{ m}.$$

and has a variance $\sigma_{S_{12}} = 8.762 \times 10^{-4} \text{ m}^2$.

The covariance matrix, C_1, for the above information is

$$C_1 = \begin{bmatrix} 4.514 \times 10^{-2} & 9.04 \times 10^{-4} & 0 & 0 \\ 9.04 \times 10^{-4} & 9.534 \times 10^{-2} & 0 & 0 \\ 0 & 0 & 8.762 \times 10^{-4} & 0 \\ 0 & 0 & 0 & 5.876 \times 10^{-10} \end{bmatrix},$$

in units of

$$C_1 = \begin{bmatrix} m^2 & m^2 & & \\ m^2 & m^2 & & \\ & & m^2 & \\ & & & rad^2 \end{bmatrix}.$$

The covariance matrix for the coordinate values has been obtained from the numerical examples used in Chapters 2 and 3 and has been converted to the mapping plane covariance matrix by formulae from, for example, Krakiwsky et al., [1977]. As mentioned in Section 4.6.1 the variance of the plane azimuth is taken to be equal to the variance of the observed astronomic azimuth and the variance of the plane distance is taken to be equal to the variance of the ellipsoid distance.

Using equation (4-27) the meridian convergence is

$$\gamma_1 = -1° 05' 29''.10 \quad ,$$

and from equation (4-8) the grid azimuth of the projected geodesic is

$$T_{12} = 136° 05' 22''.35 \quad .$$

The approximate coordinates (from equations (4-29) and (4-30)) are

$$X_2^a = 587\ 586.841 \quad m \ ,$$

$$Y_2^a = 339\ 019.232 \quad m.$$

The $(T-t)_{12}$ correction (using equation (4-31)) is

$$(T-t)_{12} = 0° \ 0' \ 0''.39$$

and the grid azimuth, t_{12}, (from equation (4-5)) is

$$t_{12} = 136° 05' 21''.96 \quad .$$

The line scale factor, $\bar{k}_{12}$, (from equation (4-33)) is

$$\bar{k}_{12} = 1.000\ 013 \quad ,$$

and using equation (4-6) the chord distance is

$$\ell_{12} = 2\ 496.516 \quad m \quad .$$

Using equations (4-39) and (4-40) the coordinates of point 2 are

$$X_2 = 587\ 586.867 \quad m \quad ,$$

$$Y_2 = 339\ 019.212 \quad m \quad .$$

The X_2, Y_2 values are converted to the geodetic coordinates ϕ_2, λ_2 (using formulae from, for example, Krakiwsky et al., [1977]) yielding

$$\phi_2 = 46° \ 41' \ 30''.973 \quad ,$$

$$\lambda_2 = 64° \ 28' \ 10''.933 \quad W \quad ,$$

which are identical to the solutions obtained in the Three Dimensional and Ellipsoidal examples (Sections 2.5.1 and 3.9.1 respectively)

Turning to the error propagation (Section 4.6.1) the Jacobian of transformation, B_1, is (from equation (4-71))

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & .69353 & -1798.549 \\ 0 & 1 & -.72042 & -1731.421 \end{bmatrix} .$$

The covariance matrix for points 1 and 2 is (from equation (4-72))

$$C_2 = \begin{bmatrix} 4.514 \times 10^{-2} & 9.04 \times 10^{-4} & 4.514 \times 10^{-2} & 9.04 \times 10^{-4} \\ 9.04 \times 10^{-4} & 9.54 \times 10^{-2} & 9.04 \times 10^{-4} & 9.54 \times 10^{-2} \\ 4.514 \times 10^{-2} & 9.04 \times 10^{-4} & 4.746 \times 10^{-2} & 2.296 \times 10^{-3} \\ 9.04 \times 10^{-4} & 9.54 \times 10^{-2} & 2.296 \times 10^{-3} & 9.755 \times 10^{-2} \end{bmatrix},$$

in which all units are in m^2.

The lower right hand (2 x 2) sub matrix is converted to the covariance matrix of the geodetic coordinates ϕ_2, λ_2 (using formula from, for example, Krakiwsky et al. [1977]) yielding

$$C_{\phi_2\lambda_2} = \begin{bmatrix} 1.024 \times 10^{-4} & 2.07 \times 10^{-6} \\ 2.07 \times 10^{-6} & 1.050 \times 10^{-4} \end{bmatrix},$$

in units of arcsec^2. It can be seen that this is equivalent to the covariance matrices derived in the Three-Dimensional and Ellipsoidal examples (Sections 2.5.1 and 3.9.1 respectively)

4.9.2 <u>Inverse Problem</u>

In the inverse problem the coordinates of points 1 and 2 along with the associated covariance matrix, C_2, are known (in this example the results of the direct problem).

The grid distance and the direct and inverse grid azimuth are (from equations (4-41), (4-42) and (4-43))

$$\ell_{12} = 2\ 496.517 \quad m \ ,$$

$$t_{12} = 136°05'\ 21''.96 \ ,$$

and

$$t_{21} = 316°\ 05'\ 21''.96.$$

The ellipsoidal distance S_{12} is (from equation (4-45))

$$S_{12} = 2\ 496.484 \text{ m} \ .$$

The direct geodetic azimuth is (from equation (4-44))

$$\alpha_{12} = 134° \ 59' \ 53''.25 \ ,$$

and then (from equations (4-46)) the inverse geodetic azimuth is

$$\alpha_{21} = 315° 00' \ 53''.71 \ .$$

The inverse problem error propagation begins with the Jacobian of transformation, B_2, (given by equation (4-76))

$$B_2 = \begin{bmatrix} 2.8857 \times 10^{-4} & 2.7780 \times 10^{-4} & -2.8857 \times 10^{-4} & -2.7780 \times 10^{-4} \\ -.69353 & .72042 & .69353 & -.72042 \end{bmatrix}$$

The covariance matrix, C_3, (from equation (4-77)) is

$$C_3 = \begin{bmatrix} 25.00 & 0 \\ 0 & 8.762 \times 10^{-4} \end{bmatrix} ,$$

in units of

$$C_3 = \begin{bmatrix} \text{arcsec}^2 & \text{m.arcsec} \\ \text{m.arcsec} & \text{m}^2 \end{bmatrix} ,$$

where the arcsec^2 have been obtained from the rad^2 by multiplication by ρ^2 and arcsec.m have been obtained from the rad.m by multiplication by ρ.

The variance of the plane azimuth is taken to be equivalent to that of the geodetic azimuth and the variance of the plane distance is taken to be equivalent to that of the ellipsoid distance (as described in Section 4.6.1).

4.10 Nova Scotia Numerical Example

4.10.1 Direct Problem

The following information is given for the solution of the direct problem and its associated error propagation.

The coordinates of point 1 are

$$X_1 = 5\ 618\ 978.072 \text{ m} \quad ,$$

$$Y_1 = 4\ 946\ 528.965 \text{ m} \quad .$$

The geodetic azimuth is (see Section 3.10.1)

$$\alpha_{12} = 224\ 59'\ 54''.01 \ .$$

The ellipsoid distance is (see Section 3.10.2).

$$S_{12} = 2\ 496.479 \text{ m}$$

and has a variance $\sigma_{S_{12}}^2 = 8.762 \times 10^{-4}\ \text{m}^2$.

The covariance matrix, C_1, for the above information is

$$C_1 = \begin{bmatrix} 4.861 \times 10^{-2} & -9.15 \times 10^{-4} & 0 & 0 \\ -9.15 \times 10^{-4} & 9.539 \times 10^{-2} & 0 & 0 \\ 0 & 0 & 8.762 \times 10^{-4} & 0 \\ 0 & 0 & 0 & 5.876 \times 10^{-10} \end{bmatrix}$$

in units of

$$C_1 = \begin{bmatrix} m^2 & m^2 & & \\ m^2 & m^2 & & \\ & & m^2 & \\ & & & \text{rad}^2 \end{bmatrix} .$$

The covariance matrix for the coordinate values has been obtained from the numerical examples used in Chapters2 and 3 and has been converted to the mapping plane covariance matrix by formulae from, for example, Krakiwsky et al., [1977]. As mentioned in Section 4.6.1 the variance of the plane azimuth is taken to be equal to the variance of the observed astronomic azimuth and the variance of the plane distance is taken to be equal to the variance of the ellipsoid distance.

Using equation (4-47) the meridian convergence is

$$\gamma_1 = 1°03'\ 15''.48 \quad ,$$

and from equation (4-8) the grid azimuth of the projected geodesic is

$$T_{12} = 223°\ 56'\ 38''.53 \quad .$$

The approximate coordinates (from equations (4-55) and (4-56)) are

$$x_2^{\,a} = 5\ 617\ 245.627 \text{ m} \quad ,$$

$$y_2^{\,a} = 4\ 944\ 731.455 \text{ m} \quad .$$

The $(T-t)_{12}$ correction (using equation (4-50)) is

$$(T-t)_{12} = -0°00'\ 0''.54 \quad ,$$

and the grid azimuth, t_{12}, (from equation (4-5)) is

$$t_{12} = 223°\ 56'\ 39''.07 \quad .$$

The line scale factor, $\bar{k}_{12}$, (from equation (4-57)) is

$$\bar{k}_{12} = 1.000\ 071\ 47$$

and using equation (4-6) the chord distance is

$$\ell_{12} = 2\ 496.657 \quad m \ .$$

Using equations (4-59) and (4-60) the coordinates of point 2 are

$$X_2 = 5\ 617\ 245.499 \quad m \ ,$$

$$Y_2 = 4\ 944\ 731.331 \quad m \ .$$

The X_2, Y_2 values are converted to the geodetic coordinates ϕ_2, λ_2 (using formula from, for example, Krakiwsky et al., [1977]) yielding

$$\phi_2 = 44° \ 38' \ 5''.925 \ ,$$

$$\lambda_2 = 63° 01' \ 20.088 \quad W \ .$$

which are identical to the solutions obtained in the Three Dimensional and Ellipsoid examples (Sections 2.6.1 and 3.10.1 respectively).

Turning to the error propagation (Section 4.6.1) the Jacobian of transformation, B_1, is (from equation (4-71))

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -.69396 & -1797.634 \\ 0 & 1 & -.72002 & 1732.574 \end{bmatrix} .$$

The covariance matrix for points 1 and 2 is (from equation (4-72))

$$C_2 = \begin{bmatrix} 4.861 \times 10^{-2} & -9.15 \times 10^{-4} & 4.861 \times 10^{-2} & -9.15 \times 10^{-4} \\ -9.15 \times 10^{-4} & 9.539 \times 10^{-2} & -9.15 \times 10^{-4} & 9.539 \times 10^{-2} \\ 4.861 \times 10^{-2} & -9.15 \times 10^{-4} & 5.093 \times 10^{-2} & -2.31 \times 10^{-3} \\ -9.15 \times 10^{-4} & 9.539 \times 10^{-2} & -2.31 \times 10^{-3} & 9.761 \times 10^{-2} \end{bmatrix}$$

in which all units are in metres^2.

The lower right hand sub matrix is converted to the covariance matrix of the geodetic coordinates ϕ_2, λ_2 (using formula from, for example, Krakiwsky et al., [1977]) yielding

$$C_{\phi_2,\lambda_2} = \begin{bmatrix} 1.024 \times 10^{-4} & -2.15 \times 10^{-6} \\ -2.15 \times 10^{-6} & 1.046 \times 10^{-4} \end{bmatrix},$$

in units of arcsec^2. It can be seen that this is equivalent to the covariance matrices derived in the Three Dimensional and Ellipsoidal examples (Sections 2.6.1 and 3.10.1 respectively).

4.10.2 Inverse Problem

In the inverse problem the coordinates of points 1 and 2 along with the associated covariance matrix, C_2, are known (in this example the results of the direct problem) .

The grid distance and the direct and inverse grid azimuths are (from equations (4-61), (4-62) and (4-63))

$$\ell_{12} = 2\ 496.657 \text{ m} \quad ,$$

$$t_{12} = 223° \ 56' \ 39''.07 \quad ,$$

and

$$t_{21} = 43° \ 56' \ 39''.07 \quad .$$

The ellipsoidal distance S_{12} is (from equation (4-65))

$$S_{12} = 2\ 496.479 \text{ m} \quad .$$

The direct geodetic azimuth is (from equation (4-64))

$$\alpha_{12} = 224° \ 59' \ 54''.01 \quad .$$

and then (from equation (4-66)) the inverse geodetic azimuth is

$$\alpha_{21} = 44° \ 58' \ 57''.73 \quad .$$

The inverse problem error propagation begins with the Jacobian of transformation, B_2, (given by equation (4-76))

$$B_2 = \begin{bmatrix} 2.8839 \times 10^{-4} & -2.7795 \times 10^{-4} & -2.8839 \times 10^{-4} & 2.7795 \times 10^{-4} \\ .69396 & .72002 & -.69396 & -.72002 \end{bmatrix}$$

The covariance matrix, C_3, (from equation (4-77)) is

$$C_3 = \begin{bmatrix} 25.00 & 0 \\ 0 & 8.762 \times 10^{-4} \end{bmatrix} ,$$

in units of

$$C_3 = \begin{bmatrix} \text{arcsec}^2 & \text{m.arcsec} \\ \text{m.arcsec} & \text{m}^2 \end{bmatrix} ,$$

where the arcsec^2 nave been obtained from the rad^2 by multiplication by ρ^2 and arcsec.m have been obtained from the rad . m by multiplication by ρ.

The variance of the plane azimuth is taken to be equivalent to that of the geodetic azimuth and the variance of the plane distance is taken to be equivalent to that of the ellipsoid distance (as described in Section 4.6.1).

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APPENDIX I

Rotation and Reflection Matrices

In Chapter 2 the use of rotation is an important consideration. The rotation matrices R_1, R_2 and R_3 each rotate a coordinate system about a certain axis. An R_1 rotation matrix rotates the Y and Z axes about the X axis (Figure A-1a). An R_2 rotation matrix rotates the X and Z axes about the Y axis (Figure A-1b). An R_3 rotation rotates the X and Y axes about the Z axis (Figure A-1c).

The positive direction of rotation for a right handed coordinate system is taken by convention to be counter-clockwise when viewed from the positive end of the axis about which the rotation takes place.

The rotation matrices are given by [Wells, 1971]

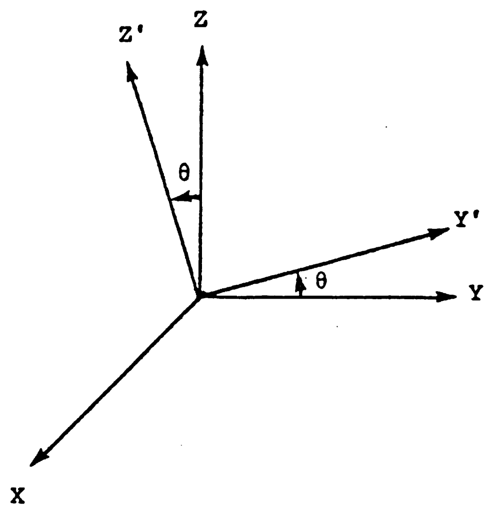
$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \qquad \text{A-1}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad \text{A-2}$$

and

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{A-3}$$

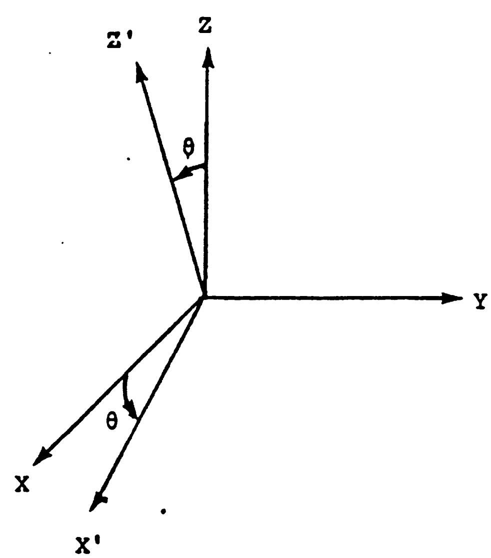
in which θ is the angle of rotation.



(a)

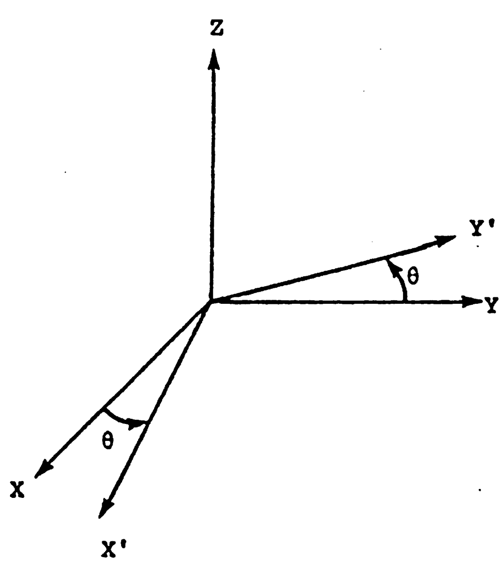
POSITIVE X ROTATION (R_1)

( )' Denotes New Position of Axis



(b)

POSITIVE Y ROTATION (R_2)



(c)

POSITIVE Z ROTATION (R_3)

Figure A-1

ROTATION MATRICES

Other important transformation matrices are the reflections P_1, P_2, and P_3. Their function is to interchange the positive and negative direction along each axis. The P_1 reflection is used on the X axis (Figure A-2a). The P_2 reflection is used on the Y-axis (Figure A-2b). The P_3 reflection is used on the Z axis (Figure A-2).

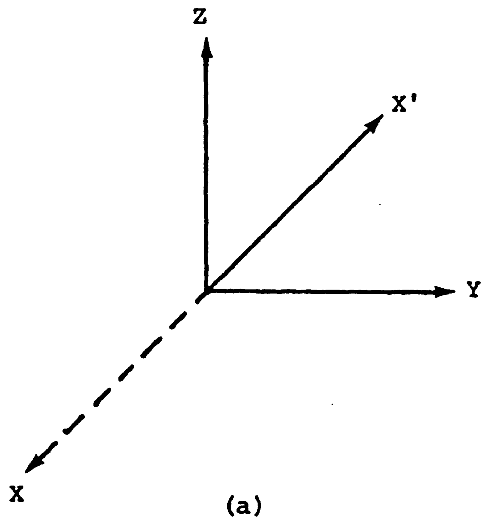
The reflection matrices are given by

$$P_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(A-4)}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(A-5)}$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \qquad \text{(A-6)}$$

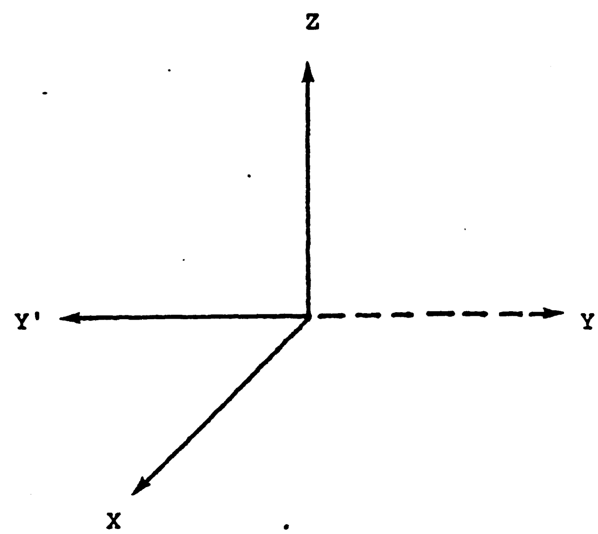
For further information about rotation and reflection matrices and their properties, the reader is referred to, for example, Wells [1971].



( )' Denotes New Position of Axis

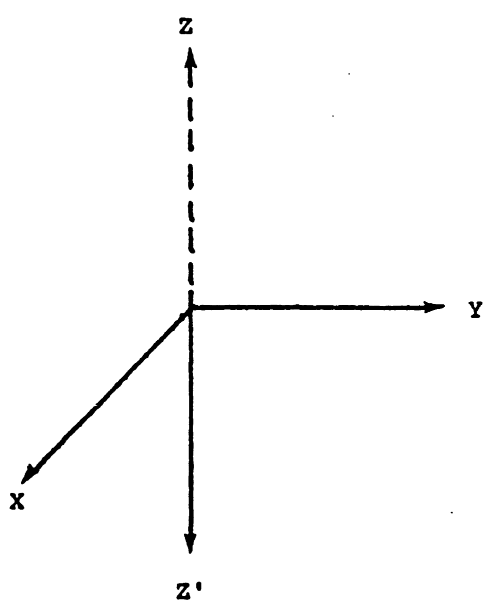
(a)

X Reflection (P_1)



(b)

Y Reflection (P_2)



Z Reflection (P_3)

Figure A-2

Reflection Matrices

APPENDIX II

Covariance Law and the Jacobian of Transformation

This appendix gives a short explanation of the covariance law of which the law of propagation of errors is a special case.

To begin with consider a random variable x that can take on an infinite number of values. Let dx be the actual error in x, which is involved in the definition of the variance of x (σ^2); namely*

$$\sigma_x^2 = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} dx^2}{n} \qquad \text{(A-7)}$$

If we do not know the actual error, but only an estimate of it - usually called the residual $v_i = x_i - \bar{x}$, where $\bar{x}$ is the sample mean, $\sum_{i=1}^{n} x_i/n$, then the sample variance is defined as

$$s^2 = \frac{\sum_{i=1}^{n} v_i^2}{n-1} \quad . \qquad \text{(A-8)}$$

Note that for s^2 to be an unbiased estimate of σ^2 we need to define it with n-1 in the denominator.

Let us now work our way up to the covariance law, in matrix form, by beginning with the simplest case: y is some function of x; namely

$$y = f(x) \quad . \qquad \text{(A-9)}$$

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* the variance can be defined in the forms of mathematical expectations - a rigorous manner which is more appealing from the mathematical point of view.

Taking the total differential of the above yields

$$dy = \frac{\partial f(x)}{\partial x} \; dx \quad . \tag{A-10}$$

Considering dy and dx to be (actual) errors, we sum the squares of n of these and then divide by n in the context of the above equation; this yields

$$\frac{\sum_{i=1}^{n} dy^2}{n} = \left(\frac{\partial f(x)}{\partial x}\right)^2 \frac{\sum_{i=1}^{n} dx^2}{n} \quad , \tag{A-11}$$

and as $n \to \infty$, we get

$$\sigma_y^{\,2} = \left(\frac{\partial f(x)}{\partial x}\right)^2 \; \sigma_x^{\,2} \; , \tag{A-12}$$

which is the formula for the <u>propagation of errors from one variable x into another variable y</u>.

Let us now take the case that y is a function of two random variables x_1 and x_2, i.e.

$$y = f(x_1, \, x_2) \quad . \tag{A-13}$$

Taking the total differential yields

$$dy = \frac{\partial f}{\partial x_1} \; dx_1 + \frac{\partial f}{\partial x_2} \; dx_2 \quad . \tag{A-14}$$

Considering dy, dx_1 and x_2 again to be actual errors, and squaring and summing the terms on both sides of the above equation results in

$$\frac{\sum_{i=1}^{n} dy^2}{n} = \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\sum_{i=1}^{n} dx_1^{\,2}}{n} + \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\sum_{i=1}^{n} dx_2^{\,2}}{n} + 2\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_2}\right) \frac{\sum_{i=1}^{n} dx_1 \, dx_2}{n} \quad ; \tag{A-15}$$

after considering $n \rightarrow \infty$, we can write

$$\sigma_y^2 = (\frac{\partial f}{\partial x_1})^2 \sigma_{x_1}^2 + (\frac{\partial f}{\partial x_2}) \sigma_{x_2}^2 + 2 (\frac{\partial f}{\partial x_1})(\frac{\partial f}{\partial x_2}) \sigma_{x_1 x_2} \quad . \tag{A-16}$$

Note the newly introduced quantity $\sigma_{x_1 \cdot x_2}$, the covariance between x_1 and x_2; namely

$$\sigma_{x_1 x_2} = \frac{\sum_{i=1}^{n} dx_1 \, dx_2}{n} \quad . \tag{A-17}$$

This quantity is zero if the errors dx_1 and dx_2 are statistically independent.

Let us expand our model such that we have two random variables y_1 and y_2 which are both a function of the same two random variables x_1 and x_2. In equation form we have:

$$y_1 = f_1 \, (x_1, x_2) \quad , \tag{A-18}$$

$$y_2 = f_2 \, (x_1, x_2) \quad .$$

Applying the concepts given above we can write the <u>variance of y_1</u> as:

$$\sigma_{y_1}^2 = (\frac{\partial f_1}{\partial x_1})^2 \sigma_{x_1}^2 + (\frac{\partial f_1}{\partial x_2})^2 \sigma_{x_2}^2 + 2 (\frac{\partial f_1}{\partial x_1}) (\frac{\partial f_1}{\partial x_2}) \sigma_{x_1 x_2} \quad ; \tag{A-19}$$

the <u>variance of y_2</u> as:

$$\sigma_{y_2}^2 = (\frac{\partial f_2}{\partial x_1})^2 \sigma_{x_1}^2 + (\frac{\partial f_2}{\partial x_2}) \sigma_{x_2}^2 + 2 (\frac{\partial f_2}{\partial x_1}) (\frac{\partial f_2}{\partial x_2}) \sigma_{x_1 x_2} \quad ; \tag{A-20}$$

and the <u>covariance between y_1 and y_2</u> as:

$$\sigma_{y_1 y_2} = \left(\frac{\partial y_1}{\partial x_1}\right)\left(\frac{\partial y_2}{\partial x_1}\right)\sigma_{x_1}{}^2 + \left(\frac{\partial y_1}{\partial x_2}\right)\left(\frac{\partial y_2}{\partial x_2}\right)\sigma_{x_2}{}^2 \tag{A-21}$$

$$+ \left(\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} + \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}\right)\sigma_{x_1 x_2} \quad .$$

The above three equations can be written in matrix form as follows:

$$C_{y_1, y_2} = J\, C_{x_1, x_2}\, J^T \quad , \tag{A-22}$$

where the covariance matrix of x_1 and x_2 is

$$C_{x_1, x_2} = \begin{bmatrix} \sigma_{x_1}{}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_2 x_1} & \sigma_{x_2}{}^2 \end{bmatrix} . \tag{A-23}$$

The Jacobian of transformation from the x's to the y's is

$$J = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} \quad , \tag{A-24}$$

and J^T is the transpose of J (above). The resultant covariance matrix for the y's is

$$C_{y_1, y_2} = \begin{bmatrix} \sigma_{y_1}{}^2 & \sigma_{y_1 y_2} \\ \sigma_{y_2 y_1} & \sigma_{y_2}{}^2 \end{bmatrix} \quad . \tag{A-25}$$

The covariance law is valid for any number of y's and x's. In the case above

$$\underset{u \times u}{C_y} = \underset{u \times n}{J} \; \underset{n \times n}{C_x} \; \underset{n \times u}{J^T} \tag{A-26}$$

where u = 2 and n = 2.

Note, the covariance law is also valid in terms of the sample variances and covariances. The only thing that changes is the interpretation of the results, which of course must be in terms of the sample values and not the true or actual variances and covariances.