A MANUAL FOR GEODETIC COORDINATE TRANSFORMATIONS IN THE MARITIME PROVINCES

D. B. THOMSON
E. J. KRAKIWSKY
R. R. STEEVES

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PREFACE

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A MANUAL FOR GEODETIC COORDINATE TRANSFORMATIONS IN THE MARITIMES

Edward J. Krakiwsky
Donald B. Thomson
Robin R. Steeves

Department of Geodesy and Geomatics Engineering
University of New Brunswick
P.O. Box 4400
Fredericton, N.B.
Canada
E3B 5A3

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PREFACE

Maximum advantage of the geodetic information of the redefined Maritime Geodetic Network will be achieved if surveyors utilize it in a correct and practical manner. The realization of this goal is seen as a "three-part" process. In the first part, one learns to deal with the information associated with a single point on the terrain. The second part, where observations are introduced, involves two terrain points. In the third and final part, one learns to deal with many terrain points and the observations amongst them, namely, a network.

This "manual" is the first of three being written to cover the above mentioned process. It was written as a guide to the use and interpretation of the geodetic information for a single terrain point, and is a complete surveyors handbook for Geodetic Coordinate Transformations in the maritime provinces. No derivations or extensive explanations of the mathematical formulae are given. The equations required to solve certain coordinate and associated error transformation problems are stated, the notation used is explained, and a numerical example is presented. A reader desiring extensive background information as to the relevance of this Manual, and a detailed explanation of the origins of the mathematical formulae, is referred to the reference material.

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PART I

INTRODUCTION

1. REDEFINITION OF THE CANADIAN GEODETIC NETWORKS

The redefinition of the Canadian geodetic networks (Figure 1.1) is one of the most extensive geodetic projects that has ever been undertaken in Canada. It is, of course, an integral part of a much larger project - the redefinition of the North American geodetic networks. There are many reasons for the redefinition project. An adequate yet brief explanation has been given by C.D. McLellan of the Geodetic Survey of Canada [1976]:

> "...our geodetic networks today fail to meet modern standards, and coordinate values in many areas do not reflect the quality of the survey work. In fact, roughly half of our primary networks fail to meet first-order standards."

Of course, in a broader context, the failure to meet internal standards is not reason enough to justify such a great undertaking. Two major underlying reasons are (i) the expanded use of coordinates in the planning, execution, and analysis of the surveying portion of numerous projects, and (ii) the need for a knowledge of the quality (e.g. variance-covariance matrix, relative and absolute confidence regions) associated with the quantitative data (e.g. coordinates, lengths and directions of lines, areas). E.J. Krakiwsky and P. Vaníček, in a paper presented at the Geodesy for Canada Conference [E.M.R., 1974], explained the need for, and use of, a homogeneous set of coordinated network points for mapping, boundary demarcation, urban management, engineering projects, hydrography, oceanography, environmental management, ecology, earthquake hazard assessment, and space research. Such extensive use of the coordinates of network points justifies the redefinition

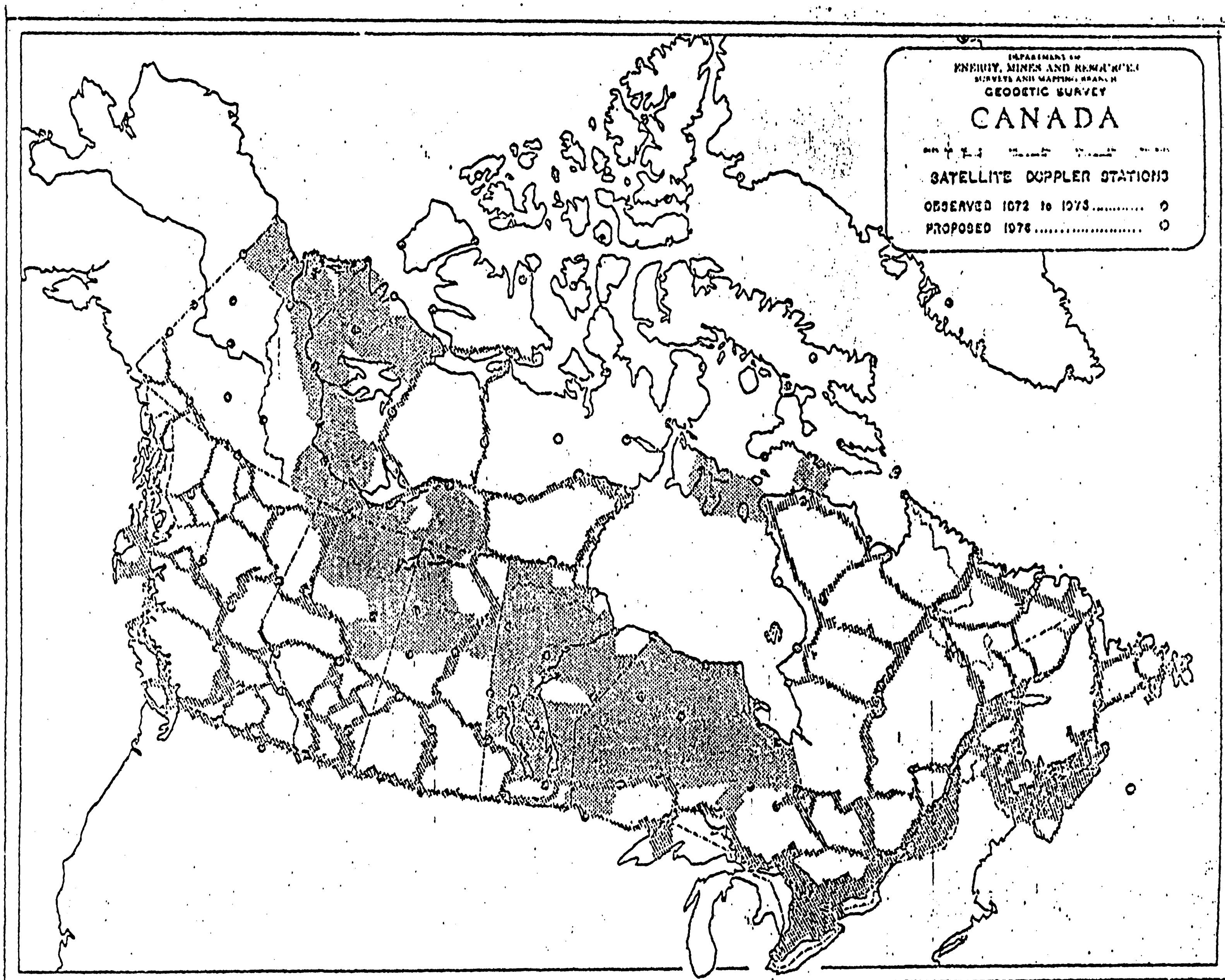

Figure 1.1

Canadian Geodetic Network

of the Canadian geodetic networks, the results of which will be a homogeneous set of coordinates and associated accuracy information that will fulfill the needs of a majority of users.

The redefinition of the Canadian geodetic networks is being carried out by the Geodetic Survey of Canada [McLellan, 1976]. While a description of the project is not warranted here, the reader should note that the most modern technologies and methodologies are being brought to bear on the project to ensure a successful solution. An excellent overview of the work is given by C.D. McLellan [1977]. More detailed information, for example, regarding the coordinate system to which network points will be related, can be found in papers, reports, etc., such as Kouba [1976].

2. REDEFINITION OF THE MARITIME GEODETIC NETWORK

The redefinition of the coordinates of the approximately 36,000 points in the Maritime geodetic network (Figure 2.1) is required for two major reasons. The first, which comes about by default, is that since the Maritime geodetic network is dependent on the national framework for the definition of the coordinate system, a redefinition of the latter necessitates similar action in the former if homogeneity is to be maintained. That the national and regional redefinition projects will be completed in the near future is not in doubt according to the following statement [Fila and Chamberlain, 1977]: "The Geodetic Survey of Canada will have its redefinition completedby January 1978 and this project (the redefinition of the Maritime geodetic network) will be completed by the end of 1978".

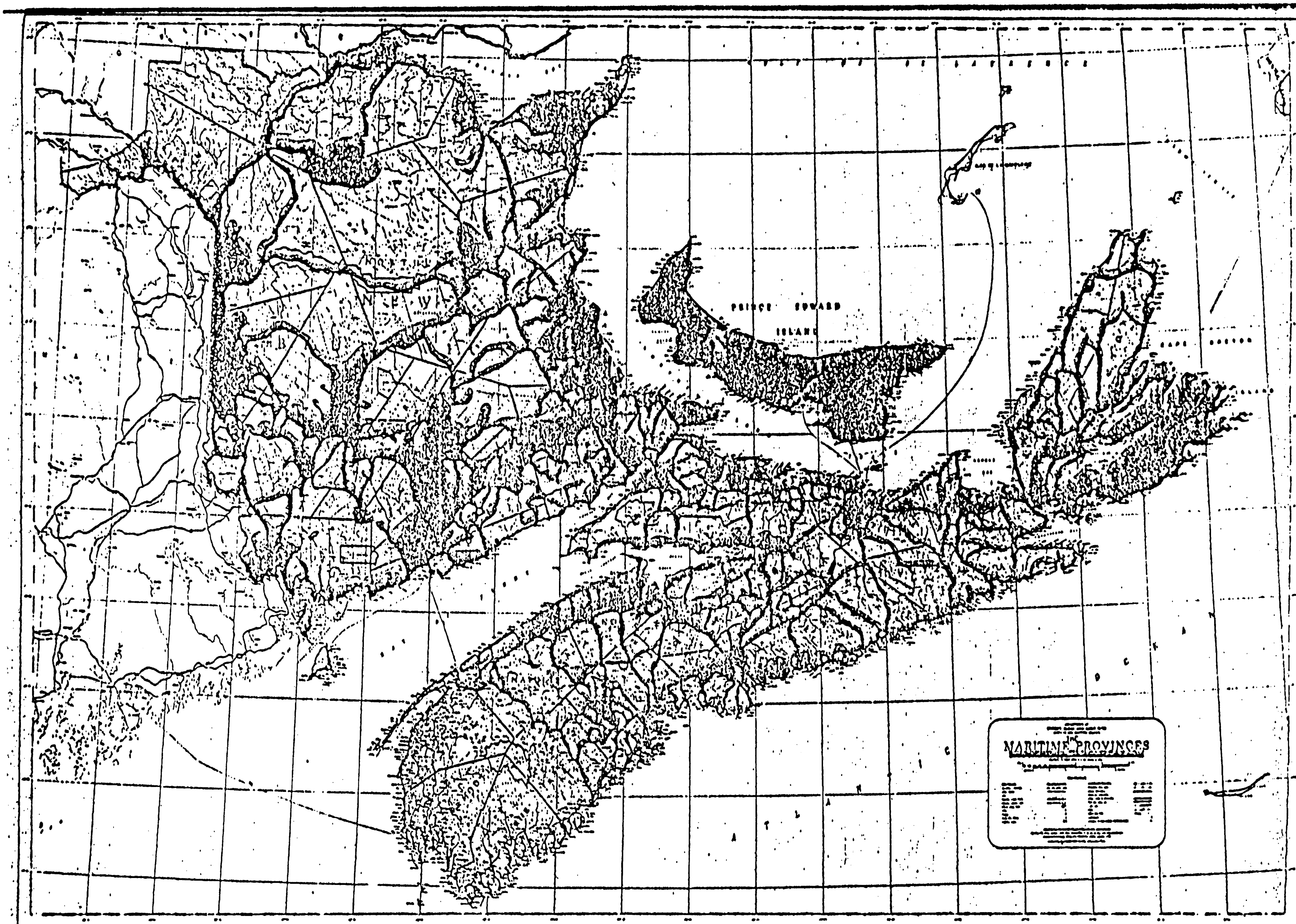

Figure 2-1

Maritime Geodetic Network

The second reason, which has greater regional consequences, is as follows: In the Maritime Provinces, the use of the national and regional networks as a position base for the program of the Land Registration and Information Services requires that a homogeneous set of coordinates and associated accuracy estimates be available. This latter point is discussed at great length in a report entitled The Maritime Cadastral Accuracy Study [McLaughlin et al., 1977]. According to L.R.I.S., the use of the redefined coordinates is not in doubt [MacIntosh, 1977]: "Making use of the plane co-ordinate system, the corners of each parcel brought under the system would be assigned the appropriate co-ordinate values. These values would govern over monuments. The boundaries of a property would be guaranteed within certain tolerances established with respect to the class of property involved."

The Maritime redefinition task is being carried out by the Surveys and Mapping Division of L.R.I.S., with assistance being given by the Systems and Planning Division of L.R.I.S. and the Department of Surveying Engineering at the University of New Brunswick. K. Fila and C. Chamberlain [1977] have outlined the overall concepts regarding project planning, data screening, mathematical models, software development, sub-network analysis, the geodetic data bank and the maintenance of the redefined geodetic system.

Of particular interest and importance to surveyors are the following:

(1) The reference surface to be used for the solution of the Maritime geodetic network will be an ellipsoid of revolution. The size, shape, position and orientation of the reference ellipsoid will be the same as that used by the Geodetic Survey of Canada in their redefinition of the national geodetic framework.

(2) Rigorous mathematical models will be used in all phases of the redefinition process (data screening, subnetwork analysis, etc.) except where external constraints must be imposed.

(3) To obtain realistic estimates of the accuracies of the redefined coordinates of points (to be expressed in terms of the variance-covariance matrix of the adjusted coordinates), all sources of errors will be taken into account.

The geodetic information required for the rigorous computation of local or regional surveys will be stored in a geodetic data bank. The following geodetic information will be of particular importance to the surveying community:

(1) The redefined geodetic latitude (ϕ) and longitude (λ) of each network point;

(2) the 2 x 2 variance-covariance matrices of the redefined coordinates;

(3) the information required to compute covariance elements amongst any coordinates of points in the network.

Certain conformal mapping plane coordinates will be used for the three Maritime Provinces and mathematical transformation software will be developed to provide:

(1) Relevant conformal mapping plane or grid coordinates;

(2) the 2 x 2 variance-covariance matrix for any set of conformal mapping plane or grid coordinates;

(3) the covariance elements amongst any conformal mapping plane or grid coordinates of points in the network.

As is obvious from the foregoing, a great deal of effort is being expended in the redefinition of the Maritime geodetic network. It is being done for the benefit of all surveyors. The coordinates, and associated accuracy information, will form part of a unified homogeneous system to which all surveys may be referenced and analyzed. Furthermore, the geodetic maintenance scheme being devised in conjunction with the redefinition task, along with the geodetic data bank, will ensure surveyors of continued access to the "best" geodetic information available in the Maritime Provinces.

PART II

COORDINATE SYSTEMS

3. REFERENCE SURFACES AND GEODETIC DATUM

3.1 Reference Surfaces

In geodesy the three main reference surfaces used for terrestrial work are the terrain, geoid and ellipsoid surfaces (see Figure 3-1). We will only briefly mention these here. For a complete treatment see, for example, Krakiwsky and Wells [1971], Bomford [1971], Vaníček and Krakiwsky [in prep.].

The terrain surface is the topographic surface of the earth. It is the surface upon which observations, such as distances, directions, astronomic azimuths, are made for the purpose of determining coordinates of terrain points. These "observed values" need to be reduced from the terrain surface to the surface (ellipsoid or conformal mapping) upon which the computations are made.

The geoid surface closely corresponds to the undisturbed mean sea level on the oceans and is the continuation of mean sea level underneath the continents. Besides being the "figure" of the earth, it serves as an intermediate surface through which reductions are made down from the terrain surface onto the computation surface.

The ellipsoid surface is a mathematical surface described by an ellipse rotated about its minor axis. It is upon this surface that computations are made employing ellipsoidal geometry. The ellipsoid surface is also mapped conformally onto a plane thus giving another surface upon which computations can be made (see chapter 4). In this chapter

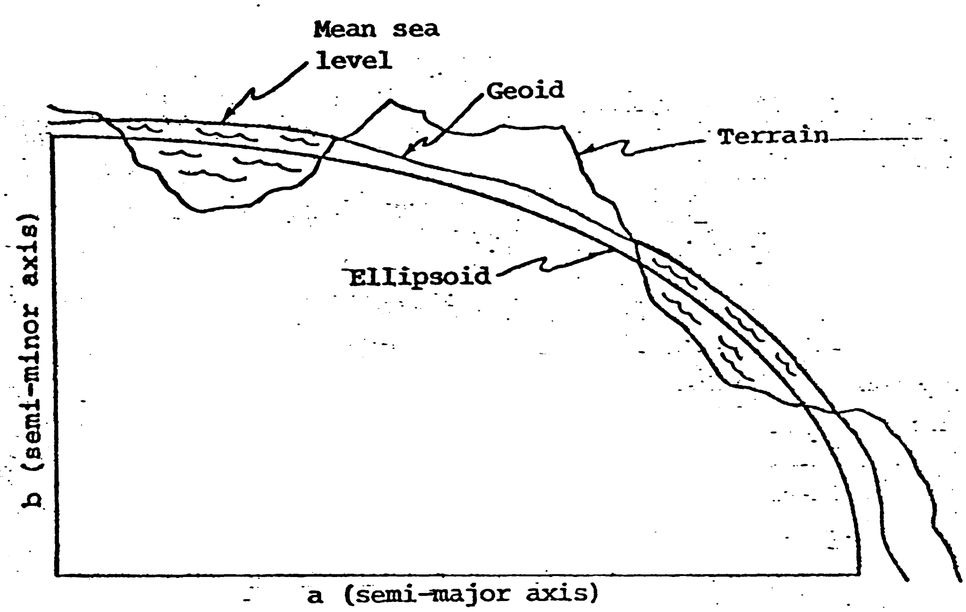

Figure 3-1

Cross-Section Showing:
Terrain, Geoid and Ellipsoid Surfaces.

we restrict the discussion up to and including the ellipsoid surface.

3.2 Geodetic Datum

A geodetic datum is a geodetic coordinate system consisting of the two parameters which define the ellipsoid and six parameters which describe the position of this elliposid within the body of the earth (see Figure 3-2). The parameter describing the size of the ellipsoid is the semi-major axis a. The shape is defined through the flattening $f = (a-b)/a$, where b is the semi-minor axis. The simplest way to give the position of the ellipsoid within the earth is by specifying three translations and three rotations. The translations (x_o, y_o, z_o) give the location of the geometric centre of the ellipsoid with respect to the centre of gravity of the earth. The rotations $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ give the orientations of the axes of the ellipsoid with respect to the axes of the ideal terrestrial system. Thus a geodetic datum is defined by the eight parameters $a, f, x_o, y_o, z_o, \varepsilon_x, \varepsilon_y, \varepsilon_z$. See the above references for an alternative way of defining a geodetic datum via the coordinates of the "initial point" of the geodetic network. At the time of writing this manual the above eight parameters were not yet chosen for the proposed redefined geodetic coordinate system.

4. CONFORMAL MAPPING SYSTEMS

In this chapter we treat the conformal mapping of the ellipsoid surface onto a plane. This surface, along with the coordinates of points on it, is mapped onto a flat two dimensional plane. This results in a plane representation of a curved earth (ellipsoid). Computations on this

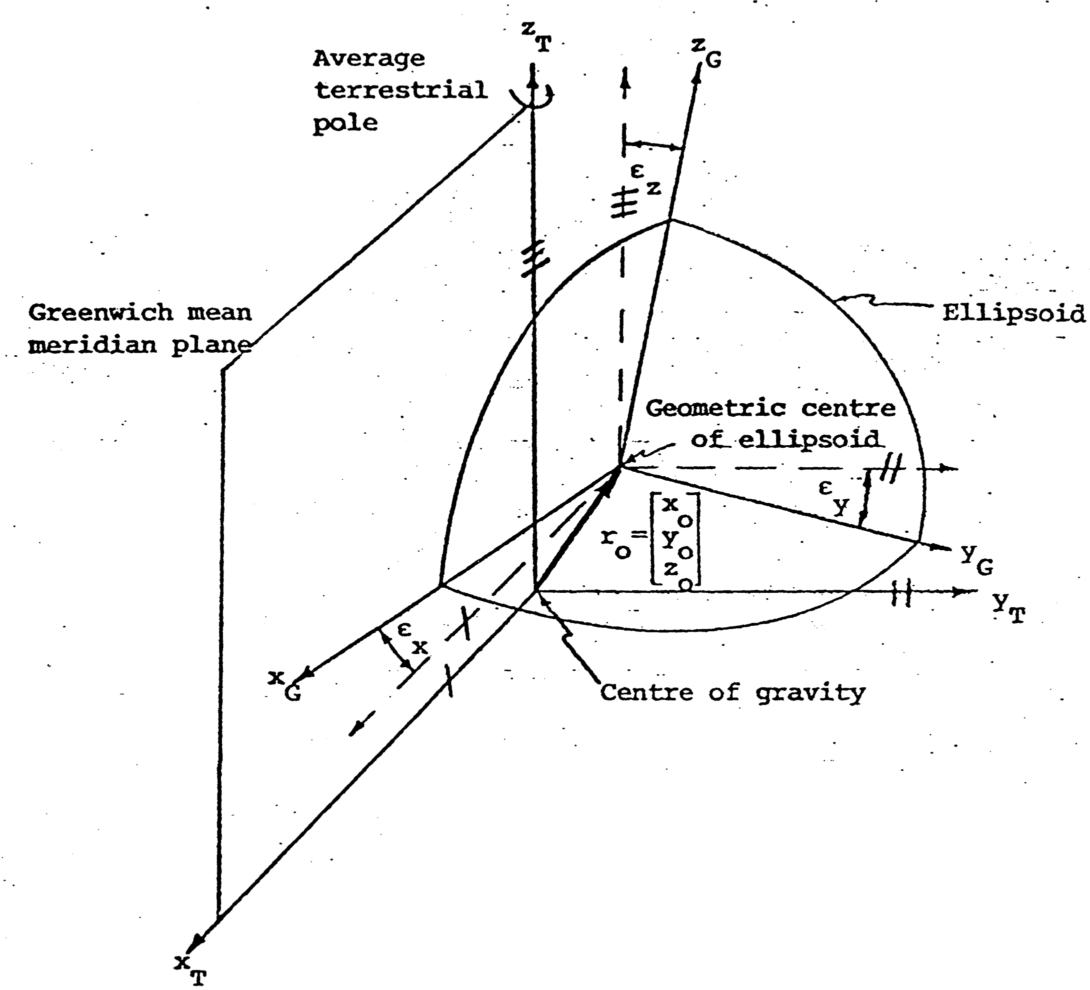

Figure 3-2

Geodetic Datum:
Position of the Ellipsoid Relative
to the Terrestrial System.

surface (conformal mapping plane) are made simpler as now we may use plane trigonometry as opposed to ellipsoidal geometry when working on the ellipsoid surface.

Mapping is a part of mathematics which deals with the mapping of one surface onto a second. We employ a very restrictive part of this general theory of mathematical mapping - that of "conformal mapping". We are interested in conformal mapping because (i) angles are preserved; and (ii) scale distortion is a minimum as compared to other types of mapping. Thus, conformal mapping is best suited to our purposes. For cartographic purposes, other types of mapping, such as equiareal, equidistance, equiazimuthal, are appropriate.

Conformal mapping of the ellipsoid surface onto a plane is given by

$$(x + i\, y) = f(\phi + i\, \lambda) \quad ,$$

where x and y are the map plane Cartesian coordinates; ϕ and λ are the geodetic latitude and longitude; f is the mapping function. Note the use of complex arithmetic. Strictly speaking, conformal mapping of the ellipsoid onto a plane cannot be given a geometric interpretation like unrolling cones and cylinders. To maintain rigour, we stick to the mathematical mapping approach.

Depending upon the conditions imposed, the particular mapping function, f, is deduced by imposing certain conditions. Theoretically, there is an infinite number of map projections that can be deduced. We are familiar with such names as Mercator, stereographic, and Transverse Mercator. In this work, we give the formulae for three conformal projections: 3° Transverse Mercator (Nova Scotia); the stereographic

(Prince Edward Island); and the stereographic (New Brunswick). Thus we will be dealing with only two mapping functions f: f_{TM} for the Transverse Mercator and f_s for the stereographic. The approach to be used is to suppress the complex arithmetic part by separating the formulae into real and imaginary parts, and dropping the imaginary identifier $i = \sqrt{-1}$ from the y part, thus producing "real" formulae for the x and y coordinates on the conformal mapping plane.

PART III

COORDINATE TRANSFORMATIONS

5. THREE DIMENSIONAL COORDINATE TRANSFORMATIONS

Equations given in this chapter are for transforming ellipsoidal coordinates (ϕ, λ, h) to three dimensional (3-D) Cartesian coordinates (X, Y, Z) referred to geocentric Cartesian system of coordinates. Before the equations are given, we provide a review of the notation used in this chapter.

5.1 Notation

a,b ≡ semi-major and semi-minor axes of the reference ellipsoid,

e ≡ first eccentricity of the ellipsoid,

$$e^2 = (a^2 - b^2)/a^2; \tag{5-1}$$

ϕ, λ ≡ ellipsoidal coordinates; latitude (positive north of the equator) and longitude (positive east of the Greenwich meridian) respectively;

h ≡ height of a point above the ellipsoid measured along the ellipsoid normal to that point;

N ≡ radius of curvature of the reference ellipsoid in the prime vertical plane, $N = a/(1-e^2 \sin^2 \phi)^{1/2}$; (5-2)

M ≡ radius of curvature of the reference ellipsoid in the meridian plane, $M = a(1-e^2)/(1-e^2 \sin^2 \phi)^{3/2}$; (5-3)

X,Y,Z ≡ Cartesian coordinates of a point referred to a geocentric Cartesian coordinate system;

X_o, Y_o, Z_o ≡ translation components from the origin of the Cartesian coordinate system (X,Y,Z) to the centre of the reference ellipsoid.

5.2 Coordinate Transformation Formulae

5.2.1 Transformation of (ϕ, λ, h) to (X, Y, Z)

The transformation of (ϕ, λ, h) ellipsoidal coordinates to (X, Y, Z) Cartesian coordinates is given by: [Krakiwsky & Wells, 1971]

$$X = X_o + (N + h) \cos \phi \cos \lambda \quad , \tag{5-4}$$

$$Y = Y_o + (N + h) \cos \phi \sin \lambda \quad , \tag{5-5}$$

$$Z = Z_o + (N(1-e^2) + h) \sin \phi \quad , \tag{5-6}$$

where, N and e^2 are computed from equations (5-2) and (5-1) respectively, and X_o, Y_o, and Z_o are pre-determined translation components. The ellipsoidal parameters a and b are chosen in this manual to be the Clarke 1866 values

$$a = 6\ 378\ 206.4 \text{ m}$$

$$b = 6\ 356\ 583.8 \text{ m} \quad .$$

5.2.2 Transformation of (X, Y, Z) to (ϕ, λ, h)

The transformation of (X, Y, Z) geocentric Cartesian coordinate to (ϕ, λ, h) ellipsoidal coordinates may be done as follows: [Krakiwsky & Wells, 1971]

Let

$$x = X - X_o \quad , \tag{5-7}$$

$$y = Y - Y_o \quad , \tag{5-8}$$

then

$$\lambda = \tan^{-1} \left(\frac{y}{x}\right) \quad . \tag{5-9}$$

Also, let

$$z = Z - Z_o \quad , \tag{5-10}$$

and

$$S = (x^2 + y^2)^{1/2} \quad . \tag{5-11}$$

Now, we find first approximations for ϕ and h,

$$\phi_1 = \tan^{-1} \left[\frac{z/S}{\left(1 - \frac{e^2 a}{a + h}\right)}\right] \quad , \tag{5-12}$$

$$h_1 = (x^2 + y^2 + z^2)^{1/2} - a \quad , \tag{5-13}$$

and then iterate the following equations to get more accurate values for ϕ and h,

$$N_i = \frac{a}{(1-e^2 \sin^2 \phi_{i-1})^{1/2}} \quad ,$$

$$h_i = \frac{S}{\cos \phi_{i-1}} - N_i \quad , \tag{5-14}$$

$$\phi_i = \tan^{-1}\left[\frac{z/S}{(1-e^2 \cdot \frac{N_i}{N_i + h_i})}\right] \quad .$$

We iterate equations (5-14) until $|h_i - h_{i-1}|$ and $|\phi_i - \phi_{i-1}|$ are smaller than some pre-determined limit.

5.2.3 Error propagation

The error propagation for the three-dimensional transformations of this chapter is done as follows [Thomson, 1976; Mikhail, 1976].

Let

$$\underline{C}_{\phi,\lambda,h} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} & \sigma_{\phi h} \\ \sigma_{\phi\lambda} & \sigma_\lambda^2 & \sigma_{\lambda h} \\ \sigma_{\phi h} & \sigma_{\lambda h} & \sigma_h^2 \end{bmatrix} \quad \text{in units of}$$

$$\begin{bmatrix} \text{seconds}^2 & \text{seconds}^2 & \text{second metres} \\ \text{seconds}^2 & \text{seconds}^2 & \text{second metres} \\ \text{second-metres} & \text{second-metres} & \text{metres}^2 \end{bmatrix} , \quad \text{(5-15}$$

$$\underline{C}_{X,Y,Z} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z^2 \end{bmatrix} \text{ metres}^2 \quad , \qquad (5\text{-}16)$$

and,

$$\underline{B} = \frac{1}{\rho''} \begin{bmatrix} -(M+h) \sin\phi \cos\lambda & -(N+h) \cos\phi \sin\lambda & \cos\phi \cos\lambda \\ -(M+h) \sin\phi \sin\lambda & (N+h) \cos\phi \cos\lambda & \cos\phi \sin\lambda \\ (M+h) \cos\phi & 0 & \sin\phi \end{bmatrix} , \qquad (5\text{-}17)$$

where $\rho'' = 206\ 264.806 \ldots$.

Then, [Thomson, 1976]

$$\underline{C}_{X,Y,Z} = \underline{B}\, \underline{C}_{\phi,\lambda,h}\, \underline{B}^t \quad , \qquad (5\text{-}18)$$

where $\underline{B}^t$ is the transpose of $\underline{B}$, and

$$\underline{C}_{\phi,\lambda,h} = \underline{B}^{-1}\, \underline{C}_{X,Y,Z}\, (\underline{B}^{-1})^t \quad , \qquad (5\text{-}19)$$

where $\underline{B}^{-1}$ is the inverse of the matrix $\underline{B}$., which is given by

$$\underline{B}^{-1} = \rho'' \begin{bmatrix} \frac{-\sin\phi \cos\lambda}{(M+h)} & \frac{-\sin\phi \sin\lambda}{(M+h)} & \frac{\cos\phi}{(M+h)} \\ \frac{-\sin\lambda}{(N+h)\cos\phi} & \frac{\cos\lambda}{(N+h)\cos\phi} & 0 \\ \cos\phi \cos\lambda & \cos\phi \sin\lambda & \sin\phi \end{bmatrix} .$$

5.3 Numerical Example

For the numerical example, we choose, arbitrarily, the point

$$\phi = 46° \; 34' \; 10''.035 \;\; N \quad ,$$

$$\lambda = 67° \; 13' \; 03''.086 \;\; W \quad , = -67° \; 13' \; 03''.086$$

$$h = 24.654 \text{ m} \quad ,$$

with a covariance matrix

$$\underline{C}_{\phi,\lambda,h} = \begin{bmatrix} 1 \times 10^{-8} & -8 \times 10^{-10} & -4 \times 10^{-9} \\ -8 \times 10^{-10} & 2 \times 10^{-8} & -6 \times 10^{-9} \\ -4 \times 10^{-9} & -6 \times 10^{-9} & 1 \times 10^{-2} \end{bmatrix} ,$$

with units as in equation (5-15).

Also we choose, arbitrarily, the translation components to be

$$X_o = - 15.000 \text{ m} \quad ,$$

$$Y_o = 150.000 \text{ m} \quad ;$$

$$Z_o = 180.000 \text{ m} \quad .$$

The transformation from (ϕ, λ, h) to (X, Y, Z) is done via equations (5-4), (5-5), and (5-6). We get

$$N = 6\;389\;620.93258 \text{ m} \quad ,$$

and

$$X = 1\;700\;993.900 \text{ m} \quad ,$$

$$Y = -4\;049\;857.257 \text{ m} \quad ,$$

$$Z = 4\;608\;985.532 \text{ m} \quad .$$

We now take these geocentric Cartesian coordinates and transform them back to ellipsoidal coordinates as follows. From equations (5-7) and (5-8) we get

$$x = X - X_o = 1\ 701\ 008.900 \text{ m} \quad ,$$

$$y = Y - Y_o = -4\ 050\ 007.257 \text{ m} \quad ,$$

and from equations (5-9), (5-10), and (5-11),

$$\lambda = \tan^{-1} \left(\frac{y}{x}\right) = -1.1731671069 \text{ radians},$$

$$= 67° \ 13' \ 03''.086 \text{ W}, \quad = -67° \ 13' \ 03''.086$$

$$z = Z - Z_o = 4\ 608\ 805.532 \text{ m} \ ,$$

and

$$S = (x^2 + y^2)^{1/2} = 4\ 392\ 720.12062 \text{ m} .$$

Now, from equations (5-12) and (5-13) we find first approximations for ϕ and h,

$$\phi_1 = \tan^{-1} \left[\frac{z/S}{(1 - \frac{e^2 a}{a + h})}\right] = 0.8127963752 \text{ radians} \quad ,$$

$$h_1 = (x^2 + y^2 + z^2)^{1/2} - a = -11\ 332.87437 \text{ m} \quad ,$$

and then iterate according to equations (5-14) ,

$$N_2 = \frac{a}{(1-e^2 \sin^2 \phi_1)^{1/2}} = 6\ 389\ 621.06409 \text{ m} \quad ,$$

$$h_2 = \frac{S}{\cos \phi_1} - N_2 = 65.48740 \text{ m} \quad ,$$

$$\phi_2 = \tan^{-1} \left[\frac{z/S}{(1-e^2 \frac{N_2}{N_2 + h_2})}\right] = 0.8127902843 \text{ radians},$$

$$N_3 = \frac{a}{(1-e^2 \sin^2 \phi_2)^{1/2}} = 6\ 389\ 620.93210 \text{ m} \quad ,$$

$$h_3 = \frac{S}{\cos\phi_2} - N_3 = 24.50772 \text{ m} \quad ,$$

$$\phi_3 = \tan^{-1}\left[\frac{z/S}{(1-e^2 \frac{N_3}{N_3 + h_3})}\right] = 0.8127903061 \text{ radians} \quad ,$$

continuing,

$$N_4 = 6\ 389\ 620.93258 \text{ m} \quad ,$$

$$h_4 = 25.654 \text{ m} \quad ,$$

$$\phi_4 = 0.8127903061 \text{ radians.}$$

The next iteration (not shown here) gives very insignificant changes in N, h and ϕ, thus we stop the iteration.

Thus we have

$$\phi = 0.8127903061 \text{ radians} \quad ,$$

$$= 46^\circ\ 34'\ 10''035 \text{ N} \quad ,$$

$$\lambda = -1.1731671069 \text{ radians}$$

$$= 67^\circ\ 13'\ 03''086 \text{ W} \quad , = -67^\circ\ 13'\ 03''086$$

$$h = 24.654 \text{ metres.}$$

For the error propagation, we have

$$M = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}} = 6\ 369\ 107.22112 \text{ m} \quad ,$$

$$N = \frac{a}{(1-e^2 \sin^2\phi)^{1/2}} = 6\ 389\ 620.93258 \text{ m} \quad ,$$

$$M + h = 6\ 369\ 131.87512 \text{ m} \quad ,$$

$$N + h = 6\ 289\ 645.58658 \text{ m} \quad ,$$

$$\sin\phi = 0.7262082639 \quad ,$$

$$\cos\phi = 0.6874747685 \quad ,$$

$$\sin \lambda = -0.9219816299 \quad ,$$

$$\cos \lambda = 0.3872336169 \quad .$$

Substituting these values in equation (8-17) gives

$$\underline{B} = \frac{1}{\rho''} \cdot \begin{bmatrix} -1791077.922 & 4050007.257 & 0.2662133411 \\ 4264456.571 & 1701008.900 & -0.6368391076 \\ 4378617.462 & 0 & 0.7262082640 \end{bmatrix}$$

and from equation (5-18)

$$\underline{C}_{X,Y,Z} = \underline{B}\, \underline{C}_{\phi,\lambda,h}\, \underline{B}^t$$

$$= \begin{bmatrix} 7.174 \times 10^{-4} & -1.686 \times 10^{-3} & 1.931 \times 10^{-3} \\ -1.686 \times 10^{-3} & 4.023 \times 10^{-3} & -4.599 \times 10^{-3} \\ 1.931 \times 10^{-3} & -4.599 \times 10^{-3} & 5.278 \times 10^{-3} \end{bmatrix} m^2$$

or from equation (5-19),

$$\underline{C}_{\phi,\lambda,h} = \underline{B}^{-1}\, \underline{C}_{X,Y,Z}\, (\underline{B}^{-1})^t$$

$$= \begin{bmatrix} 1 \times 10^{-8} & -8 \times 10^{-10} & -4 \times 10^{-9} \\ -8 \times 10^{-10} & 2 \times 10^{-8} & -6 \times 10^{-9} \\ -4 \times 10^{-9} & -6 \times 10^{-9} & 1 \times 10^{-2} \end{bmatrix} ,$$

with units as in equation (5-15).

6. NEW BRUNSWICK STEREOGRAPHIC DOUBLE PROJECTION

6.1 Notation

The symbols used in this chapter are listed here for convenience:

a, b ≡ semi-major and semi-minor axes of the reference ellipsoid;

e ≡ first eccentricity of ellipsoid; $e^2 = (a^2 - b^2)/a^2$.

ϕ, λ ≡ ellipsoidal coordinates, latitude (positive north of equator) and longitude (positive east of Greenwich) respectively.

χ, Λ ≡ spherical coordinates (on the conformal sphere), latitude (positive north) and longitude (positive east) respectively.

X, Y ≡ N.B. stereographic grid coordinates; X is positive east; Y is positive north.

$\phi_o, \lambda_o, \chi_o, \Lambda_o, X_o, Y_o$ ≡ coordinates of the origin of the projection.

$\Delta\Lambda$ = spherical longitude of point to be mapped minus the spherical longitude of the origin ($\Delta\Lambda = \Lambda - \Lambda_o$).

N ≡ radius of curvature of the ellipsoid in the prime vertical plane ($N = a/(1 - e^2 \sin^2 \phi)^{1/2}$).

M ≡ radius of curvature of the ellipsoid in the meridian plane ($M = a(1 - e^2)/(1 - e^2 \sin^2 \phi)^{3/2}$).

R ≡ radius of the conformal sphere ($R = (MN)^{1/2}$ evaluated at ϕ_o).

k_o ≡ scale factor at the origin of the projection.

6.2 Coordinate Transformation Formulae

When transforming ϕ, λ coordinates to New Brunswick X, Y stereographic grid coordinates rigorously, we must first transform the ϕ, λ coordinates to χ, Λ coordinates, then transform the χ, Λ coordinates to X, Y. Conversely, when tranforming X, Y stereographic coordinates to ϕ, λ, we first transform X, Y to χ, Λ, then χ, Λ to ϕ, λ.

6.2.1 Transformation from ϕ, λ to New Brunswick stereographic grid coordinates X, Y

The transformation from ϕ, λ to χ, Λ is done as follows [Jordan/Eggert, 1948; Thomson et al., 1977]:

$$\chi = 2\left\{\tan^{-1}\left[c_2\left\{\tan\left(45° + \frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2}\right\}^{c_1}\right] - 45°\right\} \qquad (6\text{-}1)$$

$$\Lambda = c_1\,\lambda \qquad (6\text{-}2)$$

where

$$c_1 = \left\{1 + \frac{e^2\cos^4\phi_o}{1-e^2}\right\}^{1/2}, \qquad (6\text{-}3a)$$

$$c_2 = \tan\left(45° + \frac{\chi_o}{2}\right)\left\{\tan\left(45° + \frac{\phi_o}{2}\right)\left(\frac{1-e\sin\phi_o}{1+e\sin\phi_o}\right)^{e/2}\right\}^{-c_1}, \qquad (6\text{-}3b)$$

and

$$\chi_o = \sin^{-1}\left[\frac{\sin\phi_o}{c_1}\right] \qquad (6\text{-}4)$$

The transformation from χ, Λ to New Brunswick stereographic X, Y is [Thomas, 1952; Thomson et al., 1977]:

$$X = X_o + \frac{2\,k_o\,R\cos\chi\sin\Delta\Lambda}{1 + \sin\chi\sin\chi_o + \cos\chi\cos\chi_o\cos\Delta\Lambda}, \qquad (6\text{-}5)$$

$$Y = Y_o + \frac{2\,k_o\,R(\sin\chi\cos\chi_o - \cos\chi\sin\chi_o\cos\Delta\Lambda)}{1 + \sin\chi\sin\chi_o + \cos\chi\cos\chi_o\cos\Delta\Lambda}, \qquad (6\text{-}6)$$

where, for the New Brunswick stereographic projection,

$\phi_o = 46°\ 30'$

$\lambda_o = -66°\ 30'$

$= 66°\ 30'$ W

$$R = 6\ 379\ 303.38 \text{ metres}$$

$$X_o = 300\ 000 \text{ metres}$$

$$Y_o = 800\ 000 \text{ metres}$$

$$k_o = 0.999912$$

and where χ_o is computed from equation (6-4) and $\Delta\Lambda = \Lambda - \Lambda_o$ where $\Lambda_o = c_1 \lambda_o$ from equation (6-2). The ellipsoidal parameters a and b are chosen, in this manual, to be the Clarke 1866 values

$$a = 6\ 378\ 206.4 \text{ m} \quad ,$$

$$b = 6\ 356\ 583.8 \text{ m} \quad ,$$

although these may not be chosen in the redefinition.

6.2.2 Transformation from X, Y to ϕ, λ

The transformation from X, Y to χ, Λ is as follows:

Let

$$X' = (X - X_o)/k_o \quad , \tag{6-7}$$

$$Y' = (Y - Y_o)/k_o \quad , \tag{6-8}$$

$$S = ((X')^2 + (Y')^2)^{1/2} \quad , \tag{6-9}$$

$$\cos \beta = \frac{X'}{S} \quad (\text{if } S = 0,\ \cos \beta = 1) \quad , \tag{6-10}$$

$$\sin \beta = \frac{Y'}{S} \quad (\text{if } S = 0,\ \sin \beta = 0) \quad , \tag{6-11}$$

and

$$\delta = 2 \tan^{-1} \left[\frac{S}{2R}\right] \quad , \tag{6-12}$$

then

$$\chi = \sin^{-1} [\sin \chi_o \cos \delta + \sin \delta \cos \chi_o \sin \beta] \quad , \tag{6-13}$$

$$\Lambda = \Lambda_o + \sin^{-1} \left[\frac{\sin \delta \cos \beta}{\cos \chi}\right] \quad , \tag{6-14}$$

where χ_o is computed from equation (6-4) and $\Lambda_o = c_1 \lambda_o$ from equation (6-2).

The transformation from χ, Λ to ϕ, λ is done via the Newton-Raphson iteration technique as follows:

let a first approximation of ϕ be;

$$\phi_1 = \chi \quad ,$$

then

$$\phi_2 = \phi_1 - \frac{f(\phi)}{f'(\phi)}\Bigg|_{\phi_1} \tag{6-15}$$

etc.,

$$\phi_n = \phi_{n-1} - \frac{f(\phi)}{f'(\phi)}\Bigg|_{\phi_{n-1}} \quad ,$$

where, from equation (6-1)

$$f(\phi) = c_2 \left[\tan(45° + \frac{\phi}{2})(\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}\right]^{c_1} - \tan(45° + \frac{\chi}{2}) = 0 \quad ,$$

(note that the value of χ which was computed from equation (6-13) does not change through the iterations!)

and after differentiation we have;

$$f'(\phi) = c_1 c_2 \left[\tan(45° + \frac{\phi}{2})(\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}\right]^{(c_1-1)} \cdot \Bigg[(\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}$$

$$\{\frac{1}{2} \sec^2(45° + \frac{\phi}{2}) - \frac{e^2 \cos \phi}{(1-e^2 \sin^2 \phi)} \cdot \tan(45° + \frac{\phi}{2})\}\Bigg] \; .$$

We iterate equations (6-15) until $|\phi_n - \phi_{n-1}|$ is less than, say .00001 seconds in double precision FORTRAN (which corresponds to less than 1 millimetre on the ellipsoid) or until it is smaller than our required accuracy.

The ellipsoidal longitude λ is computed from (see equation (6-2)):

$$\lambda = \frac{\Lambda}{c_1} \tag{6-16}$$

6.2.3 Error propagation: New Brunswick double stereographic projection

When transforming ϕ, λ coordinates to New Brunswick stereographic grid coordinates, we may also wish to transform the ϕ, λ covariance matrix to obtain the covariance matrix of the New Brunswick grid coordinates, X, Y; and conversely. To do this, we propagate errors through equations (6-1), (6-2), (6-5), and (6-6) according to the covariance law [Wells and Krakiwsky, 1971; Mikhail, 1976]. This results in the following [Thomson et al., 1977]:

If we let

$$\underline{C}_{X,Y} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix},$$

denote the covariance matrix of the New Brunswick stereographic grid coordinates in metres^2, and

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\phi\lambda} & \sigma_\lambda^2 \end{bmatrix},$$

denote the covariance matrix of the ellipsoidal coordinates of the same point in arc seconds squared we have

$$\underline{C}_{X,Y} = \underline{B}\,\underline{C}_{\phi,\lambda}\,\underline{B}^t , \qquad (6\text{-}17)$$

where

$$\underline{B} = \begin{bmatrix} \dfrac{-m.u}{n\rho''} & \dfrac{c_1 p}{n\rho''}\cos\chi \\[2ex] \dfrac{p.u}{n\rho''} & \dfrac{c_1 m}{n\rho''}\cos\chi \end{bmatrix}, \qquad (6\text{-}18)$$

and $$\rho'' = 206\ 264.806.....$$

$$c_1 = \{1 + \frac{e^2 \cos^4 \phi_o}{1-e^2}\}^{1/2} \quad \text{(repeated) (6-3a)}$$

$$c_2 = \tan(45° + \frac{\chi_o}{2}) \{\tan(45° + \frac{\phi_o}{2}) (\frac{1-e \sin \phi_o}{1+e \sin \phi_o})^{e/2}\}^{-c_1} \quad \text{(repeated) (6-3b)}$$

$$m = 2 k_o R \sin \Delta\Lambda(\sin \chi + \sin \chi_o), \quad (6\text{-}19)$$

$$n = (1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2, \quad (6\text{-}20)$$

$$p = 2 k_o R\{\cos \Delta\Lambda(1 + \sin \chi \sin \chi_o) + \cos \chi \cos \chi_o\}, \quad (6\text{-}21)$$

$$u = \frac{c_1 c_2 \{q.\tan r\}^{(c_1-1)} . q\{\frac{1}{2} \sec^2 r - \frac{e^2 \cos \phi \tan r}{(1-e^2 \sin^2 \phi)}\}}{1 + c_2^2 \{q . \tan r\}^{2c_1}}, \quad (6\text{-}22)$$

where

$$r = 45° + \frac{\phi}{2}, \quad (6\text{-}23)$$

and

$$q = \{\frac{1-e \sin \phi}{1+e \sin \phi}\}^{e/2} \quad . \quad (6\text{-}24)$$

For the inverse propagation of errors, i.e., propagating errors through the transformation (X, Y) to (ϕ,λ) we have;

$$\underline{C}_{\phi,\lambda} = \underline{B}^{-1} \underline{C}_{X,Y} (\underline{B}^{-1})^t, \quad (6\text{-}25)$$

where

$$\underline{B}^{-1} = \begin{bmatrix} \frac{mn . \rho''}{u(m^2 + p^2)} & \frac{pn . \rho''}{u(m^2 + p^2)} \\ \frac{pn . \rho''}{C_1 \cos \chi(m^2+p^2)} & \frac{mn . \rho''}{C_1 \cos \chi(m^2 + p^2)} \end{bmatrix}.$$

6.3 Numerical Example

For the numerical example, we choose a point with

$$\phi = 47° \; 03' \; 24''.644 \;\; N \quad ,$$

$$\lambda = 65° \; 29' \; 03''.453 \;\; W \quad ,$$

$$= -65° \; 29' \; 03''.453 \; .$$

6.3.1 (ϕ, λ) to New Brunswick (X, Y)

First, we compute c_1 from equation (6-3),

$$c_1 = \{1 + \frac{e^2 \cos^4 \phi_o}{1-e^2}\}^{1/2} \quad ,$$

with $e^2 = \frac{a^2 - b^2}{a^2} = 0.006768658$

and, $\phi_o = 46° \; 30' \; 00''$.

We get,

$$c_1 = 1.0007647244 \quad .$$

Next, from Equation (6-4)

$$\chi_o = \sin^{-1} [\frac{\sin \phi_o}{c_1}] \quad ,$$

$$\chi_o = 46° \; 27' \; 13''.974 \quad .$$

Now, from equation (6-3b)

$$c_2 = \tan(45° + \frac{\chi_o}{2}) \{\tan(45° + \frac{\phi_o}{2}) (\frac{1-e \sin \phi_o}{1+e \sin \phi_o})^{e/2}\}^{-c_1}$$

$$C_2 = 1.0030525528 \quad .$$

Next we have, from equation (6-1)

$$\chi = 2\{\tan^{-1} [c_2\{\tan(45° + \frac{\phi}{2}) (\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}\}^{c_1}] - 45°\}.$$

and with $= 47^\circ\ 03'\ 24''.644$ N ,

$$\chi = 47^\circ\ 00'\ 35''.490 \quad \text{N} .$$

Further, from equation (6-2),

$$\Lambda = c_1 \lambda$$

and with $\lambda = 65^\circ\ 29'\ 03''.453$ W, $(= -65^\circ\ 29'\ 03''.453)$

$$\Lambda = 65^\circ\ 32'\ 03''.726 \ \text{W} , \quad (= -65^\circ\ 32'\ 03''.726) .$$

Finally, we compute the grid coordinates from equations (6-5) and (6-6), i.e.

$$X = X_o + \frac{2\, k_o\, R \cos\chi \sin\Delta\Lambda}{1 + \sin\chi \sin\chi_o + \cos\chi \cos\chi_o \cos\Delta\Lambda}$$

$$Y = Y_o + \frac{2\, k_o\, R(\sin\chi \cos\chi_o - \cos\chi \sin\chi_o \cos\Delta\Lambda)}{1 + \sin\chi \sin\chi_o + \cos\chi \cos\chi_o \cos\Delta\Lambda}$$

where

$$X_o = 300\ 000 \text{ metres} ,$$

$$Y_o = 800\ 000 \text{ metres} ,$$

$$R = 6\ 379\ 303.376 \text{ metres} ,$$

$$k_o = 0.999912 ,$$

and

$$\Delta\Lambda = \Lambda - \Lambda_o = 65^\circ\ 32'\ 03''.726 \ \text{W} - \Lambda_o ,$$

$$= -65^\circ\ 32'\ 03''.726 - \Lambda_o$$

where, from equation (6-2)

$$\Lambda_o = c_1\, \lambda_o \quad (\lambda_o = 66^\circ\ 30' \ \text{W} ,$$

$$= -66^\circ\ 30')$$

$$\Lambda_o = 66^\circ\ 33'\ 03''.072\ \text{W}, = -66^\circ\ 33'\ 03''.072.$$

Thus,

$$\Delta\Lambda = -\ 65^\circ\ 32'\ 03''.726 - (-66^\circ\ 33'\ 03''.072)$$

$$\Delta\Lambda = +\ 01^\circ\ 00'\ 59.346'' \quad .$$

Finally, we get,

$$X = 300\ 000 + 77\ 164.887 = 377\ 164.887 \quad \text{m}$$

$$Y = 800\ 000 + 62\ 395.774 = 862\ 395.774 \ \text{m} \quad .$$

6.3.2 New Brunswick (X, Y) to (ϕ, λ)

Now we will take the computed grid coordinates

$$X = 377\ 164.887 \quad \text{m}$$

$$Y = 862\ 395.774 \quad \text{m}$$

and transform them to ϕ, λ.

From equations (6-7), (6-8), and (6-9), we have

$$X' = (X - X_o)/k_o$$

$$= 77\ 171.67809 \quad \text{m}$$

$$Y' = (Y - Y_o)/k_o$$

$$= 62\ 401.26488 \ \text{m}$$

$$s = ((X')^2 + (Y')^2)^{1/2}$$

$$= 99\ 244.07165 \quad \text{m}$$

Further, we have from equations (6-10) and (6-11),

$$\cos\beta = \frac{X'}{s} = 0.7775948407 \quad ,$$

$$\sin\beta = \frac{Y'}{s} = 0.6287656667 \quad ,$$

and from equation (6-12),

$$\delta = 2\tan^{-1}\left[\frac{s}{2R}\right] \quad ,$$

$$\delta = 0° \ 53' \ 28''.830 \quad .$$

From the above and equations (6-13) and (6-14), we have

$$\chi = \sin^{-1}[\sin \chi_o \cos \delta + \sin \delta \cos \chi_o \sin \beta] \quad ,$$

$$\Lambda = \Lambda_o + \sin^{-1} [\frac{\sin \delta \cos \beta}{\cos \chi}] \quad ,$$

$$\chi = 47° \ 00' \ 35''.490$$

$$\Lambda = 66° \ 33' \ 03''.072 \ W + 1° \ 00' \ 59''.340 \quad ,$$

$$= -66° \ 33' \ 03''.072 + 1° \ 00' \ 59''.340$$

$$\Lambda = 65° \ 32' \ 03''.726 \quad W$$

$$= -65° \ 32' \ 03''.726 \quad .$$

Next, we compute λ from equation (6-16)

$$\lambda = \frac{\Lambda}{c_1}$$

$$\lambda = 65° \ 29' \ 03''.453 \quad W \ ,$$

$$= -65° \ 29' \ 03''.453$$

and ϕ via the iteration process of equations (6-15) as follows:

$$\phi_1 = \chi = 47° \ 00' \ 35''.490 \quad ,$$

then

$$\phi_2 = \phi_1 - \frac{f(\phi)}{f'(\phi)} \bigg|_{\phi_1}$$

where

$$f(\phi)\bigg|_{\phi_1} = c_2[\tan(45° + \frac{\phi_1}{2}) (\frac{1-e \sin \phi_1}{1+e \sin \phi_1})^{e/2}]^{c_1} - \tan (45° + \frac{\chi}{2})$$

$$f'(\phi)\Big|_{\phi_1} = c_1 c_2 [\tan(45° + \frac{\phi_1}{2}) \left(\frac{1-e \sin \phi_1}{1+e \sin \phi_1}\right)^{e/2}]^{(c_1-1)} . [(\frac{1-e \sin \phi_1}{1+e \sin \phi_1})^{e/2}$$

$$\{\frac{1}{2} \sec^2 (45° + \frac{\phi_1}{2}) - \frac{e^2 \cos \phi_1}{(1-e^2 \sin^2 \phi_1)} . \tan (45° + \frac{\phi_1}{2})\}] \quad .$$

We get

$$f(\phi)\Big|_{\phi_1} = -0.0030461313 \quad ,$$

$$f'(\phi)\Big|_{\phi_1} = 3.7106130921 \quad ,$$

and

$$\phi_2 = 47° \; 00' \; 35''.490 - (-0° \; 02' \; 49''.326)$$

$$\phi_2 = 47° \; 03' \; 24''.816 \quad .$$

Now

$$\phi_3 = \phi_2 - \frac{f(\phi)}{f'(\phi)}\Big|_{\phi_2} \quad .$$

We get

$$f(\phi)\Big|_{\phi_2} = 0.0000031825 \quad ,$$

$$f'(\phi)\Big|_{\phi_2} = 3.7183708003 \quad ,$$

and

$$\phi_3 = 47° \; 03' \; 24''.816 - 00° \; 00' \; 0''.168 \quad ,$$

$$\phi_3 = 47° \; 03' \; 24''.648 \quad .$$

Further

$$\phi_4 = \phi_3 - \frac{f(\phi)}{f'(\phi)}\Big|_{\phi_3} ,$$

and

$$f(\phi)\Big|_{\phi_3} = 0.34659 \times 10^{-11} \quad ,$$

$$f'(\phi)\Big|_{\phi_3} = 3.7183626988 \quad ,$$

and

$$\phi_4 = 47° \; 03' \; 24''.648 - 00° \; 00' \; 0''.004 \quad .$$

Finally we have,

$$\phi_5 = \phi_4 - \frac{f(\phi)}{f'(\phi)}\Big|_{\phi_4} \quad ,$$

and

$$f(\phi)\Big|_{\phi_4} = 0 \quad ,$$

which gives,

$$\left.\frac{f(\phi)}{f'(\phi)}\right|_{\phi_4} = 0 \quad .$$

Thus, $\phi_5 = \phi_4$

and,

$$\phi = 47° \ 03' \ 24\overset{''}{.}644 \quad .$$

Thus, we have performed the direct and inverse transformations of coordinates.

6.3.3 Error propagation

We have, from equation (6-17)

$$\underline{C}_{X,Y} = \underline{B}\,\underline{C}_{\phi,\lambda}\,\underline{B}^t$$

where, $\underline{C}_{\phi,\lambda}$ is given as, say

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\phi\lambda} & \sigma_\lambda^2 \end{bmatrix} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \sec^2$$

$\underline{B}$ is given by equation (6-18) as

$$\underline{B} = \begin{bmatrix} \dfrac{-m.u}{n\rho''} & \dfrac{c_1}{n\rho''}\ p \ \cos\chi \\ \dfrac{p.u}{n\rho''} & \dfrac{c_1}{n\rho''}\ m \cos\chi \end{bmatrix} ,$$

where

$$\rho'' = 206\ 264.806$$

$$c_1 = \left\{ 1 + \frac{e^2 \cos^4 \phi_o}{1-e^2} \right\}^{1/2} = 1.0007647244 \quad ,$$

$$m = 2\,k_o\,R \sin \Delta\Lambda\,(\sin\chi + \sin\chi_o) = 329585.7828\ ,$$

$$n = (1 + \sin\chi \sin\chi_o + \cos\chi \cos\chi_o \cos\Delta\Lambda)^2 = 3.999515991\ ,$$

$$p = 2\,k_o\,R\{\cos\Delta\Lambda(1 + \sin\chi \sin\chi_o) + \cos\chi \cos\chi_o\} = 25511295.3578,$$

$$u = \frac{c_1 c_2 \{q.\tan r\}^{(c_1-1)} . q\{\frac{1}{2}\sec^2 r - \frac{e^2 \cos\phi \tan r}{(1-e^2 \sin^2\phi)}\}}{1 + c_2^2\{q\ .\ \tan r\}^{2c_1}} = 0.4992440249$$

where r and q are computed from equations (6-23) and (6-24) respectively.

Thus we have

$$\underline{C}_{X,Y} = \underline{B}\ \underline{C}_{\phi,\lambda}\ \underline{B}^t$$

$$= (\frac{1}{\rho''}) \cdot \begin{bmatrix} -41\ 140.9113 & 4\ 352\ 715.07 \\ 3\ 184\ 475.77 & 56\ 233.6402 \end{bmatrix} \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \cdot$$

$$(\frac{1}{\rho''}) \cdot \begin{bmatrix} -41\ 140.9113 & 3\ 184\ 475.77 \\ 4\ 352\ 715.07 & 56\ 233.6402 \end{bmatrix}$$

$$\underline{C}_{X,Y} = \begin{bmatrix} 8.9000 \times 10^{-6} & 3.4486 \times 10^{-7} \\ 3.4486 \times 10^{-7} & 2.3918 \times 10^{-6} \end{bmatrix} m^2\ .$$

The inverse propagation of errors is simply,

$$\underline{C}_{\phi,\lambda} = \underline{B}^{-1}\ \underline{C}_{X,Y}\ (B^{-1})^t$$

$$= \rho'' \cdot \begin{bmatrix} -4.0562 \times 10^{-9} & 3.1397 \times 10^{-7} \\ 2.2970 \times 10^{-7} & 2.9676 \times 10^{-9} \end{bmatrix} \begin{bmatrix} 8.9000 \times 10^{-6} & 3.4486 \times 10^{-7} \\ 3.4486 \times 10^{-7} & 2.3918 \times 10^{-6} \end{bmatrix} \cdot$$

$$\rho'' \cdot \begin{bmatrix} -4.0562 \times 10^{-9} & 2.2970 \times 10^{-7} \\ 3.1397 \times 10^{-7} & 2.9676 \times 10^{-9} \end{bmatrix}$$

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \sec^2$$

7. PRINCE EDWARD ISLAND STEREOGRAPHIC DOUBLE PROJECTION

7.1 Notation

The symbols used in this chapter are listed here for convenience:

a, b ≡ semi-major and semi-minor axes of the reference ellipsoid:

e ≡ first eccentricity of ellipsoid; $e^2 = (a^2 - b^2)/a^2$.

ϕ, λ ≡ ellipsoidal coordinates, latitude (positive north of equator) and longitude (positive east of Greenwich) respectively.

χ Λ ≡ spherical coordinates (on the conformal sphere), latitude (positive north) and longitude (positive east) respectively.

X, Y ≡ P.E.I. stereographic grid coordinates; X is positive east; Y is positive north.

$\phi_o, \lambda_o, \chi_o, \Lambda_o, X_o, Y_o$ ≡ coordinates of the origin of the projection.

$\Delta\Lambda$ ≡ spherical longitude of point to be mapped minus the spherical longitude of the origin ($\Delta\Lambda = \Lambda - \Lambda_o$) .

N ≡ radius of curvature of the ellipsoid in the prime vertical plane

$(N = a/(1 - e^2 \sin^2 \phi)^{1/2})$.

M ≡ radius of curvature of the ellipsoid in the meridian plane

$(M = a(1 - e^2)/(1 - e^2 \sin^2 \phi)^{3/2})$.

R ≡ radius of the conformal sphere ($R = (MN)^{1/2}$ evaluated at ϕ_o).

k_o ≡ scale factor at the origin of the projection.

7.2 Coordinate Transformation Formulae

When transforming ϕ, λ coordinates to Prince Edward Island X, Y stereographic grid coordinates rigorously, we must first transform the ϕ,λ coordinates to χ,Λ coordinates, then transform the χ,Λ coordinates to X, Y. Conversely, when transforming X, Y stereographic coordinates to ϕ, λ, we first transform X, Y to χ, Λ, then χ,Λ to ϕ, Λ.

7.2.1 Transformation from ϕ, λ to Prince Edward Island Stereographic grid coordinates X, Y

The transformation from ϕ, λ to χ,Λ is done as follows [Jordan/Eggbert, 1948; Thomson et al., 1977]:

$$\chi = 2\{\tan^{-1}[c_2\{\tan(45° + \frac{\phi}{2})(\frac{1-e \sin\phi}{1+e \sin\phi})^{e/2}\}^{c_1}] - 45°\} \quad . \qquad (7\text{-}1)$$

$$\Lambda = c_1 \lambda \quad . \qquad (7\text{-}2)$$

where

$$c_1 = \{1 + \frac{e^2 \cos^4 \phi_o}{1-e^2}\}^{1/2} \qquad (7\text{-}3a)$$

$$c_2 = \tan(45° + \frac{\chi_o}{2}) \{\tan(45° + \frac{\phi_o}{2}) (\frac{1-e \sin \phi_o}{1+e \sin \phi_o})^{e/2}\}^{-c_1} . \quad (7\text{-}3b)$$

With ϕ, λ, χ, Λ all in radians, and

$$\chi_o = \sin^{-1} [\frac{\sin \phi_o}{c_1}] \quad (7\text{-}4)$$

The transformation from χ, Λ to Prince Edward Island Stereographic X, Y is [Thomas, 1952; Thomson et al., 1977]:

$$X = X_o + \frac{2 k_o R \cos \chi \sin \Delta\Lambda}{1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda} , \quad (7\text{-}5)$$

$$Y = Y_o + \frac{2 k_o R(\sin \chi \cos \chi_o - \cos \chi \sin \chi_o \cos \Delta\Lambda)}{1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda} , \quad (7\text{-}6)$$

where, the Prince Edward Island stereographic projection,

$\phi_o = 47° 15' 00"$

$\lambda_o = 63° 00' 00"$ W

$= -63° 00' 00"$

$R = 6\ 379\ 869.43$ metres,

$X_o = 700\ 000$ metres

$Y_o = 400\ 000$ metres

$k_o = 0.999912$,

and where χ_o is computed from equation (7-4) and $\Delta\Lambda = \Lambda - \Lambda_o$ where $\Lambda_o = c_1 \lambda_o$ from equation (7-2). The ellipsoidal parameters a and b are chosen, in this manual, to be the Clarke 1866 values;

$a = 6\ 378\ 206.4$ m

$b = 6\ 356\ 583.8$ m

although these may not be chosen in the redefinition.

7.2.2 Transformation from X, Y to ϕ, λ

The transformation from X, Y to χ, Λ is as follows:

Let

$$X' = (X - X_o)/k_o \quad , \qquad (7\text{-}7)$$

$$Y' = (Y - Y_o)/k_o \quad , \qquad (7\text{-}8)$$

$$S = ((X')^2 + (Y')^2)^{1/2} \quad , \qquad (7\text{-}9)$$

$$\cos \beta = \frac{X'}{S} \quad (\text{if } S = 0, \cos \beta = 1) \quad , \qquad (7\text{-}10)$$

$$\sin \beta = \frac{Y'}{S} \quad (\text{if } S = 0, \sin \beta = 0) \quad , \qquad (7\text{-}11)$$

and

$$\delta = 2 \tan^{-1} [\frac{S}{2R}] \quad , \qquad (7\text{-}12)$$

then

$$\chi = \sin^{-1} [\sin \chi_o \cos \delta + \sin \delta \cos \chi_o \sin \beta] \quad , \qquad (7\text{-}13)$$

$$\Lambda = \Lambda_o + \sin^{-1} [\frac{\sin \delta \cos \beta}{\cos \chi}] \quad , \qquad (7\text{-}14)$$

where χ_o is computed from equation (7-4) and $\Lambda_o = c_1 \lambda_o$ from equation (7-2).

The transformation from χ, Λ to ϕ, λ is done via the Newton-Raphson iteration technique as follows:

let a first approximation of ϕ be,

$$\phi_1 = \chi \quad ,$$

then

$$\phi_2 = \phi_1 - \frac{f(\phi)}{f'(\phi)} \bigg|_{\phi_1} \qquad (7\text{-}15)$$

etc.,

$$\phi_n = \phi_{n-1} - \left.\frac{f(\phi)}{f'(\phi)}\right|_{\phi_{n-1}} ,$$

where from equation (7-1)

$$f(\phi) = c_2[\tan(45° + \frac{\phi}{2}) (\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}]^{c_1} - \tan(45° + \frac{\chi}{2}) = 0 ;$$

(note that the value of χ which was computed from equation (7-13) does not change through the iterations!)

and after differentiation we have

$$f'(\phi) = c_1 c_2[\tan(45° + \frac{\phi}{2}) (\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}]^{(c_1-1)} \cdot [(\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}$$

$$\{\frac{1}{2} \sec^2(45° + \frac{\phi}{2}) - \frac{e^2 \cos \phi}{(1-e^2 \sin^2 \phi)} \cdot \tan(45° + \frac{\phi}{2})\}] .$$

We iterate equations (7-15) until $|\phi_n - \phi_{n-1}|$ is less than, say .00001 seconds in double precision FORTRAN (which corresponds to less than 1 millimetre on the ellipsoid) or until it is smaller than our required accuracy.

The ellipsoidal longitude λ is computed from (see equation (7-2)):

$$\lambda = \frac{\Lambda}{c_1} . \qquad (7\text{-}16)$$

7.2.3 Error propagation: Prince Edward Island double stereographic projection

When transforming ϕ, λ coordinates to Prince Edward Island stereographic grid coordinates, we may also wish to transform the ϕ, λ covariance matrix to obtain the covariance matrix of the Prince Edward Island grid coordinates, X, Y; and conversely. To do this, we propagate errors through equations (7-1), (7-2), (7-5), and (7-6) according to the covariance law [Wells and Krakiwsky, 1971; Mikhail, 1976]. This results in the following [Thomson et al., 1977]:

If we let

$$\underline{C}_{X,Y} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix},$$

denote the covariance matrix of the Prince Edward Island stereographic grid coordinates in metres^2, and

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\phi\lambda} & \sigma_\lambda \end{bmatrix},$$

denote the covariance matrix of the ellipsoidal coordinates of the same point in arc seconds squared, we have

$$\underline{C}_{X,Y} = \underline{B}\,\underline{C}_{\phi,\lambda}\,\underline{B}^t, \qquad (7\text{-}17)$$

where

$$\underline{B} = \begin{bmatrix} \frac{-m.u}{n\,\rho''} & \frac{c_1 p}{n\,\rho''}\cdot\cos\chi \\ \frac{p.u}{n\,\rho''} & \frac{c_1 m}{n\,\rho''}\cdot\cos\chi \end{bmatrix}, \qquad (7\text{-}18)$$

and

$$\rho'' = 206\ 264.806\ldots.$$

$$c_1 = \{1 + \frac{e^2\cos^4\phi_o}{1-e^2}\}^{1/2}, \qquad \text{(repeated)}(7\text{-}3a)$$

$$c_2 = \tan(45° + \frac{\chi_o}{2})\{\tan(45° + \frac{\phi_o}{2})(\frac{1-e\sin\phi_o}{1+e\sin\phi_o})^{e/2}\}^{-c_1}, \qquad \text{(repeated)}(7\text{-}3b)$$

$$m = 2\,k_o\,R\,\sin\Delta\Lambda\,(\sin\chi + \sin\chi_o) \quad , \tag{7-19}$$

$$n = (1 + \sin\chi\,\sin\chi_o + \cos\chi\,\cos\chi_o\,\cos\Delta\Lambda)^2 \quad , \tag{7-20}$$

$$p = 2\,k_o\,R\,\{\cos\Delta\Lambda(1 + \sin\chi\,\sin\chi_o) + \cos\chi\,\cos\chi_o\} \quad , \tag{7-21}$$

$$u = \frac{c_1 c_2 \{q \cdot \tan r\}^{(c_1 - 1)} \cdot q\{\frac{1}{2}\sec^2 r - \frac{e^2 \cos\phi\,\tan r}{(1-e^2\sin^2\phi)}\}}{1 + c_2^2\{q \cdot \tan r\}^{2c_1}} \quad , \tag{7-22}$$

where

$$r = 45° + \frac{\phi}{2} \quad , \tag{7-23}$$

and

$$q = \{\frac{1-e\,\sin\phi}{1+e\,\sin\phi}\}^{e/2} \tag{7-24}$$

For the inverse propagation of errors, i.e., propagation errors through the transformation (X, Y) to (ϕ, λ) we have;

$$\underline{C}_{\phi,\lambda} = \underline{B}^{-1}\,\underline{C}_{X,Y}\,(\underline{B}^{-1})^t \quad , \tag{7-25}$$

where

$$\underline{B}^{-1} = \begin{bmatrix} \dfrac{mn \cdot \rho''}{u(m^2+p^2)} & \dfrac{pn \cdot \rho''}{u(m^2+p^2)} \\ \\ \dfrac{pn \cdot \rho''}{c_1 \cos\chi(m^2+p^2)} & \dfrac{mn \cdot \rho''}{c_1 \cos\chi(m^2+p^2)} \end{bmatrix} .$$

7.3 Numerical Example

For the numerical example, we choose a point with

$$\phi = 46°\ 42'\ 28''.147\ \text{N} \quad ,$$

$$\lambda = 64°\ 29'\ 34''.014\ \text{W} \quad ,$$

$$= -\ 64°\ 29'\ 34''.014 \quad .$$

7.3.1 (ϕ, λ) to Prince Edward Island (X, Y)

First, we compute c_1 from equation (7-3a)

$$c_1 = \{1 + \frac{e^2 \cos^4 \phi_o}{1-e^2}\}^{1/2} \quad ,$$

with $e^2 = \frac{a^2 - b^2}{a^2} = 0.006768658$,

and $\phi_o = 47° \ 15' \ 00"$.

We get

$$c_1 = 1.0007231600 \quad .$$

Next, from equation (7-4)

$$\chi_o = \sin^{-1} [\frac{\sin \phi_o}{c_1}] \quad ,$$

$$\chi_o = 47° \ 12' \ 18''.816 \quad .$$

Now, from equation (7-3b)

$$c_2 = \tan(45° + \frac{\chi_o}{2}) \{\tan(45° + \frac{\phi_o}{2}) (\frac{1-e \sin \phi_o}{1+e \sin \phi_o})^{e/2}\}^{-c_1}$$

$$c_2 = 1.0031559673 \quad .$$

Next we have, from equation (7-1),

$$\chi = 2\{\tan^{-1} [c_2\{\tan(45° + \frac{\phi}{2}) (\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}\}^{c_1}] - 45°\}$$

and, with $\phi = 46° \ 42' \ 28''.147$ N ,

$$\chi = 46° \ 39' \ 50''.118 \quad \text{N.}$$

Further, from equation (7-2),

$$\Lambda = c_1 \lambda \quad ,$$

and, with $\lambda = 64° \ 29' \ 34''.014$ W, ($= -64° \ 29' \ 34''.014$) , 64° 32' 21.912 -64 32 21.912

Λ = ~~63° 02' 44".010~~ W (= ~~-63° 02' 44".010~~) .

Finally, we compute the grid coordinates from equations (7-5) and (7-6), i.e.

$$X = X_o + \frac{2\, k_o\, R \cos\chi \sin\Delta\Lambda}{1 + \sin\chi \sin\chi_o + \cos\chi \cos\chi_o \cos\Delta\Lambda} ,$$

$$Y = Y_o + \frac{2\, k_o\, R(\sin\chi \cos\chi_o - \cos\chi \sin\chi_o \cos\Delta\Lambda)}{1 + \sin\chi \sin\chi_o + \cos\chi \cos\chi_o \cos\Delta\Lambda} ,$$

where

$$X_o = 700\ 000 \text{ metres} ,$$

$$Y_o = 400\ 000 \text{ metres},$$

$$R = 6\ 379\ 869.43 \text{ metres} ,$$

$$k_o = 0.999912 ,$$

and

$$\Delta\Lambda = \Lambda - \Lambda_o = 64°\ 32'\ 21''.912 \ \ W - \Lambda_o ,$$
$$= -\ 64°\ 32'\ 21''.912 - \Lambda_o ,$$

where, from equation (7-2)

$$\Lambda_o = c_1\, \lambda_o \qquad (\lambda_o = 63°\ 00'\ 00''\ \ W ,$$
$$= -63°\ 00'\ 00'') ,$$

$$\Lambda_o = 63°\ 02'\ 44''.016 \ \ W$$
$$= -63°\ 02'\ 44''.016 .$$

Thus $\Delta\Lambda = -64°\ 32'\ 21''.912 - (-63°\ 02'\ 44''.016) ,$

$$= -\ 1°\ 29'\ 37''.896 .$$

Finally, we get

$$X = 700\ 000 + (-114\ 144.554) = 585\ 855.446 \text{ m} ,$$

$$Y = 400\ 000 + (-59\ 182.240) = 340\ 817.760 \text{ m} .$$

7.3.2 Prince Edward Island (X, Y) to (ϕ, λ)

Now, we will take the computed grid coordinates

$$X = 585\ 855.446 \text{ m} ,$$

$$Y = 340\ 817.760 \text{ m} ,$$

and transform them to ϕ, λ.

From equations (7-7), (7-8) and (7-9), we have

$$X' = (X - X_o)/k_o$$

$$= -\ 114\ 154.59918 \text{ m} ,$$

$$Y' = (Y - Y_o)/k_o$$

$$= -\ 59\ 187.44830 \text{ m} ,$$

$$S = ((X')^2 + (Y')^2)^{1/2}$$

$$= 128\ 586.26112 \text{ m} .$$

Further, we have from equations (7-10) and (7-11),

$$\cos\beta = \frac{X'}{S} = -\ 0.8877666882 ,$$

$$\sin\beta = \frac{Y'}{S} = -\ 0.4602937186 ,$$

and from equation (7-12),

$$\delta = 2\tan^{-1}\left[\frac{S}{2R}\right] ,$$

$$\delta = 1°\ 09'\ 17''.124 .$$

From the above and equations (7-13) and (7-14), we have

$$\chi = \sin^{-1}[\sin\chi_o \cos\delta + \sin\delta\cos\chi_o\sin\beta] ,$$

$$\Lambda = \Lambda_o + \sin^{-1}\left[\frac{\sin\delta\cos\beta}{\cos\chi}\right] ,$$

$$\chi = \text{~~45~~ } 46°\ 39'\ 50''.118 ,$$

$$\Lambda = 63° \; 02' \; 44''016 \text{ W} \pm 1° \; 29' \; 37''896 \quad ,$$

$$= -63° \; 02' \; 44''016 \mp 1° \; 29' \; 37''896 \quad ,$$

$$\Lambda = 64° \; 32' \; 21''912 \quad \text{W}$$

$$= -64° \; 32' \; 21''912 \quad .$$

Next we compute λ from equation (7-16)

$$\lambda = \frac{\Lambda}{c_1} \quad ,$$

$$\lambda = 64° \; 29' \; 34''014 \quad \text{W}$$

$$= -64° \; 29' \; 34''014 \quad ,$$

and ϕ via the iteration process of equations (7-15) as follows:

$$\phi_1 = \chi = 46° \; 39' \; 50''118 \quad ,$$

then

$$\phi_2 = \phi_1 - \frac{f(\phi)}{f'(\phi)}\bigg|_{\phi_1} \quad ,$$

where

$$f(\phi)\Big|_{\phi_1} = c_2[\tan(45° + \frac{\phi_1}{2}) \left(\frac{1-e \sin \phi_1}{1+e \sin \phi_o}\right)^{e/2}]^{c_1} - \tan (45° + \frac{\chi}{2}) \quad ,$$

$$f'(\phi)\Big|_{\phi_1} = c_1 c_2[\tan(45° + \frac{\phi_1}{2}) \left(\frac{1-e \sin \phi_1}{1+e \sin \phi_1}\right)^{e/2}]^{(c_1-1)} \cdot [\left(\frac{1-e \sin \phi_1}{1+e \sin \phi_1}\right)^{e/2}$$

$$\{\frac{1}{2} \sec^2(45° + \frac{\phi_1}{2}) - \frac{e^2 \cos \phi_1}{(1-e^2 \sin^2 \phi_1)} \cdot \tan (45° + \frac{\phi_1}{2})\}] \quad .$$

We get

$$f(\phi)\Big|_{\phi_1} = - 0.0028024951 \quad ,$$

$$f'(\phi)\Big|_{\phi_1} = 3.654418141 \quad ,$$

and

$$\phi_2 = 46° \; 39' \; 50''.118 \; - \; (- \; 0° \; 02' \; 38''.172) \;,$$

$$\phi_2 = 46° \; 42' \; 28''.290 \;.$$

Now,

$$\phi_3 = \phi_2 - \frac{f(\phi)}{f'(\phi)} \bigg|_{\phi_2} \;.$$

we get,

$$f(\phi) \Big|_{\phi_2} = 0.000002538 \;,$$

$$f'(\phi) \Big|_{\phi_2} = 3.661491622 \;,$$

and

$$\phi_3 = 46° \; 42' \; 28''.290 \; - \; 00° \; 00' \; 00''.143 \;.$$

$$\phi_3 = 46° \; 42' \; 28''.147 \;.$$

Further

$$\phi_4 = \phi_3 - \frac{f\,(\phi)}{f'(\phi)} \bigg|_{\phi_3} \;,$$

and

$$f(\phi) \Big|_{\phi_3} = 0.25324 \times 10^{-11} \;,$$

$$f'(\phi) \Big|_{\phi_3} = 3.6614847824 \;,$$

and

$$\phi_4 = 46° \; 42' \; 28''.147 \; - \; 00° \; 00' \; 0''.000 \;,$$

$$\phi = 46° \; 42' \; 28''.147 \;.$$

Thus, we have performed the direct and inverse transformations of coordinates.

7.3.3 Error propagation

We have, from equation (7-17)

$$\underline{C}_{X,Y} = \underline{B}\,\underline{C}_{\phi,\lambda}\,\underline{B}^t \; ,$$

where, $\underline{C}_{\phi,\lambda}$ is given as, say;

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\phi\lambda} & \phi_\lambda^2 \end{bmatrix} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \sec^2 .$$

$\underline{B}$ is given by equations (7-18) as

$$\begin{bmatrix} -m \dfrac{u}{n\rho''} & \dfrac{c_1}{n\rho''}\, p \cdot \cos\chi \\ p \cdot \dfrac{u}{n\rho''} & \dfrac{c_1}{n\rho''}\, m \cdot \cos\chi \end{bmatrix} ,$$

where

$$\rho'' = 206\ 264.806 \ldots..$$

$$c_1 = \{1 + \frac{e^2 \cos^4 \phi_o}{1-e^2}\}^{1/2} = 1.007231600 \; ,$$

$$m = 2\, k_o\, R \sin \Delta\Lambda(\sin\chi + \sin\chi_o) = -485994.7876 \; ,$$

$$n = (1 + \sin\chi \sin\chi_o + \cos\chi \cos\chi_o \cos\Delta\Lambda)^2 = 3.999187676 \; ,$$

$$p = 2\, k_o R\{\cos\Delta\Lambda(1 + \sin\chi \sin\chi_o) + \cos\chi\cos\chi_o\} = 25\ 510\ 011.895 \; ,$$

$$u = \frac{c_1 c_2 \{q . \tan r\}^{(c_1-1)} \; . \; q\{\frac{1}{2}\sec^2 r - \dfrac{e^2 \cos\phi \tan r}{(1-e^2 \sin^2 \phi)}\}}{1 + c_2^2 \{q \; . \; \tan \; r\}^{2c_1}} = 0.4991688128 \; ,$$

where r and q are computed from equations (7-23) and (7-24) respectively.

Thus we have

$$\underline{C}_{X,Y} = \underline{B}\,\underline{C}_{\phi,\lambda}\,\underline{B}^t$$

$$= \left(\frac{1}{\rho''}\right) \begin{bmatrix} 60\ 660.6793 & 4\ 380\ 785.02 \\ 3\ 184\ 097.22 & -83\ 458.945 \end{bmatrix} \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \cdot$$

$$\frac{1}{\rho''} \begin{bmatrix} 60\ 660.6793 & 3\ 184\ 097.22 \\ 4\ 380\ 785.02 & -83\ 458.945 \end{bmatrix},$$

$$\underline{C}_{X,Y} = \begin{bmatrix} 9.0325 \times 10^{-6} & 1.3572 \times 10^{-7} \\ 1.3572 \times 10^{-7} & 2.3763 \times 10^{-6} \end{bmatrix} m^2 .$$

The inverse propagation of errors is simply

$$\underline{C}_{\phi,\lambda} = \underline{B}^{-1}\,\underline{C}_{X,Y}\,(\underline{B}^{-1})^t$$

$$= \rho'' \begin{bmatrix} 5.9810 \times 10^{-9} & 3.1395 \times 10^{-7} \\ 2.2819 \times 10^{-7} & -4.3472 \times 10^{-9} \end{bmatrix} \begin{bmatrix} 9.0325 \times 10^{-6} & 1.3572 \times 10^{-7} \\ 1.3572 \times 10^{-7} & 2.3763 \times 10^{-6} \end{bmatrix} \cdot$$

$$\rho'' \begin{bmatrix} 5.9810 \times 10^{-9} & 2.2819 \times 10^{-7} \\ 3.1395 \times 10^{-7} & -4.3472 \times 10^{-9} \end{bmatrix},$$

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \text{arc sec}^2 .$$

8. THE NOVA SCOTIA 3° TRANSVERSE MERCATOR PROJECTION

The Nova Scotia map projection is a Transverse Mercator projection in zones 3° of longitude in width. The province is covered by two zones numbered 4 and 5. The limits of zone 4 are from longitude 60° 00' 00" W to 63° 00' 00" W, with a central meridian at 61" 30' 00" W. The limits of zone 5 are from longitude 63° 00' 00" W to 66° 00' 00" W, with a central meridian at 64° 30' 00" W.

Before giving the equations for coordinate transformations and error propagation, it may be convenient to review the notation used in this chapter.

8.1 Notation

a, b ≡ semi-major and semi-minor axes of the reference ellipsoid;

e ≡ first eccentricity of the reference ellipsoid,

$$e^2 = (a^2 - b^2)/a^2 \qquad (8\text{-}1)$$

ϕ, λ ≡ ellipsoidal coordinates: latitude (positive north of the equator) and longitude (positive east of the Greenwich meridian) respectively.

λ_o ≡ longitude of the central meridian, (positive east of Greenwich).

$\Delta\lambda$ ≡ longitude of a particular point minus the longitude of the central meridian, i.e. $\Delta\Lambda = \lambda - \lambda_o$ (8-2)

X, Y ≡ grid coordinates of the Nova Scotia 3° Transverse Mercator projection easting and northing respectively.

X_o ≡ grid coordinate value adopted for the central meridian in order to avoid negative coordinates. X_o is sometimes referred to as the "false easting".

k_o ≡ point scale factor at the central meridian. For Nova Scotia,

$$k_o = 0.99990. \qquad (8\text{-}3)$$

S_ϕ ≡ meridian arc length from the equator to latitude ϕ.

8.2 Coordinate Transformation Formulae.

8.2.1 Transformation of (ϕ,λ) to Nova Scotia (X, Y) Grid-Coordinates

In order to compute the Y coordinate from a given (ϕ, λ) we must first compute the meridian arc length from [Bomford, 1971; Krakiwsky, 1973]:

$$S_\phi = a[A_o\phi - A_2\sin\ 2\phi + A_4\sin 4\phi\ - A_6\sin\ 6\phi\ + A_8\sin 8\phi] \qquad (8\text{-}4)$$

where, ϕ is the latitude of the point in degrees (and decimals of degrees)

$$A_o = (1 - \frac{1}{4}e^2 - \frac{3}{64}\ e^4 - \frac{5}{256}\ e^6 - \frac{175}{16384}\ e^8)\ \frac{\pi}{180}\ ,$$

$$A_2 = \frac{3}{8}(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 - \frac{455}{4096}\ e^8)\ ,$$

$$A_4 = \frac{15}{256}(e^4 + \frac{3}{4}e^6 - \frac{77}{128}\ e^8)\ , \qquad (8\text{-}5)$$

$$A_6 = \frac{35}{3072}(e^6 - \frac{41}{32}e^8)\ ,$$

and

$$A_8 = -\frac{315}{131072}\ e^8\ ,$$

where, $\pi = 3.141592653....$

Now, we can compute the (X, Y) grid coordinates from [Thomas, 1952; Krakiwsky, 1973],

$$X = X_o + k_o N[\Delta\lambda \cdot \cos\phi + \frac{\Delta\lambda^3 \cos^3\phi}{6}(1 - t^2 + \eta^2)$$

$$+ \frac{\Delta\lambda^5 \cos^5\phi}{120}(5 - 18t^2 + t^4 + 14\eta^2 - 58t^2\eta^2 + 13\eta^4 + 4\eta^6 - 64\eta^4t^2 - 24\eta^6t^2)$$

$$+ \frac{\Delta\lambda^7 \cos^7\phi}{5040}(61 - 479t^2 + 179t^4 - t^6)] \quad , \qquad (8\text{-}6)$$

$$Y = k_o \cdot S_\phi + k_o N[\frac{\Delta\lambda^2}{2}\sin\phi\cos\phi + \frac{\Delta\lambda^4}{24}\sin\phi\cos^3\phi(5 - t^2 + 9\eta^2 + 4\eta^4)$$

$$+ \frac{\Delta\lambda^6}{720}\sin\phi\cos^5\phi(61 - 58t^2 + t^4 + 270\eta^2 - 330t^2\eta^2 + 445\eta^4 + 324\eta^6$$

$$- 680\eta^4t^2 + 88\eta^8 - 600\eta^6t^2 - 192\eta^8t^2)$$

$$+ \frac{\Delta\lambda^8}{40320}\sin\phi\cos^7\phi(1385 - 311t^2 + 543t^4 - t^6)] \quad , \qquad (8\text{-}7)$$

where, $\Delta\lambda$ is in radians ($\Delta\lambda = \lambda - \lambda_o$) ;

λ_o = 61° 30' 00" W if computing in zone 4;

λ_o = 64° 30' 00" W if computing in zone 5;

$t = \tan\phi$;

$\eta^2 = (\frac{a^2 - b^2}{b^2})\cos^2\phi$;

$N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}}$;

a = 6 378 206.4 m)
) (See note on next page.)
b = 6 356 583.8 m)

X_o = 4 500 000 m if computing in zone 4;

X_o = 5 500 000 m if computing in zone 5;

k_o = 0.99990 ; and

e^2 is computed from equation (8-1).

NOTE: The ellipsoidal parameters used in this manual are for the Clarke 1866 reference ellipsoid as above. In the redefinition a different ellipsoid may be chosen with different parameters a and b.

8.2.2 Transformation of Nova Scotia (X, Y) Grid Coordinates to (ϕ,λ) Coordinates

In order to compute ϕ from a given (X, Y) we first compute the latitude (ϕ') which corresponds to a meridian arc length of $S_{\phi'} = Y$. The latitude ϕ' is usually referred to as the "footpoint latitude" and is computed as follows [Krakiwsky, 1973; Bomford, 1971].

We take, as a first approximation to ϕ',

$$\phi'_1 = \frac{Y}{a} \quad .$$

Then, we use the Newton-Raphson iteration technique to compute a more accurate value as follows:

$$\phi'_2 = \phi'_1 - \frac{f(\phi')}{f'(\phi')}\bigg|_{\phi'_1} \quad ,$$

$$\phi'_3 = \phi'_2 - \frac{f(\phi')}{f'(\phi')}\bigg|_{\phi'_2} \quad , \qquad (8\text{-}8)$$

etc., until $|\phi'_n - \phi'_{n-1}|$ is smaller than some pre-determined limit, where from equation (8-4)

$$f(\phi') = a[A_o\phi' - A_2 \sin 2\phi' + A_4 \sin 4\phi' - A_6 \sin 6\phi' + A_8 \sin 8\phi'] - Y = 0 \; , \qquad (8\text{-}9)$$

and after differentiation

$$f'(\phi') = a[A_o - 2\,A_2 \cos 2\phi' + 4\,A_4 \cos 4\phi' - 6\,A_6 \cos 6\phi' + 8\,A_8 \cos 8\phi'] , \qquad (8\text{-}10)$$

where the coefficients A_i, i=0, 2, 4, 6, 8 are computed from equations (8-5). From the iteration of equations (8-8) then, we get,

$$\phi' = \phi'_n .$$

Now, we can compute the (ϕ,λ) ellipsoidal coordinates from the given (X, Y) grid coordinates from [Thomas, 1952; Krakiwsky, 1973].

Let

$$x = \frac{X - X_o}{k_o} ; \quad y = \frac{Y}{k_o} ,$$

then;

$$\phi = \phi' - \frac{t_1 x^2}{2 . M_1 N_1} + \frac{t_1 x^4}{24\ M_1 N_1^3} (5 + 3t_1^2 + \eta_1^2 - 4\eta_1^4 - 9\eta_1^2 t_1^2)$$

$$- \frac{t_1 x^6}{720\ M_1 N_1^5} (61 - 90t_1^2 + 46\eta_1^2 + 45t_1^4 - 252t_1^2\eta_1^2 - 3\eta_1^4 + 100\eta_1^6 - 66t_1^2\eta_1^4 -$$

$$90t_1^4\eta_1^2 + 88\eta_1^8 + 225t_1^4\eta_1^4 + 84t_1^2\eta_1^6 - 192t_1^2\eta_1^8)$$

$$+ \frac{t_1 x^8}{40320\ M_1 N_1^7} (1385 + 3633t_1^2 + 4095t_1^4 + 1575t_1^6) , \qquad (8\text{-}11)$$

$$\lambda = \lambda_o + [\frac{x}{N_1} - \frac{x^3}{6\ N_1^3} (1 + 2t_1^2 + \eta_1^2)$$

$$+ \frac{x^5}{120\ N_1^5} (5 + 6\eta_1^2 + 28t_1^2 - 3\eta_1^4 + 8t_1^2\eta_1^2 + 24t_1^4 - 4\eta_1^6 + 4t_1^2\eta_1^4 + 24t_1^2\eta_1^6)$$

$$- \frac{x^7}{5040\ N_1^7} (61 + 662t_1^2 + 1320t_1^4 + 720t_1^6)]/\cos \phi' . \qquad (8\text{-}12)$$

where, $t_1 = \tan \phi'$,

$$n_1^2 = (\frac{a^2 - b^2}{b^2}) \cos^2 \phi' ,$$

$$N_1 = \frac{a}{(1-e^2 \sin^2 \phi')^{1/2}} ,$$

$$M_1 = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi')^{3/2}} ,$$

X_o = 4 500 000 m if computing in zone 4,

X_o = 5 500 000 m if computing in zone 5,

k_o = 0.99990 ,

λ_o = 61° 30' 00" W = -61° 30' 00" if computing in zone 4,

λ_o = 64° 30' 00" W = -61° 30' 00" if computing in zone 5, and

e^2 is computed from equation (8-1).

8.2.3 Error Propagation in Nova Scotia Coordinate Transformations

As well as transforming (ϕ,λ) to Nova Scotia (X, Y) grid coordinates (or (X, Y) grid coordinates to (ϕ, λ)), we may also wish to transform any variance-covariance information. We do this as follows (see for example, Mikhail [1976]; Wells and Krakiwsky [1971]).

If we let

$$\underline{C}_{X,Y} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \text{metres}^2 \qquad (8\text{-}13)$$

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} \\ \sigma_{\phi\lambda} & \sigma_\lambda^2 \end{bmatrix} \sec^2 , \qquad (8\text{-}14)$$

be the covariance matrices of the (X, Y) grid coordinates and (ϕ, λ) ellipsoidal coordinates respectively, and

$$\underline{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \tag{8-15}$$

where,

$$\begin{aligned} b_{11} &= (21\ 670\ \Delta\lambda \cos\phi - 6\ 390\ 000 \cdot \Delta\lambda \cdot \sin\phi) \frac{1}{\rho''}, \\ b_{12} &= (6\ 390\ 000 \cdot \cos\phi) \frac{1}{\rho''}, \\ b_{21} &= (6\ 370\ 000) \frac{1}{\rho''}, \\ b_{22} &= (6\ 390\ 000\ \Delta\lambda \sin\phi \cos\phi) \cdot \frac{1}{\rho''}, \end{aligned} \tag{8-16}$$

where $\rho'' = 206\ 264.806....$

then,

$$\underline{C}_{X,Y} = \underline{B}\ \underline{C}_{\phi,\lambda}\ \underline{B}^t, \tag{8-17}$$

where $\underline{B}^t$ is the transpose of $\underline{B}$ and

$$\underline{C}_{\phi,\lambda} = \underline{B}^{-1}\ \underline{C}_{X,Y}\ (\underline{B}^{-1})^t, \tag{8-18}$$

where

$$\underline{B}^{-1} = \begin{bmatrix} \dfrac{\rho'' \cdot b_{22}}{b_{11}b_{22}-b_{21}b_{12}} & \dfrac{-\rho'' \cdot b_{12}}{b_{11}b_{22}-b_{21}b_{12}} \\ \dfrac{-\rho'' \cdot b_{21}}{b_{11}b_{22}-b_{21}b_{12}} & \dfrac{\rho''\ b_{11}}{b_{11}b_{22}-b_{21}b_{12}} \end{bmatrix}.$$

It should be noted here that equations (8-16) are accurate enough so that the transformations (8-17) and (8-18) are good to three significant figures, which should be sufficient for all practical purposes.

8.3 Numerical Example

For the numerical example we have chosen a point on the meridian common to both zone 4 and zone 5, i.e., let

$$\phi = 44° \; 39' \; 03''.123 \;\; N,$$

$$\lambda = 63° \; 00' \; 0''.000 \; W = -63° \; 00' \; 00'' \; .$$

We will compute grid coordinates for this point in both zones and then transform both sets of grid coordinates back to ϕ,λ.

8.3.1 (ϕ, λ) to (X, Y)

First, we compute the meridian arc length from the equator to $\phi = 44° \; 39' \; 02''.123$ N from equation (8-4). We get, from equations (8-5);

$$A_o = (0.9983056819) \cdot \frac{\pi}{180} \quad ,$$

$$A_2 = 0.2542555420 \times 10^{-2} \quad ,$$

$$A_4 = 0.2698010543 \times 10^{-5} \quad ,$$

$$A_6 = 0.3502448582 \times 10^{-8} \quad ,$$

$$A_8 = -0.5044416192 \times 10^{-11} \quad ,$$

and from equation (8-4)

$$S_\phi = 4\;945\;928.90006 \text{ m} \quad .$$

Now, we compute grid coordinates from equations (8-6) and (8-7) for both zones.

Zone 4

$$X_o = 4\;500\;000 \text{ m} \; ,$$

$$k_o = 0.99990 \quad ,$$

$$\Delta\lambda = -1^\circ\ 30'\ 00''$$

$$= -0.0261799388 \text{ radians} \quad ,$$

$$t = 0.9878866397 \quad ,$$

$$\eta^2 = 0.34489174 \times 10^{-2} \quad ,$$

$$N = 6\ 388\ 894.64974 \text{ m} \quad ,$$

and we get

$$X = 4\ 381\ 021.928 \text{ m} \quad ,$$

$$Y = 4\ 946\ 528.965 \text{ m} \quad .$$

<u>Zone 5</u>

$$X_o = 5\ 500\ 000 \text{ m} \quad ,$$

$$k_o = 0.99990 \quad ,$$

$$\Delta\lambda = +1^\circ\ 30'\ 00''$$

$$= 0.0261799388 \text{ radians} \quad ,$$

$$t = 0.9878866397 \quad ,$$

$$\eta^2 = 0.34489174 \times 10^{-2} \quad ,$$

$$N = 6\ 388\ 894.64974 \text{ m} \quad ,$$

and we get

$$X = 5\ 618\ 978.072 \text{ m} \quad ,$$

$$Y = 4\ 946\ 528.965 \text{ m} \quad .$$

8.3.2 <u>(X, Y) to (φ, λ)</u>

To transform these computed grid coordinates back to (φ, λ) ellipsoidal coordinates, we must first compute the footpoint latitude of the point in each zone. Using the iteration technique of equations (8-8) we get:

<u>Zone 4</u>

$$\phi_1' = \frac{Y}{a} = 0.7756136061 \text{ radians} \quad ,$$

$$\phi'_2 = 0.7756136061 - (-0.0038630671) = .7794766732 \text{ radians} \quad ,$$

$$\phi'_3 = .7794766732 - (0.7599776152 \times 10^{-7}) = .7794765972 \text{ radians} \; ,$$

$$\phi'_4 = .7794765972 - (-0.1495125943 \times 10^{-15}) = .7794765972 \text{ radians} \quad .$$

This computed change (.1495 ... x 10^{-15} radians) is equivalent to approximately 3 x 10^{-11} seconds which corresponds to less than 1 micrometre on the ellipsoid, thus we stop the iterations!

Thus,

$$\phi' = .7794765972 \text{ radians} = 44° \; 39' \; 38''.586 \quad .$$

Zone 5

$$\phi'_1 = \frac{Y}{a} = 0.7756136031 \text{ radians} \quad ,$$

$$\phi'_2 = 0.7756136061 - (-0.0038630671) = .7794766732 \text{ radians},$$

$$\phi'_3 = .7794766732 - (0.7599776152 \times 10^{-7}) = .7794765972 \text{ radians} \quad ,$$

and again the next iteration gives a change in ϕ' of about 1 micrometre (1 x 10^{-6}), thus we stop the iteration which gives;

metre

$$\phi' = .7794765972 \text{ radians} = 44° \; 39' \; 38''.586 \quad .$$

Now, we compute (ϕ, λ) from equations (8-11) and (8-12) for each zone.

Zone 4

$$X_o = 4\,500\,000 \text{ m} \quad ,$$

$$k_o = 0.99990 \quad ,$$

$$x = \frac{X - X_o}{k_o} = \frac{4\,381\,021.928 - 4\,500\,000}{0.99990} = -\,118\,989.9712 \text{ m} \; .$$

$$y = \frac{Y}{k_o} = \frac{4\ 946\ 528.965}{0.99990} = 4\ 947\ 023.6669 \quad m \quad ,$$

$$t_1 = 0.9882264479 \quad ,$$

$$\eta_1^2 = 0.344774457 \times 10^{-2} \quad ,$$

$$N_1 = 6\ 388\ 898.37977 \quad m \quad ,$$

$$M_1 = 6\ 366\ 946.76625 \quad m \quad ,$$

$$\lambda_o = 61° \ 30' \ 00'' \quad W$$

$$= -61° \ 30' \ 00'' \quad ,$$

and we get

$$\phi = 0.7793046517 \text{ radians} = 44° \ 39' \ 03.''123 \quad ,$$

$$\lambda = -\ 1.099557429 \text{ radians} = 63° \ 00' \ 0.''0000 \ W$$

$$= -\ 63° \ 00' \ 0.''000 \quad .$$

<u>Zone 5</u>

$$X_o = 5\ 500\ 000 \ m \quad ,$$

$$k_o = 0.99990 \quad ,$$

$$x = \frac{X - X_o}{k_o} = \frac{5\ 618\ 978.072 - 5\ 500\ 000}{.99990} = 118\ 989.9712 \quad ,$$

$$y = \frac{Y}{k_o} = \frac{4\ 946\ 528.965}{.99990} = 4\ 947\ 023.6669 \quad ,$$

$$t_1 = 0.9882264479 \quad ,$$

$$\eta_1^2 = 0.34477457 \times 10^{-2} \quad ,$$

$$N_1 = 6\ 388\ 898.37977 \quad ,$$

$$M_1 = 6\ 366\ 946.76625 \quad ,$$

$$\lambda_o = 64^\circ\ 30'\ 00''\ W \quad ,$$

and we get

$$\phi = 0.7793046517 \text{ radians} = 44^\circ\ 39'\ 03''.123\ N \quad ,$$

$$\lambda = -\ 1.099557429 \text{ radians} = 63^\circ\ 00'\ 0''.000\ W$$

$$= -\ 63^\circ\ 00'\ 0''.000 \quad .$$

8.3.3 <u>Error propagation</u>

The error propagation is carried out by simply using equations (8-17) and (8-18).

If we choose,

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \text{ arc seconds squared} \quad ,$$

we get:

<u>Zone 4</u>

$$b_{11} = 0.568032735 \quad ,$$

$$b_{12} = 22.03895738 \quad ,$$

$$b_{21} = 30.88263149 \quad ,$$

$$b_{22} = -\ 0.405491826 \quad ,$$

and, from equation (8-17),

$$\underline{C}_{X,Y} = \begin{bmatrix} 9.738 \times 10^{-6} & 5.410 \times 10^{-7} \\ 5.410 \times 10^{-7} & 9.521 \times 10^{-6} \end{bmatrix} m^2 \quad ,$$

and, from equation (8-18),

$$\underset{\sim}{C}_{\phi,\lambda} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \text{ arc seconds squared .}$$

Zone 5

$$b_{11} = -0.568032735 ,$$

$$b_{12} = 22.03895738 ,$$

$$b_{21} = 30.88263149 ,$$

$$b_{22} = 0.405491826 ,$$

and, from equation(8-17),

$$\underset{\sim}{C}_{X,Y} = \begin{bmatrix} 9.698 \times 10^{-6} & 5.476 \times 10^{-7} \\ 5.476 \times 10^{-7} & 9.561 \times 10^{-6} \end{bmatrix} m^2 ,$$

and from equation (8-18)

$$\underset{\sim}{C}_{\phi,\lambda} = \begin{bmatrix} 1 \times 10^{-8} & 8 \times 10^{-10} \\ 8 \times 10^{-10} & 2 \times 10^{-8} \end{bmatrix} \text{ arc seconds squared.}$$

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APPENDIX I

Glossary of Terms

Variance (σ^2: sigma squared): Variance is a measure of dispersion in a sample. When referring to an adjusted coordinate, it is used as a statistical measure of the reliability of the value obtained from the adjustment.

Standard Deviation (σ: sigma): The standard deviation of an adjusted coordinate is the positive square root of the corresponding variance.

Covariance (σ_{xy}: sigma, subscript xy): Covariance is an expression of the degree to which two variables are linearly related. When referring to a pair (e.g. x and y) of coordinate values, it is a measure of the statistical interdependence of the two values.

APPENDIX II

The Greek Alphabet

Alpha	Α	α
Beta	Β	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	Ε	ε
Zeta	Ζ	ζ
Eta	Η	η
Theta	Θ	θ
Iota	Ι	ι
Kappa	Κ	κ
Lambda	Λ	λ
Mu	Μ	μ
Nu	Ν	ν
Xi	Ξ	ξ
Omicron	Ο	ο
Pi	Π	π
Rho	Ρ	ρ
Sigma	Σ	σ
Tau	Τ	τ
Upsilon	Υ	υ
Phi	Φ	φ
Chi	Χ	χ
Psi	Ψ	ψ
Omega	Ω	ω

APPENDIX III

Conversions Between Angular Units

The angular units considered here are degrees (or degrees, minutes, seconds) and radians.

One radian is defined as the angle at the centre of a circle of radius r subtended by an arc on the circumference of length r. This is illustrated in Figure A-2. From elementary geometry we know that the total length C of the circumference of a circle of radius r is given by:

$$C = 2\pi r \quad (\pi = 3.14159265\ldots) \quad ,$$

and since an arc of length r subtends one radian at the centre of the circle, we must have 2π radians subtended at the centre by the entire circumference.

One degree is defined as the angle at the centre of a circle subtended by 1/360 of the circumference. Thus the entire circumference subtends an angle of 360 degrees at the centre of the circle.

Thus we have defined the angle at the centre of the circle subtended by the entire circumference in two ways. Therefore, if we denote this angle by α we can write:

$$\alpha = 2\pi \text{ radians} = 360 \text{ degrees} \quad .$$

From this we have,

$$1 \text{ radian} = \frac{360}{2\pi} \text{ degrees} \quad ,$$

or

$$\boxed{\begin{aligned} 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \\ &= 57.2957795\ldots \text{ degrees} \end{aligned}} \quad .$$

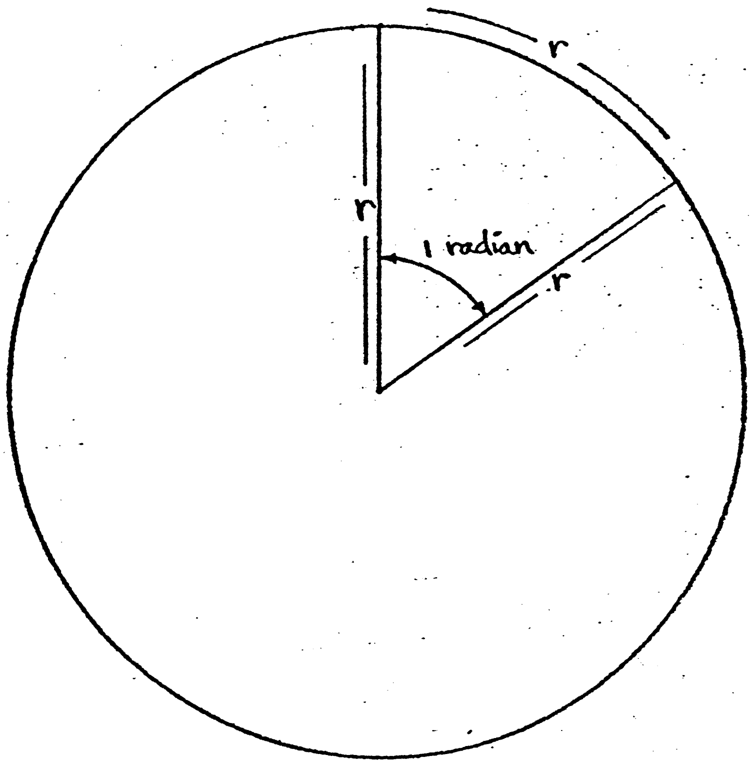

FIGURE A-2

Definition of a Radian

Also,

$$1 \text{ degree} = \frac{2\pi}{360} \text{ radians} \quad ,$$

or,

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$

$$= 0.017453292 \ldots\ldots \text{ radians} \quad .$$

Thus, to change an angle expressed in degrees to the same angle expressed in radians we simply multiply by $\frac{\pi}{180}$. Conversely, to change an angle expressed in radians to the same angle expressed in degrees we multiply by $\frac{180}{\pi}$.

EXAMPLE:

$$10^\circ\ 15'\ 05''.1 = 10.2514166\ldots \text{ degrees}$$

$$= (10.2514166\ldots)\left(\frac{3.14159265\ldots}{180}\right) \text{ radians}$$

$$= 0.17892097\ldots \text{ radians} \quad .$$

Here the angle was given in degrees, minutes and seconds. We first change this to degrees (and decimals of a degree) by knowing that: 1° (one degree) = 60' (sixty minutes); 1' (one minute) = 60" (sixty seconds). Thus,

$$10^\circ\ 15'\ 05''.1 = \left(10 + \frac{15}{60} + \frac{5.1}{3600}\right) = 10.2514166\ldots \text{ degrees.}$$

Next we change the angle expressed in degrees to the same angle expressed in radians by multiplying by $\frac{\pi}{180}$.