THE STEREOGRAPHIC DOUBLE PROJECTION

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PREFACE

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PREFACE

The work done in preparing this report was funded, under contract, by the Land Registration and Information Services, Surveys and Mapping Division.

This report is not meant to cover all aspects of conformal mapping. The theory of conformal mapping is adequately covered in several referenced texts. This report concerns itself only with the theory of the stereographic double projection, particularly as it may apply to use in the Maritime provinces of New Brunswick and Prince Edward Island. The approach used herein is analytical; no numerical examples regarding coordinate transformations, plane survey computations, etc., are included. For numerical examples, the reader is referred to a manual entitled <u>Geodetic Coordinate Transformations in the Maritimes</u> [1977].

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1. INTRODUCTION

As is well known, the mathematical figure that is a most convenient representation of the size and shape of the earth is a biaxial ellipsoid (ellipsoid of revolution). Traditional geodetic computations are carried out on this surface. When one wishes to perform the same computations on a plane, it is necessary to map the ellipsoidal information - points, angles, lines, etc. - on a plane mapping surface. A convenient mapping for geodetic purposes is a conformal mapping in which ellipsoidal angles are preserved on the mapping plane.

The stereographic projection of a sphere on a plane is credited to Hipparchus (c. 150 B.C.), the same man to whom we are indebted for plane and spherical trigonometry. This mapping has the following properties [Grossmann, 1964]:

(i) it is a perspective projection whose perspective centre is the antipodal point of the point at which the plane is tangent to the sphere;

(ii) it is an azimuthal conformal projection;

(iii) isoscale lines are concentric circles about the origin of the projection;

(iv) great circles are projected as circles.

However, we are interested in the conformal mapping of a biaxial ellipsoid on a plane. There is no mapping of an ellipsoid to a plane that possesses all of the characteristics of the stereographic mapping of a sphere to a plane [Grossmann, 1964]. The stereographic projection of an ellipsoid of revolution can be approached in two different ways:

(i) a double projection, in which the biaxial ellipsoid is conformally mapped to a sphere, which is then"stereographically projected" to a plane;

(ii) a "quasi-stereographic" mapping is obtained directly in which one of the properties of the spherical stereographic projection is rigorously retained while the others are only approximately fulfilled.

The approach described in this report is one of a double projection. Ellipsoidal data is mapped conformally on a conformal sphere. Then, a second conformal mapping of the spherical data to the plane completes the process. Since the two mappings are conformal, the result is a conformal mapping of ellipsoidal data on a plane.

2. THE CONFORMAL SPHERE

In the development of the stereographic "double projection" of the ellipsoid to a plane, the conformal sphere is introduced as a necessary intermediate mapping surface. Once the ellipsoidal information is mapped on the conformal sphere, the application of the geometric and trigonometric principals for the stereographic perspective projection of a sphere to a plane can be applied. The end result is a conformal mapping of the ellipsoid on the plane.

Note that the projection of the ellipsoid on the conformal sphere, as developed by Gauss [Jordan/ Eggert, 1948], is a conformal mapping. Further, the meridians and parallels on the ellipsoid map as meridians and parallels on the sphere.

2.1 Direct (ϕ, $\lambda \to \chi$, Λ) Mapping

A differential length of a geodesic on the ellipsoidal surface is given by (for example Krakiwsky [1973])

$$ds^2 = N^2 \cos^2 \phi \left(\frac{M^2}{N^2} \sec^2 \phi \, d\phi^2 + d\lambda^2\right) \quad , \tag{1}$$

where N and M are the prime vertical and meridian radii of curvature of the ellipsoid at the point of interest designated by the geodetic latitude ϕ and geodetic longitude λ. Similarly, a differential length of a corresponding arc of a great circle on a sphere of radius R is given as (for example Jordan/Eggert [1948])

$$dS^2 = R^2 \cos^2 \chi \left(\sec^2 \chi \, d\chi^2 + d\Lambda^2\right) \quad , \tag{2}$$

in which χ and Λ are the spherical latitude and longitude respectively.

Now, the condition that the mapping of the ellipsoid on the sphere be conformal can be expressed as (for example Richardus and Adler [1972])

$$\frac{E}{e} = \frac{G}{g} = \text{constant} \quad , \tag{3}$$

where E, G, and e, g are known as the first Gaussian fundamental quantities of the sphere and the ellipsoid respectively. For the development here, (3) is re-written as

$$\frac{\left(\frac{\partial \chi}{\partial \phi}\right)^2 E'}{e} = \frac{\left(\frac{\partial \Lambda}{\partial \lambda}\right)^2 G'}{g} = k^2 \quad , \tag{4}$$

in which [Richardus and Adler, 1973]

$$\begin{aligned} E' &= R^2; \quad G' = R^2 \cos^2 \chi \quad ; \\ e &= M^2; \quad g = N^2 \cos^2 \phi \quad . \end{aligned} \tag{5}$$

Substituting (5) in (4) yields

$$\frac{\left(\frac{\partial \chi}{\partial \phi}\right)^2 R^2}{M^2} = \frac{\left(\frac{\partial \Lambda}{\partial \lambda}\right)^2 R^2 \cos^2 \chi}{N^2 \cos^2 \phi} = k^2 \quad , \tag{6}$$

or

$$\frac{R}{M} \frac{\partial \chi}{\partial \phi} = \frac{R \cos \chi}{N \cos \phi} \frac{\partial \Lambda}{\partial \lambda} = k \quad , \tag{7}$$

where the constant k is referred to as the point scale factor.

Now, we want the mapping to obey the condition

$$\chi = f(\phi) \quad , \tag{8}$$

thus

$$\frac{\partial \Lambda}{\partial \lambda} = c_1 \tag{9}$$

or

$$\partial \Lambda = c_1 \, \partial \lambda \quad , \tag{10}$$

in which c_1 is some constant. Then, (7) can be re-written as

$$\frac{\partial \chi}{\partial \phi} = c_1 \frac{M \cos \chi}{N \cos \phi} = k \quad . \tag{11}$$

The solution of this differential equation (11) is given by (for example H.J. Heuvelink [1918])

$$\boxed{\tan(\frac{\Pi}{4} + \frac{\chi}{2}) = c_2\{\tan(\frac{\Pi}{4} + \frac{\phi}{2})(\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}\}^{c_1}} \quad , \tag{12}$$

in which e is the first eccentricity of the ellipsoid and c_2 is an integration constant.

Now, since c_1 is a constant, and since the constant of integration would only mean a change in the choice of a zero meridian, equation (10) yields directly

$$\boxed{\Lambda = c_1 \lambda} \quad . \tag{13}$$

It follows that if c_1, ϕ, χ are known, the point scale factor k could be computed from (7) as

$$\boxed{k = c_1 \frac{R \cos \chi}{N \cos \phi}} \quad . \tag{14}$$

The problem to be addressed now is the evaluation of the constants c_1, c_2, and R in order that one might obtain solutions for equations (12), (13), and (14). The method of solution given here is one in which, for a particular differentially small region, the deviation of the point scale factor k from 1 shall be a minimum. While this stipulation is not mandatory, it is done for convenience and as an ideal case.

The point (differentially small region) of interest on the ellipsoid - commonly called <u>the origin</u> - is designated by (ϕ_o, λ_o), and its counterpart on the sphere by (χ_o, Λ_o). At this point, it is required that k = 1. Now the scale factor can be expressed as

$$k = f(\chi) \quad , \tag{15}$$

and when expanded as a Taylor series (including terms to order 2), to be evaluated at the origin, yields

$$k = f(\chi)_o + \left(\frac{d\ f(\chi)}{d\chi}\right)_o d\chi + \left(\frac{d^2\ f(\chi)}{d^2\chi}\right)_o \frac{d\ \chi^2}{2} + \ldots \quad . \tag{16}$$

In the above expression

$$f(\chi)_o = k_o \quad , \tag{17}$$

where k_o is the point scale factor referred to the origin (ϕ_o, λ_o) or (χ_o, Λ_o). Now, with the conditions expressed above, one must have

$$\begin{aligned} k_o &= 1 \quad , \\ \frac{dk}{d\chi_o} &= 0 \quad , \\ \frac{d^2k}{d\ \chi_o^2} &= 0 \quad . \end{aligned} \tag{18}$$

Evaluation of these derivatives yields (for example Heuvelink [1918])

$$\frac{dk}{d\chi_o} = \frac{\sin\ \phi_o - c_1\ \sin\ \chi_o}{c_1\ \cos\ \chi_o} \quad , \tag{19}$$

$$\frac{d^2k}{d\chi_o^2} = \frac{N_o}{M_o} \frac{\cos^2\phi_o}{c_1^2\ \cos^2\chi_o} - \frac{1}{\cos^2\chi_o} + \frac{\sin\ \phi_o\ \sin\ \chi_o}{c_1\ \cos^2\ \chi_o} \quad . \tag{20}$$

Since $k_o = 1$, then directly from (14)

$$k_o = 1 = c_1 \frac{R\ \cos\ \chi_o}{N_o\ \cos\ \phi_o} \quad . \tag{21}$$

Setting (19) equal to zero (as per the condition expressed by (18)), one then gets

$$\frac{\sin\ \phi_o - c_1\ \sin\ \chi_o}{c_1\ \cos\ \chi_o} = 0 \quad , \tag{22}$$

or

$$\boxed{\sin\ \phi_o = c_1\ \sin\ \chi_o} \quad . \tag{23}$$

Then, setting (20) equal to zero, multiplying each term by $c_1^2 \cos^2 \chi_o$, and re-writing yields

$$c_1^2 - c_1 \sin \phi_o \sin \chi_o = \frac{N_o}{M_o} \cos^2 \phi_o \quad . \tag{24}$$

Now, replacing $\sin \phi_o$ by $c_1 \sin \chi_o$ gives

$$c_1^2 - c_1^2 \sin^2 \chi_o = \frac{N_o}{M_o} \cos^2 \phi_o \quad , \tag{25}$$

or

$$c_1 \cos \chi_o = \cos \phi_o \sqrt{\frac{N_o}{M_o}} \quad . \tag{26}$$

Now, squaring each of (23) and (26) and equating them yields

$$c_1^2 \sin^2 \chi_o + c_1^2 \cos^2 \chi_o = \sin^2 \phi_o + \cos^2 \phi_o \frac{N_o}{M_o} \quad , \tag{27}$$

or

$$c_1^2 = \sin^2 \phi_o + \cos^2 \phi_o \frac{N_o}{M_o} \quad . \tag{28}$$

Rewriting (28) as

$$c_1^2 = \sin^2 \phi_o + \cos^2 \phi_o + \cos^2 \phi_o \left(\frac{N_o}{M_o} - 1 \right) \quad , \tag{29}$$

gives

$$c_1^2 = 1 + \cos^2 \phi_o \frac{N_o}{M_o} - \cos^2 \phi_o \quad . \tag{30}$$

However, since

$$N_o = \frac{a}{(1-e^2 \sin^2 \phi_o)^{1/2}} \quad \text{and} \quad M_o = \frac{a(1-e^2)}{(1-e^2 \sin^2 \phi_o)^{3/2}} \quad , \tag{31}$$

then

$$\frac{N_o}{M_o} = 1 + \frac{e^2}{1-e^2} \cos^2 \phi_o \quad . \tag{32}$$

Replacing $\frac{N_o}{M_o}$ in (30) and manipulating terms yields

$$\boxed{c_1 = \left(1 + \frac{e^2}{1-e^2} \cos^4 \phi_o\right)^{1/2}} \; . \tag{33}$$

Now, to compute R, one proceeds as follows. Recall that for the case at hand, $k_o = 1$, therefore from (21)

$$R = \frac{N_o \cos \phi_o}{c_1 \cos \chi_o} \; . \tag{34}$$

From (26)

$$c_1 = \frac{\cos \phi_o}{\cos \chi_o} \sqrt{\frac{N_o}{M_o}} \; . \tag{35}$$

Then, substituting for c_1 in (34) one gets

$$\boxed{R = \sqrt{N_o M_o}} \; . \tag{36}$$

Finally, the constant c_2 is evaluated via a re-written version of equation (12), namely

$$\boxed{c_2 = \tan\left(\frac{\Pi}{4} + \frac{\chi_o}{2}\right) \left\{\tan\left(\frac{\Pi}{4} + \frac{\phi_o}{2}\right) \left(\frac{1-e \sin \phi_o}{1+e \sin \phi_o}\right)^{e/2}\right\}^{-c_1}} \; . \tag{37}$$

In summary, the direct transformation, with $k_o = 1$, proceeds as follows. Select an origin (ϕ_o, λ_o). Compute the radius of the conformal sphere using (36), then the constant c_1 by (33). Then, χ_o can be evaluated by (23) and Λ_o from (13). Finally, the constant c_2 is computed using (37). The evaluation of further (χ, Λ) from any (ϕ, λ) is then carried out using equations (12) and (13) using the already computed constants c_1, c_2, k_o, and R.

Recall again that this development is for $k_o = 1$. If one were to select k_o equal to another value, then R would change as is obvious from the expression obtained combining (21) and (26) with $k_o \neq 1$, namely

$$R = k_o \sqrt{M_o N_o} \; . \tag{38}$$

2.2 Inverse ($\chi, \Lambda \rightarrow \phi, \lambda$) Mapping

It is assumed for this section that the fundamental definition of coordinates is in terms of a biaxial ellipsoid. Furthermore, it is assumed that the required constants - c_1, c_2, k_o, R - have been computed as outlined in section 2.1.

The transformation of conformal spherical longitude to geodetic (ellipsoidal) longitude is given by a simple re-arrangement of terms in equation (13) namely

$$\boxed{\lambda = \Lambda/c_1} \quad . \tag{39}$$

The solution for ϕ from χ cannot be solved for directly from (12). It can however be solved for via the Newton-Raphson iteration technique. Briefly, one proceeds as follows. Given a non-linear function $f(x) = 0$, select an initial approximate value $x_{(n-1)}$. Evaluate the iterative improved solutions x_n from

$$\boxed{x_n = x_{n-1} - \frac{f(x)}{f'(x)} \, , \qquad n=0,\ 1,\ 2,\ \ldots} \, , \tag{40}$$

in which $f'(x)$ is the first derivative of the function $f(x)$. The process is continued until $|x_n - x_{n-1}| < \varepsilon$, where ε is a predetermined limit.

In the case $\chi \rightarrow \phi$, one starts with $\phi = \chi$ as a first approximation. The equation (12) is re-written as

$$f(\phi) = c_2 [\tan(\frac{\Pi}{4} + \frac{\phi}{2}) \ (\frac{1-e \sin \phi}{1+e \sin \phi})^{e/2}]^{c_1} - \tan(\frac{\Pi}{4} + \frac{\chi}{2}) = 0 \tag{41}$$

The first derivative with respect to ϕ, $f'(\phi)$ is

$$f'(\phi) = c_1 c_2 [\tan(\frac{\Pi}{4} + \frac{\phi}{2}) (\frac{1-e\sin\phi}{1+e\sin\phi})^{e/2}]^{(c_1-1)} [(\frac{1-e\sin\phi}{1+e\sin\phi})^{e/2}$$

$$\{\frac{1}{2} \sec^2(\frac{\Pi}{4} + \frac{\phi}{2}) - \frac{e^2 \cos \phi}{1-e^2\sin^2\phi} \tan(\frac{\Pi}{4} + \frac{\phi}{2})\}] \quad . \tag{42}$$

The iterative process (40) should be continued until ε is less than 1/100,000 of a second of arc (less than 1 mm).

2.3 Summary

The conformal mapping of a biaxial ellipsoid on a conformal sphere has been developed. While most equations are of a general nature, the reader is again cautioned that the final results yielding $R = \sqrt{N_o M_o}$ was based on $k_o = 1$.

The choice of constants - c_1, c_2, R - can lead to several alternative conformal mappings. The constant c_1 is a function of only ellipsoidal eccentricity e and geodetic latitude ϕ_o of the origin. It can be seen, using equation (33), that at the equator ($\phi_o = 0°$)

$$c_1 = \left(\frac{1}{1-e^2}\right)^{1/2} \quad , \tag{43}$$

and from equation (37)

$$c_2 = \tan\left(\frac{\pi}{4} + \frac{\chi_o}{2}\right) \quad . \tag{44}$$

Furthermore, if $k_o = 1$, then from (34)

$$R = \frac{a(1-e^2)^{1/2}}{\cos \chi_o} \quad , \tag{45}$$

since at $\phi_o = 0$, $N_o = a$ and $\cos \phi_o = 1$.

At the pole ($\phi_o = 90°$), one gets from (33) that

$$c_1 = 1 \quad . \tag{46}$$

However, the problem now arises that c_2 is undefined at the pole (check with $\phi_o = 90°$ in equation (37)). For a treatment of the polar stereographic projection of the ellipsoid on a plane, the reader is referred to, for example, Krakiwsky [1973], pp. 92-94.

In this report, only the oblique case will be dealt with hereafter. Furthermore, the treatment will be limited to the case in which the ellipsoid is conformally mapped on a sphere and vice-versa, such that $R = \sqrt{M_o N_o}$ and $k_o = 1$, with c_1 and c_2 being determined by equations (33) and (37) respectively.

3. CONFORMAL MAPPING OF THE CONFORMAL SPHERE ON A PLANE

This process - the mapping of the sphere on a plane - is what is commonly referred to in the literature as the "stereographic projection". In our case, where we are interested in the conformal mapping of the ellipsoid on a plane, it is the second step of the "double projection" process.

3.1 Geometric Properties

As has been mentioned previously, this part of the mapping process is commonly referred to as a perspective projection. The perspective point P (Figure 1) is diametrically opposite the origin O. To project any point Q on the sphere to the mapping plane, one need only draw a line from P through Q to obtain the projected point Q' (Figure 1). It should be noted that Figure 1 depicts the situation in which the mapping plane is tangent to the conformal sphere at the origin O, while in Figure 2, the mapping plane becomes a secant plane. As will be shown later, the difference between the two cases is expressed analytically as a change in the scale factor at the origin (k'_o).

The stereographic projection (sphere to plane) is generally treated such that three distinct cases arise as to the position of mapping plane relative to the sphere. The three are the normal (polar)(see Figure 3), transverse (see Figure 4), and oblique (see Figure 5). The origin (O) of the normal (polar) stereographic is one of the poles of the conformal sphere with the opposite pole as the perspective point.

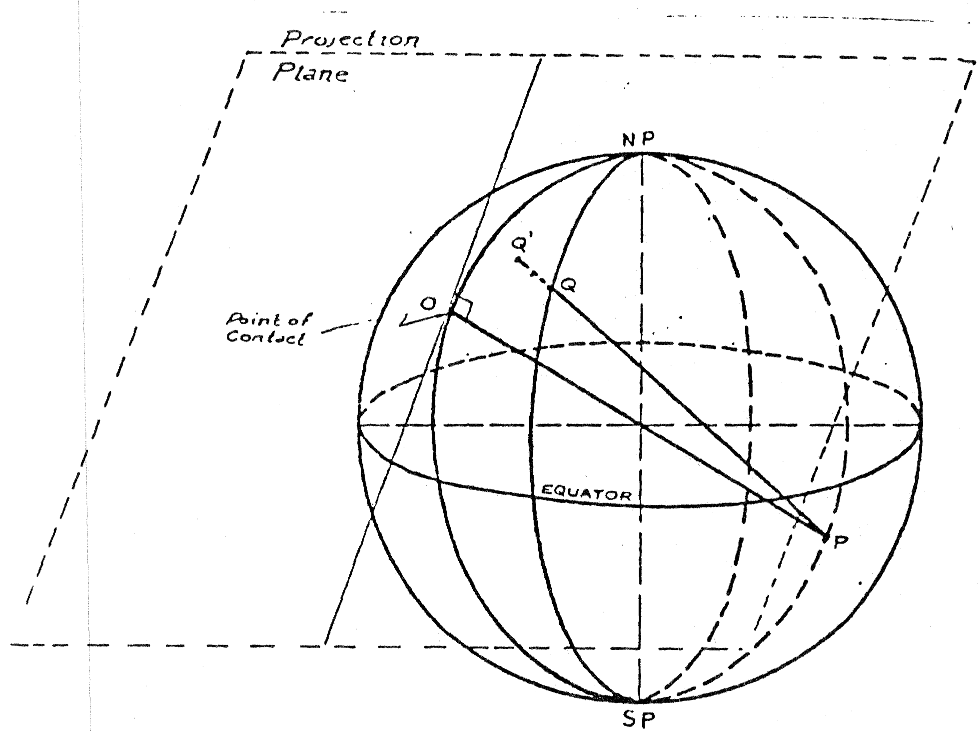


Figure 1

Stereographic (Spherical) Projection (Tangent Plane)

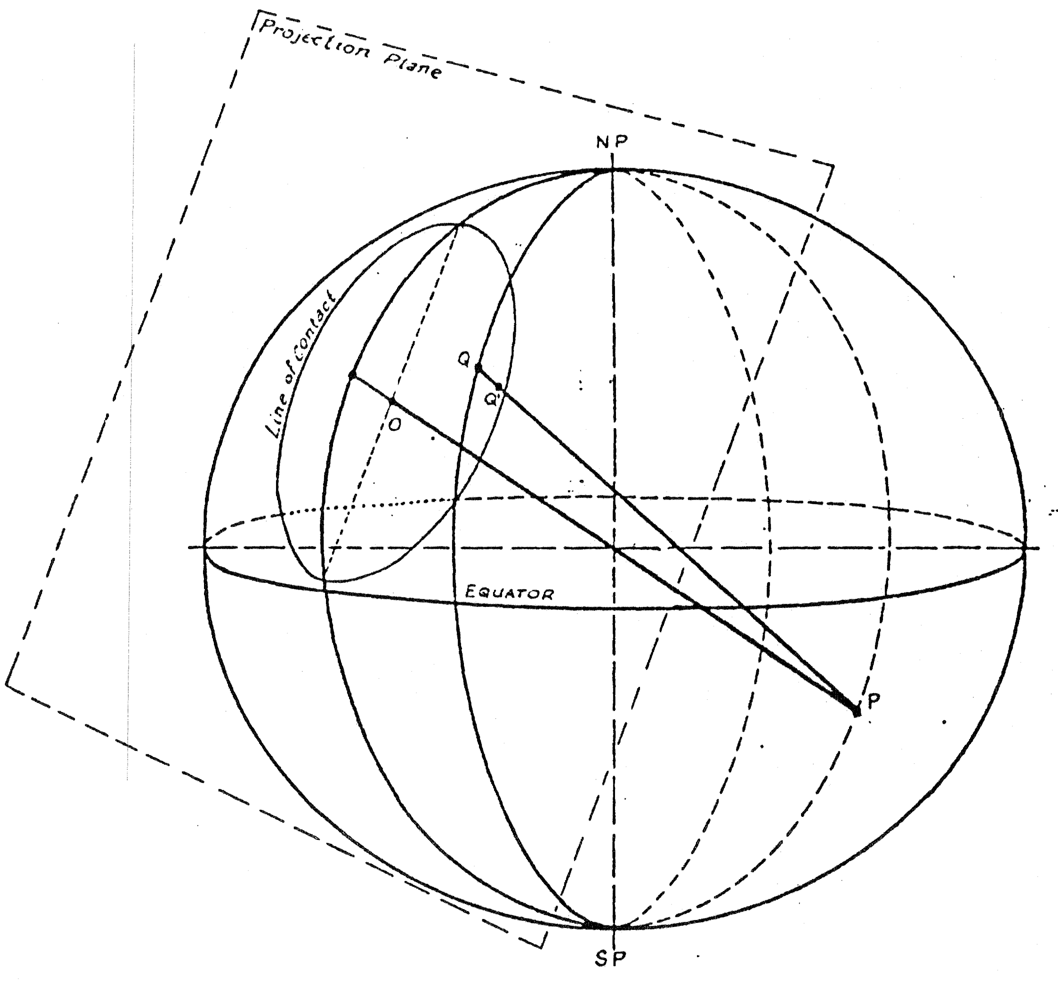
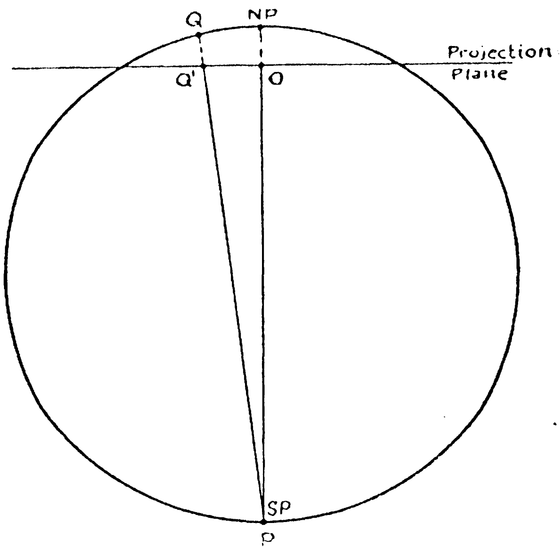
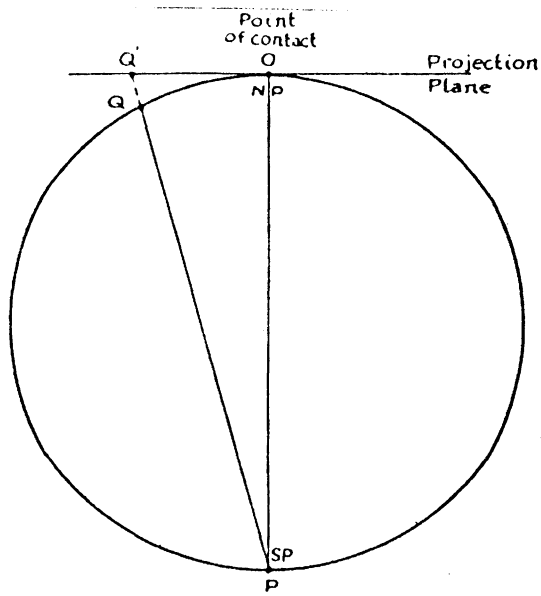


Figure 2

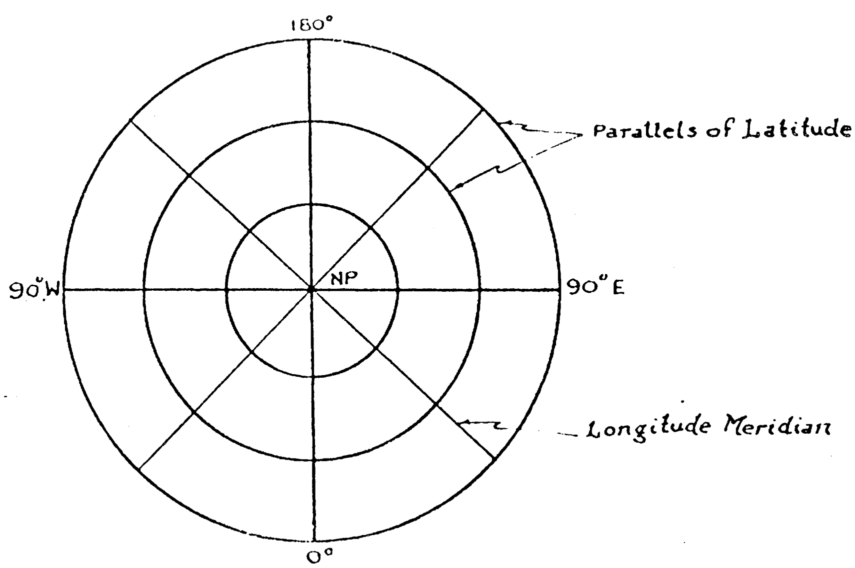
Stereographic (Spherical) Projection (Secant Plane)



(a) Secant Plane



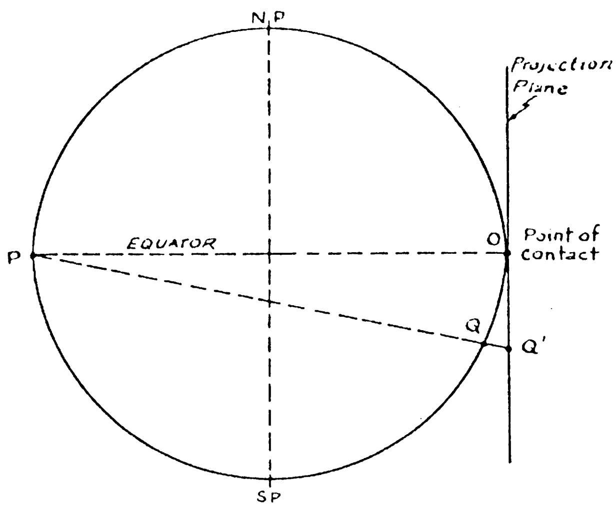
(b) Tangent Plane



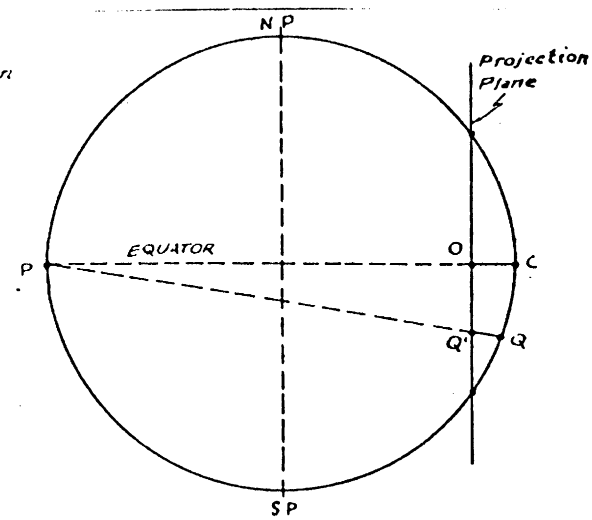
(c) Projected Meridians and Parallels

Figure 3

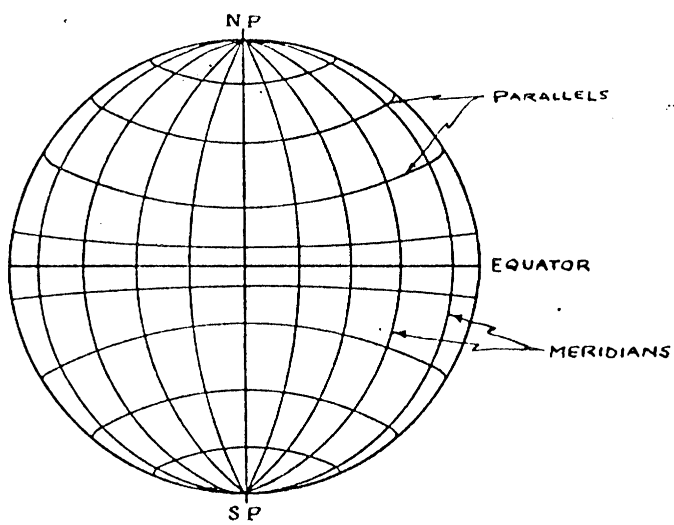
Normal (Polar) Stereographic Spherical Projection



(a) Tangent Plane



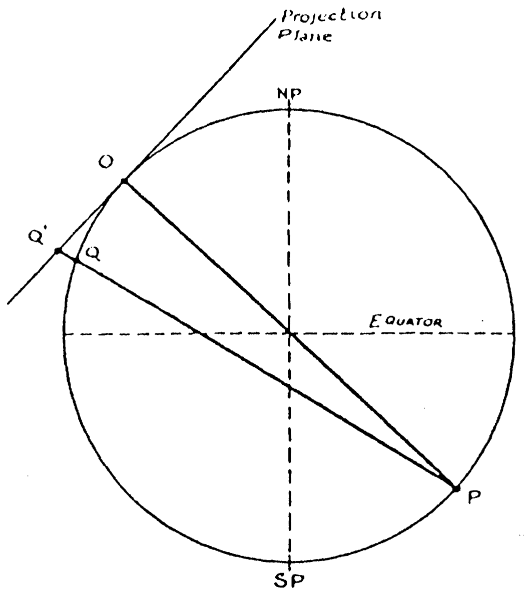
(b) Secant Plane



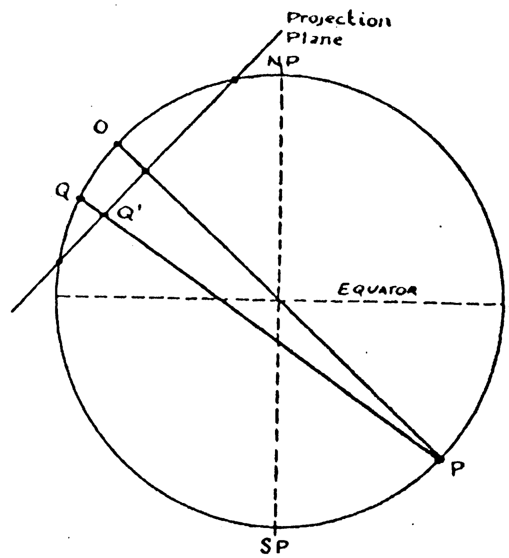
(c) Mapping of Parallels and Meridians

Figure 4

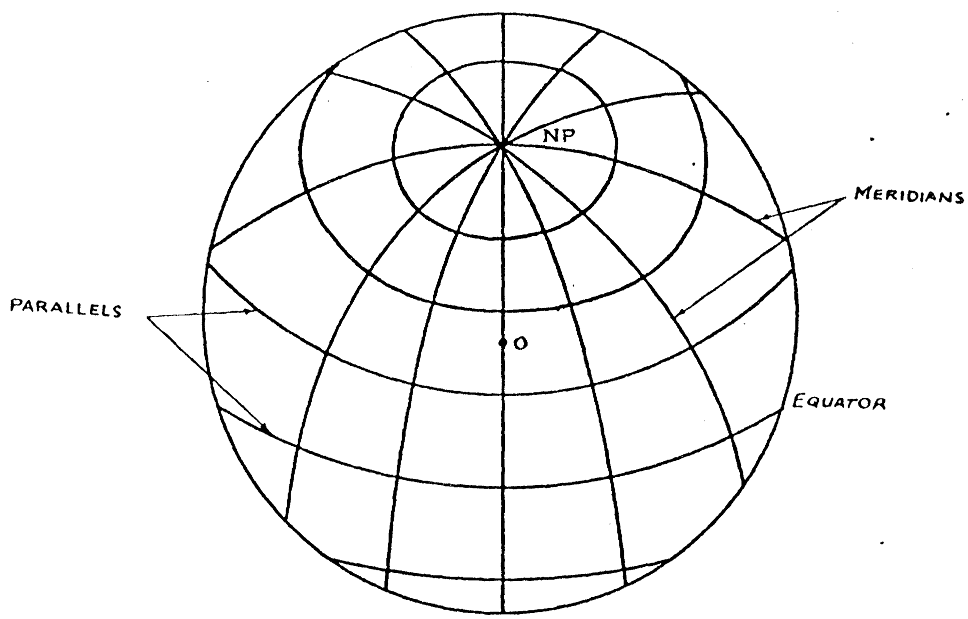
Transverse Stereographic Spherical Projection



(a) Tangent Plane



(b) Secant Plane



(c) Mapping of Parallels and Meridians

Figure 5

Oblique Stereographic Spherical Projection

Parallels of latitude map as concentric circles, while longitude meridians map as straight lines radiating from the origin (Figure 3(c)) [Thomas, 1952; Richardus and Adler, 1972]. The transverse stereographic has its origin at some point on the spherical equator (Figure 4). The spherical equator then maps as a straight line, as does the meridian through the origin (Figure 4(c)). All other meridians and parallels map as parts of circles [Thomas, 1952; Richardus and Adler, 1972]. The choice of the origin for an oblique stereographic mapping is arbitrary. The meridian through the origin maps as a straight line. All other meridians and parallels map as circles (Figure 5(c)) [Thomas, 1952; Richardus and Adler, 1972].

3.2 Direct (χ, $\Lambda \rightarrow x$, y) Mapping

Figure 5 shows the elements of interest as they appear on the conformal sphere. As before, the point O is the origin and Q is the point to be mapped. Figure 6 depicts the section of the conformal sphere containing O, Q, C (centre of the sphere), and P (perspective point). In Figure 7, a plan view of the mapping plane and the sphere (along the line O C P) is given.

From Figure 8, it can be seen that the plane (x, y) mapping coordinates of the projected point Q' are given by

$$\begin{aligned} x &= s \cos \beta \quad , \\ y &= s \sin \beta \quad , \end{aligned} \tag{47}$$

in which, from Figure 7, s is given as

$$s = 2 R \tan \delta/2 \quad . \tag{48}$$

Now, using respectively the cosine and sine laws of spherical trigonometry in the spherical triangle O N Q (Figure 6), one finds

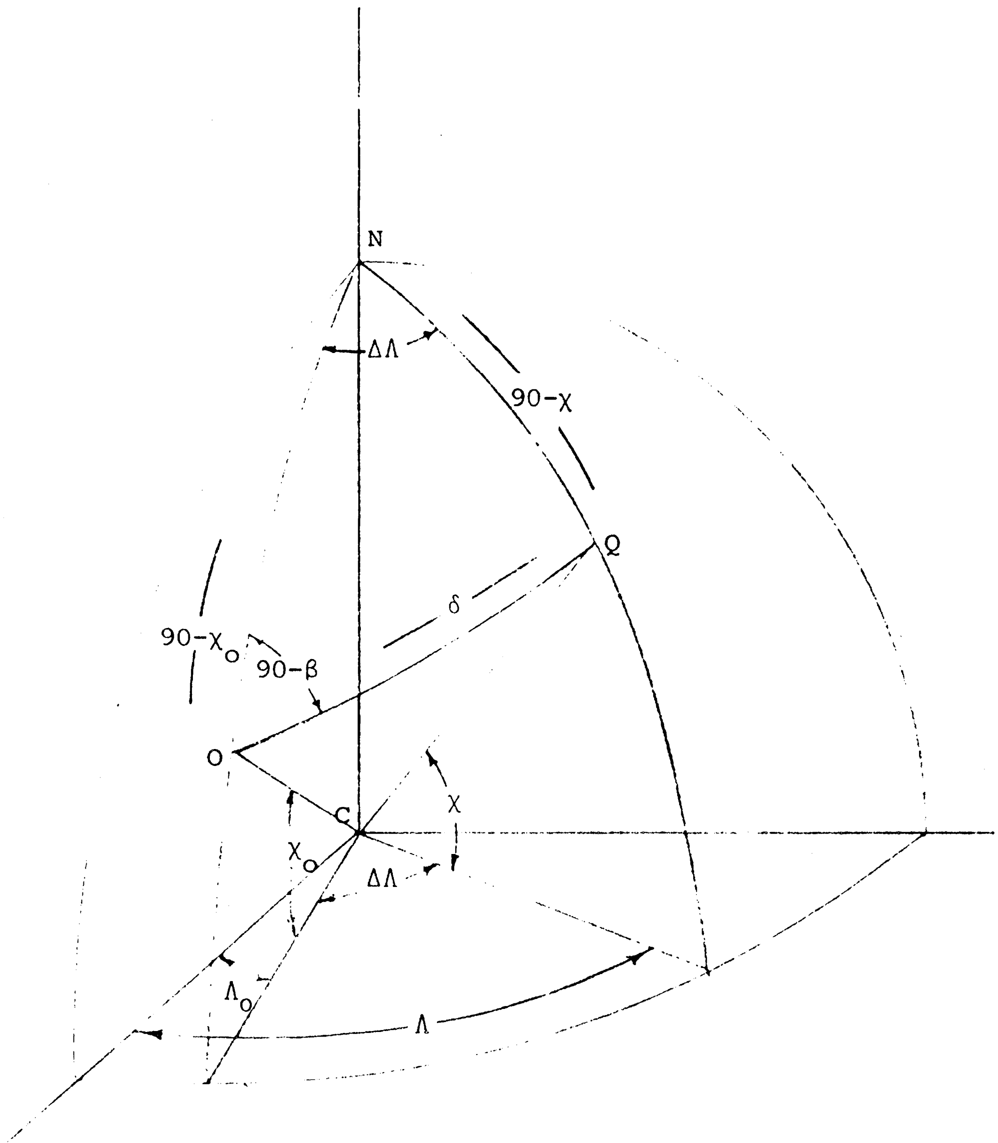
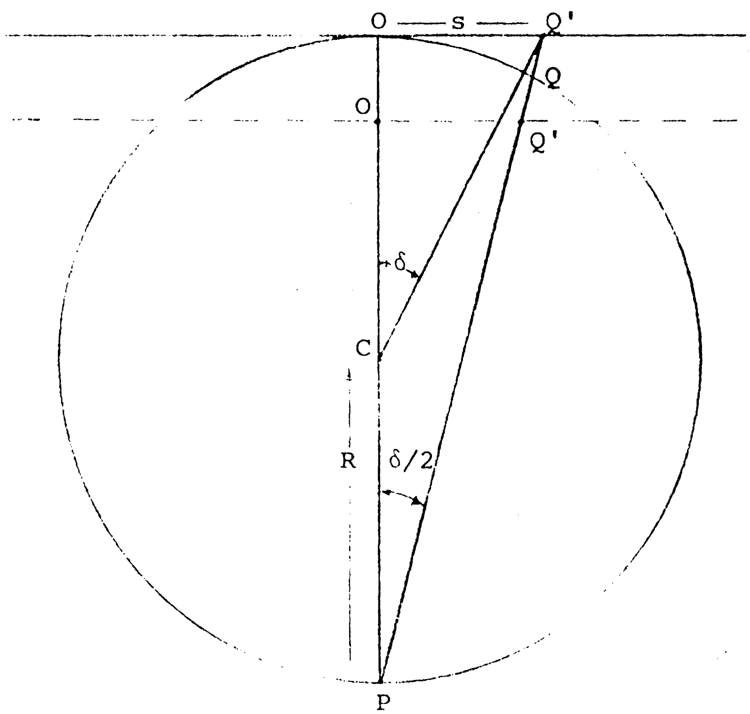


Figure 6

Mapping of the Conformal Sphere on a Plane

Figure 7

Geometric Interpretation of the Stereographic Projection (Sphere to Plane)



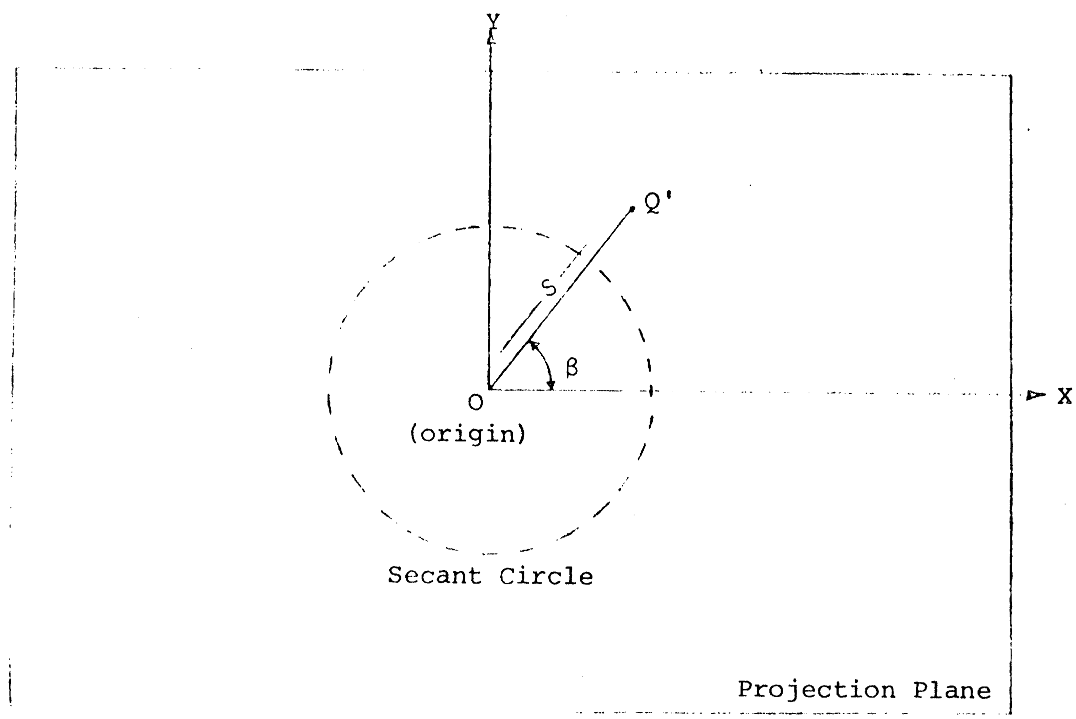


Figure 8

Plan View of Stereographic Projection (Sphere to Plane)

$$\cos\delta = \cos(90 - \chi_o)\cos(90 - \chi) + \sin(90 - \chi_o)\sin(90 - \chi)\cos \Delta\Lambda \ , \quad (49)$$

and

$$\sin(90 - \beta) = \sin(90 - \chi) \frac{\sin \Delta\Lambda}{\sin\delta} \ , \quad (50)$$

in which

$$\Delta \Lambda = \Lambda - \Lambda_o \ . \quad (51)$$

The above equations (49) and (50), finally yield

$$\cos \delta = \sin \chi_o \sin \chi + \cos \chi_o \cos \chi \cos \Delta \Lambda \ , \quad (52)$$

$$\cos \beta = \cos \chi \frac{\sin \Delta\Lambda}{\sin\delta} \ . \quad (53)$$

Now, using the trigonometric identity

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos\theta} \ , \quad (54)$$

substituting in equation (48) for $\tan \delta/2$, thence in (47) for s, yields

$$x = \frac{2 R \sin \delta}{1 + \cos \delta} \cdot \frac{\cos \chi \sin \Delta\Lambda}{\sin\delta} \ , \quad (55)$$

or

$$\boxed{x = 2 R \frac{\cos \chi \sin \Delta\Lambda}{1 + \sin \chi_o \sin \chi + \cos \chi_o \cos \chi \cos \Delta\Lambda}} \ . \quad (56)$$

Using the spherical triangle O Q N (Figure 6), and the five parts rule of spherical trigonometry, one gets

$$\cos(90-\beta)\sin\delta = \cos(90-\chi)\sin(90-\chi_o) - \cos(90-\chi_o)\sin(90-\chi)\cos \Delta\Lambda \ , \quad (57)$$

or

$$\sin \beta = \frac{\sin\chi \cos \chi_o - \sin \chi_o \cos \chi \cos \Delta\Lambda}{\sin \delta} \ . \quad (58)$$

Substituting for $\sin \beta$ and s in (47) now yields

$$y = 2 R \frac{\sin \delta}{1+\cos \delta} \cdot \frac{\sin \chi \cos \chi_o - \cos \chi \sin \chi_o \cos \Delta\Lambda}{\sin \delta} \ , \quad (59)$$

and finally

$$\boxed{y = 2 R \frac{\sin \chi \cos \chi_o - \cos \chi \sin \chi_o \cos \Delta\Lambda}{1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda}} \ . \quad (60)$$

These equations - (56) and (60) - for the mapping of a point from the conformal sphere to a plane are identical to those found in the literature (for example, Jordan/Eggert [1948]; Richardus and Adler [1972]).

To close this section, it is left to show:

(i) that the mapping is conformal;

(ii) how one deals with a scale factor other than $k'_o = 1$ at the origin (χ_o, Λ_o);

(iii) how one computes grid coordinates.

Thomas [1952] shows that for mapping any surface on a plane, the mapping is conformal if

$$F = 0 \quad , \qquad \frac{G}{E} = \frac{U}{V} \quad , \tag{61}$$

in which E, F and G are the Gaussian fundamental quantities, defined here as

$$E = \left(\frac{\partial x}{\partial \chi}\right)^2 + \left(\frac{\partial y}{\partial \chi}\right)^2 \quad ,$$

$$F = \frac{\partial x}{\partial \chi}\frac{\partial x}{\partial \Lambda} + \frac{\partial y}{\partial \chi}\frac{\partial y}{\partial \Lambda} \quad , \tag{62}$$

$$G = \left(\frac{\partial x}{\partial \Lambda}\right)^2 + \left(\frac{\partial y}{\partial \Lambda}\right)^2 \quad .$$

U and V are any functions such that

$$U = U(\Lambda) \quad , \qquad V = V(\chi) \quad . \tag{63}$$

Evaluating the derivatives, one obtains

$$\frac{\partial x}{\partial \chi} = 2\ R \sin \Delta\Lambda \left(\frac{-\ (\sin \chi + \sin \chi_o)}{(1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2}\right) \quad , \tag{64}$$

$$\frac{\partial x}{\partial \Lambda} = 2\ R \cos \chi \left(\frac{\cos \Delta\Lambda\ (1 + \sin \chi \sin \chi_o) + \cos \chi \cos \chi_o}{(1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2}\right) \quad , \tag{65}$$

$$\frac{\partial y}{\partial \chi} = 2\ R\ \left(\frac{\cos \Delta\Lambda (1 + \sin \chi \sin \chi_o) + \cos \chi \cos \chi_o}{(1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2}\right) \quad , \tag{66}$$

$$\frac{\partial y}{\partial \Lambda} = 2\ R \sin \Delta\Lambda \cos \chi \ \left(\frac{\sin \chi + \sin \chi_o}{(1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2}\right) \ . \tag{67}$$

Substituting in (61) yields

$$F = 0 \quad ,$$

and

$$\frac{G}{E} = \frac{U}{V} = \frac{1}{\cos^2 \chi} \quad , \tag{68}$$

proving that the mapping, represented by equations (56) and (60), is conformal. Finally, since the mapping of the ellipsoid to the conformal sphere was developed as a conformal mapping (section 2.1), and since the mapping of the conformal sphere to the plane has been proven to be conformal, then the entire mapping - ellipsoid to plane - through a double projection process, is conformal.

Thus far, it has been assumed that the mapping plane has been tangent to the conformal sphere at the origin. This infers a scale factor of unity ($k_o' = 1$) at the origin. In the case where k_o' takes on a value other than unity, the mapped coordinates must change. It can be shown (for example, Jordan/Eggert [1948]) that the resulting mapping coordinates are given by

$$\boxed{\begin{aligned} x &= k_o' \left[2\ R \frac{\cos \chi \sin \Delta\Lambda}{1 + \sin \chi_o \sin \chi + \cos \chi_o \cos \chi \cos\Delta\Lambda}\right] \\ y &= k_o' \left[2\ R \frac{\sin \chi \cos \chi_o - \cos \chi \sin \chi_o \cos \Delta\Lambda}{1 + \sin \chi_o \sin \chi + \cos \chi_o \cos \chi \cos \Delta\Lambda}\right] \end{aligned}} \ ,\ . \tag{69}$$

The development of an expression for k' is given in section 4.1.

Finally, the topic of "grid" coordinates must be dealt with. The origin of the plane coordinate system for the stereographic projection is the ellipsoidal point (ϕ_o, λ_o), or equivalently the spherical point (χ_o, Λ_o). The orientation of the coordinate system axes is given in Figure 7. To avoid negative coordinates, it is common practice that some "false" coordinates - other than $x_o = 0$, $y_o = 0$ - be assigned to the origin of the plane coordinate system. The shifted plane coordinate system is commonly referred to as the mapping "grid". Designating grid coordinates by X, Y, the arbitrary origin coordinates (sometimes referred to as the "false" origin) by x_o, y_o, and the mapped coordinates by x, y (from equations (69)), then the following relationship is established:

$$\boxed{\begin{aligned} X &= x_o + x \\ Y &= y_o + y \end{aligned}} \quad \begin{aligned} &, \\ &. \end{aligned} \tag{70}$$

It should be noted that when dealing with the direct $(\chi, \Lambda \rightarrow x, y)$ or inverse $(x, y \rightarrow \chi, \Lambda)$ mappings, one operates with the mapping coordinates (x, y).

3.3 Inverse $(x, y \rightarrow \chi, \Lambda)$ Mapping

Applying the cosine law to the spherical triangle O N Q (Figure 6), yields

$$\cos(90 - \chi) = \cos(90 - \chi_o)\cos \delta + \sin(90 - \chi_o)\sin \delta \cos(90 - \beta) , \tag{71}$$

or

$$\boxed{\sin \chi = \sin \chi_o \cos \delta + \cos \chi_o \sin \delta \sin \beta} \quad . \tag{72}$$

This equation is used directly for the solution of χ for a point of interest since, from Figure 8,

$$\beta = \sin^{-1} (\frac{y}{s}) \quad , \tag{73}$$

$$s = (x^2 + y^2)^{1/2} \quad , \tag{74}$$

and from Figure 7

$$\delta = 2 \tan^{-1} \frac{s}{2R} \quad . \tag{75}$$

Similarly, the sine law yields

$$\boxed{\sin \Delta\Lambda = \frac{\cos \beta \sin \delta}{\cos \chi}} \quad . \tag{76}$$

Finally,

$$\boxed{\Lambda = \Lambda_o + \Delta\Lambda} \quad .$$

3.4 Summary

Special cases of the spherical mapping equations are easily obtained from the general ones given in section 3.3. For example, when $\chi_o = 90°$ (polar), then (69) reduces to

$$\begin{aligned} x &= k_o' \left[2 R \frac{\cos \chi \sin \Delta\Lambda}{1 + \sin \chi}\right] \quad , \\ y &= k_o' \left[2 R \frac{- \cos \chi \cos \Delta\Lambda}{1 + \sin \chi}\right] \quad . \end{aligned} \tag{77}$$

Similarly, when $\chi_o = 0°$ (transverse), then (69) becomes

$$\begin{aligned} x &= k_o' \left[2 R \frac{\cos \chi \sin \Delta\Lambda}{1 + \cos \chi \cos \Delta\Lambda}\right] \quad , \\ y &= k_o' \left[\frac{\sin \chi}{1 + \cos \chi \cos \Delta\Lambda}\right] \end{aligned} \tag{78}$$

The analytical expressions for the conformal mapping of ellipsoidal points on a plane via the stereographic double projection process have now been completely developed. In brief, one maps any ellipsoidal point (ϕ, λ) to its conformal spherical counterpart (χ, Λ) via equations (12) and (13). Mapping plane coordinates (x, y) are then computed from (χ, Λ) using (69), and finally, grid coordinates are obtained via (70).

The remainder of this report deals with two items:

(i) the reduction of ellipsoidal data - directions, distances, azimuths - to the mapping plane so that plane computations can be carried out correctly;

(ii) the propagation of the variance-covariance information associated with the (ϕ, λ)'s to equivalent information associated with the (x, y)'s.

4. SCALE FACTOR, MERIDIAN CONVERGENCE, (T - t)

The conformal mapping of ellipsoidal points, designated by geodetic coordinates (ϕ, λ), to the plane, designated by coordinates (x, y), using the stereographic double projection technique, was given in Chapters 2 and 3. To carry out computations on the mapping plane (e.g. direct problem, intersection, resection), ellipsoidal quantities must be similarly mapped. In this chapter, the mapping of ellipsoidal distances, azimuths, and directions (or angles) are treated.

4.1 Scale Factor

In section 2.1, the point scale factor for the mapping of ellipsoidal data to the conformal sphere was shown to be given by equation (14), namely

$$\boxed{k = c_1 \frac{R \cos \chi}{N \cos \phi}} \, .$$

The point scale factor for the mapping of the conformal sphere on the plane is derived as follows. In Figure 9, the element of arc dS is projected to the mapping plane as ds. The arc element dS can be expressed as

$$dS = R \, d \, \delta \, . \tag{79}$$

Using plane trigonometry, the length ds is determined as follows. The triangle POQ' yields the length PQ' by

$$PQ' = \frac{2 \, R}{\cos \, \delta/2} \, . \tag{80}$$

The length of the side Q'A in the triangle PQ'A is then

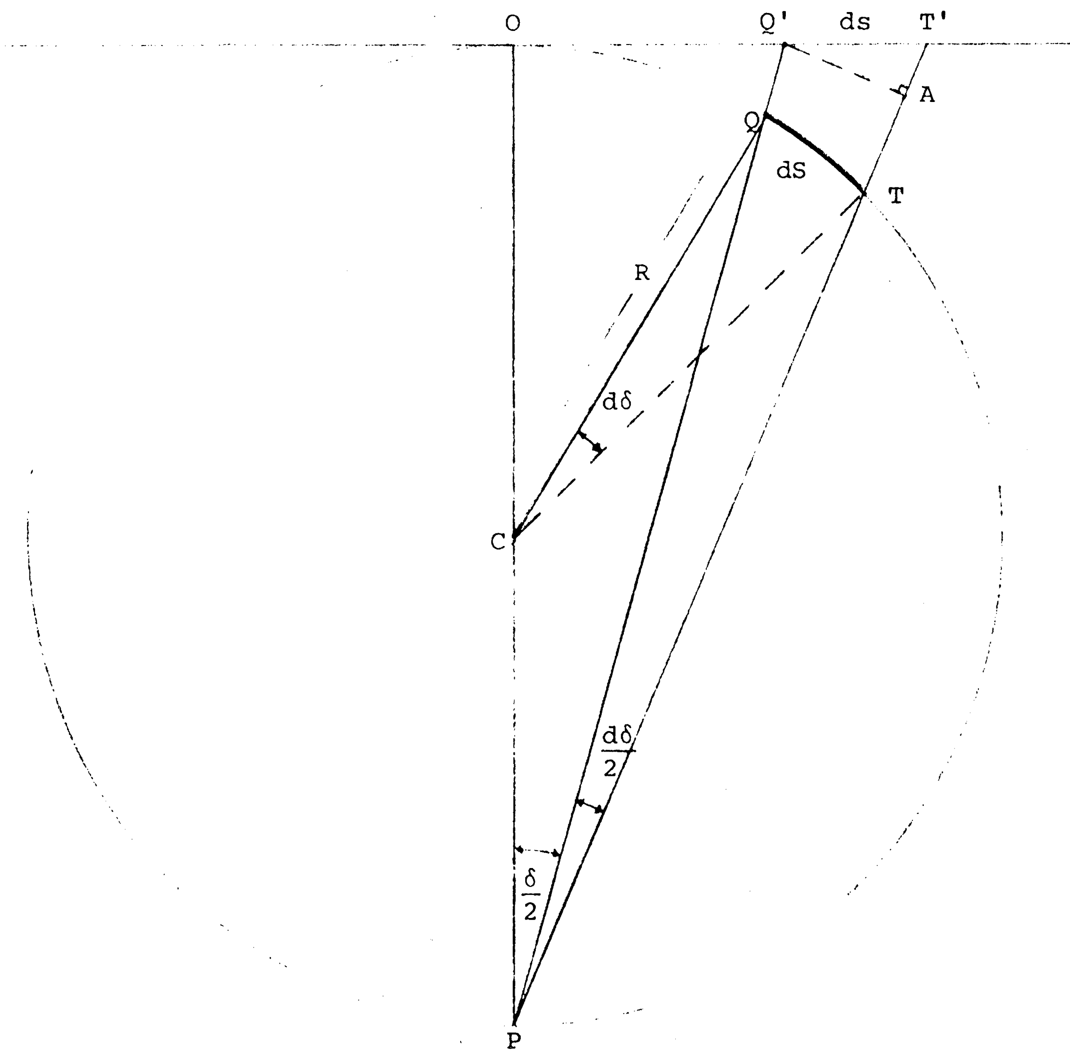


Figure 9

Sphere to Plane Point Scale Factor

$$Q'A = PQ' \cdot \frac{d\delta}{2} = \frac{R\ d\ \delta}{\cos\ \delta/2} \quad . \tag{81}$$

Now, from Q'AT'

$$ds = \frac{Q'A}{\cos\ \delta/2} = \frac{R\ d\ \delta}{\cos^2\ \delta/2} \quad . \tag{82}$$

The scale factor is then expressed as

$$k' = \frac{ds}{dS} = \frac{R\ d\ \delta/\cos^2\ \delta/2}{R\ d\ \delta} \quad ,$$

or

$$k' = \frac{1}{\cos^2\ \delta/2} \quad . \tag{83}$$

From equation (52),

$$\cos\ \delta = \sin\ \chi_o\ \sin\ \chi + \cos\ \chi_o\ \cos\ \chi\ \cos\ \Delta\Lambda \quad ,$$

and we know that

$$\cos^2\ \delta/2 = (\frac{1 + \cos\ \delta}{2}) \quad .$$

Equation (83) in its final form is then

$$\boxed{k' = \frac{2}{1 + \sin\ \chi_o\ \sin\ \chi + \cos\ \chi_o\ \cos\ \chi\ \cos\ \Delta\Lambda}} \quad . \tag{84}$$

If the scale factor at the origin of the mapping (χ_o, Λ_o) is other than unity - say an arbitrary value k'_o - then the general expression for the point scale factor is

$$\boxed{k' = k'_o\ (\frac{2}{1 + \sin\ \chi_o\ \sin\ \chi + \cos\ \chi_o\ \cos\ \chi\ \cos\Delta\Lambda})} \quad . \tag{85}$$

The point scale factor (m) for the double projection - ellipsoid to plane - is obtained simply as

$$\boxed{m = k \cdot k'}$$

or

$$\boxed{m = \frac{2\ c_1\ R}{N} \cdot \frac{\cos\ \chi}{\cos\ \phi(1 + \sin\ \chi_o\ \sin\ \chi + \cos\ \chi_o\ \cos\ \chi\ \cos\ \Delta\Lambda)}} \quad . \tag{86}$$

Again, if one has a scale factor at the origin that is designated to be other than unity, one need only multiply the value obtained from (86) by the designated value m_o.

A simpler approach to the problem is as follows. It is known that the scale factor is given by (for example, Krakiwsky [1973])

$$m^2 = \frac{(\frac{\partial x}{\partial \lambda})^2 + (\frac{\partial y}{\partial \lambda})^2}{N^2 \cos^2 \phi} \quad . \tag{87}$$

Now

$$\begin{aligned} \frac{\partial x}{\partial \lambda} &= \frac{\partial x}{\partial \Lambda} \cdot \frac{\partial \Lambda}{\partial \lambda} \quad , \\ \frac{\partial y}{\partial \lambda} &= \frac{\partial y}{\partial \Lambda} \cdot \frac{\partial \Lambda}{\partial \lambda} \quad . \end{aligned} \tag{88}$$

From (9) we know that

$$\frac{\partial \Lambda}{\partial \lambda} = c_1 \quad ,$$

and from (65) and (67), $\partial x/\partial \Lambda$ and $\partial y/\partial \Lambda$ are given. Substitution in (87) and a re-arrangement of terms yields

$$\boxed{m = \frac{2\, c_1\, R}{N} \;\frac{\cos \chi}{\cos \phi (1 + \sin \chi_o \sin \chi + \cos \chi_o \cos \chi \cos \Delta\Lambda)}} \quad . \tag{89a}$$

The above equation is identical to that derived previously.

It can be shown that the point scale factor for the mapping of ellipsoidal data to the conformal sphere, given by equation (14), can be neglected for any practical purposes. That is, the factor,

$$\frac{c_1\, R \cos \chi}{N \cos \phi} \quad ,$$

in equation (89a) causes changes in m in the order of 1×10^{-7}. Thus, for practical purposes, the point scale factor for the double projection may be computed from equation (85). With this in mind, we derive an expression for the point scale factor in terms of the grid coordinates as follows.

From equations (47), (48), (69), and (70) we get,

$$X - x_o = 2k'_o \; R \tan \delta/2 \cos \beta \quad ,$$

$$Y - y_o = 2k'_o \; R \tan \delta/2 \sin \beta \quad ,$$

which gives

$$\tan^2 \delta/2 = \frac{(X - x_o)^2 + (Y - y_o)^2}{4(k'_o)^2 \; R^2} \quad .$$

Now from (83), (84), and (85) and neglecting the scale factor from the ellipsoid to the conformal sphere,

$$m \simeq k' = k'_o \left(\frac{1}{\cos^2 \delta/2}\right) = k'_o \left(\frac{\cos^2 \delta/2 + \sin^2 \delta/2}{\cos^2 \delta/2}\right) \quad ,$$

or

$$m \simeq k'_o \; (1 + \tan^2 \delta/2) \quad ,$$

or, substituting the previous expression for $\tan^2 \delta/2$,

$$m \simeq k'_o + \frac{(X - x_o)^2 + (Y - y_o)^2}{4 \; k'_o \; R^2} \quad . \tag{89b}$$

This equation gives the point scale factor from ellipsoid to plane accurate to approximately 1×10^{-7} for an area the size of New Brunswick.

The topic of a line scale factor is often raised when dealing with conformal mapping. In many cases, expressions are developed - in terms of mapped coordinates x, y - for use in surveying computations. However, as has been shown by several authors (for example, Jordan/Eggert [1948]), these expressions are nothing more than functions of the point scale factors at the end and mid-points of a line. For example, Jordan/Eggert [1948] develop the expression for a line ij as

$$\frac{1}{m_{ij}} = \frac{S}{s} = \frac{1}{6}\left(\frac{1}{m_i} + \frac{4}{m_m} + \frac{1}{m_j}\right) \quad , \tag{90}$$

in which S is the ellipsoidal distance, s the plane distance, and m_i, m_m, m_j are the point scale factors at i, the mid-point of the line (m), and j respectively.

One final point about scale should be noted - the isoscale lines on the conformal mapping plane are not concentric circles about the origin. This property is adequately explained, for example, by Grossmann [1964], p.221.

4.2 Meridian Convergence

Meridian convergence (γ) for a conformal projection is given by, for example, Krakiwsky [1973]

$$\tan \gamma = \frac{\partial y/\partial \lambda}{\partial x/\partial \lambda} \quad . \tag{91}$$

From equations (88), then (9), (65) and (67) respectively, one gets

$$\tan \gamma = \frac{2\, c_1\, R \cos \chi \dfrac{\sin \Delta\Lambda(\sin \chi + \sin \chi_o)}{(1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2}}{2\, c_1\, R \cos \chi \dfrac{\cos \chi \cos \chi_o + (1 + \sin \chi \sin \chi_o) \cos \Delta\Lambda}{(1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2}} \quad , \tag{92}$$

or when reduced

$$\boxed{\tan \gamma = \frac{\sin \Delta\Lambda\ (\sin \chi + \sin \chi_o)}{\cos \chi \cos \chi_o + (1 + \sin \chi \sin \chi_o) \cos \Delta\Lambda}} \quad . \tag{93}$$

It can easily be shown, using equation (91) that meridian convergence is equal to zero when mapping the ellipsoid on the conformal sphere. While this fact, on its own, is of no great consequence, it is important if one wishes to derive the (T - t) correction for the mapping of the ellipsoid on a conformal sphere.

4.3 (T - t)

The (T - t) correction is important for survey computations on the plane. It is needed in the reduction of a geodetic azimuth and of an ellipsoidal direction (see Figure 10). For the conformal stereographic double projection, the rigorous determination of the (T - t) correction requires two parts:

(i) that caused by the mapping of the ellipsoidal data on a conformal sphere;

(ii) that caused by the mapping of the conformal spherical data on a mapping plane.

Let us deal with (i) first.

Jordan/Eggert [1948] pp. 256-260 present a rigorous derivation of the ellipsoidal to spherical (T - t). The result of this is

$$(T - t) = \frac{2\,k_i \sin \alpha_{ij} + k_j \sin \alpha_{ji}}{3} \cdot \frac{s}{R} \quad , \tag{94}$$

in which k_i, k_j are defined for the end points of a line ij by

$$k = \rho'' \cdot \frac{(e')^2 \cos^2 \phi}{(1 + (e')^2 \cos^2 \phi)^{1/2}} \cdot \tan \phi \cdot \frac{\chi^{\circ 2}}{\rho^{\circ 2}} \quad . \tag{95}$$

In equations (94) and (95) above, α_{ij} and α_{ji} are the direct and inverse geodetic azimuths, s the plane length of the line ij, R the radius of the conformal sphere, e' the second eccentricity of the ellipsoid and ρ'' = 206264.806... and ρ° = 57.29578.... Using this information, one obtains (T - t) in seconds of arc.

The development of the expression for the ellipsoid to conformal sphere (T - t) has not been presented here. The main reason for this is that for any region over which one may apply the stereographic double projection, this portion of the total (T - t) correction is insignificant.

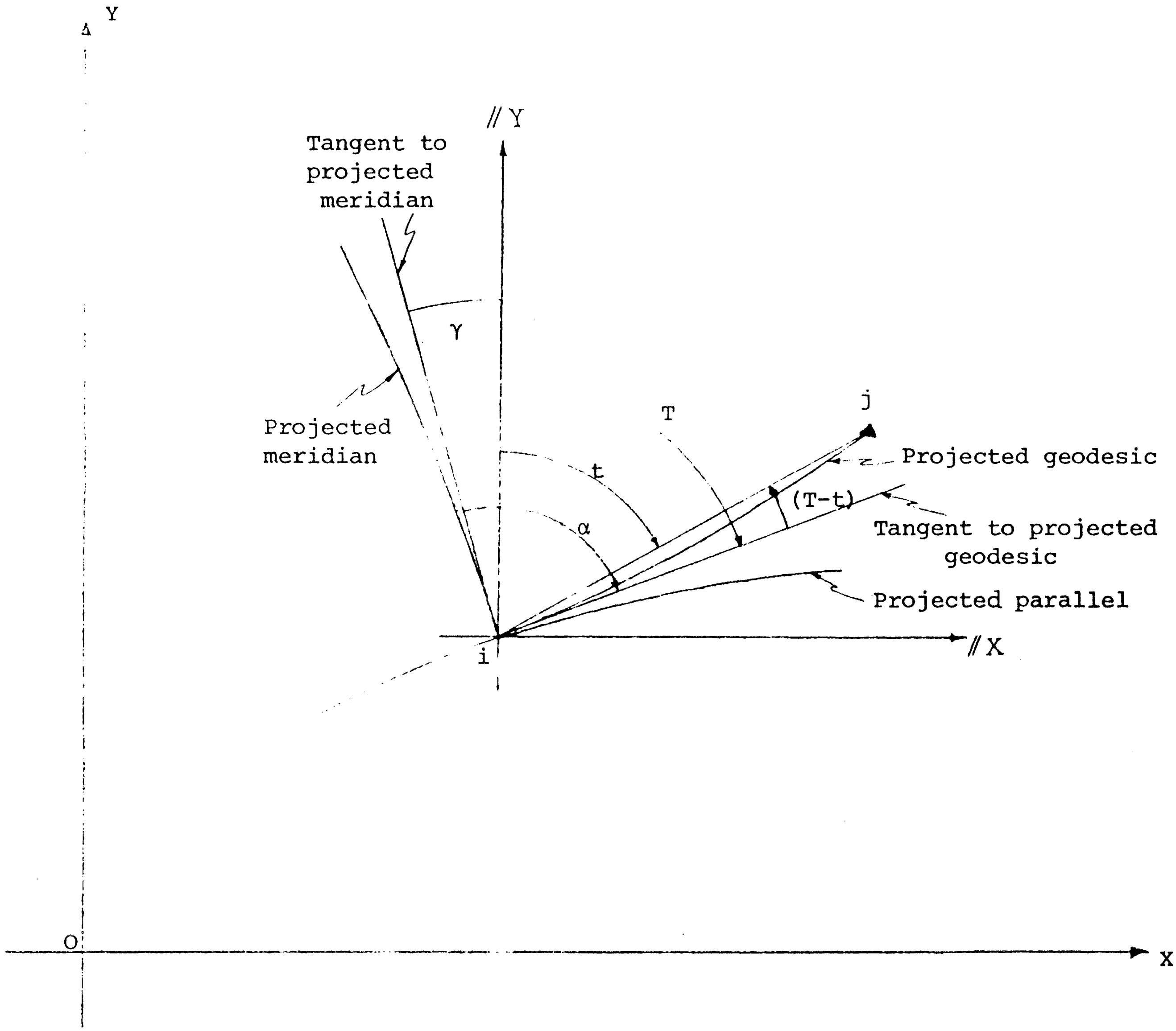


Figure 10

Meridian Convergence and (T-t)

For example, in Hungary, where the extent of the territory is approximately 5° in longitude and 4° in latitude, this portion of the (T - t) has not been used [Jordan/Eggert, 1948]. Furthermore, if one computes the following example

$$\phi_1 = 49°30', \quad \phi_2 = 50°30', \quad \Delta\lambda = 1° \quad ,$$

one finds

$$(T - t)_{12} = + 0''.019, \quad (T - t)_{21} = 0''.015 \quad .$$

Clearly, for the two provinces in Canada that use the stereographic double projection, these corrections are insignificant and can be safely neglected.

The second part of (T - t) - sphere to plane - is of greater concern. An adequate derivation is given as follows.

In Figure 12 we see that the lines O'P' and O'Q' are true images of the lines OP and OQ in Figure 11. This is because the great circles radiating from the origin of the stereographic projection map as straight lines. The great circle from P to Q however, maps as an arc of a circle. As a result of its being a circle, we have that $(T - t)_{PQ} = - (T - t)_{QP}$ and since the projection is conformal the sum of the angles is the same in the spherical triangle and the projected triangle. Therefore we have that

$$(T - t)_{PQ} = \varepsilon/2 \quad , \tag{96}$$

where ε is the spherical excess of the triangle OPQ.

From spherical trigonometry

$$\cot (\varepsilon/2) = \cot(\sphericalangle POQ) + \frac{\cot (\delta_{OP/2}) \cot (\delta_{OQ/2})}{\sin (\sphericalangle POQ)} \quad , \tag{97}$$

therefore

$$\cot (T - t) = \cot (\beta_{OP} - \beta_{OQ}) + \frac{\cot(\delta_{OP/2}) \cot (\delta_{OQ/2})}{\sin (\beta_{OP} - \beta_{OQ})} \quad . \tag{98}$$

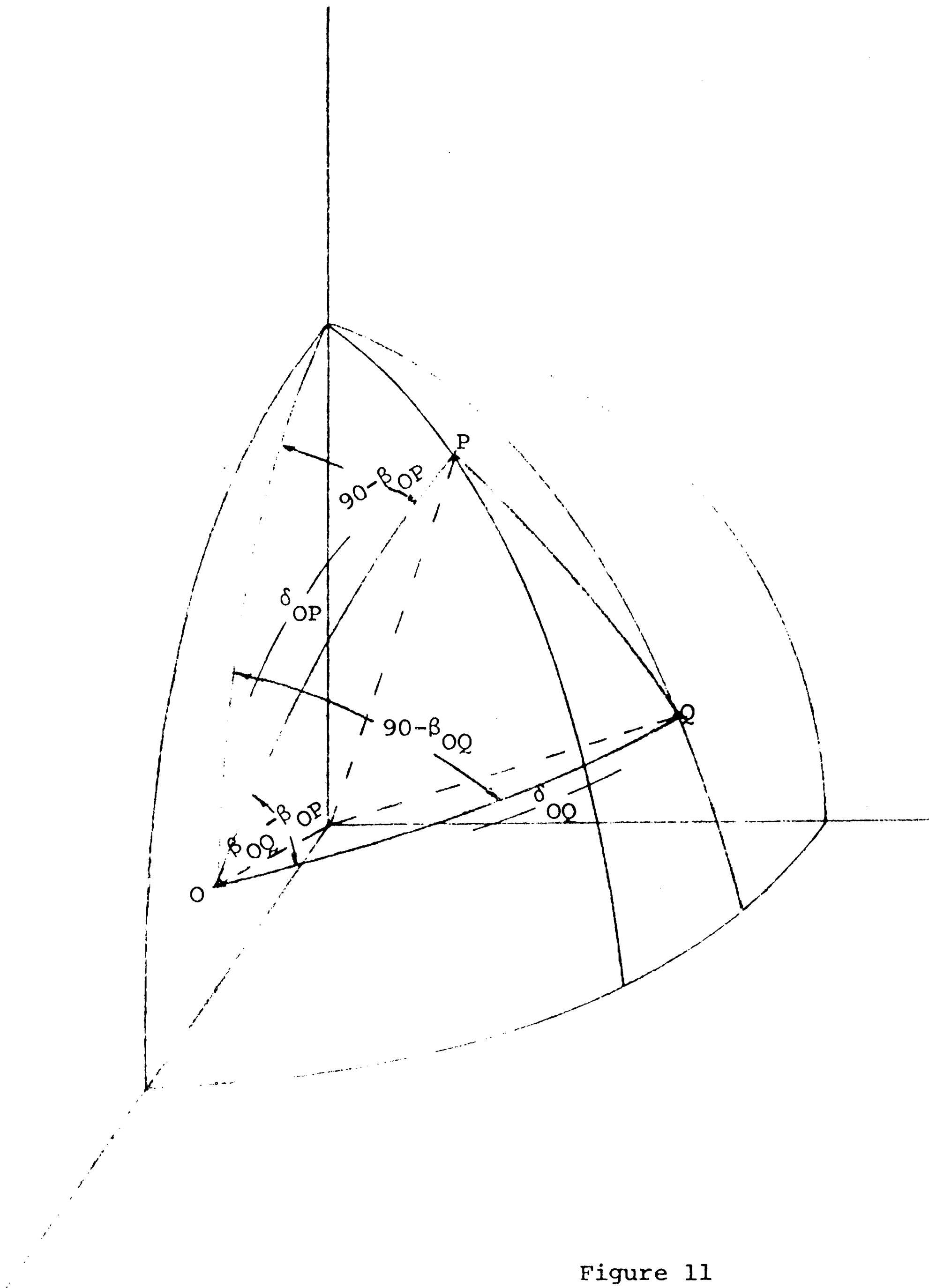


Figure 11

Spherical Triangle OPQ

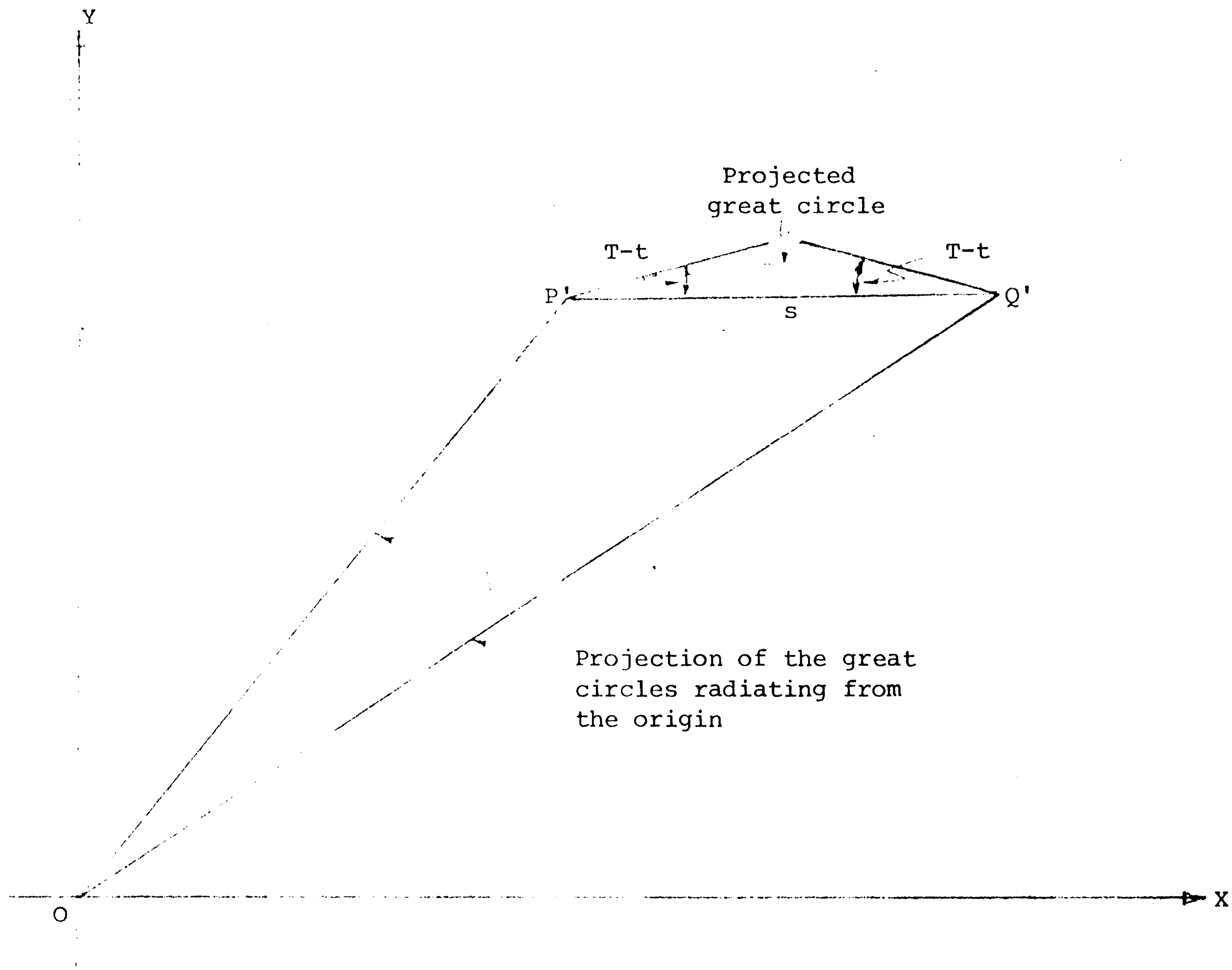


Figure 12

(T-t) Sphere to Plane

From the derivation of the inverse coordinate transformations, we have

$$\beta_{OP} = \sin^{-1} \frac{y_P}{s_P} \quad , \tag{99}$$

$$\beta_{OQ} = \sin^{-1} \frac{y_Q}{s_Q} \quad , \tag{100}$$

$$\frac{\delta_{OP}}{2} = \tan^{-1} \left(\frac{s_P}{2k_o'R}\right) \quad , \tag{101}$$

$$\frac{\delta_{OQ}}{2} = \tan^{-1} \left(\frac{s_Q}{2k_o'R}\right) \quad . \tag{102}$$

Using the trigonometric identities

$$\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A} \quad , \tag{103}$$

and

$$\cot (\sin^{-1} A) = \frac{\sqrt{1 - A^2}}{A} \quad , \tag{104}$$

then

$$\cot (\beta_{OP} - \beta_{OQ}) = \frac{\dfrac{\sqrt{1 - \dfrac{y_P^2}{s_P^2}}}{y_P/s_P} \cdot \dfrac{\sqrt{1 - \dfrac{y_Q^2}{s_Q^2}}}{y_Q/s_Q} + 1}{\dfrac{\sqrt{1 - \dfrac{y_Q^2}{s_Q^2}}}{y_Q/s_Q} - \dfrac{\sqrt{1 - \dfrac{y_P^2}{s_P^2}}}{y_P/s_P}} \quad . \tag{105}$$

or

$$\cot (\beta_{OP} - \beta_{OQ}) = \frac{\dfrac{s_P \sqrt{\dfrac{s_P^2 - y_P^2}{s_P^2}}}{y_P} \cdot \dfrac{s_Q \sqrt{\dfrac{s_Q^2 - y_Q^2}{s_Q^2}}}{y_Q} + 1}{\dfrac{s_Q \sqrt{\dfrac{s_Q^2 - y_Q^2}{s_Q^2}}}{y_Q} - \dfrac{s_P \sqrt{\dfrac{s_P^2 - y_P^2}{s_P^2}}}{y_P}} \quad . \tag{106}$$

Noting that $\sqrt{s^2 - y^2} = x$, (107)

then

$$\cot\,(\beta_{OP} - \beta_{OQ}) = \frac{\frac{x_P}{y_P}\,\frac{x_Q}{y_Q} + 1}{\frac{x_Q}{y_Q} - \frac{x_P}{y_P}} \quad , \tag{108}$$

or

$$\cot\,(\beta_{OP} - \beta_{OQ}) = \frac{x_P\,x_Q + y_P\,y_Q}{x_Q\,y_P - y_Q\,x_P} \quad . \tag{109}$$

If we now use the trigonometric identities

$$\cot\,(\tan^{-1} A) = 1/A \quad , \tag{110}$$

$$\sin\,(A - B) = \sin A \cos B - \cos A \sin B \quad , \tag{111}$$

$$\sin\,(\sin^{-1} A) = A \quad , \tag{112}$$

$$\cos\,(\sin^{-1} A) = \sqrt{1 - A^2} \quad , \tag{113}$$

the second term in equation (98) becomes

$$\frac{\cot\,(\delta_{OP})\,\cot\,(\delta_{OQ})}{\sin\,(\beta_{OP} - \beta_{OQ})} = \frac{\frac{(2k'_o\,R)^2}{s_P\,s_Q}}{\frac{y_P}{s_P}\sqrt{1 - \frac{y_Q^2}{s_Q^2}} - \frac{y_Q}{s_Q}\sqrt{1 - \frac{y_P^2}{s_P^2}}} \quad ,$$

$$= \frac{\frac{(2k'_o\,R)^2}{s_P\,s_Q}}{\frac{y_P\,x_Q}{s_P\,s_Q} - \frac{y_Q\,x_P}{s_Q\,s_P}} \quad ,$$

$$= \frac{(2k'_o R)^2}{y_P\,x_Q - y_Q\,x_P} \quad . \tag{114}$$

Therefore,

$$\cot\,(T-t)_{PQ} = \frac{x_P\,x_Q + y_P\,y_Q + (2k'_o R)^2}{x_Q\,y_P - x_P\,y_Q} \quad , \tag{115}$$

or

$$(T - t)_{PQ} = \cot^{-1}\left(\frac{x_P\ x_Q + y_P\ y_Q + (2k_o'R)^2}{x_Q\ y_P - x_P\ y_Q}\right) \quad . \tag{116}$$

Since this equation is unstable for (T - t) small, which it always is, it is re-written as

$$(T - t)_{PQ} = \tan^{-1}\left(\frac{x_Q\ y_P - x_P\ y_Q}{x_P\ x_Q + y_P\ y_Q + (2k_o'R)^2}\right) \quad . \tag{117}$$

In summary, it should be noted that for rigorous corrective procedure, the (T - t) for the stereographic double projection would be the sum of the results of equations (94) and (117). However, as has been previously stated, the ellipsoid to sphere (T - t) can be safely neglected, thus leaving the user to cope only with the (T - t) for sphere to plane represented analytically by equation (117).

5. ERROR PROPAGATION IN COORDINATE TRANSFORMATIONS

When transforming ϕ, λ coordinates to plane (X, Y) grid coordinates we may also wish to transform the ϕ, λ covariance matrix to obtain the X, Y covariance matrix. The rigorous development of this transformation as well as the inverse transformation (i.e. the transformation of the X, Y covariance matrix to the ϕ, λ covariance matrix) is the subject of this chapter. Readers not familiar with basic matrix algebra and the theory of propagation of variances and covariances are referred to Mikhail [1976].

Let

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma_{\phi}^2 & \sigma_{\phi,\lambda} \\ \sigma_{\phi,\lambda} & \sigma_{\lambda}^2 \end{bmatrix} , \tag{118}$$

and

$$\underline{C}_{X,Y} = \begin{bmatrix} \sigma_{X}^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_{Y}^2 \end{bmatrix}$$

denote the covariance matrices of the ϕ, λ and the X, Y coordinates respectively. Then, from the theory of propagation of variances and covariances [Mikhail, 1976]:

$$\underline{C}_{X,Y} = \begin{bmatrix} \frac{\partial X}{\partial \phi} & \frac{\partial X}{\partial \lambda} \\ \frac{\partial Y}{\partial \phi} & \frac{\partial Y}{\partial \lambda} \end{bmatrix} \cdot \underline{C}_{\phi,\lambda} \cdot \begin{bmatrix} \frac{\partial X}{\partial \phi} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial X}{\partial \lambda} & \frac{\partial Y}{\partial \lambda} \end{bmatrix} \tag{120}$$

or

$$\underline{C}_{X,Y} = \underline{J}\, \underline{C}_{\phi,\lambda}\, \underline{J}^t \tag{121}$$

where the matrix $\underline{J}$ (we denote matrices here by an underlined uppercase letter) is

$$\underline{J} = \begin{bmatrix} \frac{\partial X}{\partial \phi} & \frac{\partial X}{\partial \lambda} \\ \frac{\partial Y}{\partial \phi} & \frac{\partial Y}{\partial \lambda} \end{bmatrix} \quad , \tag{122}$$

and $\underline{J}^t$ denotes the transpose of $\underline{J}$.

To carry out the transformation of equation (121) then, we must determine the elements of $\underline{J}$, i.e. we must determine $\frac{\partial X}{\partial \phi}$, $\frac{\partial X}{\partial \lambda}$, $\frac{\partial Y}{\partial \phi}$ and $\frac{\partial Y}{\partial \lambda}$.

We note that

$$\frac{\partial X}{\partial \phi} = \frac{\partial X}{\partial \chi}\frac{\partial \chi}{\partial \phi} + \frac{\partial X}{\partial \Lambda}\frac{\partial \Lambda}{\partial \phi} \quad , \tag{123}$$

$$\frac{\partial X}{\partial \lambda} = \frac{\partial X}{\partial \chi}\frac{\partial \chi}{\partial \lambda} + \frac{\partial X}{\partial \Lambda}\frac{\partial \Lambda}{\partial \lambda} \quad , \tag{124}$$

$$\frac{\partial Y}{\partial \phi} = \frac{\partial Y}{\partial \chi}\frac{\partial \chi}{\partial \phi} + \frac{\partial Y}{\partial \Lambda}\frac{\partial \Lambda}{\partial \phi} \quad , \tag{125}$$

and

$$\frac{\partial Y}{\partial \lambda} = \frac{\partial Y}{\partial \chi}\frac{\partial \chi}{\partial \lambda} + \frac{\partial Y}{\partial \Lambda}\frac{\partial \Lambda}{\partial \lambda} \quad , \tag{126}$$

for the stereographic double projection. Thus, we must determine expressions for the partial derivatives $\frac{\partial X}{\partial \chi}$, $\frac{\partial X}{\partial \Lambda}$, $\frac{\partial Y}{\partial \chi}$, $\frac{\partial Y}{\partial \Lambda}$, $\frac{\partial \chi}{\partial \phi}$, $\frac{\partial \chi}{\partial \lambda}$, $\frac{\partial \Lambda}{\partial \phi}$ and $\frac{\partial \Lambda}{\partial \lambda}$.

From equations (69) and (70) we have

$$X = x_o + \frac{2\,k_o'\,R\cos\chi\sin\Delta\Lambda}{1 + \sin\chi_o\sin\chi + \cos\chi_o\cos\chi\cos\Delta\Lambda} \tag{127}$$

$$Y = y_o + \frac{2\,k_o'\,R(\sin\chi\cos\chi_o - \cos\chi\sin\chi_o\cos\Delta\Lambda)}{1 + \sin\chi_o\sin\chi + \cos\chi_o\cos\chi\cos\Delta\Lambda} \tag{128}$$

from which

$$\frac{\partial X}{\partial \chi} = 2\,k_o'\,R\sin\Delta\Lambda\left(\frac{-(\sin\chi + \sin\chi_o)}{(1 + \sin\chi\sin\chi_o + \cos\chi\cos\chi_o\cos\Delta\Lambda)^2}\right) \tag{129}$$

$$\frac{\partial X}{\partial \Lambda} = 2\ k_o'\ R \cos\chi\left(\frac{\cos\Delta\Lambda(1 + \sin\chi\sin\chi_o) + \cos\chi\cos\chi_o}{(1 + \sin\chi\sin\chi_o + \cos\chi\cos\chi_o\cos\Delta\Lambda)^2}\right) \tag{130}$$

$$\frac{\partial Y}{\partial \chi} = 2\ k_o'\ R\left(\frac{\cos\Delta\Lambda(1 + \sin\chi\sin\chi_o) + \cos\chi\cos\chi_o}{(1 + \sin\chi\sin\chi_o + \cos\chi\cos\chi_o\cos\Delta\Lambda)^2}\right) \tag{131}$$

$$\frac{\partial Y}{\partial \Lambda} = 2\ k_o'\ R \sin\Delta\Lambda\cos\chi\left(\frac{\sin\chi + \sin\chi_o}{(1 + \sin\chi\sin\chi_o + \cos\chi\cos\chi_o\cos\Delta\Lambda)^2}\right). \tag{132}$$

From equations (12) and (13) we have

$$\chi = 2\{\tan^{-1}[c_2\{\tan(\frac{\Pi}{4} + \frac{\phi}{2})(\frac{1 - e\sin\phi}{1 + e\sin\phi})^{e/2}\}^{c_1}] - \frac{\Pi}{4}\}, \tag{133}$$

$$\Lambda = c_1\ \lambda \tag{134}$$

from which we get

$$\frac{\partial \chi}{\partial \phi} = \frac{2c_1c_2\ \{q\tan r\}^{(c_1-1)} \cdot q\{\frac{1}{2}\sec^2 r - \frac{e^2\cos\phi\tan r}{(1 - e^2\sin^2\phi)}\}}{1 + c_2^2\ \{q\tan r\}^{2c_1}}, \tag{135}$$

in which

$$q = (\frac{1 - e\sin\phi}{1 + e\sin\phi})^{e/2}, \tag{136}$$

and

$$r = (\frac{\Pi}{4} + \frac{\phi}{2}). \tag{137}$$

Also

$$\frac{\partial \chi}{\partial \lambda} = 0 \tag{138}$$

$$\frac{\partial \Lambda}{\partial \phi} = 0 \tag{139}$$

and

$$\frac{\partial \Lambda}{\partial \lambda} = c_1. \tag{140}$$

Now, from equations (123) to (126) we have

$$\frac{\partial X}{\partial \phi} = -\frac{m}{n} \cdot u \quad , \tag{141}$$

$$\frac{\partial X}{\partial \lambda} = c_1 \frac{P}{n} \cos \chi \ , \tag{142}$$

$$\frac{\partial Y}{\partial \phi} = \frac{P}{n} \cdot u \quad , \tag{143}$$

and

$$\frac{\partial Y}{\partial \lambda} = c_1 \frac{m}{n} \cos \chi , \tag{144}$$

where

$$m = 2\ k_o'\ R \sin \Delta\Lambda(\sin \chi + \sin \chi_o) \quad , \tag{145}$$

$$n = (1 + \sin \chi \sin \chi_o + \cos \chi \cos \chi_o \cos \Delta\Lambda)^2 \quad , \tag{146}$$

$$P = 2\ k_o'\ R\ (\cos \Delta\Lambda(1 + \sin \chi \sin \chi_o) + \cos \chi \cos \chi_o) \quad , \tag{147}$$

$$u = \frac{2c_1 c_2\ q\{\tan r\}^{(c_1 - 1)} \cdot q\{ \dfrac{\sec^2 r}{2} - \dfrac{e^2 \cos \phi \tan r}{(1 - e^2 \sin^2 \phi}\}}{1 + c_2^2\{q \tan r\}^{2c_1}} \quad . \tag{148}$$

Substituting equations (141), (142), (143), and (144) into equation (122) gives

$$\underline{J} = \begin{bmatrix} -\dfrac{m}{n} \cdot u & c_1 \cdot \dfrac{P}{n} \cdot \cos \chi \\ \\ \dfrac{P}{n} \cdot u & c_1 \cdot \dfrac{m}{n} \cdot \cos \chi \end{bmatrix} \quad , \tag{149}$$

and finally we obtain $\underline{C}_{X,Y}$ from equation (121). For the inverse transformation (i.e. transform $\underline{C}_{X,Y}$ to get $\underline{C}_{\phi,\lambda}$) we have

$$\underline{C}_{X,Y} = \underline{J}\ \underline{C}_{\phi,\lambda}\ \underline{J}^t \quad , \qquad \text{(repeated) (121)}$$

$$\underline{J}^{-1}\ \underline{C}_{X,Y} = \underline{J}^{-1}\ \underline{J}\ \underline{C}_{\phi,\lambda}\ \underline{J}^t \quad , \tag{150}$$

$$\underline{J}^{-1}\,\underline{C}_{X,Y} = \underline{C}_{\phi,\lambda}\,\underline{J}^{t} \quad , \tag{151}$$

$$\underline{J}^{-1}\,\underline{C}_{X,Y}\,(J^{t})^{-1} = \underline{C}_{\phi,\lambda}\,\underline{J}^{t}\,(\underline{J}^{t})^{-1} \quad , \tag{152}$$

thus

$$\underline{C}_{\phi,\lambda} = \underline{J}^{-1}\,\underline{C}_{X,Y}\,(\underline{J}^{t})^{-1} \quad . \tag{153}$$

It should be noted here that certain approximations may be made in equations (145) to (148) to obtain sufficient precision in equations (121) and (153). This is left for the reader. Also, if one wishes to propagate variances for more than one point, a hyper-matrix equation may be formulated from equation (121) or (153) as follows.

Let $\underline{C}_{\phi,\lambda}$ be the covariance matrix of <u>two</u> points, i.e.

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \sigma^2_{\phi_1} & \sigma_{\phi_1\lambda_1} & \sigma_{\phi_1\phi_2} & \sigma_{\phi_1\lambda_2} \\ \sigma_{\phi_1\lambda_1} & \sigma^2_{\lambda_1} & \sigma_{\lambda_1\phi_2} & \sigma_{\lambda_1\lambda_2} \\ \sigma_{\phi_1\phi_2} & \sigma_{\lambda_1\phi_2} & \sigma^2_{\phi_2} & \sigma_{\phi_2\lambda_2} \\ \sigma_{\phi_1\lambda_2} & \sigma_{\lambda_1\lambda_2} & \sigma_{\phi_2\lambda_2} & \sigma^2_{\lambda_2} \end{bmatrix} \quad . \tag{154}$$

Then we may write:

$$\underline{C}_{X,Y} = \begin{bmatrix} \sigma^2_{X_1} & \sigma_{X_1Y_1} & \sigma_{X_1X_2} & \sigma_{X_1Y_2} \\ \sigma_{X_1Y_1} & \sigma^2_{Y_1} & \sigma_{Y_1X_2} & \sigma_{Y_1Y_2} \\ \sigma_{X_1X_2} & \sigma_{Y_1X_2} & \sigma^2_{X_2} & \sigma_{X_2Y_2} \\ \sigma_{X_1Y_2} & \sigma_{Y_1Y_2} & \sigma_{X_2Y_2} & \sigma^2_{Y_2} \end{bmatrix} \quad , \tag{155}$$

$$= \underline{J}\,\underline{C}_{\phi,\lambda}\,\underline{J}^{t} \quad , \tag{156}$$

where now

$$\underline{J} = \begin{bmatrix} \frac{\partial X_1}{\partial \phi_1} & \frac{\partial X_1}{\partial \lambda_1} & \frac{\partial X_1}{\partial \phi_2} & \frac{\partial X_1}{\partial \lambda_2} \\ \frac{\partial Y_1}{\partial \phi_1} & \frac{\partial Y_1}{\partial \lambda_1} & \frac{\partial Y_1}{\partial \phi_2} & \frac{\partial Y_1}{\partial \lambda_2} \\ \frac{\partial X_2}{\partial \phi_1} & \frac{\partial X_2}{\partial \lambda_1} & \frac{\partial X_2}{\partial \phi_2} & \frac{\partial X_2}{\partial \lambda_2} \\ \frac{\partial Y_2}{\partial \phi_1} & \frac{\partial Y_2}{\partial \lambda_1} & \frac{\partial Y_2}{\partial \phi_2} & \frac{\partial Y_2}{\partial \lambda_2} \end{bmatrix} . \tag{157}$$

But of course

$$\frac{\partial X_1}{\partial \phi_2} = \frac{\partial X_1}{\partial \lambda_2} = \frac{\partial Y_1}{\partial \phi_2} = \frac{\partial Y_1}{\partial \lambda_2} = \frac{\partial X_2}{\partial \phi_1} = \frac{\partial X_2}{\partial \lambda_1} = \frac{\partial y_2}{\partial \phi_1} = \frac{\partial Y_2}{\partial \lambda_1} = 0 \quad . \tag{158}$$

Thus

$$\underline{J} = \begin{bmatrix} \frac{\partial X_1}{\partial \phi_1} & \frac{\partial X_1}{\partial \lambda_1} & 0 & 0 \\ \frac{\partial Y_1}{\partial \phi_1} & \frac{\partial Y_1}{\partial \lambda_1} & 0 & 0 \\ 0 & 0 & \frac{\partial X_2}{\partial \phi_2} & \frac{\partial X_2}{\partial \lambda_2} \\ 0 & 0 & \frac{\partial Y_2}{\partial \phi_2} & \frac{\partial Y_2}{\partial \lambda_2} \end{bmatrix} = \begin{bmatrix} \underline{J}_1 & \underline{O} \\ \underline{O} & \underline{J}_2 \end{bmatrix} , \tag{159}$$

and

$$\underline{C}_{X,Y} = \begin{bmatrix} \underline{J}_1 & \underline{O} \\ \underline{O} & \underline{J}_2 \end{bmatrix} \underline{C}_{\phi,\lambda} \begin{bmatrix} \underline{J}_1^t & \underline{O} \\ \underline{O} & \underline{J}_2^t \end{bmatrix} , \tag{160}$$

where $\underline{C}_{\phi,\lambda}$ and $\underline{C}_{X,Y}$ are of the form given in equations (154) and (155) respectively. The inverse transformation of equation (160) is given by:

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \underline{J}_1 & \underline{O} \\ \underline{O} & \underline{J}_2 \end{bmatrix}^{-1} \underline{C}_{X,Y} \begin{bmatrix} \underline{J}_1^t & \underline{O} \\ \underline{O} & \underline{J}_2^t \end{bmatrix}^{-1} . \qquad (161)$$

It should be noted here that any <u>covariance between separate stations</u> is being propagated in equations (160) and (161) where if we use equations (121) and (153) for each station separately, we are ignoring this covariance information. Equations (160) and (161) may be expanded for any number of stations in a similar way. For example, for three stations;

$$\underline{C}_{\phi,\lambda} = \begin{bmatrix} \underline{J}_1 & \underline{O} & \underline{O} \\ \underline{O} & \underline{J}_2 & \underline{O} \\ \underline{O} & \underline{O} & \underline{J}_3 \end{bmatrix}^{-1} \underline{C}_{X,Y} \begin{bmatrix} \underline{J}_1^t & \underline{O} & \underline{O} \\ \underline{O} & \underline{J}_2^t & \underline{O} \\ \underline{O} & \underline{O} & \underline{J}_3^t \end{bmatrix}^{-1} , \qquad (162)$$

where $\underline{C}_{\phi,\lambda}$ and $\underline{C}_{X,Y}$ are now covariance matrices of dimension six by six.

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