GRAVITY FIELD AND LEVELLED HEIGHTS IN CANADA

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March 1977

TECHNICAL REPORT NO. 41

PREFACE

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GRAVITY FIELD AND LEVELLED HEIGHTS

IN CANADA

by

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PREFACE

This report is an unaltered printing of the author's doctoral thesis, of the same title, submitted to the U.N.B. Graduate School in Febraury, 1977.

The principal thesis advisor and research supervisor for this work was Dr. Petr Vaníček. During the last few months, while Dr. Vaníček was abroad on Sabbatical Leave, Dr. Edward J. Krakiwsky ably filled in.

Details of the financial support and the assistance rendered by others are given in the ACKNOWLEDGEMENTS.

ABSTRACT

Modern trends in geodesy demand an increased accuracy of relative heights and height changes. Precise spirit levelling is known to be the most accurate method available to meet such requirements. It is also known that unique height determination can be made only by taking into account the convergence and irregularities of the equipotential surfaces of the earth's actual gravity field. In the context of levelling, this is accomplished by supplementing the spirit levelling with actual gravity values observed along levelling routes.

In Canada, and the U.S.A., because of the lack of actual gravity values (during the period of building up and extending the levelling networks) the normal gravity was used instead to define the heights. The normal gravity values were computed along levelling routes from a simplified mathematical model of the earth.

Two systems of heights - orthometric and dynamic - are used in Canada, both taking into account only the broadest features of the gravity field expressed via the computed normal gravity. This implies the neglect of the effect of local irregularities of the actual gravity field on the defined heights, which results in systematic distortions of the computed heights.

The study contained herein focuses on the investigation of the influence of actual gravity variations (anomalies) on heights currently used in Canada. These influences are referred to here as "GRAVITY CORRECTIONS", GC's. The GC's are to be added to the existing height differences (based on normal gravity) to obtain the corresponding rigorous height differences based on actual gravity.

The GC's for three systems of heights - Dynamic, Helmert and Vignal - are modelled in terms of practically obtainable quantities: free-air gravity anomalies, observed heights and latitudes of levelling bench marks along the levelling routes. Although the developed formulae for the GC's can be readily used for the evaluation of these corrections, tables are provided to facilitate field estimation of the GC's.

Results based on real data indicate that the GC's can be evaluated with adequate reliability, whether we use observed or predicted gravity anomalies, for all three systems of heights under investigation. This reliability is characterized by the small standard deviations associated with the GC's compared to the magnitude of the corrections themselves.

The behaviour of GC's along real levelling lines and loops are investigated and compared to the corresponding standard error, $\sigma_{\Delta h}$, of precise levelling as specified in the Canadian specifications for vertical control [Surveys and Mapping Branch, 1961; Boal, 1971b; Surveys and Mapping Branch, 1973]. The results show that the influence of the GC's on the derived heights of most of the bench marks along the tested lines and loops is significant.

A computational approach, based on the least-squares surface fitting techniques, is proposed for the prediction of GC's. The aim of this approach is to treat the problem of GC's in two-dimensions, so as to enable one to determine the geographical areas in Canada where actual gravity influence on heights is significant and should be taken into account. Gravity corrections within each 1° x 1° block, that vary with direction, have been predicted for the entire country, using real gravity data supplied by the Earth Physics Branch, Ottawa. Obtained results - compared to a prespecified significance criterion of 0.14 mm/km (10% of the standard deviation of a height difference in the Canadian Precise Level Net, CPLN) - reveal the significance of the GC's, in practically all the Canadian areas, at least in the direction of its maximum value. In many cases, the GC even exceeds the standard error of precise levelling, especially in Helmert system.

Based on the analysis and results of this study, it appears necessary to begin basing the heights in Canada on actual gravity in order to maintain the standard of accuracy required for the CPLN. Such procedure was recommended by the International Association of Geodesy as early as 1950 [IAG, 1950]. This has become feasible since the coverage of the Canadian territory with gravity observations has become sufficiently dense. The information and findings contained in this thesis should thus contribute to the forthcoming new adjustment and analysis of the CPLN, as planned by the Geodetic Survey of Canada for early 1980's.

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My association with my fellow graduate students, at U.N.B., created an ideal working environment, while this and related studies were accomplished. Those to whom I owe special mentioning are Drs. Pierre Gagnon, Charles L. Merry, Donald B. Thomson and David E. Wells. Working and studying with them was always a pleasure and continuous education.

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The excellent job done by Mrs. Susan Biggar and Mrs. Debbie Smith in typing this thesis, and by Mrs. Debbie Jordan and Mrs. Virginia Patterson in drafting the illustrations, is sincerely appreciated.

Last but not the least, I dedicate this thesis to my wife, Samira, in appreciation for her understanding, cooperation, encouragement and vital support, which significantly contributed to the success of this study. For her patience, sacrifice and fruitful care of our son, Sameh, I shall be forever indebted and grateful.

CHAPTER 1

INTRODUCTION

1.1 Concept of Heights and Height Datums

The objective of this introductory section is to introduce the concept of heights and their close connection with a complete geodetic positioning system. It also includes definitions of some of the related terms associated with height datums, in view of the classical and modern geodetic theories. Such terms are used frequently within the text of this thesis.

It is well known that in order to completely define the location of any point on the earth's surface, it is necessary to determine its three coordinates referred to a three-dimensional (3-D) coordinate system. In case of the geodetic coordinate system, the two-dimensional (2-D) geographical coordinates (latitude ϕ and longitude λ) of the point in question - referred to a chosen reference ellipsoid - are determined. In addition, its elevation, h, referred to an arbitrary or natural surface known as the "height datum" [e.g.: U.S. Dept. of Commerce, 1961], is needed. Consequently, such elevation h of the terrain point is known as its "height", which constitutes the third dimension of a complete 3-D geodetic position [Hotine, 1969]. Thus, the height of a point A (see Figure 1-1) is usually defined as the distance between an equipotential surface through the point in question and the corresponding equipotential surface representing the height datum, measured along the line of force or along its tangent [Mueller and Rockie, 1966].

In practice (e.g. in North America and other parts of the world) the 3-D geodetic position has been conventionally split into two parts and treated separately. The first, dealing with ϕ and λ only, is termed horizontal control. The second part, dealing with h only, is referred to as vertical control. The associated reference surfaces in both cases are usually known as geodetic datums. Discussions concerning various geodetic datums can be found, e.g. in Jones [1973] and Thomson [1976]. Definitions connected with height datums only, for vertical control, are dealt with herein.

The height datum used in North America for vertical control [Christodoulidis et. al., 1973] has been chosen to be the geoid. The height h above the geoid is known as the orthometric height (Figure 1-1).

The geoid is defined as that particular equipotential surface of the earth's actual gravity field which most nearly coincides with the undisturbed mean sea level (MSL) [Mueller and Rockie, 1966; Rapp, 1973; Lelgemann, 1976]. This is so because it is found useful, in practice, that the zero height contour should lie close to MSL [Bomford, 1971]. More sophisticated definitions of the geoid, which are considered outside the scope of this study, are given in many recent publications [e.g.: Rapp, 1973; Lelgemann, 1976]. It is reported by Rapp [1975] that the above definition of the geoid is now undergoing examination and refinement because of the anticipated direct measurements to the geoid from satellites by satellite altimetry.

The reference ellipsoid, which is usually the datum for horizontal control networks, closely approximates the geoid [Heiskanen and Moritz, 1967]. The separation N* (Figure 1-1) between these two surfaces is known as the geoidal height (undulation), which is taken as

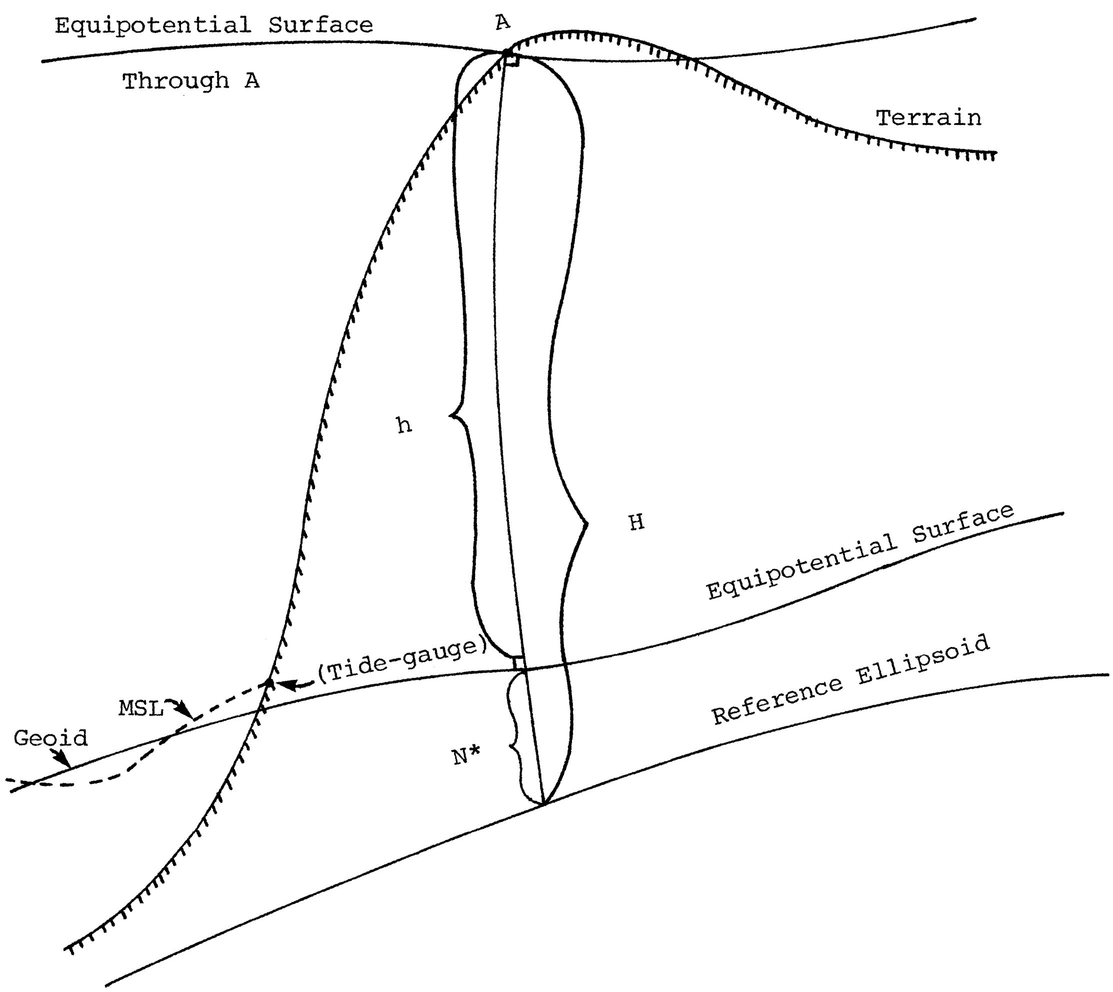


FIGURE 1-1

Classical Height Datum

positive when the geoid is above the ellipsoid [e.g. Merry, 1975]. Many studies have been done to design the best technique for a more accurate determination of the quantity N*, however the achieved accuracy of N* is still in the order of few metres [e.g.: Rapp, 1973; Merry, 1975; John, 1976]. The sum of the orthometric height h and the geoidal height N* is known as the ellipsoid height H (height above the ellipsoid) [e.g. Krakiwsky and Wells, 1971]. This investigation considers only the orthometric part h, above the geoid.

Details about the determination of MSL, and its role as a vertical control datum, from tide-gauge records can be found in literature [e.g. Simonsen, 1966; Bomford, 1971; Lennon, 1974]. It is now known that MSL is not completely coincident with the geoid, since it varies due to spatial variations in temperature, pressure, salinity and other parameters [Bomford, 1971]. This causes MSL to depart from an equipotential surface [Vaníček, 1972] by an amount estimated to be less than two metres [Lisitzin and Pattulo, 1961; Lelgemann, 1976]. Therefore, the adoption of MSL as an approximation to the geoid to serve as a height datum, or the choice of the geoid as the datum for studying MSL variations, is one of the fundamental geodetic problems which is not settled yet [Rapp, 1973; Lelgemann, 1976].

In modern geodetic theories, the height datum is either the quasigeoid, as used by Molodenskii [Molodenskii et. al., 1962], or the mean earth ellipsoid, as used by Hirvonen [Hirvonen, 1960]. The resulting heights in both cases are known as normal heights h^N (Figure 1-2 a,b).

Molodenskii's normal heights are referred to the quasigeoid, (Figure 1-2a). The quasigeoid is a purely mathematical surface without any physical meaning that departs from the geoid by at most a few metres

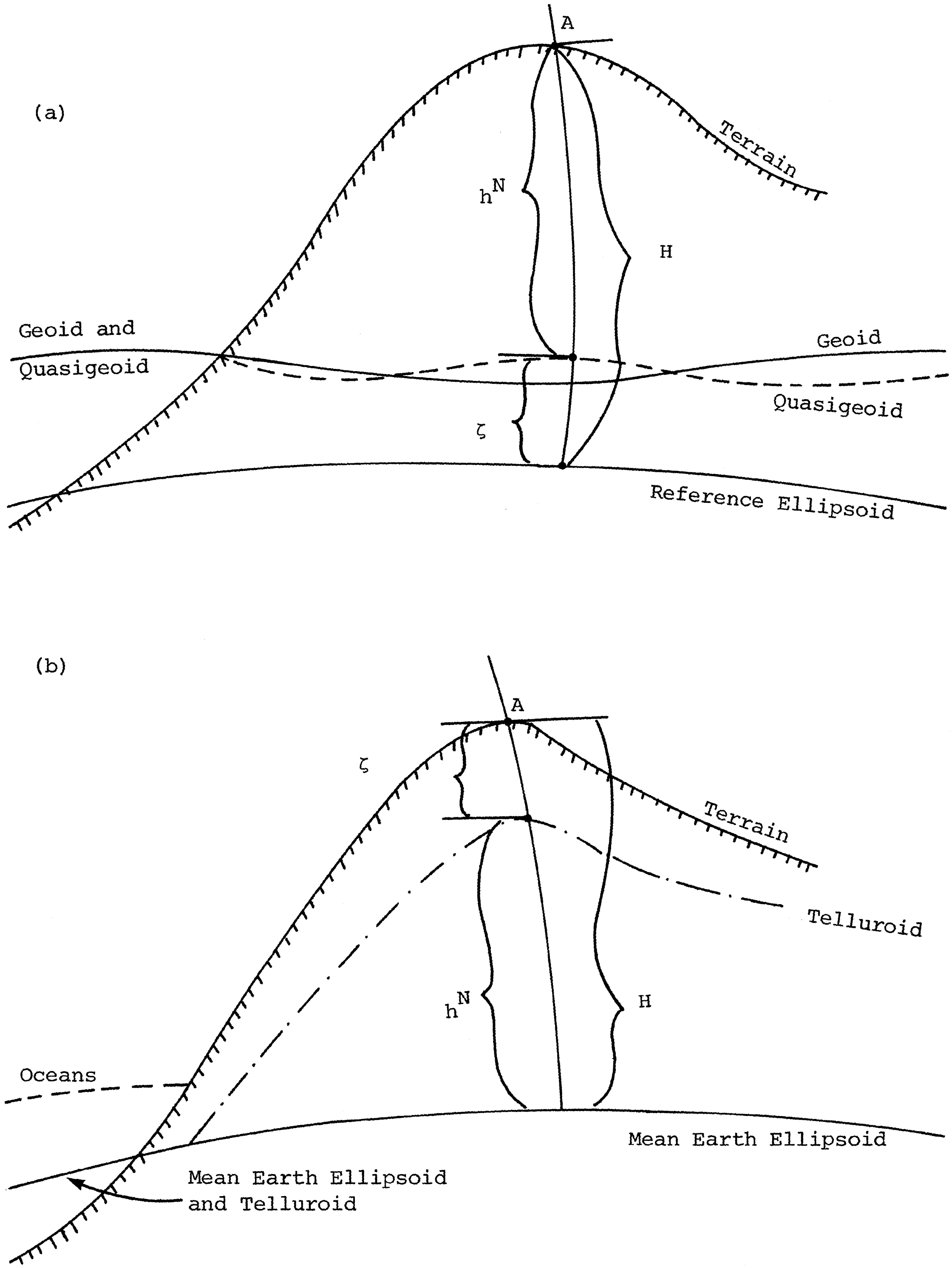


FIGURE 1-2

Modern Height Datums

under the continents and coincides with it on the seas [Müller, 1960]. It is the geometric locus of all points whose distances from the physical surface of the earth are equal to their Molodenskii normal heights. The quasigeoid is not generally an equipotential surface.

The Hirvonen normal heights are measured from the mean earth ellipsoid towards the terrain, generating another surface called telluroid by Hirvonen. Characteristics of the mean earth ellipsoid can be found e.g., in Heiskanen and Moritz [1967], and Vaníček [1971]. The telluroid (Figure 1-2b) is a continuous surface without physical meaning coinciding with the ellipsoid on the oceans; again, the telluroid is generally not an equipotential surface. On the islands and continents it displays the details of the topographic irregularities [Hirvonen, 1960]. The telluroid can be also interpreted as the geometric locus of all points whose distances above the mean earth ellipsoid are equal to their respecitve Hirvonen normal heights, and whose normal potential equals the actual potential on the terrain.

The difference between the ellipsoid height H and the normal height h^N is known as the height anomaly ζ. In case of Molodenskii heights, ζ is the separation between the quasigeoid and the mean earth ellipsoid (Figure 1-2a). In case of Hirvonen heights, ζ is the separation between the telluroid and the physical surface of the earth. The quantity ζ plays a similar role as the quantity N*, in classical geodesy, in computing the ellipsoid height H. Techniques for the determination of ζ in the above two cases are given in Molodenskii et al. [1962] and Hirvonen [1960], respectively. In this study, the normal height h^N only is of interest.

1.2 Needs For Accurate Heights - Spirit Levelling

Accurate heights are needed for a multitude of applications. In this section, only the basic practical and scientific geodetic applications are considered. Some of these applications do not require very accurate heights. Others demand high precision relative heights. Both categories are summarized. Then, the method of spirit levelling for relative height determination is outlined, because it is the only tool to meet the current requirements of high accuracy.

From the basic definitions, given in the previous section, it can be seen that the orthometric (or normal) height is a vital quantity needed to compute the ellipsoid height H. The latter quantity, H, is essential for many geodetic applications, amongst which are the following:

1. Reduction of observed distances and directions from the surface of the earth to the computational reference ellipsoid, for rigorous computations of horizontal geodetic networks [e.g.: Thomson et. al., 1974; Mutajwaa, 1976];
2. 3-D geodetic computations [e.g. Heiskanen and Moritz, 1967; Hotine, 1969];
3. Relating the 3-D satellite coordinates with corresponding 2-D terrestrial coordinates [e.g. Krakiwsky and Wells, 1971; Thomson, 1976].

The above particular practical applications do not require very accurate heights h, since the accuracy of H is directly affected by the obtained accuracy of N* (or ζ) which is of the order of few metres.

In view of modern trends of geodesy, one can notice the increasing demand for precise estimates of relative heights and height changes [Surveys and Mapping Branch, 1974; Holdahl, 1974; Vaníček, 1974; Vaníček, 1976a]. Such quantities are useful for investigating and providing the principal evidence on many scientific questions [Bomford, 1971; Clark and Jackson, 1973] concerning the following problems:

1. Vertical crustal movements and related studies [e.g. Kukkamäki, 1955; Korhonen, 1961; Holdahl and Morrison, 1973; Christodoulidis, 1973; Vaníček, 1975; Vaníček, 1976b];
2. MSL time and space variations and its validity as a height datum [e.g.: Braaten and McCombs, 1963; Lennon, 1974; Sharaf Eldin, 1975; Vaníček, 1976c];
3. Earthquake prediction [e.g. Ellingwood, 1969], and search for safe locations for nuclear power sites [Holdahl, 1976];
4. Hydraulic and other related engineering investigations [e.g. Coordinating Committee on the IGLD-55, 1961];
5. Deformation of engineering structures [e.g. Penman and Charles, 1971];
6. Inclusion of precise height information into a more precise 3-D adjustment of geodetic networks [e.g. Stolz and Gilliland, 1969].

To achieve the above requirements, the precise spirit levelling is known to be the most accurate method to use for relative height determination [Wassef, 1959; Krakiwsky, 1965; Heiskanen and Moritz, 1967; Bomford, 1971]. To utilise the full potential of spirit levelling, all the systematic corrections - usually neglected in the past because of the involved computational effort and sophistication in their modelling - to the measured height differences from spirit levelling should now be applied [e.g. Holdahl, 1974]. Also, the appropriate way of assessment of weights for the observed height differences on the basis of the actual obtained discrepancies of repeated measurements should be considered [Wassef, 1959; Vaníček et. al., 1972].

Precise levelled height differences, like any other measureable physical quantities, are influenced by both systematic and random errors. The main sources of systematic errors affecting precise spirit levelling are outlined in the next section. On the other hand, the random levelling errors are not discussed here any further. It suffices here, for the purpose of this study, to mention that the accumulated standard deviations of height differences derived from spirit levelling which are deduced from the actually obtained discrepancies between forward and backward levellings of each levelling section [e.g. Wassef, 1959; Peterson, 1970; Boal, 1971, Vaníček et. al., 1972] can be considered as a measure of the influence of random errors in precise levelling. This accumulated standard deviation is conventionally the basic quantity against which the influence of any systematic error affecting levelling results is compared. Hence, this custom will be also followed here.

1.3 Systematic Errors in Spirit Levelling

The systematic errors in precise spirit levelling are discussed in details in literature dealing with the subject matter [e.g.: Braaten et. al., 1950; Kukkamäki, 1950; Entin, 1959; Kowakzyk, 1968; Bomford, 1971; Clark and Jackson, 1973; Holdahl, 1974]. Nevertheless, this section is intended to serve as a guide to the reader. It outlines the main systematic errors, along with brief explanation whenever necessary. It also gives the appropriate references to investigations concerning each respective source of error. Finally, it singles out the particular error which is of interest in this study.

The precise spirit levelling field operations involve the levelling of levelling instrument (level) and then reading a vertical scale on a levelling rod (staff) back and forth along a levelling route, under certain atmospheric conditions. The systematic errors inherent in such operation can be divided into two categories [Bomford, 1971]:

(i) Errors in the first category accumulate with the distance and height differences along the levelling route. They can be mostly eliminated either by following certain techniques in the process of reading fore and back sights along the line, or by calibrating the used instruments, modelling the errors and accounting for them. This category includes errors due to: Collimation; earth curvature; symmetrical refraction (these errors affect the equidistant fore and back sights equally); systematic sinking or rising of the levelling instrument between observing fore and back staves; systematic sinking or rising of the staff between its use as a fore staff and as a back staff; errors of staff length or systematic errors in its subdivisions and non-verticality of the staff.

(ii) Errors in the second category accumulate with the height difference, average heights and relative positions between bench marks along the levelling line. The errors in this category cannot be eliminated by special observing techniques. Therefore, such errors require reliable estimates of their influences, which must be taken into account in the rigorous computations of heights of vertical control points. Since these errors are of some interest to us in this study, they will be enumerated here:

1. Tidal effect, which is caused by the different attraction of the moon and sun at the centre of the earth and at the observing station on the earth's surface [Bomford, 1971]. This phenomenon results in a slight variation of the direction of gravity with time at the observing station from its mean. This problem is also complicated by yielding of the earth itself, and by the attraction and loading of ocean tides

[Lennon, 1974]. The theory, computation and implications of the tidal correction as applied to precise levelling results have been discussed in several publications [e.g. Kukkamäki, 1949; Braaten et al., 1950; Jensen, 1950; Rune, 1950b; Simonsen, 1950; Egedal and Simonsen, 1955; Simonsen, 1966; Holdahl, 1974];

2. Errors associated with the unequal refraction for the fore and back sights [Kukkamäki, 1950; Simonsen, 1955; Strusinsky, 1959; Hytönen, 1967; Straub, 1973; Holdahl, 1974]. This will occur especially when the levelling route runs on a hilly or undulating terrain. There, the height of the line of sight above the ground will differ considerably even for equidistant fore and back sights. This situation results in a residual refraction between the fore and back sights, which is left over after eliminating the symmetrical part of the refraction, and influences the levelled height difference;

3. The abrupt movements of the earth's crust due to earthquakes [Braaten et. al., 1950], and the well known secular vertical crustal activities - uplifts or subsidences. These affect the bench marks, and result in differences between the bench marks' instantaneous heights and the previously determined heights [Kukkamäki, 1955; Korhonen, 1961; Holdahl and Morrison, 1973; Holdahl, 1975; Vaníček, 1975; Vaníček, 1976b];

4. The use of MSL, as determined from tide-gauge observations at various coastal locations, as a datum for heights (with fixed values equal to zero). This situation causes errors in the adjustment and analysis of precise levelling networks [Lisitzin and Pattulo, 1961; Braaten and McCombs, 1963; Simsonen, 1966; Ellingwood, 1969; Dohler, 1970; Christodoulidis et. al, 1973; Lennon, 1974; Lelgemann, 1976; Vaníček, 1976c], due to the sea surface slope which gives different values of

MSL at different locations. Furthermore, there are problems connected with maintaining and operating the tide-gauge itself [Lennon, 1974].

5. Suspected errors due to the differential illumination or solar radiation or both on the levelling rod. In other words, due to variations in the heating rate of the invar strips on the staff. Such systematic error, if it exists, would accumulate in the North-South (N-S) direction and would tend to be zero in the East-West (E-W) direction [e.g. Egedal, 1950; Egedal and Simonsen, 1955; Edge, 1959; Bomford, 1971; Balazs, 1975; Kakkuri, 1975]. This implies that levelling lines running N-S or S-N are influenced systematically by this type of error. For this reason, and also on the basis of the N-S discrepancies between the results of geodetic and oceanographic levellings, such error was hypothesized to exist as one of the as yet unknown systematic errors in precise levelling needed to be investigated and accounted for. Some precise levelling data show a slope of MSL from North to South, which is opposite to the oceanographic findings [Balazs, 1975; Fisher, 1975]. Several studies are now underway by both oceanographers and geodesists to solve this problem of N-S slope of MSL [Fisher, 1975].

6. Errors associated with the neglect of local irregularities of the earth's gravity field (gravity anomalies) when correcting the observed height differences for the global convergence of the equipotential surfaces. Such errors occur when, in the correction formulae, the normal value of gravity acceleration is used instead of the actual value (e.g.: Rune, 1950; Vignal and Kukkamäki, 1954; Bursa, 1958; Braaten and McCombs, 1963; Krakiwsky, 1966; Christodoulidis and Vaníček, 1972; Holdahl, 1974; Nassar and Vaníček, 1975].

The above six systematic errors, inherent in precise spirit levelling can, in most cases, exceed the accumulated standard errors achieved from precise levelling [e.g.: Braaten et. al., 1950; Holdahl, 1974], and may cause significant regional distortions in the basic national precise levelling netowrk [e.g. Vaníček, 1970]. This is why research in this area is being done, considering both existing and hypothesized errors. The optimum goal of such research is to investigate their existence, significance, modelling, computations and practical applications.

The largest source of errors in the height determinations in North America from precise spirit levelling is thought to be the last one [Holdahl, 1974]. The investigations contained herein focuses on this particular source and is introduced in more details in the subsequent sections.

1.4 Gravity Field and Spirit Levelling

The purpose of this section is to present the connection between the observed geometrical height differences from spirit levelling and the actual gravity field of the earth. To begin with, some introductory remarks concerning the earth's actual gravity field are given. Then the connection between the spirit levelling operations and the characteristics of the gravity field is discussed.

The earth's gravity field is the resultant of two vector fields: the gravitation acceleration, due to the attraction of earth masses; and the centrifugal acceleration, due to the earth's rotation. Instead of using this vector field, it has been found convenient, in geodesy, [e.g.: Heiskanen and Vening-Meinesz 1958; Heiskanen and Moritz, 1967; Vaníček, 1971] to represent the earth's gravity field by a scalar potential field, and by equipotential (geopotential) surfaces (sometimes referred to as level surfaces). A geopotential surface is an equipotential surface of the earth's actual gravity field on which the potential W, due to gravitational and centrifugal effects, is constant and usually can be expressed as W = const. The geoid, as defined in section 1.1, is one of these geopotential surfaces.

The geopotential surfaces are irregular, not parallel and generally converging from the equator towards the pole, (Figure 1-3). The direction of the gravity vector g is tangential to the curved lines of force (plumblines), which are everywhere perpendicular to the geopotential surfaces (Figure 1-3). The gravity direction is usually referred to as the direction of plumbline or the direction of the vertical. The difference of potential, dW, between two adjacent geopotential surfaces, W = const. and W + dW = const. (Figure 1-3), is a constant value representing the work done to move a unit mass from one geopotential surface to the other, and is given by:

$$dW = - g_i dh_i$$

where g_i is the magnitude of the gravity vector and dh_i is the differential distance separating the two geopotential surfaces in question at a certain location i. Since the value of g varies from one place to another while dW does not, one can see that the separation dh is not a constant value. For instance, it is maximum at the equator, dh_e, and minimum at the pole, dh_p (Figure 1-3). Such geopotential surfaces are physical but intangible reality which affect the field of surveying in several ways, one of which is the process of spirit levelling.

In spirit levelling one uses a mixture of a geometrical instrument (the staff) and a gravitational instrument (the level) [Jackson, 1963; Clark and Jackson, 1973]. The bubble axis of the latter is aligned tangentially to the local geopotential surface (Figure 1-4a), while the cross hair reads a geometrical height difference, dh, on the staff.

According to section 1.1, the height h_A of point A is defined as the distance between A and the geoid, measured along the actual plumbline of A. On the other hand, due to the nature of spirit levelling,

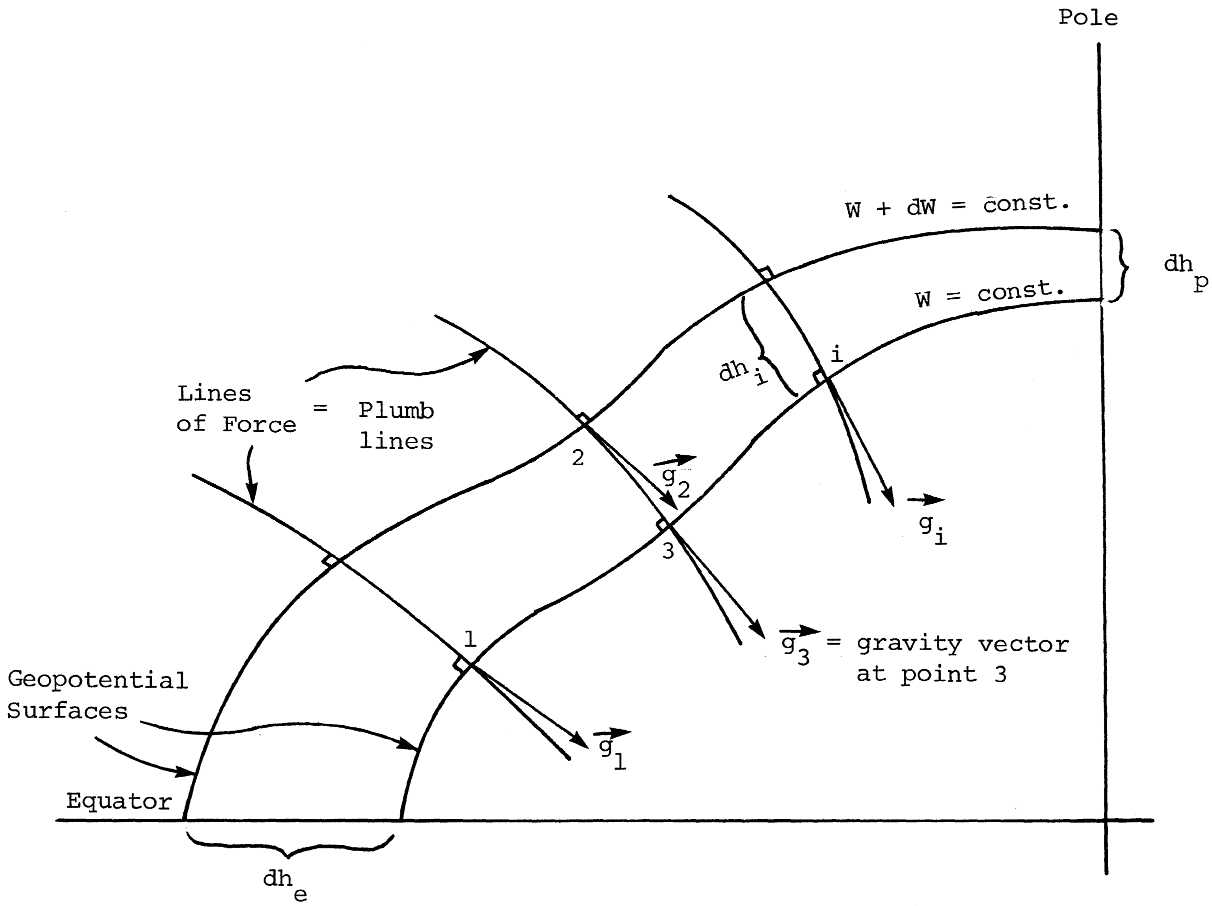


FIGURE 1-3

System of Geopotential Surfaces

and Plumblines

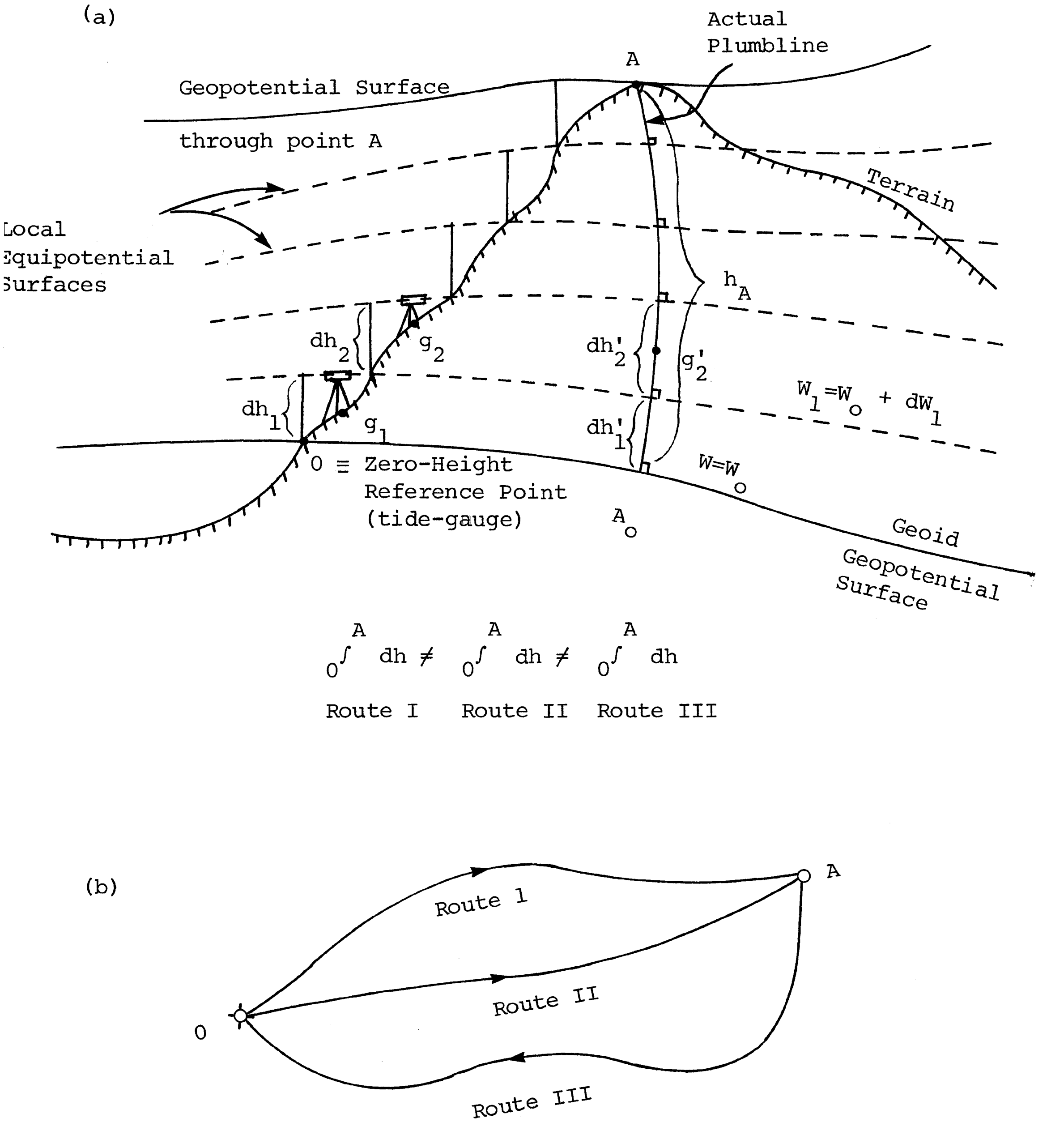


FIGURE 1-4

Observed Heights From Spirit Levelling

the measured height difference (sometimes referred to as instrumental or levelled height difference) gives the geometric distance, dh, between local geopotential surfaces above the earth surface, as opposed to their separation, dh', within the earth's crust underneath the terrain point A (Figure 1-4a). Realizing that $h_A = \int_{A_o}^{A} dh'$, and examining Figure 1-4a, one notices the fact that the sum of the measured differential distances dh between the zero-height reference point 0 and the point A, in question, does not equal the height h_A, simply because $dh_i \neq dh'_i$. Consequently, this sum depends on the selected path of the levelling route between points O and A, and generally acquires a different value for each different route connecting the same two points as illustrated in Figure 1-4b. In other words, if a levelling route goes over a closed circuit (loop), this sum will not generally be zero, i.e.:

$$\oint dh \neq 0.$$

On the other hand, if the observed gravity values, g, are incorporated with the levelled height differences, dh, over the levelling loop, and the differences in potential, dW = gdh, are used instead of dh only, such loop misclosure vanishes. This is so because the potential of the earth's gravity field has a unique value for each point. Hence, it gives theoretically a zero misclosure over a closed loop, which can be written as:

$$\oint g\,dh = \oint dW = 0 \quad .$$

This approach is the basic concept behind the use of the so-called "geopotential numbers" (to be explained in details in section 2.1) as a common quantity for the unique definition of any system of heights [Krakiwsky, 1965; Heiskanen and Moritz, 1967].

An alternative approach, which was found convenient in practice, is to apply appropriate corrections to the observed instrumental height differences to account for the non-parallelism of geopotential surfaces. These corrections have to be expressed in terms of actual gravity values at the consecutive bench marks along the followed levelling route in order to yield precise and unique heights of terrain points above the chosen datum [e.g.: Vignal and Kukkamäki, 1954; Heiskanen and Moritz, 1967; Vaníček et. al., 1972].

To conclude this section, one can say that the spirit levelling results without being supplemented with gravity measurements are not useful from the rigorous point of view [Heiskanen and Moritz, 1967], since they lead in general to a misclosure, and consequently do not define the heights of terrain points uniquely.

1.5 The Problem to be Investigated and its Background

In this section, a detailed discussion of the background of the problem to be investigated herein is given. Appropriate references dealing with the subject of heights and gravity are listed. The specific problem of interest is introduced and defined. Results of previous investigations of the problem are commented upon. In addition, reasoning and motivations behind the present study are presented, with particular emphasis on the Canadian precise levelling network.

In the previous section we commented on the necessity of correcting the levelled height differences due to the non-parallelism of the geopotential surfaces. This has to be performed in terms of observed gravity for a unique definition of the sought heights. Canada and the

United States of America are, however, among the countries where the gravity survey was not detailed enough to compute such required corrections [Cannon, 1929; Rappeleye, 1948; Rapp, 1961; Krakiwsky, 1965; Vaníček et. al., 1972] at the time of last adjustment of their networks. This seemed natural, since no useful equipment for rapid and exact determination of gravity at the physical surface of the earth existed until the 1930's [Vykutil, 1964]. Such a situation led to the substitution of "normal gravity", γ, for the actual gravity, g, in the observed height correction-equations.

The values of γ are computed via a theoretical formula based on an adopted mean earth ellipsoid, which is closely approximating the geoid. Among other characteristics, this ellipsoid should have the same mass, centre of gravity and angular velocity as the actual earth. Sometimes, in practice, such an ellipsoid is referred to as the "normal earth". The mean earth ellipsoid generates what is known as the normal gravity field, which can be represented by the system of normal equipotential surfaces usually denoted by U = const., and the normal lines of force (normal plumblines). One of them, denoted by U_o = const., is defined to be coincident with the surface of the ellipsoid itself, and in addition, postulated to acquire the same potential value of the geoid W_o = const. (i.e. one can write: $U_o = W_o$ = const [e.g. Heiskanen and Moritz, 1967]).

In reality, the actual values of gravity may differ considerably from the calculated values of normal gravity. Consequently, the geopotential surfaces will have undulations, as opposed to the idealized regular normal equipotential surfaces. This is due to the local or regional topographical and geological irregularities [Clark and Jackson, 1973]. In other words, the use of actual, instead of normal, gravity will reflect the local circumstances under which the levelling was performed (e.g. showing the effect of some existing local anomalous mass [Krakiwsky, 1965]).

The use of normal gravity to correct the levelled height differences accounts for the effect of the overall convergence of equipotential surfaces (i.e. the long wave latitudinal features only). This implies that the differences between actual and normal gravity, which represent the local irregularities of the actual gravity field (i.e. the short wave features) are neglected. These neglected differences, usually less than 200 mgals [Vaníček, 1970], may be referred to as the neglected gravity anomalies [Rune, 1950a]. The influence of these anomalies on the heights currently utilized in Canada constitutes the backbone of the problem under investigation herein. Heights defined on the basis of computed normal gravity only depend on an adopted formula of reference and to that extent they are approximate quantities not representing the reality [Clark and Jackson, 1973]. They can be considered, however, as lower order heights from the point of view of rigor [Heiskanen and Moritz, 1967; Vaníček, 1972], since they are meant to approximate the proper heights (based on actual gravity) and generally produce closing errors [Ramsayer, 1965b].

A vast number of publications treating problems related to heights and gravity have appeared. Among the main investigators in this domain (listed here in chronological order) are: Helmert [1890]; Rune [1950a]; Ledersteger [1954]; Vignal [1954]; Vignal and Kukkamäki [1954]; Bursa [1958]; Baeschlin [1960a]; Müller [1960]; Schneider [1960]; Rapp [1961]; Molodenskii et. al. [1962]; Vignal and Simonsen [1962]; Braaten and McCombs [1963]; Weidauer [1963]; Krakiwsky and Mueller [1965, 1966]; Ramsayer [1965 a,b]; Heiskanen and Moritz [1967]; Meuller et. al [1968];

Vaníček [1970]; Bomford [1971]; Laflamme [1971]; Christodoulidis and Vaníček [1972]; Vaníček et. al. [1972]; Pardalis [1973] and Holdahl [1973, 1974, 1975a]. Many specific works by these authors will be found referenced in corresponding sections within the text of this thesis.

Most of the above listed authors devoted their studies to either one or more of the following aspects: The basic foundation of the theory of heights and the role of actual gravity field into their unique definition; spacing between and location of gravity stations along levelling routes; the advantages and disadvantages of each known system of heights as recommended for a particular country; the adjustment procedures of precise levelling networks utilizing both spirit levelling and observed gravity data; extensive analysis of errors and required precisions in levelling and gravity observables to minimize the errors of resulting heights. Some of them have also investigated the influence of actual gravity (as opposed to the employed normal gravity) on heights of bench marks along selected levelling profiles. The present study, as pointed out earlier (section 1.3), concentrates only on this very last problem.

All the previous investigators dealing with the problem on hand have followed exactly the same approach, namely, studying individual lines or loops. Their method was to compute the heights of all bench marks along the chosen profile twice - once with normal and once with actual gravity values. Then, the two sets of results were compared either graphically or numerically. This procedure has been employed in many parts of the World, especially in most of the European countries and North America. Individual cases of significant influence of neglected gravity irregularities along the tested levelling profiles were reported in, e.g., [Rune, 1950a; Rapp, 1961; Braaten and McCombs, 1963; Krakiwsky, 1966; Christodoulidis and Vaníček, 1972; Vaníček et al., 1972; Holdahl, 1973]. Also, cases of insignificant influence were reported in, e.g. [Weidauer, 1963; Ramsayer, 1965b; Bomford, 1971]. Such analysis is basically one-dimensional, and as such it is unable to reveal anything concerning the neighbourhood of the tested line. This is why most of these investigations have proved to be inconclusive.

On the international level, realizing the consequences of neglecting the gravity anomalies on heights used in investigating scientific problems related to geosciences and also to support the future technological developments in geodesy (outlined in section 1.2), the International Association of Geodesy (IAG) has passed a resolution recommending that all member nations should observe gravity values at each bench mark of their levelling networks and compute their heights properly on the basis of actual instead of normal gravity [IAG, 1950, Vignal and Kukkamäki, 1954]. Although it was the aim of this resolution to settle the controversial argument concerning the significance and consideration of gravity influence, it was found that various opinions about this problem still existed. While practising geodesists maintain that the effects of local gravity irregularities on heights are negligible, and hence in many cases it is doubtful whether the expensive determination of actual gravity at bench marks is justified, theoreticians keep showing that the effects are significant, and thus cannot be neglected.

In Canada, two systems of heights are used - orthometric and dynamic, both defined on the basis of normal gravity because of the lack of gravity observations during the period of building-up and extending the Canadian precise level network prior to 1945 [GSC, 1972]. At present, the situation is different. With the rapid extension of regional gravity coverage in Canada occuring at the same time as the availability of electronic digital computers, extensive geodetic investigations using gravity data can now be undertaken [Shimazu, 1962; Hamilton, 1963a; Nagy, 1963; Buck and Tanner, 1972; Nagy, 1973; Nagy, 1974; Valliant, 1975].

The Earth Physics Branch (EPB) carried out some studies about the geophysical and geodetic uses of gravity data for Canada [Shimazu, 1962]. However, only the computations of deflections of the vertical and geoidal heights have been considered. There is no attempt, in that publication, to even mention the implications of gravity data on the used precise levelling operations. This was surprising, at least in the author's opinion, since Hamilton [1960] had already reported cases of levelling loops with unusually large misclosures. The neglected gravity anomalies were suspected to be the reason. However, subsequent studies gave no definite answers. In addition, several localized studies undertaken by the Geodetic Survey of Canada (GSC) into this problem have proved inconclusive. The GSC [1972] is planning to recompute post 1944 precise levelling with observed gravity at all bench marks and compare this with previous adjustments (based on normal gravity) to investigate the distortions due to the neglect of gravity anomalies. In the spring of 1964 a program to establish gravity values at bench marks throughout Canada was initiated by the Gravity Division of the EPB. Results of this project, which so far covers only one area in Eastern Ontario, are documented in Hamilton and Buchan [1965] for 619 bench marks of precise and secondary level network. Furthermore, preliminary experiments carried out at the Department of Surveying Engineering, University of New Brunswick (UNB), over some levelling profiles in Alberta showed significant gravity influence on their heights [Christodoulidis and Vaníček, 1972].

Based on the general background presented above, the following statement defines, in summary, the specific problem of the current investigation. This study focuses on investigating the influence of neglected actual gravity irregularities (gravity anomalies) on the height differences - derived from spirit levelling and defined on the basis of normal gravity, with particular interest within the Canadian territory. This influence, which can be viewed as the difference between the height difference of a levelling section based on actual gravity and the corresponding height difference based on normal gravity, will be termed "GRAVITY CORRECTION" throughout this thesis. Thus, the "GRAVITY CORRECTION" dealt with in this study accounts for the difference between normal gravity (which has been already accounted for in the Canadian precise level network) and the actual gravity, i.e., it is solely due to the neglected "gravity anomalies".

In addition to the aforementioned reasoning, the forthcoming readjustment of North American Geodetic Networks is another motivation behind undertaking this research. At present, the emphasis of the GSC is directed towards the horizontal networks. Efforts have been made to investigate the effects of the neglect of local irregularities of the gravity field on the reduction procedures of observed distances and directions from the terrain to the computational reference ellipsoid [e.g.: Thomson et. al., 1974; Mutajwaa, 1976]. On the other hand, the GSC has the intention and plans to readjust the vertical control net [Vaníček, 1970; McLellan, 1974; Young, 1974] in the near future, perhaps in the early 1980's [Young, 1975]. Hence, it appeared to be a necessity to study the influence of the aforementioned "GRAVITY CORRECTIONS" on the heights currently used in Canada. The contribution of this research should be relevant to the planned new adjustment of the Canadian precise levelling network (CPLN).

1.6 Objectives and Methodology of Investigation

On the basis of the presented controversy concerning the "GRAVITY CORRECTIONS", it was felt that many questions needed to be answered and many points needed to be clarified. These are:

- Significance of the corrections within a certain geographical area of interest;
- Simplest mathematical modelling of these corrections and methods of their computations;
- Reliability of the corrections and feasibility of their practical applications;
- Behaviour of the corrections and their influences along levelling profiles (lines and loops);
- Data requirements for the numerical evaluation of the corrections, especially gravity observations at bench marks for a levelling route located in an area where gravity data (e.g. gravity maps) are already available.

Accordingly, this thesis has two main objectives:

1. Formulation of a mathematical model for the gravity correction (for three systems of heights: Dynamic, Helmert and Vignal) in terms of practically obtainable quantities;

2. Determination of the geographical areas where the gravity corrections are significant and must be taken into account in order to maintain the standard of accuracy required for the first-order levelling newtowk.

To achieve the first objective, the mathematical background of the theory of heights based on actual gravity and their counterparts based on normal gravity had to be carefully documented. Then, the gravity corrections had to be formulated in terms of free-air gravity anomalies, levelled heights, and latitudes of the benchmarks (reasons to be stated later).

To accomplish the second objective, the method of investigation had to be based on a two-dimensional treatment of the problem. This methodology differs considerably from the approach used by other researchers who had treated the problem in one-dimension. The technique used involves two major steps. First, 2-D approximating polynomials are used to model both the gravity and height fields, given through observed values at discrete points within a unit block. Second, a 1-D polynomial is generated to approximate each of gravity and height profiles along simulated levelling lines radiating, in all required directions, from the centre of the block under investigation. This approach permits one to compute the gravity correction along these simulated lines, to display their pattern and to examine their significance in each block and each direction. When such analysis is repeated for adjacent blocks, one has the opportunity of studying the significance of the gravity corrections over the whole region. This holds true providing that the region is sufficiently densely covered with gravity and height data.

1.7 Scope and Summary of Contributions

The nature and characteristics of the problem on hand dictated the format of the present thesis. The purpose of each Chapter is briefly described below.

Chapter 2 gives the basic foundations of the theory of rigorous heights, based on actual gravity. It also shows the appropriate way of incorporating spirit levelling results with observed gravity data in the form of actual geopotential numbers as a common basis for defining any system of heights. It considers three systems of heights: Dynamic, Orthometric (Helmert) and Normal (Vignal).

Chapter 3 deals with the theory of approximate heights based on normal gravity, as used in Canada. Approximate orthometric and dynamic systems are discussed. The adopted approach for the practical computations in both systems is demonstrated. Differences in values of normal gravity given by the Canadian adopted (USC&GS) formula and the 1967 International formula for normal gravity are tabulated, for Canadian latitudes, in Appendix I.

Chapter 4 is devoted to the derivation of formulae for the gravity corrections, in case of Dynamic, Helmert and Vignal systems of heights. This is based on the formulations stated in chapters 2 and 3. Expressions for the precision estimates of the gravity corrections are also developed.

Chapter 5 is included herein for the sake of completeness. It discusses, in general terms, both precise levelling and gravity data coverage in Canada, with reference to the historical and present status in each case. The availability and format of the EPB point gravity data file is explained in Appendix II.

Chapter 6 analyses the significance and application of gravity corrections to actual levelling lines and loops. Tables for practical evaluation of the gravity corrections for Vignal, Dynamic and Helmert systems are provided in Appendix III.

In Chapter 7, the second main objective of this study is attained. Development, testing and conclusions associated with the proposed technique are given. Results of its application to the Canadian territory are documented in several external Appendices (see Table of Contents).

Chapter 8 summarises the findings of this investigation. Conclusions based on the obtained results are given. Recommendations for related future studies are stated.

Finally, a list of 180 references, quoted within the text of the present thesis, is compiled in alphabetical order. Five internal Appendices are included. For reasons of volume, the remaining five Appendices are external.

This research has resulted in several contributions which are relevant to the problem of the influence of neglected gravity anomalies on heights. Six of these contributions are considered, in the author's opinion, to be the most significant. These are summarized below:

1. Development of rigorous formulae for the "GRAVITY CORRECTIONS" (Chapter 4), as well as their precision estimates. These formulae are in terms of practically obtainable quantities, namely: free-air gravity anomalies, observed heights and scaled latitudes at consecutive bench marks along the followed levelling route. The formulae are derived for Dynamic, Helmert and Vignal systems of heights. These corrections can be evaluated and readily added to the existing height differences to achieve the corresponding height differences based on actual gravity.

2. Construction of Tables (Appendix III) to facilitate the practical computations of "Gravity Corrections", in case of Vignal, Dynamic and Helmert systems. These Tables can be easily used even in the field without the need for pocket calculators. The needed arguments are observed heights (from levelling field books) and free-air gravity anomalies (e.g. from a gravity anomaly contour map).
3. A complete discussion (Chapter 5) of precise levelling and gravity data coverage in Canada, including its historical background, present status, future plans and format of available data. This should contribute valuable information for related future investigations.
4. A thorough discussion (Chapter 6) of the significance and practical application of the developed gravity corrections. Clarification of the controversial argument about the gravity correction accumulations, cancellations and effects along levelling lines and loops.
5. Development of a tool (based on surface fitting techniques) that allows one to determine the geographical area (with available gravity and height information) where the gravity correction is significant and in which direction should it be taken into account within the area of interest. This technique has been applied to all areas in Canada covered by sufficient point gravity and height data. Results for all of Canada are given in the five external Appendices (available from the thesis supervisor).
6. Development and documentation of two computer program packages: LOOPGC and AREAGC. The first is designed basically to compute the accumulated gravity corrections and their standard deviations along levelling lines or loops, and to display them for comparison against the corresponding accumulated standard errors achieved in precise

levelling, considering Helmert, Vignal and Dynamic Systems. The second package is written specifically for the purpose of predicting areas with significant gravity corrections, in Canada, using the available EPB new gravity data file based on the 1967 system. Both programs can be obtained from the Surveying Engineering Computer Library, U.N.B.

CHAPTER 2

HEIGHT SYSTEMS BASED ON ACTUAL GRAVITY

In this Chapter, the concept of geopotential numbers based on actual gravity is introduced. The fundamental definitions of rigorous heights based on actual geopotential numbers are given. Three systems of heights: dynamic, orthometric and normal, are considered. The normal height system is treated herein, in spite of its absence among the systems in use at present in Canada, because it has been proposed by Vaníček et. al [1972] to be adopted for Canada as a more modern system of heights.

2.1 Actual Geopotential Numbers

Actual geopotential numbers are basic to any definition of height [e.g.: Heiskanen and Moritz, 1967]. The actual geopotential number C_A of a terrain point A (Figure 2.1) represents the amount of work needed to lift a unit mass, along any route on or inside the earth, from the geoid to the point A. In other words, C_A is the negative potential difference between the geopotential surface, $W=W_A$, through the terrain point A, and the particular reference geopotential surface, $W=W_o$, the geoid (section 1.1). It is defined as:

$$C_A = -(W_A - W_o) = -(-\int_o^A g dh) = -(-\int_{A_o}^A g' dh'), \qquad (2\text{-}1)$$

where g is the actual (observed) gravity on the earth's surface along the levelling path from O on the geoid to the terrain point A, and dh is the projection of a differential path increment onto the vertical.

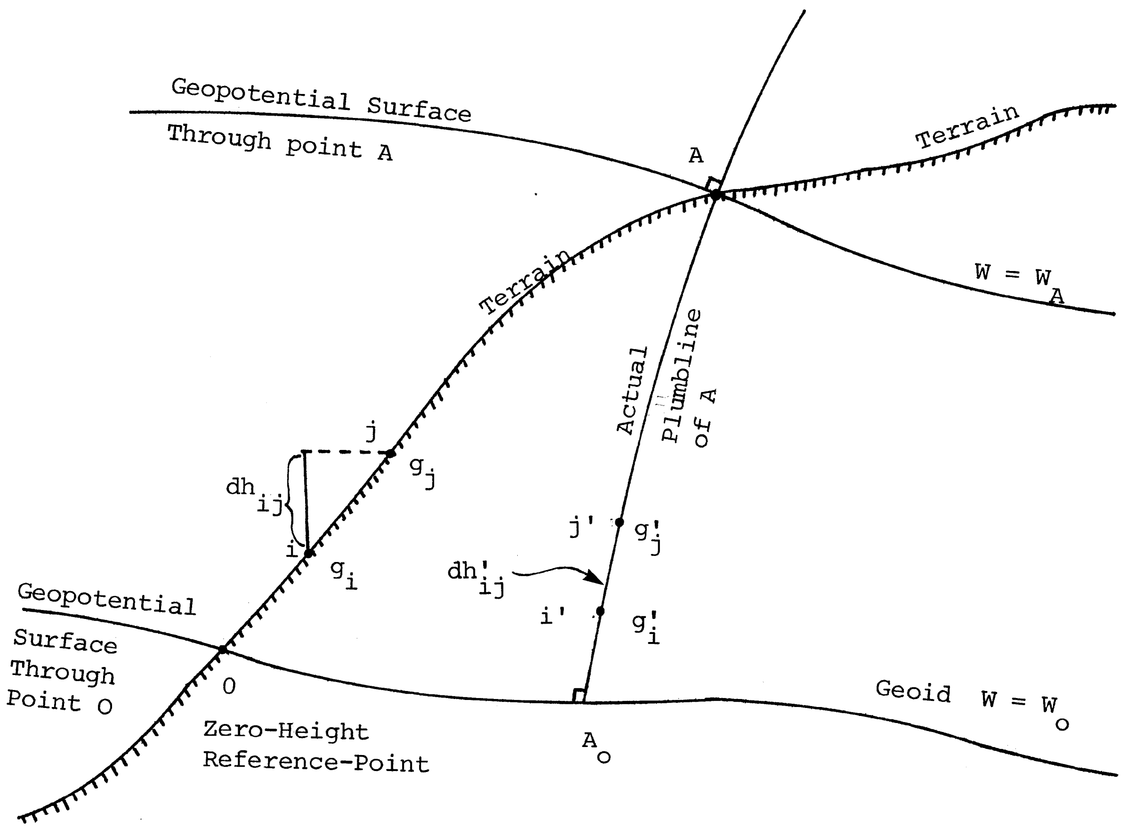


FIGURE 2-1

Concept of Actual Geopotential Number

g' and dh' are the corresponding quantities along the plumbline of A (inside the earth, see section 1.4). The reason for the negative sign outside the parentheses in equation (2-1) is to make the geopotential numbers increase upwards from the geoid to be consistent with the concept of heights.

The geopotential numbers as obtained from the levelled height differences and enroute gravity values can be used as a natural measure of heights because they define heights of individual points uniquely [e.g.: Krakiwsky, 1965]. The geopotential number is positive above the geoid, negative below it, constant everywhere on the same geopotential surface and equals zero for the geoid. They are usually measured in geopotential units (g.p.u.) [Mueller and Rockie, 1966; Bomford, 1971], where: 1 g.p.u. = 1 kgal - metre. The reason for such choice is to make the numerical value of the geopotential numbers approximately equal to the heights of the corresponding points above sea level in metres.

One also speaks about the actual geopotential number difference ΔC_{AB} between two terrain points A and B, which is given by:

$$\Delta C_{AB} = C_B - C_A = \int_A^B g \, .dh \, , \qquad (2\text{-}2)$$

where the integration is carried out along the route following the levelling line on the terrain between the two points A and B. This is the only possible approach since the actual gravity g' cannot be observed inside the earth. In practice, however, a continuous profile of observed gravity g and height h, along the levelling path A to B, is not available. Thus, the integral in equation (2-2) cannot be rigorously evaluated. Consequently, this integral has to be replaced by summation over a set of discrete points along the line AB where gravity and height difference are observed [e.g. Baeschlin, 1960b; Krakiwsky, 1965]. The evaluation of ΔC_{AB} will be then approximate only. The following formula is usually used:

$$\Delta C_{AB} \doteq \sum_{i=A}^{i=B-1} \bar{g}_{ij} \cdot \Delta h_{ij}, \qquad (2-3)$$

Where: $j = i + 1$, (2-4a)

$$\bar{g}_{ij} = \frac{1}{2} (g_i + g_j) , \qquad (2-4b)$$

$$\Delta h_{ij} = h_j - h_i . \qquad (2-4c)$$

Here, Δh_{ij} is the levelled (observed) height difference between the two adjacent points i and j; g_i and g_j are actual values at i and j. For ΔC to be in g.p.u., Δh has to be in metres and $\bar{g}$ in kgals.

In practice, if g is not immediately available at each bench mark i, j, ... , it can be interpolated from the existing gravity data in the surrounding area either by least-squares surface fitting techniques [Vaníček, 1970; Vaníček et. al., 1972], or by graphical interpolation from available gravity maps [Krakiwsky, 1966]. However, it can be noticed from equation (2-3), that in order to obtain an adequate accuracy of ΔC, the spacing between adjacent points (i and j) has to be appropriately close. The allowed spacing varies with terrain and with the degree of variability of the gravity field [Baeschlin, 1960a; Levallois, 1964; Krakiwsky, 1965; Ramsayer, 1965a]. Equation (2-3) thus indicates that the actual geopotential number difference is a quantity that can be measured by field procedures (spirit levelling and gravity observations) and then computed to at least the same accuracy as other geodetic quantities [Jackson, 1963]. An extensive analysis of required precisions in the observed spirit levelling height differences and enroute gravity values, to ensure minimum errors in the computed actual geopotential number differences, can be found in Krakiwsky [1966].

Theoretically, the actual geopotential number differences around a closed circuit should have a zero closure [Heiskanen and Moritz, 1967]. Consequently, it has been recommended by the IAG, since 1954, to perform the adjustment of precise levelling networks in terms of geopotential numbers, before transforming them to the adopted system of heights [Ellingwood, 1969]. This has been already executed in case of the United European Levelling Net (UELN) [Alberda et. al., 1960; Kääriäinen, 1960; Alberda, 1963].

2.2 Dynamic Heights

The dynamic height h_A^D (based on actual gravity) of a terrain point A is defined [e.g. Heiskanen and Moritz, 1967] as:

$$h_A^D = \frac{C_A}{G} \tag{2-5}$$

where C_A is the actual geopotential number of the point A in g.p.u., and G is a "reference gravity" value. This reference gravity is usually taken as the normal gravity on the mean earth ellipsoid (described in section 1.5) computed for an adopted "reference latitude" ϕ_R, in kgal. G is generally selected close to the average value of gravity for the area in question. For instance, it has been suggested [Vaníček et. al., 1972] that $G = \gamma_{o,50°}$ should be used for Canada, where $\gamma_{o,50°}$ is the normal gravity on the ellipsoid, computed from the 1967 International formula , e.g. [IAG, 1971] . The "reference gravity" G can be also regarded as a metric scale factor to convert the geopotential number, in g.p.u., to metres [Krakiwsky, 1965]. In practice, the dynamic height is usually referred to as the "dynamic number".

The reference surface (height datum) for the dynamic system of heights is the geoid (defined in section 1.1), $W=W_o$, Figure 2-2. It can be seen that although h_A^D is expressed in length units, it does not represent the length ℓ_A of the plumbline of A between A and the geoid. From equation (2-5), it is obvious that h^D is constant for all points located on the same geopotential surface (e.g. $W=W_A$, in Figure 2-2), whereas the geometrical separations ℓ between the two equipotential surfaces $W=W_A$ and $W=W_o$ is generally different (see section 1.4) at each location (A, i, j, k, ...). Therefore, it can be noticed that the dynamic height does not depict the actual geometric deviations of the physical surface of the earth from the geoid.

The dynamic height difference between two points A and B can be written as:

$$\Delta h_{AB}^D = h_B^D - h_A^D = \frac{C_B}{G} - \frac{C_A}{G} = \frac{\Delta C_{AB}}{G} \quad . \tag{2-6}$$

An alternative way to compute Δh_{AB}^D, found convenient in practice, is to express the dynamic height difference as:

$$\Delta h_{AB}^D = \Delta h_{AB} + DC_{AB} \ , \tag{2-7}$$

that is by adding the quantity DC_{AB} known as "dynamic correction" to the levelled height difference Δh_{AB}. Here Δh_{AB} is given by:

$$\Delta h_{AB} = \sum_{i=A}^{B-1} \Delta h_{ij}, \ j = i + 1 \tag{2-8}$$

The dynamic correction DC_{AB} (based on actual gravity) is given by the following formula [e.g. Vaníček, 1972]:

$$DC_{AB} = \sum_{i=A}^{B-1} \frac{\bar{g}_{ij} - G}{G} \Delta h_{ij} \quad , \tag{2-9}$$

where j, $\bar{g}_{ij}$ and Δh_{ij} have been defined earlier (equations 2-4). The

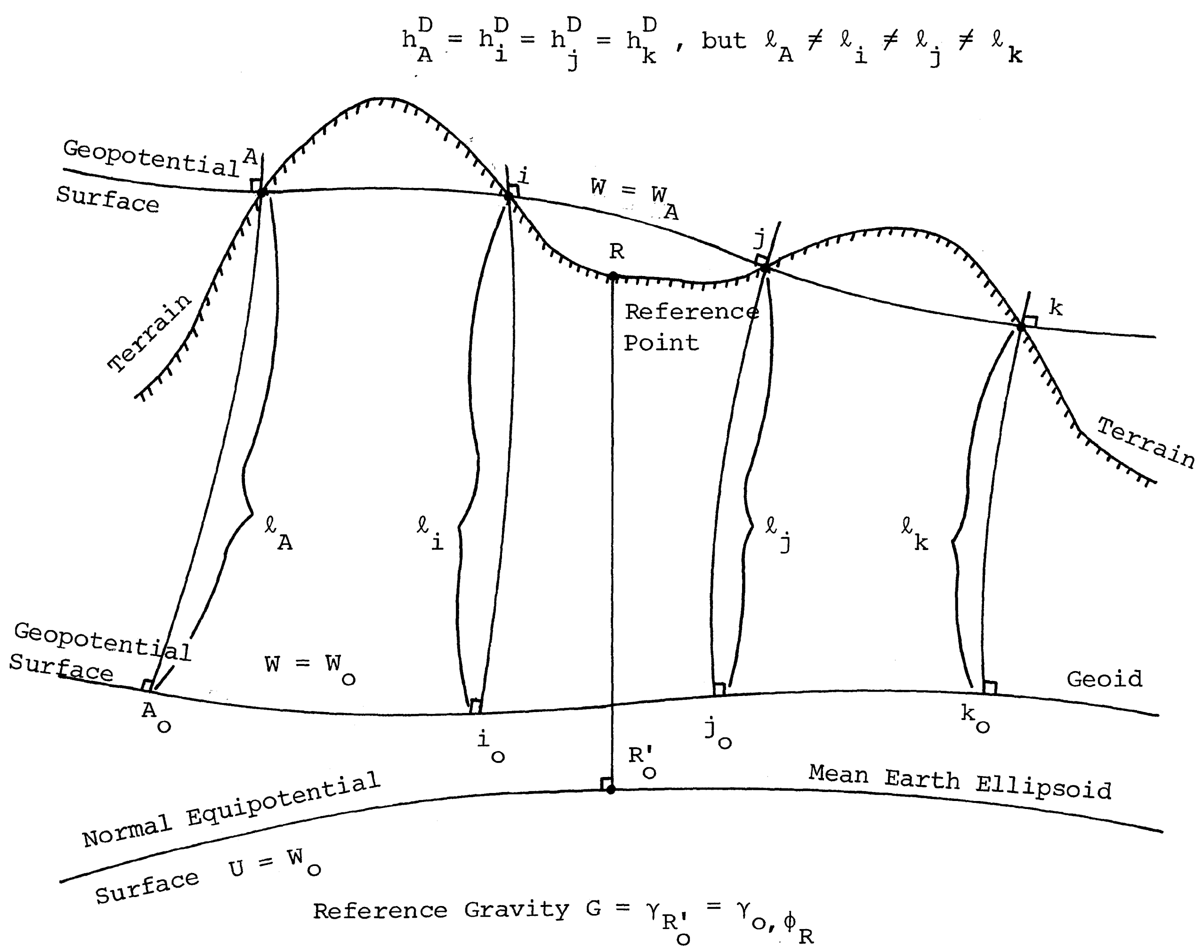


FIGURE 2-2

Concept of Dynamic Heights

Based on Actual Gravity

above equation is not necessarily carried over every set-up taken in levelling between points A and B, to obtain adequate accuracy. This will depend, again, on the admissible spacing of gravity measurements along the levelling route according to the type of the terrain [e.g. Ramsayer, 1965b].

In subsequent developments, only one levelling section will be considered. A levelling section, in this context, can be defined as the segment of a levelling line between two consecutive permanent bench marks, whose height difference was established using precise levelling. In the current Canadian specifications, the length of levelling sections is on the average of the order of 1 km [e.g. Peterson, 1970]. Hence, when dealing with only one levelling section between points i and j, equation (2-9) becomes:

$$DC_{ij} = \frac{\bar{g}_{ij}}{G} \Delta h_{ij} - \Delta h_{ij} \quad . \tag{2-10}$$

2.3 Orthometric Heights

The orthometric height h^O_A of a point A on the terrain (see Figure 2-3) is the distance between the point A and the geoid surface $W=W_o$, measured along the true plumbline of A. This implies that the geoid is again the reference surface (height datum) for this system of heights, and hence the orthometric heights can be viewed as the actual deviations of the terrain (physical surface of the earth) from the geoid. h^O_A is defined by:

$$h^O_A = \frac{C_A}{\bar{g}'_A} \quad , \tag{2-11}$$

where C_A is the actual geopotential number of A in g.p.u. and $\bar{g}'_A$ is the mean actual gravity along the true plumbline of A from the geoid to the

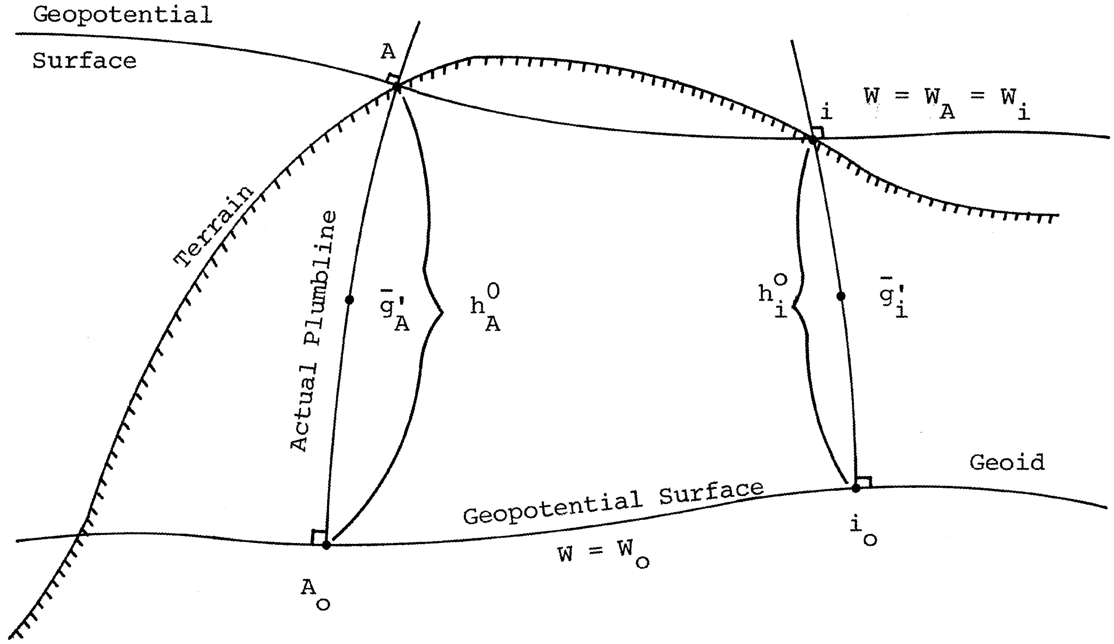


FIGURE 2-3

Orthometric Heights

Based on Actual Gravity

terrain in kgals, for h_A^O in metres.

The mean gravity $\bar{g}_A'$ cannot be determined rigorously, from the theoretical point of view, because the actual mass density distribution within the earth (i.e. along the true plumbline of A) is not known [Rune, 1950a; Hirvonen, 1960]. Therefore, one has to adopt a hypothesis regarding the variation of actual gravity along the true plumbline [Baeschlin, 1960a]. Once an attempt has been made to obtain $\bar{g}_A'$ as close as possible to the reality (resulting in an orthometric height that comes as close as possible to its true value), it is usual to refer to the height obtained from such an attempt as a "rigorous" orthometric height [Krakiwsky, 1965]. It should be clear that even the rigorous orthometric heights are not exact, due to the involved hypothesis, especially concerning the density [Molodenskii et. al., 1962].

Associated with the computation of rigorous orthometric height systems, one usually hears the names: Niethammer; Mader; Helmert; Mueller; ... etc. [Krakiwsky, 1965]. All differ in their assumptions and methodology of estimating the value of $\bar{g}_A'$ (in equation 2-11). Of all the aforementioned approaches to the computation of rigorous orthometric heights, the method of "Helmert" is the only one which is known to be the most widely used in practice [Krakiwsky, 1965; Vaníček, 1972]. Hence, the remainder of this section deals only with Helmert orthometric heights.

The Helmert orthometric height h_A^H of a point A on the physical surface of the earth is expressed as:

$$h_A^H = \frac{C_A}{\bar{g}_A'^{\,H}} \qquad (2\text{-}12)$$

where $\bar{g}_A'^{\,H}$ is the Helmert approximation to the mean value of actual gravity along the true plumbline of A between the geoid and the terrain. The Helmert

[1890] formula for $\bar{g}'^{H}_{A}$ is based on the application of the mean value theorem and the use of Poincaré - Pray's hypothesis concerning the gravity gradient along the plumbline [see, e.g. Vaníček, 1972]. This formula reads:

$$\bar{g}'^{H}_{A} = g_{A} + 0.0424\ h_{A} \quad , \tag{2-13}$$

in which g_A is the observed surface gravity on the terrain at A, and h_A is the observed height of A above sea level usually deduced from the spirit levelling results before adjustment. It should be noted that the units of the second term in equation (2-13) are mgals for h_A in metres. Realizing that $\bar{g}'^{H}$ generally varies from point to point, it follows that Helmert heights of points located on the same geopotential surface will be different (see Figure 2-3).

In practice, since the levelling process gives the levelled height difference Δh_{AB} (equation 2-8), it is again convenient to express the Helmert orthometric height difference Δh^{H}_{AB} between points A and B as:

$$\Delta h^{H}_{AB} = h^{H}_{B} - h^{H}_{A} = \Delta h_{AB} + HC_{AB} \quad , \tag{2-14}$$

where HC_{AB} is known as the rigorous orthometric correction (based on actual gravity). It will be referred to here as "Helmert correction". The Helmert correction can be evaluated from the following formula [e.g. Vaníček, 1972]:

$$HC_{AB} = \sum_{i=A}^{B-1} \frac{\bar{g}_{ij}-G}{G}\ \Delta h_{ij} + \frac{\bar{g}'^{H}_{A}-G}{G}\ h_{A} - \frac{\bar{g}'^{H}_{B}-G}{G}\ h_{B} \quad , \tag{2-15}$$

in which: j, Δh_{ij} and $\bar{g}_{ij}$ are defined in equations (2-4); G is the reference gravity for the dynamic heights mentioned before (or any other); $\bar{g}'^{H}_{A}$ and $\bar{g}'^{H}_{B}$ are the values of Helmert's approximate mean gravity along the true plumblines at A and B as given by equation (2-13); and h_A , h_B are approximate heights of A and B (e.g. derived from observed levelled differences).

The first term in equation (2-15) is nothing else but the dynamic correction DC_{AB}, based on actual gravity, as given by equation (2-9). Hence, equation (2-15) can be rewritten as:

$$HC_{AB} = DC_{AB} + \frac{\bar{g}'^H_A - G}{G} h_A - \frac{\bar{g}'^H_B - G}{G} h_B \, . \qquad (2\text{-}16)$$

This indicates that the Helmert orthometric correction can be interpreted as the sum of dynamic corrections for the open loop A_oABB_o, where A and B are the terrain points and A_o and B_o are their projections on the geoid surface [Heiskanen and Moritz, 1967].

Similar to the previous section, when dealing with only one levelling section between points i and j, equation (2-16) becomes:

$$HC_{ij} = DC_{ij} + \frac{\bar{g}'^H_i - G}{G} h_i - \frac{\bar{g}'^H_j - G}{g} h_j \quad . \qquad (2\text{-}17)$$

Substitution for DC_{ij} from equation (2-10) into (2-17) gives:

$$HC_{ij} = \frac{1}{G} [\bar{g}'^H_i h_i - \bar{g}'^H_j h_j + \bar{g}_{ij} \Delta h_{ij}] . \qquad (2\text{-}18)$$

Recalling that: $h_j = h_i + \Delta h_{ij}$, equation (2-18) can be reformulated (for subsequent developments) as:

$$HC_{ij} = \frac{1}{G} [h_i (\bar{g}'^H_i - \bar{g}'^H_j) + \Delta h_{ij} (\bar{g}_{ij} - \bar{g}'^H_j)] . \qquad (2\text{-}19)$$

2.4 Normal Heights

The theory of normal heights, their practical advantages, precise computations and freedom from any hypothesis concerning the actual density distribution within the earth crust have excited great scientific and practical interests among geodesists in different countries [e.g. Müller, 1960; Schneider, 1960; Weidauer, 1963; Yeremeyev, 1965; Krakiwsky and Mueller, 1966; Pick, 1970; Vaníček et.al., 1972; Wolf, 1974]. The normal heights are not meant to describe the heights above the geoid, like the previously discussed systems do. Instead, they relate the terrain points to another reference surface (height datum) and are closely tied to the modern geodetic theories - Molodenskii's and Hirvonen's in particular [Vaníček, 1972]. Hence, one usually hears the names "Molodenskii's normal heights" and "Hirvonen's normal heights".

The normal height h_A^N of a terrain point A (based on actual gravity) is defined [e.g. Krakiwsky, 1965; Heiskanen and Moritz, 1967; Vaníček, 1972] as:

$$h_A^N = \frac{C_A}{\bar{\gamma}'_A} \quad , \tag{2-20}$$

where $\bar{\gamma}'_A$ is the mean value of normal gravity along the normal plumbline of A (Figure 2-4) between the mean earth ellipsoid (point A'_o), where $U=U_{A'_o}=W_o$, and the point A' (inside the earth under A), where the normal potential $U_{A'}$ has the same value as the actual geopotential W_A at the corresponding point A on the terrain. Here, one may notice that the actual geopotential number C_A, defined by equation (2-1), can also be written as:

$$C_A = -(W_A - W_o) = -(U_{A'} - U_{A'_o}) \quad . \tag{2-21}$$

For h^N in metres, C has to be in g.p.u. and $\bar{\gamma}'$ has to be in kgal. Again, since both C and $\bar{\gamma}'$ are unique for each point, it follows that the normal height system (based on actual gravity) defines the heights of terrain points uniquely. Moreover, realizing that $\bar{\gamma}'$ varies only with latitude and height, one can see that the points which lie in the same geopotential surface (W = const.) and on the same

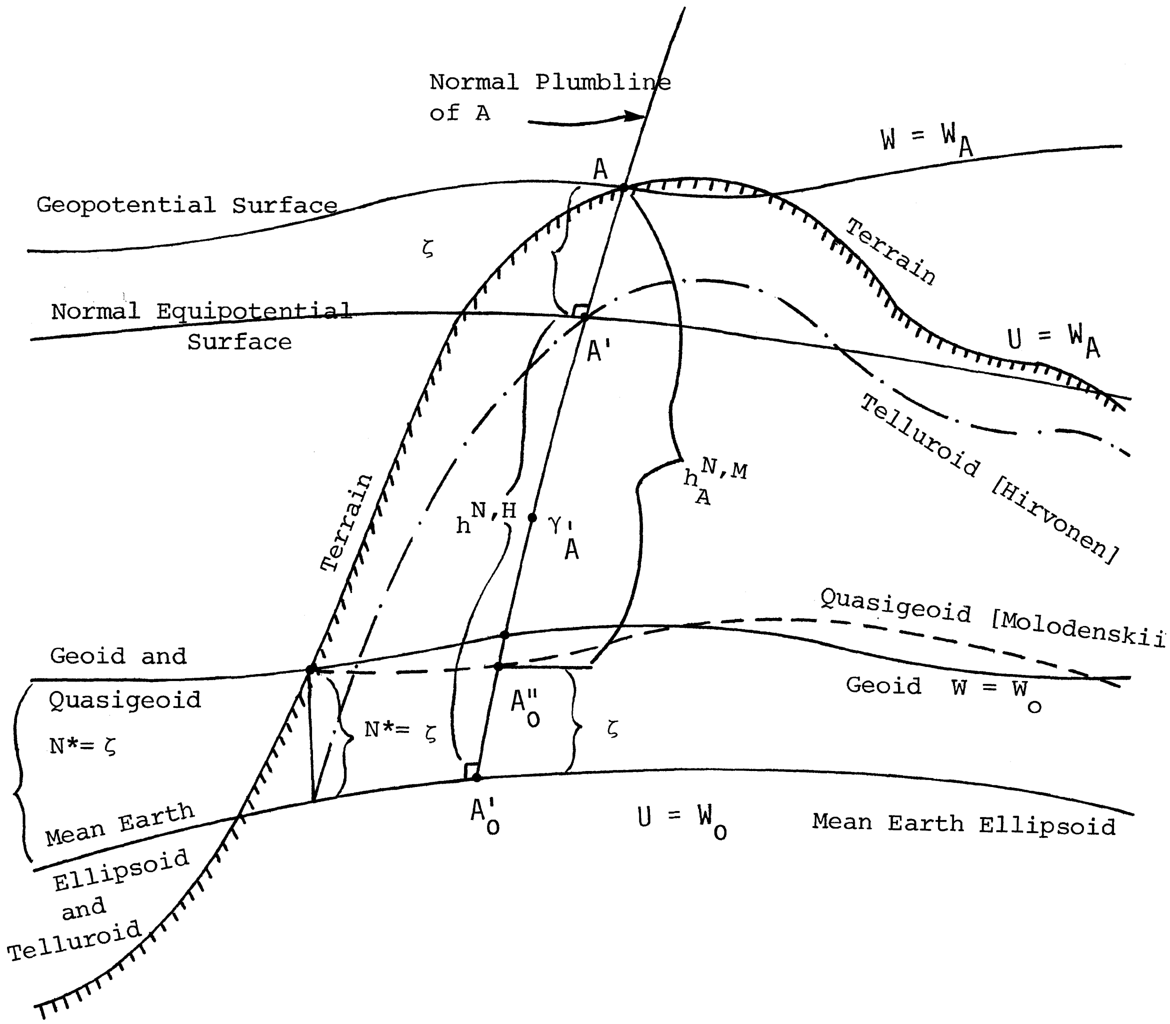


FIGURE 2-4

Molodenskii and Hirvonen Normal

Heights based on Actual Gravity

parallel of latitude (ϕ = const.) will have equal normal heights [Pick, 1970]. This is not generally the case in the orthometric height system.

Concerning the zero-height reference surface for the system of normal heights h^N, there are two alternatives. The first is the approach used in practice whereby we measure h^N from the physical surface of the earth along the corresponding normal plumbline (to point A''_o, e.g., in Figure 2-4). Consequently, the locus of h^N defines the height datum which is a mathematical surface (not generally an equipotential surface) known as "quasigeoid", whose properties are discussed in section 1.1. The quasigeoid was introduced by Molodenskii in the late 1940's [Krakiwsky, 1965], and hence the normal heights referred to it are known as "Molodenskii's normal heights" $h^{N,M}$ (Figure 2-4). The second is a theoretical approach based on reckonning h^N from the surface of the mean earth ellipsoid, as the height datum, along the normal plumbline (to point A', e.g., in Figure 2-4). Thus the locus of h^N above the ellipsoid defines another mathematical surface (not generally an equipotential surface) known as "telluroid", whose characteristics are outlined in section 1.1. This theoretical approach has been followed extensively by Hirvonen [1960], and hence the normal heights referring to the ellipsoid and generating the telluroid are usually known as "Hirvonen's normal heights" $h^{N,H}$ (Figure 2-4). The remaining length (see Figure 2-4) of the normal plumbline of A between A''_o and A'_o in case of Molodenskii's height, or between A' and A in case of Hirvonen's height, is called "height anomaly" ζ. The height anomaly ζ, in the modern geodetic theories, plays the same role as the geoidal height N*, in the classical theories. Note, in Figure 2-4, that on the oceans $N^* = \zeta$. Further elaboration on ζ is considered beyond the scope of this study.

Besides the different geometrical interpretation, Molodenskii's and Hirvonen's normal heights differ also according to the approach followed to compute $\bar{\gamma}'_A$ (in equation 2-20). Molodenskii's formula for $\bar{\gamma}'_A$ [Molodenskii et al., 1962; Vaníček, 1972] reads:

$$\bar{\gamma}'^{M}_A = \gamma_{o,A} - \frac{\gamma_{o,A}}{a} [1 + m + f \cos 2 \phi_A] h_A \quad , \qquad (2\text{-}21)$$

and Hirvonen's formula [Hirvonen, 1960] for $\bar{\gamma}'_A$ reads:

$$\bar{\gamma}'^{H}_A = \gamma_{o,A} - \frac{\gamma_{o,A}}{(a^2 b)^{1/3}} [1 + m + \frac{1}{2} e^2 - \frac{3}{2} e^2 \sin^2\phi_A + 10^{-4}] h_A \quad , \qquad (2\text{-}22)$$

where: a, b, f, e and m are respectively the semi-major axis, semi-minor axis, flattenning, first eccentricity, and a constant $\simeq$ 0.0033 of the mean earth ellipsoid. Here, $\gamma_{o,A}$ is again the normal gravity on the ellipsoid for latitude ϕ_A and h_A is the observed height of A.

On the other hand, the numerical values of both Molodenskii's and Hirvonen's normal heights are so close, that they are practically identical [e.g. Krakiwsky, 1965].

In spite of the fact that Molodenskii's heights are used, at present, in the USSR and some other countries of Eastern Europe, they are still not very popular in other parts of the World. For instance, they have not been introduced yet in North America. This could be due to psychological reasons because the quasigeoid, as a mathematical surface without any physical meaning compared to the geoid being a natural surface does not appeal to the users of heights. In case of Hirvonen's heights the theory [Hirvonen, 1960] implies the replacement of the physical surface of the earth by his telluroid, which makes it even more difficult for the users of heights to accept the concept. Nevertheless, Hirvonen's theory is useful and could be used for theoretical investigations related to modern gravimetric geodesy.

In addition, there is another well-known system of heights introduced by Vignal in the early 1950's [Yeremeyev, 1965]. This system has already proved, from both theoretical and practical analysis viewpoints, to be adequate as a system of gravimetrically corrected precise levelling heights [Krakiwsky, 1965]. At present, it is being used in France and other Western European countries, and has been adopted for the unification of the UELN [Vaníček et al., 1972]. Recently, Vignal system was proposed by Krakiwsky and Mueller [1966] to be adopted for the U.S. first-order heights. Also, it was recommended by Vaníček et al. [1972] to be utilized as a more modern system of heights for Canada.

In view of classical and modern geodetic theories, the physical interpretation of the definition of Vignal height is not unique. This results, in practice, in different names for it. In the classical way, it is usually treated as "approximate orthometric height", e.g. [Baeschlin, 1960a; Krakiwsky, 1965; Mueller and Rockie, 1966]. In the modern theories, it is regarded as a "normal height", e.g.: [Vykutil, 1964; Yeremeyev, 1965; Simonsen, 1966; Vaníček, 1972]. The second view, shared by the author, is based on the fundamental definition of normal heights as presented earlier.

Therefore, Vignal height system is classified, in the present study, as belonging to the systems of normal heights. It was felt that an effort should be made to clarify some of the controversial concepts that led to this ambiguity.

The above mentioned ambiguities are due to Vignal himself. There are contradictions between his original objective and the finally used definition of his height [Yeremeyev, 1965]. The original aim of Vignal, when he first introduced his height system in 1952, was to refer it to the geoid, and he names it "Orthodynamic height" [Vignal, 1954; Ledersteger, 1954; Baeschlin, 1960a]. The Vignal height h_A^V ($h_A^{o,v}$ in Figure 2-5a) of a terrain point A is given by (see section 2.3):

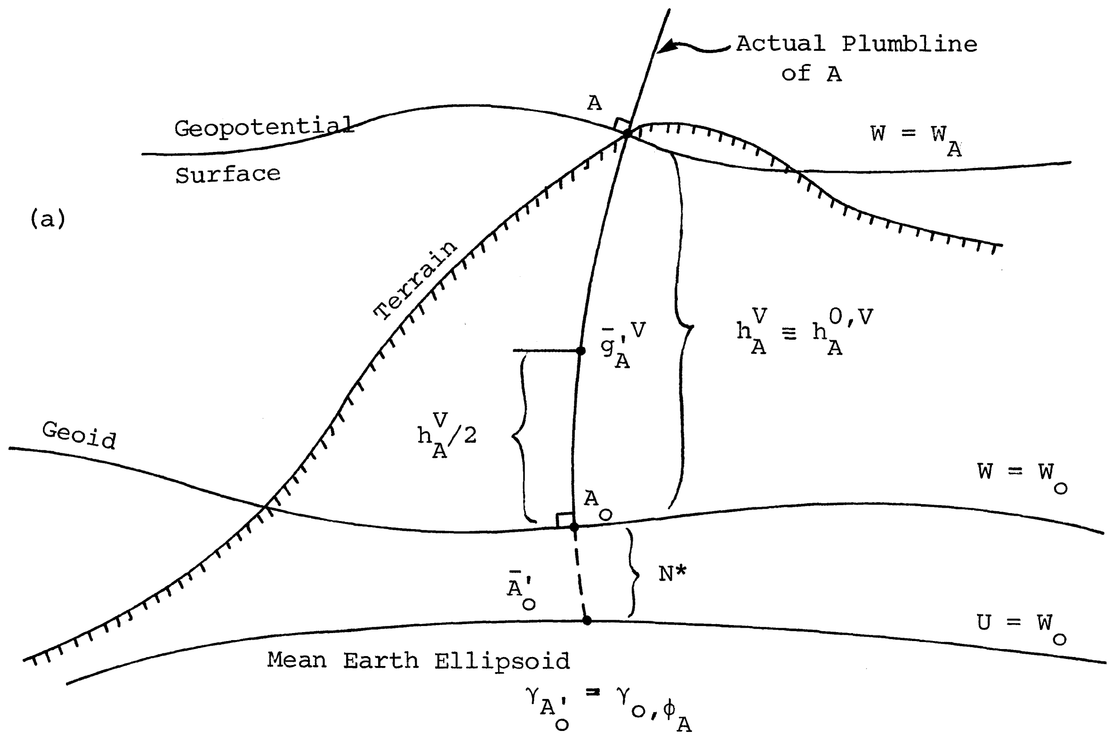
$$h_A^V = \frac{C_A}{\bar{g}_A'^V} \quad , \tag{2-23}$$

where $\bar{g}_A'^V$ is the Vignal approximation to the mean value of actual gravity along the true plumbline of A between the geoid and the terrain (see Figure 2-5a).

The original intention of Vignal was to make his "orthodynamic" height to differ as little as possible from the corresponding levelled height, to serve as practical height. Consequently, he suggested that $\bar{g}_A'^V$ be computed by practical methods, which are not necessarily rigorous but should give results adequately close to the true value $\bar{g}_A'$. Thus, Vignal proposed the following expression to compute $\bar{g}_A'^V$ at the midpoint of the plumbline between A and its projection A_o on the geoid, i.e.:

$$\bar{g}_A'^V = \frac{1}{2} (\gamma_{A_o} + \gamma_A) \quad , \tag{2-24}$$

where γ_{A_o} and γ_A are the normal gravity values at points A_o and A, respectively. For computing γ_A, he used the formula:



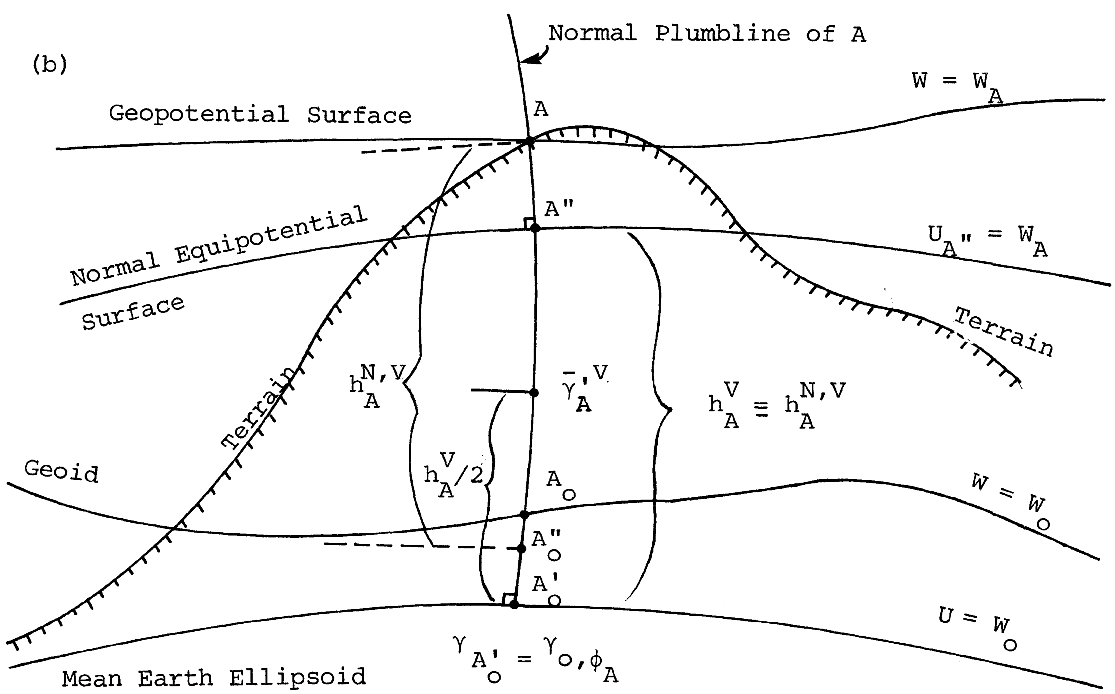


FIGURE 2-5

Contradictions between the Objective and the

Definition of Vignal Height

$$\gamma_A = \gamma_{o,A} - 0.3086\ h_A^V , \tag{2-25}$$

in which $\gamma_{o,A}$ is the normal gravity on the mean earth ellipsoid, computed for latitude ϕ_A. Vignal ended up with the following formula for computing the sought $\bar{g}'^V_A$ [e.g. Laflamme, 1971]:

$$\bar{g}'^V_A = \gamma_{o,A} - 0.3086\ (\frac{h_A^V}{2}) . \tag{2-26}$$

By examining equation (2-26), one discovers that $\bar{g}'^V_A$ is nothing else but the normal gravity propagated upward through "free-air" along the normal plumbline of A, to the point $\frac{h_A^V}{2}$ above the ellipsoid, as shown in Figure 2-5b. It should be thus denoted by $\bar{\gamma}'^V_A$. Hence, equation (2-26) can be rewritten, by replacing h_A^V by the levelled height h_A of A without any detrimental effect on the result [e.g. Vaníček, 1972], as:

$$\bar{g}'^V_A = \bar{\gamma}'^V_A = \gamma_{o,A} - 0.1543\ h_A . \tag{2-27}$$

Therefore, equation (2-23) should be written as:

$$h_A^V = \frac{C_A}{\bar{\gamma}'^V_A} . \tag{2-28}$$

Based on this discovery, and the definition of normal heights as stated at the beginning of this section, it can be argued that the resulting h_A^V from equation (2-28) should be classified as a "normal height", and denoted by $h_A^{N,V}$ as in Figure 2-5b. This $h^{N,V}$ can be again interpreted either theoretically as measured from the mean earth ellipsoid, or practically as measured from the physical surface of the earth to the corresponding height datum. In the latter case, strictly speaking the datum is neither the geoid nor the quasigeoid but much closer to the quasigeoid. On the other hand however, from the practical point of view, the Vignal (or any other) normal height can be considered as an approximation to the orthometric height [e.g. Vaníček, 1972]. Hence, Vignal height may be also regarded from the practical point of view as "approximate orthometric height" referred to the geoid. The main point here is that all the height systems discussed so far are all based on actual gravity, and thus possess the same characteristic quality of defining the heights of terrain points uniquely.

It has been found that the numerical value of Vignal height is very close to the corresponding value of Molodenskii's height [Vignal and Simonsen, 1962; Krakiwsky, 1965; Vaníček et al., 1972]. Both are practically identical. This can be verified as follows. The Molodenskii's $\bar{\gamma}'^{M}_{A}$ (equation 2-21) can be expressed in the first approximation as:

$$\bar{\gamma}'^{M}_{A} \doteq \gamma_{o,A} - \frac{\gamma_{o,A}}{a} h_A \quad . \qquad (2\text{-}29)$$

Further, it can be shown [e.g. Vaníček, 1972] that $(\frac{\gamma_{o,A}}{a})$ is approximately equal to one half the free-air gradient (i.e. $\frac{1}{2}$ (o.3086) mgal/m). Hence equation (2-29) can be written as:

$$\bar{\gamma}'^{M}_{A} \doteq \gamma_{o,A} - 0.1543\ h_A \quad . \qquad (2\text{-}30)$$

Comparison of equations (2-30) and (2-27) indicates that Vignal's $\bar{\gamma}'^{V}$ is the first approximation of Molodenskii's $\bar{\gamma}'^{M}$. Numerically, the maximum difference between $\bar{\gamma}'^{V}$ (equation 2-27) and $\bar{\gamma}'^{M}$ (equation 2-21) is at the pole, and is of the order of 0.1 mgal per each 1 km height. This means that Vignal height is the first approximation of Molodenskii height, and in most cases they are numerically very close. Yet, Vignal system is more popular due to the simplicity of its computations.

The Vignal normal height difference Δh^V_{AB} (based on actual gravity) between two points A and B on the levelling route, can be expressed as:

$$\Delta h^V_{AB} = \Delta h_{AB} + VC_{AB} \quad , \tag{2-31}$$

where Δh_{AB} is given by equation (2-8) and VC_{AB} is known as the "Vignal Correction". This correction is given by the following formula (e.g. Vaníček, 1972]:

$$VC_{AB} = \sum_{i=1}^{B-1} \frac{\bar{g}_{ij} - G}{G} \Delta h_{ij} + \frac{\bar{\gamma}'^V_A - G}{G} h_A - \frac{\bar{\gamma}'^V_B - G}{G} h_B \; , \tag{2-32}$$

in which $\bar{\gamma}'^V_A$ and $\bar{\gamma}'^V_B$ are computed from equation (2-27) for points A and B , and the other symbols are as defined before. Equation (2-32) can be also written as (see equation 2-9):

$$VC_{AB} = DC_{AB} + \frac{\bar{\gamma}'^V_A - G}{G} h_A - \frac{\bar{\gamma}'^V_B - G}{G} h_B \; . \tag{2-33}$$

Dealing with only one levelling section between points i and j, equation (2-33) becomes:

$$VC_{ij} = DC_{ij} + \frac{\bar{\gamma}'^V_i - G}{G} h_i - \frac{\bar{\gamma}'^V_j - G}{G} h_j \; . \tag{2-34}$$

After substituting for DC_{ij} from equation (2-10) and rearranging the terms, one gets:

$$VC_{ij} = \frac{1}{G} [h_i \, (\bar{\gamma}_i'^{V} - \bar{\gamma}_j'^{V}) + \Delta h_{ij} \, (\bar{g}_{ij} - \bar{\gamma}_j'^{V})] . \tag{2-35}$$

To compute the normal gravity values needed in the present investigation, the most up-to-date system of reference for the earth's gravity field (adopted by the IAG, 15th General Assembly in Moscow in 1971) is used. This system is known as the "Geodetic Reference System 1967" (GRS67) [IAG, 1971; Levallois, 1972]. In this system, the formula for computing the normal gravity value γ_o on the mean earth ellipsoid (referred to as the "1967 International formula" for normal gravity) reads [e.g. Vaníček, 1971; Levallois, 1972]:

$$\gamma_o = 978031.8 \; [1 + 0.005\,3024 \sin^2 \phi - 0.0000059 \sin^2 2\phi] \text{ mgal} . \tag{2-36}$$

In most practical applications, the latitude ϕ is referred to a local reference ellipsoid for horizontal geodetic networks. The errors introduced in equation (2-36), by using ϕ reckoned on the local reference ellipsoid instead of the mean earth ellipsoid, is at the most a few tenths of a milligal [Vaníček, 1972]. This error does not have significant influence.

A closing remark to this section is now in order. The reader should keep in mind that all the normal height systems presented in this section are rigorous heights based on actual (observed) gravity. These should not be confused with other approximate systems of heights based on (computed) normal gravity, that will be discussed in the next chapter.

CHAPTER 3

HEIGHT SYSTEMS BASED ON NORMAL GRAVITY

(AS USED IN CANADA)

In this Chapter the dynamic and orthometric height systems used in Canada are discussed. Both systems are defined on the basis of normal gravity, i.e. the gravity values computed from a simplified mathematical model of the earth. When normal instead of actual gravity is used, the Helmert's and Vignal's definitions introduced in the previous chapter are exactly equivalent and lead to the same expression for the orthometric height, which will be referred to here as the "orthometric height based on normal gravity", $\tilde{h}^O$ (the "telda" above h is introduced here to distinguish it from its counterpart based on actual gravity). Similarly, the dynamic height computed on the basis of normal gravity will be referred to as "dynamic height based on normal gravity", and denoted by $\tilde{h}^D$.

As explained, in section 1.5, these heights based on normal gravity have been used in North America since the start of precise levelling work, e.g. [Bowie and Avers, 1914; Cannon, 1929; Cannon, 1935; Rappleye, 1948]. This approach was originally adopted because of the lack of knowledge of the actual gravity field, i.e. the insufficiency of observed gravity values along the levelling loops. It may be worthwhile mentioning here that these heights have been also introduced in almost all countries, and meant to serve as the so-called practical heights [Vykutil, 1964]. Many countries have already started to define their height systems on the basis of actual gravity, in accordance with the IAG recommendations since 1950. Canada and the United States of America are among the countries where heights based on normal gravity are still used exclusively [Krakiwsky, 1965; Vaníček, 1972]. In the sequel the orthometric height is discussed before the dynamic height, since the latter has to be computed from the former.

Most of the notations and definitions contained herein are taken from the following references: Bowie and Avers [1914]; Cannon [1929]; Rappleye [1948]; Heiskanen and Vening-Meinesz [1958]; Geodetic Survey of Canada [1960]; Coordinating Committee on the IGLD-55 [1961]; Krakiwsky [1965]; Mueller and Rockie [1966]; Kowalczyk [1968]; Vaníček [1972]; and Jones [1973].

3.1 Geopotential Numbers Based on Normal Gravity

The geopotential number (based on normal gravity) $\tilde{C}_A$ of any terrain point A (see Figure 3-1) is the negative potential difference between the two normal equipotential surfaces: $U = U_A$, passing through A, and $U = U_O$, passing through a zero-elevation adopted reference point (defining MSL) [Mueller and Rockie, 1966]. $\tilde{C}_A$ can be also defined as the amount of work needed to transport a unit mass in the normal gravity field from a point at sea level to the terrain point A, which is given by Krakiwsky [1965] as:

$$\tilde{C}_A = -(U_A - U_O) = \int_{\tilde{O}}^{A} \gamma dh = \int_{\tilde{A}_O}^{A} \gamma' dh', \qquad (3\text{-}1)$$

where dh is an infinitesimal height increment along the normal plumbline and γ is the normal gravity value on the terrain; and dh' and γ' are the corresponding quantities inside the earth. The normal gravity γ used here is usually obtained so that the gravity value γ_O computed on

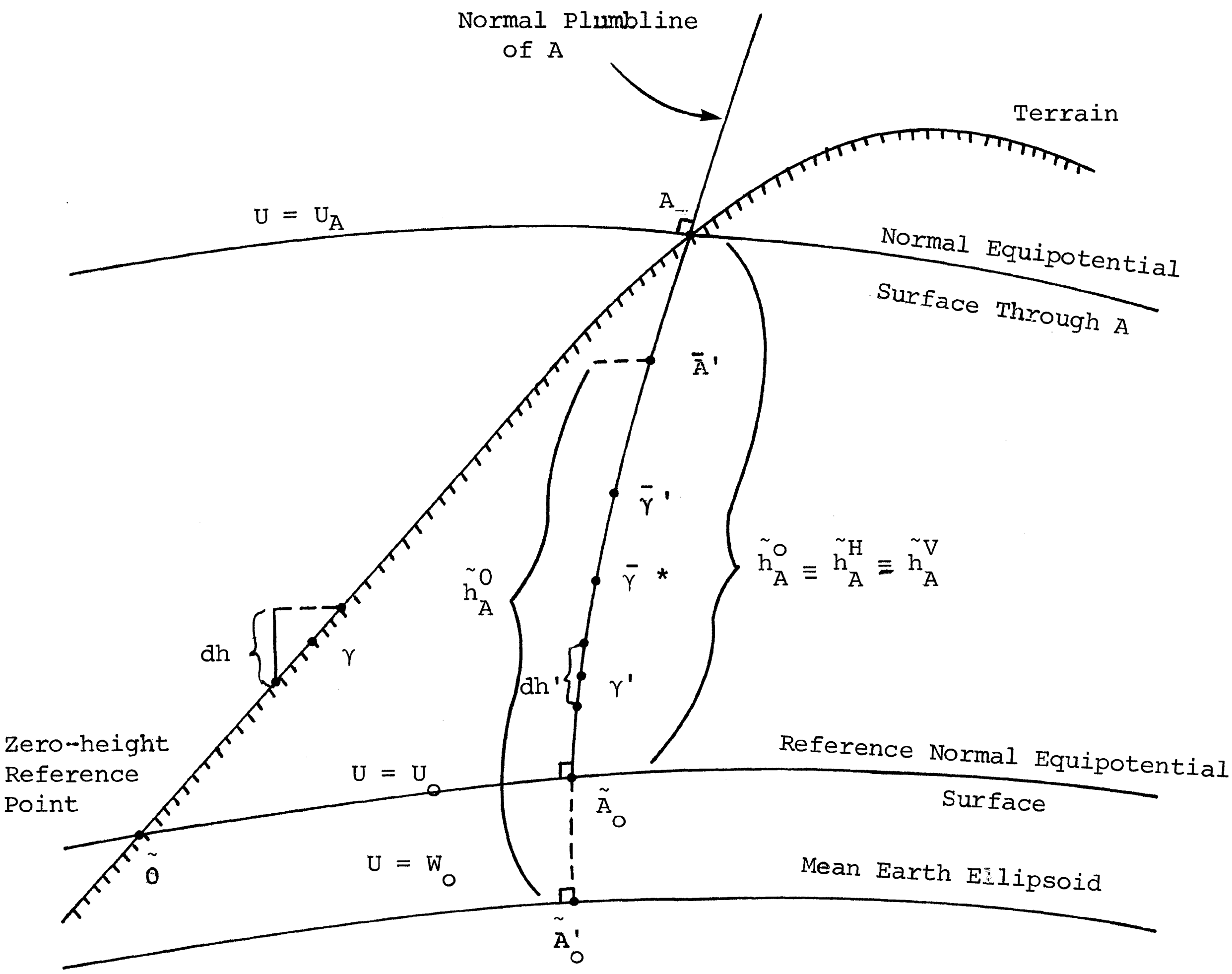


FIGURE 3-1

Geopotential Number and Orthometric Heights

Based on Normal Gravity

the surface of the ellipsoid, using for instance equation (2-36), is propagated upward to the terrain by a negative free-air correction, e.g. [Heiskanen and Vening-Meinesz, 1958].

In Canada, the formula for γ_o differs slightly from the 1967 International formula [Cannon, 1929; GSC, 1960; Jones, 1973]. This formula was developed by the United States Coast and Geodetic Survey (USC&GS) as early as 1907 [Bowie and Avers, 1914; Rappleye, 1948], and subsequently adopted by the Geodetic Survey of Canada (GSC) [Cannon, 1929]. The USC&GS formula for normal gravity has the same form as the Cassini's formula (see e.g. [Heiskanen and Moritz, 1967]) but slightly different coefficients. It reads:

$$\gamma^*_{o,A} = \gamma^*_{o,45°} \, [1 - \alpha \cos 2\phi_A + \beta \cos^2 2\phi_A], \tag{3-2}$$

where $\gamma^*_{o,A}$ is the normal gravity at the ellipsoid surface computed for the latitude ϕ_A and $\gamma^*_{o,45°}$ is the adopted value for the normal gravity on the ellipsoid at latitude of 45° - used also as the reference gravity for height computations (corresponding to G in the previous chapter) - and is given by:

$$\gamma^*_{o,45°} = 980\ 624 \text{ mgal}. \tag{3-3}$$

The coefficients α and β are given by:

$$\alpha = 0.002\ 644 \text{ unitless}, \tag{3-4a}$$

$$\beta = 0.000\ 007 \text{ unitless}. \tag{3-4b}$$

The use of the asterisk (*) as a superscript for γ is necessary, in this context, to distinguish between the normal gravity γ computed from the 1967 formula (equation 2-36) and the USC&GS formula (equation 3-2).

It might be of interest to mention here that the value of α, as given above, was also used to compute the orthometric correction based on normal gravity for the Polish levelling [Kowalczyk, 1968].

Reformulating equation (3-2) to get it in the same form as equation (2-36), the following result is obtained (see Appendix I for details):

$$\gamma^*_{o,A} = 978038.095\ [1 + 0.005\ 302\ \sin^2 \phi_A - 0.000\ 007\ \sin^2 2\ \phi_A)\text{mgal}. \tag{3-5}$$

The evaluation of the difference in the normal gravity computed from the two formulae (2-36) and (3-5) for different latitudes is then easy and the results are shown in Appendix I. For Canadian latitudes, the differences are between -5.005 mgal (at latitude 47°N) and -5.843 mgal (at latitude 82°N), always negative. The effect of these differences on the computed height differences will be discussed in the next chapter.

The USC&GS has also developed a formula for the normal gravity γ^*_A propagated from the surface of the ellipsoid to the terrain point A. It is given by Bowie and Avers [1914] and reads:

$$\gamma^*_A = \gamma^*_{o,A} - \gamma^*_{0,45°}\ K\ h_A\ , \tag{3-6}$$

where the second term on the RHS is termed as the "normal gravity correction" (analogous to the free-air correction). It accounts for the decrease of gravity with height. h_A is the levelled height of A in metres and K is given by:

$$K = K'\ [1 + d\ \cos 2\phi_A - c\ h_A], \tag{3-7}$$

where: $K' = 3.147\ .\ 10^{-7}$ unitless, (3-8a)

$d = 7.1\ .\ 10^{-4}$ unitless, (3-8b)

and: $c = 2.3 \cdot 10^{-7}$ unitless, (3-8c)

in which h_A has to be, again, in metres.

Substituting then γ^*_A, from equation (3-6), for γ' in equation (3-1) and performing the integration along the normal plumbline of A, the following expression for $\tilde{C}_A$ is obtained:

$$\tilde{C}_A = \int_0^{\tilde{h}^o_A} \gamma^*_A \, dh' = \gamma^*_{o,45°} \, [(1 - \alpha \cos 2\phi_A + \beta \cos^2 2\phi_A) \, \tilde{h}^o_A - \frac{K' (\tilde{h}^o_A)^2}{2} (1 + d \cos 2\phi_A - \frac{2}{3} c\tilde{h}^o_A)] , \quad (3\text{-}9)$$

where $\tilde{h}^o_A$ is the orthometric height of point A based on normal gravity (see Figure 3-1). The geopotential number difference $\Delta\tilde{C}_{AB}$ (based on normal gravity) between the two terrain points A and B can be computed from the following formula:

$$\Delta\tilde{C}_{AB} \doteq \sum_{i=1}^{B-1} \bar{\gamma}^*_{ij} \, \Delta h_{ij} , \; j = i + 1, \quad (3\text{-}10)$$

where: $\bar{\gamma}^*_{ij} = \frac{1}{2} (\gamma^*_i + \gamma^*_j)$ and Δh_{ij} is the levelled height difference between i and j. Here, γ^*_i and γ^*_j are computed using equation (3-6).

3.2 USC&GS Orthometric Heights

The orthometric height (based on normal gravity) $\tilde{h}^o_A$ of a terrain point A is defined [Krakiwsky, 1965] as the distance measured along the normal plumbline of A between the two normal equipotential surfaces: $U = U_A$ and $U = U_o$, as illustrated in Figures 3-1. Hence $\tilde{h}^o_A$ can be computed from:

$$\tilde{h}^o_A = \frac{\tilde{C}_A}{\bar{\gamma}'} . \quad (3\text{-}11a)$$

where $\tilde{C}_A$ is given by equation (3-1) and $\bar{\gamma}'$ is the mean value of normal gravity along the normal plumbline of A between A and $\tilde{A}_o$ (see Figure 3-1).

On the other hand, the USC&GS definition for $\tilde{C}_A$ is given by equation (3-9). This implies [Krakiwsky, 1965] that $\bar{\gamma}'_A$ can be sufficiently approximated by the mean value of normal gravity $\bar{\gamma}'^*_A$ along the plumb-line of A between the geocentric ellipsoid (point $\tilde{A}_o$) and a point, $\tilde{A}$, $\tilde{h}^o_A$ above the ellipsoid. Hence the expression for $\bar{\gamma}'_A$ will be:

$$\bar{\gamma}'_A \doteq \bar{\gamma}'^*_A = \gamma^*_{o,A} - 0.3086 \left(\frac{h_A}{2}\right) , \qquad (3\text{-}11b)$$

where $\gamma^*_{o,A}$ is given by equation (3-5). The resulting $\tilde{h}^o_A$ in this case is referred to here as the "USC&GS orthometric height".

According to Krakiwsky [1965, page 109], the orthometric height based on normal gravity as defined above, was formulated by Helmert. Thus, $\tilde{h}_A^{\,o}$ may be also denoted by $\tilde{h}_A^{\,H}$. In addition, it can be seen that $\bar{\gamma}'^*$, as defined by equation (3-11b), is practically equivalent to Vignal definition of $\bar{\gamma}'^V$, as given by equation (2-27). As a result, the following concluding statement can be made. The USC&GS (Helmert) definition of orthometric height $\tilde{h}^O$ ($\tilde{h}^H$ based on normal gravity) is equivalent to Vignal definition of normal height $\tilde{h}^V$ (based on normal gravity). Both definitions will be denoted here by $\tilde{h}^o$ (Figure 3-1). Consequently, one can see that Vignal normal heights based on normal gravity have been already used in North America.

The orthometric height difference $\Delta\tilde{h}^o_{AB}$ between points A and B is in practice computed from the following formula:

$$\Delta\tilde{h}^{\,o}_{AB} = \Delta h_{AB} + \tilde{OC}_{AB} , \qquad (3\text{-}12)$$

where $\tilde{OC}_{AB}$ is called the "orthometric correction based on normal gravity" and Δh_{AB} is the levelled height difference.

The expression for $\tilde{OC}_{AB}$, used by the GSC [Cannon, 1929; GSC, 1960; Konecny, 1965; Laflamme, 1971; Jones, 1973], is the one developed by Bowie and Avers [1914]. It can be derived in the following manner. Firstly, it is known that $\tilde{C}_A$ is constant for all points located on the same normal equipotential surface $U = U_A$. Thus, an equation of the normal equipotential surface going through A can be written using equation (3-9) as:

$$\tilde{h}^o_A - (\alpha \cos 2\phi_A - \beta \cos^2 2\phi_A)\, \tilde{h}^o_A - K' \frac{(\tilde{h}^o_A)^2}{2} (1 + d \cos 2\phi_A - \frac{2}{3} c\tilde{h}^o_A) = $$
$$= \text{const.} \qquad (3\text{-}13)$$

The next step is to differentiate equation (3-13) and to neglect terms smaller than the errors in levelling. According to Bowie and Avers [1914, page 52], the resulting expression for the differential change of the orthometric height at point A reads:

$$d\tilde{h}^o_A = -2\tilde{h}^o_A\, \alpha \sin 2\phi_A \,[1 + (\alpha - \frac{2\beta}{\alpha}) \cos 2\phi_A]\, d\phi_A \,. \qquad (3\text{-}14)$$

In the differential environment of the point A, the variation $d\tilde{h}^o_A$ can be considered equivalent to the difference between the observed height and the orthometric height, because the observed height difference is "sufficiently close" to the geopotential number difference (up to a scale factor). Hence, one can write:

$$d\tilde{h}^o_A = d\, \tilde{OC}_A, \qquad (3\text{-}15)$$

where $\tilde{OC}$ is the orthometric correction (based on normal gravity). Assuming the validity of equations (3-14) and (3-15), even for finite differences $\Delta\tilde{h}^o$, $\Delta\phi$ and $\Delta\tilde{OC}$, the final expression for the orthometric correction is obtained as:

$$\tilde{OC}_{ij} = - 2\bar{h}_{ij}\, \alpha \sin 2\bar{\phi}_{ij} \,[1 + (\alpha - \frac{2\beta}{\alpha}) \cos 2\bar{\phi}_{ij}]\, \Delta\phi_{ij} \,. \qquad (3\text{-}16)$$

It is valid only for individual levelling sections (as defined in section 2.2) between each pair of consecutive bench marks i and j. Here the quantities $\bar{h}_{ij}$ (the average levelled height); $\bar{\phi}_{ij}$ (the average latitude) and $\Delta\phi_{ij}$ (the difference in latitude) between points i and j are computed from:

$$\bar{h}_{ij} = \frac{1}{2}(h_i + h_j); \qquad (3\text{-}17a)$$

$$\bar{\phi}_{ij} = \frac{1}{2}(\phi_i + \phi_j); \qquad (3\text{-}17b)$$

$$\Delta\phi_{ij} = \phi_j - \phi_i. \qquad (3\text{-}17c)$$

The coefficients α and β are defined by equations (3-4a) and (3-4b). When dealing with an entire levelling line between points A and B, for instance the one given by equation (3-12), the total orthometric correction for the line is given by:

$$\tilde{OC}_{AB} = \sum_{i=1}^{B-1} \tilde{OC}_{ij} \quad . \qquad (3\text{-}18)$$

It is now worthwhile to examine equation (3-16). It can be easily seen that the orthometric correction, $\tilde{OC}_{ij}$, is mainly dependent on the latitudes of the two points i and j and goes to zero for any two points on the same latitude. On the other hand, it is known that the observed height difference, Δh_{ij}, is path (route) dependent, as explained in section 1.4, which of course holds true even for the two points of the same latitude. Therefore, it can be concluded that the orthometric heights computed from the orthometric correction based on normal gravity are generally not unique, and also route dependent.

In their publication, Bowie and Avers [1914] provide tables to simplify the computation of $\tilde{OC}_{ij}$. For this purpose, they rewrote equation (3-16) in the following form:

$$\tilde{OC}_{ij} = - C \bar{h}_{ij} \quad \Delta\phi_{ij}, \qquad (3\text{-}19)$$

where

$$C = + 2 \alpha \sin 2\bar{\phi}_{ij} \; [1 + (\alpha - \frac{2\beta}{\alpha}) \cos 2\bar{\phi}_{ij}] \qquad (3\text{-}20)$$

is obtained from the aforementioned tables, arranged with $\bar{\phi}_{ij}$ as the argument to the nearest tenth of a degree of arc. Using C from the tables, $\Delta\phi_{ij}$ has to be expressed in minutes of arc and the units of $\tilde{OC}_{ij}$ are the same as the units of $\bar{h}_{ij}$ [e.g. Konecny, 1965; LaFlamme, 1971].

3.3 Reformulation of the USC&GS Orthometric Correction

The purpose of this section is to reformulate equation (3-16), for the USC&GS orthometric correction, $\tilde{OC}_{ij}$ (based on normal gravity), so as to make it suitable for the subsequent developments.

Differentiating the USC&GS formula for normal gravity on the ellipsoid, i.e. equation (3-2), with respect to ϕ, gives:

$$\frac{d\gamma^*_o}{d\phi} = \gamma^*_{o,45°} \; (2\alpha \sin 2\phi - 4\beta \cos 2\phi \sin 2\phi) \; . \qquad (3\text{-}21)$$

By considering a levelling section between two points i and j, and replacing differentials by finite differences, equation (3-21) can be rewritten as:

$$\frac{\Delta\gamma^*_{o,ij}}{\gamma^*_{o,45°} \; \Delta\phi_{ij}} = 2 \alpha \sin 2\bar{\phi}_{ij} - 4\beta \sin 2\bar{\phi}_{ij} \cos 2\bar{\phi}_{ij}, \qquad (3\text{-}22)$$

where:

$$\Delta\gamma^*_{o,ij} = \gamma^*_{o,j} - \gamma^*_{o,i} \quad , \qquad (3\text{-}23)$$

and the remaining symbols are as defined earlier. From equation (3-16) the following expression can be obtained:

$$\frac{\tilde{OC}_{ij}}{\bar{h}_{ij}\ \Delta\phi_{ij}} = -\ 2\ \alpha\ \sin\ 2\bar{\phi}_{ij} - 2\alpha^2\ \sin\ 2\bar{\phi}_{ij}\ \cos\ 2\bar{\phi}_{ij} + \\ + 4\beta\ \sin\ 2\bar{\phi}_{ij}\ \cos\ 2\bar{\phi}_{ij}. \tag{3-24}$$

Combining equations (3-22) and (3-24), one gets:

$$\frac{\tilde{OC}_{ij}}{\bar{h}_{ij}\ \Delta\phi_{ij}} = -\ \frac{\Delta\gamma^*_{o,ij}}{\gamma^*_{o,45°}\ \Delta\phi_{ij}} - 2\alpha^2\ \sin\ 2\ \bar{\phi}_{ij}\ \cos\ 2\bar{\phi}_{ij}. \tag{3-25}$$

For reasons mentioned at the beginning of section 3.1, $\gamma^*_{o,45°}$ will be referred to from now on as the reference gravity, and the symbol "G" will be used for it as in the previous Chapter. Further, realizing that $2\ \sin\theta\ \cos\theta = \sin\ 2\theta$, equation (3-25) becomes

$$\tilde{OC}_{ij} = -\ \frac{\bar{h}_{ij}}{G}\ \Delta\gamma^*_{o,ij} - \bar{h}_{ij}\ \Delta\phi_{ij}\ \alpha^2\ \sin\ 4\bar{\phi}_{ij}\ . \tag{3-26}$$

A closer look at the second term on the RHS of equation (3-26) should now be taken. To compute its magnitude, an extreme case of $\bar{\phi}_{ij} = 67\overset{\circ}{.}5$, $\Delta\phi_{ij} = 1$ arcmin (corresponding to $\simeq$ 2 km), and $\bar{h}_{ij} = 2$ km, is considered. Then, $\sin\ 4\ \bar{\phi}_{ij}$ will equal to -1 and the sought numerical value of the second term will be approximately -0.0036 mm. Comparing this result to the expected (specified) accuracy of precise levelling which is estimated to be between 0.5 to 1.5 mm per 1 km [Peterson, 1970; Boal, 1971a,b; Holdahl, 1974; U.S. Dept. of Commerce, 1974], it can be easily seen that the contribution of the second term on the RHS of equation (3-26) is generally negligible. Accordingly, equation (3-26) becomes:

$$\tilde{OC}_{ij} = -\ \frac{\bar{h}_{ij}}{G}\ \Delta\gamma^*_{o,ij}\ . \tag{3-27}$$

Recalling that $\bar{h}_{ij} = h_i + \frac{1}{2} \Delta h_{ij}$ and $\Delta\gamma^*_{o,ij} = \gamma^*_{o,j} - \gamma^*_{o,i}$, and then substituting these relations into equation (3-27), the sought reformulated expression for $\tilde{OC}_{ij}$ can be finally written as:

$$\tilde{OC}_{ij} = \frac{1}{G} [h_i (\gamma^*_{o,i} - \gamma^*_{o,j}) + \Delta h_{ij} (\bar{\gamma}^*_{o,ij} - \gamma^*_{o,j})] , \tag{3-28}$$

where

$$\bar{\gamma}^*_{o,ij} = \frac{1}{2} (\gamma^*_{o,i} + \gamma^*_{o,j}) . \tag{3-29}$$

By comparison, one can see that equation (3-29), for the USC&GS orthometric correction $\tilde{OC}_{ij}$ (based on normal gravity) has the same form as equation (2-19) for the Helmert orthometric correction HC_{ij} (based on actual gravity), and as equation (2-35) for the Vignal normal correction VC_{ij} (based on actual gravity).

3.4 USC&GS Dynamic Heights

The dynamic height $\tilde{h}_A^D$ (based on normal gravity) of a terrain point A is defined [Vykutil, 1964; Krakiwsky, 1965; Mueller and Rockie, 1966] as:

$$\tilde{h}_A^D = \frac{\tilde{C}_A}{\tilde{G}} , \tag{3-30}$$

where $\tilde{C}_A$ is the geopotential number of A, based on normal gravity (defined by equation (3-9) for the USC&GS), and $\tilde{G}$ is the "reference gravity" taken as the normal gravity on the geocentric ellipsoid, i.e. $\gamma_{o,R}$, for the adopted reference latitude ϕ_R. Recall that the USC&GS definition, used also in Canada, uses $\phi_R = 45°$ and consequently, $\tilde{G} = \gamma^*_{0,45} = G$ (as stated before) whose numerical value is given by equation (3-3). Combining equations (3-9) and (3-30), the USC&GS dynamic height (the term "dynamic number" is frequently used for it) $\tilde{h}_A^D$ is given as:

$$\tilde{h}_A^D = \tilde{h}_A^o - (\alpha \cos 2\phi_A - \beta \cos^2 2\phi_A)\, \tilde{h}_A^o - $$
$$ -K' \frac{(\tilde{h}_A^o)^2}{2} (1 + d \cos 2\phi_A - \frac{2}{3} c\tilde{h}_A^o), \qquad (3\text{-}31)$$

in which the last two terms represent a "dynamic correction" to the orthometric height $\tilde{h}_A^o$. This situation explains the reason for discussing the USC&GS orthometric heights (section 3.2) before their corresponding dynamic heights.

Bowie and Avers [1914] provide again tables to simplify the computation of dynamic heights from equation (3-31). For such purpose, they rewrote equation (3-31) in the following form:

$$\tilde{h}_A^D = \tilde{h}_A^o - D_1 h_A - D_2\, (h_A)^2 , \qquad (3\text{-}32)$$

where:

$$D_1 = (\alpha \cos 2\phi_A - \beta \cos^2 2\phi_A) , \qquad (3\text{-}33a)$$

and

$$D_2 = \frac{K'}{2} (1 + d \cos 2\phi_A - \frac{2}{3} ch_A) . \qquad (3\text{-}33b)$$

D_1 and D_2 are obtained from the tables for arguments ϕ_A and (ϕ_A, h_A), respectively. It can be noticed that the orthometric height $\tilde{h}_A^o$ is replaced, in the above formulae, by the corresponding levelled height h_A. This approximation is justified because it is used into corrective terms only.

Considering a levelling section between two points i and j, the dynamic height difference $\Delta\tilde{h}_{ij}^D$ between them is given by (using equation 3-32):

$$\Delta\tilde{h}_{ij}^D = \tilde{h}_j^D - \tilde{h}_i^D = (\tilde{h}_j^o - \tilde{h}_i^o) - D_1^{ij}\, (h_j - h_i) - D_2^{ij}\, (h_j^2 - h_i^2), \qquad (3\text{-}34)$$

where D_1^{ij} and D_2^{ij} are the average values of D_1 and D_2 computed from equations (3-33a) and (3-33b) by putting $\phi_A = \frac{1}{2} (\phi_i + \phi_j)$ and $h_A = \frac{1}{2} (h_i + h_j)$.

In order to compute the dynamic height difference, $\Delta \tilde{h}_{ij}^D$, from the levelled height difference, Δh_{ij}, the "dynamic correction" $\tilde{DC}_{ij}$ (based on normal gravity), has to be added to Δh_{ij}, that is:

$$\Delta \tilde{h}_{ij}^D = \Delta h_{ij} + \tilde{DC}_{ij} \quad . \tag{3-35}$$

The USC&GS formula for $\tilde{DC}_{ij}$ is stated by Bowie and Avers [1914] as:

$$\tilde{DC}_{ij} = - (D_1^{ij} + K \, \bar{h}_{ij}) \, \Delta h_{ij} \quad , \tag{3-36}$$

where $K = 3.147 \, . \, 10^{-7}$ and the other symbols are defined above. D_1^{ij} can be again obtained from Bowie and Avers tables for argument $\bar{\phi}_{ij}$. The units of $\tilde{DC}_{ij}$ will be the same as those of $\bar{h}_{ij}$ and Δh_{ij} e.g. metres [Konecny, 1965].

Substituting equation (3-36) into equation (3-35) and comparing the result to equation (3-34), it can be seen that Bowie and Avers [1914] have made the following approximations:

$$\tilde{h}_j^o - \tilde{h}_i^o \doteq h_j - h_i = \Delta h_{ij} \quad , \tag{3-37a}$$

$$D_2 \doteq K \, / \, 2 \quad . \tag{3-37b}$$

These approximations imply an error in computing $\tilde{DC}_{ij}$ from equation (3-36) of the order of the orthometric correction (based on normal gravity). This fact may explain why this approach is not used in practice. In Canada, the dynamic heights (based on normal gravity) are computed by adding the dynamic correction to the corresponding orthometric heights using equation (3-32) and Bowie and Avers tables [Cannon, 1929; GSC, 1960; Dohler, 1970]. However, as already explained in the previous section, it can be again seen that dynamic heights based on normal gravity are not generally unique, but route dependent. The system of USC&GS dynamic heights has been used exclusively in the establishment of the International Great Lakes Datum of 1955 (IGLD-55) [Coordinating Committee on the IGLD-55, 1961; Ramsayer, 1965b; Ropes, 1965].

3.5 Reformulation of the USC&GS Dynamic Correction

The main idea in this section is to obtain an alternative form for the USC&GS dynamic correction applied in Canada, ready to be used in the subsequent developments.

First, equation (3-32) for dynamic height is rewritten for the point A as:

$$\tilde{h}_A^D = \tilde{h}_A^o - (D_1 + D_2 h_A)\, h_A \; . \tag{3-38}$$

Then, from equations (3-2) and (3-33a), the quantity D_1 can be expressed as:

$$D_1 = \frac{G - \gamma^*_{o,A}}{G} \; . \tag{3-39}$$

At this point, one of Bowie and Avers approximations given by equation (3-37b) can be introduced. It can be seen that such approximation is equivalent to taking $\frac{2}{3}\, ch_A \doteq ch_A$ in equation (3-33b). The effect of this on the computed $\tilde{h}_A^D$ will be of the order of 0.01 mm per 1 km of height which appears to be admissible. From equation (3-6), the following expression for K can be obtained:

$$K = \frac{\gamma^*_{o,A} - \gamma^*_A}{G\ h_A} , \qquad (3\text{-}40)$$

and consequently:

$$D_2 \doteq \frac{K}{2} = \frac{\gamma^*_{o,A} - \gamma^*_A}{2\ G\ h_A} . \qquad (3\text{-}41)$$

Substituting equations (3-39) and (3-41) into equation (3-38) gives:

$$\tilde{h}^D_A = \tilde{h}^o_A - h_A + \frac{h_A}{2G} (\gamma^*_{o,A} + \gamma^*_A) . \qquad (3\text{-}42)$$

Considering a levelling section between two points i and j, a similar expression to (3-42) can be written for $\tilde{h}^D_i$ and $\tilde{h}^D_j$. The dynamic height difference $\Delta\tilde{h}^D_{ij}$ between them can be hence obtained as:

$$\Delta\tilde{h}^D_{ij} = \tilde{h}^D_j - \tilde{h}^D_i = \Delta\tilde{h}^o_{ij} - \Delta h_{ij} + \frac{1}{2G} [h_j\ (\gamma^*_{o,j}, + \gamma^*_j) - h_i\ (\gamma^*_{o,i} + \gamma^*_i)] . \qquad (3\text{-}43)$$

Realizing that $\Delta\tilde{h}^o_{ij}$ is given by equation (3-12) as: $\Delta\tilde{h}^o_{ij}=\Delta h_{ij}+\tilde{OC}_{ij}$, where $\tilde{OC}_{ij}$ is expressed by equation (3-27), equation (3-43) can be rewritten as:

$$\Delta\tilde{h}^D_{ij} = \Delta h_{ij} + \{- \frac{h_{ij}}{G} \Delta\gamma^*_{o,ij} - \Delta h_{ij} + \frac{1}{2G} [h_j\ (\gamma^*_{o,j} + \gamma^*_j) -$$

$$- h_i\ (\gamma^*_{o,i} + \gamma^*_i)] \} . \qquad (3\text{-}44)$$

Comparing equations (3-44) and (3-35) reveals that the entire second term on the RHS of (3-44) can be regarded as the dynamic correction $\tilde{DC}_{ij}$ (based on normal gravity) to the observed elevation difference Δh_{ij}. It involves only minor approximations and is consistent with the approach adopted by Canada.

Accordingly, $\tilde{DC}_{ij}$ will be given by:

$$\tilde{DC}_{ij} = \frac{1}{G} \{- \bar{h}_{ij} \, \Delta\gamma^*_{o,ij} + \frac{1}{2} [h_j \, (\gamma^*_{o,j} + \gamma^*_j) - h_i \, (\gamma^*_{o,i} + \gamma^*_i)]\} - - \Delta h_{ij} . \tag{3-45}$$

Recalling that the difference between the normal gravity on the ellipsoid and its corresponding value on the terrain can be approximated (consistent with the USC&GS approach for computing the geopotential numbers based on normal gravity) by the free-air correction [see, e.g. Heiskanen and Vening-Meinesz, 1958]. Hence, one can write the following relation at point i:

$$\gamma^*_i = \gamma^*_{o,i} - 0.3086 \, h_i \, . \tag{3-46}$$

A similar expression to (3-46) can be written for point j, and subsequent substitution in equation (3-45) leads finally to the sought reformulated expression for $\tilde{DC}_{ij}$ that reads:

$$\tilde{DC}_{ij} = \frac{\bar{\gamma}^*_{ij}}{G} \, \Delta h_{ij} - \Delta h_{ij} \quad , \tag{3-47}$$

where:

$$\bar{\gamma}^*_{ij} = \bar{\gamma}^*_{o,ij} - 0.3086 \, \bar{h}_{ij} \, , \tag{3-48}$$

and the remaining symbols are as defined earlier.

The comparison of equation (3-47) for the USC&GS dynamic correction $\tilde{DC}_{ij}$ (based on normal gravity), and equation (2-10) for the dynamic correction DC_{ij} (based on actual gravity), reveals that both are in the same form.

CHAPTER 4

CORRECTIONS TO HEIGHTS BASED ON NORMAL GRAVITY (DUE TO THE IRREGULARITIES OF ACTUAL GRAVITY FIELD)

To begin with, we recall that we have seen two approaches leading to the heights based on observed gravity. The first is by computing the actual geopotential numbers and transforming them to heights by dividing by the appropriate gravity value. This approach was proposed for the USA by Krakiwsky and Meuller [1966] and by Mueller et al. [1968], and in Canada by Vaníček et al. [1972]. The second approach is based on correcting the levelled height differences by adding corrections based on actual gravity. These corrections differ according to the particular height system adopted, i.e. Dynamic, Helmert or Vignal, as we have seen in chapter 2.

The problem on hand is somewhat different. Here, old established, already existing, levelling lines and loops which have been computed on the basis of normal gravity only are considered. This, in fact, means that only the local actual gravity irregularities are not taken into account. In particular, we shall focus on the heights currently used in Canada, i.e. dynamic, $\tilde{h}^D$, and orthometric, $\tilde{h}^O$.

In this chapter, the first main objective of this study is attained. Corrections, to the used dynamic and orthometric heights, reflecting the effect of the neglected gravity irregularities ("Gravity corrections", to be defined in section 4.1) are formulated. The development of these corrections will be given (in section 4.2) for each of the three rigorous height systems in question, Dynamic h^D, Helmert h^H and Vignal h^V, according to the following scheme. With the appropriate gravity correction, for each case, one can obtain Δh^D from $\Delta\tilde{h}^D$ and both Δh^H and Δh^V from $\Delta\tilde{h}^O$.

The computations involving the rigorous expressions are practically possible only with the aid of computers. Therefore, section 4.3 of this chapter will contain an attempt to simplify the rigorous expressions for the gravity corrections to suit the desk or pocket calculator computations. Finally, section 4.4 gives the expressions for the estimated standard deviations (precisions) of the formulated gravity corrections, again for each of the three height systems under investigation.

4.1 Definition and Motivation

The "Gravity Correction" GC, as formulated here (for only one levelling section, as described in section 2.2, and a particular height system), is explicitly defined as the correction or influence due to the neglected actual gravity irregularities, as applied to the corresponding height difference $\Delta\tilde{h}$ presently used in Canada. The algebraic addition of the computed GC and the existing $\Delta\tilde{h}$ will produce the corresponding rigorous height difference Δh, appropriately based on actual gravity. This definition holds true for each of the three height systems under consideration.

We recall that the concept of defining the heights on the basis of normal gravity, as adopted in Canada, implies that instead of using the observed gravity value g on the surface of the earth, an approximate value γ is adopted. This γ is the normal gravity value γ_o computed on the geocentric ellipsoid and then propagated (using Bowie and Avers gradient of normal gravity) to the height h above the ellipsoid. Accordingly, it can be realized that the neglected difference $g - \gamma$, at each bench mark, which has been referred to, here so far, as the actual gravity irregularity, is nothing else but basically the corresponding free-air gravity anomaly Δg^F (based on the free-air gradient of gravity) plus other minor terms (coming from the differences between the involved gravity gradients, and negligible compared to Δg^F). This follows from the definition of the free-air anomaly [e.g. Heiskanen and Vening-Meinesz, 1958; Mueller and Rockie, 1966] as the difference between the actual gravity on a geopotential surface and the normal gravity on the corresponding normal equipotential surface.

Thus, the free-air gravity anomaly is naturally one of the independent variables in the formula for the gravity correction. Expressing the influence of gravity irregularities on heights in terms of free-air anomalies was also found convenient in practice, and used by several authors [e.g. Bursa, 1958; Schneider, 1960; Weidauer, 1963; Vykutil, 1964] who investigated Molodenskii's normal heights based on actual gravity.

In addition to the above logical motivation, the following reasons are considered, in the author's opinion (based on the geophysical, geodetic and practical computation viewpoints), as justifications for the choice of the free-air anomalies, Δg^F, instead of, for instance, the corresponding absolute values of the observed gravity, g:

1. From the geophysical point of view, Δg^F is linked with the mass irregularities. It is simply a measure of gravitational difference between the irregular mass distribution within the real earth and the regular mass distribution within the normal earth (mean earth ellipsoid) [Wilcox, 1974]. Hence, Δg^F should reflect the local circumstances under which the levelling instrument (level) was performing, e.g. the effect of some local anomalous mass;
2. Δg^F is usually available within any area covered by point gravity data. This is because the free-air gravity anomaly is, by far, the most widely used anomaly for geodetic purposes, due to its simplicity and advantages over other types of anomalies [Heiskanen and Moritz, 1967; Vaníček, 1972];
3. One of the author's goals in the present investigation is to provide tables to facilitate the practical computations of the formulated gravity corrections, such that they can be used anywhere even in the field. Such stipulation requires the availability of a gravity map to perform a graphical interpolation of gravity data at the bench mark of interest. The production of gravity maps is considered one of the main features of any well-designed gravity data processing system (see, e.g. section 5.2.3). In practice, however, the graphical representation of the earth's gravity field within an area is customarily depicted by the gravity anomaly maps (either Bouguer or free-air), and not by the observed point gravity values as such. Anomaly contour maps with 5 mgal contour interval have been already used by several researchers dealing with heights and gravity, e.g. [Schneider, 1960; Rapp, 1961; Krakiwsky, 1966].

Nowadays, the high speed computers and automatic plotting machines make it possible to compile local gravity anomaly maps with any desired contour interval (e.g. 1 mgal) [Derenyi, 1965; Konecny, 1970; Wilcox et al., 1974; Estes; 1975; Nagy, 1976]. Even in certain circumstances where the available anomaly contour maps are of Bouguer type, the transformation from Bouguer to free-air anomaly value is straight forward and simple [Vykutil, 1964].

4.2 Rigorous Expressions for the Gravity Corrections

The gravity correction as described in the previous section is the difference between the height difference based on actual gravity and the corresponding height difference based on normal gravity. This is exactly equivalent to the difference between the correction to the levelled height difference based on actual gravity and the corresponding correction based on normal gravity. The latter approach will be used in the subsequent developments.

4.2.1 Dynamic Gravity Correction

The dynamic gravity correction GC^D_{ij} to the height difference $\tilde{\Delta h}^D_{ij}$ (based on normal gravity) of a levelling section between i and j is defined as:

$$GC^D_{ij} = \Delta h^D_{ij} - \tilde{\Delta h}^D_{ij} \quad . \tag{4-1}$$

Alternatively, substitution from equation (2-7) and (3-35) into the above equation gives:

$$GC^D_{ij} = DC_{ij} - \tilde{DC}_{ij} \quad . \tag{4-2}$$

Here the actual dynamic correction DC_{ij} is given by equation (2-10), i.e.:

$$DC_{ij} = \frac{\bar{g}_{ij}}{G} \Delta h_{ij} - \Delta h_{ij} \ . \tag{4-3}$$

The dynamic correction $\tilde{DC}_{ij}$ (based on normal gravity) is given by equation (3-47), i.e.:

$$\tilde{DC}_{ij} = \frac{\bar{\gamma}^{*}_{ij}}{G} \Delta h_{ij} - \Delta h_{ij} \ . \tag{4-4}$$

Substituting into equation (4-2), the dynamic gravity correction becomes:

$$GC^{D}_{ij} = \frac{\Delta h_{ij}}{G} [\bar{g}_{ij} - \bar{\gamma}^{*}_{ij}] \ . \tag{4-5}$$

To express the RHS of equation (4-5) in terms of the free-air gravity anomalies, we first define the free-air anomaly Δg^{F}. Δg^{F}_{i} at the terrain point i is defined [e.g.: Heiskanen and Vening-Meinesz, 1958; Vaníček, 1972] as:

$$\Delta g^{F}_{i} = g_{i} + 0.3086 \ h_{i} - \gamma_{o,i} \ , \tag{4-6}$$

in mgal, where g_i is the observed gravity on the terrain in mgal, and h_i is the levelled height in metres. Here, $\gamma_{o,i}$ is the normal gravity on the mean earth ellipsoid based, for instance, on the 1967 International formula (equation 2-36), in mgal. A similar expression to (4-6) can be written for point j. Consequently $\bar{g}_{ij}$ in equation (4-5) can be expressed as:

$$\bar{g}_{ij} = \overline{\Delta g}^{F}_{ij} + \bar{\gamma}_{o,ij} - 0.3086 \ \bar{h}_{ij} \ , \tag{4-7}$$

where:

$$\overline{\Delta g}^F_{ij} = \frac{1}{2} (\Delta g^F_i + \Delta g^F_j) \quad , \tag{4-8}$$

$$\bar{\gamma}_{o,ij} = \frac{1}{2} (\gamma_{o,i} + \gamma_{o,j}) \quad , \tag{4-9}$$

$$\bar{h}_{ij} = \frac{1}{2} (h_i + h_j) \quad . \tag{4-10}$$

Next, by substituting from equations (4-7) and (3-48) into equation (4-5), one gets:

$$GC^D_{ij} = \frac{\Delta h_{ij}}{G} [\overline{\Delta g}^F_{ij} + (\bar{\gamma}_{o,ij} - \bar{\gamma}^*_{o,ij})] \quad , \tag{4-11}$$

in which $\overline{\gamma^*_{o,ij}}$ is the average value of the normal gravity on the geocentric ellipsoid as computed from the USC&GS formula given in equation (3-2).

The difference between the normal gravity based on the 1967 International formula and the corresponding value based on the USC&GS formula for the same point i can be denoted (Appendix I) as:

$$\delta\gamma_{o,i} = \gamma_{o,i} - \gamma^*_{o,i} \quad . \tag{4-12}$$

Similar expression for $\delta\gamma_{o,j}$ can be written at point j. Then, the substitution into equation (4-11) leads finally to the sought rigorous formula for the dynamic gravity correction that reads:

$$GC^D_{ij} = \frac{\Delta h_{ij}}{G} [\overline{\Delta g}^F_{ij} + \overline{\delta\gamma}_{o,ij}] \quad , \tag{4-13}$$

where:

$$\overline{\delta\gamma}_{o,ij} = \bar{\gamma}_{o,ij} - \bar{\gamma}^*_{o,ij} = \frac{1}{2} (\delta\gamma_{o,i} + \delta\gamma_{o,j}) \quad . \tag{4-14}$$

Referring to Appendix I, it can be seen that $\delta\gamma_{o,ij}$ depends on the latitudes ϕ_i and ϕ_j.

It should be kept in mind that the dynamic gravity correction GC^D_{ij}, computed from equation (4-13), is to be added to the used dynamic height difference $\tilde{\Delta h}^D_{ij}$ to get the corresponding rigorous dynamic height difference Δh^D_{ij}. The physical units of GC^D_{ij} will be metres for Δh_{ij} in metres and G, $\overline{\Delta g}^F_{ij}$, $\overline{\delta\gamma}_{o,ij}$ in mgal.

Evidently, the dynamic gravity correction given by equation (4-13) is a function of levelled heights, geographical latitudes, and free-air gravity anomalies at both ends of the levelling section. Generally, GC^D_{ij} can be written as:

$$GC^D_{ij} = f^D\ (h_i,\ h_j,\ \phi_i,\ \phi_j,\ \Delta g^F_i,\ \Delta g^F_j)\ , \qquad (4\text{-}15)$$

where f, in this context, denotes the functional relationship.

4.2.2 Helmert Gravity Correction

The Helmert gravity correction GC^H_{ij} to the height difference $\tilde{\Delta h}^o_{ij}$ (based on normal gravity) can be written as:

$$GC^H_{ij} = \Delta h^H_{ij} - \tilde{\Delta h}^o_{ij}\ , \qquad (4\text{-}16)$$

where Δh^H_{ij} is the Helmert rigorous height difference. Substitution from equations (2-13) and (3-12) into the above equation gives:

$$GC^H_{ij} = HC_{ij} - \tilde{OC}_{ij}. \qquad (4\text{-}17)$$

We recall that the Helmert or actual orthometric correction HC_{ij} is given by equation (2-19), i.e.:

$$HC_{ij} = \frac{1}{G}\ [h_i\ (\bar{g}'^H_i - \bar{g}'^H_j) + \Delta h_{ij}\ (\bar{g}_{ij} - \bar{g}'^H_j)]\ , \qquad (4\text{-}18)$$

and the orthometric correction $\tilde{OC}_{ij}$ based on normal gravity is given by equation (3-28), i.e.:

$$\tilde{OC}_{ij} = \frac{1}{G} [h_i(\gamma^*_{o,i} - \gamma^*_{o,j}) + \Delta h_{ij} (\overline{\gamma}^*_{o,ij} - \gamma^*_{o,j})] . \qquad (4-19)$$

Substituting into equation (4-17), the Helmert gravity correction becomes:

$$GC^H_{ij} = \frac{1}{G} \{h_i[(\overline{g}'^H_i - \gamma^*_{o,i}) - (\overline{g}'^H_j - \gamma^*_{o,j})] +$$

$$+ \Delta h_{ij} [(\overline{g}_{ij} - \overline{\gamma}^*_{o,ij}) - (\overline{g}'^H_j - \gamma^*_{o,j})]\} . \qquad (4-20)$$

To express the RHS of equation (4-20) in terms of the free-air anomalies, the following steps can be followed. First, $\overline{g}'^H_i$ can be expressed from equations (2-12) and (4-6), as:

$$\overline{g}'^H_i = \Delta g^F_i + \gamma_{o,i} - 0.2662\ h_i \ . \qquad (4-21)$$

Further, using equation (4-12), one can write:

$$\overline{g}'^H_i - \gamma^*_{o,i} = \Delta g^F_i + \delta\gamma_{o,i} - 0.2662\ h_i \ , \qquad (4-22)$$

which holds true also for point j, with the appropriate subscripts. Next, the use of equations (4-7) and (4-14) provides:

$$\overline{g}_{ij} - \overline{\gamma}^*_{o,ij} = \overline{\Delta g}^F_{ij} + \overline{\delta\gamma}_{o,ij} - 0.3086\ \overline{h}_{ij} \ . \qquad (4-23)$$

Then, by substituting the relationship (4-22) and (4-23) into equation (4-20) for the Helmert Gravity correction, one obtains:

$$GC^H_{ij} = \frac{1}{G} \{h_i \, [-\Delta\Delta g^F_{ij} - \Delta\delta\gamma_{o,ij} + 0.2662 \, \Delta h_{ij}] +$$

$$+ \Delta h_{ij} \, [(\overline{\Delta g}^F_{ij} + \overline{\delta\gamma}_{o,ij} - 0.3086 \, \bar{h}_{ij}) -$$

$$- (\Delta g^F_j + \delta\gamma_{o,j} - 0.2662 \, h_j)] \} \, , \qquad (4\text{-}24)$$

where:

$$\Delta\Delta g^F_{ij} = \Delta g^F_j - \Delta g^F_i \quad , \qquad (4\text{-}25)$$

$$\Delta\delta\gamma_{o,ij} = \delta\gamma_{o,j} - \delta\gamma_{o,i} \quad , \qquad (4\text{-}26)$$

and the remaining symbols are defined earlier.

The simple algebraic manipulation of equation (4-24) leads finally to the sought rigorous formula for the Helmert gravity correction which reads:

$$GC^H_{ij} = - \frac{\bar{h}_{ij}}{G} \, [\Delta\Delta g^F_{ij} + \Delta\delta\gamma_{o,ij} - 0.2238 \, \Delta h_{ij}] \, . \qquad (4\text{-}27)$$

Here again, it should be noted that the Helmert gravity correction GC^H_{ij}, computed from equation (4-27), is to be added to the used orthometric height difference $\Delta\tilde{h}^o_{ij}$ to obtain the corresponding rigorous Helmert orthometric height difference Δh^H_{ij}. The units here are metres and milligals.

Similar to the dynamic gravity correction, it is also obvious here that the Helmert gravity correction is a function of levelled heights, geographical latitudes and free-air gravity anomalies of the two bench marks, i.e.;

$$GC^H_{ij} = f^H \, (h_i, \, h_j, \, \phi_i, \, \phi_j, \, \Delta g^F_i, \, \Delta g^F_j), \qquad (4\text{-}28)$$

where f, again, denotes the functional relationship.

4.2.3 Vignal Gravity Correction

The Vignal gravity correction GC^V_{ij} to the height difference $\Delta\tilde{h}^o_{ij}$ (based on normal gravity) can be expressed as:

$$GC^V_{ij} = \Delta h^V_{ij} - \Delta\tilde{h}^o_{ij} \quad , \tag{4-29}$$

where Δh^V_{ij} is the Vignal rigorous normal height difference. Substitution from equations (2-24) and (3-12) into the above equation yields:

$$GC^V_{ij} = VC_{ij} - \tilde{OC}_{ij} \quad . \tag{4-30}$$

Here Vignal or actual normal correction VC_{ij} is given by equation (2-35), i.e.:

$$VC_{ij} = \frac{1}{G} [h_i \; (\bar{\gamma}'^V_i - \bar{\gamma}'^V_j) + \Delta h_{ij} \; (\bar{g}_{ij} - \bar{\gamma}'^V_j)] \quad , \tag{4-31}$$

and the orthometric correction $\tilde{OC}_{ij}$ based on normal gravity is given by equation (3-28), i.e.:

$$\tilde{OC}_{ij} = \frac{1}{G} [h_i \; (\gamma^*_{o,i} - \gamma^*_{o,j}) + \Delta h_{ij} (\bar{\gamma}^*_{o,ij} - \gamma^*_{o,j})] . \tag{4-32}$$

Substituting into equation (4-30), the Vignal gravity correction becomes:

$$GC^V_{ij} = \frac{1}{G} \; \{h_i [(\bar{\gamma}'^V_i - \gamma^*_{o,i}) - (\bar{\gamma}'^V_j - \gamma^*_{o,j})] +$$

$$+\Delta h_{ij} \; [(\bar{g}_{ij} - \bar{\gamma}^*_{o,ij}) - (\bar{\gamma}'^V_j - \gamma^*_{o,j})]\} \; . \tag{4-33}$$

To express the RHS of equation (4-33) in terms of free-air gravity anomalies, we use equations (2-22) and (4-12) to get the following expression:

$$\bar{\gamma}_i^{\prime V} - \gamma^*_{o,i} = \delta\gamma_{o,i} - 0.1543\, h_i \,, \tag{4-34}$$

which can be also written for point j, with the proper subscripts. Then, by substituting equations (4-23) and (4-34) into equation (4-33) for the Vignal gravity correction, one finds that:

$$GC^V_{ij} = \frac{1}{G} \{h_i \,[- \Delta\delta\gamma_{o,ij} + 0.1543\, \Delta h_{ij}] + $$
$$+ \Delta h_{ij} [(\overline{\Delta g}^F_{ij} + \overline{\delta\gamma}_{o,ij} - 0.3086\, \bar{h}_{ij}) - $$
$$- (\delta\gamma_{o,i} - 0.1543\, h_j)]\} \,, \tag{4-35}$$

in which all the terms are as defined before.

Further algebraic manipulations of equation (4-35) result in the following rigorous formula for the Vignal gravity correction:

$$GC^V_{ij} = \frac{1}{G} [\Delta h_{ij}\, \overline{\Delta g}^F_{ij} - \Delta\delta\gamma_{o,ij}\, \bar{h}_{ij}] \,, \tag{4-36}$$

in which the units are metres and milligals. The GC^V_{ij} is to be added to the used orthometric height difference $\Delta\tilde{h}^O_{ij}$ to get the corresponding rigorous Vignal normal height difference Δh^V_{ij}.

Generally, equation (4-36) can be written as:

$$GC^V_{ij} = f^V (h_i, h_j, \phi_i, \phi_j, \Delta g^F_i, \Delta g^F_j), \tag{4-37}$$

where f denotes the functional relationship.

At this point, it is worth noting that there is an alternative approach, given in Appendix V, to deriving the above rigorous formulae for the gravity corrections. The approach in Appendix V can be regarded as an independent check on the correctness of the formulae derived herein. In addition, the correctness of the formulae for the gravity corrections has been checked numerically by the author [Nassar, 1976] in the following manner. Firstly, the height differences have been computed on the basis of observed gravity using the formulae given in chapter 2. Secondly, the same height differences have been computed on the basis of normal gravity by applying the USC&GS corrections as presented in chapter 3. Finally, the difference between these two sets of results have been compared with the corresponding gravity corrections, as computed from the formulae developed here, and have been found identical.

As a closing remark to this section, one may notice from the rigorous expressions that the gravity correction most seriously influenced by the difference $\delta\gamma_o$ (between the 1967 International and the USC&GS formulae for normal gravity) is the first, i.e. the dynamic. We can also see that the difference between the values of GC^D (equation 4-13) and GC^V (equation 4-36) is solely due to the effect of $\delta\gamma_o$. This means that the expressions for both the dynamic and Vignal gravity corrections will be identical if the adopted formula for normal gravity is the 1967 formula (i.e. $\delta\gamma_o = 0$). The same result can be obtained by examining equation (2-34) for Vignal correction VC_{ij} (based on actual gravity). If we rewrite this equation again for $\tilde{VC}_{ij}$ (based on normal gravity), we will discover that:

$$GC^V_{ij} = VC_{ij} - \tilde{VC}_{ij} = DC_{ij} - \tilde{DC}_{ij} = GC^D_{ij} ,$$

i.e. both dynamic and Vignal gravity corrections are equivalent. Such an interesting result may explain why Vignal called his height system "orthodynamic", as mentioned in section 2.4.

4.3 Approximate Expressions for the Gravity Corrections

The rigorous expressions for the gravity corrections (formulae 4-13, 4-27 and 4-36) have been derived in the previous section. The purpose of this section is to attempt to simplify these expressions, by making some reasonable approximations. The sought approximate expressions for the gravity corrections are meant to suit desk or pocket calculating machines.

To compute the magnitude of each term in the rigorous expressions and examine its significance, extreme values of h, Δh, Δg that may conceivably occur in Canada will be considered. In the subsequent discussion we will be dealing with a levelling section of 1 km length having the following characteristics: $\bar{h}_{ij} = 4$ km, $\Delta h_{ij} = 200$ m, $\overline{\Delta g}^F_{ij} = 200$ mgal, $\Delta\Delta g^F_{ij} = 10$ mgal and $\overline{\delta\gamma}_{o,ij} = 6$ mgal (see Appendix I). Referring to Appendix I, it can be seen that $\Delta\delta\gamma_o$ between two points, say 0.25 degrees of arc apart in latitude ($\simeq$25 km), is of the order of 0.005 mgal. This means that for the above stipulated levelling section in the direction of meridian we get: $\Delta\delta\gamma_{o,ij} \doteq 0.0002$ mgal. In addition, we know that the reference gravity G is in the order of 10^6 mgal. The individual terms in the formula for gravity correction whose contribution is less than 0.01 mm in absolute value will be considered insignificant and thus neglected (see section 7.1 for justification).

4.3.1 Dynamic Gravity Correction

First, we recall that the rigorous formula for the dynamic gravity correction GC^D_{ij} is given by equation (4-13). The examination of the RHS of (4-13), for the extreme values stated above, indicates that the first term is about 33 times larger than the second term, in absolute value. The latter, denoted here by ε^D, i.e.:

$$\varepsilon^D = \frac{1}{G} \overline{\delta\gamma}_{o,ij} \Delta h_{ij} \quad , \tag{4-38}$$

will be approximately equal to 1.2 mm. Hence, the effect of ε^D on the computed GC^D_{ij} is obviously not negligible. The effect of the first term is thus not neglible either.

However, the difference $\overline{\delta\gamma}_{o,ij}$ of normal gravity given by the 1967 International formula (equation 2-36) and the USC&GS formula (equation 3-2) can be expressed approximately as follows:

$$\overline{\delta\gamma}_{o,ij} \doteq a_o + a_1 \sin^2 \bar{\phi}_{ij} + a_2 \sin^2 2\bar{\phi}_{ij}, \tag{4-39}$$

where: $a_o = -\ 6.295$ mgal , (4-40a)

$a_1 = 0.358$ mgal , (4-40b)

$a_2 = 1.076$ mgal , (4-40c)

and $\bar{\phi}_{ij}$ is the average latitude of the levelling section. Consequently, an approximate expression for GC^D_{ij} can be obtained by substituting equation (4-39) into equation (4-13) to get:

$$GC^D_{ij} \doteq \frac{\Delta h_{ij}}{G} [\overline{\Delta g}^F_{ij} + a_o + a_1 \sin^2 \bar{\phi}_{ij} + a_2 \sin^2 2\bar{\phi}_{ij}]. \tag{4-41}$$

Assuming that $\bar{\phi}_{ij} = 45°$, the last two terms in equation (4-41) will contribute 0.04 mm and 0.2 mm, respectively, to the computed GD^D_{ij}. These effects must again be considered significant.

4.3.2 Helmert Gravity Correction

The rigorous formula for the Helmert gravity correction GC^H_{ij} is given by equation (4-27). Examination of the RHS of (4-27) for the assumed extreme values of the involved quantities reveals that the magnitude of the first and last terms are 40 mm and 179.04 mm respectively. Obviously neither effect can be neglected. The influence of the second (middle) term, denoted here by ε^H, i.e.:

$$\varepsilon^H = \frac{1}{G} \Delta\delta\gamma_{o,ij} \bar{h}_{ij} \quad , \tag{4-42}$$

is of the order of 0.0008 mm. This suggests that ε^H can be safely neglected. Hence, equation (4-27) becomes approximately:

$$GC^H_{ij} \doteq - \frac{\bar{h}_{ij}}{G} [\Delta\Delta g^F_{ij} - 0.2238 \; \Delta h_{ij}] \quad . \tag{4-43}$$

4.3.3 Vignal Gravity Correction

The rigorous formula for the Vignal gravity correction is given by equation (4-36). The effect of the first term on the RHS is the same as the effect of the first term of (4-13). This effect was found significant (40 mm, section 4.3.1) and cannot be neglected. On the other hand, the effect of the second term of (4-36) on GC^V_{ij} is the same as ε^H (equation 4-42) which was found negligible. Consequently, equation (4-36) becomes approximately:

$$GC^V_{ij} \doteq \frac{\Delta h_{ij} \; \overline{\Delta g}^F_{ij}}{G} \quad . \tag{4-44}$$

It may be worth mentioning here that the formula (4-44) is found to be identical to the actual gravity correction term for Molodenskii's normal height difference Δh^M [Bursa, 1958; Schneider, 1960; Weidauer, 1963; Vykutil, 1964]. The term denoted by K_g was originally derived by Bursa and represents the contribution of the local irregularities in the actual gravity field to the corresponding computed height difference. This in fact can be regarded as an independent check on the correctness of the formula for Vignal gravity correction, as developed herein.

4.4 Estimated Precisions of the Gravity Corrections

This section represents an attempt to get estimates of precision (standard deviations) for the computed gravity corrections. The basic motivation behind this attempt is to determine the degree of reliability of these corrections, which has been questioned in the context of justifying their evaluation and practical applications [Boal, 1972]. It is understood that it would be questionable to look for corrections whose own standard deviations (uncertainties) are larger in magnitude than the corrections themselves. Therefore, the main objective here is not to obtain the most accurate estimates, but only the order of magnitude of precision. Thus, it was decided not to venture into the complications regarding correlations between the involved original observables, such as the levelled heights and/or the observed gravity values.

The gravity correction for each system of heights will be dealt with separately. This means that the correlation between the gravity corrections for the different systems is not going to be studied either. Furthermore, the errors associated with geographical positions (latitude is of concern) will be considered negligible.

The process of propagating the variances of the observed heights and gravity values and obtaining the resulting variance associated with the gravity corrections involves several intermediate steps. To begin with let us summarize the rigorous formulae for the three kinds of gravity corrections (see section 4.2):

$$GC_{ij}^{D} = \frac{\Delta h_{ij}}{G} \, [\, \overline{\Delta g}_{ij}^{F} + K_D] \, , \qquad (4-45)$$

$$GC^{H}_{ij} = \frac{\bar{h}_{ij}}{G} [-\Delta\Delta g^{F}_{ij} - K_H + 0.2238\ \Delta h_{ij}], \tag{4-46}$$

$$GC^{V}_{ij} = \frac{1}{G} [\Delta h_{ij}\ \overline{\Delta g}^{F}_{ij} - K_H\ \bar{h}_{ij}] \ , \tag{4-47}$$

where: $K_D = \overline{\delta\gamma}_{o,ij}$ and $K_H = \Delta\delta\gamma_{o,ij}$ are treated here as errorless, since both are functions of only latitude which is considered errorless (having zero variances).

By looking at the above three expressions, we can see that the gravity corrections are functions of both errorless constants and variables containing errors. It is evident that any error committed in the variable quantities would incur an error in the computed gravity correction. Therefore, for investigating the error we take into account the variable quantities only. These variables in equations (4-45), (4-46) and (4-47), are: Δh_{ij}, $\overline{\Delta g}^{F}_{ij}$, $\bar{h}_{ij}$ and $\Delta\Delta g^{F}_{ij}$. These four quantities are generally correlated, i.e. their covariance matrix, say Σ_L, is fully populated, since they are all derived from the same primary observables: h_i, h_j, g_i and g_j.

There are two approaches to obtain the variances of the three gravity corrections that both lead to the same answer. The first is by applying the covariance law [e.g. Wells and Krakiwsky, 1971; Vaníček, 1973] on equations (4-45), (4-46) and (4-47), respectively, and taking into account the full covariance matrix Σ_L mentioned above. The second is by rewriting the formulae for the gravity corrections in terms of the levelled heights and observed gravities at points i and j (i.e. the primary variables) and then applying the law of propagation of errors on each formula separately. The second approach will be used here, because it gives the final formulae in a more simple way.

Equations (4-45), (4-46) and (4-47) can be rewritten as:

$$GC^D_{ij} = \frac{1}{G}(h_j - h_i)\ [\frac{1}{2}(g_i - \gamma_{o,i} + c_1 h_i) + \frac{1}{2}(g_j - \gamma_{o,j} + c_1 h_j) + K_D]\ , \qquad (4\text{-}48)$$

$$GC^H_{ij} = \frac{1}{2G}(h_i + h_j)\ [(g_i - \gamma_{o,i} + c_1 h_i) - (g_j - \gamma_{o,j} + c_1 h_j) - -K_H + c_2(h_j - h_i)]\ , \qquad (4\text{-}49)$$

$$GC^V_{ij} = \frac{1}{2G}[(h_j - h_i)\ \{(g_i - \gamma_{o,i} + c_1 h_i) + (g_j - \gamma_{o,j} + c_1 h_j)\} - -K_H\ (h_i + h_j)]\ , \qquad (4\text{-}50)$$

where: $c_1 = 0.3086$ and $c_2 = 0.2238$, both in mgal/m. We notice, that each of the above three expressions can be written as:

$$GC = f\ (h_i,\ h_j,\ g_i,\ g_j,\ \phi_i,\ \phi_j), \qquad (4\text{-}51)$$

where f denotes the functional relationship (see equations 4-15, 4-28, and 4-37), and ϕ_i, ϕ_j are considered errorless.

The law of propagation of errors is used to compute the variance σ^2_x of a function $x = f\ (\ell_1,\ \ell_2,\ \dots\ \ell_n)$ from the variances $\sigma_{\ell_i}^{\ 2}$ of ℓ_i which are uncorrelated (zero covariances). This law can be stated as follows [e.g. Vaníček, 1973]:

$$\sigma^2_x = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \ell_i}\right)^2 \sigma_{\ell_i}^{\ 2} \quad . \qquad (4\text{-}52)$$

When applying (4-52) on (4-51), we get:

$$\sigma_{GC}^2 = (\frac{\partial f}{\partial h_i})^2 \sigma_{h_i}^2 + (\frac{\partial f}{\partial h_j})^2 \sigma_{h_j}^2 + (\frac{\partial f}{\partial g_i})^2 \sigma_{g_i}^2 + (\frac{\partial f}{\partial g_j})^2 \sigma_{g_j}^2 \quad . \tag{4-53}$$

By applying equation (4-53) to each of the equations (4-48) (4-49) and (4-50), respectively, the following expressions for the variances σ_D^2, σ_H^2 and σ_V^2, of the corresponding gravity corrections, are obtained:

$$\sigma_D^2 = \frac{1}{G^2} [(\overline{\Delta g}_{ij}^F + K_D - 0.1543\ \Delta h_{ij})^2 \sigma_{h_i}^2 + (\overline{\Delta g}_{ij}^F + K_D + 0.1543\ \Delta h_{ij})^2 \sigma_{h_j}^2 + (0.5\ \Delta h_{ij})^2 (\sigma_{g_i}^2 + \sigma_{g_j}^2)], \tag{4-54}$$

$$\sigma_H^2 = \frac{1}{G^2} [0.5\ \Delta\Delta g_{ij}^F - 0.0848\ \bar{h}_{ij} - 0.1119\ \Delta h_{ij})^2 \sigma_{h_i}^2 + (0.5\ \Delta\Delta g_{ij}^F + 0.0848\ \bar{h}_{ij} - 0.1119 \Delta h_{ij})^2 \sigma_{h_j}^2 + \bar{h}_{ij}^2 (\sigma_{g_i}^2 + \sigma_{g_j}^2)] , \tag{4-55}$$

$$\sigma_V^2 = \frac{1}{G^2} [(\overline{\Delta g}_{ij}^F - 0.1543\ \Delta h_{ij})^2 \sigma_{h_i}^2 + (\overline{\Delta g}_{ij}^F + 0.1543\ \Delta h_{ij})^2 \sigma_{hj}^2 + (0.5\ \Delta h_{ij})^2 (\sigma_{g_i}^2 + \sigma_{g_j}^2)] , \tag{4-56}$$

in which the physical units are metres and milligals. The quantity K_D, in equation (4-54), is adequately approximated by equation (4-39). On the other hand, we notice that K_H has disappeared from the expressions (4-55) and (4-56). This is because its magnitude is so small (see section 4.3) that it has also a neglible influence on the above expressions for the variances.

At this point, if we go back again to the developed formulae (rigorous or approximate) for the gravity corrections, we find out that these corrections are evaluated from the levelled heights and free-air gravity anomalies. This suggests that the derived expressions for their variances can be also written in terms of the variances $\sigma^2_{\Delta g^F_i}$ and $\sigma^2_{\Delta g^F_j}$ of the free-air anomalies at points i and j. First, from the definition of Δg^F (sections 4.1 and 4.2) and the law of propagation of errors, we can write:

$$\sigma^2_{\Delta g^F_i} = \sigma^2_{g_i} + (0.3086)^2 \, \sigma^2_{h_i} \, . \tag{4-51}$$

Similar expression holds for point j. Hence, by substitution in equations (4-54), (4-55) and (4-56), we get:

$$\begin{aligned}\sigma^2_D = \frac{1}{G^2} \{ & [(\overline{\Delta g}^F_{ij} + K_D)^2 - 0.3086 \, \Delta h_{ij} \, (\overline{\Delta g}^F_{ij} + K_D)] \, \sigma^2_{h_i} + \\ & + [(\overline{\Delta g}^F_{ij} + K_D)^2 + 0.3086 \, \Delta h_{ij} \, (\overline{\Delta g}^F_{ij} + K_D)] \, \sigma^2_{h_j} + \\ & + 0.25 \, \Delta h^2_{ij} \, (\sigma^2_{\Delta g^F_i} + \sigma^2_{\Delta g^F_j}) \} \, ,\end{aligned} \tag{4-58}$$

$$\begin{aligned}\sigma^2_H = \frac{1}{G^2} \{ & [0.5 \, \Delta\Delta g^F_{ij} - 0.1119 \, \Delta h_{ij})^2 - 0.1696 \, \bar{h}_{ij} \, (0.5 \Delta\Delta g^F_{ij} - \\ & - 0.1119 \, \Delta h_{ij}) - 0.0880 \, \bar{h}^2_{ij}] \, \sigma^2_{h_i} + \\ & + [(0.5 \, \Delta\Delta g^F_{ij} - 0.1119 \, \Delta h_{ij})^2 + 0.1696 \, \bar{h}_{ij} (0.5 \, \Delta\Delta g^F_{ij} - \\ & - 0.1119 \, \Delta h_{ij}) - 0.0880 \, \bar{h}^2_{ij}] \, \sigma^2_{h_j} + \\ & + \bar{h}^2_{ij} \, (\sigma^2_{\Delta g^F_i} + \sigma^2_{\Delta g^F_j}) \} \, ,\end{aligned} \tag{4-59}$$

$$\sigma_V^2 = \frac{1}{G^2} \{[(\overline{\Delta g}^F_{ij})^2 - 0.3086\ \Delta h_{ij}\ \overline{\Delta g}^F_{ij}]\ \sigma^2_{h_i} +$$

$$+ [(\overline{\Delta g}^F_{ij})^2 + 0.3086\ \Delta h_{ij}\ \overline{\Delta g}^F_{ij}]\ \sigma^2_{h_j} +$$

$$+ 0.25\ \Delta h^2_{ij}\ (\sigma^2_{\Delta g^F_i} + \sigma^2_{\Delta g^F_j})\}. \tag{4-60}$$

The expressions (4-58), (4-59) and (4-60) can be simplified if we assume for a particular levelling section: $\sigma_{h_i} = \sigma_{h_j} = \sigma_h$ and $\sigma_{\Delta g^F_i} = \sigma_{\Delta g^F_j} = \sigma_{\Delta g^F}$. In this case, that can, however, be seldom used in practice, the variances of the gravity corrections become:

$$\sigma_D^2 = \frac{2}{G^2} [(\overline{\Delta g}^F_{ij} + K_D)^2\ \sigma^2_h + 0.25\ \Delta h^2_{ij}\ \sigma^2_{\Delta g^F}], \tag{4-61}$$

$$\sigma_H^2 = \frac{2}{G^2} \{[(0.5\ \Delta\Delta g^F_{ij} - 0.1119\ \Delta h_{ij})^2 - 0.0880\ \bar{h}^2_{ij}]\ \sigma^2_h +$$

$$+ \bar{h}^2_{ij}\ \sigma^2_{\Delta g^F}\}, \tag{4-62}$$

$$\sigma_V^2 = \frac{2}{G^2} [(\overline{\Delta g}^F_{ij})^2\ \sigma^2_h + 0.25\ \Delta h^2_{ij}\ \sigma^2_{\Delta g^F}]. \tag{4-63}$$

The square-root of the computed variance of the gravity correction is its estimated standard deviation in metres.

CHAPTER 5

GENERAL DISCUSSION OF DATA COVERAGE IN CANADA

This chapter is included here for the sake of completeness. It is meant to serve as a link connecting both theoretical and practical aspects of heights and gravity. The former has been discussed in detail in the previous three chapters. On the other hand, the practical aspects, associated with the application and feasibility of the gravity influence on the height systems adopted in Canada, will be dealt with in the next two chapters. Both the precise levelling data coverage and the gravity data coverage are considered. In both cases, the discussion will be in general terms, such that it includes a historical background, present status and future plans, and the format of available data for the users.

It should be mentioned that the information presented herein has been compiled from all possible source materials (publications, internal reports and private communications) that have reached the author from: The Geodetic Survey of Canada, GSC, Surveys and Mapping Branch, S&M; and the Gravity and Geodynamics Division, GGD, Earth Physics Branch, EPB, (previously called the Dominion Observatory, DO). Both the S&M and EPB are agencies of the Dept. of Energy, Mines and Resources, EMR. Nevertheless, there may be some other information that has inevitably escaped the author's search.

5.1 Precise Levelling Data

The establishment and maintenance of the Canadian Precise Level Net, CPLN, is the responsibility of the Vertical Control Section within the GSC. Their main aim is to provide and maintain a national precise vertical control network on a single acceptable datum for the whole country. This network is meant to serve all public needs as well as geodetic investigations and other scientific research connected with vertical control [GSC, 1960; Young, 1975].

5.1.1 Historical Background

The first precise levelling work was initiated in 1883, to connect St. Lawrence River area at Montreal with MSL datum on the Atlantic. However, this work was hindered due to lack of funds until 1906, and was completed in 1907. In 1906, the Precise Levelling Section of the GSC was formed, and the first bench mark BM-GSC-No. 1 was established on September 21, 1906 at Sherbrooke, Québec. In the same year, the GSC started the precise levelling operations in Québec and the Maritimes.

In 1908-1910, the DO established a precise levelling line at the Alaska - Yukon boundary. The results are published in Nelles [1913]. Following this, the Topographical Survey of Calgary established many precise levelling lines in the west. In addition to the steadily increasing work of the GSC on precise levelling, several precise levelling extensions have been established by other different organizations and private industries whenever needed. For details about the used instrumentation and field work procedures, see e.g. GSC [1960].

In 1919, the first work was done on the adjustment of the CPLN [Cannon, 1929]. The rod correction was applied to the work done prior to 1923, since the wooden rods were replaced by the invar rods. Meanwhile, an effort has been made to connect the inland lake levels with the precise levelling system. All lines of the CPLN had the orthometric correction (based on normal gravity, see section 3.2) applied to them before being used in the adjustment. The weights of the observed elevation differences, needed in the adjustment, were taken as the reciprocal of the lengths of the lines. Finally, the adjustment of the CPLN was completed in 1928 [Cannon, 1929]. Sea level values at Halifax, Yarmouth and Father's Point on the Atlantic; at Vancouver and Prince Rupert on the Pacific; and on Rouses Point on the Canadian - United States international boundary (for which a standard elevation was temporarily agreed on by both countries) were held fixed. In this adjustment different techniques were used like: observation equations, condition equations, and differential adjustment to show the effects of the new additions to the network.

After 1929, the United States performed one of their adjustments, referred to as the "1929 Special Adjustment". This adjustment was based on all sea level tidal stations of both countries on Atlantic and Pacific coasts and including all the CPLN. However, the results of this adjustment were not adopted by the Canadians at that time [Cannon, 1935]. The GSC preferred to work with their published results of 1928 [Christodoulidis et al., 1973].

The recent adjustments of the CPLN started in 1929 were finished in 1934-35 by one adjustment labelled "D" [Cannon, 1935].

The final adjustment "D" consisted of a simultaneous adjustment (by the method of least-squares) of all the orthometrically corrected loops in the net. In this adjustment the values for bench marks controlling the primary gauging stations were held fixed [Cannon, 1935; GSC, 1960]. The results of adjustment "D" were found basically the same as those of the already published 1928 adjustment. Consequently, the GSC has decided to retain the published elevations, without any changes. However, these published elevations are referred to by the GSC as resulting from the "1929 General Adjustment" [Young, 1976]. It was not until 1935 that the MSL, based on the tide-gauges at Halifax, Yarmouth and Father's Point on the Atlantic, and Prince Rupert and Caulfield Cove on the Pacific (although used before), was officially adopted to be the datum for vertical control operations in Canada [GSC, 1960].

Everytime new work is added to the CPLN, theoretically the best method is to readjust the entire net. However, this is not practically done, since users like to keep their elevations fixed as long as possible. Hence, when new levelling produces a situation where two or more loops are formed, a least-squares adjustment is made holding the differences between junction points whose elevations have already been published, as fixed and fitting the new levelling sections to them.

In 1950, the GSC started a new adjustment of the entire CPLN. The preparatory work for this new adjustment was arranged so that it could be the basis of future solutions [Jones, 1956]. This adjustment was done with and then without the sea level values being fixed, and was completed in 1952. However, it was decided, again, that there was no reason to consider changing the published elevations to the 1952 values. Accordingly, the "1929 General Adjustment" (originally the 1928) remains, until now, the basis of vertical control in the entire country [McLellan, 1974].

5.1.2 Present Status and Future Plans

The present extent of the first order vertical control (CPLN) is shown in Figure 5-1 [McLellan, 1974]. In the southern part of the country, the levelling lines follow transportation routes and are concentrated in areas of high population density. The network presently consists of over 98,000 km of levelling with over 40,000 bench marks. Currently, about 1600 new bench marks are being established each year [Canadian National Committee for IUGG, 1975]. Several lines extend also into the hinterlands to reach major developing areas. In most cases the levellings to these northerly points consist of spur or branch lines, whose pattern is very sparse. The lack of roads or railways makes it practically impossible to form closed loops in these areas. Nevertheless, about 20% of the current levellings are extended to the unsettled areas [Young, 1975].

Over 140 permanent guaging stations distributed along the Canadian coasts and on inland waters of the Great Lakes - St. Lawrence River System are presently operated by the Tides and Water Levels Section of the Hydrographic Branch, Dept. of the Environment. The water level data are processed regularly and published annually by the Marine Environmental Data Service, Ocean and Aquatic Affairs, Dept. of the Environment. The stability of bench marks controlling

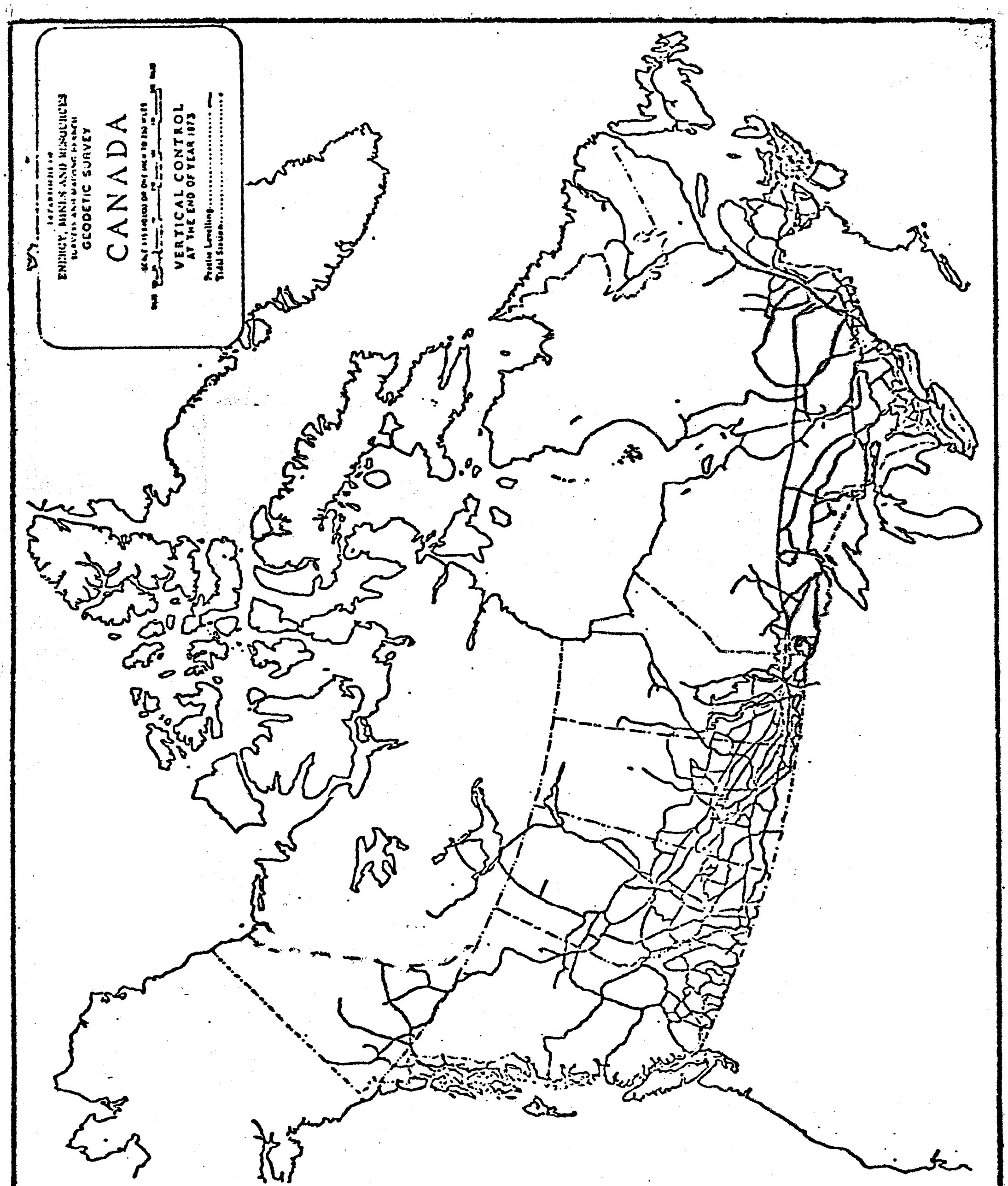


FIGURE 5-1

Present Extent of the Canadian

Precise Level Net (CPLN)

(From: McLellan [1974])

the gauging stations is checked annually by precise levelling [Canadian National Committee for IUGG, 1975]. A catalogue is available showing the description and elevations of bench marks at all permanent and temporary gauging sites.

A general evaluation of the quality of the CPLN has been attempted by the following researchers: Vaníček [1970]; Boal [1971a]; Vaníček et al. [1972] and Christodoulidis et al. [1973]. The main problems connected with the network were found to include:

1. The use of MSL at five tidal gauging stations as a datum with fixed values equal to zero [Cannon, 1929; Cannon, 1935]. This situation does not represent the reality, as mentioned in section 1.3, and thus adversely influences the network adjustment;
2. The use of unrealistic weighting scheme, as stated in the previous section, which does not reflect, in most cases, the real situation based on the statistical analysis of actually obtained discrepancies. For details about this, see e.g. [Vaníček, 1970; Vaníček et al., 1972];
3. A disregard of the systematic influences (see section 1.3) due to unsymmetrical atmospheric refraction, tidal effect and predicted crustal movements. More details can be found in [Holdahl, 1974];
4. The errors associated with using two different kinds of levelling instruments as well as two different kinds of rods pre and post 1923. More elaboration on this can be found, e.g. in [Boal, 1971a; Murakami and Boal, 1971];
5. Neglecting the actual gravity anomalies and defining the heights on the basis of adopted normal gravity only. This problem

constitutes the backbone of the investigation contained herein.

In spite of the above problems associated with the published heights of the CPLN (based on the 1929 general adjustment), such heights can still serve well the local surveying and limited technical projects. On the other hand, these published heights may prove inadequate for scientific investigations of problems related to earth sciences and modern technology [Vaníček et al., 1972; Holdahl, 1974]. A more thorough evaluation of the CPLN and extensive study of the influences of the above problems on the network should be continued to prepare for the proposed more rigorous and up-to-date new adjustment of the CPLN (see section 1.5).

The intention of the GSC over the next decade [McLellan, 1974] is to strengthen the existing network and to establish new lines of precise levelling. The new specifications for precise levelling operations are given, e.g. in [S&M, 1973]. The future plan is that the lines in the south be spaced not more than 60 km apart, 15 km in densely populated areas and about 5 km in urban centres. An average of 8,000 km of levellings a year for 10 years will be required to strengthen and complete the proposed network of 140,000 km. The program for future work includes also the extension and densification of precise levelling net into the hinterlands [S&M, 1972]. The suggested spacing between first-order lines north of latitude 60°N is not to exceed 300 km. This is a major task and it is at a stage of only long range planning.

Furthermore, a preliminary evaluation of the CPLN has indicated that many of the main lines need to be re-established

[McLellan, 1974]. Consequently, a program of extensive relevellings of old lines has been undertaken. This is to increase bench marks density or to improve accuracy or both. At present, about 10,000 km of the existing net has been relevelled [Young, 1975]. In addition, the systematic relevellings every 40 years are planned for the purpose of studying the secular crustal movements and related problems.

5.1.3 Availability of Data

All the levelling data for the vertical control network, containing the 1929 General Adjustment and the subsequent work fitted into it, are now contained in quadrangle booklets. The information is available in 505 booklets each covering an area of 1° x 1° (including maps of 1:500,000), in 63 booklets each covering an area of 0°.5 x 0°.5 (including maps of 1:250,000), and in 13 special booklets covering mainly city areas where bench mark density is high (including large scale maps showing all the bench marks) [Canadian National Committee for IUGG, 1975]. These published booklets can be updated every year, depending on the new data and required changes.

A program, prepared by the GSC, is now underway to produce the levelling booklets by a computer assisted typewriter. This implies that all required data are stored on magnetic tape. Once the contents of the quadrangle booklet are entered on tape, additions, deletions and corrections can easily be made without retyping the whole document manually.

Meanwhile, a decision was made by the GSC [McLellan, 1974] to develop a computer based data file for all geodetic stations. This file is intended to contain all the necessary information at the station, including the originally observed data which may be called out from the file for further computations and analysis. This is quite an involved task, since it requires that data contained originally only in the field books have to be screened and entered into the data file. To the author's knowledge, this file was already initiated and contains now information for about 20,000 stations of the horizontal control network, out of the final number of 400,000 stations. It has not been started yet for the vertical control network [Young, 1976].

5.2 Gravity Data

The establishment and maintenance of the Canadian National Gravity Net, CNGN, is the responsibility of the Gravity and Geodynamics Division, GGD of the EPB, Dept. of EMR. Their main objectives [Valliant, 1975] include mapping the gravity field in Canada and its coastal waters, and maintaining the Canadian national gravity library for data distribution on both the national and international levels.

5.2.1 Historical Background

It was internationally agreed [Miller, 1931] that Potsdam, Germany, be adopted as the base station to which gravity stations in all countries throughout the World be referred, and thus forming the so-called "Potsdam System". The value of gravity g = 981.274 gals, determined from pendulum absolute gravity observations, at the chosen site in Potsdam [Uotila, 1960; Hamilton, 1963b] was adopted at the 16th General Conference of the IAG in 1909.

The establishment of the CNGN has undergone three stages [Garland, 1953]. First, the adoption of a gravity value at some selected fundamental national base for the entire country, which must be related to Potsdam System. Secondly, the densification of the net, which is normally composed of two types of gravity stations: Control and detailed [Tanner and Buck, 1964]. The control stations should be pendulum stations, or surveyed with gravimeters using the base looping method [e.g. EPB, 1975], interconnected to each other and tied to pendulum stations and to the national reference point. The detailed station can be any single gravimeter observation tied to a control station. The intervals between control stations vary from 40 km in populated areas to 150 km in uninhabited northern areas [Tanner and Buck, 1964]. The spacing between detailed stations is about 5-15 km [Tanner, 1967; Nagy, 1974]. Details concerning the instructions, instrument adjustment and field procedures for the establishment of different categories of gravity stations in Canada can be found in EPB [1975]. Then, after the gravity observations are made, an adjustment is performed, basically for the primary control stations, to define the national gravity net on a single datum, e.g. Potsdam System.

Ottawa (a pier situated in the southwest corner of the basement of the D O building) was chosen to be the Canadian national reference base for all the Canadian gravity work [McDiarmid, 1915]. A direct gravity connection, using pendulum observations, between Ottawa and Potsdam was made during the summer of 1928 by the DO [Miller, 1931]. A value of g = 980.622 gal was finally adopted for Ottawa [Miller and Hughson, 1936] relative to the Potsdam system.

The gravity measurements in Canada date from the pendulum observations made by Putnam of the USC&GS at Sydney, Nova Scotia, in 1896 [Garland, 1953]. The DO began its activities as early as 1902, but the first useful gravity work was undertaken only in 1914 [Miller and Hughson, 1936]. Thompson [1959] felt that the first really reliable gravity measurements performed by the DO in Canada were made around 1921 with a Mendenhall pendulum apparatus, which has a precision of about 2 mgals.

The DO undertook the task of absolute or relative gravity measurements using pendulum observations at selected sited through the country for the following purposes [Thompson, 1959]:

1. To provide a regional network of fundamental gravity values for the control and adjustment of future gravimetric surveys;
2. For the precise calibration of gravimeters;
3. For determining gravity at places wide apart, where long travel times are necessary;
4. To provide accurate measurements of gravity differences between international sites, and ensure that the gravity standards in Canada are consistent with the World network.

The gravimeter observations started in Canada in 1944 [Garland, 1953]. During the years 1944-1951, the DO carried out extensive regional gravity surveys with gravimeters. Various types of gravimeters (e.g. Worden and LaCoste & Romberg instruments) have been used for checking their performance and capabilities. In 1952, a system of primary gravimeter base stations, that were connected and tied to all pendulum stations including the national gravity base station, was established [Innes and Thompson, 1953].

In 1952-1953, the Cambridge pendulum apparatus was used to establish a series of pendulum values in North America [Innes, 1954]. In the late 1950's the GGD (EPB) pendulum apparatus was designed, which is capable of relative gravity determinations consistent to the order of $\pm$ 0.2 mgal [Winter and Valliant, 1960; Tanner, 1967]. The design and operation of the instrument are documented in Valliant [1971a]. From 1968-1970, the GGD pendulum apparatus was used to establish control gravity stations, for the Canadian gravimeter net, with an estimated precision of about 0.08 mgal [Valliant, 1969; Valliant, 1971b].

The work of the GGD of the EPB has been expanding steadily. The number of gravity observations in Canada has increased almost exponentially [Innes, 1957; Hamilton, 1960]. Gravimeters need to be calibrated against a known standard before and after each field season [Hamilton, 1963b]. The North American and Ottawa - Washington are two established long standardization lines to serve unified calibration for all instruments [Innes, 1958; Innes et al., 1960; Uotila, 1960]. In addition to the GGD work, many detailed surveys are carried out each year by public institutions, research foundations and mining and oil exploration industries mostly for gravimetric exploration [Hamilton, 1963a].

In all the gravity work discussed above, the positions (latitude ϕ and longitude λ) of the gravity stations have been scaled from the National Topographic Map Series of the largest scale available in each case. In most cases ϕ and λ have been scaled to the nearest tenth minute of arc [Garland and Tanner, 1957], and in some cases to 0.01 minute of arc [Tanner and Buck, 1964; Hamilton and Buchan, 1965]. On the other hand, obtaining reliable elevations for the gravity stations essential for gravity anomaly computations, was and is a considerable problem. Wherever possible, gravimeter readings are taken at bench marks or other well defined points of known elevations. But as much of Canada is not covered yet with benchmarks, such requirement cannot be always met and consequently elevations must be determined in many cases by barometric altimetry [Tanner and Buck, 1964]. In reality the elevations of the gravity stations have been obtained from various available sources, including spirit levelling, trigonometric levelling, barometric altimetry and others. This situation resulted in assigning various error estimates to these elevations depending on the way they were acquired.

The assigned precision to the obtained elevation varies from 3 cm (spirit levelling) to 5 m (altimetry). Even worse, there are many cases where the given elevation has either unknown or undefined source. In this case, the error estimate can go beyond 30 m (see Appendix II for more details).

5.2.2 Present Status and Future Plans

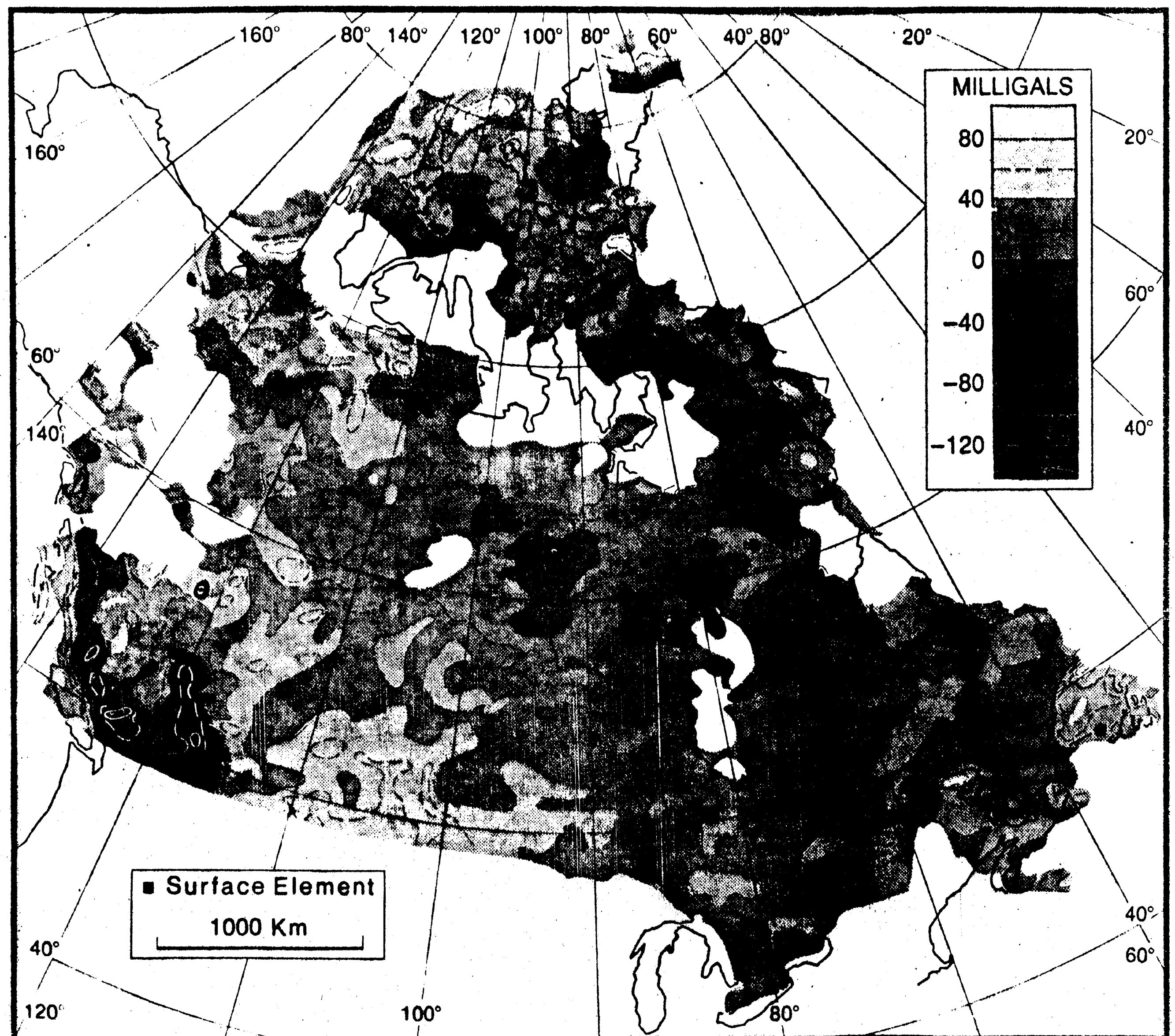
Prior to May 15, 1974, all the gravity anomaly values computed and released by the GGD of the EPB have been based on the observed gravity values as referred to Potsdam system, and the 1930 International formula for normal gravity on the geocentric ellipsoid. The 1930 formula was adopted at the IUGG Meeting in Stockholm in 1930, and reads [e.g. Hamilton, 1963b; Heiskanen and Moritz, 1967]:

$$\gamma_o = 978.049\ [1 + 0.005\ 2884 \sin^2 \phi - 0.000\ 0059 \sin^2 2\phi] \text{ gals}, \tag{5-1}$$

in which ϕ is the latitude of the terrain point in question.

The free-air gravity anomaly map, based on the 1930 International system, is given by Nagy [1973] and reproduced in Figure 5-2. The distribution of gravity stations used for this map, up to and including the 1970 data, is shown in Figure 5-3. It can be seen that significant gaps in the coverage in certain areas still existed. In addition, some areas have unevenly distributed point gravity data [Nagy, 1973]. Nevertheless, the area covered by precise levelling (Figure 5-1) is very well covered by gravity data at a density of about 11 km. Therefore, it is possible to interpolate the gravity anomalies at bench marks of interest, as first shown by Vaníček et al. [1972], and further stated in [Nagy, 1973; Valliant, 1975].

Recent advances in the instrumentation and techniques of modern gravity determination have shown that the Potsdam reference gravity value (adopted in 1909) was significantly different (14 mgals higher) from its correct value as known now. Also, with the increased number of observations and investigations, the coefficients of the 1930 formula for normal gravity have been recomputed more precisely. This situation raised the question of adopting a new value for gravity at Potsdam, as well as introducing a revised formula for normal gravity, at the IUGG meeting in 1967. The 1967 International formula for normal gravity (defined by equation 2-36) and a value of 981.260 gals



MILLIGALS
80
40
0
-40
-80
-120
Surface Element
1000 Km
160°
80°
140°
120°
100°
80°
60°
40°
80°
20°
60°
40°
120°
100°
80°

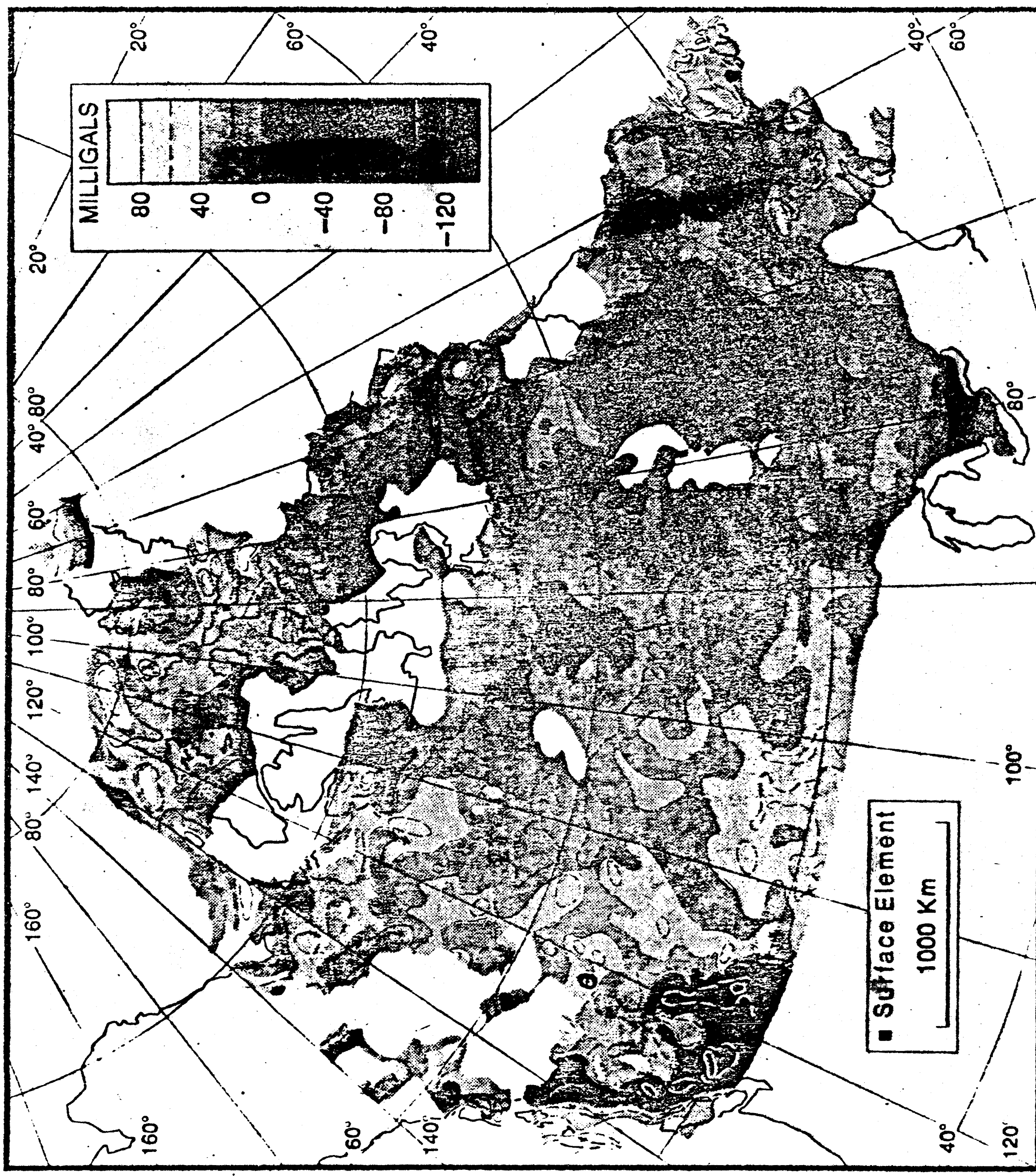


FIGURE 5-2

The 1973 Free-Air Anomaly Map of Canada

(From: Nagy [1973])

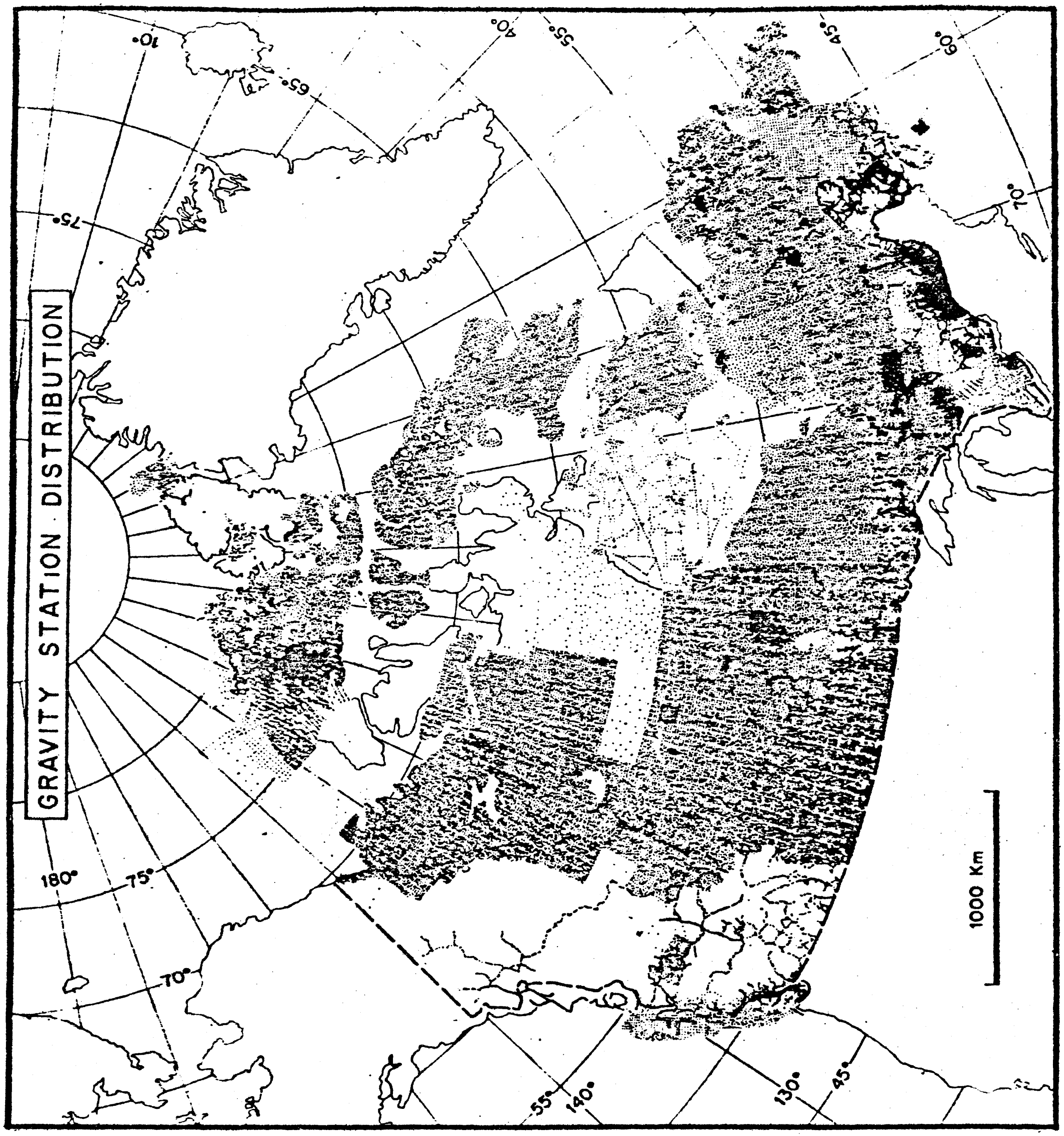


FIGURE 5-3

Plot of Gravity Stations Used in

Preparation of the 1973 Free-Air Anomaly

Map of Canada

(From: Nagy [1973])

for Potsdam reference gravity constitute a new system (adopted at the IUGG meeting in Moscow in 1971) referred to as the "Geodetic Reference System 1967", GRS67, [IAG, 1971]. It has been recommended [e.g. Morelli and Honkasalo, 1975] that all gravity anomalies throughout the world should be referred to the new GRS67. For convenience, Levallois [1972] has published a table for approximate conversion of gravity anomalies from the 1930 to the new 1967 International system.

In the meantime, another major event on the international level concerning gravity data base has occured. A worldwide international gravity network has been established of some 2,000 fundamental stations throughout the World. Twenty-four of these stations are in Canada, see Figure 5-4. The final adjustment of the network based on the new GRS67 was completed in 1974, and the results have been adopted and referred to as "The International Gravity Standardization Net 1971", IGSN71, [Morelli et al., 1974; Valliant, 1975]. The IGSN71 is claimed to be accurate to ± 0.1 mgal [Nagy, 1974].

In 1974, the GGD switched completely to the new GRS67. Further an adjustment of the CNGN, which consists of some 3,500 control stations including the 24 IGSN71 stations mentioned earlier, was performed to relate the Canadian net to the International net. It was labelled "The 1974 adjustment of the CNGN" [Valliant, 1975]. The difference between gravity values in Canada adjusted on the old 1930 and the new 1967 systems is 14-16 mgals, depending on the latitude [Buck, 1975]. From May 15, 1974, all gravity anomalies released by the GGD are based on the 1967 International formula for normal gravity and the observed gravity values are referred to the 1974

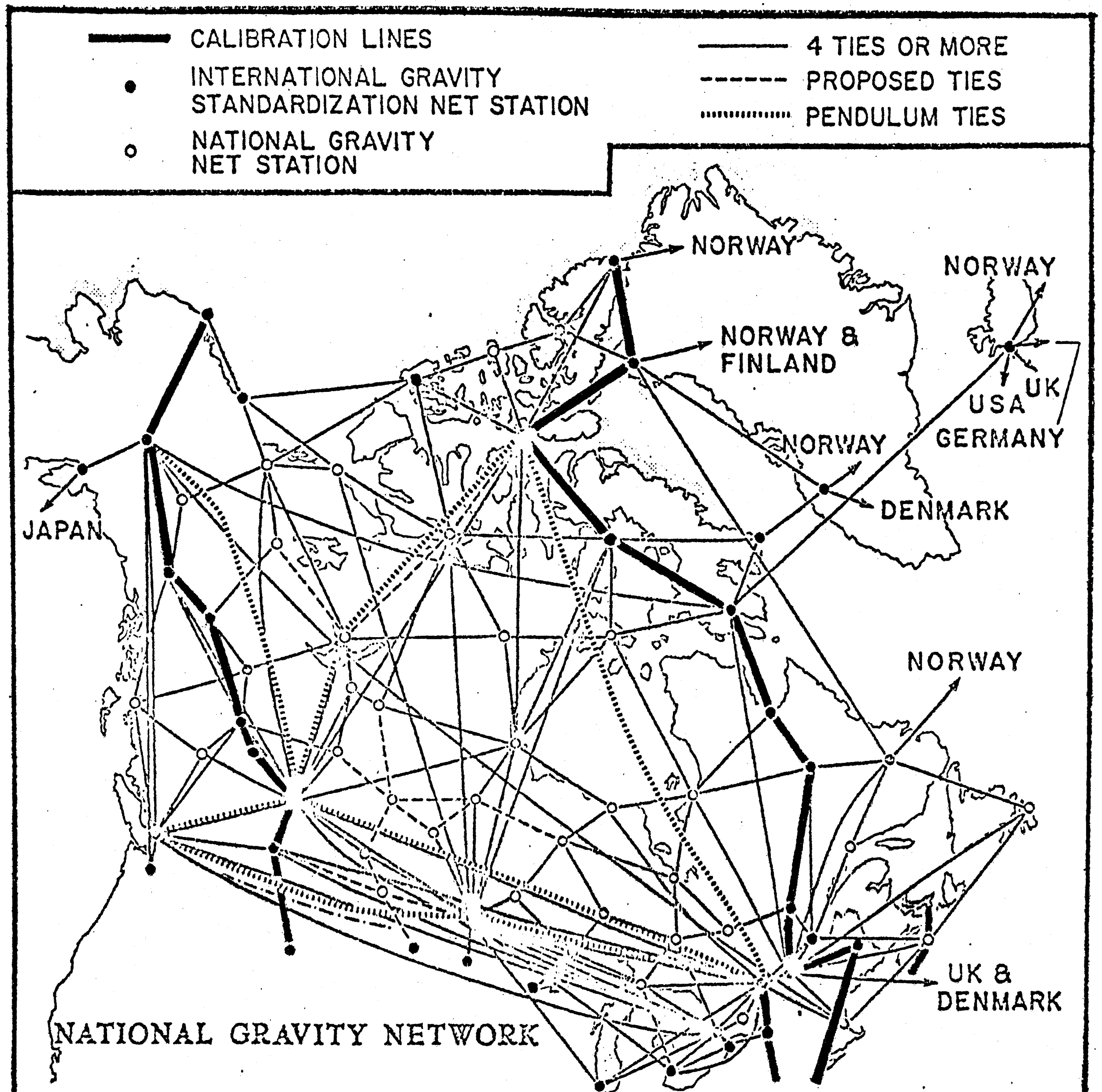


FIGURE 5-4

Canadian Contribution To The

IGSN71

(From: EPB [1975])

adjustment of the CNGN [Buck 1975].

The present status of the point gravity data coverage in Canada is given in Figure 5-5 [Nagy, 1974], which consists of about 350,000 gravity observations. Such amount of data, based on the new system explained before, has been utilized by Nagy [1974] to construct the most recent Bouguer anomaly map of Canada which possesses an accuracy of $\pm$ 2 mgals. Comparison of Figures 5-3 and 5-5 reveals that significant gaps were covered with gravity observations in the period 1970 - 1974.

The GGD is currently planning [Nagy, 1976] to undertake in the near future the task of data preparation and screening for the compilation and production of free-air gravity anomaly contour maps (based on the most up-to-date new system) for the entire country. This project is scheduled to start in the Fall of 1976, with the initial emphasis on small scale maps with 5 mgal contour intervals. There is no immediate intention of producing large scale maps with contour intervals smaller than 5 mgal. Nevertheless, such a program of producing free-air anomaly contour maps,is a major step towards the practical applications of gravity anomalies to precise levelling work.

For convenience of the users who received gravity data prior to May 15, 1974, the GGD devised an empirical formula for conversion of gravity anomalies, Δg, from the old 1930 to the new 1967 system [Nagy, 1974; Buck, 1975; Valliant, 1975] that reads:

$$\Delta g_{1967} - \Delta g_{1930} = [-0.95 - 13.6 \sin^2\phi + 0.05\phi] \text{ mgal}, \qquad (5\text{-}2)$$

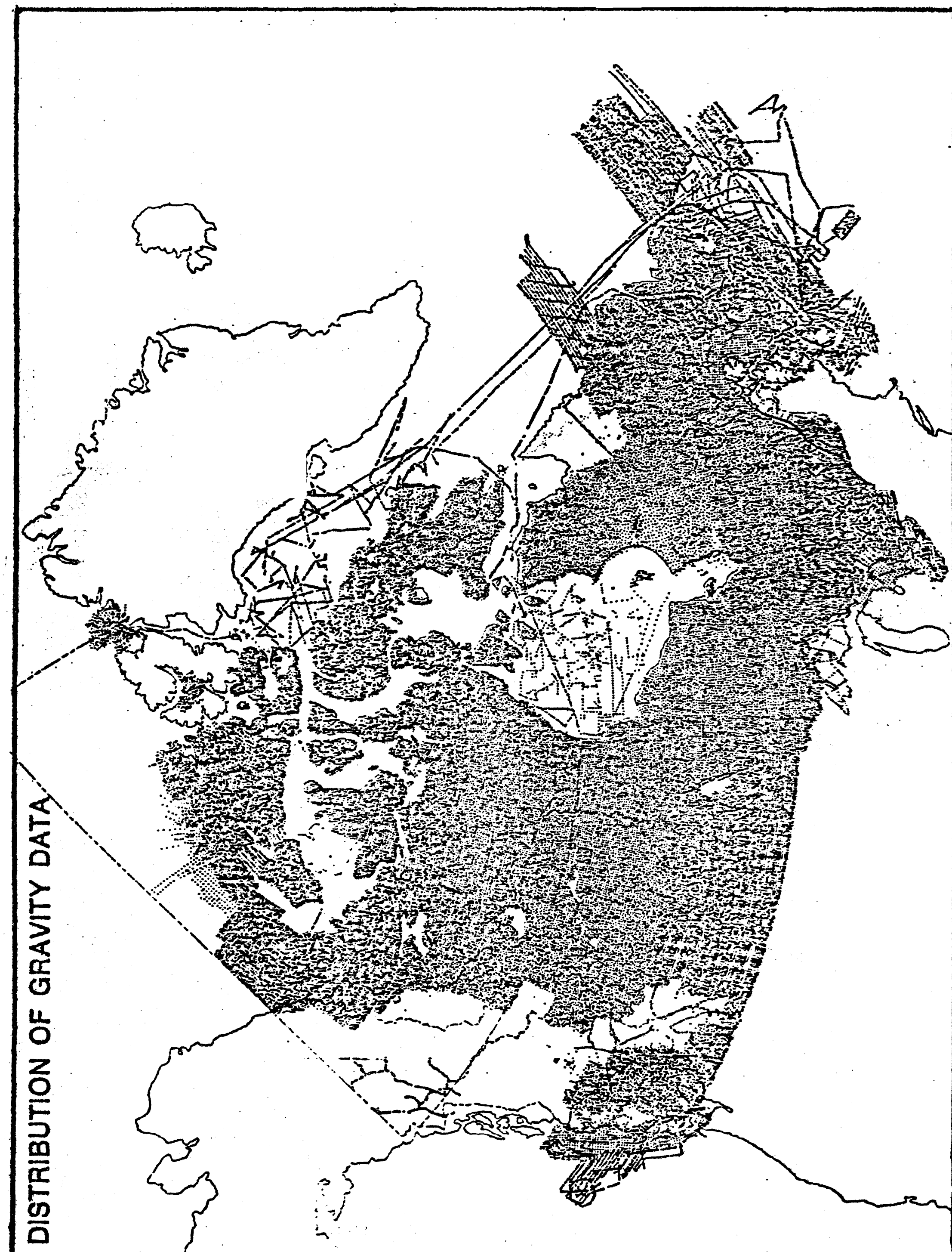


FIGURE 5-5

Plot of Gravity Stations Used in Preparation

of the 1974 Bouguer Anomaly Map of Canada

(From: Nagy [1974])

where ϕ is the station latitude in degrees. It may be worthwhile here to have a closer look at this conversion formula. Equation (5-2) is derived basically from the standard correction formula given in the GRS67 [IAG, 1971, p. 74], i.e.:

$$\Delta g_{1967} - \Delta g_{1930} = [3.2 - 13.6 \sin^2\phi] \text{ mgal}, \qquad (5\text{-}3)$$

supplemented by a corrective term (denoted here by, dg_{slope}) for the slope difference between the old and the 1974 adjustments of the CNGN. Equation (5-3) consists of two terms: the first is the difference between the normal gravity value on the ellipsoid as computed from the 1967 formula (equation 2-36) and the 1930 formula (equation 5-1) expressed as [IAG, 1971, p. 74]:

$$\gamma_{1967} - \gamma_{1930} = [-17.2 + 13.6 \sin^2\phi] \text{ mgal} , \qquad (5\text{-}4)$$

which is claimed to be accurate to 0.1 mgal. The second term is a change of -14 mgal in the absolute observed gravity value at the International reference pier in Potsdam.

An expression for dg_{slope} mentioned above can be then obtained by subtracting equation (5-3) from equation (5-2):

$$dg_{slope} = [-4.15 + 0.05\phi] \text{ mgal}, \qquad (5\text{-}5)$$

in which ϕ has to be in degrees of arc. In Canada, for $\phi = 40° - 80°$ N, dg_{slope} varies between -2.15 and -0.15 mgal, over a range of 2 mgal. This range is consistent with the aforementioned corresponding range of the differences of adjusted gravity values between the old and the new systems. The expression (5-2) is claimed to be accurate to a few tenths of a milligal over small areas (a few hundred square kilometres). However, over larger areas it has a limiting accuracy of about 0.75 mgal [Nagy, 1974; Buck, 1975]. This is because the corrective term dg_{slope} takes into account the change of slope between the old and new CNGN in the latitude direction only and disregards the change in the longitude direction.

At present, the determination and evaluation of the gravity field in Canada constitutes a tremendous problem with its own rights. No evaluation of the gravity data has been attempted so far [Merry, 1975]. For instance, it is well known that the precision of the gravity anomaly is influenced by the standard errors inherent in the observed gravity and in the elevation of the station. The analysis of the adjustment of the primary control network showed that gravity values in the net are of high quality and accurate to $\pm$ 0.05 mgal relative to the datum defined by absolute measurements [Tanner and Gibb, 1971]. On the other hand, the free-air reduction introduces significant errors in the computed gravity anomalies due to the lack of adequate height information, as explained in the previous section. Also, the process of collecting, editing and storing gravity data sets is not entirely free from blunders.

5.2.3 Availability of Data

The two major sources of gravity data are the EPB and the Atlantic Geoscience Centre; both of them are agencies of the Dept. of EMR. Other provincial agencies, Universities and Petroleum and Mineral explorations agencies make important contributions to the present gravity data coverage in Canada [Nagy, 1974]. The GGD of the EPB acts however as the collecting and coordinating centre for gravity measurements made by the EPB itself as well as by other Canadian institutions.

In order to better carry out the above responsibility, including the distribution of gravity data both nationally and internationally, a complete computer-oriented system for processing, handling, and reduction of gravity measurements has been developed at the GGD around 1960. The main features of this system are described by Tanner and Buck [1964]. The main outcome of the system is a file of basic gravity data for use in geophysics and geodesy.

The existing gravity data in Canada (about 350,000 gravity observations, based on the new system) are now available in digitized form (computer data files). These files (punched cards or magnetic tapes) are supplied to the user, on request, by the GGD's Data Centre, Ottawa. Details concerning the storage and retrieval system of the gravity data files are given in Buck and Tanner [1972].

In 1973, a point gravity file, containing $\simeq$ 90,000 gravity observations and based on the 1930 and Potsdam reference systems, was obtained from the GGD. This file was used at the Dept. of Surveying Engineering, UNB in previous investigations [Merry and Vaníček, 1974; Merry, 1975; Nassar and Vaníček, 1975; Nassar, 1975b]. The new file (based on the new 1967 system) used extensively in the current study was obtained from the GGD in January, 1975. This new file, which contains only about 270,000 gravity observations (the rest of the 350,000 observations are probably classified, belonging to some private agencies), is referred to in the present report as the

"EPB File". Description, format and use of the EPB File are given in details in Appendix II. As mentioned before, all gravity data released by the GGD after May 15, 1974, are based on the GRS67 and the IGSN71 standard systems for theoretical (normal) and observed gravity values, respectively.

Since the conventional way of depicting the gravity field, before adopting the digital form, has been in the form of maps, the GGD has also undertaken the program of publishing regional gravity maps. Maps at the scale of 1:500,000 are being published in a new series known as the "Gravity Map Series of the GGD". Bouguer gravity anomaly contour maps until 1973 are at a contour interval of 5 mgals and are based on the old 1930 system [Nagy, 1976]. The latest Bouguer anomaly map for Canada published in 1974 [Nagy, 1974] (based on the new 1967 system) is also available from the Canada Map Office, Ottawa.

So far, only one free-air anomaly contour map, at the scale of 1:1,000,000, was published in 1974 for the Hamilton Inlet [Nagy, 1976]. It was reported in Valliant [1975] that the GGD expects custom contouring of gravity data to be available as a "standard feature" of the GGD storage and retrieval system of the national gravity data and control station data files within 1976. This new feature, which is scheduled to start in the Fall of 1976, gives future promise as far as the free-air anomaly contouring is concerned.

CHAPTER 6

APPLICATION OF GRAVITY CORRECTIONS TO ACTUAL LEVELLING LINES AND LOOPS

The mathematical models for the gravity corrections have been derived in chapter 4. The present status of gravity data coverage in Canada has been presented in section 5.2 with the near future prospects of producing free-air anomaly contour maps for the entire country. On the basis of both chapters 4 and 5, one can start seriously considering the application of these corrections to the levelling lines and loops established by precise levelling operations.

This chapter is devoted to the discussion of the practical computations and application of the gravity corrections. Results of an investigation of the behaviour of gravity corrections along real levelling lines and loops using the best available data is given. Finally, conclusions based on the obtained results and other findings are presented. Thus this chapter should help clarifying some of the questions raised and discussed in section 1.6 regarding the feasibility of using the gravity corrections in practice.

6.1 Computational Aspects

The computations of gravity corrections for a levelling section between points i and j, using the rigorous expressions (4-13), (4-27) and (4-36), can be easily programmed for a computer evaluation. The basic input data to the program are: levelled heights

h_i and h_j; scaled latitudes ϕ_i and ϕ_j; and free-air anomalies Δg^F_i and Δg^F_j. The levelled heights are readily available from levelling field books. The latitudes are usually scaled off the available maps (e.g. National Topographic Map series, see section 5.2.1). The possible means of obtaining the free-air gravity anomalies are outlined in section 6.1.1.

For computing the accuracy estimates of the gravity corrections (to examine their reliability) the accuracy estimates of the heights and of the anomalies are also needed. For convenience, section 6.1.3 provides tables to facilitate the approximate evaluation of the gravity corrections.

6.1.1 Sources of Gravity Anomaly Data

The free-air gravity anomaly at any bench mark i along the levelling line can be obtained from one of the following sources:

1. By direct observation of actual gravity value g_i at the bench mark i. The anomaly Δg^F_i is obtained then as follows:

$$\Delta g^F_i = g_i - \gamma_{o,i} + 0.3086\ h_i, \tag{6-1}$$

   where $\gamma_{o,i}$ is the normal gravity (computed e.g. from equation 2-36) and h_i is the levelled elevation. The units here are mgals and metres.

2. By using the observed gravity value g_k at another point "k" not far apart from i and reducing it to i by the appropriate free-air correction [Heiskanen and Vening-Meinesz, 1958; Krakiwsky, 1965], i.e.:

$$g_i = g_k + 0.3086\ (h_k - h_i)\ , \tag{6-2}$$

where h_k is the elevation of k in metres. Δg_i^F can be then evaluated from equation (6-1) after substitution from equation (6-2);

3. By using the least-squares interpolation (prediction) techniques to obtain the best fitting surface to the free-air anomaly field surrounding the bench mark i (see, e.g. [Vaníček et al., 1972]) from the available point gravity data in the area of interest. This approach is particularly suitable for flat areas [Moritz, 1963]. In mountainous areas, it is recommended [e.g. Rapp, 1964] to predict Bouguer anomaly instead, and then transform it to free-air anomaly (see No. 5, below);
4. By using graphical interpolation from a free-air anomaly map, with contour interval less than five milligals, e.g. [Schneider, 1960; Rapp, 1961; Konecny, 1970]. This technique again is better suited for flat areas. It was reported by Derenyi [1965] that with the available point gravity data, it is possible to produce free-air anomaly maps covering the area with existing levelling loops at a scale of 1:100,000 with 1 milligal contour interval;
5. By using graphical interpolation from the available Bouguer anomaly maps. This approach is feasible even in mountainous regions [e.g. Vykutil, 1964]. The Bouguer gravity anomalies are known to be less correlated with heights than the free-air anomalies [Uotila, 1960; Moritz, 1963; Vykutil, 1964; Krakiwsky, 1966]. Transformation from the interpolated Bouguer anomaly Δg_i^B, to the corresponding free-air anomaly, Δg_i^F, is achieved by the following simple relationship [Rapp, 1964; Vykutil, 1964]:

$$\Delta g_i^F = \Delta g_i^B + 0.1119\ h_i, \qquad (6\text{-}3)$$

in milligals for the height h_i in metres. It may be worth mentioning here that this approach has proved feasible and comparable with the corresponding least-squares surface fitting technique using two-dimensional approximating polynomial [e.g. John, 1976].

The evaluation, analysis and comparison of the above techniques is a major subject on its own and much research has been done into it. A comprehensive treatment is hence considered outside the scope of the present investigation. For further details, the reader is referred to [Moritz, 1963; Rapp, 1964; Heiskanen and Moritz, 1967; Moritz, 1969; Wilcox, 1974].

6.1.2 Reliability of the Gravity Corrections

The reliability (precision) of the gravity corrections has to be examined first to justify the effort involved in their application in practice. This is usually done by computing the standard deviation of the gravity correction and comparing it to the magnitude of the correction itself. As mentioned before, it would be questionable to look for a correction whose standard deviation is larger in magnitude than the correction itself.

We recall, from section 4.4, that the variances (standard deviations) of the three different kinds of gravity corrections (dynamic, Helmert, Vignal) are computed from equations (4-58), (4-59) and (4-60). From these equations, it can be seen that the standard deviations σ_h and $\sigma_{\Delta g^F}$ of the levelled heights and free-air anomalies at both ends of the levelling section are needed, among other quantities.

The standard deviation σ_h can be reasonably estimated from previous experience and analysis of levelling networks. The GGD of the EPB has adopted certain criteria for assigning accuracy estimates σ_h to the individual heights of gravity stations. They reflect the different acquisition procedures for heights (see Appendix 11 for more details). In case of heights based on spirit levelling, the GGD considers $\sigma_h = 0.03$ m [Hamilton and Buchan, 1965; Buck, 1975]. This value seems to be unrealistic as far as the absolute heights, of bench marks, above the adopted datum are concerned. This is due to all kinds of problems connected with the CPLN, as outlined in section 5.1.2. These problems may result in uncertainties in the heights that may overshadow the gravity corrections, in which case, the computation of gravity correction would be questionable. Dealing with these problems, which involve systematic errors discussed in section 1.3, constitute a complete thesis on its own. For our purpose here, in order to investigate the influence of the gravity corrections on heights, we have to assume that all other influences do not exist. Thus, the value of $\sigma_h = 0.03$ m will be accepted here as a reasonable measure of the internal consistency of the relative heights of bench marks within the network, and will be used as accuracy estimate for the levelled heights involved in the subsequent computations.

As far as $\sigma_{\Delta g^F}$ is concerned, there are two possibilities. Either $\sigma_{\Delta g^F}$ is available as a by-product of the least-squares prediction technique, when predicting the free-air anomaly Δg^F, or there is no estimate available for $\sigma_{\Delta g^F}$. The latter is usually the case when computing the anomaly from observed gravity value or when interpolating the anomaly value from maps. In case of unavailable estimate for the variance of the gravity anomaly in question it can be computed from the following formula:

$$\sigma^2_{\Delta g^F} = (0.05)^2 + (0.3086)^2 \ \sigma^2_h \ , \qquad (6\text{-}4)$$

which is obtained by applying the law of propagation of errors on equation (6-1), and treating γ_o as errorless. Here the standard deviation of the observed gravity is taken as $\sigma_g = 0.05$ mgal, in accordance with [Hamilton and Buckan, 1965; Tanner and Gibb, 1971; Vaníček et al., 1972].

The dynamic, Helmert and Vignal gravity corrections, along with their standard deviations, have been computed for several real levelling sections. Table 6-1 shows the results, for a selected sample of four sections, as compiled from a computer output. For the first two sections, gravity values at the bench marks were observed [Hamilton and Buchan, 1965]. For the last two sections, predicted free-air anomaly values at the bench marks were used. The technique for predicting the anomalies is described in Vaníček et al. [1972], and the data used is from the EPB file.

The obtained results for the first two sections (using gravity observed at bench marks) reveal the high reliability of all three kinds of gravity corrections; their standard deviations are small compared to the magnitude of the corrections. On the other hand, when using predicted gravity anomalies at bench marks, we have obtained two distinctly different results. The first shows adequate reliability of all three kinds of gravity corrections. The second shows adequate reliability of only the dynamic and Vignal gravity corrections. The reliability of Helmert gravity correction seems

TABLE 6-1

Reliability of Gravity Corrections

(Sample Results)

		LEVELLING SECTIONS			
		FROM	TO	FROM	TO
ϕ (N)		44° 18' 28"	44° 19' 12"	45° 54' 37"	45° 54' 49"
λ (W)		78° 18' 02"	78° 18' 15"	77° 04' 33"	77° 04' 37"
h (m)		193.43	215.68	140.54	111.53
σ_h (m)		0.03	0.03	0.03	0.03
Δg^F (mgal)		-20.55	-19.73	-33.38	-36.53
$\sigma_{\Delta g^F}$ (mgal)		0.05	0.05	0.05	0.05
S (Km)		1.40		0.36	
α (deg)		348°		347°	
Gravity Corrections GC and σ_{GC} (mm)	GC^D	-0.5709		1.1825	
	σ_D	0.0014		0.0021	
	GC^H	0.8681		-0.4297	
	σ_H	0.0148		0.0091	
	GC^V	-0.4570		1.0344	
	σ_V	0.0012		0.0019	

TABLE 6-1 (Cont'd)

	LEVELLING SECTIONS			
	FROM	TO	FROM	TO
ϕ (N)	51° 02' 18"	51° 02' 18"	49° 33' 18"	49° 33' 42"
λ (W)	114° 06' 30"	114° 05' 36"	114° 19' 36"	114° 21' 30"
h (m)	1098.93	1051.07	1215.60	1237.27
σ_h (m)	0.03	0.03	0.03	0.03
Δg^F (mgal)	-17.55	-17.37	-17.22	-17.42
$\sigma_{\Delta g^F}$ (mgal)	0.74	0.72	4.67	4.49
S (km)	1.05		2.41	
α (deg)	90°		288°	
Gravity Corrections GC and σ_{GC} (mm): GC^D	1.0971		-0.4935	
σ_D	0.0253		0.0717	
GC^H	-11.9377		6.3160	
σ_H	1.1320		8.1022	
GC^V	0.8520		-0.3827	
σ_V	0.0251		0.0717	

to be questionable.

The reason for this is the relatively high standard deviation $\sigma_{\Delta g^F}$ of the predicted anomaly value, which is about two orders of magnitude larger than the standard deviation of the observed gravity values (0.05 mgal). This illustrates the fact that the accuracy of gravity anomalies is much impaired by the poor determination of heights, even though the gravity observations are of a very high quality.

The reason for the variance σ_H^2 alone being seriously affected (compared with the dynamic and Vignal systems) can be verified by examining equations (4-58), (4-59) and (4-60). In case of σ_D^2 or σ_V^2, the variance $\sigma^2_{\Delta g^F}$ is multiplied by 0.25 Δh^2. In case of σ_H^2, the $\sigma^2_{\Delta g^F}$ is multiplied by $\bar{h}^2$ which is in this case much larger than Δh^2.

The application of dynamic gravity correction based on predicted anomalies has been argued by Boal [1972]. Boal stated that the uncertainty introduced into the dynamic heights by using interpolated values of gravity is of the same magnitude as the corresponding gravity corrections. Thus he recommended that the present computation of dynamic heights based on normal gravity only should be continued until observed gravity values are available at bench marks.

To clarify the above arguement, let us now seek the condition for the uncertainty of the gravity correction (due to the uncertainty of the free-air anomalies) to be smaller in magnitude than the contribution of the anomalies to the computed gravity correction. This condition can be obtained by examining equations

(4-45), (4-46), (4-47), (4-61), (4-62) and (4-63). In case of the dynamic and Vignal systems, the condition is: $0.7\ \sigma_{\Delta g^F} < \overline{\Delta g}^F$, which is met, for instance, for all the four levelling sections given in Table 6-1. For the Helmert system, the condition is: $1.4\ \sigma_{\Delta g^F} < \Delta\Delta g^F$, which is not met in the case of the last two levelling sections.

Accordingly, we can see that all three kinds of the gravity corrections can be computed with adequate precision, whether we use observed gravity or predicted anomaly values, providing that the above condition concerning the precision of the used anomaly is satisfied. If the available free-air anomaly is adequately reliable ($\sigma_{\Delta g^F} < \Delta g^F$), then it can be guaranteed that the computed dynamic and Vignal gravity corrections are also adequately reliable. Otherwise, if the precision of the anomaly is questionable (i.e. $\sigma_{\Delta g^F} > \Delta g^F$), the gravity corrections should not be computed. However, even with reliable anomalies, the reliability of Helmert gravity correction is not easily predictable. This is so because GC^H is a function of the difference of gravity anomalies (rather than their average value) which increases the requirements on the accuracy of the involved free-air anomalies.

At this point, one can see that the argument given by Boal does not seem to be valid providing that the predicted gravity anomalies (using for instance the EPB File) have precisions adequate in the sense discussed above. It is generally known that predicted values are not as accurate as the observed values of gravity. However, if the corrections using predicted gravity are adequately reliable, they should be applied. The gain in accuracy of resulting dynamic heights is evident.

6.1.3 Gravity Correction Tables

Before the era of computers and pocket calculators, it has traditionally been found convenient in practice to use tables, especially when dealing with corrective terms. An example of such tables, mentioned in chapter 3, is Bowie and Avers tables for computing the orthometric and dynamic corrections based on normal gravity. Even now with the calculating machines being widely used everywhere, the tables may still be useful (especially for inadequately trained parties) in eliminating the need of punching lengthy numbers and using calculating machines.

Therefore, it was decided to provide here a set of tables to facilitate the practical (approximate) computations of the gravity corrections. The gravity correction tables can, in the author's opinion, serve several purposes. Firstly, the gravity correction tables will complement the Bowie and Avers tables mentioned above. In other words, the two sets of tables will form a complete package for correcting the levelled height differences for the influences of normal gravity and actual gravity irregularities. Such a package leads to rigorous heights defined on the basis of actual gravity, and can be used on occasions where machine computations are either not available or not desired. Also, the gravity correction tables can be used for quick checking on the magnitude and/or the general trend of any gravity correction for different combinations of arguments. Finally, providing a gravity anomaly contour map is available for the area surrounding the observed levelling route, these tables give one the opportunity to evaluate and apply the gravity corrections even in the field, by the precise levelling field parties, during the course of observations.

The gravity correction tables are contained in Appendix III, and are accompanied with detailed instructions illustrating their usage. For reasons given in Appendix III, the sequence of presenting the tables is: Vignal, Dynamic and then Helmert. The tables were computed on the basis of the approximate expressions for the GC's (equations 4-41, 4-43, and 4-44). The value of $\gamma^*_{o,45°}$ = 980 624.0 mgal was taken for the reference gravity G, which is the value presently used in Canada. The GC's are tabulated for different values of observed height-difference Δh, average height $\bar{h}$, free-air anomaly $\overline{\Delta g}^F$ over the levelling section under consideration. In addition, it was found convenient to break down the formula for the Helmert gravity correction GC^H into two terms that can be found from separate tables. An effort has been made to arrange the tables in such a way as to allow the user to perform only a simple linear interpolation for any combination of arguments.

To obtain any gravity correction from the tables, one needs only two arguments. One is either the levelled height difference Δh or the average height $\bar{h}$, which is usually available from the field book. The other is either the average free-air anomaly $\overline{\Delta g}^F$ or the anomaly difference $\Delta\Delta g^F$, which can be obtained by anyone of the means outlined in section 6.1.1. Once the gravity correction is obtained (with the appropriate sign), it is added algebraically to the height difference based on normal gravity to yield the gravimetrically corrected height difference.

To close this section up, let us, as an example, evaluate the gravity corrections for one levelling section, using the tables in Appendix III. To be able to compare the results, let us consider the second levelling section in Table 6-1 for which the corrections have been evaluated by the computer. For this section we have the following information:

$$\bar{h} = \frac{1}{2}\,(140.54 + 111.53) = 126.035 \text{ m},$$

$$\Delta h = 111.53 - 140.54 = -\,29.01 \text{ m},$$

$$\overline{\Delta g}^F = \frac{1}{2}\,(-33.38 - 36.53) = -34.955 \text{ mgal},$$

$$\Delta\Delta g^F = -36.53 - (-33.38) = -3.15 \text{ mgal} \quad .$$

The Vignal gravity correction GC^V is obtained from Table III-1 for arguments $\overline{\Delta g}^F$ and Δh as:

$$GC^V = +\,1.035 \text{ mm}.$$

The dynamic gravity correction GC^D is obtained from Table III-1 as explained in section III-2. First, a value of -5.007 mgal for $\overline{\delta\gamma}_o$, obtained from Table I-1 (for latitude $\phi \doteq 46°$), is added to $\overline{\Delta g}^F$. Then, the resultant (-39.962 mgal) is used along with Δh as arguments to enter Table III-1 and get:

$$GC^D = 1.183 \text{ mm}.$$

The first term in the Helmert gravity correction, GC^H, equation is obtained from Table III-1 as + 0.402 mm, for arguments $\Delta\Delta g^F$ and $\bar{h}$. The second term is obtained from Table III-2 as -0.833 mm, for arguments $\bar{h}$ and Δh. Thus, GC^H is given as:

$$GC^H = +\,0.402 - 0.833 = -\,0.431 \text{ mm}$$

Comparison of these results with the corresponding values in Table 6-1 reveals a very good agreement for all the three kinds of gravity corrections. Therefore, it can be concluded that the gravity correction tables given in Appendix III give, with appropriate linear interpolation, the gravity corrections with adequate precision.

6.2 Test Results Using Actual Data

We recall from section 1.6 that one of the questions, which needs to be clarified, is whether the gravity corrections will accumulate or cancel as one goes along an entire levelling line or loop. The answer of course depends on many factors: the relative geographical location of the levelling sections constituting the line; the nature of the elevation profile along the line; the characteristics of the actual gravity field along the line; and on the overall length of the line.

This section presents an attempt to give some quantitative answers to the above question, based on actual data. The idea here is to compute the accumulated gravity corrections for some selected lines and loops, and compare them (numerically or graphically) with the corresponding accumulated standard error in precise levelling, based on the Canadian standards of accuracy. At the same time, we want to determine the confidence intervals for the gravity corrections.

The Canadian specifications for vertical control [GSC, 1960; S&M Branch, 1961; S&M Branch, 1973] state that the allowable discrepancy Δ between corresponding forward and backward runnings in precise levelling is not to exceed 4 mm $\sqrt{S(km)}$, in absolute value. The GSC has conducted some studies on the accidental observational errors in the CPLN [Peterson, 1970; Boal, 1971a; Boal, 1971b] in an attempt to interpret the specifications in terms of actually achieved confidence intervals of the height differences Δh. The obtained results indicate that the specified allowable limit of 4 mm is met in 85% of cases, i.e.

that it is equivalent to 1.5 σ_Δ, [Boal, 1971b], where σ_Δ is the standard deviation of the discrepancy Δ, standardized for 1 km by dividing by square-root of the line length.

It is known that:

$$\Delta = F - B, \tag{6-5}$$

$$\Delta h = \frac{1}{2} (F + B), \tag{6-6}$$

where F denotes the forward and B the backward runnings. Assuming that F and B are independent and both having the same standard deviation σ, the law of propagation of errors [Braaten et al., 1950; Vaníček, 1973] applied on (6-5) and (6-6) yields:

$$\sigma_\Delta^2 = (\sigma_F^2 + \sigma_B^2) = 2\sigma^2 \;, \tag{6-7}$$

$$\sigma_{\Delta h}^2 = (\tfrac{1}{2})^2 \; (\sigma_F^2 + \sigma_B^2) = \frac{1}{2}\sigma^2 \;, \tag{6-8}$$

from which one gets:

$$\sigma_{\Delta h} = \frac{1}{2}\sigma_\Delta \;. \tag{6-9}$$

From the above discussion, it can be seen that:

$$\sigma_\Delta = (\frac{4}{1.5}) \text{ mm}, \tag{6-10}$$

and consequently by substituting in equation (6-9), we get:

$$\sigma_{\Delta h} \doteq 1.33 \text{ mm} \;, \tag{6-11}$$

standardized for 1 km. This value of $\sigma_{\Delta h}$ is used in our investigation herein as a representative value for the contribution of accidental errors in the CPLN.

It may be of interest to mention here that in the United States the specifications for precise vertical control [Holdahl, 1974; U.S. Dept. of Commerce, 1974] stipulate that the standard error $\sigma_{\Delta h}$ is not to exceed 0.7 mm $\sqrt{S(\text{km})}$. This value corresponds approximately to the International specifications for random error contribution (0.6 mm per 1 km) in precise level nets [Baeschlin, 1960a; Krakiwsky, 1965]. The total error (including random and systematic parts), according to the International specifications, should not exceed 1.08 mm per 1 km [Baeschlin, 1960a].

Using equation (6-11), the standard error of a height difference Δh_{oi} derived from Precise levelling work in Canada can be written as:

$$\sigma_{\Delta h_{oi}} = 1.33 \text{ mm} \sqrt{S_{oi} \text{ (km)}} \quad , \qquad (6\text{-}12)$$

where S_{oi} is the sum of the lengths of the segments of the levelling line up to the bench mark "i", starting from the initial point "o" of the line. Thus, we have:

$$\Delta h_{oi} = \sum_{j=0}^{i-1} \Delta h_{jk} \quad , \qquad (6\text{-}13)$$

$$S_{oi} = \sum_{j=0}^{i-1} S_{jk} \quad , \qquad (6\text{-}14)$$

where: $k = j + 1$, Δh_{jk} and S_{jk} are the levelled height difference and the length of the section between consecutive bench marks j and k.

Similarly, the accumulated gravity correction is given by:

$$GC_{oi} = \sum_{j=0}^{i-1} GC_{jk} \quad . \qquad (6\text{-}15)$$

The standard deviation of the GC_{oi} can be obtained by propagating the standard deviations of individual section corrections using equation (6-15) which gives:

$$\sigma_{GC_{oi}} = [\sum_{j=0}^{i-1} \sigma^2_{GC_{jk}}]^{1/2} \quad . \qquad (6\text{-}16)$$

By doing this, we neglect the correlation between the GC_{jk}'s.

A computer program called LOOPGC has been developed by the author to compute the accumulated quantities stated above. The program uses the rigorous formulae given in section 4.2 to compute

GC_{jk} in equation (6-15). The computation of the variances $\sigma^2_{GC_{jk}}$ in equation (6-16) are based on the expressions developed in section 4.4. Additional details about the LOOPGC program can be obtained from the Surveying Engineering Computer Library, UNB.

Let us now outline the sources of data for the selected lines and loops. The discussion of data coverage in Canada (chapter 5) indicated marked lack of observed gravity values at the bench marks. The only significant source of gravity data observed at bench marks is the project undertaken by the GGD of the EPB in 1964 in the area of Eastern Ontario [Hamilton and Buchan, 1965]. Realizing the fact that this area is of a relatively low elevation (less than 300 m on the average), it was felt that relying solely on these test data may show inconclusive results. In order to be able to draw more realistic conclusions one must have test data from various locations having different characteristics. Therefore, it was decided to consider only one line and one loop from Eastern Ontario, and search for other sources of relevant data, even outside Canada, that would serve our purpose.

From the author's search in the literature, two sources of reliable data have been found relevant. The first is Rapp's M.Sc. Thesis [1961] containing observed gravity values at bench marks along a first-order line in West Germany. The second is Krakiwsky's publication [1966] that gives observed gravity values at bench marks along a first-order loop in West Germany. Both the line and the loop are of medium elevation.

To incorporate also some levelling routes of high elevations, two lines from a first-order levelling loop in Alberta have been used. Since observed gravity values at the bench marks are not available for the Alberta loop, the predicted free-air anomalies are used instead. The reason behind our choice of Alberta loop is that this loop has been previously investigated by Vaníček [1970], Christodulidis and Vaníček [1972] and Vaníček et al. [1972]. This gives us the opportunity of checking our results.

The descriptions, computations and results associated with the selected lines and loops will be discussed in the following two sections. It should be noted here that even the German line and loop will be analysed to the Canadian standards of accuracy, since similar characteristics may be encountered in Canada as well.

6.2.1 Behaviour of Gravity Corrections Along Test Lines

The results obtained from four real test lines are discussed in this section. The first two are based on observed gravity at bench marks and the last two are based on predicted free-air anomalies. The computer output from LOOPGC program containing the pertinent information as well as detailed computations of accumulated gravity corrections and their standard deviations for the four lines is given in Appendix IV.

The first line, labelled here as "Line No. 4", was selected in Eastern Ontario [Hamilton and Buchan, 1965]. The location of the line (about 240 km long) is shown in Figure 6-1. The elevation along the line varies from 76 m to 402 m, and the free-air anomaly (based on the 1967 system) ranges between +9 mgal and -27 mgal. It should

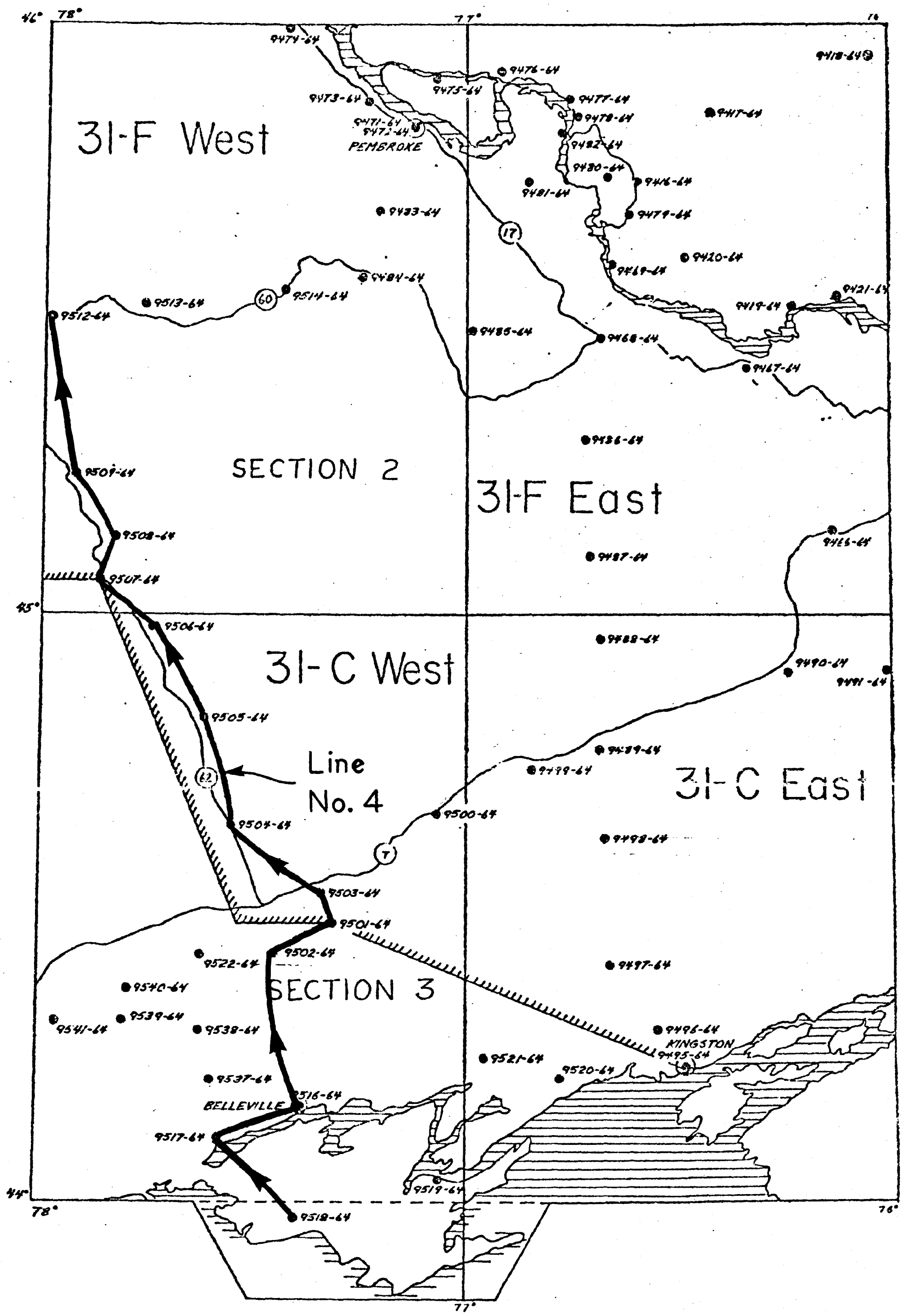


FIGURE 6-1

Location of Test Line No. 4 (Eastern Ontario).

(From: Hamilton and Buchan [1965].

be noted here that the free-air anomalies, Δg^F, given in Hamilton and Buchan [1965] were computed on the basis of the 1930 system. Thus, before using these anomalies in our computations, they have been converted to the 1967 new system using the GGD empirical formula (equation 5-2) [Valliant, 1975], i.e.:

$$\Delta g^F_{1967} = \Delta g^F_{1930} + (-0.95 - 13.6 \sin^2\phi + 0.05\phi\ ) , \qquad (6\text{-}17)$$

mgal for ϕ in degrees of arc. The standard deviations σ_g and σ_h were estimated as $\sigma_g = 0.05$ mgal and $\sigma_h = 0.03$ m by Hamilton and Buchan.

The graphical display of the accumulated gravity corrections GC_{oi} (dynamic, Helmert and Vignal) against the corresponding accumulated standard error $\sigma_{\Delta h_{oi}}$ of precise levelled height difference, at each bench mark along the line, is depicted in Figure 6-2.

The second test line, labelled here as "Line No. 8", was used by Rapp [1961], in his investigation of the different orthometric heights. The line (about 101 km long) is located in West Germany and extends from Munich southwards to the border of Austria. It constitutes a part of the UELN, and its geographical location is shown in Figure 6-3. The free-air anomalies taken from Rapp [1961] are again based on the 1930 system. The corresponding 1967 values have been obtained from the IAG devised formula (equation 5-3) [IAG, 1971], i.e.:

$$\Delta g^F_{1967} = \Delta g^F_{1930} + (3.2 - 13.6 \sin^2\phi) \text{ mgal.} \qquad (6\text{-}18)$$

There are no accuracy estimates for g and h available in Rapp [1961]. However, it was stated that both heights and gravity observations were

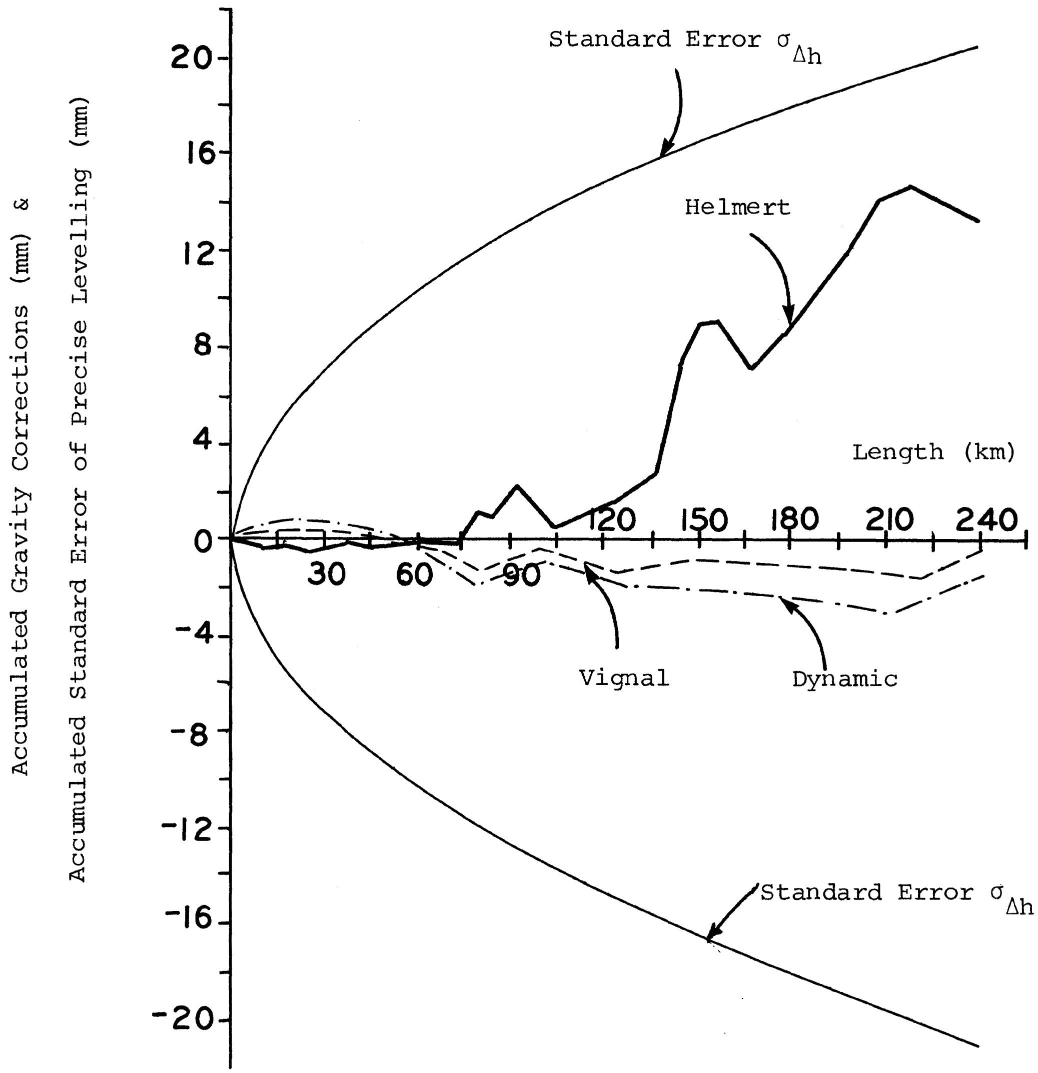


FIGURE 6-2

Accumulated Gravity Corrections for Test Line No. 4

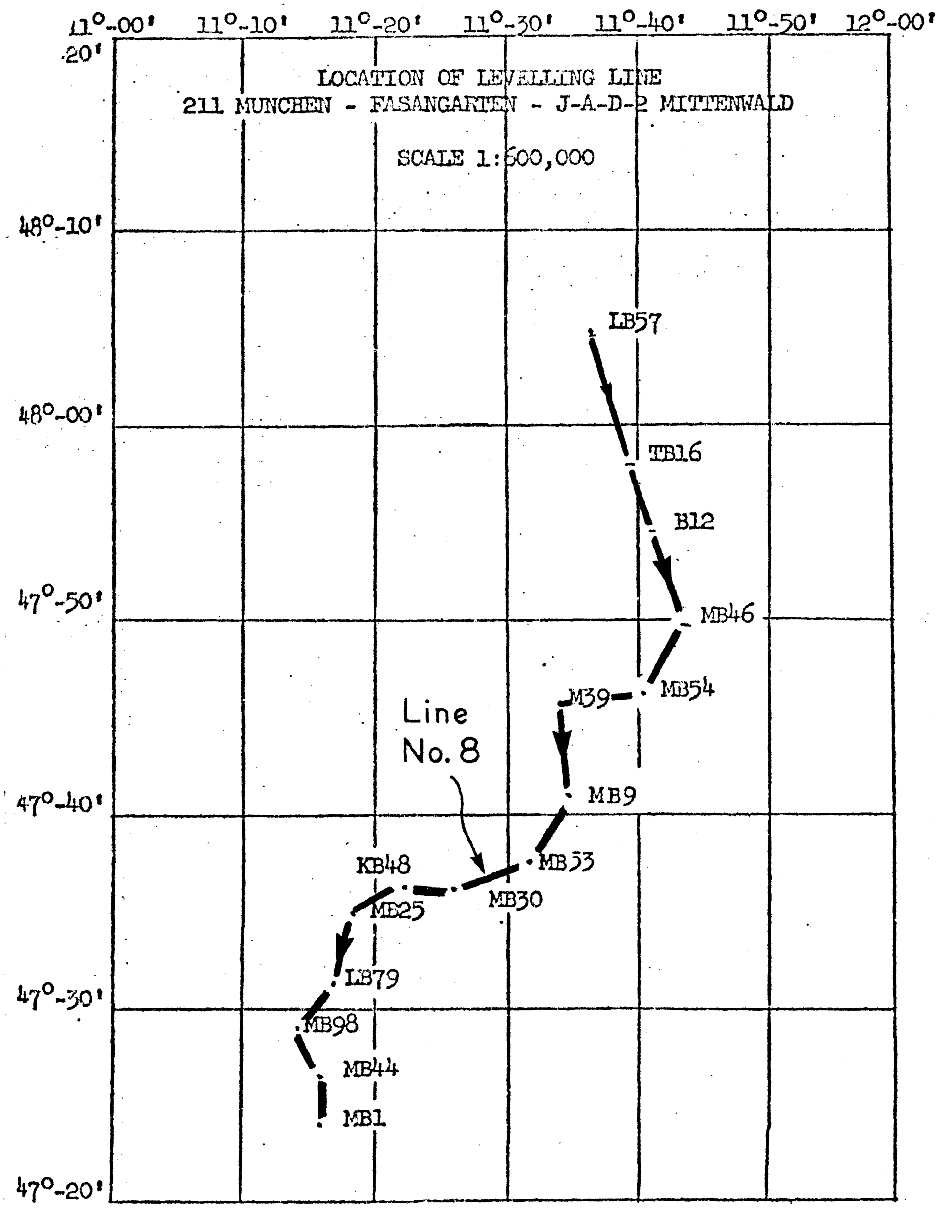


FIGURE 6-3

Location of Test Line No. 8 (Germany-Austria, Part of UELN)

(From: Rapp [1961])

of high quality. Hence, the values $\sigma_g = 0.05$ mgal and $\sigma_h = 0.03$ m have been used also in the computations of this line. The elevation along the line ranges between 546 m and 950 m, and the anomalies vary from -13 mgal to -36 mgal. The plot of accumulated gravity corrections and standard error of precise levelling for this line is given in Figure 6-4.

It is worth mentioning here that the accumulated Helmert gravity correction of 80 mm (Figure 6-4) agrees with results given by Rapp. This can be considered as an independent check on the correctness of the formula for the Helmert gravity correction derived here.

The third and fourth test lines are parts (about 40 km and 15 km, respectively) of the Alberta loop. These two lines are labelled here as "Line No. 9" and "Line No. 10", respectively, as shown in Figure 6-5. The free-air anomaly and its resulting standard deviation at each bench mark of these two lines, were predicted using the technique described by Vaníček et al. [1972] and gravity data from the EPB file (Appendix II). Here again, the value of $\sigma_h = 0.03$ m was adopted as before. The elevation along Line No. 9 varies from 1135 m to 1314 m and the free-air anomaly from -3 mgal to -18 mgal. For Line No. 10, the elevation ranges between 1047 m and 1156 m, and the anomaly is -17 mgal on the average. The corresponding plots of accumulated gravity corrections and precise levelling standard errors are shown in Figures 6-6 and 6-7.

Comparison of the accumulated dynamic and Vignal gravity corrections (Figures 6-6 and 6-7) with the corresponding results

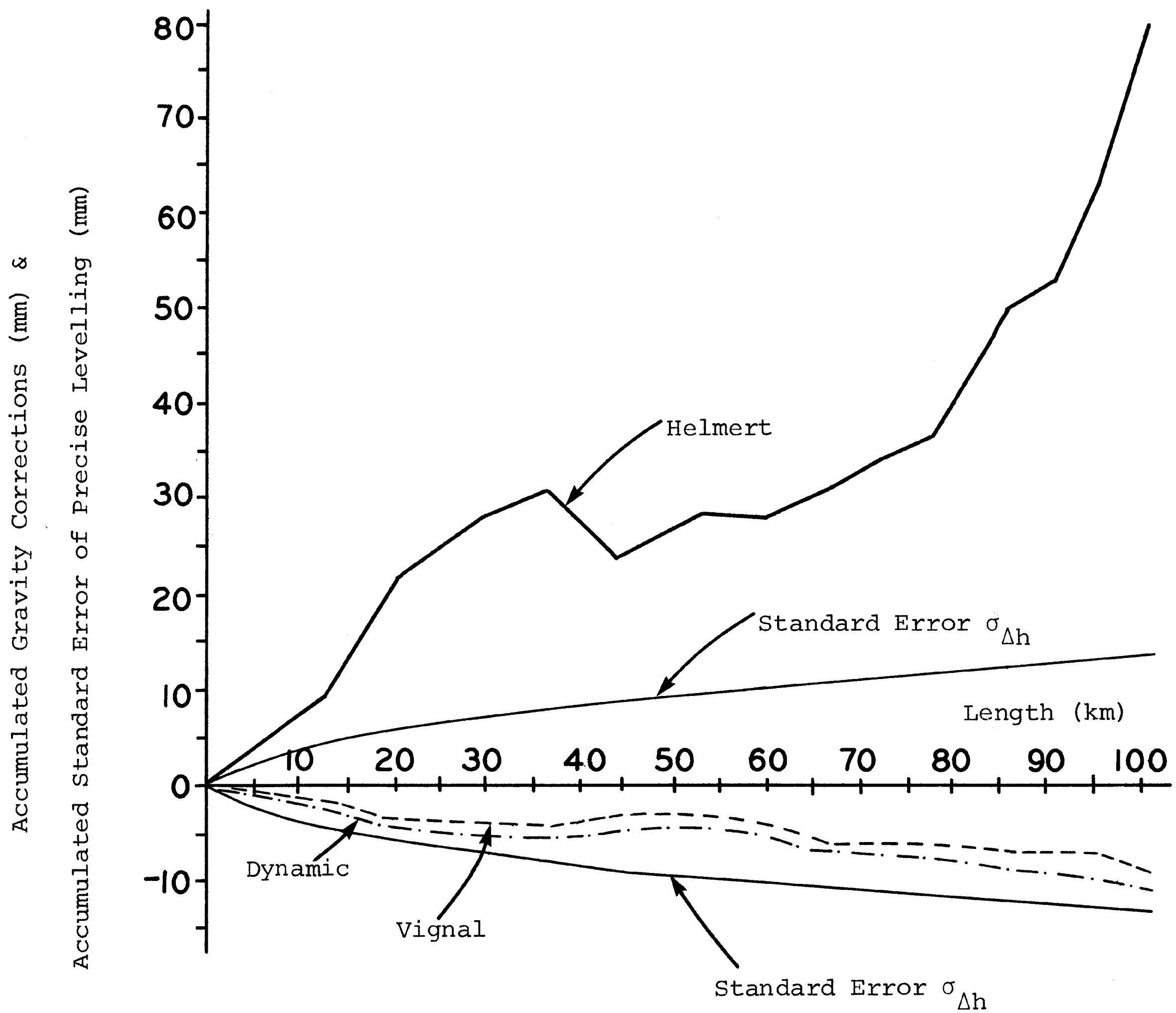


FIGURE 6-4

Accumulated Gravity Corrections for Test Line No. 8.

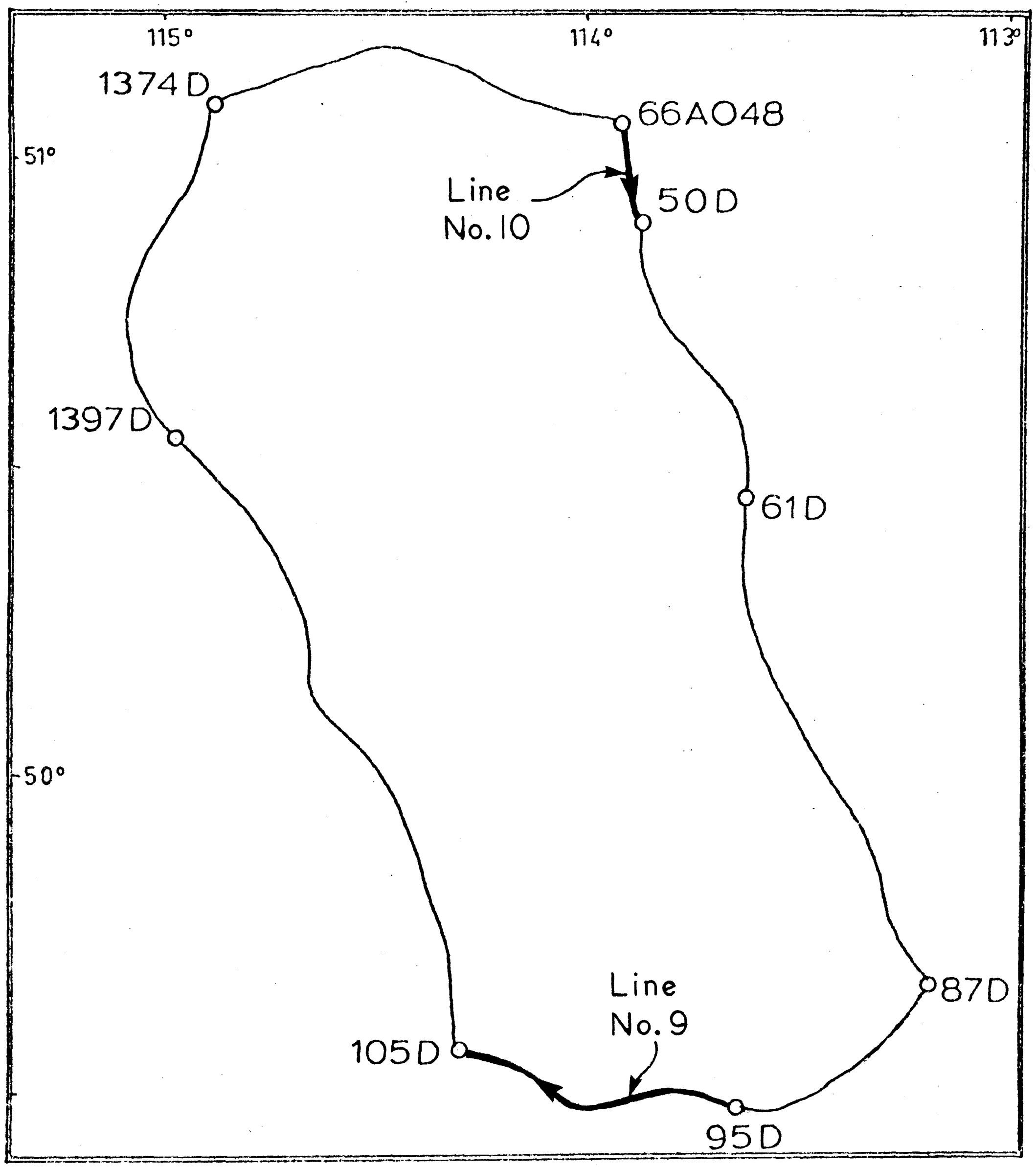


FIGURE 6-5

Location of Test Lines Nos. 9 and 10 (Parts of Alberta Loop).

(From: Vaníček et al. [1972])

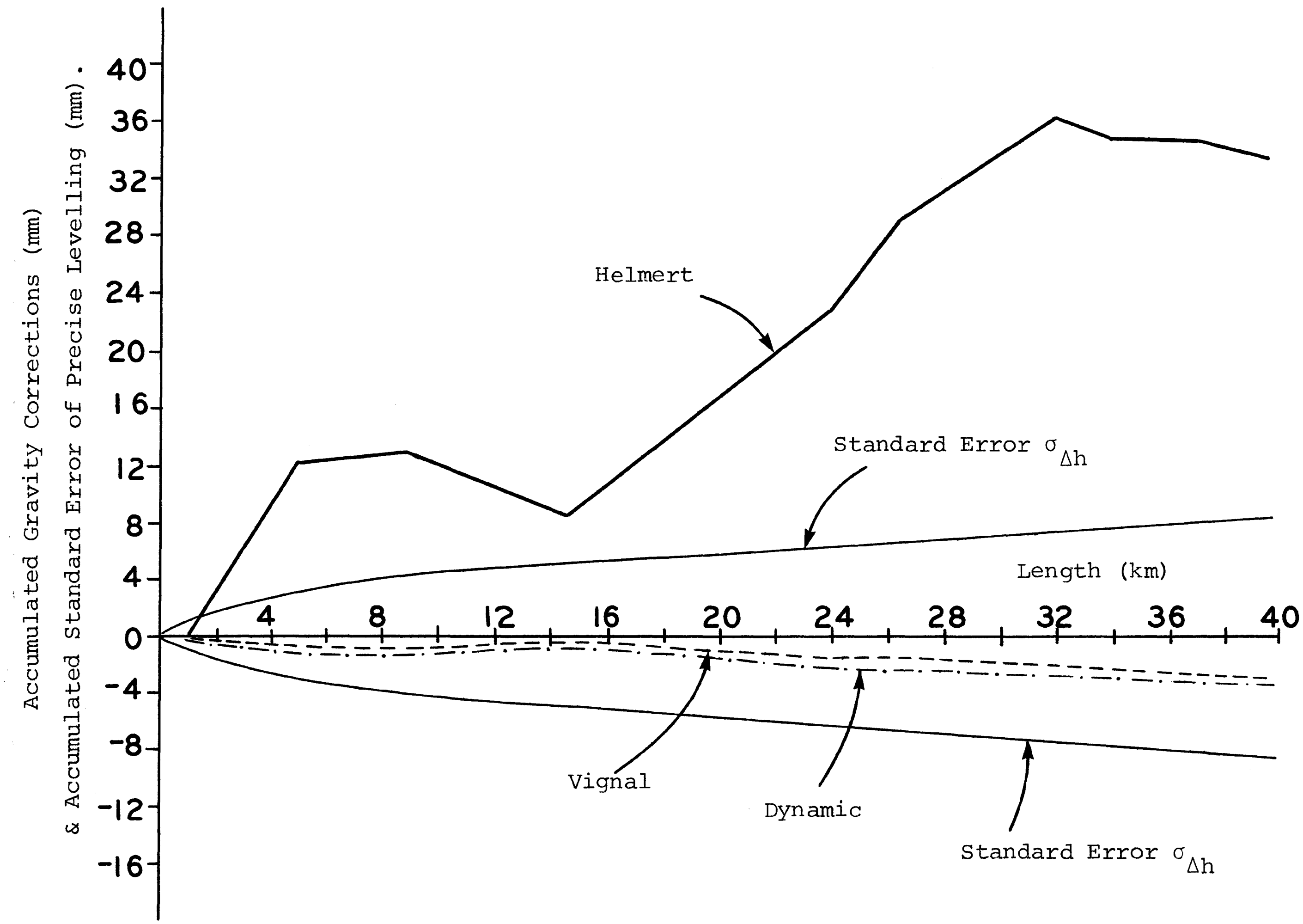


FIGURE 6.6

Accumulated Gravity Corrections For Test Line No. 9.

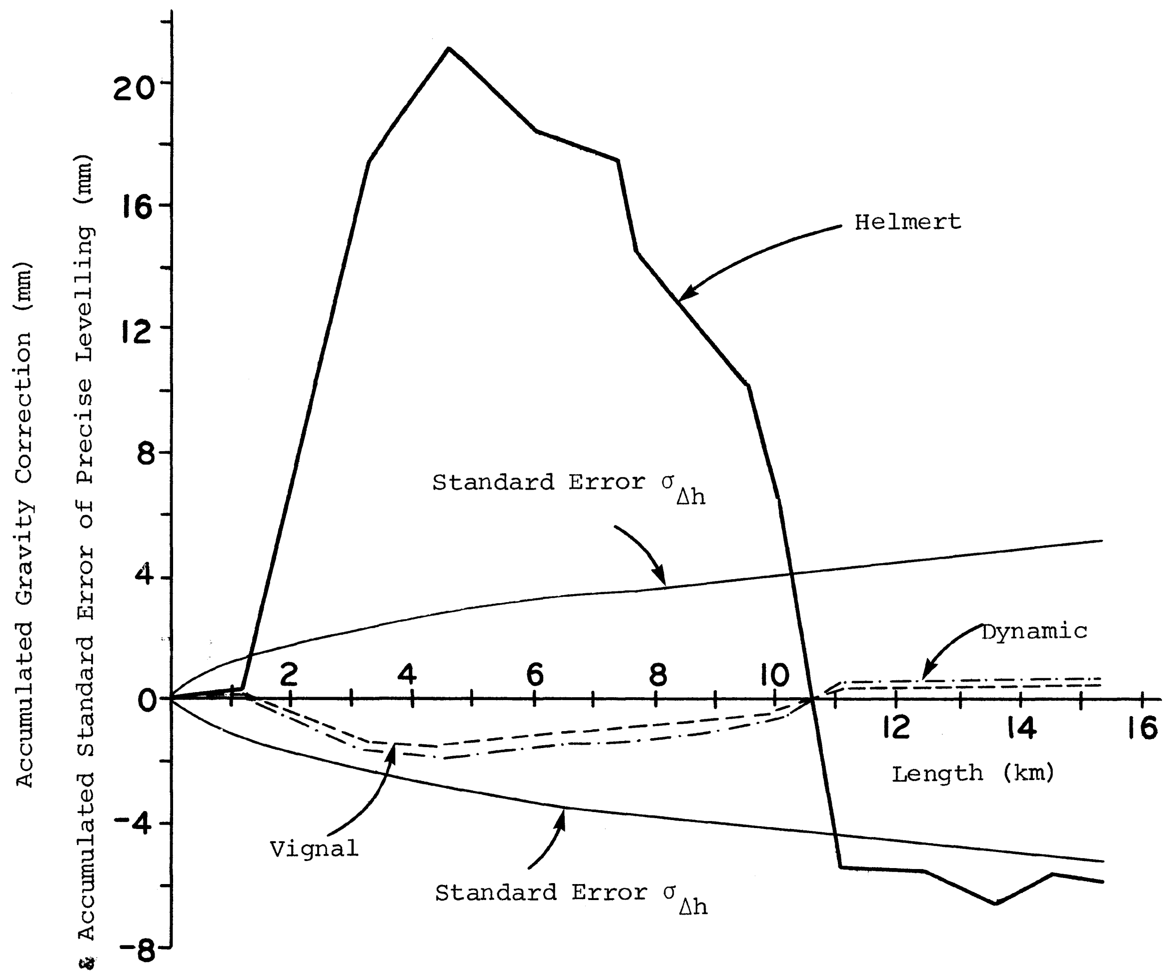


FIGURE 6-7

Accumulated Gravity Corrections for Test Line No. 10.

given in Vaníček et al. [1972] (who were investigating only dynamic and Vignal heights) shows the following. There is a slight difference in Vignal gravity correction, which can be attributed to the different gravity fields used in both cases. Here we are using the new EPB gravity file, while Vaníček et al. used the old gravity file (see section 5.2.3). On the other hand, the disagreement in the dynamic gravity correction is found significant (one order of magnitude). According to Appendix V and the discussion given at the end of section 4.2, we have made sure that our formulae for the gravity corrections are correct. This suggests that the values of either kind of dynamic heights of Alberta loop reported in Vaníček et al. [1972] are in error.

From the graphical display of the obtained results (Figures 6-2, 6-4, 6-6 and 6-7), we can notice the following about the behaviour of the gravity corrections along the tested levelling lines:

1. The accumulation of the gravity corrections along levelling lines do not generally cancel out;
2. The behaviour of both the accumulated dynamic and Vignal gravity corrections, GC^D_{oi} and GC^V_{oi}, along levelling lines is almost the same, with the former being larger in magnitude than the latter. Both corrections are, however, within the allowable limits of accumulated standard error $\sigma_{\Delta h_{oi}}$ in precise levelling;
3. The behaviour of the accumulated Helmert gravity correction, GC^H_{oi} is the most abrupt and most pronounced, compared to the dynamic and Vignal systems;
4. At several locations along levelling lines, the GC^H_{oi} is much larger in magnitude than the corresponding $\sigma_{\Delta h_{oi}}$;

5. In spite of the fact that accumulated gravity correction over the entire levelling line may be less than $\sigma_{\Delta h}$ over the line, it is obvious that the heights of several intermediate bench marks are significantly influenced by the gravity correction. Such influence is more pronounced again in case of Helmert system. This can be easily noticed from the differences in slope of the two curves representing GC_{oi} and $\sigma_{\Delta h_{oi}}$. In other words, for some levelling sections along the line, the gravity correction is larger than the expected $\sigma_{\Delta h}$. Thus relying only on the heights of the two ends of a levelling line (which may not be significantly affected by the irregularities of the gravity field) and neglecting what is happening along the entire line is not a valid argument. A levelling line could be several hundred kilometres long, and new levelling extensions could be initiated from one of the intermediate bench marks of the line whose height could be significantly affected by the lack of application of gravity correction.

From the numerical results tabulated in Appendix IV, the following remarks apply to the standard deviation of the accumulated gravity corrections:

1. For test lines No. 4 and No. 8 (computed on the basis of observed gravity), the reliability of the accumulated gravity corrections is high (having very small standard deviations). Even at the 95% probability level, the accumulated gravity correction is significantly different from zero, and thus must be taken into account. This holds true for all three kinds of gravity corrections.

2. The test lines No. 9 and No. 10 (computed on the basis of predicted anomalies) show adequate reliability of the accumulated dynamic and Vignal gravity corrections even at the 95% probability level. For most sections of the two lines, the accumulated Helmert gravity corection is different from zero at the 68% probability level. Few sections indicate, however, a questionable reliability of the corresponding GC^H. Nevertheless, on the basis of the discussion given in section 6.1.2, these results may be still considered satisfactory since the predicted anomalies are obtained with adequate precision ($\sigma_{\Delta g^F} < \Delta g^F$).

At this stage, we can see that the EPB File (Appendix II) proved to be adequate for prediction of free-air anomalies at the bench marks, of the two investigated lines in Alberta loop, for the evaluation of gravity corrections with sufficient precision. This may be or may not be the case in other parts of the country. The investigation of the quality and adequacy of the EPB File for such an application is a major task that needs to be carried out in the future. As mentioned earlier, the use of predicted gravity is not as accurate as using observed gravity values, but still renders the accuracy of heights better than neglecting the gravity corrections altogether. This implies that for the levelling lines running into areas covered by sufficiently dense and reliable point gravity and height information, the observed gravity values at bench marks are not necessarily required. The predicted anomalies with adequate reliability can be used instead.

6.2.2 Behaviour of Gravity Corrections Along Test Loops

The results obtained from two real levelling loops are presented in this section. Both loops have been computed on the basis of gravity values observed at bench marks. Details about the pertinent data as well as the accumulated gravity correction computations are tabulated in Appendix IV, for both loops.

The first test loop (about 254 km), labelled here as "Loop No. 5" is located in Eastern Ontario [Hamilton and Buchan, 1965], as illustrated in Figure 6-8. The anomaly varies from -2 mgal to -30 mgal, and the elevation from 42 m to 424 m along the loop. The plot of accumulated gravity corrections and standard error in levelling is shown in Figure 6-9.

The second test loop (about 389 km) is labelled here as "Loop No. 6". It is a part of the UELN in West Germany and was used by Krakiwsky [1966] in his analysis and comparison of various systems of orthometric heights. The loop (see Figure 6-10) is located at the S-W border of East Germany and the N-W border of Czechoslovakia, and extends southward towards Austria. Along the loop, the anomaly varies from -12 mgal to +70 mgal, and the elevation ranges between 237 m and 560 m. Figure 6-11 depicts the accumulated gravity corrections and standard error in precise levelling along the loop.

All the comments and remarks stated in the previous section about the behaviour of the gravity corrections along levelling lines hold true as well for the levelling loops. In addition, the following can be noticed. The gravity corrections do not cancel out round a closed loop, and they generally produce a loop closure. The obtained loop closures (see Figures 6-9 and 6-11) for the dynamic, Helmert

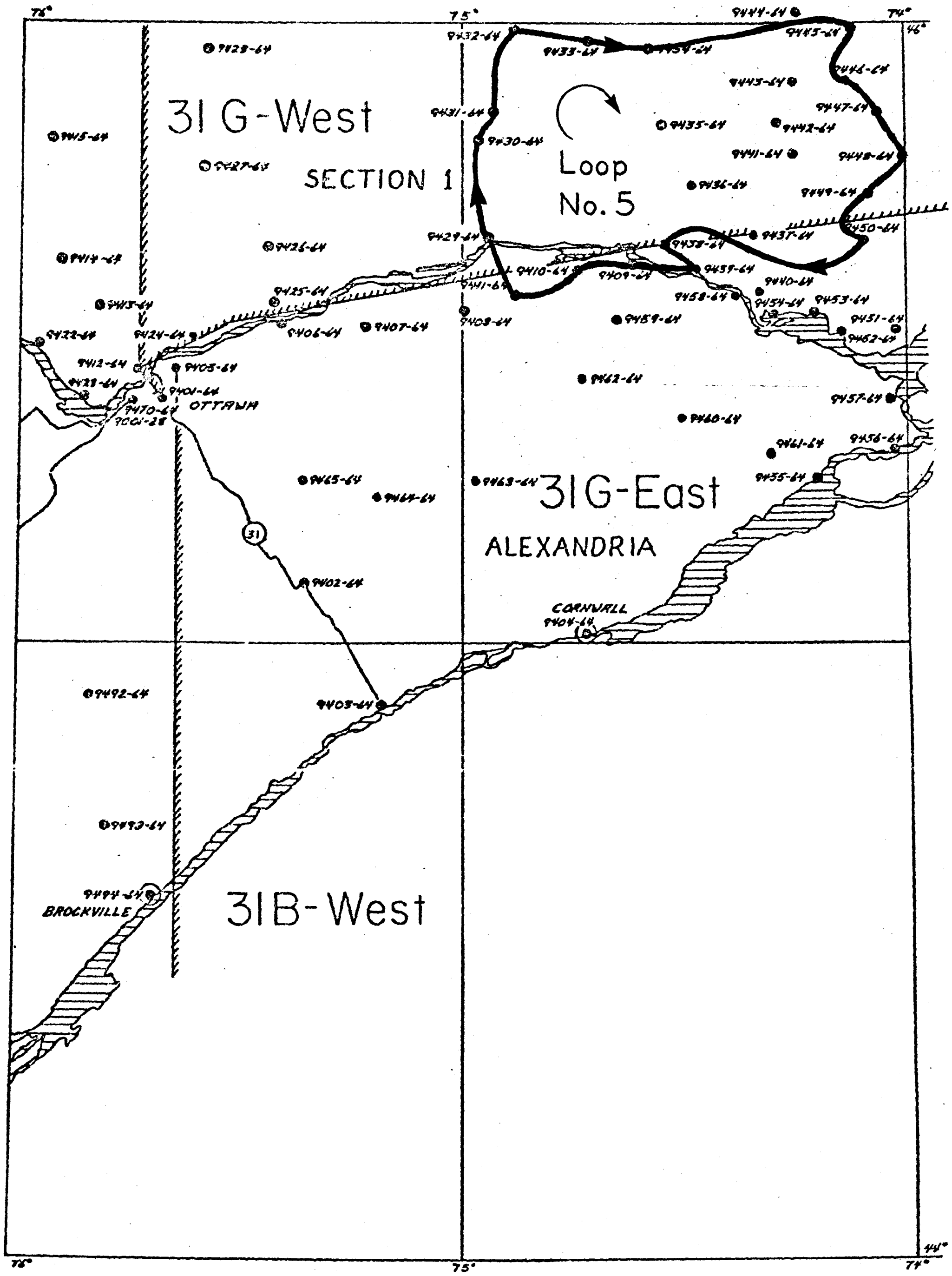


FIGURE 6-8

Location of Test Loop No. 5 (Eastern Ontario).

(From: Hamilton and Buchan [1965])

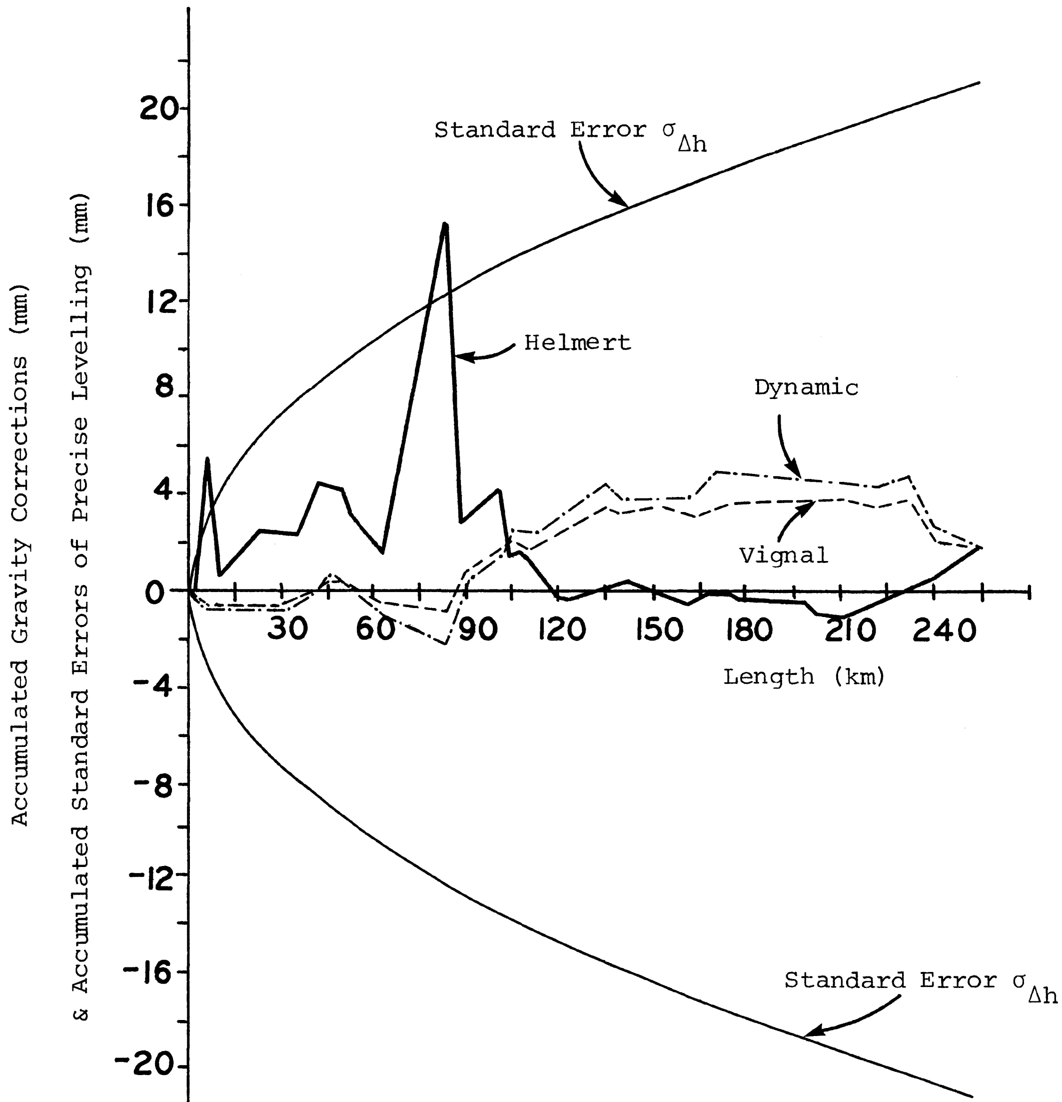


FIGURE 6-9

Accumulated Gravity Corrections for Test Loop No. 5.

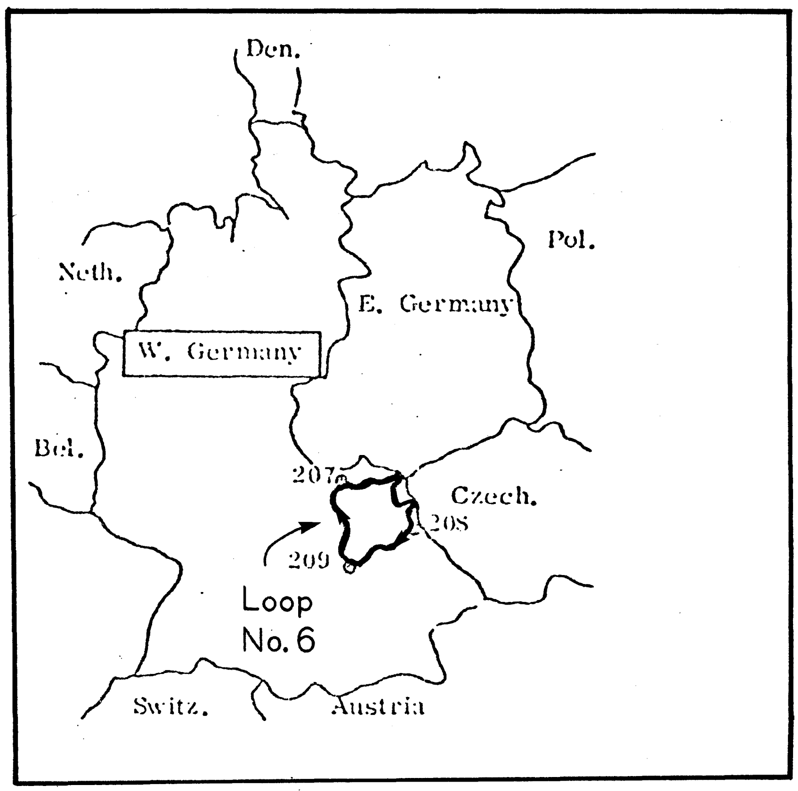


FIGURE 6-10

Location of Test Loop No. 6 (West Germany - Part of UELN)

(From: Krakiwsky [1966]).

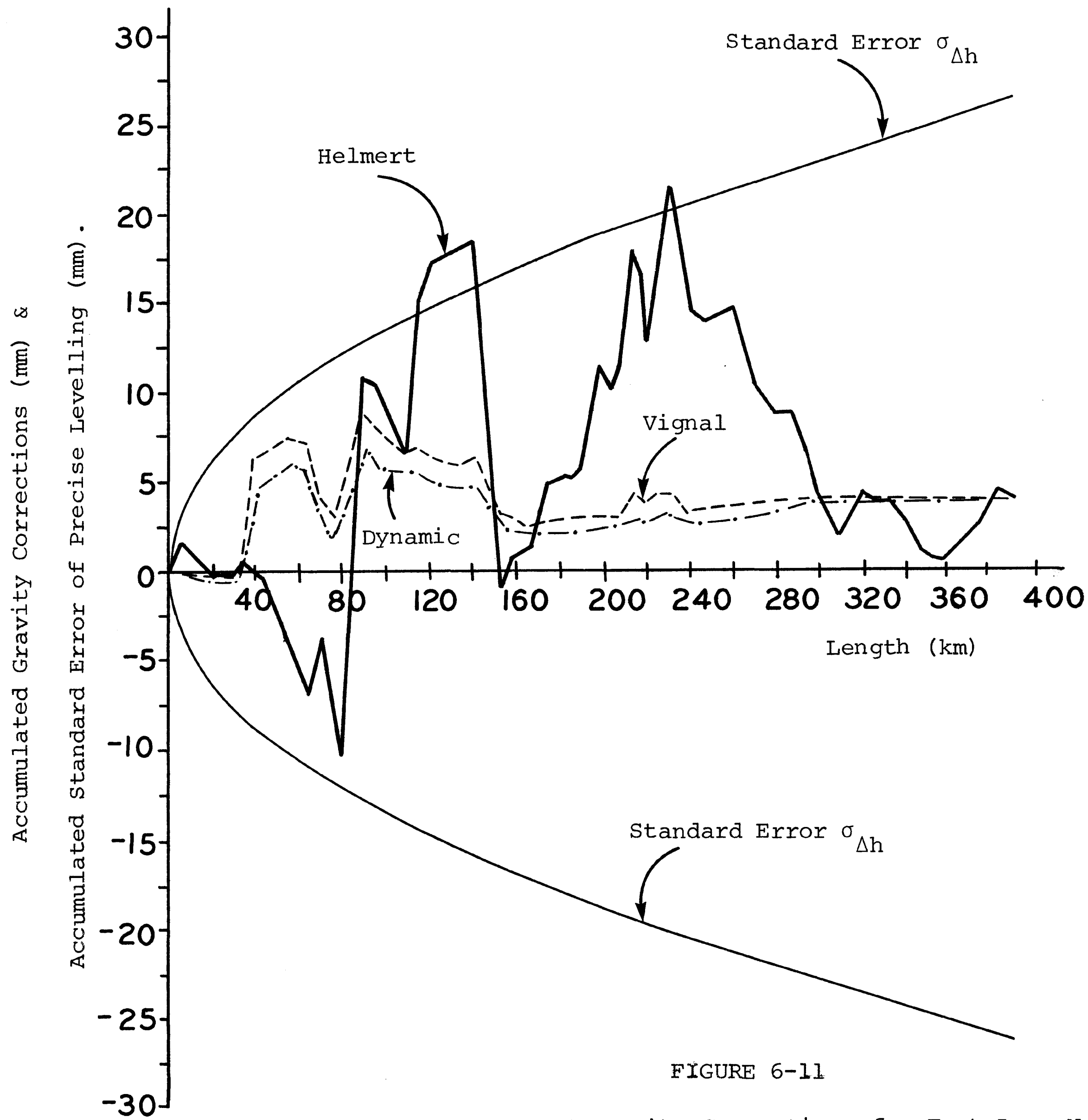


FIGURE 6-11

Accumulated Gravity Corrections for Test Loop No. 6.

and Vignal systems are all exactly the same. Such a result is not surprising. It illustrates the fact that all systems of height which take into account the actual gravity have the same characteristic quality of defining the heights of terrain points uniquely. This implies that such rigorous height systems theoretically produce zero closure round a closed loop. Hence, the loop closures of approximately 2 mm (Figure 6-9) and 4 mm (Figure 6-11) are solely due to the influence of actual gravity irregularities round the tested loops.

CHAPTER 7

PREDICTION OF GRAVITY CORRECTIONS

The questions concerning the practical computation and influence of the gravity corrections, as applied to actual levelling lines and loops, have received attention in the previous chapter. The last question now is: where in Canada are these corrections significant and hence should be taken into account? To answer this question is the second main objective of the present study, and is dealt with in this chapter. A significance criterion is set-up first. Then a computational technique is presented which allows one to identify the areas of significant gravity corrections. Finally, sample results and conclusions based on actual data are given.

7.1 Significance of Gravity Corrections

The decision as to what is and what is not significant is of course always open to discussion. As an illustration of an arbitrarily selected criterion for the significance of the gravity corrections, we can cite, for instance, Weidauer [1963] who has chosen a limiting value of the GC of 1 mm per 1 km (i.e. $\frac{GC}{\sqrt{S}}$ = 1 mm/km). As a result of this choice, most of his computations and comparisons showed that the gravity corrections are insignificant. This finding had justified, in his view, the neglect of actual gravity in his simplified expressions for the normal heights.

It seems more logical to us to compare the accumulated gravity corrections along a levelling line with the corresponding accumulated standard error expected in precise levelling, as we have done in section 6.2. Thus we shall follow this line of reasoning in our study. It is well known that the standard error, $\sigma_{\Delta h}$, in precise levelling propagates with the square root of the distance along the line. On the other hand, the gravity correction, as a systematic quantity, propagates differently. From our limited experience with actual levelling lines and loops, computed in the previous chapter, it seems rather difficult to predict how the gravity corrections are going to accumulate. In a very long run, the gravity effect accumulates randomly, but in short runs it does so linearly. In order to be on the safe side for all cases, we must assume a linear accumulation. Thus, considering only a part of the levelling line, the accumulated GC may be smaller than the corresponding $\sigma_{\Delta h}$. However, dealing with the entire line, the accumulated GC along the line could be significantly greater than $\sigma_{\Delta h}$. Therefore, the significance criterion for the gravity correction should be taken as a fraction of $\sigma_{\Delta h}$, above which the gravity correction is considered significant.

In our present study, the significance criterion for the gravity corrections will be taken as equal to 10% of the standard error $\sigma_{\Delta h}$, which is consistent with the widely spread custom. This criterion was originally set-up by Baeschlin [1960a], where he stated that the error in the actual geopotential number due to the gravity error should not exceed $\frac{1}{10}$ of the standard error in precise levelling. The same criterion was further used by Levallois [1964] and Ramsayer [1965a] in investigating the frequency of gravity measurements along the levelling lines for the European levelling networks.

We recall from section 6.2 that the standard error $\sigma_{\Delta h} = 1.33$ mm per 1 km (see equation 6-11) is typical for the CPLN. Using the above criterion, 10% of $\sigma_{\Delta h}$ is 0.133 mm/km, or approximately 0.14 mm/km. Consequently, the absolute value of the gravity correction greater than 0.14 mm per 1 km will be considered significant in Canada, and should be thus added to the existing height differences (based on normal gravity).

It should be noted here that the statement "Gravity correction greater than 0.14 mm/km must be considered significant" is not to be confused with the statement "It may be questionable to look for a gravity correction whose resulting standard deviation is larger in magnitude than the correction itself" (see sections 4.4 and 6.1.2). In the former, we are comparing the magnitude of the gravity correction against the corresponding standard error in precise levelling, and we are talking about "significance" of the gravity correction which depends on the magnitude of gravity. In the latter statement the gravity correction is compared with its own standard deviation resulting from propagating the standard deviations of the quantities involved in the evaluation of the correction. Here, we are talking about the "reliability" of the gravity correction, which depends basically on the gravity coverage available.

7.2 Computational Technique

It was decided to develop the computational technique for predicting the gravity corrections using the rigorous rather than the approximate expressions derived in Chapter 4. The reason for this decision is to be able to use this technique also in other parts of the world without much modification. If the technique was to be used outside North America, the values of some constants, coming from the difference $\delta\gamma_o$ between the actually used and the 1967 International formulae for normal gravity, in the developed software have to be changed. Outside North America, the actual $\delta\gamma_o$ may seriously affect the Helmert and Vignal gravity corrections.

The computational technique presented herein was first proposed by Nassar and Vaníček [1975], and tested further by Nassar [1975b] using gravity data based on the 1930 International system. Before outlining the technique in detail, we introduce some convenient approximations in the rigorous formulae for the gravity corrections by using approximate expressions for the quantities $\delta\gamma_o$ and $\Delta\delta\gamma_o$.

We have seen that the quantity $\delta\gamma_o$ can be approximated by equation (4-39), i.e.:

$$\delta\gamma_o \doteq a_o + a_1\sin^2\phi + a_2\sin^2 2\phi \ . \qquad (7\text{-}1)$$

The coefficients a_o, a_1 and a_2 are given for Canada (and the United States) by equations (4-40) in mgal. For other countries, where the actually used normal gravity formula can be reformulated in the same form as the 1967 International formula, an expression for $\delta\gamma_o$ similar to equation (7-1) can be obtained. The resulting values of the coefficients a_o, a_1 and a_2, in this case, may of course differ from those given by equations (4-40). The change of $\delta\gamma_o$ with latitude can be obtained by differentiating equation (7-1) with respect to ϕ. We get:

$$\frac{d\delta\gamma_o}{d\phi} = 2a_1 \sin\phi \cos\phi + 4a_2 \sin 2\phi \cos 2\phi ,$$

which can be rewritten as:

$$d\delta\gamma_o = (a_1 \sin 2\phi + 2a_2 \sin 4\phi)\, d\phi \quad . \tag{7-2}$$

For a levelling section between two sufficiently close points i and j, equation (7-2) can be approximated by:

$$\Delta\delta\gamma_{o,ij} \doteq (a_1 \sin 2\bar{\phi}_{ij} + 2a_2 \sin 4\bar{\phi}_{ij}) \Delta\phi_{ij} , \tag{7-3}$$

where $\Delta\delta\gamma_{o,ij}$ is defined by equation (4-26) and $\Delta\phi_{ij} = \phi_j - \phi_i$. Equation (7-3) can be viewed as transforming the difference of normal gravities to the corresponding difference in latitudes of i and j.

The convenient expression for the dynamic gravity correction GC^D_{ij} has already been given by equation (4-41). However, for the reader's convenience we copy it here again, i.e.:

$$GC^D_{ij} = \frac{\Delta h_{ij}}{G} [\overline{\Delta g}^F_{ij} + (a_o + a_1 \sin^2 \bar{\phi}_{ij} + a_2 \sin^2 2\bar{\phi}_{ij})] \quad . \tag{7-4}$$

The corresponding expressions for the Helmert and Vignal gravity corrections can be obtained by substituting equation (7-3) into equations (4-27) and (4-36) respectively. We get:

$$GC^H_{ij} = \frac{\bar{h}_{ij}}{G} [-\Delta\Delta g^F_{ij} - (a_1 \sin 2\bar{\phi}_{ij} + 2a_2 \sin 4\bar{\phi}_{ij}) \Delta\phi_{ij} + 0.2238\ \Delta h_{ij}] , \tag{7-5}$$

$$GC^V_{ij} = \frac{1}{G} [\Delta h_{ij} \overline{\Delta g}^F_{ij} - (a_1 \sin 2\bar{\phi}_{ij} + 2a_2 \sin 4\bar{\phi}_{ij}) \Delta\phi_{ij} \bar{h}_{ij}] \quad . \tag{7-6}$$

7.2.1 Outline of the proposed technique

We have seen that the gravity corrections are dependent not only on the characteristics of the height and gravity fields within the area of concern, but also on the direction of the levelling line. Hence, the main idea here is to compute the accumulated gravity correction along simulated levelling lines in different directions within the area of interest. In each direction, the results are then to be compared to the predetermined significance criterion of 0.14 mm/km.

To start with, let us consider a 1° x 1° block as the (unit) area of interest. The choice of the appropriate size of the block will be discussed in the next section. This block is frequently referred to here as 1° x 1° cell or, for brevity, a "cell". Moreover, we shall deal only with straight simulated levelling lines AB of length s and azimuth α radiating from the centre of the cell. This implies that A will coincide with the centre of the block, whose latitude and longitude is denoted by ϕ_o and λ_o, respectively, and B will be the end of the line running across the cell (see Figure 7-1).

The position of any point, e.g. "i", within the cell may be expressed in geodetic coordinates (ϕ, λ). However, as it has been customarily found convenient, the local orthogonal Cartesian coordinates (x, y), shown in Figure 7-1, are used instead of (ϕ, λ) in the subsequent developments. It is worth pointing out here that the final results do not depend on the adopted coordinate system. The mathematical relationship between the two systems , mentioned above, will be defined as:

$$x_i = \rho_{om} (\phi_i - \phi_o) , \qquad (7\text{-}7)$$

$$y_i = \rho_{om} \cos \phi_o (\lambda_o - \lambda_i) , \qquad (7\text{-}8)$$

where λ is taken positive west and ρ_{om} is the mean radius of curvature of the reference ellipsoid (Clark 1866 for the North America) computed

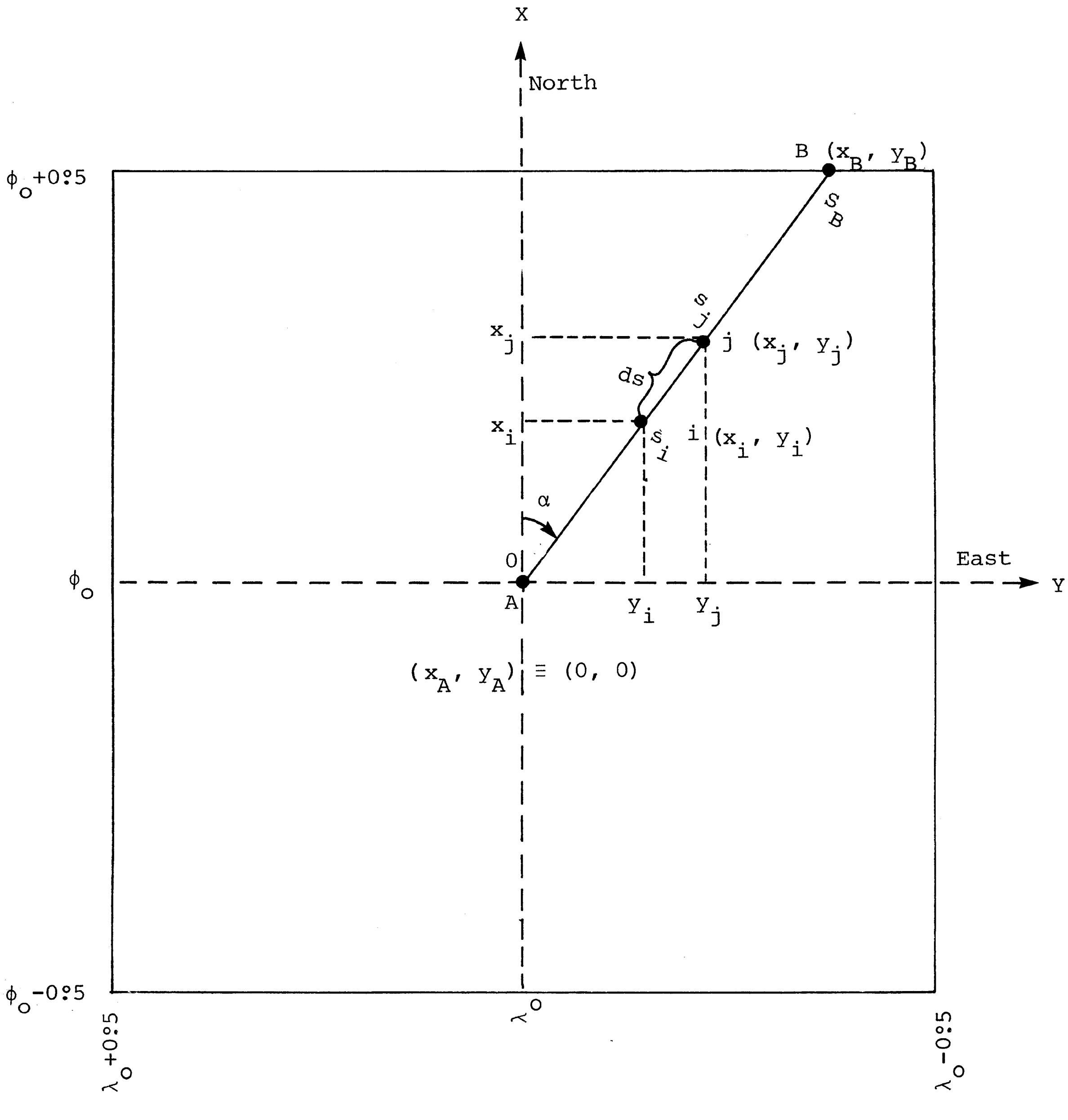


FIGURE 7-1

Unit Area: 1° x 1° Cell.

at the centre of the cell. ρ_{om} is usually computed from the following formulae [e.g. Krakiwsky and Wells, 1971]:

$$\rho_{om} = \sqrt{M_o N_o} \quad , \tag{7-9a}$$

$$M_o = \frac{a(1-e^2)}{(1-e^2 \sin^2 \phi_o)^{3/2}} \quad , \tag{7-9b}$$

$$N_o = \frac{a}{(1-e^2 \sin^2 \phi_o)^{1/2}} \quad , \tag{7-9c}$$

$$e^2 = \frac{a^2 - b^2}{a^2} \quad , \tag{7-9d}$$

where, for Clark 1866 ellipsoid, a = 6378.2064 km and b = 6356.5838 km. It should be noted here that equations (7-7) and (7-8) are the simplest mapping $(\phi, \lambda) \rightarrow (x, y)$. These two relationships can be further simplified by using a mean value for the radius of the earth instead of ρ_{om}.

At this point, one more convenient approximation can be used. This is to replace $\bar{\phi}_{ij}$ by ϕ_o in equations (7-4), (7-5) and (7-6), for all the points i and j within the cell. This approximation will result in negligible error whose extreme value is less than 5% of the corrective terms ε^D (equation 4-38) or ε^H (equation 4-42). Hence, the gravity corrections for a levelling section between i and j can be rewritten as:

$$GC^D_{ij} = \frac{\Delta h_{ij}}{G} [\overline{\Delta g}^F_{ij} + k_1] \quad , \tag{7-10}$$

$$GC^H_{ij} = \frac{\bar{h}_{ij}}{G} [-\Delta\Delta g^F_{ij} + k_2 \Delta\phi_{ij} + 0.2238 \; \Delta h_{ij}] \quad , \tag{7-11}$$

$$GC^V_{ij} = \frac{1}{G} [\Delta h_{ij} \overline{\Delta g}^F_{ij} + k_2 \Delta\phi_{ij}\bar{h}_{ij}] \ . \qquad (7\text{-}12)$$

Here k_1 and k_2 are constant for each cell given by:

$$k_1 = a_o + a_1 \sin^2\phi_o + a_2 \sin^2 2\phi_o \ , \qquad (7\text{-}13)$$

$$k_2 = -(a_1 \sin 2\phi_o + 2a_2 \sin 4\phi_o) \ . \qquad (7\text{-}14)$$

It is important for the subsequent developments to keep in mind that the physical units of the gravity corrections computed from equations (7-10), (7-11) and (7-12) are metres for $\bar{h}$ and Δh in metres, Δg^F, G, k_1, k_2 in mgal and $\Delta\phi$ in radians.

The gravity corrections accumulated along an entire hypothetical levelling line, between points A and B, composed of several sections ij are expressed by equation (6-9). These accumulated gravity corrections can be more exactly evaluated by replacing the sum in equation (6-9) by an integral so that we obtain (see equations 4-15, 4-28 and 4-37):

$$GC^D_{AB} = \int_o^s f^D(\Delta g^F_i, \Delta g^F_j, h_i, h_j, \phi_i, \phi_j) \, ds \ , \qquad (7\text{-}15)$$

$$GC^H_{AB} = \int_o^s f^H(\Delta g^F_i, \Delta g^F_j, h_i, h_j, \phi_i, \phi_j) \, ds \ , \qquad (7\text{-}16)$$

$$GC^V_{AB} = \int_o^s f^V(\Delta g^F_i, \Delta g^F_j, h_i, h_j, \phi_i, \phi_j) \, ds \ . \qquad (7\text{-}17)$$

Along the simulated levelling profiles, we further formulate the analytical expressions f^D, f^H and f^V in terms of polar coordinates s and α. Once this is done, the evaluation of the integrals for a particular azimuth and distance is straightforward. To express the subintegral functions in terms of α and s, we take the following steps:

1. Using the free-air anomalies and heights within the cell extracted from the EPB File, we seek the best least-squares fitting surface (2-D polynomial) $\Delta\tilde{g}^F$ to the free-air anomaly field and another surface $\tilde{h}$ for the height. This step produces a vector of coefficients, B or C, for each surface;
2. Then, the variations of the free-air anomaly and height along a straight levelling line of azimuth α are approximated by 1-D polynomial, whose coefficients are functions of the corresponding coefficients B or C determined in (1);
3. Inserting the 1-D polynomials for Δg^F and h into the analytical formulae for the gravity corrections (equations 7-10, 7-11 and 7-12) and transforming the geographic latitudes into distance and azimuth, we end up with differential formulae for the f's, in equations (7-15), (7-16) and (7-17), ready to be integrated.

A separate section will be devoted to the discussion of each of the above steps in detail.

7.2.2 Approximation of gravity anomaly and height fields

The least-squares approximation is known to be the best method of prediction [Cheney, 1966] in the sense that it yields the smallest mean-square error. Throughout this section we will be talking only about the prediction of the free-air anomaly field and only state the corresponding formulae for the prediction of the height field.

If there is a sufficient number of observed data within the cell, the free-air anomaly (or height) field can be approximated by a surface. This surface can be described by, for instance, a 2-D mixed algebraic polynomial whose coefficients are determined in such a way as to best fit the observed data in the least-squares sense. This technique was proposed by Nagy [1963] and further used in [Nagy, 1973; Merry and Vaníček, 1974; Merry, 1975] to predict anomalies within cells of different sizes and using different orders of the approximating polynomials. A variation of the same technique was used in [Vaníček et al., 1972] to represent a reduced gravity field, from which the reduced gravity at any point in the interpolation area can be predicted and the corresponding surface gravity obtained. The 2-D algebraic polynomials are used for smooth surface approximation because they combine the advantages of simple computations with high flexibility [Kubik, 1971].

Using the above technique, the free-air anomaly at any point i within the cell can be predicted by the following polynomial:

$$\Delta\tilde{g}^F(x_i, y_i) = \sum_{j=1}^{m} b_j \psi_j (x_i, y_i) , \tag{7-18}$$

where ψ's can be arbitrarily chosen linearly independent functions (base functions) of the position of i, and b's are the sought best fitting coefficients [Cheney, 1966; Vaníček and Wells, 1972]. Since the mixed algebraic functions $x^k y^\ell$, k, ℓ = 0, 1, ..., r are particularly simple to deal with, they are used here, where r is usually referred to as the order of the approximating polynomial.

The smaller the order "r" is, the smoother the approximating surface; it then indicates only the general trend of the approximated field. On the other hand, a higher order surface depicts the local irregularities of the field, and as such is more precise in estimating the gravity correction in the desired direction. Unfortunately, the higher order polynomial is seldom usable due to the amount of data needed to solve for the polynomial coefficients. The data coverage in the EPB file is usually not sufficiently dense. However, as our purpose here is to predict areas with significant gravity corrections, and not to produce accurate values of gravity corrections for practical use, a higher order approximation of the gravity (or height) field is not necessary. For simplicity, the subsequent developments of the prediction technique will be shown for a third order polynomial, i.e. $r = 3$, that has $m = 16$ coefficients. Also, as mentioned before, a 1° x 1° cell is taken as the interpolation area. The choice of the appropriate order "r" and the size of the basic cell will be discussed later on the basis of actually obtained results.

Taking the above restrictions into account, equation (7-18) becomes:

$$\Delta\tilde{g}^F (x_i, y_i) = \sum_{\substack{k=0 \\ \ell=0}}^{3} b_{k\ell} x_i^k y_i^\ell \quad , \tag{7-19}$$

where the b's are, again, the 16 coefficients to be determined. A similar expression can be then written for the approximating polynomial to the height field, i.e.:

$$\tilde{h} (x_i, y_i) = \sum_{\substack{k=0 \\ \ell=0}}^{3} c_{k\ell} \, x_i^k \, y_i^\ell \, , \tag{7-20}$$

where c's are another 16 coefficients, generally different from the b's.

It will be found handy to rewrite equations (7-19) and (7-20) using matrix notation:

$$\Delta\tilde{g}^F (x_i, y_i) = \Psi_i^T B , \tag{7-21}$$

$$\tilde{h} (x_i, y_i) = \Psi_i^T C , \tag{7-22}$$

where Ψ_i is the vector of the mixed algebraic base functions evaluated for (x_i, y_i), and B and C are the vectors of coefficients. The superscript T indicates the transposition operation in matrix algebra. We thus have:

$$\Psi_i^T = [1, y_i, y_i^2, y_i^3, x_i, x_i y_i, x_i y_i^2, x_i y_i^3, x_i^2, x_i^2 y_i, x_i^2 y_i^2, x_i^2 y_i^3, x_i^3, x_i^3 y_i, x_i^3 y_i^2, x_i^3 y_i^3], \tag{7-23}$$

$$B^T = [b_{00}, b_{01}, b_{02}, b_{03}, b_{10}, b_{11}, b_{12}, b_{13}, b_{20}, b_{21}, b_{22}, b_{23}, b_{30}, b_{31}, b_{32}, b_{33}] , \tag{7-24}$$

$$C^T = [c_{00}, c_{01}, c_{02}, c_{03}, c_{10}, c_{11}, c_{12}, c_{13}, c_{20}, c_{21}, c_{22}, c_{23}, c_{30}, c_{31}, c_{32}, c_{33}] . \tag{7-25}$$

To determine the unknown coefficients B and C, observation equations of the following form can be written for each point i within the cell for which the anomaly Δg_i^F and the height h_i are known:

$$\Delta\tilde{g}^F (x_i, y_i) + v_{g_i} = \Delta g_i^F , \tag{7-26}$$

$$\tilde{h} (x_i, y_i) + v_{h_i} = h_i . \tag{7-27}$$

Here v_{g_i} and v_{h_i} are the residuals. Substituting equations (7-21) and (7-22) into (7-26) and (7-27), for all the "n" data points within the cell, we get:

$$V_g = L_g - AB \ , \tag{7-28}$$

$$V_h = L_h - AC \ , \tag{7-29}$$

where: V_g is the vector of anomaly residuals, V_h is the vector of height residuals, L_g is the vector of known anomalies and L_h is the vector of known heights - all having n elements. The matrix A, known as Vandermonde's or design matrix, is composed of 16 column vectors of functional values for each of the 16 base functions evaluated at the known n data points. In other words, the n rows of A are the Ψ_i^T given by equation (7-23) for i = 1, 2, ..., n.

The systems (7-28) and (7-29) can be solved for the unknown coefficients B and C using the least-squares technique [Vaníček and Wells, 1972], providing that the number "n" of available data points is equal to or greater than 16. However, before doing that we have to decide what weights are to be assigned to the observed quantities L_g and L_h.

In the least-squares approximation, the weights are characterized by an arbitrarily selected weight function, w(x, y), which has to be non-negative over the interpolation area. This w is usually chosen, as a function of position of data points, to serve as a measure of the degree of precision or relative importance of the observed values in determining the coefficients of the approximating polynomial. This means that w can be selected in such a way as to provide different degrees of goodness of fit in the desired regions within the interpolation area.

For our purpose, however, we seek a uniform (homogeneous) least-squares fitting to the approximated field (gravity anomaly or height). Thus, from the point of view of relative importance of data points, the individual weights $w(x_i, y_i)$ could be assigned equally to all observed point values. On the other hand, we already have information about the degree of precision of the observed quantities (L_g and L_h), characterized by the individual standard deviations $\sigma_{\Delta g^F}$ and σ_h. In effect, the weights $w(x_i, y_i)$ can be defined in the following manner. Taking the individual anomalies Δg_i^F uncorrelated (see section 4.3), we can write the weight matrix W_g of the anomalies L_g in the following form:

$$W_g = \text{diag}\ [w_g(x_1, y_1),\ w_g(x_2, y_2),\ \ldots\ w_g(x_n, y_n)] \quad . \tag{7-30}$$

The individual weights $w_g(x_i, y_i)$ are computed as:

$$w_g(x_i, y_i) = \frac{1}{\sigma^2_{\Delta g_i^F}} \quad , \tag{7-31}$$

where $\sigma^2_{\Delta g_i^F}$ is given by equation (4-57), and the a priori variance factor σ_o^2 [e.g.: Wells and Krakiwsky, 1971; Vaníček, 1973] is assumed to be one. Similarly, the weight matrix W_h for the observed heights L_h will be:

$$W_h = \text{diag}\ [w_h(x_1, y_1),\ w_h(x_2, y_2),\ \ldots\ w_h(x_n, y_n)], \tag{7-32}$$

where:

$$w_h(x_i, y_i) = \frac{1}{\sigma^2_{h_i}} \quad . \tag{7-33}$$

In the above context, the model error (lack of fit) can be computed point-wise as the discrepancy (residuals V_g or V_h) between the best-fitting polynomial and the approximated surface. The significance of these residuals is magnified and will be inherent in the estimated a posteriori variance factor $\hat{\sigma}_o^2$ (to be defined later). Consequently, the resulting value of $\hat{\sigma}_o^2$ can serve as a measure of the lack of fit (i.e. the degree of roughness of the approximated field).

It may be worth noting that, when taking the standard deviation of the observed gravity values as 0.05 mgal, the following equation is valid:

$$W_g^{-1} = (0.05)^2\, I + (0.3086)^2 W_h^{-1} \,, \tag{7-34}$$

where I is the identity matrix and W_g^{-1} is in mgal-squared for W_h^{-1} in metre-squared. More sophisticated models for correlated anomalies and heights could be also treated [Moritz, 1963; Rapp, 1964; Heiskanen and Moritz, 1967; Moritz, 1969; Wilcox, 1974], but such a treatment would not be warranted within the present context.

The application of the least-squares condition on equation (7-28) yields the following normal equations for the least-squares estimate $\hat{B}$:

$$N_g\, \hat{B} = U_g \,, \tag{7-35}$$

where:

$$N_g = A^T\, W_g\, A \,, \tag{7-36}$$

$$U_g = A^T\, W_g\, L_g \,. \tag{7-37}$$

Analogously, we obtain the normal equations for $\hat{C}$ as:

$$N_h\, \hat{C} = U_h \,, \tag{7-38}$$

where:

$$N_h = A^T\, W_h\, A \,, \tag{7-39}$$

$$U_h = A^T\, W_h\, L_h \,. \tag{7-40}$$

Both matrices N_g and N_h (known as Gram's matrices) are positive definite and regular if the mixed algebraic base functions are linearly independent on the data set within the cell. We then get $\hat{B}$ from:

$$\hat{B} = N_g^{-1} U_g . \quad (7\text{-}41)$$

Similarly, the least-squares estimate $\hat{C}$ is given by:

$$\hat{C} = N_h^{-1} U_h . \quad (7\text{-}42)$$

In accordance with adjustment convention [e.g. Wells and Krakiwsky, 1971; Vaníček, 1973], computing the weights from equations (7-31) and (7-33) implies that the a priori variance factor σ_o^2 is equal to 1. The corresponding a posteriori variance factor $\hat{\sigma}_o^2$ can be evaluated in case of the anomalies from:

$$(\hat{\sigma}_o^2)_g = \frac{V_g^T W_g V_g}{df} , \quad (7\text{-}43)$$

where df is the number of degrees of freedom given as:

$$df = n - 16 . \quad (7\text{-}44)$$

In case of the heights we get:

$$(\hat{\sigma}_o^2)_h = \frac{V_h^T W_h V_h}{df} . \quad (7\text{-}45)$$

Finally, we can compute estimates for the covariance matrices $\Sigma_{\hat{B}}$ and $\Sigma_{\hat{C}}$ of the estimated vectors $\hat{B}$ and $\hat{C}$. The following expressions apply:

$$\hat{\Sigma}_{\hat{B}} = (\hat{\sigma}_o^2)_g N_g^{-1} , \quad (7\text{-}46)$$

$$\hat{\Sigma}_{\hat{C}} = (\hat{\sigma}_o^2)_h N_h^{-1} . \quad (7\text{-}47)$$

We may remark that the first element in the vector $\hat{B}$ is nothing else but the estimated (predicted) value of the free-air anomaly at the centre of the cell. Its estimated standard deviation is given by the square-root of the first element on the diagonal of $\Sigma_{\hat{B}}$. A similar statement holds true for the predicted height.

A subroutine called APPROX, based on the formulations presented in this section, was written to solve the least-squares approximation problem using the 2-D mixed algebraic approximating polymomial of an arbitrary order "r". More information about this subroutine will be given in section 7.3.1. For complete documentation, see Nassar [1975a]. Tables 7-1 and 7-2 respectively show the predicted free-air anomaly, $\Delta\tilde{g}^F$, and predicted height, $\tilde{h}$, at ten bench marks where gravity and height had been observed. These results were obtained using r = 3 and a 1° x 1° cell. The used data were extracted from the EPB File.

The examination of Tables 7-1 and 7-2 would indicate that the predicted values of the anomaly and height have adequate reliability characterized by small estimated standard deviations. However, the differences between the observed and the predicted values differ considerably from one location to another.

Also, the values of the a posteriori variance factor $\hat{\sigma}_o^2$, shown in the last column of Tables 7-1 and 7-2, seems to be quite large. Ideally, the $\hat{\sigma}_o^2$ being an a posteriori variance of a unit weight should come close to one. We have seen (equations 7-43 and 7-45) that the value of $\hat{\sigma}_o^2$ depends not only on the estimated residuals, V, but also on the a priori weights, W of the observed quantities. In our case, however, the effect of a priori weights on $\hat{\sigma}_o^2$ is minor compared to the effect of the resulting residuals. In other words, there is an "overflow" into the predicted residuals which produces very large values of $\hat{\sigma}_o^2$.

The large values of the residuals could be attributed, in

TABLE 7-1

Comparison of Observed and Predicted Anomalies.

Location of Bench Mark				Free-air Anomaly (mgal)			$\sigma_{\Delta \tilde{g}^F}$ (mgal)	No. of Data Points	$(\hat{\sigma}_o^2)_g$ (unitless)
ϕ	(N)	λ	(W)	observed Δg^F	predicted $\Delta \tilde{g}^F$	difference $\delta \Delta g^F$ *			
44°	00!32	77°	30!51	-27.01	-21.71	-5.30	1.20	280	608
44	08.96	77	34.84	-24.58	-21.44	-3.14	1.07	377	435
44	09.96	77	22.60	-23.39	-21.88	-1.51	1.24	374	471
44	31.57	77	20.14	-13.57	-13.18	-0.39	0.35	410	64
45	02.69	77	46.40	- 2.70	- 2.49	-0.21	0.39	660	34
45	16.16	77	59.07	- 3.22	- 3.46	0.24	0.41	725	21
49	36.40	114	25.80	-15.78	-11.20	-4.58	3.07	84	1098
51	02.30	114	05.60	-20.28	-17.37	-2.91	0.72	166	214
51	02.30	114	04.30	-24.45	-17.39	-7.06	0.72	165	210
51	02.50	114	04.10	-24.77	-17.41	-7.36	0.72	167	210

* $\delta \Delta g^F = \Delta g^F - \Delta \tilde{g}^F$

TABLE 7-2

Comparison of Observed and Predicted Heights.

Location of Bench Mark				Height (metres)					
ϕ	(N)	λ	(W)	Observed h	Predicted $\tilde{h}$	Difference δh *	$\sigma_{\tilde{h}}$ (metres)	No. of Data Points	$(\hat{\sigma}_o^2)_h$ (unitless)
44°	00!32	77°	30!51	76.75	74.91	1.84	1.23	280	65
44	08.96	77	34.84	95.40	94.73	0.67	1.45	377	79
44	09.96	77	22.60	93.27	93.54	-0.27	1.66	374	84
44	31.57	77	20.14	156.09	174.76	-18.67	1.23	410	77
45	02.69	77	46.40	328.91	341.77	-12.86	2.63	660	155
45	16.16	77	59.07	401.54	406.75	-5.21	2.83	725	99
49	36.40	114	25.80	1287.41	1351.24	-63.83	26.50	84	8009
51	02.30	114	05.60	1051.07	1121.70	-70.63	5.94	166	1415
51	02.30	114	04.30	1050.83	1113.15	-62.32	5.90	165	1390
51	02.50	114	04.10	1048.82	1111.99	-63.17	5.88	167	1386

* $\delta h = h - \tilde{h}$

this context, either to systematic errors (blunders of some kind) in the used point anomaly and height data, or to "model errors" due to too high a degree of smoothness of the selected surface for approximating the anomaly (or height) field. The former reason may be ruled out since it is reported [Buck, 1975; John, 1976] that all the possible detectable blunders have been eliminated from the observed data on the EPB File. Even if there are some undetected blunders in the file, they are not going to contribute significantly to the obtained values of $\hat{\sigma}_o^2$. This leaves us with the latter explanation, i.e. the model errors, which must be considered the main cause of the resulting large values of $\hat{\sigma}_o^2$. The value of $\hat{\sigma}_o^2$ can in effect be regarded as a measure of the roughness of the approximated field (anomaly or height).

The above discussion indicates that the chosen third order 2-D polynomial is too smooth a surface to approximate either the anomaly or the height fields. This suggests that the order "r" of the approximating polynomial should be increased. Table 7-3 shows the values of $\hat{\sigma}_o^2$ for both the anomaly and height fields, as obtained from six typical 1° x 1° cells, using r = 3, 4 and 5, respectively. From these results, we notice that in most cases there is an average decrease of about 30% in the value of $\hat{\sigma}_o^2$ when the order "r" is increased by one.

Unfortunately, we practically cannot increase r for two main reasons: the lack of data and the computer cost to solve for the polynomial coefficients. The best way to overcome the first problem would be to use a different value of "r" for each individual cell, depending on the number and distribution of available data points. This was not done in the present study because the increased computer

TABLE 7-3

Variation of the A Posteriori Variance Factor

With the Order of the Approximating Polynomial.

Location of the 1°x1° Cell		No. of Data Points	$(\hat{\sigma}_o^2)_g$			$(\hat{\sigma}_o^2)_h$		
ϕ_o (N)	λ_o (W)		r = 3	r = 4	r = 5	r = 3	r = 4	r = 5
50°.5	70°.5	49	4.65	3.38	1.76	22.74	14.92	8.52
51.5	72.5	52	5.78	5.16	4.47	23.09	12.12	4.21
52.5	71.5	50	14.23	12.00	10.54	30.29	22.38	24.62
47.5	81.5	50	8.67	5.03	5.48	2.44	2.19	1.41
52.5	104.5	99	62.81	50.14	41.12	163.70	155.17	97.50
53.5	103.5	58	107.20	69.87	56.54	194.80	110.40	70.61

cost was not deemed warranted. It was decided to seek only the smooth features of the anomaly and height fields (by using r = 3) and hope that the predicted gravity corrections will still be meaningful, because of the averaging power of the polynomial.

Moreover, in order to obtain a reliable solution [Vaníček et al., 1972; Merry, 1975] it is necessary to have at least one data point in each quadrant around the centre of the cell under consideration. If this condition is not met, no matter how many data points are available within the cell, no solution is performed. Such a situation may occur, even for cells larger than 0°.5 x 0°.5 in size, since the present distribution of point gravity data is, in some cases, very irregular [Nagy, 1973; Merry, 1975]. The adverse distribution of data points within the cell could be one of the reasons influencing $\hat{\sigma}_o^2$ [John, 1976].

In order to decide upon the appropriate size of the basic cell, the third order polynomial was fitted to the anomaly and height fields within selected 2° x 2°, 1° x 1° and 0°.5 x 0°.5 cells. The resulting values of $\hat{\sigma}_o^2$ are given in Table 7-4. From these results, we notice that, in most cases, the value of $\hat{\sigma}_o^2$ is decreased by almost one order of magnitude when the size of the basic cell is halved. This is consistent with the results given in Table 7-3, since the effect of halving the size of the cell is approximately equivalent to doubling the order "r" of the polynomial. Thus, from Table 7-4, it can be seen that the third order approximating polynomial seems to be well suited for cells of 0°.5 x 0°.5 or smaller. However, in several areas the number of available data points within 0°.5 x 0°.5 cells is not sufficient to get a solution. Because of the insufficient data coverage and

TABLE 7-4

Variation of the A Posteriori Variance Factor

With the Size of the Cell Using r = 3.

Location of the Cell		Size of the Cell								
		2° x 2°			1° x 1°			0°5 x 0°5		
ϕ_o (N)	λ_o (W)	#df*	$(\hat{\sigma}_o^2)_g$	$(\hat{\sigma}_o^2)_h$	#df*	$(\hat{\sigma}_o^2)_g$	$(\hat{\sigma}_o^2)_h$	#df*	$(\hat{\sigma}_o^2)_g$	$(\hat{\sigma}_o^2)_h$
47°5	80°5	2260	400	8100	98	30	15	2	2.5	39.0
48.5	82.5	746	326	407	193	30	21	43	4.3	7.3
49.5	82.5	731	136	276	359	34	17	151	12.6	8.5
50.5	100.5	420	1064	4542	66	151	522	4	4.3	75.3
52.5	104.5	395	289	648	83	63	164	5	10.3	10.3
53.5	103.5	270	199	467	65	95	81	4	19.0	13.0

* #df = No. of degrees of freedom

= No. of data points - 16 (polynomial coefficients).

irregular data distribution in some cases, we have decided to stay with the 1° x 1° cells in this study.

In summary, the third order 2-D polynomial and 1° x 1° cells are used in the subsequent developments. We should keep in mind, however, that because of this choice, the resulting surface fittings to the free-air anomaly and height fields must be regarded as indicative of the general features only. This seems to be adequate for the purpose of predicting the areas with significant gravity corrections, using the present EPB File.

7.2.3 Profiles of gravity and height fields

In the previous section the equations (7-21 and 7-22) for predicting the values of free-air anomaly and height at any point i, given by Cartesian coordinates (x_i, y_i), within the 1° x 1° cell have been given. Now these equations will be reformulated to give the predicted values of Δg^F and h along a profile, i.e. the simulated levelling line (see Figure 7-1), as a function of the distance s_i from the centre of the cell in a selected azimuth α.

Such reformulation is aided by using the polar coordinates obtained through the following transformation from Cartesian coordinates:

$$x_i = s_i \cos \alpha \ , \tag{7-48}$$

$$y_i = s_i \sin \alpha \ . \tag{7-49}$$

Substituting for x_i, y_i in equations (7-21) and (7-22), we find terms containing up to 6th power of s_i. Since we are only looking for the general features of the profiles of anomaly and height fields, it was felt that polynomials of the 4th order in s_i would be adequate.

Consequently, equations (7-21) and (7-22) are then expressed as follows:

$$\Delta\tilde{g}^F(x_i, y_i) = \Delta\tilde{g}^F(s_i, \alpha) \doteq \sum_{j=0}^{4} p_j(\alpha)\, s_i^j \quad , \tag{7-50}$$

$$\tilde{h}(x_i, y_i) = \tilde{h}(s_i, \alpha) \doteq \sum_{j=o}^{4} q_j(\alpha)\, s_i^j \quad , \tag{7-51}$$

where p's and q's are some new coefficients, functions of only α in the cell. In matrix notation, we can rewrite equations (7-50) and (7-51) as:

$$\Delta\tilde{g}^F(s_i, \alpha) = S^T(s_i)\, P(\alpha) \quad , \tag{7-52}$$

$$\tilde{h}(s_i, \alpha) = S^T(s_i)\, Q(\alpha) \quad , \tag{7-53}$$

where:

$$S^T(s_i) = [1,\ s_i,\ s_i^2,\ s_i^3,\ s_i^4] \quad , \tag{7-54}$$

$$P^T(\alpha) = [p_o,\ p_1,\ p_2,\ p_3,\ p_4] \quad , \tag{7-55}$$

$$Q^T(\alpha) = [q_o,\ q_1,\ q_2,\ q_3,\ q_4] \quad . \tag{7-56}$$

The vectors P and Q can be expressed in terms of vectors B and C (see equations 7-41 and 7-42) and the chosen azimuth α. Substituting equations (7-48) and (7-49) into equations (7-21) and (7-22), rearranging the terms and neglecting higher powers of s_i, we obtain:

$$P(\alpha) = M(\alpha)\, B \quad , \tag{7-57}$$

$$Q(\alpha) = M(\alpha)\, C \quad . \tag{7-58}$$

The matrix M is given by:

$$M(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin\alpha & 0 & 0 & \cos\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin^2\alpha & 0 & 0 & \cos\alpha\sin\alpha & 0 & 0 & \cos^2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^3\alpha & 0 & 0 & \cos\alpha\sin^2\alpha & 0 & 0 & \cos^2\alpha\sin\alpha & 0 & 0 & \cos^3\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\alpha\sin^3\alpha & 0 & 0 & \cos^2\alpha\sin^2\alpha & 0 & 0 & \cos^3\alpha\sin\alpha & 0 & 0 \end{bmatrix} \tag{7-59}$$

We note that B and C are functions of the anomaly and height data within the cell, and as such they differ from one cell to the other. On the other hand, M is a function of azimuth α only, and thus remains the same for all the cells.

7.2.4 Differential formulae for the gravity corrections

Let us now take the following differential relations to hold at point i on the levelling line (Figure 7-2):

$$ds \doteq s_j - s_i \ , \tag{7-60a}$$

$$d\phi \doteq \phi_j - \phi_i \ , \tag{7-60b}$$

$$d\Delta\tilde{g}^F \doteq \Delta\tilde{g}^F_j - \Delta\tilde{g}^F_i \ , \tag{7-60c}$$

$$d\tilde{h} \doteq \tilde{h}_j - \tilde{h}_i \ , \tag{7-60d}$$

for j = i+1. Differentiating equations (7-7), (7-48), (7-52) and (7-53), we can write:

$$d\phi = \frac{dx}{\partial x(\phi)/\partial\phi} = \left(\frac{\cos\alpha}{\rho_{om}}\right) ds \ , \tag{7-61}$$

and for a fixed azimuth α:

$$d\Delta\tilde{g}^F = \frac{\partial\Delta\tilde{g}^F(s)}{\partial s} ds = (p_1+2p_2 s+3p_3 s^2+4p_4 s^3) \ ds, \tag{7-62}$$

$$d\tilde{h} = \frac{\partial\tilde{h}(s)}{\partial s} ds = (q_1+2q_2 s+3q_3 s^2+4q_4 s^3) \ ds \ . \tag{7-63}$$

Referring to equations (7-10), (7-11) and (7-12) and using equations (7-61), (7-62) and (7-63), we can write:

$$\Delta h_{ij} \doteq d\tilde{h} = [q_1+2q_2 s_i+3q_3 s_i^2+4q_4 s_i^3] \ ds \ , \tag{7-64}$$

$$\Delta\Delta g^F_{ij} \doteq d\Delta\tilde{g}^F = [p_1+2p_2 s_i+3p_3 s_i^2+4p_4 s_i^3] \ ds \ , \tag{7-65}$$

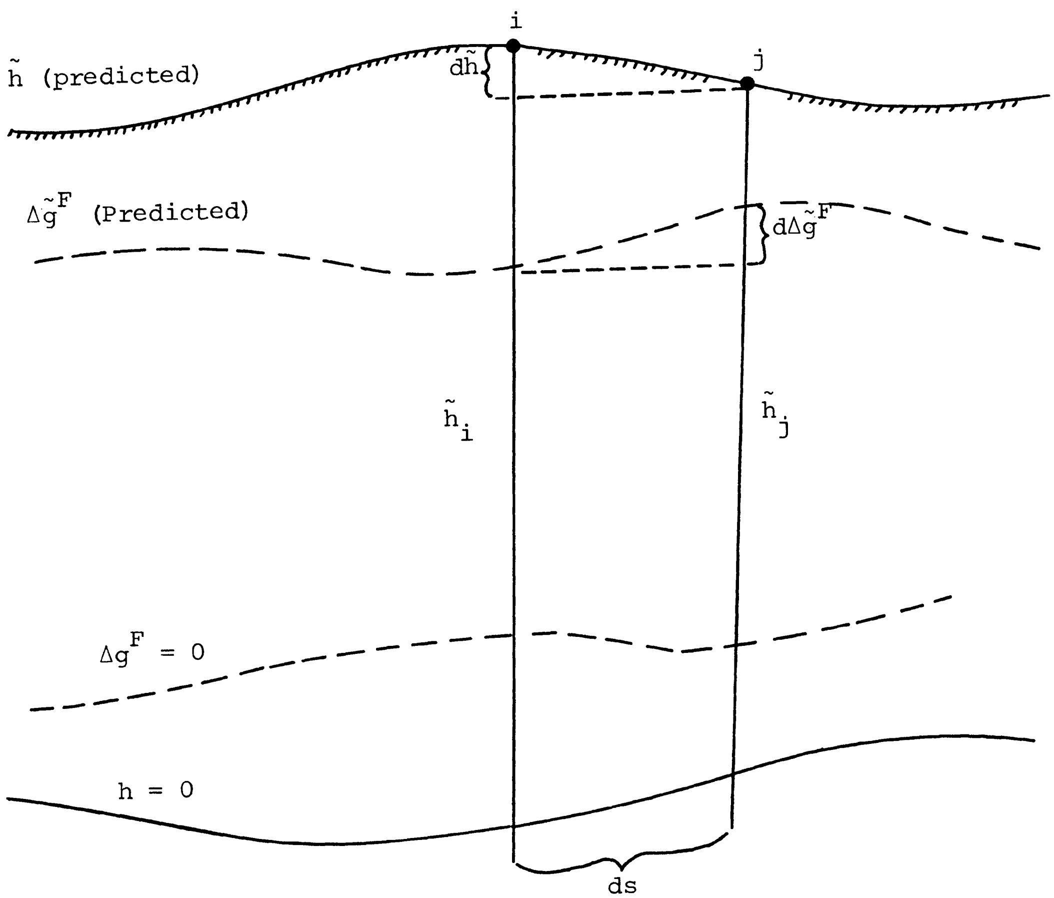


FIGURE 7-2

Differential Environment of Point i on

The Levelling Line.

$$\Delta\phi_{ij} \doteq d\phi = k_3 \, ds \ , \tag{7-66}$$

$$\bar{h}_{ij} \doteq \tilde{h}_i + \frac{1}{2} d\tilde{h} \ , \tag{7-67}$$

$$\overline{\Delta g}^F_{ij} \doteq \Delta\tilde{g}^F_i + \frac{1}{2} d \, \Delta\tilde{g}^F \ . \tag{7-68}$$

In these expressions, we have:

$$k_3 = \cos \alpha/\rho_{om} \ , \tag{7-69}$$

$$\tilde{h}_i = [q_o + q_1 s_i + q_2 s_i^2 + q_3 s_i^3 + q_4 s_i^4] \ , \tag{7-70}$$

$$\Delta\tilde{g}^F_i = [p_o + p_1 s_i + p_2 s_i^2 + p_3 s_i^3 + p_4 s_i^4] \ , \tag{7-71}$$

and $d\tilde{h}$, $d\Delta\tilde{g}^F$ are given by equations (7-63) and (7-62), respectively.

Substituting now the above differential expressions (equations 7-64 to 7-71) into equations (7-10), (7-11) and (7-12), and neglecting terms with second order differentials ds^2, we get the following linear differential equations for the three kinds of gravity corrections under investigation:

$$dGC^D = D^T J \, ds = f^D(s) \, ds, \tag{7-72}$$

$$dGC^H = H^T J \, ds = f^H(s) \, ds, \tag{7-73}$$

$$dGC^V = V^T J \, ds = f^V(s) \, ds. \tag{7-74}$$

Here D, H and V are vectors of eight components each, and J is given as:

$$J^T = [s^\ell; \ \ell = 0, 1, \ldots, 7] \ . \tag{7-75}$$

The vectors D, H and V are obtained from the following expressions:

$$D = (Z_1 + Z_2)Q \ , \tag{7-76}$$

$$H = Z_3 \; (Z_4 P + Z_5 Q + E) \; , \tag{7-77}$$

$$V = (Z_1 + Z_6) \; Q \; , \tag{7-78}$$

where P and Q are given by eqautions (7-57) and (7-58), respectively. Note here that V is a vector of coefficients, and thus is not to be confused with the vectors of residuals V_g and V_h (equations 7-28 and 7-29).

E in equation (7-77) is expressed as:

$$E^T \; (\alpha, \; \phi_o) = \frac{1}{G} \; [k_2, \; k_3, \; 0, \; 0, \; 0] \; , \tag{7-79}$$

a function of the azimuth α and the central latitude ϕ_o of the cell (see equations 7-14 and 7-69). G, in the above equation, is the reference normal gravity value used throughout the thesis. The matrices Z_1 and Z_2 in equation (7-76) are given as:

$$Z_1(\alpha, \; \Delta g^F) = \begin{bmatrix} 0 & P_o & 0 & 0 & 0 \\ 0 & P_1 & 2P_o & 0 & 0 \\ 0 & P_2 & 2P_1 & 3P_o & 0 \\ 0 & P_3 & 2P_2 & 3P_1 & 4P_o \\ 0 & P_4 & 2P_3 & 3P_2 & 4P_1 \\ 0 & 0 & 2P_4 & 3P_3 & 4P_2 \\ 0 & 0 & 0 & 3P_4 & 4P_3 \\ 0 & 0 & 0 & 0 & 4P_4 \end{bmatrix} \; , \tag{7-80}$$

$$Z_2(\phi_o) = \frac{k_1}{G} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \qquad (7\text{-}81)$$

We can see that the matrix Z_1 is a function of the free-air anomaly field within the cell and the azimuth of the profile, whereas the matrix Z_2 is a function of the latitude ϕ_o only. The matrices Z_3, Z_4 and Z_5, in equation (7-77) are expressed as follows:

$$Z_3\ (\alpha, h) = \begin{bmatrix} q_o & 0 & 0 & 0 \\ q_1 & 2q_o & 0 & 0 \\ q_2 & 2q_1 & 3q_o & 0 \\ q_3 & 2q_2 & 3q_1 & 4q_o \\ q_4 & 2q_3 & 3q_2 & 4q_1 \\ 0 & 2q_4 & 3q_3 & 4q_2 \\ 0 & 0 & 3q_4 & 4q_3 \\ 0 & 0 & 0 & 4q_4 \end{bmatrix} , \qquad (7\text{-}82)$$

$$Z_4 = \frac{1}{G} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} , \qquad (7\text{-}83)$$

$$Z_5 = -0.2238 \, Z_4 \; . \tag{7-84}$$

We note that matrix Z_3 is a function of the height field within the cell and the azimuth of the profile, while matrices Z_4 and Z_5 are constant for all cells. Finally, the matrix Z_6, in equation (7-78) is given as:

$$Z_6(\alpha, \phi_o) = \frac{k_2 k_3}{G} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \tag{7-85}$$

which is, again, a function of α and ϕ_o.

7.2.5 Integration of the gravity corrections

The purpose of this section is to develop expressions for the accumulated gravity corrections over the length s of the profile (simulated levelling line) in the desired azimuth α. This can be done by integrating the differential equations for the gravity corrections over the distance s. We start with rewriting equations (7-15), (7-16) and (7-17) as follows:

$$GC^D_{AB} = \int_o^s f^D(\Delta\tilde{g}^F(s), \tilde{h}(s), \phi(s)) \, ds \; , \tag{7-86}$$

$$GC^H_{AB} = \int_o^s f^H(\Delta\tilde{g}^F(s), \tilde{h}(s), \phi(s)) \, ds \; , \tag{7-87}$$

$$GC^V_{AB} = \int_o^s f^V(\Delta\tilde{g}^F(s),\ \tilde{h}(s),\ \phi(s))\ ds\ . \tag{7-88}$$

These equations can be further simplified as:

$$GC^D_{AB} = \int_o^s f^D\ (s)\ ds\ , \tag{7-89a}$$

$$GC^H_{AB} = \int_o^s f^H\ (s)\ ds\ , \tag{7-89b}$$

$$GC^V_{AB} = \int_o^s f^V\ (s)\ ds\ , \tag{7-89c}$$

where, of course, the subintegral functions depend on the cell and on the azimuth of the profile.

Substituting from equations (7-72), (7-73) and (7-74) into equations (7-89), and integrating with respect to s, we get the following final expressions for the accumulated gravity corrections as functions of s:

$$GC^D_{AB} = (D^T\ F\ J)s\ , \tag{7-90}$$

$$GC^H_{AB} = (H^T\ F\ J)s\ , \tag{7-91}$$

$$GC^V_{AB} = (V^T\ F\ J)s\ . \tag{7-92}$$

Here, F is a constant diagonal matrix resulting from the integration, given by:

$$F = \text{diag}\ [1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8}]\ . \tag{7-93}$$

We should keep in mind that the heights must be expressed in metres, the anomalies in milligals and the integration distance in kilometres. The units of the gravity corrections computed from equations (7-90) to (7-92) are metres. The examination of these three equations reveals that the product (F J)s, a column vector of eight elements and a function of the distance s only, is common to all of them. Let us denote this product by R and write:

$$R = (FJ)s = s(FJ) \quad . \tag{7-94}$$

By substituting in equations (7-90) to (7-92), we obtain:

$$GC^{D}_{AB} = D^{T}R = R^{T}D \ , \tag{7-95}$$

$$GC^{H}_{AB} = H^{T}R = R^{T}H \ , \tag{7-96}$$

$$GC^{V}_{AB} = V^{T}R = R^{T}V \ . \tag{7-97}$$

From the above development, we notice that the three vectors D, H and V, of eight coefficients each, are functions of the following:

1. The mean latitude "ϕ_o" of the 1° x 1° cell;
2. The estimated coefficients "$\hat{B}$" of the best-fitting surface to the free-air gravity anomaly field within the cell;
3. The estimated coefficients "$\hat{C}$" of the best-fitting surface to the height field within the cell;
4. The direction in which the simulated levelling line runs, i.e. the azimuth "α".

We note also that these three vectors are all independent of the length "s" of the levelling line.

Putting together now all the pertinent equations and denoting the variables in subscripts, we get the final set of equations for the accumulated gravity corrections as follows:

$$GC^{D}_{AB} = R^{T}_{\{s\}} \left(Z_{1_{\{\Delta g^{F}_{\alpha}\}}} + Z_{2_{\{\phi_o\}}}\right) M_{\{\alpha\}} \hat{C}_{\{h\}} \quad , \qquad (7\text{-}98)$$

$$GC^{H}_{AB} = R^{T}_{\{s\}} Z_{3_{\{h_{\alpha}\}}} \left(Z_4 M_{\{\alpha\}} \hat{B}_{\{\Delta g^{F}\}} + Z_5 M_{\{\alpha\}} \hat{C}_{\{h\}} + E_{\{\phi_{o\alpha}\}}\right), \qquad (7\text{-}99)$$

$$GC^{V}_{AB} = R^{T}_{\{s\}} \left(Z_{1_{\{\Delta g^{F}_{\alpha}\}}} + Z_{6_{\{\phi_{o\alpha}\}}}\right) M_{\{\alpha\}} \hat{C}_{\{h\}} \, . \qquad (7\text{-}100)$$

For better orientation, we give below the equation-number for each of the above involved quantities:

R .... (7-94)

Z_1 .... (7-80)

Z_2 .... (7-81)

Z_3 .... (7-82)

Z_4 .... (7-83)

Z_5 .... (7-84)

Z_6 .... (7-85)

M .... (7-59)

$\hat{B}$ .... (7-41)

$\hat{C}$ .... (7-42)

E .... (7-79)

7.2.6 Remarks on the gravity corrections

Let us now examine the final expressions for the accumulated gravity corrections, as derived in the previous section, (equations 7-98 to 7-100). Two remarks seem worth mentioning here:

1. The difference between the dynamic and the Vignal gravity corrections

is due to the difference of matrices Z_2 and Z_6. This difference arises only from the difference $\delta\gamma_o$ between the USC&GS formula and the 1967 International formula for normal gravity, and is relatively small. From equations (7-13), (7-14), (7-81) and (7-85), we can see that both Z_2 and Z_6 will be null matrices if $\delta\gamma_o = 0$. This would result in the equivalence of the dynamic and Vignal gravity corrections, which again confirms what we have stated in section 4.2;

2. The Helmert gravity correction seems to be more sensitive to the variation of height along the levelling line than the dynamic and Vignal gravity corrections. This can be verified by referring to the expressions for the gravity corrections developed in Chapter 4 where Helmert gravity correction is a function of average height as well as height difference, while the Vignal and dynamic gravity corrections are functions of height difference only. This also confirms our findings in section 6.2 concerning the behaviour of Helmert gravity correction along actual lines and loops.

A subroutine called GCAFAZ (see [Nassar, 1975a] for documentation) was written to evaluate the accumulated gravity corrections from equations (7-98) to (7-100). In designing this subroutine, the intention was to evaluate the variations of gravity corrections with azimuth for each cell under consideration. Realizing that matrices Z_1 and Z_3 are functions of vectors $\hat{B}$ and $\hat{C}$, we can see (equations 7-98 to 7-100) that the basic input to GCAFAZ are the vectors $\hat{B}$ and $\hat{C}$ of coefficients of best-fitting surfaces to the free-air anomaly and the height fields. All the other terms can then be computed within the subroutine for a given ϕ_o and azimuth α. $\hat{B}$ and $\hat{C}$ are evaluated by the subroutine APPROX.

As a final remark here, the gravity correction for a line B'B across the entire 1° x 1° cell through its centre A (see Figure 7-3) is given by the following expression:

$$GC_{B'B} = GC_{AB}(\alpha) - GC_{AB'}(\alpha+180°) ,$$

where both terms on the RHS are furnished by GCAFAZ subroutine.

7.3 Results

In this section, the programming considerations associated with the proposed technique for predicting the gravity corrections are discussed. Graphical display of sample results showing the variation of gravity corrections with azimuth are presented. These results are based on actual data from the new EPB File.

We have seen in section 5.2.3 that the old EPB File (used in previous investigations, e.g. [Nassar and Vaníček, 1975]) contains about 90,000 point anomaly values. On the other hand, the new File, used in the present study, has about 270,000 values. Thus, we were hoping that the new file would better serve our investigation and would provide refined results. However, it was discovered that about two-thirds of the point anomaly values on the new file are associated with gravity observations made at sea. Realizing that there are not and will not be any levelling lines at sea, the use of the aforementioned sea data is irrelevant for our purpose. Consequently, we decided to extract and file only the data on land from the EPB new file (see section II-2) to generate a master file (containing about 110,000

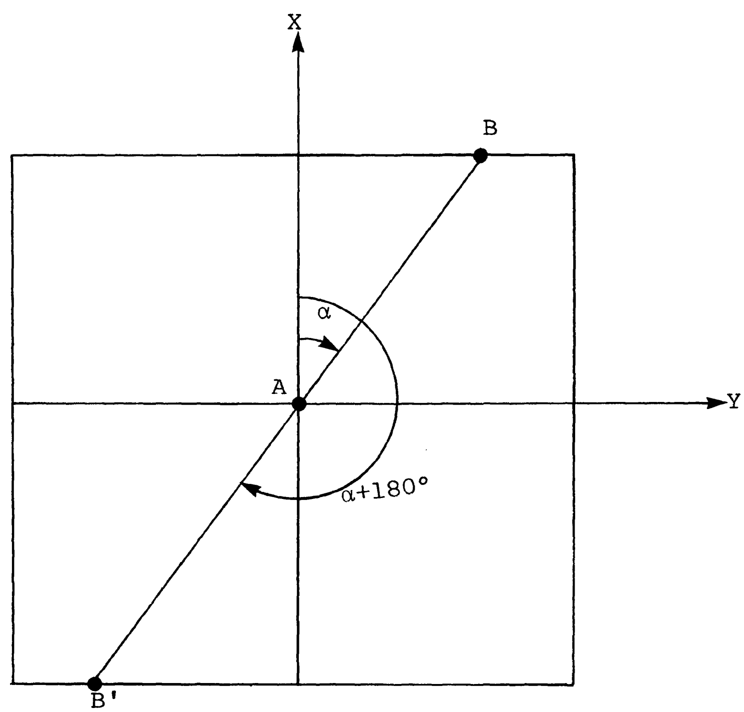


FIGURE 7-3

Gravity Correction Across The Entire Cell.

point anomaly values) for our use here.

The above arrangement implies that the computations within cells which are located around Canadian coasts or inland waters could be affected in either one of two ways. First, the cell may be dismissed altogether because of insufficient number of land data points remaining. Secondly, the land data points within the cell may have adverse distribution which may influence the results. However, realizing that the aforementioned master file contains only about one-third of the entire data points on the EPB new file, the computer time needed for reading and handling the associated data sets is reduced significantly.

7.3.1 Programming considerations

A computer program package AREAGC has been developed by the author for the purpose of studying the variation of gravity corrections (dynamic, Helmert, Vignal) with azimuth at any desired location in Canada. This program is a modified version of the earlier program LEVAGRAV [Nassar, 1975b]. The program has been written in standard FORTRAN [Cress et al., 1968; I.B.M., 1970]. The algorithms used in the program are based on the technique presented earlier in this chapter. AREAGC program expects to find available, on tape or on disc, the seven overlapping data sets discussed in Appendix II. These data sets contain the free-air anomaly and height data as well as other information relevant to the present study. Details and documentation of the program are available at the Surveying Engineering Computer Library, U.N.B.

AREAGC program package is composed of the main program and several subroutines. Of these subroutines, there are two main ones: the first, APPROX, is for approximating the anomaly and height fields within the cell; and the second, GCAFAZ, is for approximating the anomaly and height profiles and computing the gravity corrections in different directions within the cell.

The main program reads the geodetic coordinates (ϕ_o, λ_o) of the centre of the cell under consideration. It then seeks the appropriate data file and stores the relevant information needed for the prediction of both the anomaly and height fields.

APPROX has been developed as a general subroutine for performing a least-squares approximation to any data set, using a 2-D mixed algebraic polynomial of an arbitrary order. The outcome is the best-fitting coefficients, their covariance matrix and the resulting a posteriori variance factor. The basic algorithm of this technique is given in section 7.2.2; however, a few particular programming considerations should be mentioned here. First, the Vandermonde's matrix A can attain a few thousand rows, depending on the number of available data points within the cell. Thus, to save storage, the individual elements of matrix A are generated, one at a time, whenever needed in the computation without the necessity of storing them, using a function subprogram. The normal equations matrix is inverted using Cholesky decomposition technique [Wells, 1973].

The use of mixed algebraic polynomials with (x, y) coordinates being expressed in kilometres, results in a matrix of normal equations that displays:

1. Elements of very large numerical values that may cause an overflow in the inversion process, i.e. the value of its determinant exceeds the maximum permissible value in the available computer;
2. Sizeable magnitude differences among its elements, which may cause the matrix to become ill-conditioned.

In order to keep the computational errors low, it is better to work with elements which are close to unity [Faddeev and Faddeeva, 1963; Wilkinson, 1963]. This is achieved by introducing appropriate scale factors.

The first problem mentioned above has been overcome by scaling the normal equations matrix as a whole, i.e. by using only one scale factor. This is done in the subroutine APPROX. The second problem can be solved by performing a column scaling of the design matrix A before setting-up the normal equations [e.g.: Nassar, 1972]. This has been accomplished, in our case, by scaling the local plane coordinates (x, y), i.e. by changing the coordinate units. This is done in the main program. In such a case, we have to apply the same coordinate scale factor to the integration distance s of the simulated profiles and to the mean radius of curvature ρ_{om}. This is taken care of in the subroutine GCAFAZ. The choice of appropriate scale factors depends mainly on the number "n" of available data points in the cell and on their distribution relative to the centre of the cell. The scale decreases with increased n and the distance from the centre.

From our experience, it was found practically impossible to come up with a unified scale factor for the coordinates of data points that could be used for all cells having different characteristics.

Instead, the appropriate scale had to be determined for each cell by trial and error. To overcome this problem of scaling, we decided to replace the subroutine APPROX by another called ORTHO. The latter subroutine uses the Gram-Schmidt's orthogonalization process [e.g. Cheney, 1966; Thompson, 1969; Vaníček and Wells, 1972] to solve for the coefficients of the best-fitting approximating polynomial. This process involves three main steps: orthogonalization of the 16 column vectors of matrix A; computing 16 auxiliary coefficients (known as Fourier coefficients) along with their variances; and finally evaluating the sought original 16 coefficients of the 3-rd order approximating polynomial along with their covariance matrix. The subroutine ORTHO is available at the Surveying Engineering Computer Library, U.N.B., and was used in several previous studies [e.g.: Christodoulidis, 1973; Merry and Vaníček, 1974; Merry, 1975]. For reasons mentioned earlier, we also use one scale factor for the (x, y) coordinates such that they are expressed in units of tens of kilometres. ORTHO eliminates the need for inverting the normal equations matrix, and thus overcomes the scaling problem; however, it is at least 50% slower than APPROX.

The subroutine GCAFAZ computes the coefficients of the polynomials approximating the free-air anomaly and height profiles. It then computes the accumulated gravity corrections (dynamic, Helmert, Vignal) over the distance of 40 km and standardizes them for 1 km. It repeats these computations for simulated levelling profiles with azimuth increasing from 0° to 360° by 5°. All the computations in GCAFAZ subroutine are performed by elements rather than by matrices to save storage, since most of the involved matrices (see sections

7.2.3 and 7.2.4) are relatively sparse.

The main output of GCAFAZ is a plot, generated via a subroutine named GCPLOT, showing the pattern of the value of the gravity correction vs. azimuth for all three height systems. In order to achieve a uniform size for all patterns, the scale of these plots, computed automatically in the subroutine, differs according to the maximum value of the gravity correction. In addition, GCAFAZ prints out a summary of results indicating the magnitude and orientation of both the maximum and minimum absolute value of all three kinds of gravity corrections within the cell. Finally, the main program plots (via GCPLOT subroutine) a sketch showing the distribution of data points within the computed 1° x 1° cell.

It should be noted here that the subroutine GCPLOT has been written on the basis of the U.N.B. Computing Centre Library plotting routines [Gujar, 1972]. Hence, before using this subroutine elsewhere, it should be modified to suit the available system. We should also mention that several error messages are issued by the AREAGC program. The user is notified of any insufficiency or poor distribution of data points, or the non-existence of gravity data on files for a particular cell.

7.3.2 Numerical results

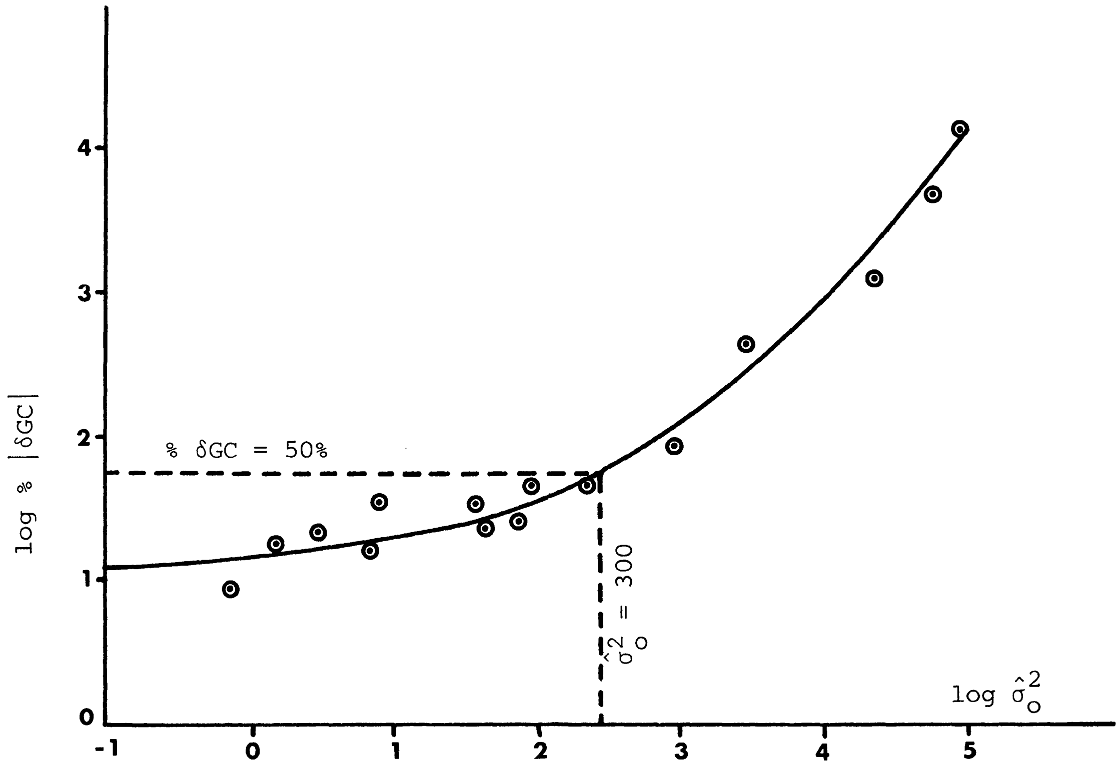
In this section the results obtained from AREAGC program are discussed. The intention was to predict the gravity corrections for the Canadian territory within each 1° x 1° cell using our technique. However, because there are no precise levelling lines in Northern Canada (see Figure 5-1), areas north of latitude 65°N were not considered.

In section 7.2.2 we have commented upon the choice of 3rd-order approximating polynomials (i.e. r=3), and upon the use of the resulting a posteriori variance factor, $\hat{\sigma}_o^2$. This factor reflects the model error, as a measure of the roughness of the approximated field (anomaly or height). By having selected r = 3 and 1° x 1° interpolation area, we seek only the smooth features of the anomaly and height fields, hoping that the predicted gravity corrections will be still meaningful. We have seen from Tables 7-1 and 7-2 that the error was in average less than 20% for the predicted free-air anomaly and less than 10% in case of the predicted height. Such results (using r = 3) are satisfactory for predicting point values centred at the origin of the cell. On the other hand, the accumulated gravity corrections along simulated levelling lines are computed using the same approximating surface for the entire cell. This means that model errors could influence the computed gravity correction significantly.

Consequently, the performance of our technique for predicting gravity corrections in areas with different degrees of roughness (i.e. with different values of $\hat{\sigma}_o^2$) had to be tested. The following procedure had been used. Several cells at different locations across the country have been selected such that they represent flat, gently rolling, hilly and mountainous regions. In each cell a hypothetical levelling line running through its centre at a different azimuth was considered. Bench marks along each line, equidistant at intervals of 0.05 degrees of arc in latitude and/or longitude were then simulated. The free-air anomaly,

Δg^F, and the height, h, were predicted (using subroutine ORTHO) at each bench mark. The predicted Δg^F and h values for each line were used as input to the LOOPGC program (see section 6.2) to compute the accumulated GC along the whole line by summation over all the segments. Finally, the results from LOOPGC were compared against the corresponding values obtained from the AREAGC program as depicted in Figure 7-4. In this Figure $\hat{\sigma}_o^2$ is taken to equal to either $(\hat{\sigma}_o^2)_g$ or $(\hat{\sigma}_o^2)_h$ whichever is larger - usually the latter. Also, the differences δGC are taken as the average difference for the three systems of heights under consideration.

Since we are looking for the order of magnitude of the GC, we can accept discrepancies in the predicted GC up to 50%. The examination of Figure 7-4 reveals that this percentage corresponds to $\hat{\sigma}_o^2 \doteq 300$. The actual results from several cells indicate that large values of $\hat{\sigma}_o^2 > 300$ occur only in mountainous areas. For flat terrain $\hat{\sigma}_o^2$ is usually less than 10; for gently rolling terrain $\hat{\sigma}_o^2 < 100$; and for moderately hilly terrain $\hat{\sigma}_o^2 < 300$. Hence, we can say that the technique for predicting the areal pattern of the GC (using r=3) gives satisfactory results for flat and gently rolling terrain which is characterised by $\hat{\sigma}_o^2 < 300$. The technique is not successful for mountainous areas, due to the resulting large model errors. Since the necessity of applying gravity corrections in mountainous areas has been well established already [e.g. Helmert, 1890; IAG, 1950; Rune, 1950a; Ledersteger, 1954; Vignal and Kukkamäki, 1954; Bursa, 1958; Baeschlin, 1960a; Krakiwsky, 1965; Ramsayer, 1965a; Holdahl, 1974; Holdahl, 1975a], this should not be considered as a major hinderance. However, if needed, the performance can be improved by selecting a higher order approximating surface for mountainous areas wherever there is enough data points to do so.



$$\% \ \delta GC = 100 \ (GC_{LOOPGC} - GC_{AREAGC})/GC_{LOOPGC}$$

$\hat{\sigma}_o^2$ = a posteriori variance factor.

FIGURE 7-4

Limitations of The Prediction Technique

All three kinds of GC's were then predicted within the 1°x1° cells across the entire country up to latitude ϕ = 65°N. The number of cells, having sufficient number of data points, is about 1000. One-hundred cells were not processed by the AREAGC, because they do not have the required distribution of at least one data point, in each quadrant around the centre of the cell. Fifty more cells were rejected after the computation due to small number of data points (less than one and half times the number of best-fitting coefficients) or due to adverse distribution of data points that resulted in unreliable estimates of predicted point anomaly or height values. Resulting standard deviation is larger than the predicted point value itself. Many of the above 150 cells are along the eastern and western shores and around Hudson Bay. Some of them are affected by the neglect of sea data in our files as mentioned earlier. The histogram showing the values of $\hat{\sigma}_o^2$ in the remaining 850 cells is given in Figure 7-5.

Considering the cut-off limit of $\hat{\sigma}_o^2$ = 300 (see Figure 7-4), the results indicate that the performance of the prediction technique has been successful in 88% of the cases (750 cells). Consequently, the 100 cells with $\hat{\sigma}_o^2 > 300$ covering mountainous regions are excluded from the listings in the external Appendices VI-X. Most of them are concentrated in Western Alberta, British Columbia and the Yukon Territory; the rest is scattered in the Maritimes, Labrador, Newfoundland and N-E Québec.

Since the results and plots for the remaining 750 cells are voluminous, they are filed as a series of five external Appendices (each of several volumes) to this thesis.

Also, because of the nature of the gravity correction to vary with azimuth (within each cell), there was not an easy or even practical

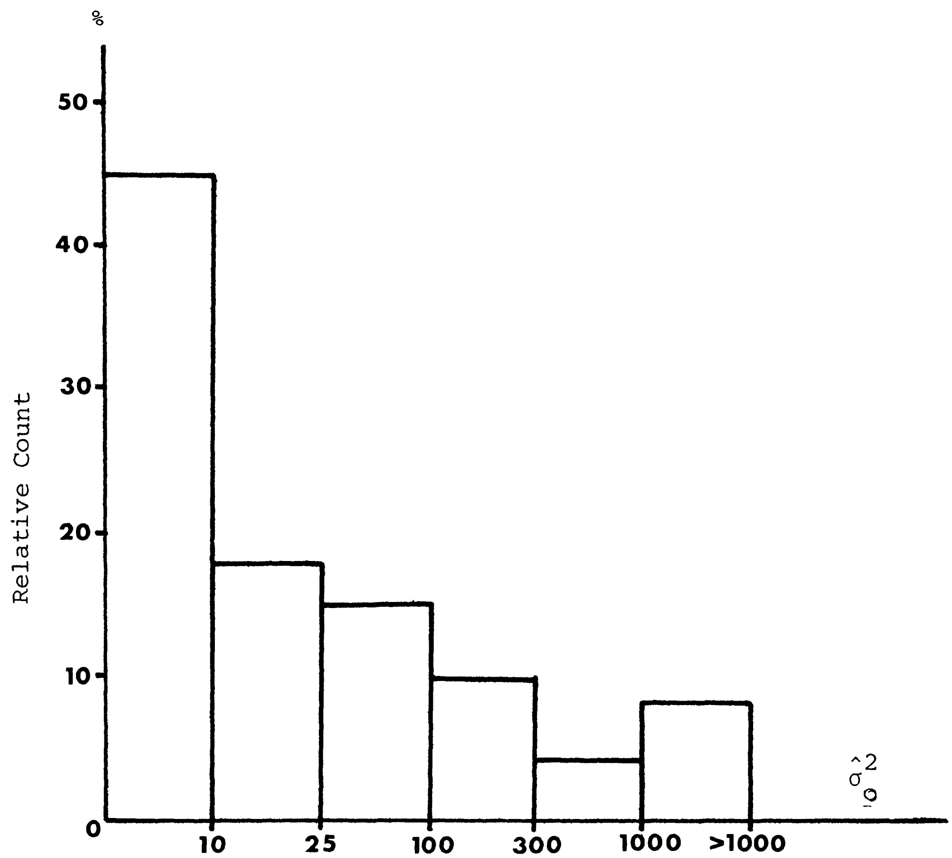


FIGURE 7-5

Histogram of The A Posteriori Variance Factor

For 850 Considered Cells

way of depicting all the gravity corrections in a single summary map for the whole country. Thus, here we give only some quantitative statistics about the significance of the gravity corrections for the different systems of heights. For illustration, some sample results, as extracted from the aforementioned external Appendices, are shown.

Figure 7-6 shows the histogram of the maximum absolute value of Helmert gravity correction in each cell. A similar plot is given in Figure 7-7 for both Vignal and Dynamic gravity corrections. These results indicate that the maximum GC^H is significant (> 0.14 mm/km) in 98% of cases and exceeds the standard error of precise levelling, $\sigma_{\Delta h}$, in 55% of cases. Further, GC^V and GC^D are significant in 92% of cases, and exceed $\sigma_{\Delta h}$ in 22% of cases.

Let us now, for illustration, show some of the obtained results. Areas (blocks) with different gravity anomaly and height characteristics are selected, so as to show different possible combinations of low or high values of anomaly and/or elevation. Four locations of such blocks of 5° x 5° containing 25 cells each, are illustrated in Figure 7-8. The first block, located in Western Ontario, covers the area of ϕ = 50°N-55°N and λ = 90°W - 95°W, and possesses relatively low anomaly and low altitude values. The patterns of the predicted gravity corrections for Helmert, Vignal and Dynamic systems are depicted in Figures 7-9, 7-10 and 7-11, respectively.

The second block, located in Southern Québec, covers the area of ϕ = 45°N - 50°N and λ = 71°W - 76°W, and is characterized by a relatively high anomaly and low elevation. The corresponding plots of the three kinds of gravity corrections are given in Figures 7-12, 7-13 and 7-14.

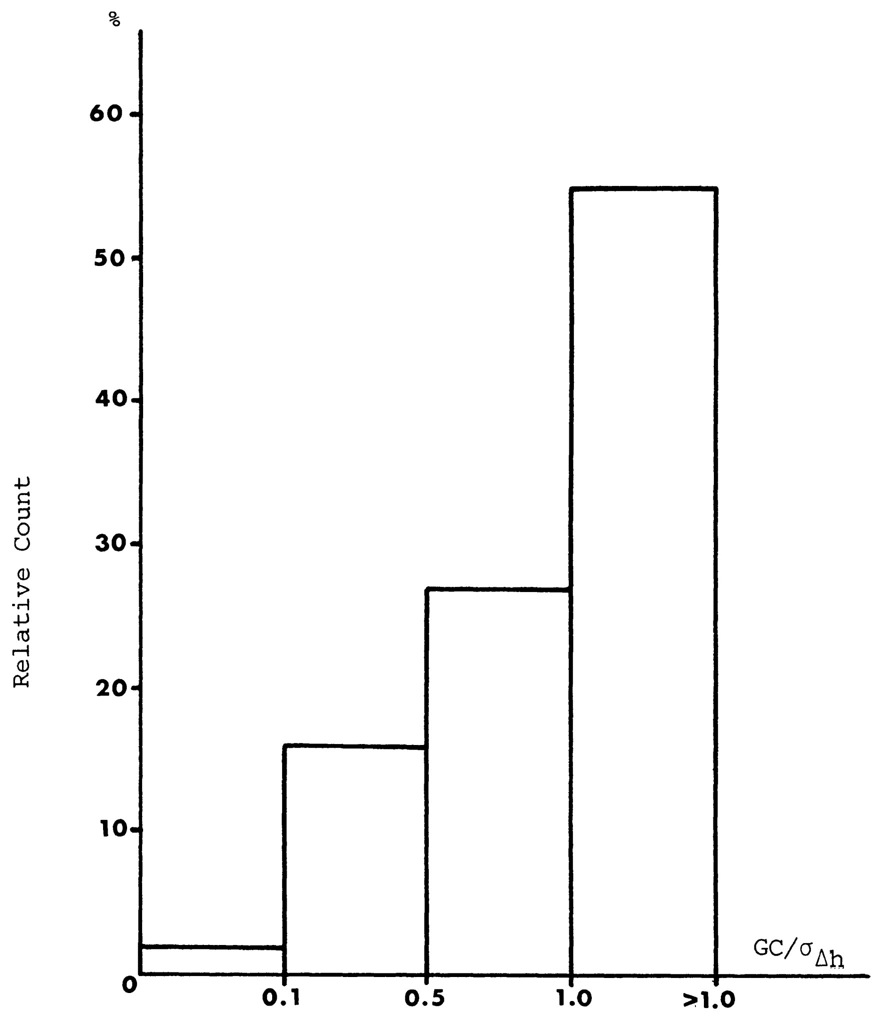


FIGURE 7-6

Histogram of Maximum Absolute Value of

Helmert Gravity Correction For The Resulting 750 Cells

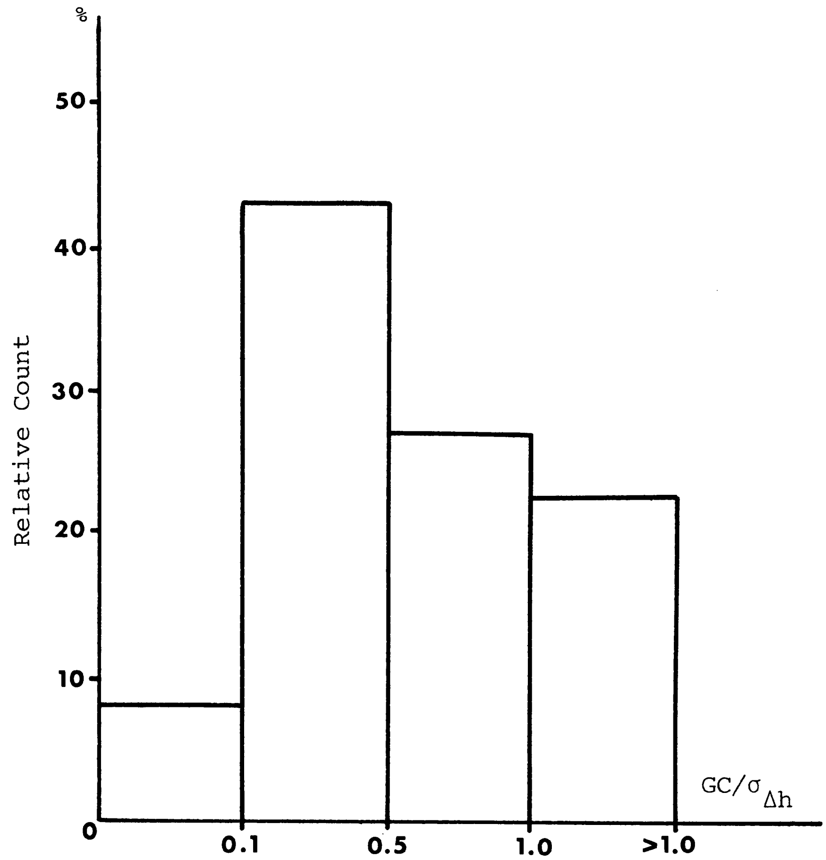


FIGURE 7-7

Histogram Of Maximum Absolute Value or Vignal And Dynamic Gravity Corrections For The Resulting 750 Cells

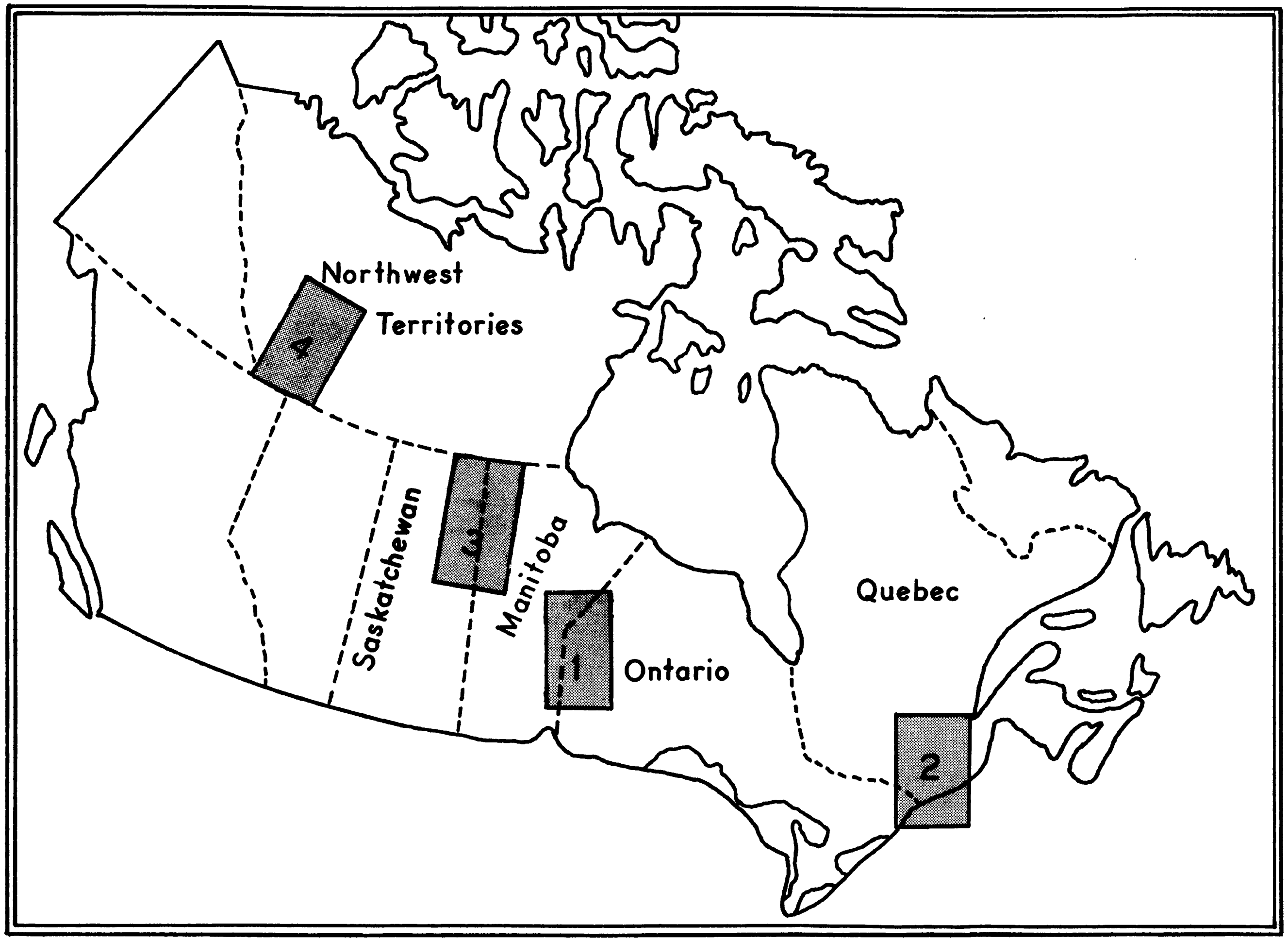
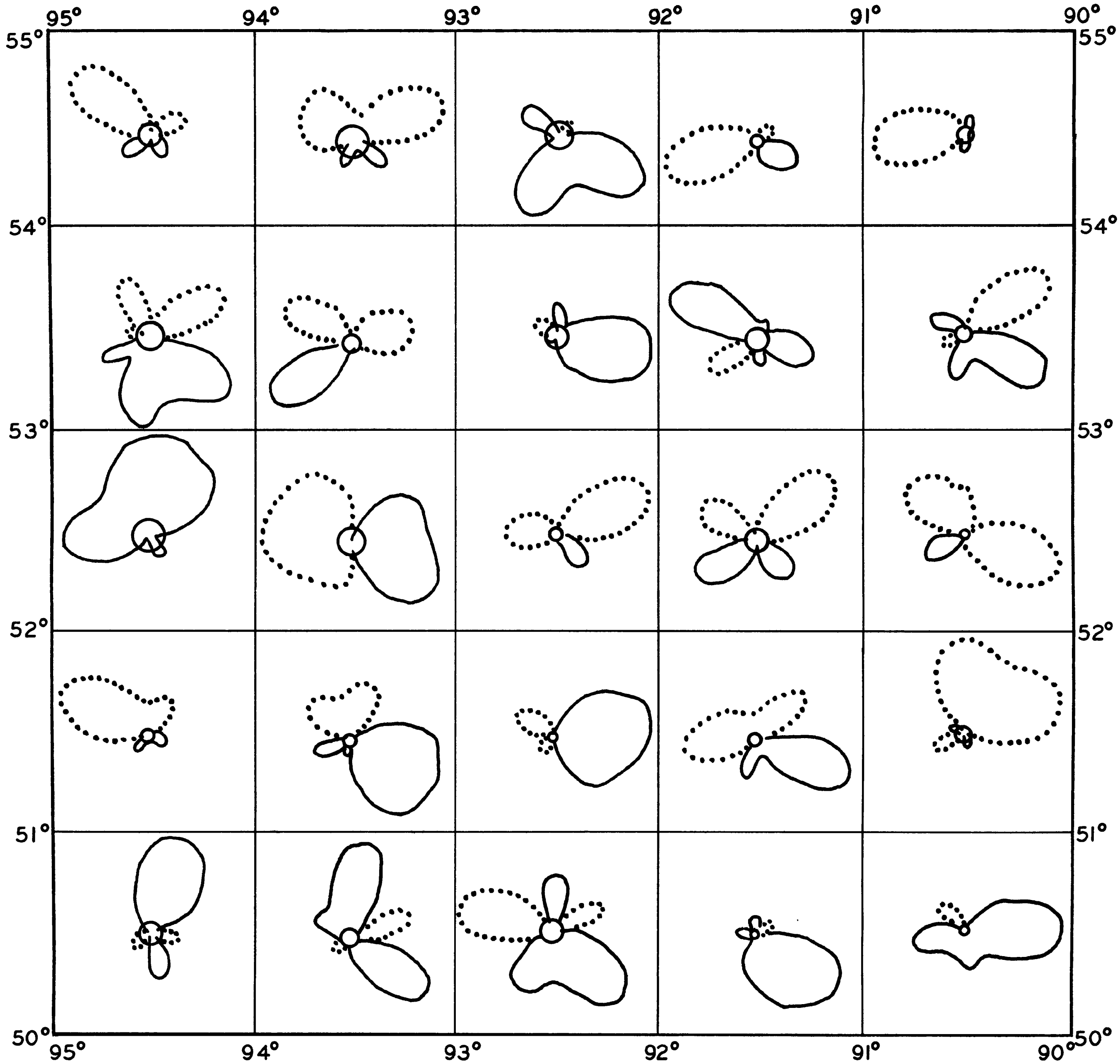


Figure 7-8

Location of Selected Areas for the Predicted Gravity Corrections



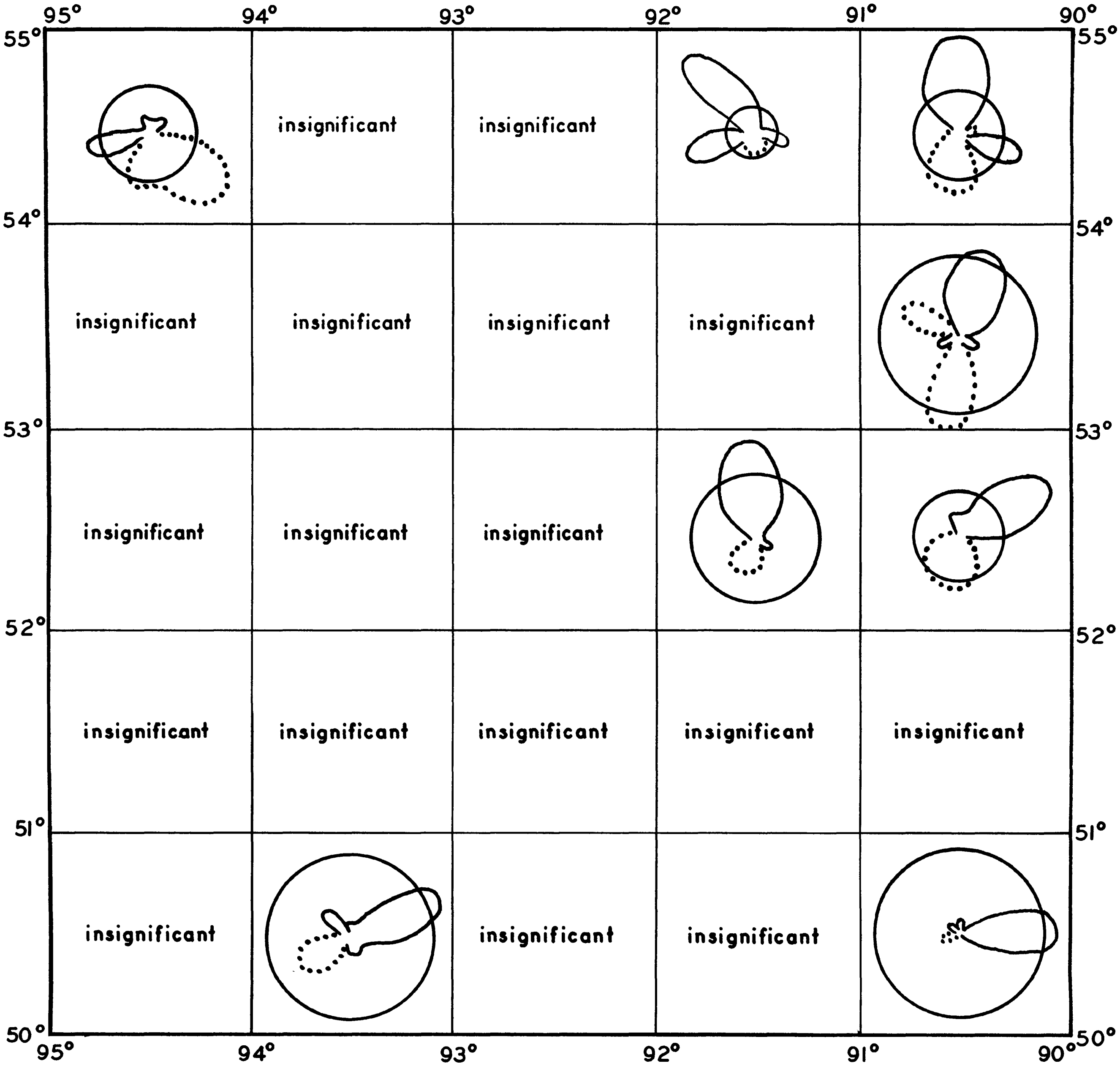
Legend: ——————— positive gravity correction

.... negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km .

Figure 7-9

Helmert Gravity Correction in Western Ontario.



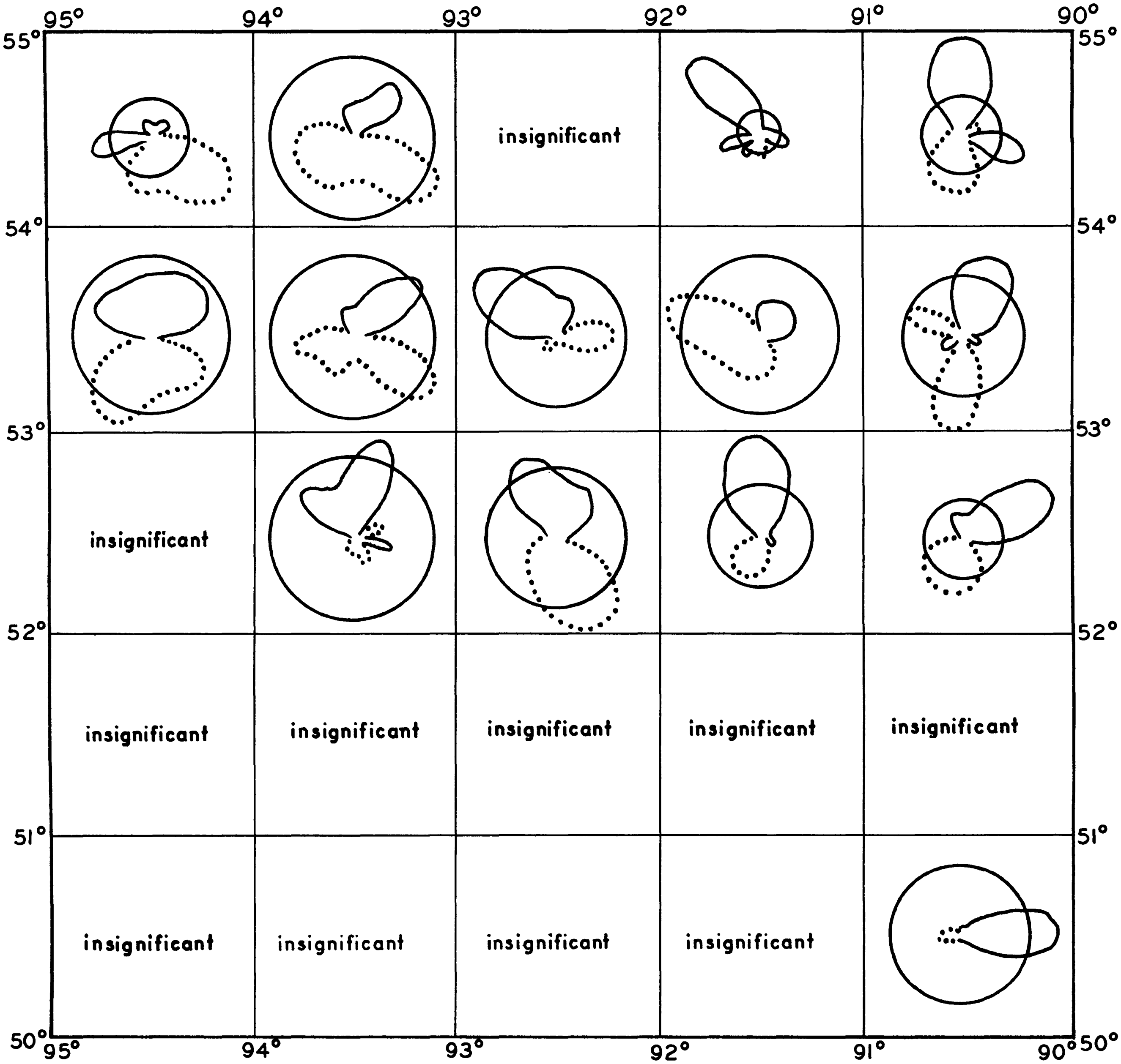
Legend: ———— positive gravity correction

············ negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-10

Vignal Gravity Correction in Western Ontario.



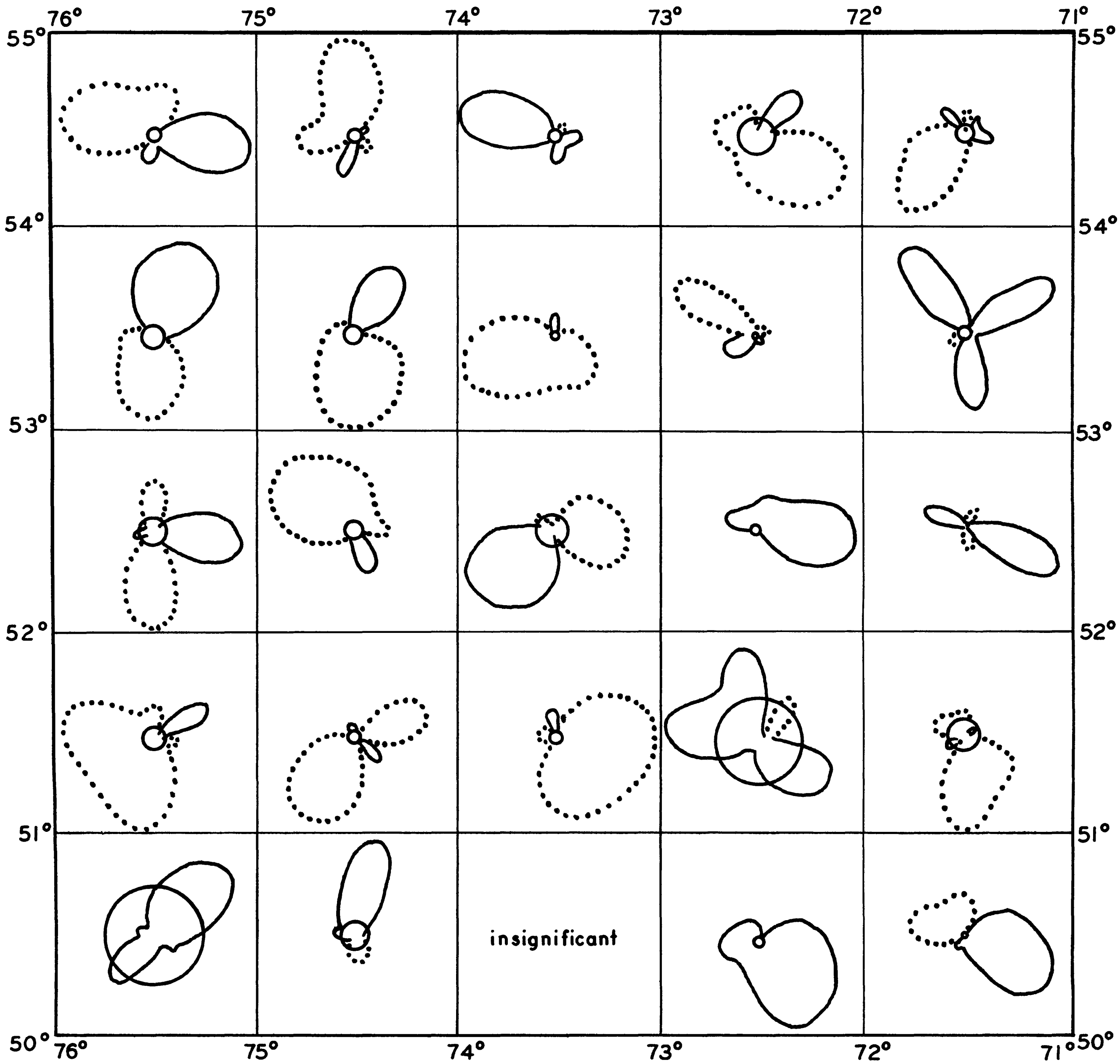
Legend: ——————— positive gravity correction

. . . . negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-11

Dynamic Gravity Correction in Western Ontario.



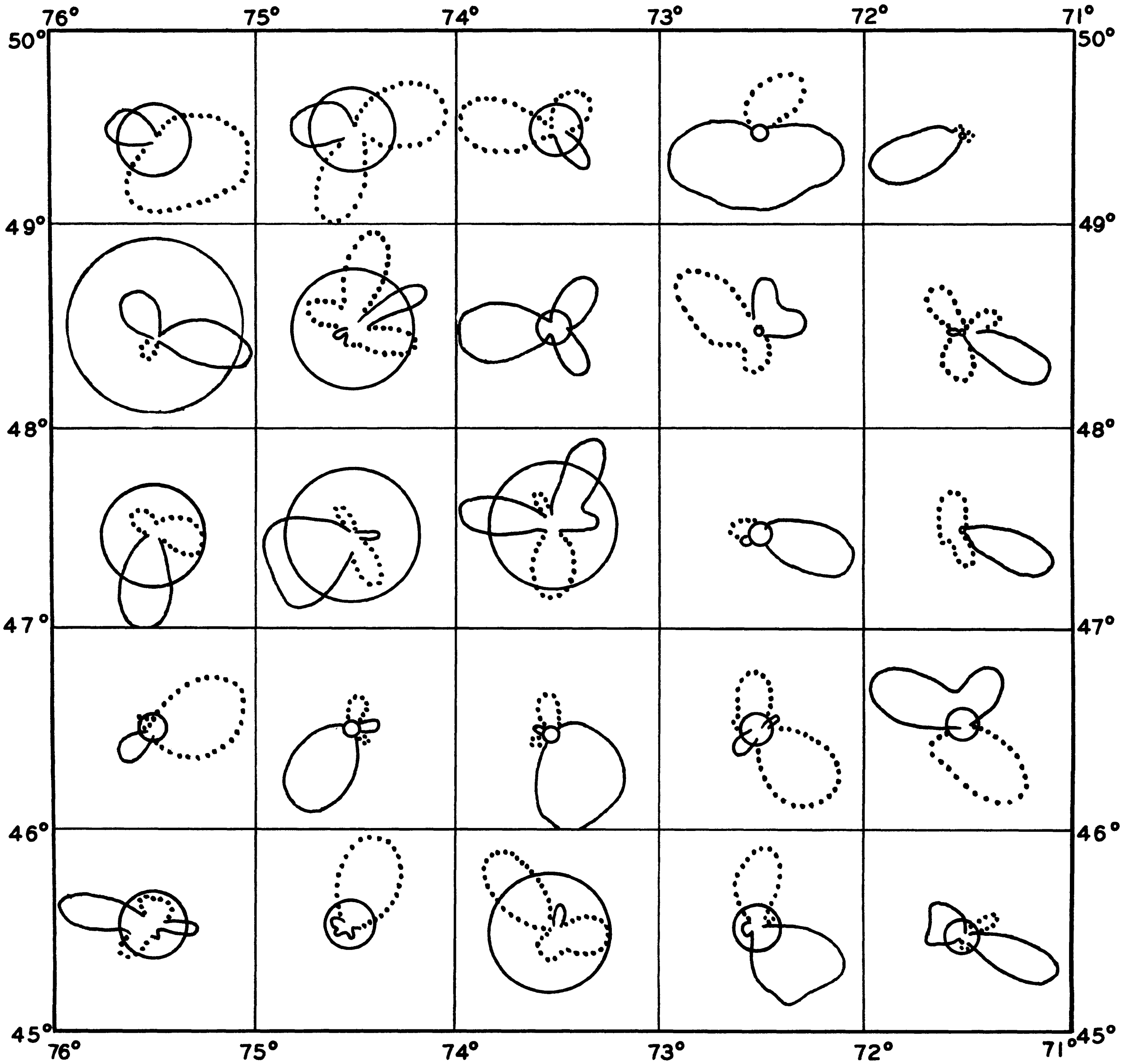
Legend: ——— positive gravity correction

. . . . negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-12

Helmert Gravity Correction in Southern Québec.



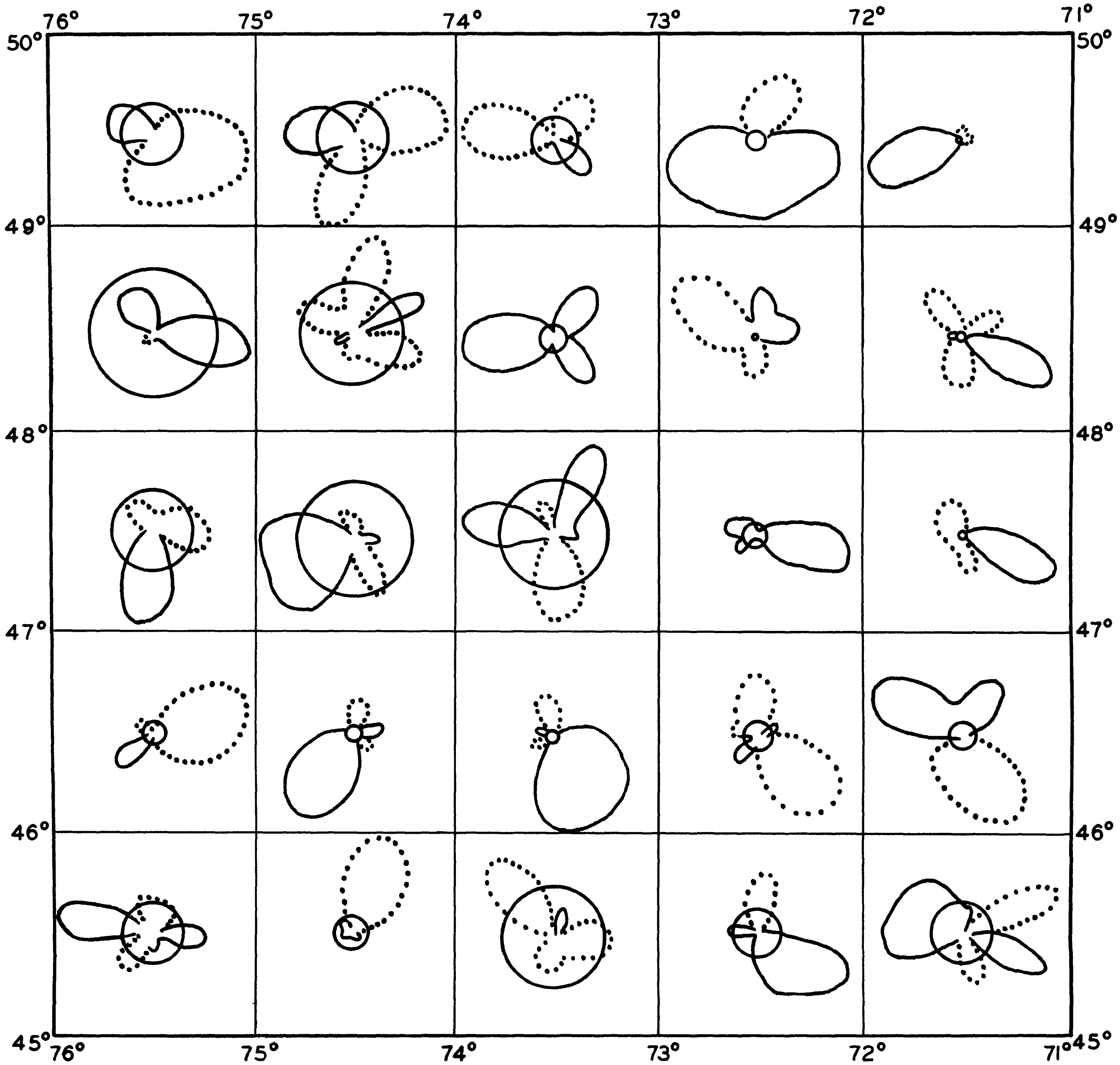
Legend: ——————— positive gravity correction

. . . . negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-13

Vignal Gravity Correction in Southern Québec.



Legend: ———————— positive gravity correction

· · · · · · · · · · · · · · negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-14

Dynamic Gravity Correction in Southern Québec.

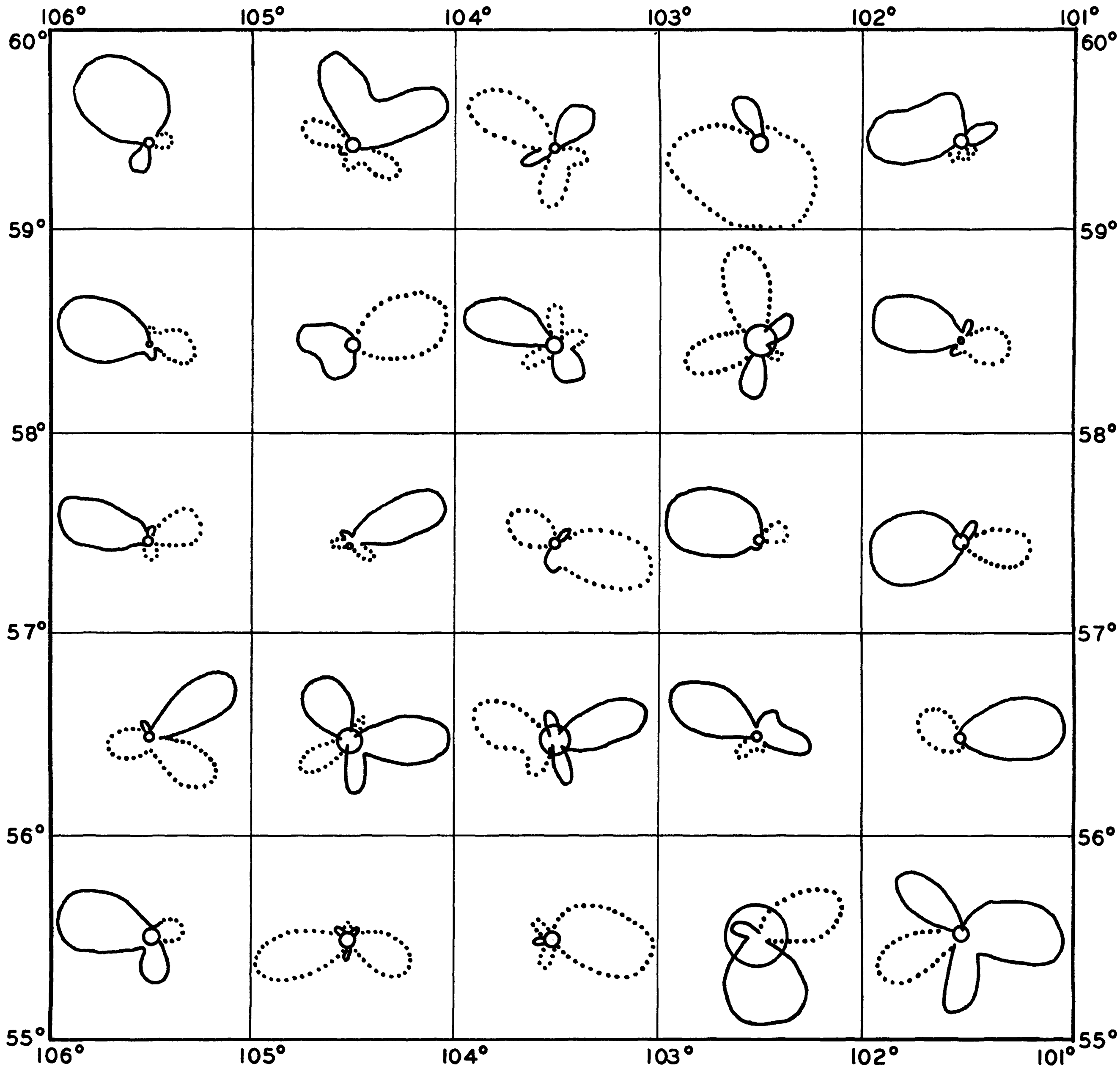
The third block, located on the border of Manitoba and Saskatchewan, covers the area of ϕ = 55°N - 60°N and λ = 101°W - 106°W, and has a relatively low anomaly and high elevations. The obtained patterns of the gravity corrections are shown in Figures 7-15, 7-16 and 7-17. The last block, located in the Northwest Territories, covers the area of ϕ = 60°N - 65°N and λ = 118°W - 123°W, and possesses both relatively high anomaly and elevations. The resulting plots of the three gravity corrections are contained in Figure 7-18, 7-19 and 7-20.

From all these Figures, it seems as if the pattern of the gravity correction, for all systems, is essentially composed of several ellipses overlapping around the centre of the cell with their major axes radiating from the centre. It is also obvious that the size and orientation of these ellipses vary among the different height systems, and from one cell to another.

In order to be able to visually inspect the significance of the gravity corrections from their pattern, a circle with radius 0.14 mm/km (refer to section 7.1) is drawn around the centre of the cell to the same scale as used for the pattern plot. The size of the circle thus varies from one cell to another. The difference between the pattern of the gravity correction and the "significance" circle along any radial line represents the amount by which the gravity correction exceeds or fall below the specified significance limit.

From the given results we can notice the following:

1. GC^H is generally larger than the other two corrections, and is significant in almost all directions within each cell. This is the case for all selected areas. In particular, in region No. 4, i.e. with



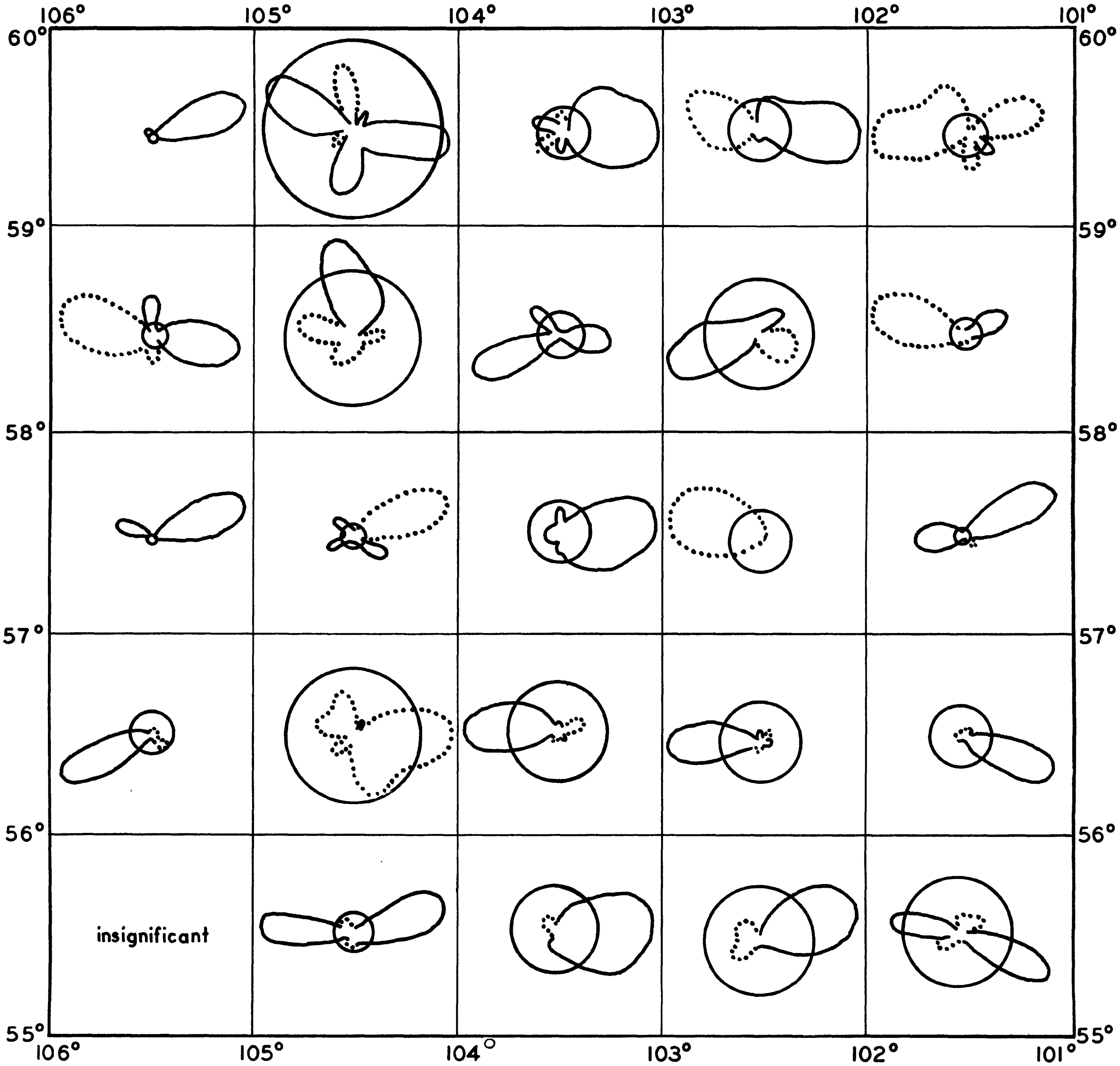
Legend: ———————— positive gravity correction

· · · · · · · · · · · · · · · negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-15

Helmert Gravity Correction in Manitoba - Saskatchewan.



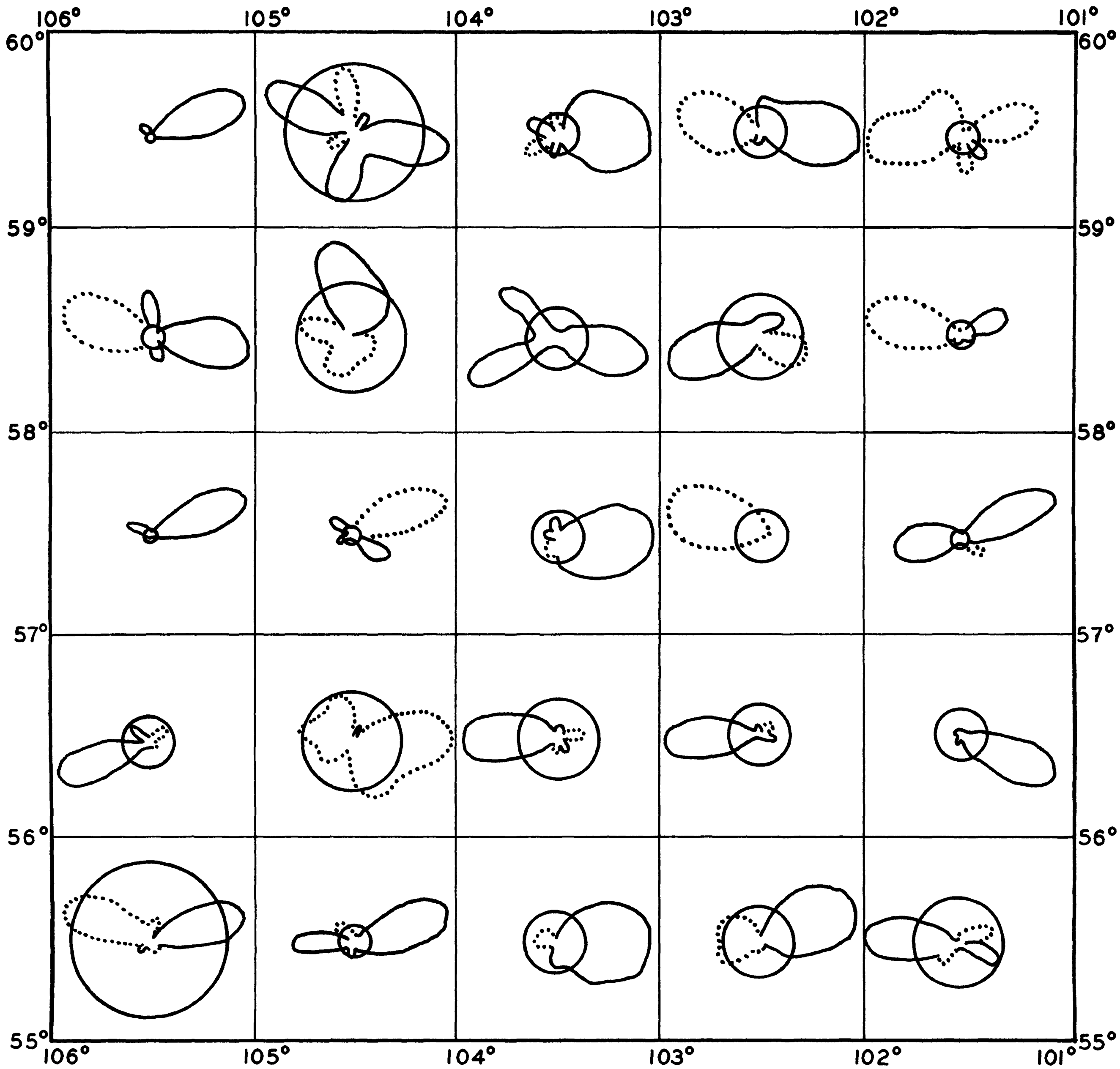
Legend: ———————— positive gravity correction

. . . . negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-16

Vignal Gravity Correction in Manitoba - Saskatchewan.



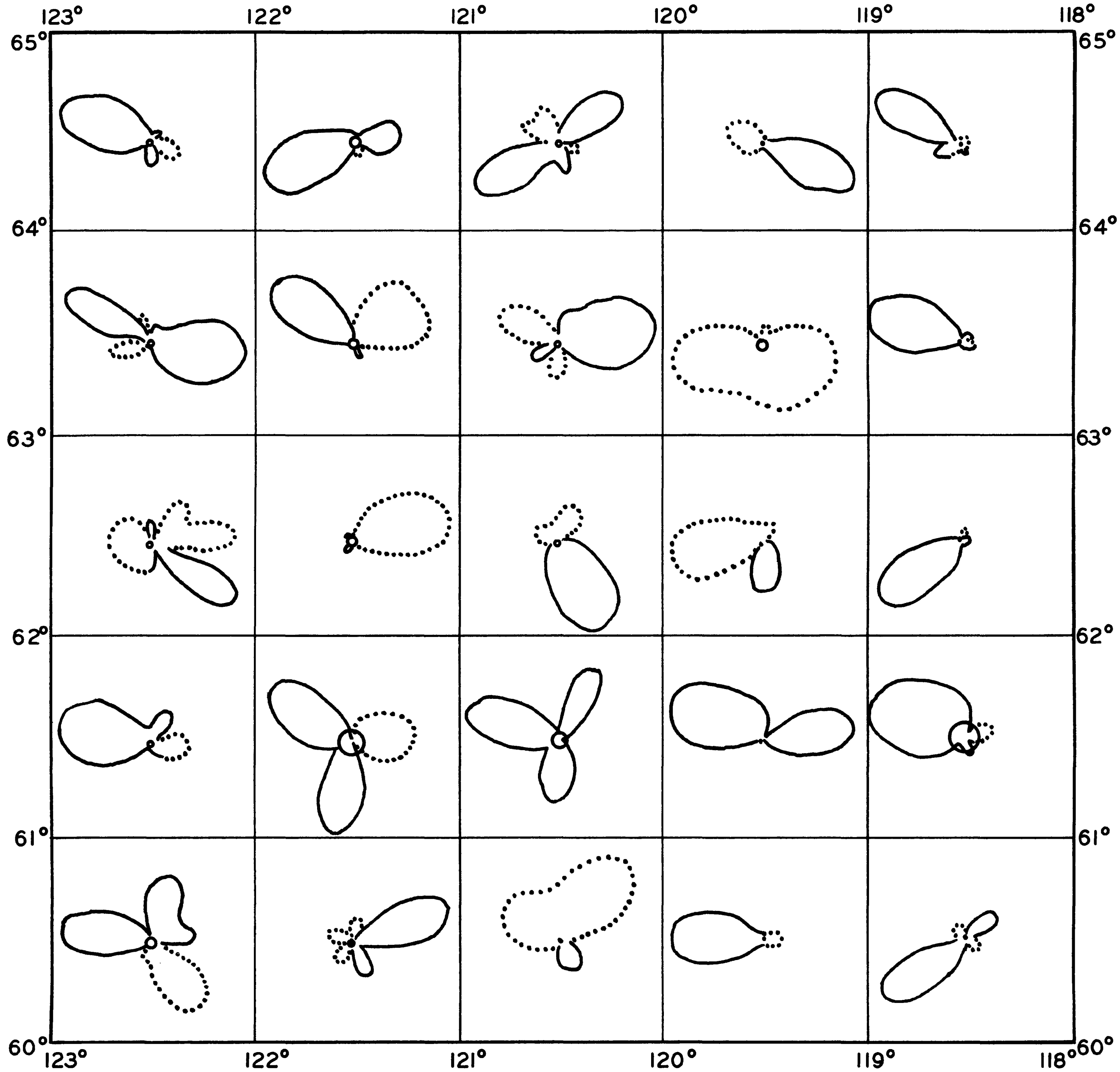
Legend: ———————— positive gravity correction

·················· negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-17

Dynamic Gravity Correction in Manitoba - Saskatchewan.



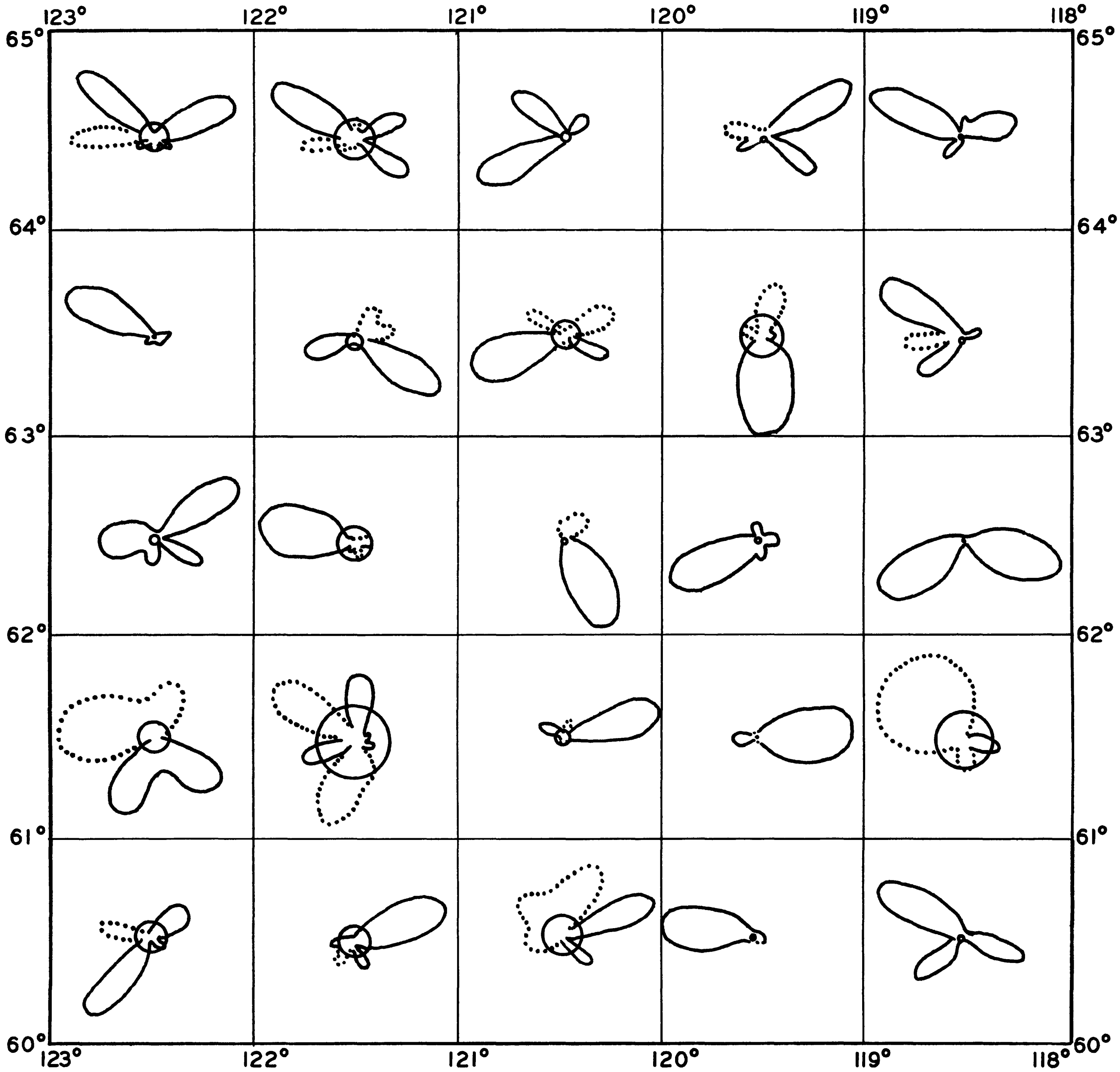
Legend: ———————— positive gravity correction

. . . . negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-18

Helmert Gravity Correction in Northwest Territories.



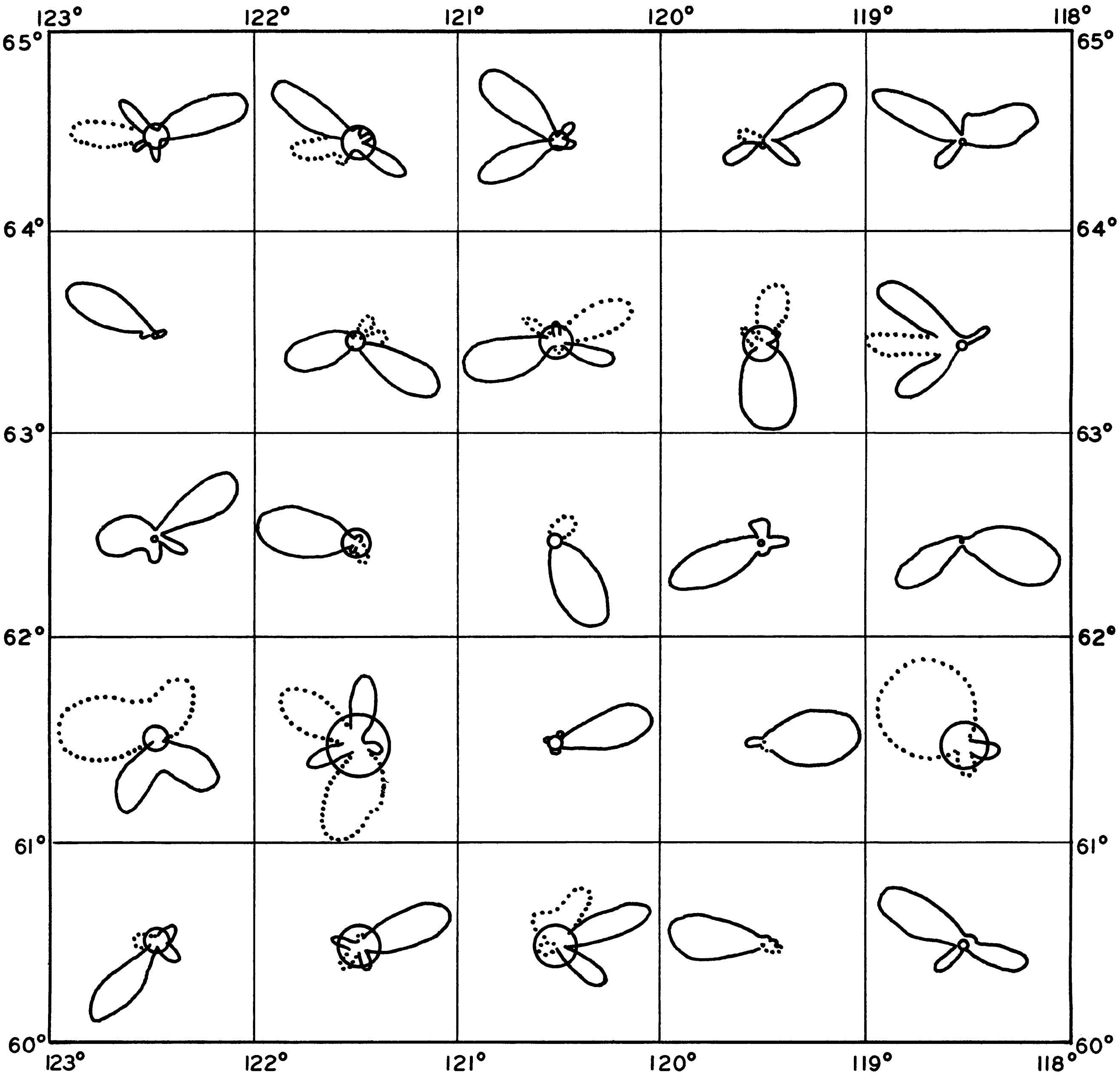
Legend: ———————— positive gravity correction

.... negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-19

Vignal Gravity Correction in Northwest Territories.



Legend: ———————— positive gravity correction

· · · · · · · · · · · · negative gravity correction

Scale: o the circle around the centre of each cell has a radius of 0.14 mm/km.

Figure 7-20

Dynamic Gravity Correction in Northwest Territories.

large Δg^F and h, the resulting GC^H is significant practically everywhere;

2. The patterns of both GC^V and GC^D are similar in shape, differing however in magnitude in some directions. This confirms the conclusions about the two systems given in Sections 4.2.3, 6.2.1 and 7.2.6;
3. About two-thirds of the region No. 1 (with low Δg^F and h) shows insignificant GC^V and GC^D. All the remaining regions indicate significant corrections in many directions within each cell, especially again within these of high Δg^F and h;
4. GC^H seems to be more sensitive to high elevations than the other two corrections, whereas GC^V and GC^D are more sensitive to high gravity anomaly;
5. In several cells, the magnitude of the maximum gravity correction, in particular for the Helmert system, exceeds even the value of the standard error of the CPLN (1.33 mm/km). This can be seen on the GC patterns wherever the significance circle (0.14 mm/km) degenerates to almost a point around the centre of the cell.

On the basis of these results, it can be concluded that, in practically all the cases, the gravity correction for all the three investigated height systems is significant in the direction of its maximum value. Furthermore, the four selected areas can be considered representative as to the general situation of flat, gently rolling and moderately hilly terrain in Canada, and thus we can generalize the above conclusion for all the Canadian territory. This in fact can be verified from Figures 7-6 and 7-7. Finally, judging from region No. 4, which is not exceedingly high or even exceedingly gravitationally disturbed, one would expect the gravity corrections to be even larger in the mountainous areas in Western Canada, and in the gravitationally disturbed areas in Eastern Canada as well as in higher latitudes (refer to Figure 5-2).

7.4 Conclusions

The patterns of predicted gravity corrections provide a means of depicting the variation of GC with location and direction of levelling lines. These patterns could be also useful, at least theoretically, for planning new levelling lines so that they will follow the directions of insignificant GC. However, practically almost all real levelling lines follow the roads, railways and other accessible routes. Thus, the direction of levelling lines would be governed in reality by accessibility in the region of interest.

The GC plots can be used in practice to determine whether the GC is significant and hence should be applied in any specific direction anywhere in Canada except mountainous areas and latitudes larger than 65°N. Considering this application one can see the advantages of treating the GC's in two-dimensions as opposed to the standard one-dimensional treatment. The technique used here enables one to scan the GC's as accumulated in any direction within certain area, and thus determine the magnitude and direction of the maximum value of GC in the area. Evidently, this is superior to the hap-hazard (not systematic) investigation of individual lines in arbitrary directions, which may or may not show significant GC but hardly can show any conclusive evidence one way or the other.

It should be reiterated here that the developed technique for areal prediction of GC's is not meant for computing gravity corrections that can be readily applied to a particular levelling section. As stated before, this has to be performed properly using the computational methods discussed in Chapter 6.

It must be noted that the predicted GC's are computed in the "mean sense", i.e. accumulated over 40 km lines. On the other hand, in our limited experiments with actual lines and loops (Chapter 6), we had some cases where although the accumulated GC was insignificant, the influence on intermediate sections was significant. Therefore, it would appear safer to consider the maximum rather than an average value of GC as an indication of significance within any cell. We can thus say that the gravity corrections are significant almost everywhere in Canada, and should be taken into account in the forthcoming new adjustment of the CPLN.

CHAPTER 8

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

In this study, the influence of actual gravity irregularities on the orthometric (Helmert) and dynamic height differences, as currently used in Canada, is investigated. This influence is referred to as "GRAVITY CORRECTION". For possible future applications, the normal (Vignal) height system is also included in the study.

First, the basic foundations of the theory of rigorous heights (based on actual gravity) and their counterparts, the approximate heights (based on normal gravity, as used in Canada) are laid down. Then, the gravity corrections, GC's, for dynamic, Helmert and Vignal systems, along with their accuracy estimates, are modelled in terms of routinely available quantities. A general discussion of precise levelling and gravity data coverage in Canada is given. The practical evaluation of the GC's for actual levelling lines and loops using real data is investigated. Tables are also provided to simplify the evaluation of GC.

A technique for predicting the GC's in two dimensions is developed to identify the geographical areas of significant GC's in Canada. Finally, results from this technique are presented and their practical implications are commented upon.

The results and conclusions of the entire investigation are summarized below:

1. The values of accumulated gravity correction around a closed loop for dynamic, Helmert and Vignal systems are shown to be identical. This illustrates the fact that all systems of heights which take into account the true gravity field, have the same characteristic quality of defining the heights of terrain points uniquely;
2. The difference between the dynamic and the Vignal gravity corrections is due solely to the effect of the difference $\delta\gamma_o$ between the normal gravity used in the evaluation of approximate height corrections and that used for computing the free-air anomaly. This implies that the expressions for both GC^D and GC^V are the same when $\delta\gamma_o = 0$;
3. The gravity corrections are not negligible when compared to the accuracy actually achieved in precise levelling. Moreover, these GC's do not cancel out around a closed loop, but generally produce a misclosure. This implies that disregarding the actual gravity irregularities in computing the heights will in many cases cause errors of the same size, if not larger, as the accumulated standard errors in precise levelling, especially in case of Helmert system;
4. The gravity corrections can be readily evaluated with sufficient accuracy, using either the developed formulae or the provided Tables. If the required values of free-air anomaly are not available from observed gravity at bench marks, the interpolated values (from anomaly contour maps) or predicted values (from surface fitting techniques) can be used;
5. Given the Canadian gravity data coverage, the program package AREAGC (using data sets extracted from the EPB gravity data bank) is capable of predicting the free-air anomaly at any bench mark along any levelling route in Canada with the exception of mountainous regions.

It also provides the accuracy estimate of predicted anomaly. The only input needed are the geographical coordinates (latitutde and longitude) of the bench mark in question;

6. The behaviour of Helmert gravity correction along levelling routes is more abrupt and its magnitude is more pronounced than that of either of the other two corrections;
7. When the free-air anomaly contour map for Canada becomes available, it should, along with the provided GC Tables, further simplify the evaluation of the gravity corrections in the field;
8. The developed technique for areal prediction of GC's, using a 2-D 3rd order approximating polynomial and a 1° x 1° interpolation area, appears to give satisfactory results for flat and gently rolling areas which are not gravitationally disturbed. These areas are recognizable by the a posteriori variance factor $\hat{\sigma}_o^2 < 300$ associated with the approximated field (gravity anomaly or height). The performance of the prediction technique, based on the averaging power of the polynomials, has been tested against the standard approach of computing the gravity correction segment by segment. The two results differ by less than 50% for $\hat{\sigma}_o^2 < 300$. However, the differences grow unreasonably large in mountainous areas (where $\hat{\sigma}^2 > 300$);
9. The gravity correction plots, which have been generated for Canadian areas up to $\phi = 65°N$, show the direction in which they are significant and should be then taken into account. The obtained results indicate that the gravity corrections are significant within more than 90% of the computed 1° x 1° cells, at least in the direction of the maximum

value. The regions with large gravity anomaly and high elevations show significant GC in almost all directions;

10. The accuracy estimates of the predicted free-air anomalies at the centre of each cell considered in this study, can be taken as an indication that the present gravity data coverage in Canada is sufficiently dense for the purpose of evaluating the GC, except in the mountainous and gravitationally disturbed areas.

Generally the above results should help in bringing to the attention of the Canadian, and the American, geodetic communities the importance of proper treatment of heights. It should prompt an action towards incorporating gravity corrections in the routine of height computations, in order to achieve the standard of accuracy required for the first-order levelling network. Since the coverage of most of the Canadian territory with gravity observations has become sufficiently dense, it appears realistic to redefine the CPLN in such a way as to have the heights based on actual gravity as recommended by the International Association of Geodesy. The advent of superior automatic levels with the anticipated reduction of observational error magnitude will further justify the routine incorporation of gravity corrections in the computational process for height determination everywhere in Canada.

An important use of the dynamic gravity correction should be in the present IGLD-55 (International Great Lakes Datum of 1955). This network, so far, utilizes the dynamic height system based on normal gravity.

The following suggestions are made for future work:

1. Gravity corrections along the same actual levelling lines and loops should be computed using "observed" anomalies and "interpolated" anomalies from contour maps. The comparison and analysis of the obtained results would help assessing the feasibility of using anomaly maps instead of observed gravity at bench marks;
2. The gravity field in the Rockies, British Columbia and Yukon Territory should be densified. The existing gaps should be eliminated so that the gravity corrections to levelling routes within these regions can be evaluated. A similar statement holds true for the Canadian Arctic in the regions of possible future extensions of the CPLN. Blocks with adverse distribution of gravity data should be densified;
3. The present technique for areal prediction of gravity corrections could be generalized to work with approximating polynomials of a higher order. This would involve a major rewriting of the subroutine GCAFAZ of AREAGC program, in addition to other minor modifications in the software. Such a step would be essential for minimizing the model errors by optimizing the order of the approximating polynomial according to the character of the field in the approximation area. This would help in investigating the gravity corrections in the mountainous and gravitationally disturbed area;
4. Other systematic errors inherent in the precise spirit levelling, which are now known and modellable, should be modelled and accounted for in the CPLN. This includes in particular, the tidal effect; residual refraction; and vertical crustal movements. Also it is recommended that future investigations should involve the sea surface topography and the suspected errors in precise levelling that cause

the systematic discrepancies between oceanographic and N-S running geodetic levellings.

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APPENDICES

APPENDIX I

THE 1967 INTERNATIONAL AND THE USC&GS

NORMAL GRAVITY FORMULAE

The purpose of this Appendix is to discuss the differences between the normal gravity values (on the ellipsoid) obtained from the 1967 International formula and from the formula used in Canada and the United States (which is called here the USC&GS formula).

The 1967 International formula for normal gravity on the ellipsoid is given by [I.A.G., 1971; Vaníček, 1971; Levallois, 1972]:

$$\gamma_o = \gamma_{o,Eq} \, [1 + c_1 \sin^2 \phi - c_2 \sin^2 2\phi], \qquad \text{(I-1)}$$

where $\gamma_{o,Eq}$ is the normal gravity on the ellipsoid at the equator ($\gamma_{o,Eq}$ = 978031.8 mgal), ϕ is the latitude and c_1, c_2 are coefficients given by:

$$c_1 = 0.005\ 3024 \ \text{ unitless}, \qquad \text{(I-2a)}$$

$$c_2 = 0.000\ 0059 \ \text{ unitless}. \qquad \text{(I-2b)}$$

The USC&GS reads [Bowie and Avers, 1914; Cannon, 1929]:

$$\gamma_o^* = \gamma_{o,45°}^* \, [1 - \alpha \cos 2\phi + \beta \cos^2 2\phi], \qquad \text{[I-3)}$$

where $\gamma_{o,45°}$ is the normal gravity on the ellipsoid at latitude $\phi = 45°$ ($\gamma_{o,45°}^*$ = 980 624 mgal) and α, β are coefficients given by:

$$\alpha = 0.002\ 644 \ \text{unitless}, \qquad \text{(I-4a)}$$

$$\beta = 0.000\ 007 \ \text{unitless}. \qquad \text{(I-4b)}$$

To derive an expression for the difference between γ_o and γ_o^*, we first reformulate equation (I-3) to have the same form as equation (I-1).

We can write:

$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi = 1 - 2 \sin^2 \phi, \tag{I-5a}$$

and $$\cos^2 2\phi = 1 - \sin^2 2\phi. \tag{I-5b}$$

By, substituting (I-5a) and (I-5b) into (I-3), we get:

$$\gamma_o^* = \gamma_{o,45°}^* \, [(1 - \alpha + \beta) + 2\, \alpha \sin^2\phi - \beta \sin^2 2\phi] \, . \tag{I-6}$$

We then compute $\gamma_{o,Eq.}^*$ from equation (I-6), for $\phi = o°$, and get:

$$\gamma_{o,Eq.}^* = \gamma_{o,45°}^* \, (1 - \alpha + \beta), \tag{I-7}$$

from which

$$\gamma_{o,45°}^* = \gamma_{o,Eq.}^* \Big/ (1 - \alpha + \beta). \tag{I-8}$$

Substituting (I-8) into (I-6), we obtain:

$$\gamma_o^* = \gamma_{o,Eq}^* \, [1 + (\frac{2\alpha}{1-\alpha+\beta}) \sin^2 \phi - (\frac{\beta}{1-\alpha+\beta}) \sin^2 2\phi] \, . \tag{I-9}$$

Evaluating $\gamma_{o,Eq}^*$ from equations (I-7), (I-4a) and (I-4b), we get:

$$\gamma_{o,Eq}^* = 978\ 038.095 \text{ mgal}. \tag{I-10}$$

Substituting (I-10), (I-4a) and (I-4b) into (I-9), we find that:

$$\gamma_o^* = 978038.095 \, [1 + 0.005302 \sin^2\phi - 0.000007 \sin^2 2\phi] \text{ mgal}. \tag{I-11}$$

We can now also rewrite equation (I-1) as:

$$\gamma_o = 978031.8 \, [1 + 0.005\ 3024 \sin^2 \phi - 0.000\ 0059 \sin^2 2\phi] \text{ mgal}. \tag{I-12}$$

The difference between γ_o and γ_o^*, denoted by $\delta\gamma_o$, i.e.

$$\delta\gamma_o = \gamma_o - \gamma_o^* \, , \tag{I-13}$$

is then found to be

$$\delta\gamma_o = [-6.295 + 0.358 \sin^2 \phi + 1.076 \sin^2 2\, \phi] \text{ mgal}. \tag{I-14}$$

The remaining part of this Appendix contains Table I-1 (computer output) showing the values of γ_o, γ_o^* and $\delta\gamma_o$, as computed from equations (I-1), (I-3) and (I-14) respectively, for different latitudes. The results indicate that for Canadian latitudes 38° - 82° N the difference $\delta\gamma_o$ is always negative, and ranging between - 5.005 mgal (for $\phi \doteq 47°$N) and - 5.843 mgal (for $\phi = 82°$N).

TABLE I-1

DIFFERENCES OF NORMAL GRAVITY VALUES BETWEEN: THE 1967 AND THE USC&GS FORMULAS

LATITUDE DEG. (+VE N)	GAMA-1967 (MGALS)	GAMA-USC&GS (MGALS)	DELTA-GAMA (MGALS)
38.00	979992.032	979997.154	-5.122
38.25	980013.988	980019.104	-5.116
38.50	980035.991	980041.101	-5.110
38.75	980058.039	980063.143	-5.104
39.00	980080.131	980085.230	-5.099
39.25	980102.264	980107.358	-5.093
39.50	980124.438	980129.526	-5.088
39.75	980146.650	980151.733	-5.083
40.00	980168.899	980173.977	-5.078
40.25	980191.183	980196.257	-5.073
40.50	980213.501	980218.569	-5.069
40.75	980235.850	980240.914	-5.064
41.00	980258.229	980263.289	-5.060
41.25	980280.637	980285.693	-5.056
41.50	980303.071	980308.123	-5.052
41.75	980325.530	980330.578	-5.048
42.00	980348.013	980353.057	-5.044
42.25	980370.517	980375.557	-5.041
42.50	980393.040	980398.077	-5.037
42.75	980415.582	980420.616	-5.034
43.00	980438.140	980443.171	-5.031
43.25	980460.713	980465.741	-5.028
43.50	980483.298	980488.324	-5.025
43.75	980505.895	980510.918	-5.023
44.00	980528.502	980533.522	-5.020
44.25	980551.116	980556.134	-5.018
44.50	980573.736	980578.752	-5.016
44.75	980596.360	980601.375	-5.014
45.00	980618.988	980624.000	-5.012
45.25	980641.615	980646.626	-5.011
45.50	980664.243	980659.252	-5.010
45.75	980686.867	980691.876	-5.008
46.00	980709.487	980714.495	-5.007
46.25	980732.102	980737.108	-5.006
46.50	980754.708	980759.714	-5.006
46.75	980777.305	980782.310	-5.005
47.00	980799.891	980804.896	-5.005
47.25	980822.464	980827.469	-5.004
47.50	980845.023	980850.027	-5.004
47.75	980867.565	980872.569	-5.005
48.00	980890.088	980895.093	-5.005
48.25	980912.593	980917.598	-5.005
48.50	980935.075	980940.081	-5.006
48.75	980957.535	980962.541	-5.007

DIFFERENCES OF NORMAL GRAVITY VALUES BETWEEN: THE 1967 AND THE USC&GS FORMULAS

LATITUDE DEG. (+VE N)	GAMA-1967 (MGALS)	GAMA-USC&GS (MGALS)	DELTA-GAMA (MGALS)
49.00	980979.969	980984.977	-5.007
49.25	981002.377	981007.386	-5.009
49.50	981024.757	981029.767	-5.010
49.75	981047.106	981052.117	-5.011
50.00	981069.424	981074.437	-5.013
50.25	981091.708	981096.723	-5.015
50.50	981113.957	981118.974	-5.016
50.75	981136.170	981141.188	-5.018
51.00	981158.343	981163.364	-5.021
51.25	981180.477	981185.500	-5.023
51.50	981202.568	981207.594	-5.026
51.75	981224.616	981229.644	-5.028
52.00	981246.619	981251.650	-5.031
52.25	981268.574	981273.608	-5.034
52.50	981290.481	981295.518	-5.037
52.75	981312.337	981317.378	-5.040
53.00	981334.142	981339.186	-5.044
53.25	981355.893	981360.940	-5.047
53.50	981377.588	981382.639	-5.051
53.75	981399.227	981404.282	-5.055
54.00	981420.807	981425.865	-5.059
54.25	981442.326	981447.389	-5.063
54.50	981463.784	981468.851	-5.067
54.75	981485.178	981490.249	-5.072
55.00	981506.506	981511.582	-5.076
55.25	981527.768	981532.849	-5.081
55.50	981548.962	981554.047	-5.086
55.75	981570.085	981575.175	-5.091
56.00	981591.136	981596.232	-5.096
56.25	981612.115	981617.215	-5.101
56.50	981633.018	981638.124	-5.106
56.75	981653.845	981658.956	-5.111
57.00	981674.593	981679.710	-5.117
57.25	981695.262	981700.385	-5.123
57.50	981715.850	981720.978	-5.128
57.75	981736.354	981741.488	-5.134
58.00	981756.774	981761.915	-5.140
58.25	981777.109	981782.255	-5.146
58.50	981797.355	981802.508	-5.153
58.75	981817.513	981822.671	-5.159
59.00	981837.579	981842.745	-5.165
59.25	981857.554	981862.726	-5.172
59.50	981877.435	981882.613	-5.178
59.75	981897.220	981902.405	-5.185
60.00	981916.909	981922.101	-5.192

TABLE I-1 (cont'd)

DIFFERENCES OF NORMAL GRAVITY VALUES BETWEEN: THE 1967 AND THE USC&GS FORMULAS

LATITUDE DEG. (+VE N)	GAMA-1967 (MGALS)	GAMA-USC&GS (MGALS)	DELTA-GAMA (MGALS)
60.25	981936.500	981941.698	-5.199
60.50	981955.990	981961.196	-5.206
60.75	981975.380	981980.593	-5.213
61.00	981994.666	981999.886	-5.220
61.25	982013.849	982019.076	-5.227
61.50	982032.925	982038.160	-5.235
61.75	982051.895	982057.137	-5.242
62.00	982070.756	982076.005	-5.249
62.25	982089.506	982094.763	-5.257
62.50	982108.145	982113.410	-5.265
62.75	982126.672	982131.944	-5.272
63.00	982145.084	982150.363	-5.280
63.25	982163.380	982168.667	-5.288
63.50	982181.558	982186.854	-5.296
63.75	982199.619	982204.922	-5.303
64.00	982217.559	982222.870	-5.311
64.25	982235.378	982240.697	-5.319
64.50	982253.074	982258.402	-5.327
64.75	982270.646	982275.982	-5.335
65.00	982288.093	982293.437	-5.344
65.25	982305.413	982310.765	-5.352
65.50	982322.605	982327.965	-5.360
65.75	982339.667	982345.035	-5.368
66.00	982356.599	982361.975	-5.376
66.25	982373.398	982378.783	-5.385
66.50	982390.064	982395.458	-5.393
66.75	982406.596	982411.998	-5.401
67.00	982422.992	982428.402	-5.410
67.25	982439.251	982444.669	-5.418
67.50	982455.371	982460.797	-5.426
67.75	982471.351	982476.786	-5.435
68.00	982487.191	982492.635	-5.443
68.25	982502.889	982508.341	-5.452
68.50	982518.443	982523.903	-5.460
68.75	982533.853	982539.322	-5.468
69.00	982549.118	982554.594	-5.477
69.25	982564.235	982569.720	-5.485
69.50	982579.204	982584.698	-5.494
69.75	982594.025	982599.527	-5.502
70.00	982608.695	982614.205	-5.510
70.25	982623.213	982628.732	-5.519
70.50	982637.579	982643.106	-5.527
70.75	982651.792	982657.327	-5.535
71.00	982665.849	982671.393	-5.544
71.25	982679.751	982685.303	-5.552

TABLE I-1 (cont'd)

DIFFERENCES OF NORMAL GRAVITY VALUES BETWEEN: THE 1967 AND THE USC&GS FORMULAS

LATITUDE DEG. (+VE N)	GAMA-1967 (MGALS)	GAMA-USC&GS (MGALS)	DELTA-GAMA (MGALS)
71.50	982693.496	982699.056	-5.560
71.75	982707.083	982712.651	-5.568
72.00	982720.511	982726.088	-5.576
72.25	982733.779	982739.364	-5.584
72.50	982746.886	982752.479	-5.593
72.75	982759.831	982765.432	-5.601
73.00	982772.613	982778.222	-5.608
73.25	982785.231	982790.847	-5.616
73.50	982797.684	982803.308	-5.624
73.75	982809.971	982815.603	-5.632
74.00	982822.091	982827.730	-5.640
74.25	982834.043	982839.690	-5.647
74.50	982845.826	982851.481	-5.655
74.75	982857.440	982863.102	-5.663
75.00	982868.883	982874.553	-5.670
75.25	982880.154	982885.832	-5.678
75.50	982891.254	982896.939	-5.685
75.75	982902.180	982907.872	-5.692
76.00	982912.932	982918.631	-5.699
76.25	982923.509	982929.216	-5.706
76.50	982933.911	982939.624	-5.713
76.75	982944.136	982949.857	-5.720
77.00	982954.184	982959.911	-5.727
77.25	982964.054	982969.788	-5.734
77.50	982973.745	982979.486	-5.741
77.75	982983.257	982989.004	-5.747
78.00	982992.588	982998.342	-5.754
78.25	983001.739	983007.499	-5.760
78.50	983010.707	983016.474	-5.766
78.75	983019.494	983025.266	-5.773
79.00	983028.097	983033.875	-5.779
79.25	983036.516	983042.301	-5.785
79.50	983044.752	983050.542	-5.790
79.75	983052.802	983058.598	-5.796
80.00	983060.666	983066.468	-5.802
80.25	983068.345	983074.152	-5.807
80.50	983075.836	983081.649	-5.813
80.75	983083.140	983088.958	-5.818
81.00	983090.256	983096.080	-5.823
81.25	983097.184	983103.012	-5.828
81.50	983103.923	983109.756	-5.833
81.75	983110.472	983116.310	-5.838
82.00	983116.831	983122.673	-5.843

APPENDIX II

EPB POINT GRAVITY DATA FILE

II.1 Description of the File

In this section, the new EPB point gravity data file (see section 5.2.3) is described. For brevity, it is referred to here as the "EPB File".

The EPB file is of a sequential type and is residing on an unlabelled, 9 track, 1600 bsi (Bytes per square inch) tape. This tape is called "BUCKO2" and is stored in slot number 1304 at the UNB Computing Centre. The data set name is "GRAV", and its pertinent characteristics are given in the following data definition (DD) cards [IBM, 1970]:

```
// GO.FTxx FOO1 DD  DSN=GRAV, UNIT=TAPE,
// LABEL = (1, NL), VOL=SER=BUCKO2,
// DCB = (RECFM = FB, LRECL=80, BLKSIZE=3200, DEN=3),
// DISP = (OLD, KEEP)
```

where xx should be replaced by the desired unit number (data reference number), which can be any integer between 01 and 25 except 5, 6, and 7 at the U.N.B. installation system.

GRAV contains 272, 567 records, one per each gravity station including data from both land and sea. The records are 80 characters long, and the data is blocked as 40 records per block. Each record contains information about the station identification, its geographical position (latitude and longitude), its elevation above M.S.L., observed gravity and free-air gravity anomaly. Table II-1 is a detailed description of the contents of one such record (80 columns).

TABLE II-1

Description of 80 Column Records of EPB File

Columns	Description	Format
1	Record Identification code, as follows: + = standard record * = header record D = deleted record	A1
2-6	Station number	I5
7-8	Year of observation [19xx]	I2
9-11	Project number	I3
12	Assumed Blank	
13-20	Latitude in decimal degrees (+ve North)	F8.5
21	Assumed Blank	
22-30	Longitude in decimal degrees (+ve West)	F9.5
31	Coordinate Factor, which gives the scale of the map from which the station coordinates were scaled, as follows: For Pre - 1969 Data / For Post - 1968 Data 0 = unknown / 1 = < 1/25,000 1 = 1/25,000 / 2 = 1/25,000 2 = 1/50,000 / 3 = 1/50,000 3 = 1/125,000 / 4 = 1/125,000 4 = 1/250,000 / 5 = 1/250,000 5 = 1/500,000 / 6 = 1/500,000 6 = Decca Survey / 7 = 1/1,000,000 7 = Other sources / 8 = > 1/1,000,000 9 = unknown	I1
32-38	Elevation of station above MSL, in Feet	F7.1
39	Assumed blank	

. . . Cont'd

TABLE II-1, Cont'd

Columns	Description	Format
40	Elevation Accuracy factor, which defines the accuracy of the station elevation, relative to the elevation datum, as follows: For Pre - 1969 Data: 0 = unknown; 1 = ± 3 ft; 2 = ± 10 ft; 3 = ± 25 ft; 4 = ± 100 ft For Post - 1968 Data: 0 = unknown; 1 = ± 0.1 ft; 2 = ± 1 ft; 3 = ± 3 ft; 4 = ± 10 ft; 5 = ± 25 ft; 6 = ± 100 ft	A1
41	Elevation Datum factor, as follows: 1 = spirit level 2 = altimeter 3 = arbitrary	I1
42-48	Depth in feet (of water or ice at the station)	F7.1
49	Depth indicator, as follows: Blank = no water or ice present I = depth of water for surface measurement U = depth of water for underwater measurement X = depth of water or thickness of ice is unknown, i.e. Bouguer anomaly will be incorrect.	A1
50	Depth accuracy factor, same definition as the elevation accuracy factor in column 40.	A1
51-59	The observed gravity in gals	F9.5
60-66	Free-air anomaly in mgals	F7.2
67-73	Bouguer anomaly in mgals, including the terrain correction	F7.2
74-78	The terrain correction in mgals, which has been added to the simple Bouguer anomaly	F5.2
79	Terrain correction uncertainty, as follows: 1 = 10% 2 = 20% 3 = > 20% 5 = not coded	I1
80	A code indicating the release status of the station	A1

The land gravity data on the EPB file is already sorted out by latitude first and then by longitude. The sea gravity data is not sorted. The current version of this file has been also used by John [1976], and some of the errors and ambiguities in the originally received tape from the EPB has been already removed by him. The clean-up of the file was executed in summer 1975, while Mr. John was working at the EPB Gravity Division, in Ottawa.

The following suggestions, concerning the EPB file, were recommended by the Gravity Division [communications with Ron Buck, 1975]:

(i) Records with a record identification code equals "D" (Column 1, see Table II-1) should be deleted;

(ii) Records with free-air anomaly values of -0.0 mgal (columns 60-66, see Table II-1) should not be used;

(iii) Records with elevation accuracy factor (column 40, see Table II-1) given as a minus sign "-", should be changed so that this factor will read one ("1").

To solve the third problem above, each elevation accuracy factor value, say CH, should be read from the EPB file using the character format "A1" as indicated in Table II-1. This value is then converted to a numerical value, say N, by using the following "magic" formula (as called by Buck):

$$N = (CH - IZ) / 16\ 777\ 216 \qquad \text{(II-1)}$$

where IZ is a character variable that has to be defined, using a DATA statement, as:

```
DATA IZ / 1HO / ,
```

which must be executed before the conversion takes place. Now, if CH takes the value "-", the formula (II-1) cannot be applied. In this case, we have to use the following two statements:

```
DATA  MINUS / '-' /
IF (CH. EQ. MINUS)  N=1
```

before applying the conversion formula (II-1).

The formula used by the Gravity Division for Computing the free-air gravity anomaly, Δg^F is:

$$\Delta g^F = g - (\gamma_o - \frac{dg}{dz} h) - (\frac{dg}{dz} - 4\pi \kappa \rho_w) d_w \quad , \qquad \text{(II-2)}$$

where: g is the observed value of gravity at the station, based on the 1974 adjustment of the National Gravity Net and the recently adopted IGSN 71 (International Gravity Standardization Net 1971) [Morelli et al., 1974]; γ_o is the normal gravity value on the ellipsoid, based on the 1967 International formula [IAG, 1971] which reads:

$$\gamma_o = 978.031\ 85\ [1 + 0.005\ 278\ 895 \sin^2 \phi + + 0.000\ 023\ 462 \sin^4 \phi] \text{ gals}; \qquad \text{(II-3)}$$

$\frac{dg}{dz}$ is the vertical gradient of normal gravity dg/dz = + 0.09406 mgal/ft = 0.3086 mgal/m of the station elevation above sea level), and h is the elevation of the gravity station above sea level. The second term in equation (II-2) applies only when g has been observed on the ocean or lake floor with an underwater gravimeter. There, κ is the universal constant of gravitation; ρ_w is the density of the water (1.03 gm/ccm); and d_w is the depth of water in feet.

It should be noted that the EPB File format is fixed-blocked, which implies that this tape has to be read with the appropriate format-control statement as indicated in Table II-1 (last column).

It is worth mentioning here that the version of the 1967 International formula for normal gravity used by the EPB (equation II-3) is the Chebychev's approximation to the closed form of normal gravity [IAG, 1971]. Equation (II-3) is claimed to be accurate to 0.004 mgal. It differs, however, from the conventional version used for the developments in Appendix I (Equation I-1) and throughout the present study. The accuracy of Equation (I-1) is limited to 0.1 mgal [IAG, 1971]. However, the differences of normal gravity values given by Equations (II-3) and (I-1), as computed by the author, were found to range between 0.01 mgal and 0.06 mgal over the Canadian latitudes.

Based on the discussion given in section 4.3, it is obvious that the aforementioned discrepancy between the two versions of the 1967 formula (≃ 0.03 mgal on the average) has negligible effect on the computation of Helmert and Vignal gravity corrections, GC^H and GC^V, respectively. In the case of Dynamic gravity correction, GC^D, the corresponding effect on the computed value of GC^D will be, in the extreme cases, in the order of 0.006 mm for a 1 km levelling section. This is again negligible, in accordance with section 4.3.

Two main reasons, however, were behind the use of the conventional version in this investigation. First, it is in the same form as the familiar 1930 International formula except with different values for its coefficients. Secondly, it is formable in the same form as the USC&GS formula for normal gravity (Equation I-3) under investigation. The latter reason helps providing a simple model for the difference between the 1967 International and the USC&GS formulae for normal gravity (Equation I-14). Such a simple model was found convenient in the development of the technique for aerial prediction of gravity corrections (Chapter 7).

II-2 Use of the File

As can be seen from Table II-1, there are lots of information on the EPB file for each gravity station. Only part of this information is relevant to the current study. In addition, the point gravity data on the file are collected for both land and sea. Therefore, this section discusses the way of using the EPB file to extract and file only the necessary data at each gravity station on land (as stated in Chapter 7) pertinent to the study of actual gravity influence on levelled heights.

For our investigation herein, we deal with data on land only, and we need the following information at each gravity station: geographical position (latitude and longitude); elevation above sea level; feee-air anomaly; accuracy of elevation and accuracy of free-air anomaly. It was decided to extract also the observed gravity for possible future needs.

To get the elevation accuracy δh, from the file, we have to compute it from the elevation accuracy factor N (explained in section II-1) along with the year of observation, according to the categories listed in Table II-1 (pertaining to column 40).

It is clear from the description of the EPB file (Table II-1) that no individual estimates for the accuracy of the gravity anomalies are readily available. Therefore, it was decided to use the formula devised by Vaníček et. al. [1972] for the variance $\sigma^2_{\Delta g^F}$ of the free-air anomaly Δg^F, which reads:

$$\sigma^2_{\Delta g^F} = [(0.05)^2 + (0.3086\ \delta h)^2 \text{ mgal}^2, \qquad \text{(II-4)}$$

for δh in metres. The first term accounts for the measurement error in observed gravity g [Hamilton and Buchan, 1965; Tanner and Gibb, 1971].

The second term is a function of the error δh in the elevation h, which depends on the way the elevation was acquired (spirit levelling, altimetry or other).

A program called "EXTRAC" was written to read the EPB tape and extract the above discussed information, considering the data on land only. Also EXTRAC deletes the suspected records, following the suggestions of the Gravity Division as explained in the previous section. EXTRAC thus creates a new relevant data file, referred to here as the "master file", by filing, on tape, one record per each gravity station containing the following information: latitude ϕ and longitude λ in degrees and decimal fractions of degrees, positive north and west respectively; observed gravity g in gal; free-air anomaly Δg^F in mgal; elevation h above MSL and elevation accuracy δh in feet (same units as the EPB File). Then the first 4000 records, as well as the total number of records stored on the created master file are printed-out. Listings and sample output of EXTRAC are shown at the end of this Appendix, which indicate that a total of 110,171 point gravity values on land are available on the EPB file.

The created master file is of a sequential type, residing on a standard-labelled, 9 track, 1600 bpi tape, called "SL1285", and stored in slot number 1285 at the U.N.B. Computing Centre. Each record is 48 characters long, and the data is blocked into 150 records per block. Table II-2 describes the contents and format of each record. The complete pertinent characteristics of the master file are given in the following data definition (DD) cards [IBM, 1970]:

```
// GO.FTxxF001 DD DSN = NEW.PTFA.CANADA.LAND.TEMPOR,
//    UNIT=TAPE,LABEL=(9, SL), VOL=SER=SL1285,
```

TABLE II-2

Description of 48 Columns Records

of the Master File

Columns	Description	Format
1-9	Station latitude ϕ, in degrees and decimal fractions of degrees $-^{+}$ ve North	F9.5
10-19	Station Longitude λ, in degrees and decimal fractions of degrees$-+$ ve west.	F10.5
20-28	Observed gravity g, in gals	F9.5
29-35	Free-air gravity anomaly Δg^{F}, in mgals	F7.2
36-42	Elevation h of station above sea level, in feet	F7.1
43-48	Elevation assigned - accuracy δh, in feet	F6.1

```
//    DCB = (RECFM=FB, LRECL=48, BLKSIZE=7200, DEN=3),
//    DISP=(OLD, KEEP)
```

where xx is explained in the previous section. Note here, also, that this master file has to be read using the appropriate format - controlled statement as explained in Table II-2 (first and last columns).

For further use, the master file was partitioned to seven, geographically overlapping, subsets of data. A program called "SUBSET" was written for this purpose, whose listings and sample output are not given in this Appendix. The geographical extent of each created subset is shown in Table II-3, as well as its name and FORTRAN reference number.

All the new seven data sets are of sequential type, and they have to be read <u>without</u> format - controlled statements. They are residing on tape "SE 1309" in slot number 1309 at the UNB Computing Centre. The required DD cards for these sets are listed as follows [IBM, 1970]:

```
//GO.FT21F001   DD   DSN=NASSAR.NEW.PTFA.LAND.EAST,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(2,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
//GO.FT22F001   DD   DSN=NASSAR.NEW.PTFA.LAND.CENTRE,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(3,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
//GO.FT23F001   DD   DSN=NASSAR.NEW.PTFA.LAND.PRAIRI,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(4,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
//GO.FT24F001   DD   DSN=NASSAR.NEW.PTFA.LAND.WESTCO,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(5,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
//GO.FT25F001   DD   DSN=NASSAR.NEW.PTFA.LAND.NWT,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(6,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
//GO.FT16F001   DD   DSN=NASSAR.NEW.PTFA.LAND.FRANK,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(7,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
//GO.FT17F001   DD   DSN=NASSAR.NEW.PTFA.LAND.GREEN,
//        UNIT=TAPE,VOL=SER=SE1309,LABEL=(8,SL),
//        DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600,DEN=3),
//        DISP=(OLD,KEEP)
```

TABLE II-3

Geographical Extent of Data Subsets

DATA SUBSET		EXTENT	
Ref. No.	NAME	ϕ (+ve N)	λ (+ve W)
21	NEW.PTFA.LAND.EAST	40°-65°	51°-75°
22	NEW.PTFA.LAND.CENTRE	42°-65°	72°-93°
23	NEW.PTFA.LAND.PRAIRI	47°-65°	90°-120°
24	NEW.PTFA.LAND.WESTCO	47°-65°	117°-142°
25	NEW.PTFA.LAND.NWT	62°-72°	60°-144°
16	NEW.PTFA.LAND.FRANK	69°-82°	65°-144°
17	NEW.PTFA.LAND.GREEN	79°-84°	50°-80°

For convenience, the seven data sets, explained above, are also duplicated on the disc "SEGEOP", at UNB Computing Centre, according to the following listed DD cards [IBM, 1970]:

```
//GO.FT21F001   DD   DSN=NASSAR.NEW.PTFA.LAND.EAST,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
//GO.FT22F001   DD   DSN=NASSAR.NEW.PTFA.LAND.CENTRE,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
//GO.FT23F001   DD   DSN=NASSAR.NEW.PTFA.LAND.PRAIRI,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
//GO.FT24F001   DD   DSN=NASSAR.NEW.PTFA.LAND.WESTCO,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
//GO.FT25F001   DD   DSN=NASSAR.NEW.PTFA.LAND.NWT,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
//GO.FT16F001   DD   DSN=NASSAR.NEW.PTFA.LAND.FRANK,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
//GO.FT17F001   DD   DSN=NASSAR.NEW.PTFA.LAND.GREEN,
//       UNIT=M2314,VOL=(PRIVATE,SER=SEGEOP),
//       DISP=(OLD,KEEP),
//       DCB=(RECFM=VBS,LRECL=48,BLKSIZE=3600)
```

```
C*********************************************************************
C*                                                                   *
C* PROGRAM     'EXTRAC' :                                            *
C* -------------------                                               *
C*                     THE PURPOSE OF THIS PROGRAM IS :-             *
C* 1-TO READ THE NEW EPB POINT GRAVITY TAPE (WHICH WAS SUPPLIED EARLY *
C*   IN 1975), AND EXTRACT THE RELEVANT INFORMATION FOR INVESTIGATING *
C*   THE EFFECT OF THE EARTH'S GRAVITY FIELD ON THE LEVELLED HEIGHTS, *
C*   NAMELY :-                                                       *
C*          1-YEAR OF OBSERVATION,                                   *
C*          2-LATITUDE IN DECIMAL DEGREES,                           *
C*          3-LONGITUDE IN DECIMAL DEGREES,                          *
C*          4-ELEVATION ABOVE M.S.L.  IN FEET,                       *
C*          5-HEIGHT ACCURACY FACTOR IN CHARACTER FORM,              *
C*          6-OBSERVED GRAVITY IN GALS,                              *
C*          7-FREE-AIR ANOMALY IN MGALS.                             *
C*                                                                   *
C* 2-TO ASSIGN THE APPROPRIATE ACCURACY (IN FEET) OF THE ELEVATION,  *
C*   ACCORDING TO THE YEAR OF OBSERVATION AND THE HEIGHT ACCURACY FACTOR *
C*   (AFTER TRANSFORMING IT FROM THE CHARACTER TO THE NUMERICAL CODE). *
C*                                                                   *
C* 3-TO PRINT OUT FOR SOME POINTS: THE LATITUDE, LONGITUDE, OBSERVED *
C*   GRAVITY, FREE-AIR ANOMALY,HEIGHT ABOVE MSL AND HEIGHT ACCURACY. *
C*                                                                   *
C* 4-TO WRITE ALL THE INFORMATION MENTIONED IN (3) ABOVE ONTO A DATA *
C*   FILE WHOSE APPROPRIATE NUMBER IS INPUTTED, FOR FUTURE USAGE.    *
C*   -CONSIDERING ONLY THE POINT GRAVITY DATA ON LAND, AND EXCLUDING *
C*   THE DATA ON SEA- SINCE IT IS IRRELEVANT TO OUR INVESTIGATION.   *
C*                                                                   *
C*********************************************************************
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SAVE/ P,AL,GRAV,DG,AHT,HTACC
      DIMENSION RWVEC(6)
      EQUIVALENCE (P,RWVEC(1))
      DATA ICHAR/'-'/
      DATA INUM/1H0/
      DATA IDELET/'D'/
C
C READ-IN THE DATA REFERENCE NUMBER (UNIT NUMBER) ASSIGNED TO THE EPB NEW
C POINT GRAVITY FILE, FROM WHICH WE WILL EXTRACT THE NEEDED INFORMATION.
C THIS CAN BE ANY INTEGER FROM 1 TO 25 EXCEPT 5,6 & 7, AT U.N.B. SYSTEM.
C
      READ(5,30) IEPB
   30 FORMAT(I2)
C
C READ-IN THE DATA REFERENCE NUMBER (UNIT NUMBER) DESIRED TO BE ASSIGNED
C TO THE DATA SET TO BE CREATED ON TAPE (OR DISC) FROM IEPB AS EXPLAINED
C ABOVE, AND IS GOING TO BE THE MASTER FILE OF INFORMATION FOR THE CURRENT
C INVESTIGATION.
C
      READ(5,30) MASTER
C
C PRINT-OUT A TITLE AND SOME HEADINGS FOR THE OUTPUT OF THE JOB.
C
      WRITE(6,15)
   15 FORMAT('1'//10X,'POINT GRAVITY AND POINT FREE-AIR ANOMALY VALUES
     1IN CANADA (FROM THE EPB NEW FILE)'/)
```

```
0013            WRITE(6,25)
0014         25 FORMAT(/,2X,'LAT.(DEG.)+VE.N',2X,'LONG.(DEG.)+VE.W',2X,'GRAV.(GALS
               1)',2X,'FREE-AIR ANOMALY (MGALS)',2X,'HEIGHT (FEET)',2X,'HEIGHT ACC
               2URACY (FEET)'/)
0015            J=0
0016            REWIND IEPB
0017          1 READ(IEPE,4,END=2)ID,NUM,IY,IP,P,AL,IC,AHT,IT,IHT,IDAT,DEP,IDEP,IDF
               *F,GRAV,DG,BA,TC,TCU,IRS
0018          4 FORMAT(A1,I5,I2,I3,F9.5,F10.5,I1,F7.1,2A1,I1,F7.1,A1,A1,F9.5,F7.2,
               *F7.2,F5.2,2A1)
          C
          C DELETING THE SEA DATA, I.E. FOR WHICH THE HEIGHT OF THE GRAVITY STATION
          C IS ASSIGNED A VALUE OF 0 FEET.
          C
0019            IF(AHT.EQ.0.0)  GO TO 1
          C
          C DELETING THE RECORDS FOR WHICH THE RECORD IDENTIFICATION CODE EQUALS 'D'
          C AND/OR THE FREE-AIR ANOMALY EQUALS '-0.0' MGALS.
          C AS RECOMMENDED BY THE GRAVITY DIVISION.
          C
0020            IF(ID.EQ.IDELET.OR.DG.EQ.-0.0)  GO TO 1
          C
          C CONVERSION OF CHARACTER CODE FOR HEIGHT ACCURACY FACTOR TO NUMERICAL
          C CODE FACTOR VIA THE SUPPLIED FORMULA BY THE EPB
          C
0021            NCODE=(IHT-INUM)/16777216
0022            IF(IHT.EQ.ICHAR)  NCODE=1
          C
          C ASSIGNING VALUES FOR THE HEIGHT ACCURACY (CONSIDERED AS STANDARD
          C DEVIATION) IN FEET ,ACCORDING TO THE YEAR OF OBSERVATION AND THE
          C SPECIFIED NUMERICAL CODE ON THE EMP GRAVITY FILE FOR THE HEIGHT
          C ACCURACY FACTOR
          C
0023            IF(IY.GE.69)  GO TO 10
          C
          C VALUES FOR THE DATA PRIOR TO '1969'
          C
0024            IF(NCODE.EQ.1)  HTACC=3.0
0025            IF(NCODE.EQ.2)  HTACC=10.0
0026            IF(NCODE.EQ.3)  HTACC=25.0
0027            IF(NCODE.EQ.4)  HTACC=100.0
0028            IF(NCCDE.EQ.0)  HTACC=125.0
0029            GO TO 20
0030         10 CONTINUE
          C
          C VALUES FOR THE DATA AFTER '1968'
          C
0031            IF(NCODE.EQ.1)  HTACC=0.1
0032            IF(NCODE.EQ.2)  HTACC=1.0
0033            IF(NCODE.EQ.3)  HTACC=3.0
0034            IF(NCODE.EQ.4)  HTACC=10.0
0035            IF(NCODE.EQ.5)  HTACC=25.0
0036            IF(NCODE.EQ.6)  HTACC=100.0
0037            IF(NCODE.EQ.0)  HTACC=125.0
0038         20 CONTINUE
0039            J=J+1
          C
          C WRITE THE RELEVANT INFORMATION ON AN INTERNAL MEDIA (TAPE OR DISC).
```

```
                C WHOSE REFERENCE NUMBER IS 'MASTER'.
                C
 0040                 WRITE(MASTER,3) (RWVEC(I),I=1,6)
 0041               3 FORMAT(F9.5,F10.5,F9.5,F7.2,F7.1,F6.1)
                C
                C PRINT-OUT THE FIRST 4000 RECORDS (POINTS) ON THE CREATED MASTER FILE.
                C
 0042                 IF(J.LE.4000) WRITE(6,35) (RWVEC(I),I=1,6)
 0043              35 FORMAT(5X,F9.5,8X,F10.5,6X,F9.5,12X,F7.2,13X,F7.1,13X,F6.1)
 0044                 GO TO 1
 0045               2 CONTINUE
                C
                C PRINT-OUT THE TOTAL NUMBER OF POINTS (RECORDS) ON THE MASTER FILE
                C WHICH IS THE TOTAL NUMBER OF AVAILABLE DATA POINTS ON CANADA LANDS ONLY.
                C
 0046                 WRITE(6,5) J
 0047               5 FORMAT(//,5X,'TOTAL NO. OF POINTS ON THIS FILE =',I8)
 0048                 STOP
 0049                 END
```

POINT GRAVITY AND POINT FREE-AIR ANOMALY VALUES IN CANADA (FROM THE EPB NEW FILE)

LAT.(DEG.)+VE.N	LONG.(DEG.)+VE.W	GRAV.(GALS)	FREE-AIR ANOMALY (MGALS)	HEIGHT (FEET)	HEIGHT ACCURACY (FEET)
40.64333	73.78333	980.20900	-15.95	15.0	125.0
40.34667	74.65500	980.16262	-24.26	138.0	125.0
40.44333	79.94500	980.09798	-2.52	1148.0	125.0
40.00000	82.86667	980.07900	-59.85	320.0	125.0
40.00333	105.26167	979.58857	-72.11	5406.8	125.0
40.78333	111.96667	979.79928	-42.20	4225.0	125.0
41.91833	80.89000	980.27200	-23.32	517.0	3.0
41.99667	80.64167	980.28040	-17.54	571.0	3.0
41.55282	81.82332	980.24468	-13.26	571.0	1.0
41.55282	81.94166	980.24338	-14.76	571.0	1.0
41.55332	81.70666	980.24378	-13.16	571.0	1.0
41.64083	81.94166	980.25098	-15.47	571.0	1.0
41.64116	81.82332	980.25258	-13.67	571.0	1.0
41.64166	81.58883	980.24178	-23.57	571.0	1.0
41.64166	81.70666	980.25488	-11.37	571.0	1.0
41.72499	81.82332	980.26189	-12.29	571.0	1.0
41.72916	81.94166	980.26069	-14.29	571.0	1.0
41.73016	81.58949	980.25539	-19.19	571.0	1.0
41.73449	81.46999	980.25119	-22.89	571.0	1.0
41.74732	81.72916	980.26779	-8.29	571.0	1.0
41.82332	81.23416	980.25989	-22.41	571.0	1.0
41.82332	81.35282	980.25859	-23.91	571.0	1.0
41.82332	81.46999	980.25939	-23.81	571.0	1.0
41.83066	81.97782	980.26929	-14.71	571.0	1.0
41.83332	81.58999	980.26229	-22.11	571.0	1.0
41.83332	81.73066	980.27509	-9.51	571.0	1.0
41.83332	81.84999	980.27259	-11.71	571.0	1.0
41.91083	81.23499	980.26630	-24.82	571.0	1.0
41.91116	81.11616	980.26500	-25.82	571.0	1.0
41.91166	81.47166	980.26650	-25.12	571.0	1.0
41.91249	81.35332	980.26600	-25.52	571.0	1.0
41.91667	81.13667	980.27830	-18.42	517.0	3.0
41.92000	81.01000	980.27500	-21.82	517.0	3.0
41.92449	81.73066	980.28150	-11.43	571.0	1.0
41.92499	81.84999	980.28020	-12.73	571.0	1.0
41.92633	81.60832	980.27810	-15.03	571.0	1.0
41.45416	82.43599	980.22417	-24.04	571.0	1.0
41.45416	82.55550	980.22007	-27.94	571.0	1.0
41.46249	82.29583	980.22877	-20.24	571.0	1.0
41.49663	82.23164	980.23197	-20.25	569.0	10.0
41.52001	82.40838	980.22998	-24.55	569.0	10.0
41.53663	82.58495	980.22288	-32.65	569.0	10.0
41.53999	82.67499	980.21968	-35.95	571.0	1.0
41.54166	82.43999	980.21178	-44.55	571.0	1.0
41.54166	82.55832	980.22658	-29.65	571.0	1.0
41.55083	82.29583	980.23548	-21.56	571.0	1.0
41.55166	82.17782	980.23848	-19.16	571.0	1.0
41.55216	82.05999	980.24298	-14.86	571.0	1.0
41.58829	82.29004	980.23808	-22.16	569.0	10.0
41.60832	82.75499	980.22288	-38.27	571.0	1.0
41.62500	82.91399	980.22298	-39.57	571.0	1.0
41.62916	82.67782	980.22838	-36.17	571.0	1.0
41.62999	82.44100	980.23898	-25.27	571.0	1.0
41.62999	82.55999	980.23888	-25.17	571.0	1.0

45.95499	72.39833	980.64310	-34.58	296.0	25.0
45.95499	72.85833	980.65760	-42.98	95.0	25.0
45.95833	72.47333	980.64260	-40.38	242.0	25.0
45.95999	72.38999	980.64260	-34.58	306.0	125.0
45.96999	72.02833	980.67830	16.02	473.0	25.0
45.97166	72.99666	980.65660	-43.68	71.0	25.0
45.98999	72.23166	980.65750	-23.49	294.0	25.0
45.98999	72.23666	980.65650	-23.49	305.0	125.0
45.99166	72.53333	980.64140	-46.19	225.0	25.0
45.99499	72.64333	980.64420	-48.69	172.0	25.0
45.01033	73.37200	980.59992	-9.33	114.0	3.0
45.01040	73.37697	980.59965	-8.30	124.0	25.0
45.01603	73.05763	980.57115	-38.70	110.0	25.0
45.01999	73.43999	980.59685	-6.90	182.0	25.0
45.02311	73.97659	980.60315	28.60	492.0	25.0
45.02473	73.39713	980.59815	-6.40	174.0	25.0
45.02666	73.94666	980.60005	26.30	507.0	25.0
45.02974	73.92468	980.59815	23.99	504.0	25.0
45.03167	73.04517	980.56987	-40.37	124.0	3.0
45.03999	73.07833	980.57385	-39.41	100.0	25.0
45.03999	73.63833	980.59565	18.49	484.0	125.0
45.04033	73.86400	980.59651	19.44	485.0	3.0
45.04297	73.03755	980.57065	-39.91	127.0	25.0
45.04333	73.82666	980.60115	13.89	379.0	25.0
45.04399	73.82677	980.60015	15.19	403.0	25.0
45.04417	73.75896	980.60165	6.49	297.0	25.0
45.04499	73.51666	980.60215	-2.61	195.0	125.0
45.04499	73.51833	980.60175	-3.21	193.0	25.0
45.04499	73.75999	980.60265	7.09	293.0	125.0
45.04550	73.58367	980.59814	-0.37	262.0	3.0
45.04553	73.52492	980.60115	-3.41	193.0	25.0
45.04594	73.78038	980.60565	9.39	283.0	25.0
45.04632	73.63943	980.59765	-1.91	250.0	25.0
45.04637	73.72146	980.60015	1.49	260.0	25.0

TOTAL NO. OF POINTS ON THIS FILE = 110171

APPENDIX III

GRAVITY CORRECTION TABLES

The aim of this Appendix is to provide tables to facilitate the application of the gravity corrections for any of the three height systems considered in this investigation - as applied to the height differences currently used in Canada. These tables are based on the approximate formulae for the gravity corrections developed in section 4.3 of this thesis. Two principal quantities are needed to get a gravity correction from these tables. The first is the levelled height difference (and/or the corresponding heights), which can be easily picked up from the precise levelling field book. The second is the free-air gravity anomaly, which can be obtained from any of the sources discussed in section 6.1.1; e.g. from a free-air anomaly contour map.

According to the characteristics of the expressions for the gravity corrections, it was found convenient to present the tables in the sequence: Vignal, dynamic and then Helmert. A separate section of this Appendix is devoted to each respective system.

III-1 Vignal Gravity Correction

The Vignal Gravity Correction GC_{ij}^{V} to the height difference $\Delta\tilde{h}_{ij}$ of a levelling section between the two bench marks i and j - corrected for the effect of normal gravity only - is computed from the expression:

$$GC_{ij}^{V} = \Delta h_{ij} \cdot \overline{\Delta g}_{ij}^{F}/G \ , \qquad \text{(III-1)}$$

where Δh_{ij} is the levelled height difference and $\overline{\Delta g}_{ij}^{F}$ is the average free-air gravity anomaly between bench marks i and j. G is the reference gravity which is taken here to be the normal gravity value on the ellipsoid, computed from the USC&GS formula, at latitude $\phi = 45^{\circ}$. This value is the presently used reference gravity in Canada, which is given as:

$$G = \gamma^{*}_{o,45^{o}} = 980624.0 \text{ mgal.} \qquad \text{(III-2)}$$

The tables for Vignal gravity correction (Table III-1) are computed for different values of Δh and $\overline{\Delta g}^{F}$ as follows: Δh from 1 m to 10 m with 1 m interval, from 10 m to 100 m with 10 m interval and finally from 100 m to 1000 m with 100 m interval; $\overline{\Delta g}^{F}$ from 1 mgal to 200 mgals with 1 mgal interval. Such arrangement, allows one to interpolate in the tables and get GC^{V}, in millimetres, for any value of Δh, in metres and $\overline{\Delta g}^{F}$, in milligals.

It should be noted here, that the tabulated values for GC^{V} are all positive, i.e. corresponding to positive Δh and positive $\overline{\Delta g}^{F}$. Therefore, recalling that:

$$\Delta h_{ij} = h_{j} - h_{i} \ , \qquad \text{(III-3)}$$

$$\overline{\Delta g}_{ij}^{F} = \frac{1}{2}\,(\Delta g_{i}^{F} + \Delta g_{j}^{F}) \ , \qquad \text{(III-4)}$$

one can determine the appropriate sign of GC_{ij}^{V}. The correction has to be added algebraically to the height difference $\Delta\tilde{h}_{ij}^{O}$ to obtain the appropriate Vignal height difference Δh_{ij}^{V} based on actual gravity.

TABLE III-1

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	1	2	3	4	5	6	7	8	9	10
1	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
2	0.002	0.004	0.006	0.008	0.010	0.012	0.014	0.016	0.018	0.020
3	0.003	0.006	0.009	0.012	0.015	0.018	0.021	0.024	0.028	0.031
4	0.004	0.008	0.012	0.016	0.020	0.024	0.029	0.033	0.037	0.041
5	0.005	0.010	0.015	0.020	0.025	0.031	0.036	0.041	0.046	0.051
6	0.006	0.012	0.018	0.024	0.031	0.037	0.043	0.049	0.055	0.061
7	0.007	0.014	0.021	0.029	0.036	0.043	0.050	0.057	0.054	0.071
8	0.008	0.016	0.024	0.033	0.041	0.049	0.057	0.065	0.073	0.082
9	0.009	0.018	0.028	0.037	0.046	0.055	0.064	0.073	0.083	0.092
10	0.010	0.020	0.031	0.041	0.051	0.061	0.071	0.082	0.092	0.102
11	0.011	0.022	0.034	0.045	0.056	0.067	0.079	0.090	0.101	0.112
12	0.012	0.024	0.037	0.049	0.061	0.073	0.086	0.098	0.110	0.122
13	0.013	0.027	0.040	0.053	0.066	0.080	0.093	0.106	0.119	0.133
14	0.014	0.029	0.043	0.057	0.071	0.086	0.100	0.114	0.128	0.143
15	0.015	0.031	0.046	0.061	0.076	0.092	0.107	0.122	0.138	0.153
16	0.016	0.033	0.049	0.065	0.082	0.098	0.114	0.131	0.147	0.163
17	0.017	0.035	0.052	0.069	0.087	0.104	0.121	0.139	0.156	0.173
18	0.018	0.037	0.055	0.073	0.092	0.110	0.128	0.147	0.165	0.184
19	0.019	0.039	0.058	0.078	0.097	0.116	0.136	0.155	0.174	0.194
20	0.020	0.041	0.061	0.082	0.102	0.122	0.143	0.163	0.184	0.204
21	0.021	0.043	0.064	0.086	0.107	0.128	0.150	0.171	0.193	0.214
22	0.022	0.045	0.067	0.090	0.112	0.135	0.157	0.179	0.202	0.224
23	0.023	0.047	0.070	0.094	0.117	0.141	0.164	0.188	0.211	0.235
24	0.024	0.049	0.073	0.098	0.122	0.147	0.171	0.196	0.220	0.245
25	0.025	0.051	0.076	0.102	0.127	0.153	0.178	0.204	0.229	0.255
26	0.027	0.053	0.080	0.106	0.133	0.159	0.186	0.212	0.239	0.265
27	0.028	0.055	0.083	0.110	0.138	0.165	0.193	0.220	0.248	0.275
28	0.029	0.057	0.086	0.114	0.143	0.171	0.200	0.228	0.257	0.286
29	0.030	0.059	0.089	0.118	0.148	0.177	0.207	0.237	0.266	0.296
30	0.031	0.061	0.092	0.122	0.153	0.184	0.214	0.245	0.275	0.306
31	0.032	0.063	0.095	0.126	0.158	0.190	0.221	0.253	0.285	0.316
32	0.033	0.065	0.098	0.131	0.163	0.196	0.228	0.261	0.294	0.326
33	0.034	0.067	0.101	0.135	0.168	0.202	0.236	0.269	0.303	0.337
34	0.035	0.069	0.104	0.139	0.173	0.208	0.243	0.277	0.312	0.347
35	0.036	0.071	0.107	0.143	0.178	0.214	0.250	0.286	0.321	0.357
36	0.037	0.073	0.110	0.147	0.184	0.220	0.257	0.294	0.330	0.367
37	0.038	0.075	0.113	0.151	0.189	0.226	0.264	0.302	0.340	0.377
38	0.039	0.078	0.116	0.155	0.194	0.233	0.271	0.310	0.349	0.388
39	0.040	0.080	0.119	0.159	0.199	0.239	0.278	0.318	0.358	0.398
40	0.041	0.082	0.122	0.163	0.204	0.245	0.286	0.326	0.357	0.408

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	1	2	3	4	5	6	7	8	9	10
41	0.042	0.084	0.125	0.167	0.209	0.251	0.293	0.334	0.376	0.418
42	0.043	0.086	0.128	0.171	0.214	0.257	0.300	0.343	0.385	0.428
43	0.044	0.088	0.132	0.175	0.219	0.263	0.307	0.351	0.395	0.438
44	0.045	0.090	0.135	0.179	0.224	0.269	0.314	0.359	0.404	0.449
45	0.046	0.092	0.138	0.184	0.229	0.275	0.321	0.367	0.413	0.459
46	0.047	0.094	0.141	0.188	0.235	0.281	0.328	0.375	0.422	0.469
47	0.048	0.096	0.144	0.192	0.240	0.288	0.336	0.383	0.431	0.479
48	0.049	0.098	0.147	0.196	0.245	0.294	0.343	0.392	0.441	0.489
49	0.050	0.100	0.150	0.200	0.250	0.300	0.350	0.400	0.450	0.500
50	0.051	0.102	0.153	0.204	0.255	0.306	0.357	0.408	0.459	0.510
51	0.052	0.104	0.156	0.208	0.260	0.312	0.364	0.416	0.468	0.520
52	0.053	0.106	0.159	0.212	0.265	0.318	0.371	0.424	0.477	0.530
53	0.054	0.108	0.162	0.216	0.270	0.324	0.378	0.432	0.486	0.540
54	0.055	0.110	0.165	0.220	0.275	0.330	0.385	0.441	0.496	0.551
55	0.056	0.112	0.168	0.224	0.280	0.337	0.393	0.449	0.505	0.561
56	0.057	0.114	0.171	0.228	0.286	0.343	0.400	0.457	0.514	0.571
57	0.058	0.116	0.174	0.233	0.291	0.349	0.407	0.465	0.523	0.581
58	0.059	0.118	0.177	0.237	0.296	0.355	0.414	0.473	0.532	0.591
59	0.060	0.120	0.180	0.241	0.301	0.361	0.421	0.481	0.541	0.602
60	0.061	0.122	0.184	0.245	0.306	0.367	0.428	0.489	0.551	0.612
61	0.062	0.124	0.187	0.249	0.311	0.373	0.435	0.498	0.560	0.622
62	0.063	0.126	0.190	0.253	0.316	0.379	0.443	0.506	0.569	0.632
63	0.064	0.128	0.193	0.257	0.321	0.385	0.450	0.514	0.578	0.642
64	0.065	0.131	0.196	0.261	0.326	0.392	0.457	0.522	0.587	0.653
65	0.066	0.133	0.199	0.265	0.331	0.398	0.464	0.530	0.597	0.663
66	0.067	0.135	0.202	0.269	0.337	0.404	0.471	0.538	0.606	0.673
67	0.068	0.137	0.205	0.273	0.342	0.410	0.478	0.547	0.615	0.683
68	0.069	0.139	0.208	0.277	0.347	0.416	0.485	0.555	0.624	0.693
69	0.070	0.141	0.211	0.281	0.352	0.422	0.493	0.563	0.633	0.704
70	0.071	0.143	0.214	0.286	0.357	0.428	0.500	0.571	0.642	0.714
71	0.072	0.145	0.217	0.290	0.362	0.434	0.507	0.579	0.652	0.724
72	0.073	0.147	0.220	0.294	0.367	0.441	0.514	0.587	0.661	0.734
73	0.074	0.149	0.223	0.298	0.372	0.447	0.521	0.596	0.670	0.744
74	0.075	0.151	0.226	0.302	0.377	0.453	0.528	0.604	0.679	0.755
75	0.076	0.153	0.229	0.306	0.382	0.459	0.535	0.612	0.688	0.765
76	0.078	0.155	0.233	0.310	0.388	0.465	0.543	0.620	0.698	0.775
77	0.079	0.157	0.236	0.314	0.393	0.471	0.550	0.628	0.707	0.785
78	0.080	0.159	0.239	0.318	0.398	0.477	0.557	0.636	0.716	0.795
79	0.081	0.161	0.242	0.322	0.403	0.483	0.564	0.644	0.725	0.806
80	0.082	0.163	0.245	0.326	0.408	0.489	0.571	0.653	0.734	0.816

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION
WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES) 1	2	3	4	5	6	7	8	9	10
81	0.083	0.165	0.248	0.330	0.413	0.496	0.578	0.661	0.743	0.826
82	0.084	0.167	0.251	0.334	0.418	0.502	0.585	0.669	0.753	0.836
83	0.085	0.169	0.254	0.339	0.423	0.508	0.592	0.677	0.762	0.846
84	0.086	0.171	0.257	0.343	0.428	0.514	0.600	0.685	0.771	0.857
85	0.087	0.173	0.260	0.347	0.433	0.520	0.607	0.693	0.780	0.867
86	0.088	0.175	0.263	0.351	0.438	0.526	0.614	0.702	0.789	0.877
87	0.089	0.177	0.266	0.355	0.444	0.532	0.621	0.710	0.798	0.887
88	0.090	0.179	0.269	0.359	0.449	0.538	0.628	0.718	0.808	0.897
89	0.091	0.182	0.272	0.363	0.454	0.545	0.635	0.726	0.817	0.908
90	0.092	0.184	0.275	0.367	0.459	0.551	0.642	0.734	0.826	0.918
91	0.093	0.186	0.278	0.371	0.464	0.557	0.650	0.742	0.835	0.928
92	0.094	0.188	0.281	0.375	0.469	0.563	0.657	0.751	0.844	0.938
93	0.095	0.190	0.285	0.379	0.474	0.569	0.664	0.759	0.854	0.948
94	0.096	0.192	0.288	0.383	0.479	0.575	0.671	0.767	0.863	0.959
95	0.097	0.194	0.291	0.388	0.484	0.581	0.678	0.775	0.872	0.969
96	0.098	0.196	0.294	0.392	0.489	0.587	0.685	0.783	0.881	0.979
97	0.099	0.198	0.297	0.396	0.495	0.593	0.692	0.791	0.890	0.989
98	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.799	0.899	0.999
99	0.101	0.202	0.303	0.404	0.505	0.606	0.707	0.808	0.909	1.010
100	0.102	0.204	0.306	0.408	0.510	0.612	0.714	0.816	0.918	1.020
101	0.103	0.206	0.309	0.412	0.515	0.618	0.721	0.824	0.927	1.030
102	0.104	0.208	0.312	0.416	0.520	0.624	0.728	0.832	0.936	1.040
103	0.105	0.210	0.315	0.420	0.525	0.630	0.735	0.840	0.945	1.050
104	0.106	0.212	0.318	0.424	0.530	0.636	0.742	0.848	0.954	1.061
105	0.107	0.214	0.321	0.428	0.535	0.642	0.750	0.857	0.964	1.071
106	0.108	0.216	0.324	0.432	0.540	0.649	0.757	0.865	0.973	1.081
107	0.109	0.218	0.327	0.436	0.546	0.655	0.764	0.873	0.982	1.091
108	0.110	0.220	0.330	0.441	0.551	0.661	0.771	0.881	0.991	1.101
109	0.111	0.222	0.333	0.445	0.556	0.667	0.778	0.889	1.000	1.112
110	0.112	0.224	0.337	0.449	0.561	0.673	0.785	0.897	1.010	1.122
111	0.113	0.226	0.340	0.453	0.566	0.679	0.792	0.906	1.019	1.132
112	0.114	0.228	0.343	0.457	0.571	0.685	0.799	0.914	1.028	1.142
113	0.115	0.230	0.346	0.461	0.576	0.691	0.807	0.922	1.037	1.152
114	0.116	0.233	0.349	0.465	0.581	0.698	0.814	0.930	1.046	1.163
115	0.117	0.235	0.352	0.469	0.586	0.704	0.821	0.938	1.055	1.173
116	0.118	0.237	0.355	0.473	0.591	0.710	0.828	0.946	1.065	1.183
117	0.119	0.239	0.358	0.477	0.597	0.716	0.835	0.954	1.074	1.193
118	0.120	0.241	0.361	0.481	0.602	0.722	0.842	0.963	1.083	1.203
119	0.121	0.243	0.364	0.485	0.607	0.728	0.849	0.971	1.092	1.214
120	0.122	0.245	0.367	0.489	0.612	0.734	0.857	0.979	1.101	1.224

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 1	2	3	4	5	6	7	8	9	10
121	0.123	0.247	0.370	0.494	0.617	0.740	0.864	0.987	1.111	1.234
122	0.124	0.249	0.373	0.498	0.622	0.746	0.871	0.995	1.120	1.244
123	0.125	0.251	0.376	0.502	0.627	0.753	0.878	1.003	1.129	1.254
124	0.126	0.253	0.379	0.506	0.632	0.759	0.885	1.012	1.138	1.265
125	0.127	0.255	0.382	0.510	0.637	0.765	0.892	1.020	1.147	1.275
126	0.128	0.257	0.385	0.514	0.642	0.771	0.899	1.028	1.156	1.285
127	0.130	0.259	0.389	0.518	0.648	0.777	0.907	1.036	1.166	1.295
128	0.131	0.261	0.392	0.522	0.653	0.783	0.914	1.044	1.175	1.305
129	0.132	0.263	0.395	0.526	0.658	0.789	0.921	1.052	1.184	1.315
130	0.133	0.265	0.398	0.530	0.663	0.795	0.928	1.061	1.193	1.326
131	0.134	0.267	0.401	0.534	0.668	0.802	0.935	1.069	1.202	1.336
132	0.135	0.269	0.404	0.538	0.673	0.808	0.942	1.077	1.211	1.346
133	0.136	0.271	0.407	0.543	0.678	0.814	0.949	1.085	1.221	1.356
134	0.137	0.273	0.410	0.547	0.683	0.820	0.957	1.093	1.230	1.366
135	0.138	0.275	0.413	0.551	0.688	0.826	0.964	1.101	1.239	1.377
136	0.139	0.277	0.416	0.555	0.693	0.832	0.971	1.109	1.248	1.387
137	0.140	0.279	0.419	0.559	0.699	0.838	0.978	1.118	1.257	1.397
138	0.141	0.281	0.422	0.563	0.704	0.844	0.985	1.126	1.267	1.407
139	0.142	0.283	0.425	0.567	0.709	0.850	0.992	1.134	1.276	1.417
140	0.143	0.286	0.428	0.571	0.714	0.857	0.999	1.142	1.285	1.428
141	0.144	0.288	0.431	0.575	0.719	0.863	1.007	1.150	1.294	1.438
142	0.145	0.290	0.434	0.579	0.724	0.869	1.014	1.158	1.303	1.448
143	0.146	0.292	0.437	0.583	0.729	0.875	1.021	1.167	1.312	1.458
144	0.147	0.294	0.441	0.587	0.734	0.881	1.028	1.175	1.322	1.468
145	0.148	0.296	0.444	0.591	0.739	0.887	1.035	1.183	1.331	1.479
146	0.149	0.298	0.447	0.596	0.744	0.893	1.042	1.191	1.340	1.489
147	0.150	0.300	0.450	0.600	0.750	0.899	1.049	1.199	1.349	1.499
148	0.151	0.302	0.453	0.604	0.755	0.906	1.056	1.207	1.358	1.509
149	0.152	0.304	0.456	0.608	0.760	0.912	1.064	1.216	1.367	1.519
150	0.153	0.306	0.459	0.612	0.765	0.918	1.071	1.224	1.377	1.530
151	0.154	0.308	0.462	0.616	0.770	0.924	1.078	1.232	1.386	1.540
152	0.155	0.310	0.465	0.620	0.775	0.930	1.085	1.240	1.395	1.550
153	0.156	0.312	0.468	0.624	0.780	0.936	1.092	1.248	1.404	1.560
154	0.157	0.314	0.471	0.628	0.785	0.942	1.099	1.256	1.413	1.570
155	0.158	0.316	0.474	0.632	0.790	0.948	1.106	1.265	1.423	1.581
156	0.159	0.318	0.477	0.636	0.795	0.954	1.114	1.273	1.432	1.591
157	0.160	0.320	0.480	0.640	0.801	0.961	1.121	1.281	1.441	1.601
158	0.161	0.322	0.483	0.644	0.806	0.967	1.128	1.289	1.450	1.611
159	0.162	0.324	0.486	0.649	0.811	0.973	1.135	1.297	1.459	1.621
160	0.163	0.326	0.489	0.653	0.816	0.979	1.142	1.305	1.468	1.632

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 1	2	3	4	5	6	7	8	9	10
161	0.164	0.328	0.493	0.657	0.821	0.985	1.149	1.313	1.478	1.642
162	0.165	0.330	0.496	0.661	0.826	0.991	1.156	1.322	1.487	1.652
163	0.166	0.332	0.499	0.665	0.831	0.997	1.164	1.330	1.496	1.662
164	0.167	0.334	0.502	0.669	0.836	1.003	1.171	1.338	1.505	1.672
165	0.168	0.337	0.505	0.673	0.841	1.010	1.178	1.346	1.514	1.683
166	0.169	0.339	0.508	0.677	0.846	1.016	1.185	1.354	1.524	1.693
167	0.170	0.341	0.511	0.681	0.851	1.022	1.192	1.362	1.533	1.703
168	0.171	0.343	0.514	0.685	0.857	1.028	1.199	1.371	1.542	1.713
169	0.172	0.345	0.517	0.689	0.862	1.034	1.206	1.379	1.551	1.723
170	0.173	0.347	0.520	0.693	0.867	1.040	1.214	1.387	1.560	1.734
171	0.174	0.349	0.523	0.698	0.872	1.046	1.221	1.395	1.569	1.744
172	0.175	0.351	0.526	0.702	0.877	1.052	1.228	1.403	1.579	1.754
173	0.176	0.353	0.529	0.706	0.882	1.059	1.235	1.411	1.588	1.764
174	0.177	0.355	0.532	0.710	0.887	1.065	1.242	1.420	1.597	1.774
175	0.178	0.357	0.535	0.714	0.892	1.071	1.249	1.428	1.606	1.785
176	0.179	0.359	0.538	0.718	0.897	1.077	1.256	1.436	1.615	1.795
177	0.180	0.361	0.541	0.722	0.902	1.083	1.263	1.444	1.624	1.805
178	0.182	0.363	0.545	0.726	0.908	1.089	1.271	1.452	1.634	1.815
179	0.183	0.365	0.548	0.730	0.913	1.095	1.278	1.460	1.643	1.825
180	0.184	0.367	0.551	0.734	0.918	1.101	1.285	1.468	1.652	1.836
181	0.185	0.369	0.554	0.738	0.923	1.107	1.292	1.477	1.661	1.846
182	0.186	0.371	0.557	0.742	0.928	1.114	1.299	1.485	1.670	1.856
183	0.187	0.373	0.560	0.746	0.933	1.120	1.306	1.493	1.680	1.866
184	0.188	0.375	0.563	0.751	0.938	1.126	1.313	1.501	1.689	1.876
185	0.189	0.377	0.565	0.755	0.943	1.132	1.321	1.509	1.698	1.887
186	0.190	0.379	0.569	0.759	0.948	1.138	1.328	1.517	1.707	1.897
187	0.191	0.381	0.572	0.763	0.953	1.144	1.335	1.526	1.716	1.907
188	0.192	0.383	0.575	0.767	0.959	1.150	1.342	1.534	1.725	1.917
189	0.193	0.385	0.578	0.771	0.964	1.156	1.349	1.542	1.735	1.927
190	0.194	0.388	0.581	0.775	0.969	1.163	1.356	1.550	1.744	1.938
191	0.195	0.390	0.584	0.779	0.974	1.169	1.363	1.558	1.753	1.948
192	0.196	0.392	0.587	0.783	0.979	1.175	1.371	1.566	1.762	1.958
193	0.197	0.394	0.590	0.787	0.984	1.181	1.378	1.575	1.771	1.968
194	0.198	0.396	0.593	0.791	0.989	1.187	1.385	1.583	1.780	1.978
195	0.199	0.398	0.597	0.795	0.994	1.193	1.392	1.591	1.790	1.989
196	0.200	0.400	0.600	0.799	0.999	1.199	1.399	1.599	1.799	1.999
197	0.201	0.402	0.603	0.804	1.004	1.205	1.406	1.607	1.808	2.009
198	0.202	0.404	0.606	0.808	1.010	1.211	1.413	1.615	1.817	2.019
199	0.203	0.406	0.609	0.812	1.015	1.218	1.421	1.623	1.826	2.029
200	0.204	0.408	0.612	0.816	1.020	1.224	1.429	1.632	1.836	2.040

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	10	20	30	40	50	60	70	80	90	100
1	0.010	0.020	0.031	0.041	0.051	0.061	0.071	0.082	0.092	0.102
2	0.020	0.041	0.061	0.082	0.102	0.122	0.143	0.163	0.184	0.204
3	0.031	0.061	0.092	0.122	0.153	0.184	0.214	0.245	0.275	0.306
4	0.041	0.082	0.122	0.163	0.204	0.245	0.286	0.326	0.367	0.408
5	0.051	0.102	0.153	0.204	0.255	0.306	0.357	0.408	0.459	0.510
6	0.061	0.122	0.184	0.245	0.306	0.367	0.428	0.489	0.551	0.612
7	0.071	0.143	0.214	0.286	0.357	0.428	0.500	0.571	0.642	0.714
8	0.082	0.163	0.245	0.326	0.408	0.489	0.571	0.653	0.734	0.816
9	0.092	0.184	0.275	0.367	0.459	0.551	0.642	0.734	0.826	0.918
10	0.102	0.204	0.306	0.408	0.510	0.612	0.714	0.816	0.918	1.020
11	0.112	0.224	0.337	0.449	0.561	0.673	0.785	0.897	1.010	1.122
12	0.122	0.245	0.367	0.489	0.612	0.734	0.857	0.979	1.101	1.224
13	0.133	0.265	0.398	0.530	0.663	0.795	0.928	1.061	1.193	1.326
14	0.143	0.286	0.428	0.571	0.714	0.857	0.999	1.142	1.285	1.428
15	0.153	0.306	0.459	0.612	0.765	0.918	1.071	1.224	1.377	1.530
16	0.163	0.326	0.489	0.653	0.816	0.979	1.142	1.305	1.468	1.632
17	0.173	0.347	0.520	0.693	0.867	1.040	1.214	1.387	1.560	1.734
18	0.184	0.367	0.551	0.734	0.918	1.101	1.285	1.468	1.652	1.836
19	0.194	0.388	0.581	0.775	0.969	1.163	1.356	1.550	1.744	1.938
20	0.204	0.408	0.612	0.816	1.020	1.224	1.428	1.632	1.836	2.040
21	0.214	0.428	0.642	0.857	1.071	1.285	1.499	1.713	1.927	2.141
22	0.224	0.449	0.673	0.897	1.122	1.346	1.570	1.795	2.019	2.243
23	0.235	0.469	0.704	0.938	1.173	1.407	1.642	1.876	2.111	2.345
24	0.245	0.489	0.734	0.979	1.224	1.468	1.713	1.958	2.203	2.447
25	0.255	0.510	0.765	1.020	1.275	1.530	1.785	2.040	2.294	2.549
26	0.265	0.530	0.795	1.061	1.326	1.591	1.856	2.121	2.386	2.651
27	0.275	0.551	0.826	1.101	1.377	1.652	1.927	2.203	2.478	2.753
28	0.286	0.571	0.857	1.142	1.428	1.713	1.999	2.284	2.570	2.855
29	0.296	0.591	0.887	1.183	1.479	1.774	2.070	2.366	2.662	2.957
30	0.306	0.612	0.918	1.224	1.530	1.836	2.141	2.447	2.753	3.059
31	0.316	0.632	0.948	1.265	1.581	1.897	2.213	2.529	2.845	3.161
32	0.326	0.653	0.979	1.305	1.632	1.958	2.284	2.611	2.937	3.263
33	0.337	0.673	1.010	1.346	1.683	2.019	2.356	2.692	3.029	3.365
34	0.347	0.693	1.040	1.387	1.734	2.080	2.427	2.774	3.120	3.467
35	0.357	0.714	1.071	1.428	1.785	2.141	2.498	2.855	3.212	3.569
36	0.367	0.734	1.101	1.468	1.836	2.203	2.570	2.937	3.304	3.671
37	0.377	0.755	1.132	1.509	1.887	2.264	2.641	3.018	3.396	3.773
38	0.388	0.775	1.163	1.550	1.938	2.325	2.713	3.100	3.488	3.875
39	0.398	0.795	1.193	1.591	1.989	2.386	2.784	3.182	3.579	3.977
40	0.408	0.816	1.224	1.632	2.040	2.447	2.855	3.263	3.671	4.079

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	10	20	30	40	50	60	70	80	90	100
41	0.418	0.836	1.254	1.672	2.091	2.509	2.927	3.345	3.763	4.181
42	0.428	0.857	1.285	1.713	2.141	2.570	2.998	3.426	3.855	4.283
43	0.438	0.877	1.315	1.754	2.192	2.631	3.069	3.508	3.946	4.385
44	0.449	0.897	1.346	1.795	2.243	2.692	3.141	3.590	4.038	4.487
45	0.459	0.918	1.377	1.836	2.294	2.753	3.212	3.671	4.130	4.589
46	0.469	0.938	1.407	1.876	2.345	2.815	3.284	3.753	4.222	4.691
47	0.479	0.959	1.438	1.917	2.396	2.876	3.355	3.834	4.314	4.793
48	0.489	0.979	1.468	1.958	2.447	2.937	3.426	3.916	4.405	4.895
49	0.500	0.999	1.499	1.999	2.498	2.998	3.498	3.997	4.497	4.997
50	0.510	1.020	1.530	2.040	2.549	3.059	3.569	4.079	4.589	5.099
51	0.520	1.040	1.560	2.080	2.600	3.120	3.641	4.161	4.681	5.201
52	0.530	1.061	1.591	2.121	2.651	3.182	3.712	4.242	4.772	5.303
53	0.540	1.081	1.621	2.162	2.702	3.243	3.783	4.324	4.864	5.405
54	0.551	1.101	1.652	2.203	2.753	3.304	3.855	4.405	4.956	5.507
55	0.561	1.122	1.683	2.243	2.804	3.365	3.926	4.487	5.048	5.609
56	0.571	1.142	1.713	2.284	2.855	3.426	3.997	4.569	5.140	5.711
57	0.581	1.163	1.744	2.325	2.906	3.488	4.069	4.650	5.231	5.813
58	0.591	1.183	1.774	2.366	2.957	3.549	4.140	4.732	5.323	5.915
59	0.602	1.203	1.805	2.407	3.008	3.610	4.212	4.813	5.415	6.017
60	0.612	1.224	1.836	2.447	3.059	3.671	4.283	4.895	5.507	6.119
61	0.622	1.244	1.866	2.488	3.110	3.732	4.354	4.976	5.598	6.221
62	0.632	1.265	1.897	2.529	3.161	3.794	4.426	5.058	5.690	6.323
63	0.642	1.285	1.927	2.570	3.212	3.855	4.497	5.140	5.782	6.424
64	0.653	1.305	1.958	2.611	3.263	3.916	4.569	5.221	5.874	6.526
65	0.663	1.326	1.989	2.651	3.314	3.977	4.640	5.303	5.966	6.628
66	0.673	1.346	2.019	2.692	3.365	4.038	4.711	5.384	6.057	6.730
67	0.683	1.366	2.050	2.733	3.416	4.099	4.783	5.466	6.149	6.832
68	0.693	1.387	2.080	2.774	3.467	4.161	4.854	5.547	6.241	6.934
69	0.704	1.407	2.111	2.815	3.518	4.222	4.925	5.629	6.333	7.036
70	0.714	1.428	2.141	2.855	3.569	4.283	4.997	5.711	6.424	7.138
71	0.724	1.448	2.172	2.896	3.620	4.344	5.063	5.792	6.516	7.240
72	0.734	1.468	2.203	2.937	3.671	4.405	5.140	5.874	6.608	7.342
73	0.744	1.489	2.233	2.978	3.722	4.467	5.211	5.955	6.700	7.444
74	0.755	1.509	2.264	3.018	3.773	4.528	5.282	6.037	6.792	7.546
75	0.765	1.530	2.294	3.059	3.824	4.589	5.354	6.119	6.883	7.648
76	0.775	1.550	2.325	3.100	3.875	4.650	5.425	6.200	6.975	7.750
77	0.785	1.570	2.356	3.141	3.926	4.711	5.497	6.282	7.067	7.852
78	0.795	1.591	2.386	3.182	3.977	4.772	5.568	6.363	7.159	7.954
79	0.806	1.611	2.417	3.222	4.028	4.834	5.639	6.445	7.250	8.056
80	0.816	1.632	2.447	3.263	4.079	4.895	5.711	6.526	7.342	8.158

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 10	20	30	40	50	60	70	80	90	100
81	0.826	1.652	2.478	3.304	4.130	4.956	5.782	6.608	7.434	8.260
82	0.836	1.672	2.509	3.345	4.181	5.017	5.853	6.690	7.526	8.362
83	0.846	1.693	2.539	3.386	4.232	5.078	5.925	6.771	7.618	8.464
84	0.857	1.713	2.570	3.426	4.283	5.140	5.996	6.853	7.709	8.566
85	0.867	1.734	2.600	3.467	4.334	5.201	6.068	6.934	7.801	8.668
86	0.877	1.754	2.631	3.508	4.385	5.262	6.139	7.016	7.893	8.770
87	0.887	1.774	2.662	3.549	4.436	5.323	6.210	7.098	7.985	8.872
88	0.897	1.795	2.692	3.590	4.487	5.384	6.282	7.179	8.076	8.974
89	0.908	1.815	2.723	3.630	4.538	5.446	6.353	7.261	8.168	9.076
90	0.918	1.836	2.753	3.671	4.589	5.507	6.424	7.342	8.260	9.178
91	0.928	1.856	2.784	3.712	4.640	5.568	6.496	7.424	8.352	9.280
92	0.938	1.876	2.815	3.753	4.691	5.629	6.567	7.505	8.444	9.382
93	0.948	1.897	2.845	3.794	4.742	5.690	6.639	7.587	8.535	9.484
94	0.959	1.917	2.876	3.834	4.793	5.751	6.710	7.669	8.627	9.586
95	0.969	1.938	2.906	3.875	4.844	5.813	6.781	7.750	8.719	9.688
96	0.979	1.958	2.937	3.916	4.895	5.874	6.853	7.832	8.811	9.790
97	0.989	1.978	2.967	3.957	4.946	5.935	6.924	7.913	8.902	9.892
98	0.999	1.999	2.998	3.997	4.997	5.996	6.996	7.995	8.994	9.994
99	1.010	2.019	3.029	4.038	5.048	6.057	7.067	8.076	9.086	10.096
100	1.020	2.040	3.059	4.079	5.099	6.119	7.138	8.158	9.178	10.198
101	1.030	2.060	3.090	4.120	5.150	6.180	7.210	8.240	9.270	10.300
102	1.040	2.080	3.120	4.161	5.201	6.241	7.281	8.321	9.361	10.402
103	1.050	2.101	3.151	4.201	5.252	6.302	7.352	8.403	9.453	10.504
104	1.061	2.121	3.182	4.242	5.303	6.363	7.424	8.484	9.545	10.605
105	1.071	2.141	3.212	4.283	5.354	6.424	7.495	8.566	9.637	10.707
106	1.081	2.162	3.243	4.324	5.405	6.486	7.567	8.648	9.728	10.809
107	1.091	2.182	3.273	4.365	5.455	6.547	7.638	8.729	9.820	10.911
108	1.101	2.203	3.304	4.405	5.507	6.608	7.709	8.811	9.912	11.013
109	1.112	2.223	3.335	4.446	5.558	6.669	7.781	8.892	10.004	11.115
110	1.122	2.243	3.365	4.487	5.609	6.730	7.852	8.974	10.096	11.217
111	1.132	2.264	3.396	4.528	5.660	6.792	7.924	9.055	10.187	11.319
112	1.142	2.284	3.426	4.569	5.711	6.853	7.995	9.137	10.279	11.421
113	1.152	2.305	3.457	4.609	5.762	6.914	8.066	9.219	10.371	11.523
114	1.163	2.325	3.488	4.650	5.813	6.975	8.138	9.300	10.463	11.625
115	1.173	2.345	3.518	4.691	5.864	7.036	8.209	9.382	10.555	11.727
116	1.183	2.366	3.549	4.732	5.915	7.098	8.280	9.463	10.646	11.829
117	1.193	2.386	3.579	4.772	5.966	7.159	8.352	9.545	10.738	11.931
118	1.203	2.407	3.610	4.813	6.017	7.220	8.423	9.627	10.830	12.033
119	1.214	2.427	3.641	4.854	6.068	7.281	8.495	9.708	10.922	12.135
120	1.224	2.447	3.671	4.895	6.119	7.342	8.566	9.790	11.013	12.237

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	10	20	30	40	50	60	70	80	90	100
121	1.234	2.468	3.702	4.936	6.170	7.403	8.637	9.871	11.105	12.339
122	1.244	2.488	3.732	4.976	6.221	7.465	8.709	9.953	11.197	12.441
123	1.254	2.509	3.763	5.017	6.272	7.526	8.780	10.034	11.289	12.543
124	1.265	2.529	3.794	5.058	6.323	7.587	8.852	10.116	11.381	12.645
125	1.275	2.549	3.824	5.099	6.373	7.648	8.923	10.198	11.472	12.747
126	1.285	2.570	3.855	5.140	6.424	7.709	8.994	10.279	11.564	12.849
127	1.295	2.590	3.885	5.180	6.475	7.771	9.066	10.361	11.656	12.951
128	1.305	2.611	3.916	5.221	6.526	7.832	9.137	10.442	11.748	13.053
129	1.315	2.631	3.946	5.262	6.577	7.893	9.208	10.524	11.839	13.155
130	1.326	2.651	3.977	5.303	6.628	7.954	9.280	10.605	11.931	13.257
131	1.336	2.672	4.008	5.344	6.679	8.015	9.351	10.687	12.023	13.359
132	1.346	2.692	4.038	5.384	6.730	8.076	9.423	10.769	12.115	13.461
133	1.356	2.713	4.069	5.425	6.781	8.138	9.494	10.850	12.207	13.563
134	1.366	2.733	4.099	5.466	6.832	8.199	9.565	10.932	12.298	13.665
135	1.377	2.753	4.130	5.507	6.883	8.260	9.637	11.013	12.390	13.767
136	1.387	2.774	4.161	5.547	6.934	8.321	9.708	11.095	12.482	13.869
137	1.397	2.794	4.191	5.588	6.985	8.382	9.779	11.177	12.574	13.971
138	1.407	2.815	4.222	5.629	7.036	8.444	9.851	11.258	12.665	14.073
139	1.417	2.835	4.252	5.670	7.087	8.505	9.922	11.340	12.757	14.175
140	1.428	2.855	4.283	5.711	7.138	8.566	9.994	11.421	12.849	14.277
141	1.438	2.876	4.314	5.751	7.189	8.627	10.065	11.503	12.941	14.379
142	1.448	2.896	4.344	5.792	7.240	8.688	10.136	11.584	13.033	14.481
143	1.458	2.917	4.375	5.833	7.291	8.750	10.208	11.666	13.124	14.583
144	1.468	2.937	4.405	5.874	7.342	8.811	10.279	11.748	13.216	14.685
145	1.479	2.957	4.436	5.915	7.393	8.872	10.351	11.829	13.308	14.787
146	1.489	2.978	4.467	5.955	7.444	8.933	10.422	11.911	13.400	14.888
147	1.499	2.998	4.497	5.995	7.495	8.994	10.493	11.992	13.491	14.990
148	1.509	3.018	4.528	6.037	7.546	9.055	10.565	12.074	13.583	15.092
149	1.519	3.039	4.558	6.078	7.597	9.117	10.636	12.156	13.675	15.194
150	1.530	3.059	4.589	6.119	7.648	9.178	10.707	12.237	13.767	15.296
151	1.540	3.080	4.620	6.159	7.699	9.239	10.779	12.319	13.859	15.398
152	1.550	3.100	4.650	6.200	7.750	9.300	10.850	12.400	13.950	15.500
153	1.560	3.120	4.681	6.241	7.801	9.361	10.922	12.482	14.042	15.602
154	1.570	3.141	4.711	6.282	7.852	9.423	10.993	12.563	14.134	15.704
155	1.581	3.161	4.742	6.323	7.903	9.484	11.064	12.645	14.226	15.806
156	1.591	3.182	4.772	6.363	7.954	9.545	11.136	12.727	14.317	15.908
157	1.601	3.202	4.803	6.404	8.005	9.606	11.207	12.808	14.409	16.010
158	1.611	3.222	4.834	6.445	8.056	9.667	11.279	12.890	14.501	16.112
159	1.621	3.243	4.864	6.486	8.107	9.728	11.350	12.971	14.593	16.214
160	1.632	3.263	4.895	6.526	8.158	9.790	11.421	13.053	14.685	16.316

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION
WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 10	20	30	40	50	60	70	80	90	100
161	1.642	3.284	4.925	6.567	8.209	9.851	11.493	13.134	14.776	16.418
162	1.652	3.304	4.956	6.608	8.260	9.912	11.564	13.216	14.868	16.520
163	1.662	3.324	4.987	6.649	8.311	9.973	11.635	13.298	14.960	16.622
164	1.672	3.345	5.017	6.690	8.362	10.034	11.707	13.379	15.052	16.724
165	1.683	3.365	5.048	6.730	8.413	10.096	11.778	13.461	15.143	16.826
166	1.693	3.386	5.078	6.771	8.464	10.157	11.850	13.542	15.235	16.928
167	1.703	3.406	5.109	6.812	8.515	10.218	11.921	13.624	15.327	17.030
168	1.713	3.426	5.140	6.853	8.566	10.279	11.992	13.706	15.419	17.132
169	1.723	3.447	5.170	6.894	8.617	10.340	12.064	13.787	15.511	17.234
170	1.734	3.467	5.201	6.934	8.668	10.402	12.135	13.869	15.602	17.336
171	1.744	3.488	5.231	6.975	8.719	10.463	12.207	13.950	15.694	17.438
172	1.754	3.508	5.262	7.016	8.770	10.524	12.278	14.032	15.786	17.540
173	1.764	3.528	5.293	7.057	8.821	10.585	12.349	14.113	15.878	17.642
174	1.774	3.549	5.323	7.098	8.872	10.646	12.421	14.195	15.969	17.744
175	1.785	3.569	5.354	7.138	8.923	10.707	12.492	14.277	16.061	17.846
176	1.795	3.590	5.384	7.179	8.974	10.769	12.563	14.358	16.153	17.948
177	1.805	3.610	5.415	7.220	9.025	10.830	12.635	14.440	16.245	18.050
178	1.815	3.630	5.446	7.261	9.076	10.891	12.706	14.521	16.337	18.152
179	1.825	3.651	5.476	7.301	9.127	10.952	12.778	14.603	16.428	18.254
180	1.836	3.671	5.507	7.342	9.178	11.013	12.849	14.685	16.520	18.356
181	1.846	3.692	5.537	7.383	9.229	11.075	12.920	14.766	16.612	18.458
182	1.856	3.712	5.568	7.424	9.280	11.136	12.992	14.848	16.704	18.560
183	1.866	3.732	5.598	7.465	9.331	11.197	13.063	14.929	16.795	18.662
184	1.876	3.753	5.629	7.505	9.382	11.258	13.134	15.011	16.887	18.764
185	1.887	3.773	5.660	7.546	9.433	11.319	13.206	15.092	16.979	18.866
186	1.897	3.794	5.690	7.587	9.484	11.381	13.277	15.174	17.071	18.968
187	1.907	3.814	5.721	7.628	9.535	11.442	13.349	15.256	17.163	19.069
188	1.917	3.834	5.751	7.669	9.586	11.503	13.420	15.337	17.254	19.171
189	1.927	3.855	5.782	7.709	9.637	11.564	13.491	15.419	17.346	19.273
190	1.938	3.875	5.813	7.750	9.688	11.625	13.563	15.500	17.438	19.375
191	1.948	3.895	5.843	7.791	9.739	11.686	13.634	15.582	17.530	19.477
192	1.958	3.916	5.874	7.832	9.790	11.748	13.706	15.663	17.621	19.579
193	1.968	3.936	5.904	7.873	9.841	11.809	13.777	15.745	17.713	19.681
194	1.978	3.957	5.935	7.913	9.892	11.870	13.848	15.827	17.805	19.783
195	1.989	3.977	5.966	7.954	9.943	11.931	13.920	15.908	17.897	19.885
196	1.999	3.997	5.996	7.995	9.994	11.992	13.991	15.990	17.989	19.987
197	2.009	4.018	6.027	8.036	10.045	12.054	14.062	16.071	18.080	20.089
198	2.019	4.038	6.057	8.076	10.096	12.115	14.134	16.153	18.172	20.191
199	2.029	4.059	6.088	8.117	10.147	12.176	14.205	16.235	18.264	20.293
200	2.040	4.079	6.119	8.158	10.198	12.237	14.277	16.316	18.356	20.395

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	100	200	300	400	500	600	700	800	900	1000
1	0.102	0.204	0.306	0.408	0.510	0.612	0.714	0.816	0.918	1.020
2	0.204	0.408	0.612	0.816	1.020	1.224	1.428	1.632	1.836	2.040
3	0.306	0.612	0.918	1.224	1.530	1.836	2.141	2.447	2.753	3.059
4	0.408	0.816	1.224	1.632	2.040	2.447	2.855	3.263	3.671	4.079
5	0.510	1.020	1.530	2.040	2.549	3.059	3.569	4.079	4.589	5.099
6	0.612	1.224	1.836	2.447	3.059	3.671	4.283	4.895	5.507	6.119
7	0.714	1.428	2.141	2.855	3.569	4.283	4.997	5.711	6.424	7.138
8	0.816	1.632	2.447	3.263	4.079	4.895	5.711	6.526	7.342	8.158
9	0.918	1.836	2.753	3.671	4.589	5.507	6.424	7.342	8.260	9.178
10	1.020	2.040	3.059	4.079	5.099	6.119	7.138	8.158	9.178	10.198
11	1.122	2.243	3.365	4.487	5.609	6.730	7.852	8.974	10.096	11.217
12	1.224	2.447	3.671	4.895	6.119	7.342	8.566	9.790	11.013	12.237
13	1.326	2.651	3.977	5.303	6.628	7.954	9.280	10.605	11.931	13.257
14	1.428	2.855	4.283	5.711	7.138	8.566	9.994	11.421	12.849	14.277
15	1.530	3.059	4.589	6.119	7.648	9.178	10.707	12.237	13.767	15.296
16	1.632	3.263	4.895	6.526	8.158	9.790	11.421	13.053	14.685	16.316
17	1.734	3.467	5.201	6.934	8.668	10.402	12.135	13.869	15.602	17.336
18	1.836	3.671	5.507	7.342	9.178	11.013	12.849	14.685	16.520	18.356
19	1.938	3.875	5.813	7.750	9.688	11.625	13.563	15.500	17.438	19.375
20	2.040	4.079	6.119	8.158	10.198	12.237	14.277	16.316	18.356	20.395
21	2.141	4.283	6.424	8.566	10.707	12.849	14.990	17.132	19.273	21.415
22	2.243	4.487	6.730	8.974	11.217	13.461	15.704	17.948	20.191	22.435
23	2.345	4.691	7.036	9.382	11.727	14.073	16.418	18.764	21.109	23.454
24	2.447	4.895	7.342	9.790	12.237	14.685	17.132	19.579	22.027	24.474
25	2.549	5.099	7.648	10.198	12.747	15.296	17.846	20.395	22.945	25.494
26	2.651	5.303	7.954	10.605	13.257	15.908	18.560	21.211	23.862	26.514
27	2.753	5.507	8.260	11.013	13.767	16.520	19.273	22.027	24.780	27.533
28	2.855	5.711	8.566	11.421	14.277	17.132	19.987	22.843	25.698	28.553
29	2.957	5.915	8.872	11.829	14.787	17.744	20.701	23.658	26.616	29.573
30	3.059	6.119	9.178	12.237	15.296	18.356	21.415	24.474	27.533	30.593
31	3.161	6.323	9.484	12.645	15.806	18.968	22.129	25.290	28.451	31.613
32	3.263	6.526	9.790	13.053	16.316	19.579	22.843	26.106	29.369	32.632
33	3.365	6.730	10.096	13.461	16.826	20.191	23.556	26.922	30.287	33.652
34	3.467	6.934	10.402	13.869	17.336	20.803	24.270	27.737	31.205	34.672
35	3.569	7.138	10.707	14.277	17.846	21.415	24.984	28.553	32.122	35.692
36	3.671	7.342	11.013	14.685	18.356	22.027	25.698	29.369	33.040	36.711
37	3.773	7.546	11.319	15.092	18.866	22.639	26.412	30.185	33.958	37.731
38	3.875	7.750	11.625	15.500	19.375	23.251	27.126	31.001	34.876	38.751
39	3.977	7.954	11.931	15.908	19.885	23.862	27.839	31.816	35.794	39.771
40	4.079	8.158	12.237	16.316	20.395	24.474	28.553	32.632	36.711	40.790

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	H E I G H T - D I F F E R E N C E S (IN METRES)									
	100	200	300	400	500	600	700	800	900	1000
41	4.181	8.362	12.543	16.724	20.905	25.086	29.267	33.448	37.629	41.810
42	4.283	8.566	12.849	17.132	21.415	25.698	29.981	34.264	38.547	42.830
43	4.385	8.770	13.155	17.540	21.925	26.310	30.695	35.080	39.465	43.850
44	4.487	8.974	13.461	17.948	22.435	26.922	31.409	35.896	40.382	44.869
45	4.589	9.178	13.767	18.356	22.945	27.533	32.122	36.711	41.300	45.889
46	4.691	9.382	14.073	18.764	23.454	28.145	32.836	37.527	42.218	46.909
47	4.793	9.586	14.379	19.171	23.964	28.757	33.550	38.343	43.136	47.929
48	4.895	9.790	14.685	19.579	24.474	29.369	34.264	39.159	44.054	48.948
49	4.997	9.994	14.990	19.987	24.984	29.981	34.978	39.975	44.971	49.968
50	5.099	10.198	15.296	20.395	25.494	30.593	35.692	40.790	45.889	50.988
51	5.201	10.402	15.602	20.803	26.004	31.205	36.405	41.606	46.807	52.008
52	5.303	10.605	15.908	21.211	26.514	31.816	37.119	42.422	47.725	53.027
53	5.405	10.809	16.214	21.619	27.024	32.428	37.833	43.238	48.642	54.047
54	5.507	11.013	16.520	22.027	27.533	33.040	38.547	44.054	49.560	55.067
55	5.609	11.217	16.826	22.435	28.043	33.652	39.261	44.869	50.478	56.087
56	5.711	11.421	17.132	22.843	28.553	34.264	39.975	45.685	51.396	57.106
57	5.813	11.625	17.438	23.251	29.063	34.876	40.688	46.501	52.314	58.126
58	5.915	11.829	17.744	23.658	29.573	35.488	41.402	47.317	53.231	59.146
59	6.017	12.033	18.050	24.066	30.083	36.099	42.116	48.133	54.149	60.166
60	6.119	12.237	18.356	24.474	30.593	36.711	42.830	48.948	55.067	61.186
61	6.221	12.441	18.662	24.882	31.103	37.323	43.544	49.764	55.985	62.205
62	6.323	12.645	18.968	25.290	31.613	37.935	44.258	50.580	56.903	63.225
63	6.424	12.849	19.273	25.698	32.122	38.547	44.971	51.396	57.820	64.245
64	6.526	13.053	19.579	26.106	32.632	39.159	45.685	52.212	58.738	65.265
65	6.628	13.257	19.885	26.514	33.142	39.771	46.399	53.027	59.656	66.284
66	6.730	13.461	20.191	26.922	33.652	40.382	47.113	53.843	60.574	67.304
67	6.832	13.665	20.497	27.330	34.162	40.994	47.827	54.659	61.491	68.324
68	6.934	13.869	20.803	27.737	34.672	41.606	48.541	55.475	62.409	69.344
69	7.036	14.073	21.109	28.145	35.182	42.218	49.254	56.291	63.327	70.363
70	7.138	14.277	21.415	28.553	35.692	42.830	49.968	57.106	64.245	71.383
71	7.240	14.481	21.721	28.961	36.201	43.442	50.682	57.922	65.163	72.403
72	7.342	14.685	22.027	29.369	36.711	44.054	51.396	58.738	66.080	73.423
73	7.444	14.888	22.333	29.777	37.221	44.665	52.110	59.554	66.998	74.442
74	7.546	15.092	22.639	30.185	37.731	45.277	52.824	60.370	67.916	75.462
75	7.648	15.296	22.945	30.593	38.241	45.889	53.537	61.186	68.834	76.482
76	7.750	15.500	23.251	31.001	38.751	46.501	54.251	62.001	69.752	77.502
77	7.852	15.704	23.556	31.409	39.261	47.113	54.965	62.817	70.669	78.521
78	7.954	15.908	23.862	31.816	39.771	47.725	55.679	63.633	71.587	79.541
79	8.056	16.112	24.168	32.224	40.280	48.337	56.393	64.449	72.505	80.561
80	8.158	16.316	24.474	32.632	40.790	48.948	57.106	65.265	73.423	81.581

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION
WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 100	200	300	400	500	600	700	800	900	1000
81	8.260	16.520	24.780	33.040	41.300	49.560	57.820	66.080	74.340	82.600
82	8.362	16.724	25.086	33.448	41.810	50.172	58.534	66.896	75.258	83.620
83	8.464	16.928	25.392	33.856	42.320	50.784	59.248	67.712	76.176	84.640
84	8.566	17.132	25.698	34.264	42.830	51.396	59.962	68.528	77.094	85.660
85	8.668	17.336	26.004	34.672	43.340	52.008	60.676	69.344	78.012	86.680
86	8.770	17.540	26.310	35.080	43.850	52.620	61.339	70.159	78.929	87.699
87	8.872	17.744	26.616	35.488	44.360	53.231	62.103	70.975	79.847	88.719
88	8.974	17.948	26.922	35.896	44.869	53.843	62.817	71.791	80.765	89.739
89	9.076	18.152	27.228	36.303	45.379	54.455	63.531	72.607	81.683	90.759
90	9.178	18.356	27.533	36.711	45.889	55.067	64.245	73.423	82.600	91.778
91	9.280	18.560	27.839	37.119	46.399	55.679	64.959	74.238	83.518	92.798
92	9.382	18.764	28.145	37.527	46.909	56.291	65.672	75.054	84.436	93.818
93	9.484	18.968	28.451	37.935	47.419	56.903	66.386	75.870	85.354	94.838
94	9.586	19.171	28.757	38.343	47.929	57.514	67.100	76.686	86.272	95.857
95	9.688	19.375	29.063	38.751	48.439	58.126	67.814	77.502	87.189	96.877
96	9.790	19.579	29.369	39.159	48.948	58.738	68.528	78.317	88.107	97.897
97	9.892	19.783	29.675	39.567	49.458	59.350	69.242	79.133	89.025	98.917
98	9.994	19.987	29.981	39.975	49.968	59.962	69.955	79.949	89.943	99.936
99	10.096	20.191	30.287	40.382	50.478	60.574	70.669	80.765	90.861	100.956
100	10.198	20.395	30.593	40.790	50.988	61.186	71.383	81.581	91.778	101.976
101	10.300	20.599	30.899	41.198	51.498	61.797	72.097	82.397	92.696	102.996
102	10.402	20.803	31.205	41.606	52.008	62.409	72.811	83.212	93.614	104.015
103	10.504	21.007	31.511	42.014	52.518	63.021	73.525	84.028	94.532	105.035
104	10.605	21.211	31.816	42.422	53.027	63.633	74.238	84.844	95.449	106.055
105	10.707	21.415	32.122	42.830	53.537	64.245	74.952	85.660	96.367	107.075
106	10.809	21.619	32.428	43.238	54.047	64.857	75.666	86.476	97.285	108.094
107	10.911	21.823	32.734	43.646	54.557	65.469	76.380	87.291	98.203	109.114
108	11.013	22.027	33.040	44.054	55.067	66.080	77.094	88.107	99.121	110.134
109	11.115	22.231	33.346	44.461	55.577	66.692	77.808	88.923	100.038	111.154
110	11.217	22.435	33.652	44.869	56.087	67.304	78.521	89.739	100.956	112.173
111	11.319	22.639	33.958	45.277	56.597	67.916	79.235	90.555	101.874	113.193
112	11.421	22.843	34.264	45.685	57.106	68.528	79.949	91.370	102.792	114.213
113	11.523	23.047	34.570	46.093	57.616	69.140	80.663	92.186	103.709	115.233
114	11.625	23.251	34.876	46.501	58.126	69.752	81.377	93.002	104.627	116.253
115	11.727	23.454	35.182	46.909	58.636	70.363	82.091	93.818	105.545	117.272
116	11.829	23.658	35.488	47.317	59.146	70.975	82.804	94.634	106.463	118.292
117	11.931	23.862	35.794	47.725	59.656	71.587	83.518	95.449	107.381	119.312
118	12.033	24.066	36.099	48.133	60.166	72.199	84.232	96.265	108.298	120.332
119	12.135	24.270	36.405	48.541	60.676	72.811	84.946	97.081	109.216	121.351
120	12.237	24.474	36.711	48.948	61.186	73.423	85.660	97.897	110.134	122.371

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 100	200	300	400	500	600	700	800	900	1000
121	12.339	24.678	37.017	49.356	61.695	74.034	86.374	98.713	111.052	123.391
122	12.441	24.882	37.323	49.764	62.205	74.646	87.087	99.528	111.970	124.411
123	12.543	25.086	37.629	50.172	62.715	75.258	87.801	100.344	112.887	125.430
124	12.645	25.290	37.935	50.580	63.225	75.870	88.515	101.160	113.805	126.450
125	12.747	25.494	38.241	50.988	63.735	76.482	89.229	101.976	114.723	127.470
126	12.849	25.698	38.547	51.396	64.245	77.094	89.943	102.792	115.641	128.490
127	12.951	25.902	38.853	51.804	64.755	77.706	90.657	103.607	116.558	129.509
128	13.053	26.106	39.159	52.212	65.265	78.317	91.370	104.423	117.476	130.529
129	13.155	26.310	39.465	52.620	65.774	78.929	92.084	105.239	118.394	131.549
130	13.257	26.514	39.771	53.027	66.284	79.541	92.798	106.055	119.312	132.569
131	13.359	26.718	40.077	53.435	66.794	80.153	93.512	106.871	120.230	133.588
132	13.461	26.922	40.382	53.843	67.304	80.765	94.226	107.687	121.147	134.608
133	13.563	27.126	40.688	54.251	67.814	81.377	94.940	108.502	122.065	135.628
134	13.665	27.330	40.994	54.659	68.324	81.989	95.653	109.318	122.983	136.648
135	13.767	27.533	41.300	55.067	68.834	82.600	96.367	110.134	123.901	137.667
136	13.869	27.737	41.606	55.475	69.344	83.212	97.081	110.950	124.818	138.687
137	13.971	27.941	41.912	55.883	69.853	83.824	97.795	111.766	125.736	139.707
138	14.073	28.145	42.218	56.291	70.363	84.436	98.509	112.581	126.654	140.727
139	14.175	28.349	42.524	56.699	70.873	85.048	99.223	113.397	127.572	141.746
140	14.277	28.553	42.830	57.106	71.383	85.660	99.936	114.213	128.490	142.766
141	14.379	28.757	43.136	57.514	71.893	86.272	100.650	115.029	129.407	143.786
142	14.481	28.961	43.442	57.922	72.403	86.883	101.364	115.845	130.325	144.806
143	14.583	29.165	43.748	58.330	72.913	87.495	102.078	116.660	131.243	145.826
144	14.685	29.369	44.054	58.738	73.423	88.107	102.792	117.476	132.161	146.845
145	14.787	29.573	44.360	59.146	73.933	88.719	103.506	118.292	133.079	147.865
146	14.888	29.777	44.665	59.554	74.442	89.331	104.219	119.108	133.996	148.885
147	14.990	29.981	44.971	59.962	74.952	89.943	104.933	119.924	134.914	149.905
148	15.092	30.185	45.277	60.370	75.462	90.555	105.647	120.739	135.832	150.924
149	15.194	30.389	45.583	60.778	75.972	91.166	106.361	121.555	136.750	151.944
150	15.296	30.593	45.889	61.186	76.482	91.778	107.075	122.371	137.667	152.964
151	15.398	30.797	46.195	61.593	76.992	92.390	107.789	123.187	138.585	153.984
152	15.500	31.001	46.501	62.001	77.502	93.002	108.502	124.003	139.503	155.003
153	15.602	31.205	46.807	62.409	78.012	93.614	109.216	124.818	140.421	156.023
154	15.704	31.409	47.113	62.817	78.521	94.226	109.930	125.634	141.339	157.043
155	15.806	31.613	47.419	63.225	79.031	94.838	110.644	126.450	142.256	158.063
156	15.908	31.816	47.725	63.633	79.541	95.449	111.358	127.265	143.174	159.082
157	16.010	32.020	48.031	64.041	80.051	96.061	112.071	128.082	144.092	160.102
158	16.112	32.224	48.337	64.449	80.561	96.673	112.785	128.898	145.010	161.122
159	16.214	32.428	48.642	64.857	81.071	97.285	113.499	129.713	145.927	162.142
160	16.316	32.632	48.948	65.265	81.531	97.897	114.213	130.529	146.845	163.161

TABLE III-1 (cont'd)

TABLE OF V I G N A L GRAVITY CORRECTION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION
WITH DIFFERENT VALUES OF HEIGHT-DIFFERENCE AND AVERAGE FREE-AIR ANOMALY

AVERAGE FREE-AIR ANOMALY (IN MGAL)	HEIGHT-DIFFERENCES (IN METRES) 100	200	300	400	500	600	700	800	900	1000
161	16.418	32.836	49.254	65.672	82.091	98.509	114.927	131.345	147.763	164.181
162	16.520	33.040	49.560	66.080	82.600	99.121	115.641	132.161	148.681	165.201
163	16.622	33.244	49.866	66.488	83.110	99.732	116.354	132.977	149.599	166.221
164	16.724	33.448	50.172	66.896	83.620	100.344	117.068	133.792	150.516	167.240
165	16.826	33.652	50.478	67.304	84.130	100.956	117.782	134.608	151.434	168.260
166	16.928	33.856	50.784	67.712	84.640	101.568	118.496	135.424	152.352	169.280
167	17.030	34.060	51.090	68.120	85.150	102.180	119.210	136.240	153.270	170.300
168	17.132	34.264	51.396	68.528	85.660	102.792	119.924	137.056	154.188	171.319
169	17.234	34.468	51.702	68.936	86.170	103.404	120.637	137.871	155.105	172.339
170	17.336	34.672	52.008	69.344	86.680	104.015	121.351	138.687	156.023	173.359
171	17.438	34.876	52.314	69.752	87.189	104.627	122.065	139.503	156.941	174.379
172	17.540	35.080	52.620	70.159	87.699	105.239	122.779	140.319	157.859	175.399
173	17.642	35.284	52.925	70.567	88.209	105.851	123.493	141.135	158.776	176.418
174	17.744	35.488	53.231	70.975	88.719	106.463	124.207	141.950	159.694	177.438
175	17.846	35.692	53.537	71.383	89.229	107.075	124.920	142.766	160.612	178.458
176	17.948	35.896	53.843	71.791	89.739	107.687	125.634	143.582	161.530	179.478
177	18.050	36.099	54.149	72.199	90.249	108.298	126.348	144.398	162.448	180.497
178	18.152	36.303	54.455	72.607	90.759	108.910	127.062	145.214	163.365	181.517
179	18.254	36.507	54.761	73.015	91.268	109.522	127.776	146.029	164.283	182.537
180	18.356	36.711	55.067	73.423	91.778	110.134	128.490	146.845	165.201	183.557
181	18.458	36.915	55.373	73.831	92.288	110.746	129.203	147.661	166.119	184.576
182	18.560	37.119	55.679	74.238	92.798	111.358	129.917	148.477	167.036	185.596
183	18.662	37.323	55.985	74.646	93.308	111.970	130.631	149.293	167.954	186.616
184	18.764	37.527	56.291	75.054	93.818	112.581	131.345	150.109	168.872	187.636
185	18.866	37.731	56.597	75.462	94.328	113.193	132.059	150.924	169.790	188.655
186	18.968	37.935	56.903	75.870	94.838	113.805	132.773	151.740	170.708	189.675
187	19.069	38.139	57.208	76.278	95.347	114.417	133.486	152.556	171.625	190.695
188	19.171	38.343	57.514	76.686	95.857	115.029	134.200	153.372	172.543	191.715
189	19.273	38.547	57.820	77.094	96.367	115.641	134.914	154.188	173.461	192.734
190	19.375	38.751	58.126	77.502	96.877	116.253	135.628	155.003	174.379	193.754
191	19.477	38.955	58.432	77.910	97.387	116.864	136.342	155.819	175.297	194.774
192	19.579	39.159	58.738	78.317	97.897	117.476	137.056	156.635	176.214	195.794
193	19.681	39.363	59.044	78.725	98.407	118.088	137.769	157.451	177.132	196.813
194	19.783	39.567	59.350	79.133	98.917	118.700	138.483	158.267	178.050	197.833
195	19.885	39.771	59.656	79.541	99.426	119.312	139.197	159.082	178.968	198.853
196	19.987	39.975	59.962	79.949	99.936	119.924	139.911	159.898	179.885	199.873
197	20.089	40.178	60.268	80.357	100.446	120.535	140.625	160.714	180.803	200.892
198	20.191	40.382	60.574	80.765	100.956	121.147	141.339	161.530	181.721	201.912
199	20.293	40.586	60.880	81.173	101.466	121.759	142.052	162.346	182.639	202.932
200	20.395	40.790	61.186	81.581	101.976	122.371	142.766	163.161	183.557	203.952

III-2 Dynamic Gravity Correction

The dynamic gravity correction GC_{ij}^{D} to the height difference $\Delta\tilde{h}_{ij}$ of a levelling section between the two bench marks i and j - corrected for the effect of normal gravity only - is computed from the expression:

$$GC_{ij}^{D} = \frac{\Delta h_{ij}}{G} [\overline{\Delta g}_{ij}^{F} + \overline{\delta\gamma}_{o,ij}] , \qquad \text{(III-5)}$$

where: $\overline{\Delta g}_{ij}^{F}$, Δh_{ij} and G are as defined in the previous section, and $\overline{\delta\gamma}_{o,ij}$ is the average difference between the 1967 International and the USC&GS formulae for normal gravity, between bench marks i and j.

A sufficiently precise value for $\overline{\delta\gamma}_{o,ij}$ can be obtained by consulting Table I-1. There, the value of $\delta\gamma_{o,i}$ is given for different latitudes (with 0.25 arc-deg. intervals) which enables one to obtain $\overline{\delta\gamma}_{o,ij}$ by interpolation into that table. This $\overline{\delta\gamma}_{o,ij}$ has to be added algebraically to $\overline{\Delta g}_{ij}^{F}$ and the resultant, in mgal, is then used, along with Δh_{ij}, in metres, to enter Table III-1 and get the corresponding dynamic gravity correction GC_{ij}^{D}, in millimetres. Note here again that the appropriate sign of the correction has to be determined manually. Finally, the value of GC_{ij}^{D} has to be added algebraically to the height difference $\Delta\tilde{h}_{ij}^{D}$ to obtain the proper dynamic height difference Δh_{ij}^{D} based on actual gravity.

III-3 Helmert Gravity Correction

The Helmert gravity correction GC^H_{ij} to the height difference $\Delta\tilde{h}_{ij}$ of a levelling section between the two bench marks i and j - corrected for the effect of normal gravity only - is computed from the following expression:

$$GC^H_{ij} = - \frac{\bar{h}_{ij}}{G} \; [\Delta\Delta g^F_{ij} - 0.2238 \; \Delta h_{ij}] \; , \qquad \text{(III-6)}$$

where: G and Δh_{ij} are as defined before, $\bar{h}_{ij}$ is the average height, above the geoid, of the levelling section and $\Delta\Delta g^F_{ij}$ is the difference of free-air gravity anomaly between bench marks i and j.

The above expression can be rewritten as:

$$GC^H_{ij} = - \left(\frac{\bar{h}_{ij} \cdot \Delta\Delta g^F_{ij}}{G}\right) + \left(0.2238 \; \frac{\bar{h}_{ij} \cdot \Delta h_{ij}}{G}\right) \quad . \qquad \text{(III-7)}$$

The form of the first term is the same as that of Vignal gravity correction. In other words, its numerical value, in millimetres, can be obtained from Table III-1 by replacing the arguments Δh_{ij} by $\bar{h}_{ij}$ and $\overline{\Delta g}^F_{ij}$ by $\Delta\Delta g^F_{ij}$, using the same units. Note the negative sign. The second term varies with the quantities $\bar{h}_{ij}$ and Δh_{ij}. Table III-2 gives the values of this term, in millimetres, for different values of $\bar{h}$ and Δh, in metres. The values of the argument Δh used here are identical to the ones used in Table III-1. The value of $\bar{h}$ in Table III-2 varies between 1 m and 10,000 m Values of the second term in GC^H_{ij}, obtained from Table III-2 are positive, corresponding to positive $\bar{h}$ and positive Δh. Both terms added together constitute the final value of Helmert Gravity Correction, GC^H_{ij}, which has to be added algebraically to the height difference $\Delta\tilde{h}_{ij}^{O}$ to get the proper Helmert height difference, Δh_{ij}^{H}, based on actual gravity.

TABLE III-2

TABLE OF VALUES OF THE SECOND-TERM OF H E L M E R T GRAVITY-CORRECTION EQUATION (IN MILLIMETRES) FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF ITS HEIGHT-DIFFERENCE AND AVERAGE HEIGHT

AVERAGE HEIGHTS (IN METRES)	HEIGHT-DIFFERENCES (IN METRES) 1	2	3	4	5	6	7	8	9	10
1	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002
2	0.000	0.001	0.001	0.002	0.002	0.003	0.003	0.004	0.004	0.005
3	0.001	0.001	0.002	0.003	0.003	0.004	0.005	0.005	0.006	0.007
4	0.001	0.002	0.003	0.004	0.005	0.005	0.006	0.007	0.008	0.009
5	0.001	0.002	0.003	0.005	0.006	0.007	0.008	0.009	0.010	0.011
6	0.001	0.003	0.004	0.005	0.007	0.008	0.010	0.011	0.012	0.014
7	0.002	0.003	0.005	0.006	0.008	0.010	0.011	0.013	0.014	0.016
8	0.002	0.004	0.005	0.007	0.009	0.011	0.013	0.015	0.016	0.018
9	0.002	0.004	0.006	0.008	0.010	0.012	0.014	0.016	0.018	0.021
10	0.002	0.005	0.007	0.009	0.011	0.014	0.016	0.018	0.021	0.023
10	0.002	0.005	0.007	0.009	0.011	0.014	0.016	0.018	0.021	0.023
20	0.005	0.009	0.014	0.018	0.023	0.027	0.032	0.037	0.041	0.046
30	0.007	0.014	0.021	0.027	0.034	0.041	0.048	0.055	0.062	0.068
40	0.009	0.018	0.027	0.037	0.046	0.055	0.064	0.073	0.082	0.091
50	0.011	0.023	0.034	0.046	0.057	0.068	0.080	0.091	0.103	0.114
60	0.014	0.027	0.041	0.055	0.068	0.082	0.096	0.110	0.123	0.137
70	0.016	0.032	0.048	0.064	0.080	0.096	0.112	0.128	0.144	0.160
80	0.018	0.037	0.055	0.073	0.091	0.110	0.128	0.146	0.164	0.183
90	0.021	0.041	0.062	0.082	0.103	0.123	0.144	0.164	0.185	0.205
100	0.023	0.046	0.068	0.091	0.114	0.137	0.160	0.183	0.205	0.228
100	0.023	0.046	0.068	0.091	0.114	0.137	0.160	0.183	0.205	0.228
200	0.046	0.091	0.137	0.183	0.228	0.274	0.320	0.365	0.411	0.456
300	0.068	0.137	0.205	0.274	0.342	0.411	0.479	0.548	0.616	0.685
400	0.091	0.183	0.274	0.365	0.456	0.548	0.639	0.730	0.822	0.913
500	0.114	0.228	0.342	0.456	0.571	0.685	0.799	0.913	1.027	1.141
600	0.137	0.274	0.411	0.548	0.685	0.822	0.959	1.095	1.232	1.369
700	0.160	0.320	0.479	0.639	0.799	0.959	1.118	1.278	1.438	1.598
800	0.183	0.365	0.548	0.730	0.913	1.095	1.278	1.461	1.643	1.826
900	0.205	0.411	0.616	0.822	1.027	1.232	1.438	1.643	1.849	2.054
1000	0.228	0.456	0.685	0.913	1.141	1.369	1.598	1.826	2.054	2.282
1000	0.228	0.456	0.685	0.913	1.141	1.369	1.598	1.826	2.054	2.282
2000	0.456	0.913	1.369	1.826	2.282	2.739	3.195	3.652	4.108	4.564
3000	0.685	1.369	2.054	2.739	3.423	4.108	4.793	5.477	6.162	6.847
4000	0.913	1.826	2.739	3.652	4.564	5.477	6.390	7.303	8.216	9.129
5000	1.141	2.282	3.423	4.564	5.706	6.847	7.988	9.129	10.270	11.411
6000	1.369	2.739	4.108	5.477	6.847	8.216	9.585	10.955	12.324	13.693
7000	1.598	3.195	4.793	6.390	7.988	9.585	11.183	12.780	14.378	15.976
8000	1.826	3.652	5.477	7.303	9.129	10.955	12.780	14.606	16.432	18.258
9000	2.054	4.108	6.162	8.216	10.270	12.324	14.378	16.432	18.486	20.540
10000	2.282	4.564	6.847	9.129	11.411	13.693	15.976	18.258	20.540	22.822

TABLE III-2 (cont'd)

TABLE OF VALUES OF THE SECOND-TERM OF H E L M E R T GRAVITY-CORRECTION EQUATION (IN MILLIMETRES)
FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF ITS HEIGHT-DIFFERENCE AND AVERAGE HEIGHT

AVERAGE HEIGHTS (IN METRES)	HEIGHT-DIFFERENCES (IN METRES) 10	20	30	40	50	60	70	80	90	100
1	0.002	0.005	0.007	0.009	0.011	0.014	0.016	0.018	0.021	0.023
2	0.005	0.009	0.014	0.018	0.023	0.027	0.032	0.037	0.041	0.046
3	0.007	0.014	0.021	0.027	0.034	0.041	0.048	0.055	0.062	0.068
4	0.009	0.018	0.027	0.037	0.046	0.055	0.064	0.073	0.082	0.091
5	0.011	0.023	0.034	0.046	0.057	0.068	0.080	0.091	0.103	0.114
6	0.014	0.027	0.041	0.055	0.068	0.082	0.096	0.110	0.123	0.137
7	0.016	0.032	0.048	0.064	0.080	0.096	0.112	0.128	0.144	0.160
8	0.018	0.037	0.055	0.073	0.091	0.110	0.128	0.146	0.164	0.183
9	0.021	0.041	0.062	0.082	0.103	0.123	0.144	0.164	0.185	0.205
10	0.023	0.046	0.068	0.091	0.114	0.137	0.160	0.183	0.205	0.228
10	0.023	0.046	0.068	0.091	0.114	0.137	0.160	0.183	0.205	0.228
20	0.046	0.091	0.137	0.183	0.228	0.274	0.320	0.365	0.411	0.456
30	0.068	0.137	0.205	0.274	0.342	0.411	0.479	0.548	0.616	0.685
40	0.091	0.183	0.274	0.365	0.456	0.548	0.639	0.730	0.822	0.913
50	0.114	0.228	0.342	0.456	0.571	0.685	0.799	0.913	1.027	1.141
60	0.137	0.274	0.411	0.548	0.685	0.822	0.959	1.095	1.232	1.369
70	0.160	0.320	0.479	0.639	0.799	0.959	1.118	1.278	1.438	1.598
80	0.183	0.365	0.548	0.730	0.913	1.095	1.278	1.461	1.643	1.826
90	0.205	0.411	0.616	0.822	1.027	1.232	1.438	1.643	1.849	2.054
100	0.228	0.456	0.685	0.913	1.141	1.369	1.598	1.826	2.054	2.282
100	0.228	0.456	0.685	0.913	1.141	1.369	1.598	1.826	2.054	2.282
200	0.456	0.913	1.369	1.826	2.282	2.739	3.195	3.652	4.108	4.564
300	0.685	1.369	2.054	2.739	3.423	4.108	4.793	5.477	6.162	6.847
400	0.913	1.826	2.739	3.652	4.564	5.477	6.390	7.303	8.216	9.129
500	1.141	2.282	3.423	4.564	5.706	6.847	7.988	9.129	10.270	11.411
600	1.369	2.739	4.108	5.477	6.847	8.216	9.585	10.955	12.324	13.693
700	1.598	3.195	4.793	6.390	7.988	9.585	11.183	12.780	14.378	15.976
800	1.826	3.652	5.477	7.303	9.129	10.955	12.780	14.606	16.432	18.258
900	2.054	4.108	6.162	8.216	10.270	12.324	14.378	16.432	18.486	20.540
1000	2.282	4.564	6.847	9.129	11.411	13.693	15.976	18.258	20.540	22.822
1000	2.282	4.564	6.847	9.129	11.411	13.693	15.976	18.258	20.540	22.822
2000	4.564	9.129	13.693	18.258	22.822	27.387	31.951	36.516	41.080	45.644
3000	6.847	13.693	20.540	27.387	34.233	41.080	47.927	54.773	61.620	68.467
4000	9.129	18.258	27.387	36.516	45.644	54.773	63.902	73.031	82.160	91.289
5000	11.411	22.822	34.233	45.644	57.056	68.467	79.878	91.289	102.700	114.111
6000	13.693	27.387	41.080	54.773	68.467	82.160	95.853	109.547	123.240	136.933
7000	15.976	31.951	47.927	63.902	79.878	95.853	111.829	127.804	143.780	159.755
8000	18.258	36.516	54.773	73.031	91.289	109.547	127.804	146.062	164.320	182.578
9000	20.540	41.080	61.620	82.160	102.700	123.240	143.780	164.320	184.860	205.400
10000	22.822	45.644	68.467	91.289	114.111	136.933	159.755	182.573	205.400	228.222

TABLE III-2 (cont'd)

TABLE OF VALUES OF THE SECOND-TERM OF H E L M E R T GRAVITY-CORRECTION EQUATION (IN MILLIMETRES)
FOR A DIFFERENTIAL LEVELLING-SECTION WITH DIFFERENT VALUES OF ITS HEIGHT-DIFFERENCE AND AVERAGE HEIGHT

AVERAGE HEIGHTS (IN METRES)	HEIGHT-DIFFERENCES (IN METRES) 100	200	300	400	500	600	700	800	900	1000
1	0.023	0.046	0.068	0.091	0.114	0.137	0.160	0.183	0.205	0.228
2	0.046	0.091	0.137	0.183	0.228	0.274	0.320	0.365	0.411	0.456
3	0.068	0.137	0.205	0.274	0.342	0.411	0.479	0.548	0.616	0.685
4	0.091	0.183	0.274	0.365	0.456	0.548	0.639	0.730	0.822	0.913
5	0.114	0.228	0.342	0.456	0.571	0.685	0.799	0.913	1.027	1.141
6	0.137	0.274	0.411	0.548	0.685	0.822	0.959	1.095	1.232	1.369
7	0.160	0.320	0.479	0.639	0.799	0.959	1.118	1.278	1.438	1.598
8	0.183	0.365	0.548	0.730	0.913	1.095	1.278	1.461	1.643	1.826
9	0.205	0.411	0.616	0.822	1.027	1.232	1.438	1.643	1.849	2.054
10	0.228	0.456	0.685	0.913	1.141	1.369	1.598	1.826	2.054	2.282
10	0.228	0.456	0.685	0.913	1.141	1.369	1.598	1.826	2.054	2.282
20	0.456	0.913	1.369	1.826	2.282	2.739	3.195	3.652	4.108	4.564
30	0.685	1.369	2.054	2.739	3.423	4.108	4.793	5.477	6.162	6.847
40	0.913	1.826	2.739	3.652	4.564	5.477	6.390	7.303	8.216	9.129
50	1.141	2.282	3.423	4.564	5.706	6.847	7.988	9.129	10.270	11.411
60	1.369	2.739	4.108	5.477	6.847	8.216	9.585	10.955	12.324	13.693
70	1.598	3.195	4.793	6.390	7.988	9.585	11.183	12.780	14.378	15.976
80	1.826	3.652	5.477	7.303	9.129	10.955	12.780	14.606	16.432	18.258
90	2.054	4.108	6.162	8.216	10.270	12.324	14.378	16.432	18.486	20.540
100	2.282	4.564	6.847	9.129	11.411	13.693	15.976	18.258	20.540	22.822
100	2.282	4.564	6.847	9.129	11.411	13.693	15.976	18.258	20.540	22.822
200	4.564	9.129	13.693	18.258	22.822	27.387	31.951	36.516	41.080	45.644
300	6.847	13.693	20.540	27.387	34.233	41.080	47.927	54.773	61.620	68.467
400	9.129	18.258	27.387	36.516	45.644	54.773	63.902	73.031	82.160	91.289
500	11.411	22.822	34.233	45.644	57.056	68.467	79.878	91.289	102.700	114.111
600	13.693	27.387	41.080	54.773	68.467	82.160	95.853	109.547	123.240	136.933
700	15.976	31.951	47.927	63.902	79.878	95.853	111.829	127.804	143.780	159.755
800	18.258	36.516	54.773	73.031	91.289	109.547	127.804	146.062	164.320	182.578
900	20.540	41.080	61.620	82.160	102.700	123.240	143.780	164.320	184.860	205.400
1000	22.822	45.644	68.467	91.289	114.111	136.933	159.755	182.578	205.400	228.222
1000	22.822	45.644	68.467	91.289	114.111	136.933	159.755	182.578	205.400	228.222
2000	45.644	91.289	136.933	182.578	228.222	273.866	319.511	365.155	410.800	456.444
3000	68.467	136.933	205.400	273.866	342.333	410.800	479.266	547.733	616.199	684.666
4000	91.289	182.578	273.866	365.155	456.444	547.733	639.022	730.310	821.599	912.888
5000	114.111	228.222	342.333	456.444	570.555	684.666	798.777	912.888	1026.999	1141.110
6000	136.933	273.866	410.800	547.733	684.666	821.599	958.533	1095.466	1232.399	1369.332
7000	159.755	319.511	479.266	639.022	798.777	958.533	1118.288	1278.043	1437.799	1597.554
8000	182.578	365.155	547.733	730.310	912.888	1095.466	1278.043	1460.621	1643.199	1825.776
9000	205.400	410.800	616.199	821.599	1026.999	1232.399	1437.799	1643.199	1848.598	2053.998
10000	228.222	456.444	684.666	912.888	1141.110	1369.332	1597.554	1825.776	2053.998	2282.220

APPENDIX IV

DETAILED COMPUTATIONS OF THE GRAVITY CORRECTIONS ALONG THE TESTED LINES AND LOOPS IN CHAPTER 6.

This Appendix contains the computer outputs from the program package LOOPGC (mentioned at the begninning of section 6.2) for all tested lines and loops described in sections 6.2.1 and 6.2.2. The information presented herein for each line or loop is composed basically of three tables a, b and c:

(a) summary of input data (known information) at each bench mark;

(b) detailed computations of the three kinds of gravity corrections (Helmert, Vignal and Dynamic) for each individual levelling section along the line or loop;

(c) summary of results - to be utilized in plotting and comparing the accumulated gravity corrections, at each bench mark, against the corresponding accumulated standard errors as expected from the precise levelling work in Canada (see section 6.2).

The sequence of these tables is identical to the one described in sections 6.2.1 and 6.2.2, i.e.

Tables IV-1a, b and c: For test line No. 4;

Tables IV-2a, b and c: For test line No. 8;

Tables IV-3a, b and c: For test line No. 9;

Tables IV-4a, b and c: For test line No. 10;

Tables IV-5a, b and c: For test loop No. 5;

Tables IV-6a, b and c: For test loop No. 6.

TABLE IV-1a

SELECTED LEVELLING LINE NO. '4'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS.D.O.,1965

TABLE OF GIVEN INFORMATION AT BENCH MARKS

B.MARK NUMBER	LATITUDE (+VE NORTH) DEG	MIN	SEC	LONGITUDE (+VE WEST) DEG	MIN	SEC	HEIGHT (METERS)	HT.ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A.ACCURACY (MGALS)
9518	43	58	15.60	77	23	39.59	100.55	0.03	-24.87	0.05
8311	44	0	19.19	77	30	30.60	76.75	0.03	-27.01	0.05
8310	44	2	10.21	77	35	33.61	78.03	0.03	-25.60	0.05
8309	44	3	16.20	77	36	15.01	77.57	0.03	-26.99	0.05
9517	44	6	28.80	77	34	33.60	84.98	0.03	-21.08	0.05
8308	44	7	16.79	77	35	30.01	83.73	0.03	-26.23	0.05
8398	44	7	37.20	77	35	39.59	84.12	0.03	-26.54	0.05
8399	44	8	57.59	77	34	50.41	95.40	0.03	-24.58	0.05
8400	44	9	36.00	77	35	7.80	95.40	0.03	-25.33	0.05
8307	44	9	51.01	77	31	59.41	93.57	0.03	-25.43	0.05
8306	44	9	51.01	77	25	46.20	93.88	0.03	-23.83	0.05
8305	44	10	34.21	77	22	57.00	85.31	0.03	-24.66	0.05
9516	44	9	57.60	77	22	36.01	93.27	0.03	-23.39	0.05
8330	44	12	4.21	77	22	39.61	99.91	0.03	-22.86	0.05
8331	44	12	55.80	77	22	58.19	101.56	0.03	-21.54	0.05
8332	44	15	10.19	77	25	7.21	109.09	0.03	-21.87	0.05
8333	44	21	40.21	77	27	8.39	139.87	0.03	-10.95	0.05
8334	44	24	53.39	77	30	25.20	188.40	0.03	-11.59	0.05
9502	44	25	26.40	77	27	1.80	187.70	0.03	-10.27	0.05
8283	44	27	18.00	77	21	19.80	179.71	0.03	-19.90	0.05
9501	44	28	42.60	77	18	45.61	147.25	0.03	-21.35	0.05
8282	44	28	40.80	77	18	20.99	146.00	0.03	-20.46	0.05
8284	44	29	40.20	77	19	26.40	156.45	0.03	-15.64	0.05
9503	44	31	34.21	77	20	8.41	156.09	0.03	-13.57	0.05
9504	44	38	42.61	77	33	10.80	253.20	0.03	2.75	0.05
8285	44	44	51.00	77	35	31.20	300.26	0.03	9.01	0.05
9505	44	49	30.00	77	37	10.81	307.91	0.03	-2.57	0.05
8286	44	50	37.21	77	39	15.01	323.45	0.03	-4.46	0.05
8287	44	52	27.01	77	42	17.39	348.02	0.03	0.17	0.05
9506	44	58	47.39	77	44	24.00	336.19	0.03	3.29	0.05
8288	45	1	17.40	77	43	32.99	334.61	0.03	-0.10	0.05
8289	45	2	41.39	77	46	23.99	328.91	0.03	-2.70	0.05
9507	45	3	16.20	77	51	18.00	328.36	0.03	-6.03	0.05
8290	45	5	16.80	77	52	52.21	333.73	0.03	-9.64	0.05
8291	45	5	45.60	77	53	0.60	338.39	0.03	-10.09	0.05
9508	45	7	46.20	77	49	41.99	359.18	0.03	-5.54	0.05
8292	45	9	27.61	77	50	33.00	401.12	0.03	-0.53	0.05
9509	45	14	17.41	77	55	4.80	395.14	0.03	-2.81	0.05
8293	45	15	30.60	77	58	4.80	397.70	0.03	-2.25	0.05
8294	45	16	9.59	77	59	4.20	401.54	0.03	-3.22	0.05
8295	45	16	39.00	77	59	30.01	402.64	0.03	-2.54	0.05
9512	45	30	8.39	77	58	43.79	316.44	0.03	-18.72	0.05

TABLE IV-1b

SELECTED LEVELLING LINE NO. '4'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS,D.O.,1965

TABLE OF COMPUTED GRAVITY CORRECTIONS
FOR EACH INDIVIDUAL LEVELLING-SECTION ALONG THE GIVEN LEVELLING-ROUTE
===

TOTAL NUMBER OF LEVELLING-SECTIONS ALONG THE LINE = 41

LEVELLING-SECTION		LENGTH	AZIMUTH	G R A V I T Y - C O R R E C T I O N S					
				HELMERT		VIGNAL		DYNAMIC	
FROM	TO	(KM)	(DEGREES)	MAGNITUDE (MM)	STANDARD (MM/KM.)	MAGNITUDE (MM)	STANDARD (MM/KM.)	MAGNITUDE (MM)	STANDARD (MM/KM.)
9518	8311	9.92	292.65	-0.2876	-0.0913	0.6297	0.1999	0.7516	0.2386
8311	8310	7.57	296.95	-0.0890	-0.0324	-0.0344	-0.0125	-0.0409	-0.0149
8310	8309	2.24	335.66	0.1024	0.0685	0.0122	0.0082	0.0146	0.0098
8309	9517	6.36	20.78	-0.3525	-0.1398	-0.1816	-0.0720	-0.2195	-0.0870
9517	8308	1.94	319.74	0.4183	0.3002	0.0301	0.0216	0.0365	0.0262
8308	8398	0.67	341.33	0.0342	0.0419	-0.0107	-0.0131	-0.0127	-0.0156
8398	8399	2.71	23.77	0.0520	0.0316	-0.2940	-0.1785	-0.3517	-0.2136
8399	8400	1.25	341.95	0.0732	0.0655	-0.0000	-0.0000	0.0000	0.0000
8400	8307	4.21	83.67	-0.0297	-0.0145	0.0473	0.0231	0.0567	0.0276
8307	8306	8.29	89.96	-0.1464	-0.0508	-0.0077	-0.0027	-0.0092	-0.0032
8306	8305	3.99	70.45	-0.1000	-0.0501	0.2117	0.1060	0.2556	0.1280
8305	9516	1.22	157.58	0.0472	0.0427	-0.1949	-0.1763	-0.2356	-0.2131
9516	8330	3.91	358.83	0.0939	0.0475	-0.1567	-0.0793	-0.1907	-0.0965
8330	8331	1.64	345.48	-0.0975	-0.0760	-0.0373	-0.0291	-0.0457	-0.0356
8331	8332	5.04	325.40	0.2161	0.0962	-0.1667	-0.0742	-0.2052	-0.0914
8332	8333	12.33	347.43	-0.5117	-0.1457	-0.5153	-0.1467	-0.6727	-0.1915
8333	8334	7.38	323.87	1.9247	0.7083	-0.5578	-0.2053	-0.8059	-0.2966
8334	9502	4.61	77.22	-0.2829	-0.1317	0.0078	0.0036	0.0114	0.0053
9502	8283	8.31	65.48	1.3934	0.4834	0.1212	0.0420	0.1621	0.0562
8283	9501	4.29	52.53	-0.9020	-0.4353	0.6761	0.3263	0.8422	0.4064
9501	8282	0.55	95.83	-0.1749	-0.2365	0.0266	0.0360	0.0330	0.0447
8282	8284	2.33	321.76	-0.3836	-0.2510	-0.1924	-0.1259	-0.2459	-0.1609
8284	9503	3.64	345.23	-0.3420	-0.1793	0.0054	0.0028	0.0073	0.0038
9503	9504	21.74	307.53	1.1299	0.2423	-0.5362	-0.1150	-1.0326	-0.2215
9504	8285	11.78	344.81	1.2052	0.3511	0.2818	0.0821	0.0414	0.0121
8285	9505	8.89	345.74	4.1199	1.3821	0.0250	0.0084	-0.0140	-0.0047
9505	8286	3.43	307.27	1.7294	0.9342	-0.0557	-0.0301	-0.1352	-0.0730
8286	8287	5.25	310.26	0.2957	0.1291	-0.0538	-0.0235	-0.1793	-0.0783
8287	9506	12.07	346.71	-2.0121	-0.5793	-0.0212	-0.0061	0.0395	0.0114
9506	8288	4.76	13.56	1.0407	0.4768	-0.0027	-0.0012	0.0055	0.0025
8288	8289	4.55	304.73	0.4461	0.2091	0.0081	0.0038	0.0373	0.0175
8289	9507	6.52	279.51	1.0754	0.4211	0.0024	0.0009	0.0052	0.0021
9507	8290	4.26	331.04	1.6226	0.7866	-0.0429	-0.0208	-0.0703	-0.0341
8290	8291	0.91	348.34	0.5124	0.5378	-0.0469	-0.0492	-0.0707	-0.0742
8291	9508	5.72	49.37	0.0385	0.0161	-0.1658	-0.0693	-0.2719	-0.1137
9508	8292	3.32	340.41	1.6946	0.9296	-0.1300	-0.0713	-0.3442	-0.1888
8292	9509	10.73	326.48	0.3847	0.1174	0.0100	0.0030	0.0407	0.0124
9509	8293	4.53	299.94	0.0027	0.0013	-0.0067	-0.0031	-0.0197	-0.0093
8293	8294	1.77	312.91	0.7464	0.5613	-0.0107	-0.0081	-0.0303	-0.0228
8294	8295	1.07	328.22	-0.1775	-0.1718	-0.0032	-0.0031	-0.0088	-0.0085
8295	9512	25.01	2.30	-1.1403	-0.2280	0.9342	0.1868	1.3750	0.2750

TABLE IV-1c

SELECTED LEVELLING LINE NO. '4'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS.D.C..1965

*** S U M M A R Y O F R E S U L T S ***

TABLE OF ACCUMULATED GRAVITY-CORRECTIONS AND THEIR ACCUMULATED STANDARD DEVIATIONS

AT EACH BENCH MARK ALONG THE GIVEN LEVELLING ROUTE

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
9518	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8311	9.92	4.1893	-0.2876	0.0064	0.6297	0.0014	0.7516	0.0016
8310	17.49	5.5623	-0.3766	0.0085	0.5953	0.0018	0.7107	0.0021
8309	19.73	5.9071	-0.2742	0.0102	0.6076	0.0022	0.7253	0.0025
9517	26.08	6.7927	-0.6267	0.0118	0.4260	0.0024	0.5058	0.0029
8308	28.03	7.0409	-0.2084	0.0132	0.4561	0.0026	0.5424	0.0031
8398	28.69	7.1240	-0.1742	0.0146	0.4454	0.0029	0.5297	0.0034
8399	31.40	7.4530	-0.1222	0.0159	0.1515	0.0031	0.1780	0.0037
8400	32.65	7.5995	-0.0491	0.0174	0.1515	0.0033	0.1780	0.0039
8307	36.86	8.0748	-0.0788	0.0187	0.1988	0.0035	0.2347	0.0041
8306	45.15	8.9370	-0.2252	0.0198	0.1912	0.0037	0.2255	0.0043
8305	49.14	9.3234	-0.3252	0.0209	0.4029	0.0038	0.4811	0.0045
9516	50.36	9.4386	-0.2780	0.0218	0.2080	0.0040	0.2455	0.0047
8330	54.27	9.7981	-0.1841	0.0229	0.0513	0.0041	0.0548	0.0049
8331	55.92	9.9454	-0.2816	0.0241	0.0140	0.0042	0.0091	0.0050
8332	60.96	10.3840	-0.0655	0.0252	-0.1527	0.0043	-0.1961	0.0052
8333	73.29	11.3862	-0.5772	0.0268	-0.6680	0.0045	-0.8688	0.0054
8334	80.68	11.9460	1.3475	0.0293	-1.2257	0.0049	-1.6747	0.0057
9502	85.29	12.2829	1.0646	0.0323	-1.2179	0.0049	-1.6633	0.0058
8283	93.60	12.8674	2.4581	0.0349	-1.0968	0.0050	-1.5013	0.0058
9501	97.89	13.1592	1.5561	0.0368	-0.4207	0.0052	-0.6591	0.0061
8282	98.44	13.1959	1.3812	0.0383	-0.3940	0.0053	-0.6261	0.0062
8284	100.78	13.3515	0.9976	0.0399	-0.5865	0.0054	-0.8720	0.0063
9503	104.42	13.5905	0.6556	0.0414	-0.5811	0.0054	-0.8646	0.0063
9504	126.16	14.9386	1.7855	0.0440	-1.1172	0.0065	-1.8973	0.0073
8285	137.94	15.6207	2.9907	0.0483	-0.8354	0.0067	-1.8559	0.0075
9505	146.83	16.1160	7.1106	0.0531	-0.8105	0.0067	-1.8699	0.0075
8286	150.26	16.3030	8.8400	0.0578	-0.8662	0.0067	-2.0051	0.0075
8287	155.50	16.5852	9.1357	0.0626	-0.9200	0.0068	-2.1844	0.0076
9506	167.57	17.2166	7.1236	0.0673	-0.9411	0.0068	-2.1448	0.0076
8288	172.33	17.4596	8.1643	0.0716	-0.9438	0.0058	-2.1393	0.0076
8289	176.89	17.6888	8.6104	0.0755	-0.9357	0.0068	-2.1020	0.0076
9507	183.41	18.0120	9.6858	0.0791	-0.9333	0.0068	-2.0968	0.0076
8290	187.66	18.2197	11.3085	0.0826	-0.9762	0.0068	-2.1670	0.0076
8291	188.57	18.2637	11.8208	0.0861	-1.0231	0.0068	-2.2378	0.0077
9508	194.29	18.5386	11.8593	0.0897	-1.1889	0.0069	-2.5097	0.0077
8292	197.61	18.6965	13.5539	0.0938	-1.3188	0.0071	-2.8539	0.0079
9509	208.35	19.1976	13.9387	0.0981	-1.3089	0.0071	-2.8132	0.0079
8293	212.88	19.4051	13.9414	0.1022	-1.3155	0.0071	-2.9329	0.0079
8294	214.65	19.4856	14.6877	0.1062	-1.3262	0.0071	-2.8632	0.0079
8295	215.71	19.5340	14.5102	0.1101	-1.3295	0.0071	-2.8721	0.0079
9512	240.72	20.6352	13.3699	0.1131	-0.3953	0.0073	-1.4970	0.0085

TABLE IV-2a

SELECTED LEVELLING LINE NO. '8'-GERMANY**AUSTRIA-PART OF THE UELN. RAPP. 1961.

TABLE OF GIVEN INFORMATION AT BENCH MARKS

B. MARK NUMBER	LATITUDE (+VE NORTH)			LONGITUDE (+VE WEST)			HEIGHT (METERS)	HT. ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A. ACCURACY (MGALS)
	DEG	MIN	SEC	DEG	MIN	SEC				
LB57	48	5	15.00	348	23	29.40	546.02	0.03	-21.33	0.05
TB16	47	58	28.20	348	20	36.60	617.40	0.03	-22.10	0.05
B[illegible]12	47	54	42.01	348	18	59.40	699.33	0.03	-22.19	0.05
MB45	47	49	54.59	348	17	33.00	726.16	0.03	-24.87	0.05
MB54	47	46	23.99	348	19	30.00	759.29	0.03	-21.06	0.05
M[illegible]39	47	46	1.20	348	25	54.01	686.99	0.03	-27.76	0.05
MB[illegible]9	47	41	3.01	348	25	18.01	678.85	0.03	-36.24	0.05
MB53	47	37	45.59	348	28	10.81	718.23	0.03	-26.82	0.05
MB37	47	36	20.99	348	33	53.39	774.59	0.03	-18.02	0.05
KB48	47	36	27.00	348	37	40.80	803.76	0.03	-15.62	0.05
MB25	47	35	7.80	348	41	30.01	803.94	0.03	-18.01	0.05
LB79	47	31	21.00	348	42	58.79	866.55	0.03	-19.50	0.05
MB98	47	29	1.21	348	45	42.84	914.70	0.03	-13.19	0.05
MB44	47	26	43.80	348	44	3.59	911.17	0.03	-24.28	0.05
MB[illegible]1	47	24	1.80	348	43	57.61	950.27	0.03	-32.97	0.05

TABLE IV-2b

SELECTED LEVELLING LINE NO. '8'-GERMANY**AUSTRIA-PART OF THE UELN. RAPP. 1961.

TABLE OF COMPUTED GRAVITY CORRECTIONS

FOR EACH INDIVIDUAL LEVELLING-SECTION ALONG THE GIVEN LEVELLING-ROUTE

TOTAL NUMBER OF LEVELLING-SECTIONS ALONG THE LINE = 14

LEVELLING-SECTION FROM	TO	LENGTH (KM)	AZIMUTH (DEGREES)	GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT STANDARD (MM/KM.)	VIGNAL MAGNITUDE (MM)	VIGNAL STANDARD (MM/KM.)	DYNAMIC MAGNITUDE (MM)	DYNAMIC STANDARD (MM/KM.)
LB57	TB16	13.06	164.08	9.9355	2.7488	-1.5811	-0.4374	-1.9453	-0.5382
TB16	BO12	7.27	163.88	12.3675	4.5864	-1.8504	-0.6862	-2.2685	-0.8413
BO12	MB46	9.06	168.56	6.3117	2.0573	-0.6437	-0.2139	-0.7805	-0.2593
MB46	MB54	6.95	200.53	2.7274	1.0349	-0.7759	-0.2944	-0.9450	-0.3586
MB54	MO39	8.03	265.01	-6.9922	-2.4680	1.7994	0.6351	2.1684	0.7654
MO39	MB09	9.24	175.34	4.6375	1.5256	0.2655	0.0873	0.3071	0.1010
MB09	MB53	7.08	210.62	-0.4272	-0.1605	-1.2662	-0.4757	-1.4671	-0.5512
MB53	MB37	7.62	249.97	2.8979	1.0501	-1.2885	-0.4669	-1.5761	-0.5711
MB37	KB48	4.75	272.26	3.3225	1.5239	-0.5003	-0.2294	-0.6491	-0.2977
KB48	MB25	5.38	242.97	1.9966	0.8610	-0.0031	-0.0014	-0.0041	-0.0018
MB25	LB79	7.25	194.85	13.1989	4.9032	-1.1974	-0.4448	-1.5169	-0.5635
LB79	MB98	5.52	218.51	4.0585	1.7281	-0.8026	-0.3417	-1.0484	-0.4464
MB98	MB44	4.73	153.89	9.5888	4.4110	0.0675	0.0311	0.0856	0.0394
MB44	MB01	5.00	178.57	16.5537	7.3996	-1.1415	-0.5103	-1.3411	-0.5995

TABLE IV-2c

SELECTED LEVELLING LINE NO. '8'-GERMANY**AUSTRIA-PART OF THE UELN. RAPP. 1961.

*** S U M M A R Y O F R E S U L T S ***

TABLE OF ACCUMULATED GRAVITY-CORRECTIONS AND THEIR ACCUMULATED STANDARD DEVIATIONS

AT EACH BENCH MARK ALONG THE GIVEN LEVELLING ROUTE

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
LB57	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
TB16	13.06	4.8073	9.9355	0.0420	-1.5811	0.0028	-1.9453	0.0029
B012	20.34	5.9977	22.3030	0.0634	-3.4315	0.0042	-4.2138	0.0043
MB46	29.39	7.2106	28.6147	0.0817	-4.0752	0.0044	-4.9943	0.0046
MB54	36.34	8.0174	31.3421	0.0977	-4.8511	0.0047	-5.9393	0.0049
M039	44.36	8.8587	24.3499	0.1108	-3.0517	0.0055	-3.7709	0.0057
MB09	53.60	9.7376	28.9874	0.1213	-2.7862	0.0057	-3.4638	0.0060
MB53	60.69	10.3611	28.5602	0.1313	-4.0524	0.0060	-4.9310	0.0064
MB30	68.30	10.9920	31.4581	0.1420	-5.3409	0.0065	-6.5071	0.0068
KB48	73.06	11.3681	34.7806	0.1530	-5.8412	0.0066	-7.1562	0.0069
MB25	78.44	11.7790	36.7772	0.1636	-5.8443	0.0066	-7.1603	0.0070
LB79	85.68	12.3110	49.9761	0.1744	-7.0417	0.0071	-8.6772	0.0074
MB98	91.20	12.7011	54.0347	0.1859	-7.8444	0.0073	-9.7256	0.0077
MB44	95.92	13.0260	63.6234	0.1972	-7.7768	0.0074	-9.6400	0.0078
MB01	100.93	13.3615	80.1771	0.2083	-8.9183	0.0076	-10.9811	0.0081

TABLE IV-3a

SELECTED LEVELLING LINE NO. '9'-PART OF ALBERTA LOOP- VANICEK ET.AL., 1972.

TABLE OF GIVEN INFORMATION AT BENCH MARKS

B.MARK NUMBER	LATITUDE (+VE NORTH) DEG	MIN	SEC	LONGITUDE (+VE WEST) DEG	MIN	SEC	HEIGHT (METERS)	HT.ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A.ACCURACY (MGALS)
95-D	49	33	24.01	114	0	29.99	1137.76	0.03	-15.49	2.49
96-D	49	33	42.01	114	1	12.00	1135.11	0.03	-16.06	2.55
97-D	49	33	54.00	114	4	30.00	1169.94	0.03	-18.66	2.88
98-D	49	34	54.01	114	7	18.01	1179.61	0.03	-17.53	3.18
99-D	49	34	54.01	114	12	11.99	1164.58	0.03	-16.79	3.47
100D	49	33	18.00	114	19	36.01	1215.60	0.03	-17.22	4.67
101D	49	33	42.01	114	21	29.99	1237.27	0.03	-17.42	4.49
102D	49	36	6.01	114	24	6.01	1282.63	0.03	-12.83	3.81
103D	49	36	24.01	114	25	48.00	1287.41	0.03	-11.20	3.07
104D	49	37	18.01	114	27	54.00	1303.26	0.03	-7.49	3.32
105D	49	38	6.00	114	29	42.00	1313.96	0.03	-3.86	3.43

TABLE IV-3b

SELECTED LEVELLING LINE NO. '9'-PART OF ALBERTA LOOP- VANICEK ET.AL., 1972.

TABLE OF COMPUTED GRAVITY CORRECTIONS
FOR EACH INDIVIDUAL LEVELLING-SECTION ALONG THE GIVEN LEVELLING-ROUTE

TOTAL NUMBER OF LEVELLING-SECTIONS ALONG THE LINE = 10

LEVELLING-SECTION FROM	TO	LENGTH (KM)	AZIMUTH (DEGREES)	GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT STANDARD (MM/KM.)	VIGNAL MAGNITUDE (MM)	VIGNAL STANDARD (MM/KM.)	DYNAMIC MAGNITUDE (MM)	DYNAMIC STANDARD (MM/KM.)
95-D	96-D	1.01	303.37	-0.0272	-0.0270	0.0427	0.0425	0.0562	0.0559
96-D	97-D	4.00	275.34	12.2195	6.1126	-0.6167	-0.3085	-0.7947	-0.3976
97-D	98-D	3.85	298.80	1.2369	0.5303	-0.1782	-0.0908	-0.2277	-0.1160
98-D	99-D	5.91	270.03	-4.9041	-2.0180	0.2630	0.1082	0.3397	0.1398
99-D	100D	9.40	251.66	14.3799	4.6896	-0.8350	-0.2886	-1.1455	-0.3736
100D	101D	2.41	287.96	6.3160	4.0703	-0.3827	-0.2466	-0.4935	-0.3180
101D	102D	5.44	324.85	7.1444	3.0626	-0.6993	-0.2997	-0.9313	-0.3992
102D	103D	2.12	285.20	-0.7325	-0.5029	-0.0586	-0.0402	-0.0831	-0.0570
103D	104D	3.03	303.42	-0.2150	-0.1235	-0.1509	-0.0867	-0.2320	-0.1333
104D	105D	2.63	304.38	-1.6489	-1.0175	-0.0618	-0.0381	-0.1166	-0.0719

TABLE IV-3c

SELECTED LEVELLING LINE NO. '9'-PART OF ALBERTA LOOP- VANICEK ET.AL., 1972.

*** S U M M A R Y O F R E S U L T S ***

TABLE OF ACCUMULATED GRAVITY-CORRECTIONS AND THEIR ACCUMULATED STANDARD DEVIATIONS

AT EACH BENCH MARK ALONG THE GIVEN LEVELLING ROUTE

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS: HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
95-D	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
96-D	1.01	1.3373	-0.0272	4.1303	0.0427	0.0049	0.0562	0.0049
97-D	5.01	2.9761	12.1923	6.1236	-0.5740	0.0685	-0.7385	0.0685
98-D	8.86	3.9585	13.4292	7.9947	-0.7522	0.0717	-0.9662	0.0717
99-D	14.76	5.1104	8.5251	9.7757	-0.4893	0.0803	-0.6265	0.0803
100D	24.17	6.5382	22.9050	12.0590	-1.3742	0.1713	-1.7720	0.1713
101D	26.57	6.8562	29.2210	14.5281	-1.7570	0.1857	-2.2654	0.1857
102D	32.02	7.5255	36.3654	16.3802	-2.4562	0.2303	-3.1967	0.2303
103D	34.14	7.7709	35.6329	17.5904	-2.5148	0.2306	-3.2798	0.2306
104D	37.17	8.1085	35.4179	18.5768	-2.6657	0.2335	-3.5118	0.2335
105D	39.79	8.3900	33.7690	19.5387	-2.7276	0.2349	-3.6284	0.2349

TABLE IV-4a

SELECTED LEVELLING LINE NO.'10'-PART OF ALBERTA LOOP- VANICEK ET.AL., 1972.

TABLE OF GIVEN INFORMATION AT BENCH MARKS

B.MARK NUMBER	LATITUDE (+VE NORTH)			LONGITUDE (+VE WEST)			HEIGHT (METERS)	HT.ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A.ACCURACY (MGALS)
	DEG	MIN	SEC	DEG	MIN	SEC				
6A04	51	5	17.99	114	11	17.99	1074.39	0.03	-17.18	0.84
0085	51	4	41.99	114	10	54.01	1074.36	0.03	-17.48	0.83
7021	51	3	47.99	114	9	47.99	1142.45	0.03	-17.56	0.78
6404	51	3	6.01	114	9	47.99	1156.69	0.03	-17.79	0.78
7120	51	2	17.99	114	9	47.99	1144.62	0.03	-17.77	0.77
7820	51	2	17.99	114	8	42.00	1141.05	0.03	-17.78	0.76
8020	51	2	17.99	114	8	24.00	1129.41	0.03	-17.78	0.75
9020	51	2	17.99	114	7	0.01	1115.32	0.03	-17.53	0.73
9120	51	2	17.99	114	6	54.00	1114.38	0.03	-17.54	0.73
9320	51	2	17.99	114	6	29.99	1098.93	0.03	-17.55	0.74
0020	51	2	17.99	114	5	35.99	1051.07	0.03	-17.37	0.72
1020	51	2	17.99	114	4	18.01	1050.83	0.03	-17.39	0.72
CITY	51	2	30.01	114	4	5.99	1048.82	0.03	-17.41	0.72
H-02	51	2	42.00	114	3	36.00	1047.32	0.03	-17.36	0.69
1025	51	2	48.01	114	4	23.99	1049.82	0.03	-17.48	0.72
50-D	51	3	0.00	114	4	59.99	1049.12	0.03	-17.43	0.72

TABLE IV-4b

SELECTED LEVELLING LINE NO.'10'-PART OF ALBERTA LOOP- VANICEK ET.AL., 1972.

TABLE OF COMPUTED GRAVITY CORRECTIONS
FOR EACH INDIVIDUAL LEVELLING-SECTION ALONG THE GIVEN LEVELLING-ROUTE

TOTAL NUMBER OF LEVELLING-SECTIONS ALONG THE LINE = 15

LEVELLING-SECTION FROM	TO	LENGTH (KM)	AZIMUTH (DEGREES)	GRAVITY-CORRECTIONS: HELMERT MAGNITUDE (MM)	HELMERT STANDARD (MM/KM.)	VIGNAL MAGNITUDE (MM)	VIGNAL STANDARD (MM/KM.)	DYNAMIC MAGNITUDE (MM)	DYNAMIC STANDARD (MM/KM.)
6A04	0085	1.21	157.24	0.3211	0.2923	0.0004	0.0004	0.0007	0.0006
0085	7021	2.11	142.38	17.3151	11.0301	-1.2167	-0.8383	-1.5652	-1.0784
7021	6404	1.30	180.00	4.0039	3.5155	-0.2567	-0.2254	-0.3294	-0.2893
6404	7120	1.48	180.00	-3.1933	-2.6212	0.2187	0.1795	0.2806	0.2304
7120	7320	1.29	89.99	-0.9185	-0.8100	0.0646	0.0570	0.0829	0.0731
7820	8020	0.35	90.00	-3.0166	-5.0938	0.2111	0.3565	0.2707	0.4571
8020	9020	1.64	89.99	-3.8932	-3.0434	0.2535	0.1982	0.3256	0.2546
9020	9120	0.12	90.00	-0.2290	-0.5692	0.0169	0.0494	0.0217	0.0635
9120	9320	0.47	90.00	-3.8917	-5.6896	0.2765	0.4042	0.3556	0.5199
9320	0020	1.05	89.99	-11.9377	-11.6381	0.8520	0.8307	1.0971	1.0695
0020	1020	1.52	89.99	-0.0371	-0.0301	0.0043	0.0035	0.0056	0.0045
1020	CITY	0.44	32.23	-0.4605	-0.5949	0.0357	0.0539	0.0460	0.0694
CITY	H-02	0.69	57.62	-0.4106	-0.4937	0.0265	0.0319	0.0341	0.0410
H-02	1025	0.95	281.25	0.7264	0.7441	-0.0444	-0.0455	-0.0572	-0.0586
1025	50-D	0.79	297.85	-0.2214	-0.2486	0.0125	0.0140	0.0161	0.0180

TABLE IV-4c

SELECTED LEVELLING LINE NO.'10'-PART OF ALBERTA LOOP- VANICEK ET.AL., 1972.

*** S U M M A R Y O F R E S U L T S ***
TABLE OF ACCUMULATED GRAVITY-CORRECTIONS AND THEIR ACCUMULATED STANDARD DEVIATIONS
AT EACH BENCH MARK ALONG THE GIVEN LEVELLING ROUTE

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS: HELMERT		VIGNAL		DYNAMIC	
			MAGNITUDE (MM)	ST. DEV. (MM)	MAGNITUDE (MM)	ST. DEV. (MM)	MAGNITUDE (MM)	ST. DEV. (MM)
6A04	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0085	1.21	1.4608	0.3211	1.2937	0.0004	0.0008	0.0007	0.0010
7021	3.31	2.4208	17.6362	1.8251	-1.2163	0.0396	-1.5645	0.0396
6404	4.61	2.8557	21.6401	2.2367	-1.4730	0.0404	-1.8940	0.0404
7120	6.09	3.2833	18.4468	2.5800	-1.2543	0.0409	-1.6133	0.0410
7820	7.38	3.6131	17.5284	2.8716	-1.1896	0.0410	-1.5304	0.0410
8020	7.73	3.6979	14.5118	3.1263	-0.9785	0.0415	-1.2597	0.0415
9020	9.37	4.0705	10.6186	3.3479	-0.7250	0.0422	-0.9341	0.0422
9120	9.48	4.0959	10.3896	3.5477	-0.7081	0.0422	-0.9123	0.0422
9320	9.95	4.1957	6.4979	3.7365	-0.4316	0.0430	-0.5567	0.0430
0020	11.00	4.4119	-5.4397	3.9042	0.4204	0.0498	0.5403	0.0499
1020	12.52	4.7067	-5.4768	4.0538	0.4248	0.0498	0.5459	0.0499
CITY	12.96	4.7885	-5.9373	4.1978	0.4605	0.0498	0.5919	0.0499
H-02	13.65	4.9146	-6.3480	4.3310	0.4870	0.0499	0.6260	0.0499
1025	14.61	5.0832	-5.6215	4.4603	0.4426	0.0499	0.5688	0.0499
50-D	15.40	5.2194	-5.8429	4.5915	0.4551	0.0499	0.5849	0.0499

TABLE IV-5a

SELECTED LEVELLING LOOP NO. '5'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS.D.O..1965

TABLE OF GIVEN INFORMATION AT BENCH MARKS

B.MARK NUMBER	LATITUDE (+VE NORTH)			LONGITUDE (+VE WEST)			HEIGHT (METERS)	HT.ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A.ACCURACY (MGALS)
	DEG	MIN	SEC	DEG	MIN	SEC				
9430	45	48	31.21	74	57	56.99	188.61	0.03	-8.97	0.05
8109	45	48	25.20	74	57	6.59	194.46	0.03	-7.59	0.05
8110	45	48	34.81	74	56	33.00	198.06	0.03	-4.74	0.05
8111	45	49	7.21	74	56	23.39	197.88	0.03	-5.85	0.05
9431	45	50	47.40	74	56	2.40	221.99	0.03	-27.57	0.05
8112	45	53	43.80	74	55	42.60	220.77	0.03	-4.83	0.05
9432	45	59	34.80	74	52	55.81	224.61	0.03	-13.79	0.05
8113	45	59	59.39	74	51	17.39	223.17	0.03	-14.54	0.05
8114	45	59	13.20	74	47	15.61	221.35	0.03	-13.24	0.05
9433	45	58	11.39	74	42	47.41	216.83	0.03	-14.60	0.05
8115	45	58	34.79	74	38	1.79	201.84	0.03	-27.53	0.05
8116	45	58	28.81	74	37	55.81	197.60	0.03	-27.23	0.05
8117	45	57	51.01	74	37	26.40	190.41	0.03	-29.52	0.05
9434	45	57	33.01	74	34	41.41	185.93	0.03	-26.99	0.05
8118	45	56	30.59	74	32	48.01	189.43	0.03	-22.26	0.05
8119	45	55	55.81	74	34	10.81	188.31	0.03	-22.22	0.05
8120	45	51	4.21	74	33	45.61	248.08	0.03	-2.45	0.05
8147	45	55	55.81	74	18	35.39	424.86	0.03	-3.36	0.05
8146	45	55	34.21	74	16	23.99	289.07	0.03	-15.37	0.05
9443	45	54	9.00	74	15	1.80	236.68	0.03	-5.07	0.05
8151	45	56	22.81	74	7	25.21	206.11	0.03	-18.04	0.05
9446	45	54	20.41	74	7	50.41	159.38	0.03	-14.11	0.05
8152	45	52	46.81	74	5	17.41	174.71	0.03	-13.11	0.05
9447	45	50	59.39	74	3	56.99	167.76	0.03	-12.24	0.05
8153	45	48	7.81	74	0	29.41	110.12	0.03	-14.65	0.05
9448	45	46	38.39	74	0	17.39	94.98	0.03	-15.16	0.05
8154	45	45	45.00	73	59	9.60	74.37	0.03	-21.46	0.05
8155	45	43	7.21	74	4	55.81	75.50	0.03	-21.28	0.05
9449	45	42	16.81	74	5	58.20	64.13	0.03	-25.39	0.05
9450	45	39	11.99	74	5	34.80	73.55	0.03	-30.91	0.05

SELECTED LEVELLING LOOP NO. '5'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS,D.O.,1965

B. MARK NUMBER	LATITUDE (+VE NORTH) DEG	MIN	SEC	LONGITUDE (+VE WEST) DEG	MIN	SEC	HEIGHT (METERS)	HT. ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A. ACCURACY (MGALS)
8156	45	40	38.39	74	11	49.81	72.36	0.03	-24.08	0.05
8159	45	39	7.20	74	13	50.41	74.04	0.03	-24.34	0.05
8157	45	39	52.20	74	17	46.79	64.34	0.03	-25.27	0.05
8158	45	39	44.39	74	18	31.79	63.34	0.03	-25.32	0.05
8129	45	39	26.39	74	20	9.60	64.34	0.03	-26.58	0.05
9438	45	38	10.21	74	19	3.00	76.99	0.03	-25.38	0.05
8139	45	36	55.48	74	21	2.99	63.09	0.03	-31.50	0.05
8138	45	35	52.19	74	20	51.61	42.98	0.03	-31.36	0.05
8135	45	35	39.59	74	27	38.99	48.49	0.03	-26.82	0.05
9439	45	35	49.20	74	28	6.60	48.40	0.03	-27.35	0.05
8133	45	37	34.21	74	36	15.01	50.41	0.03	-21.67	0.05
8180	45	36	37.80	74	36	20.99	45.14	0.03	-21.56	0.05
9409	45	36	31.79	74	36	53.39	44.32	0.03	-20.42	0.05
8130	45	38	37.21	74	40	33.60	57.82	0.03	-9.12	0.05
8131	45	38	33.00	74	43	14.99	58.06	0.03	-4.24	0.05
9410	45	36	0.61	74	44	0.60	50.02	0.03	-7.89	0.05
8042	45	35	32.39	74	46	5.41	53.89	0.03	-6.80	0.05
8043	45	35	5.39	74	48	3.60	45.51	0.03	-12.26	0.05
8044	45	34	20.39	74	50	1.21	51.76	0.03	-14.72	0.05
8045	45	33	56.41	74	50	30.59	52.27	0.03	-16.60	0.05
9411	45	33	33.59	74	52	45.59	70.87	0.03	-20.22	0.05
8132	45	38	48.59	74	51	55.19	51.24	0.03	-21.84	0.05
9429	45	39	2.41	74	56	24.00	49.74	0.03	-24.09	0.05
8106	45	40	52.79	74	56	37.21	158.50	0.03	-7.92	0.05
8107	45	45	20.41	74	58	47.39	160.69	0.03	-11.20	0.05
8108	45	46	36.59	74	58	51.60	166.63	0.03	-12.98	0.05
9430	45	48	31.21	74	57	56.99	188.61	0.03	-8.97	0.05

TABLE IV-5b

SELECTED LEVELLING LOOP NO. 'S'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS,C.C.,1965

TABLE OF COMPUTED GRAVITY CORRECTIONS
FOR EACH INDIVIDUAL LEVELLING-SECTION ALONG THE GIVEN LEVELLING-ROUTE

TOTAL NUMBER OF LEVELLING-SECTIONS ALONG THE LINE = 56

LEVELLING-SECTION FROM	TO	LENGTH (KM)	AZIMUTH (DEGREES)	GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT STANDARD (MM/KM.)	VIGNAL MAGNITUDE (MM)	VIGNAL STANDARD (MM/KM.)	DYNAMIC MAGNITUDE (MM)	DYNAMIC STANDARD (MM/KM.)
9430	8109	1.10	99.67	-0.0138	-0.0131	-0.0494	-0.0470	-0.0793	-0.0755
8109	8110	0.78	67.74	-0.4092	-0.4623	-0.0226	-0.0256	-0.0410	-0.0463
8110	8111	1.02	11.72	0.2162	0.2139	0.0010	0.0010	0.0019	0.0019
8111	9431	3.13	8.33	5.8039	3.2825	-0.4109	-0.2324	-0.5340	-0.3020
9431	8112	5.46	4.48	-5.1954	-2.2228	0.0201	0.0086	0.0264	0.0113
8112	9432	11.42	18.32	2.2294	0.6598	-0.0365	-0.0108	-0.0561	-0.0166
9432	8113	2.25	70.27	0.0983	0.0655	0.0207	0.0138	0.0280	0.0187
8113	8114	5.40	105.30	-0.3857	-0.1660	0.0259	0.0112	0.0352	0.0152
8114	9433	6.08	178.26	0.0776	0.0315	0.0641	0.0260	0.0871	0.0353
9433	8115	6.19	83.27	2.0440	0.8215	0.3222	0.1295	0.3987	0.1603
8115	8116	0.22	145.11	-0.2543	-0.5361	0.1183	0.2494	0.1399	0.2951
8116	8117	1.33	151.51	0.1342	0.1164	0.2082	0.1806	0.2449	0.2125
8117	9434	3.60	98.87	-0.6781	-0.3576	0.1291	0.0681	0.1520	0.0801
9434	8118	3.11	128.27	-0.7557	-0.4285	-0.0880	-0.0499	-0.1059	-0.0600
8118	8119	2.08	238.96	-0.0547	-0.0379	0.0256	0.0177	0.0313	0.0217
8119	8120	9.02	176.54	-1.4236	-0.4740	-0.7519	-0.2504	-1.0572	-0.3520
8120	8147	21.59	65.27	13.8891	2.9891	-0.5242	-0.1128	-1.4268	-0.3071
8147	8146	2.91	103.24	-6.6909	-3.9232	1.2974	0.7607	1.9907	1.1673
8146	9443	3.17	146.04	-5.9057	-3.3161	0.5462	0.3067	0.8137	0.4569
9443	8151	10.67	67.18	1.3828	0.4233	0.3601	0.1102	0.5163	0.1580
8151	9446	3.82	188.18	-2.6804	-1.3718	0.7659	0.3920	1.0045	0.5141
9446	8152	4.39	131.21	0.4150	0.1982	-0.2128	-0.1016	-0.2911	-0.1390
8152	9447	3.74	152.38	-0.4245	-0.2194	0.0899	0.0464	0.1253	0.0648
9447	8153	6.94	139.75	-1.4860	-0.5642	0.7903	0.3000	1.0846	0.4118
8153	9448	2.77	174.62	-0.3017	-0.1812	0.2302	0.1383	0.3076	0.1847
9448	8154	2.21	138.37	0.1464	0.0986	0.3847	0.2591	0.4899	0.3299
8154	8155	8.93	236.98	0.0057	0.0019	-0.0246	-0.0082	-0.0303	-0.0102
8155	9449	2.06	220.94	0.1113	0.0775	0.2706	0.1885	0.3286	0.2290
9449	9450	5.73	174.93	0.5355	0.2237	-0.2704	-0.1130	-0.3185	-0.1331

TABLE IV-5b (continued)

SELECTED LEVELLING LOOP NO. '5'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS.D.O. .1965

LEVELLING-SECTION FROM	TO	LENGTH (KM)	AZIMUTH (DEGREES)	GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	STANDARD (MM/KM.)	VIGNAL MAGNITUDE (MM)	STANDARD (MM/KM.)	DYNAMIC MAGNITUDE (MM)	STANDARD (MM/KM.)
9450	8156	8.55	288.23	-0.5283	-0.1807	0.0333	0.0114	0.0394	0.0135
8156	8159	3.84	222.85	0.0478	0.0244	-0.0414	-0.0211	-0.0499	-0.0255
8159	8157	5.30	285.21	-0.0873	-0.0379	0.2452	0.1065	0.2947	0.1280
8157	8158	1.00	256.10	-0.0114	-0.0114	0.0260	0.0259	0.0311	0.0310
8158	8129	2.19	255.31	0.0966	0.0653	-0.0266	-0.0130	-0.0318	-0.0215
8129	9438	2.76	148.48	0.1173	0.0706	-0.3351	-0.2018	-0.3997	-0.2407
9438	8139	3.48	228.42	0.2147	0.1152	0.4031	0.2162	0.4740	0.2543
8139	8138	1.97	172.81	-0.2507	-0.1786	0.6447	0.4594	0.7475	0.5326
8138	8135	8.84	267.52	-0.1542	-0.0519	-0.1637	-0.0551	-0.1918	-0.0645
8135	9439	0.67	296.38	0.0252	0.0308	0.0025	0.0031	0.0030	0.0037
9439	8133	11.07	287.08	-0.2637	-0.0793	-0.0503	-0.0151	-0.0606	-0.0182
8133	8180	1.75	184.25	-0.0625	-0.0473	0.1162	0.0880	0.1432	0.1083
8183	9409	0.73	255.19	-0.0604	-0.0709	0.0176	0.0207	0.0218	0.0256
9409	8130	6.14	309.09	-0.4313	-0.1740	-0.2034	-0.0821	-0.2724	-0.1099
8130	8131	3.50	267.89	-0.2851	-0.1525	-0.0017	-0.0009	-0.0029	-0.0016
8131	9410	4.81	191.87	0.1020	0.0465	0.0498	0.0227	0.0909	0.0414
9410	8042	2.84	252.16	-0.0119	-0.0071	-0.0290	-0.0172	-0.0488	-0.0289
8042	8043	2.69	251.99	0.1816	0.1106	0.0815	0.0496	0.1243	0.0757
8043	8044	2.90	241.43	0.1912	0.1122	-0.0859	-0.0504	-0.1179	-0.0692
8044	8045	0.98	220.72	0.2119	0.2144	-0.0088	-0.0039	-0.0114	-0.0116
8045	9411	3.01	256.48	0.3635	0.2095	-0.3680	-0.2121	-0.4630	-0.2668
9411	8132	9.79	6.40	-0.1729	-0.0553	0.4210	0.1346	0.5213	0.1666
8132	9429	5.84	274.22	0.0987	0.0408	0.0350	0.0145	0.0426	0.0176
9429	8106	3.42	355.20	0.8669	0.4688	-1.7750	-0.9598	-2.3304	-1.2602
8106	8107	8.73	341.20	0.6143	0.2079	-0.0214	-0.0073	-0.0326	-0.0110
8107	8108	2.35	357.78	0.5181	0.3377	-0.0733	-0.0478	-0.1036	-0.0675
8108	9430	3.73	18.43	0.1656	0.0857	-0.2459	-0.1273	-0.3582	-0.1854

TABLE IV-5c

SELECTED LEVELLING LOOP NO. '5'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS,C.O.,1965

*** S U M M A R Y O F R E S U L T S ***

TABLE OF ACCUMULATED GRAVITY-CORRECTIONS AND THEIR ACCUMULATED STANDARD DEVIATIONS

AT EACH BENCH MARK ALONG THE GIVEN LEVELLING ROUTE

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
9430	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8109	1.10	1.3974	-0.0138	0.0138	-0.0494	0.0004	-0.0793	0.0006
8110	1.89	1.8273	-0.4230	0.0198	-0.0720	0.0005	-0.1203	0.0008
8111	2.91	2.2685	-0.2068	0.0244	-0.0711	0.0006	-0.1184	0.0009
9431	6.04	3.2674	5.5970	0.0287	-0.4819	0.0013	-0.6524	0.0016
8112	11.50	4.5099	0.4016	0.0329	-0.4618	0.0015	-0.6260	0.0019
9432	22.92	6.3667	2.6310	0.0366	-0.4984	0.0015	-0.6820	0.0020
8113	25.17	6.6720	2.7294	0.0400	-0.4777	0.0017	-0.6540	0.0021
8114	30.56	7.3525	2.3437	0.0431	-0.4518	0.0018	-0.6188	0.0023
9433	36.64	8.0508	2.4213	0.0459	-0.3877	0.0019	-0.5317	0.0024
8115	42.83	8.7044	4.4653	0.0484	-0.0656	0.0022	-0.1330	0.0028
8116	43.06	8.7272	4.2110	0.0505	0.0527	0.0025	0.0070	0.0031
8117	44.39	8.8608	4.3452	0.0524	0.2609	0.0028	0.2519	0.0034
9434	47.98	9.2127	3.6671	0.0541	0.3900	0.0031	0.4038	0.0037
8118	51.09	9.5067	2.9114	0.0558	0.3020	0.0032	0.2979	0.0040
8119	53.17	9.6985	2.8566	0.0574	0.3276	0.0034	0.3293	0.0041
8120	62.19	10.4888	1.4331	0.0595	-0.4243	0.0041	-0.7279	0.0048
8147	83.79	12.1740	15.3221	0.0643	-0.9485	0.0077	-2.1548	0.0080
8146	86.69	12.3836	8.6313	0.0693	0.3489	0.0091	-0.1640	0.0095
9443	89.87	12.6080	2.7256	0.0718	0.8950	0.0094	0.6496	0.0097
8151	100.54	13.3356	4.1084	0.0736	1.2551	0.0094	1.1659	0.0098
9446	104.35	13.5865	1.4280	0.0748	2.0210	0.0096	2.1704	0.0100
8152	108.74	13.8690	1.8430	0.0757	1.8082	0.0097	1.8793	0.0100
9447	112.48	14.1057	1.4186	0.0767	1.8981	0.0097	2.0046	0.0101
8153	119.42	14.5342	-0.0675	0.0774	2.6884	0.0099	3.0892	0.0103
9448	122.19	14.7020	-0.3691	0.0777	2.9186	0.0100	3.3968	0.0104
8154	124.40	14.8341	-0.2227	0.0780	3.3033	0.0100	3.8867	0.0105
8155	133.33	15.3573	-0.2170	0.0782	3.2787	0.0101	3.8564	0.0105
9449	135.39	15.4754	-0.1057	0.0783	3.5493	0.0101	4.1850	0.0106
9450	141.12	15.7995	0.4298	0.0785	3.2789	0.0102	3.8665	0.0107

TABLE IV-5c (continued)

SELECTED LEVELLING LOOP NO. '5'-EASTERN ONTARIO-GRAVITY AT BENCH MARKS,D.O.,1965

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
8156	149.66	16.2708	-0.0986	0.0787	3.3123	0.0103	3.9059	0.0108
8159	153.50	16.4782	-0.0507	0.0788	3.2709	0.0103	3.8560	0.0109
8157	158.80	16.7604	-0.1380	0.0790	3.5161	0.0104	4.1507	0.0110
8158	159.81	16.8132	-0.1494	0.0791	3.5420	0.0105	4.1818	0.0110
8129	162.00	16.9280	-0.0528	0.0793	3.5154	0.0105	4.1500	0.0111
9438	164.76	17.0716	0.0644	0.0794	3.1803	0.0106	3.7503	0.0112
8139	168.23	17.2507	0.2791	0.0796	3.5834	0.0107	4.2243	0.0113
8138	170.20	17.3514	0.0284	0.0797	4.2281	0.0108	4.9718	0.0115
8135	179.04	17.7962	-0.1258	0.0798	4.0644	0.0109	4.7800	0.0116
9439	179.71	17.8293	-0.1006	0.0798	4.0670	0.0109	4.7830	0.0116
8133	190.78	18.3702	-0.3643	0.0799	4.0167	0.0110	4.7224	0.0117
8180	192.52	18.4541	-0.4268	0.0800	4.1329	0.0110	4.8656	0.0118
9409	193.25	18.4889	-0.4872	0.0801	4.1505	0.0111	4.8874	0.0118
8130	199.39	18.7805	-0.9185	0.0801	3.9471	0.0111	4.6150	0.0119
8131	202.89	18.9445	-1.2037	0.0803	3.9455	0.0111	4.6121	0.0119
9410	207.70	19.1676	-1.1016	0.0803	3.9952	0.0111	4.7030	0.0119
8042	210.54	19.2983	-1.1136	0.0804	3.9662	0.0111	4.6542	0.0119
8043	213.23	19.4214	-0.9320	0.0805	4.0477	0.0111	4.7785	0.0119
8044	216.14	19.5532	-0.7408	0.0806	3.9618	0.0111	4.6606	0.0120
8045	217.11	19.5973	-0.5289	0.0807	3.9530	0.0112	4.6492	0.0120
9411	220.13	19.7327	-0.1653	0.0808	3.5849	0.0112	4.1862	0.0121
8132	229.91	20.1666	-0.3382	0.0809	4.0059	0.0113	4.7075	0.0121
9429	235.75	20.4210	-0.2396	0.0810	4.0409	0.0113	4.7501	0.0122
8106	239.17	20.5685	0.6273	0.0814	2.2659	0.0120	2.4196	0.0129
8107	247.90	20.9405	1.2416	0.0822	2.2445	0.0120	2.3870	0.0129
8108	250.25	21.0397	1.7597	0.0830	2.1712	0.0120	2.2834	0.0129
9430	253.98	21.1959	1.9253	0.0840	1.9253	0.0121	1.9253	0.0130

TABLE IV-6a

SELECTED LEVELLING LOOP NO. '6'-WEST GERMANY-PART OF THE UELN. KRAKIWSKY.1966.

TABLE OF GIVEN INFORMATION AT BENCH MARKS

B.MARK NUMBER	LATITUDE (+VE NORTH) DEG	MIN	SEC	LONGITUDE (+VE WEST) DEG	MIN	SEC	HEIGHT (METERS)	HT.ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A.ACCURACY (MGALS)
0207	50	8	52.19	348	55	57.61	272.37	0.03	-4.91	0.05
LB04	50	9	16.81	348	49	33.60	274.05	0.03	-11.93	0.05
LB32	50	7	25.21	348	40	57.00	285.87	0.03	-2.85	0.05
KB18	50	6	12.60	348	35	9.60	297.94	0.03	0.76	0.05
LB07	50	7	52.21	348	28	14.41	319.13	0.03	3.84	0.05
KB35	50	10	15.60	348	21	51.59	535.23	0.03	53.73	0.05
TB35	50	11	58.81	348	12	7.20	556.32	0.03	65.87	0.05
TB09	50	16	13.19	348	9	10.19	553.94	0.03	70.33	0.05
TB17	50	19	12.61	348	5	7.80	502.34	0.03	51.89	0.05
TB60	50	15	59.40	348	3	52.81	481.55	0.03	60.87	0.05
TB22	50	9	9.61	348	2	55.79	590.52	0.03	46.08	0.05
TB20	50	4	51.60	348	1	29.39	559.80	0.03	40.31	0.05
TB28	50	0	25.20	347	55	21.00	541.66	0.03	42.86	0.05
R049	49	58	19.81	347	49	52.21	547.45	0.03	29.51	0.05
R025	49	57	7.81	347	45	19.19	520.05	0.03	18.88	0.05
R012	49	52	34.21	347	39	40.79	505.33	0.03	14.60	0.05
RE15	49	49	23.99	347	43	37.20	531.05	0.03	18.94	0.05
TB06	49	43	54.01	347	49	39.00	419.31	0.03	34.24	0.05
B059	49	41	34.19	347	50	4.20	401.63	0.03	26.52	0.05
KB16	49	37	45.59	347	51	31.21	388.92	0.03	21.89	0.05
TB69	49	35	1.21	347	50	58.81	390.42	0.03	17.94	0.05
0208	49	32	25.19	347	50	15.00	377.16	0.03	9.94	0.05
MB24	49	32	48.59	347	56	22.81	394.66	0.03	13.00	0.05
MB34	49	32	55.21	347	59	52.80	403.56	0.03	15.33	0.05
MB14	49	32	44.99	348	3	16.20	411.91	0.03	15.73	0.05
TB28	49	30	37.80	348	6	6.01	438.33	0.03	10.93	0.05
MB08	49	28	18.01	348	8	31.81	381.94	0.03	-4.50	0.05
TB24	49	26	44.99	348	8	25.19	373.56	0.03	-2.50	0.05
NT21	49	25	45.01	348	11	16.80	407.67	0.03	1.03	0.05
LB31	49	24	7.20	348	14	28.79	537.21	0.03	16.52	0.05

TABLE IV-6a (continued)

SELECTED LEVELLING LOOP NO. '6'-WEST GERMANY-PART OF THE USLN. KRAKIWSKY,1966.

B.MARK NUMBER	LATITUDE (+VE NORTH) DEG	MIN	SEC	LONGITUDE (+VE WEST) DEG	MIN	SEC	HEIGHT (METERS)	HT.ACCURACY (METERS)	FREE-AIR ANOM (MGALS)	F.A.A.ACCURACY (MGALS)
NT11	49	22	55.20	348	17	58.20	501.91	0.03	11.90	0.05
LB21	49	22	3.00	348	18	49.79	429.46	0.03	3.38	0.05
NT14	49	21	24.01	348	23	49.20	522.03	0.03	13.28	0.05
LB33	49	19	21.00	348	28	4.19	552.63	0.03	13.93	0.05
LB39	49	16	49.80	348	33	38.99	423.89	0.03	-1.14	0.05
LB28	49	12	36.61	348	32	47.40	413.49	0.03	-2.32	0.05
TE16	49	11	24.61	348	39	1.80	426.26	0.03	-0.29	0.05
MB11	49	12	4.79	348	42	45.61	424.98	0.03	-1.29	0.05
MB30	49	11	23.39	348	48	26.39	379.57	0.03	-2.98	0.05
D239	49	11	28.21	348	48	37.19	379.63	0.03	-2.87	0.05
MB44	49	13	3.00	348	51	18.61	353.69	0.03	-5.74	0.05
KB10	49	16	2.39	348	55	7.79	325.89	0.03	-7.51	0.05
MB39	49	19	55.20	348	58	37.20	325.53	0.03	-7.90	0.05
MB29	49	22	48.61	348	58	9.59	304.59	0.03	-6.06	0.05
MB92	49	26	46.79	349	0	35.39	299.31	0.03	2.11	0.05
MB20	49	30	47.99	349	1	30.00	291.30	0.03	6.52	0.05
MB13	49	33	15.01	349	0	32.40	282.52	0.03	0.31	0.05
MB47	49	36	5.40	348	59	35.41	278.74	0.03	-4.67	0.05
MB43	49	39	46.19	348	57	34.20	262.34	0.03	-7.08	0.05
B041	49	43	13.19	348	56	12.59	258.96	0.03	-7.09	0.05
KB14	49	46	23.99	348	57	57.60	252.95	0.03	-3.92	0.05
MB98	49	48	55.19	349	0	28.80	249.72	0.03	1.14	0.05
NT65	49	51	51.59	349	2	57.01	249.27	0.03	3.36	0.05
B005	49	53	44.99	349	6	43.20	237.53	0.03	0.55	0.05
TB13	49	58	19.81	349	6	49.79	245.58	0.03	-2.20	0.05
TB26	50	1	8.40	349	3	58.79	248.96	0.03	-5.22	0.05
TB51	50	6	13.79	348	59	51.61	274.20	0.03	-6.43	0.05
D207	50	8	52.19	348	55	57.61	272.37	0.03	-4.91	0.05

TABLE IV - 6b

SELECTED LEVELLING LOOP NO. '6'-WEST GERMANY-PART OF THE UELN. KRAKIWSKY,1966.

TABLE OF COMPUTED GRAVITY CORRECTIONS
FOR EACH INDIVIDUAL LEVELLING-SECTION ALONG THE GIVEN LEVELLING-ROUTE
==

TOTAL NUMBER OF LEVELLING-SECTIONS ALONG THE LINE = 57

LEVELLING-SECTION		LENGTH	AZIMUTH	GRAVITY - CORRECTIONS					
				HELMERT		VIGNAL		DYNAMIC	
FROM	TO	(KM)	(DEGREES)	MAGNITUDE (MM)	STANDARD (MM/KM.)	MAGNITUDE (MM)	STANDARD (MM/KM.)	MAGNITUDE (MM)	STANDARD (MM/KM.)
0207	LB04	7.66	84.26	2.0608	0.7445	-0.0144	-0.0052	-0.0230	-0.0083
LB04	LB32	10.82	108.52	-1.8359	-0.5581	-0.0892	-0.0271	-0.1496	-0.0455
LB32	KB18	7.26	107.97	-0.2690	-0.0998	-0.0129	-0.0048	-0.0746	-0.0277
KB18	LB07	8.80	69.49	0.5215	0.1758	0.0497	0.0167	-0.0537	-0.0198
LB07	KB35	8.80	59.72	-0.6650	-0.2242	6.3433	2.1386	5.2383	1.7661
KB35	TB35	12.02	74.56	-4.1312	-1.1913	1.2864	0.3710	1.1784	0.3398
TB35	TB09	8.61	24.03	-2.8221	-0.9619	-0.1648	-0.0562	-0.1529	-0.0521
TB09	TB17	7.33	40.85	3.7072	1.3691	-3.2156	-1.1876	-2.9519	-1.0902
TB17	TB60	6.15	166.03	-6.8353	-2.7559	-1.1954	-0.4819	-1.0889	-0.4390
TB60	TB22	12.71	174.89	21.4110	6.0052	5.9417	1.6665	5.3849	1.5103
TB22	TB20	8.15	167.84	-0.6471	-0.2266	-1.3537	-0.4740	-1.1963	-0.4189
TB20	TB28	11.02	138.27	-3.7103	-1.1176	-0.7693	-0.2317	-0.6763	-0.2037
TB28	R049	7.61	120.57	8.1341	2.9487	0.2135	0.0774	0.1841	0.0667
R049	R025	5.88	112.21	2.4454	1.0085	-0.6761	-0.2788	-0.5360	-0.2210
R025	R012	10.82	141.35	0.5160	0.1569	-0.2516	-0.0765	-0.1761	-0.0535
R012	RB15	7.54	218.81	0.7475	0.2722	0.4397	0.1601	0.3084	0.1123
RB15	TB06	12.50	215.41	-19.5328	-5.5239	-3.0303	-0.8570	-2.4590	-0.6954
TB06	R059	4.35	186.67	1.5757	0.7555	-0.5478	-0.2627	-0.4574	-0.2193
B059	KB16	7.28	193.89	0.7216	0.2675	-0.3139	-0.1164	-0.2488	-0.0922
KB16	TB69	5.12	172.70	1.7021	0.7522	0.0302	0.0134	0.0227	0.0100
TB69	0208	4.90	169.64	1.9695	0.8897	-0.1885	-0.0852	-0.1207	-0.0545
0208	MR24	7.43	275.62	0.3373	0.1237	0.2046	0.0751	0.1152	0.0423
MR24	MB34	4.23	272.80	-0.1415	-0.0688	0.1286	0.0625	0.0831	0.0404
MB34	MB14	4.10	265.60	0.6147	0.3035	0.1323	0.0653	0.0896	0.0442
MB14	TB28	5.21	221.01	4.6411	2.0341	0.3591	0.1574	0.2242	0.0983
TB28	MB08	5.22	214.21	1.1757	0.5145	-0.1851	-0.0810	0.1030	0.0451
MB08	TB24	2.88	177.34	-1.4916	-0.8794	0.0299	0.0176	0.0727	0.0429
TB24	NT21	3.92	241.83	1.6328	0.8244	-0.0256	-0.0129	-0.1998	-0.1009
NT21	LB31	4.91	232.04	6.5061	2.9362	1.1592	0.5232	0.4976	0.2246

TABLE IV-6b (continued)

SELECTED LEVELLING LOOP NO. '6'-WEST GERMANY-PART OF THE UELN. KRAKIWSKY.1966.

LEVELLING-SECTION FROM	LEVELLING-SECTION TO	LENGTH (KM)	AZIMUTH (DEGREES)	GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT STANDARD (MM/KM.)	VIGNAL MAGNITUDE (MM)	VIGNAL STANDARD (MM/KM.)	DYNAMIC MAGNITUDE (MM)	DYNAMIC STANDARD (MM/KM.)
LB31	NT11	4.77	242.25	-1.7400	-0.7964	-0.5116	-0.2342	-0.3312	-0.1516
NT11	LB21	1.92	212.84	-3.6510	-2.6353	-0.5645	-0.4075	-0.1944	-0.1403
LB21	NT14	6.16	258.76	5.2464	2.1137	0.7862	0.3167	0.3133	0.1262
NT14	LB33	6.40	233.59	3.3976	1.3432	0.4244	0.1678	0.2682	0.1060
LB33	LB09	8.22	235.41	-6.8431	-2.3868	-0.8394	-0.2928	-0.1817	-0.0634
LB09	LB28	7.89	172.40	-0.4923	-0.1752	0.0182	0.0065	0.0714	0.0254
LB28	TB16	7.90	253.68	0.3568	0.1270	-0.0170	-0.0061	-0.0822	-0.0292
TB16	MB11	4.70	285.34	0.3108	0.1434	0.0011	0.0005	0.0076	0.0035
MB11	MB30	7.02	259.53	-3.4773	-1.3127	0.0990	0.0374	0.3309	0.1249
MB30	0209	0.26	304.27	-0.0372	-0.0723	-0.0002	-0.0003	-0.0005	-0.0010
0209	MB44	4.39	311.88	-1.0988	-0.5246	0.1139	0.0544	0.2463	0.1176
MB44	KB10	7.23	320.11	-1.5417	-0.5736	0.1878	0.0699	0.3297	0.1227
KB10	MB39	8.34	329.56	0.1042	0.0361	0.0030	0.0010	0.0047	0.0016
MB39	MB09	5.39	5.94	-2.0995	-0.9046	0.1491	0.0643	0.2560	0.1103
MB09	MB92	7.92	338.24	-2.8772	-1.0222	0.0107	0.0038	0.0375	0.0133
MB92	MB20	7.53	351.61	-1.8695	-0.6812	-0.0352	-0.0128	0.0057	0.0021
MB20	MB13	4.69	14.30	1.2421	0.5737	-0.0305	-0.0141	0.0142	0.0066
MB13	MB47	5.39	12.26	1.1835	0.5099	0.0085	0.0036	0.0277	0.0119
MB47	MB43	7.24	19.61	-0.3463	-0.1287	0.0983	0.0365	0.1820	0.0676
MB43	B041	6.60	14.34	-0.2002	-0.0779	0.0245	0.0095	0.0417	0.0162
B041	KB14	6.26	340.38	-1.1775	-0.4707	0.0338	0.0135	0.0644	0.0257
KB14	MB88	5.56	327.10	-1.4822	-0.6283	0.0046	0.0020	0.0211	0.0089
MB88	NT65	6.20	331.50	-0.5904	-0.2371	-0.0010	-0.0004	0.0013	0.0005
NT65	B005	5.72	307.83	0.0450	0.0188	-0.0234	-0.0098	0.0366	0.0153
B005	TB13	8.49	359.11	1.1231	0.3854	-0.0067	-0.0023	-0.0479	-0.0164
TB13	TB26	6.22	33.16	0.9502	0.3809	-0.0127	-0.0051	-0.0301	-0.0121
TB26	TB51	10.64	27.50	1.8295	0.5609	-0.1496	-0.0459	-0.2788	-0.0855
TB51	0207	6.75	43.50	-0.5375	-0.2069	0.0107	0.0041	0.0199	0.0077

TABLE IV-6c

SELECTED LEVELLING LOOP NO. '6'-WEST GERMANY-PART OF THE UELN. KRAKIWSKY.1966.

*** S U M M A R Y O F R E S U L T S ***

TABLE OF ACCUMULATED GRAVITY-CORRECTIONS AND THEIR ACCUMULATED STANDARD DEVIATIONS

AT EACH BENCH MARK ALONG THE GIVEN LEVELLING ROUTE

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
0207	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
LB04	7.66	3.6815	2.0608	0.0197	-0.0144	0.0004	-0.0230	0.0006
LB32	18.48	5.7182	0.2249	0.0282	-0.1035	0.0007	-0.1725	0.0009
KB18	25.74	6.7481	-0.0441	0.0352	-0.1165	0.0008	-0.2471	0.0011
LB07	34.55	7.8173	0.4774	0.0417	-0.0668	0.0011	-0.3058	0.0013
KB35	43.34	8.7563	-0.1876	0.0519	6.2765	0.0081	4.9325	0.0081
TB35	55.37	9.8966	-4.3187	0.0651	7.5629	0.0086	6.1109	0.0085
TB09	63.98	10.6380	-7.1408	0.0765	7.3981	0.0091	5.9579	0.0089
TB17	71.31	11.2311	-3.4336	0.0855	4.1825	0.0096	3.0061	0.0095
TB60	77.46	11.7055	-10.2688	0.0925	2.9872	0.0100	1.9172	0.0098
TB22	90.17	12.6295	11.1421	0.1003	8.9288	0.0110	7.3021	0.0103
TB20	98.33	13.1882	10.4951	0.1086	7.5751	0.0112	6.1058	0.0109
TB28	109.35	13.9078	6.7848	0.1156	6.8058	0.0114	5.4295	0.0111
R049	116.96	14.3836	14.9189	0.1221	7.0194	0.0115	5.6136	0.0112
R025	122.84	14.7406	17.3643	0.1281	6.3433	0.0116	5.0776	0.0112
R012	133.66	15.3761	17.8803	0.1333	6.0917	0.0116	4.9015	0.0113
RB15	141.20	15.8038	18.6278	0.1385	6.5314	0.0117	5.2100	0.0113
TB06	153.70	16.4887	-0.9050	0.1427	3.5012	0.0125	2.7510	0.0121
B059	158.05	16.7204	0.6706	0.1457	2.9534	0.0125	2.2936	0.0121
KB16	165.32	17.1009	1.3923	0.1485	2.6395	0.0126	2.0449	0.0122
TB69	170.44	17.3637	3.0943	0.1511	2.6697	0.0126	2.0676	0.0122
0208	175.34	17.6115	5.0638	0.1536	2.4812	0.0126	1.9469	0.0122
MR24	182.77	17.9807	5.4011	0.1561	2.6858	0.0127	2.0621	0.0122
MB34	187.00	18.1874	5.2596	0.1588	2.8144	0.0127	2.1452	0.0122
MB14	191.10	18.3858	5.8743	0.1615	2.9466	0.0127	2.2347	0.0123
TB28	196.31	18.6345	10.5154	0.1644	3.3057	0.0128	2.4589	0.0123
MB08	201.53	18.8807	11.6911	0.1670	3.1206	0.0129	2.5620	0.0125
TB24	204.40	19.0150	10.1995	0.1692	3.1505	0.0129	2.6347	0.0125
NT21	208.33	19.1966	11.8324	0.1716	3.1249	0.0130	2.4349	0.0125
LB31	213.24	19.4215	18.3385	0.1749	4.2841	0.0138	2.9325	0.0134

TABLE IV-6c (continued)

SELECTED LEVELLING LOOP NO. '6'-WEST GERMANY-PART OF THE UELN, KRAKIWSKY,1966.

BENCH-MARK NUMBER	ACCUMULATED LENGTH (KM)	ACCUMULATED ST. ERROR (MM)	*ACCUMULATED* GRAVITY-CORRECTIONS HELMERT MAGNITUDE (MM)	HELMERT ST. DEV. (MM)	VIGNAL MAGNITUDE (MM)	VIGNAL ST. DEV. (MM)	DYNAMIC MAGNITUDE (MM)	DYNAMIC ST. DEV. (MM)
NT11	218.01	19.6377	16.5986	0.1789	3.7726	0.0139	2.6013	0.0135
LB21	219.93	19.7239	12.9476	0.1820	3.2080	0.0142	2.4069	0.0137
NT14	226.09	19.9983	18.1940	0.1852	3.9942	0.0146	2.7202	0.0142
LB33	232.49	20.2793	21.5916	0.1893	4.4186	0.0146	2.9884	0.0142
LB09	240.71	20.6347	14.7485	0.1925	3.5792	0.0154	2.8067	0.0150
LB28	248.60	20.9702	14.2562	0.1949	3.5974	0.0154	2.8781	0.0150
TB16	256.50	21.3007	14.6131	0.1972	3.5804	0.0154	2.7959	0.0150
MB11	261.20	21.4949	14.9239	0.1996	3.5814	0.0154	2.8035	0.0150
MB30	268.21	21.7817	11.4466	0.2017	3.6804	0.0155	3.1344	0.0151
0209	268.48	21.7925	11.4094	0.2036	3.6802	0.0155	3.1339	0.0151
MB44	272.87	21.9698	10.3106	0.2053	3.7941	0.0155	3.3803	0.0151
KB10	280.09	22.2588	8.7689	0.2067	3.9819	0.0156	3.7100	0.0152
MB39	288.44	22.5879	8.8731	0.2081	3.9849	0.0156	3.7147	0.0152
MB09	293.82	22.7978	6.7736	0.2093	4.1340	0.0156	3.9708	0.0152
MB92	301.75	23.1032	3.8963	0.2104	4.1448	0.0156	4.0083	0.0152
MB20	309.28	23.3898	2.0269	0.2115	4.1096	0.0156	4.0140	0.0152
MB13	313.97	23.5664	3.2689	0.2125	4.0791	0.0156	4.0282	0.0152
MB47	319.35	23.7677	4.4524	0.2135	4.0875	0.0156	4.0559	0.0152
MB43	326.59	24.0357	4.1061	0.2144	4.1359	0.0156	4.2379	0.0152
B041	333.20	24.2774	3.9059	0.2152	4.2104	0.0156	4.2797	0.0153
KB14	339.45	24.5043	2.7284	0.2160	4.2441	0.0156	4.3440	0.0153
MB88	345.02	24.7043	1.2462	0.2168	4.2488	0.0156	4.3651	0.0153
NT65	351.22	24.9254	0.6558	0.2175	4.2478	0.0156	4.3664	0.0153
BC05	356.94	25.1274	0.7008	0.2182	4.2244	0.0156	4.4030	0.0153
TB13	365.43	25.4246	1.8239	0.2189	4.2178	0.0156	4.3551	0.0153
TB26	371.65	25.6402	2.7741	0.2196	4.2051	0.0156	4.3250	0.0153
TB51	382.29	26.0046	4.6036	0.2205	4.0554	0.0156	4.0462	0.0153
0207	389.04	26.2331	4.0661	0.2213	4.0661	0.0157	4.0661	0.0153

APPENDIX V

ALTERNATIVE DERIVATION OF THE RIGOROUS EXPRESSIONS FOR THE GRAVITY CORRECTIONS

The purpose of this Appendix is to derive the expressions for the gravity corrections via an approach alternative to the one followed in section 4.2

The dynamic correction DC_{ij} (based on actual gravity) to the levelled height difference can be expressed [e.g. Vaníček, 1972] as:

$$DC_{ij} = \frac{\bar{g}_{ij}}{G} \Delta h_{ij} - \Delta h_{ij} \quad . \qquad \text{(V-1)}$$

Using the normal gravity γ^* (computed from the USC&GS formula - adopted in Canada), the corresponding expression for $\tilde{DC}_{ij}$ (based on normal gravity) will be:

$$\tilde{DC}_{ij} = \frac{\bar{\gamma}^*_{ij}}{G} \Delta h_{ij} - \Delta h_{ij} \quad . \qquad \text{(V-2)}$$

Hence, the dynamic gravity correction is:

$$GC^D_{ij} = DC_{ij} - \tilde{DC}_{ij} = \frac{\Delta h_{ij}}{G} (\bar{g}_{ij} - \bar{\gamma}^*_{ij}) . \qquad \text{(V-3)}$$

Realizing that:

$$\overline{\Delta g}^F_{ij} = \bar{g}_{ij} - \bar{\gamma}_{ij} \qquad ,$$

where γ is referred to the 1967 formula, and assuming that the vertical gradients of γ and γ^* are very close such that we can write:

$$\bar{\gamma}^*_{ij} = \bar{\gamma}_{ij} - \overline{\delta\gamma}_{o,ij} ,$$

equation (V-3) becomes:

$$GC^D_{ij} = \frac{\Delta h_{ij}}{G} [\overline{\Delta g}^F_{ij} + \overline{\delta\gamma}_{o,ij}] \ , \qquad \text{(V-4)}$$

which is identical to equation (4-13).

Similarily, the Helmert correction HC_{ij} (based on actual gravity) can be expressed [e.g. Vaníček, 1972] as:

$$HC_{ij} = DC_{ij} + \frac{\bar{g}'^H_i - G}{G} h_i - \frac{\bar{g}'^H_j - G}{G} h_j \ , \qquad \text{(V-5)}$$

where:

$$\bar{g}'^H_i = g_i + 0.0424 \ h_i \ . \qquad \text{(V-6)}$$

The corresponding expression for $\tilde{HC}_{ij}$, based on normal gravity, will be:

$$\tilde{HC}_{ij} = \tilde{DC}_{ij} + \frac{\bar{\gamma}'^*_i - G}{G} h_i - \frac{\bar{\gamma}'^*_j - G}{G} h_j \ , \qquad \text{(V-7)}$$

where:

$$\bar{\gamma}'^*_i = \gamma^*_{o,i} - 0.1543 \ h_i \ . \qquad \text{(V-8)}$$

The Helmert gravity correction can be thus written as:

$$GC^H_{ij} = HC_{ij} - \tilde{HC}_{ij} = GC^D_{ij} + \frac{1}{G} [(\bar{g}'^H_i - \bar{\gamma}'^*_i) h_i - (\bar{g}'^H_j - \bar{\gamma}'^*_j) h_j] \ . \qquad \text{(V-9)}$$

Realizing that:

$$\Delta g^F_i = g_i - \gamma_{o,i} + 0.3086 \ h_i \ ,$$

and using equations (V-6) and (V-8), we can write:

$$(\bar{g}'^H_i - \bar{\gamma}'^*_i) = \Delta g^F_i + \delta\gamma_{o,i} - 0.1119 \ h_i \ . \qquad \text{(V-10)}$$

In addition, we note that:

$$h_i = \bar{h}_{ij} - \frac{\Delta h_{ij}}{2} \ ,$$

and

$$h_j = \bar{h}_{ij} + \frac{\Delta h_{ij}}{2} \ .$$

Thus, substitution in (V-9) provides:

$$GC^{H}_{ij} = GC^{D}_{ij} + \frac{1}{G} [(\bar{h}_{ij}(-\Delta\Delta g^{F}_{ij} - \Delta\delta\gamma_{o,ij} + 0.2238\ \Delta h_{ij}) - \Delta h_{ij}(\overline{\Delta g}^{F}_{ij}+\overline{\delta\gamma}_{o,ij})] . \tag{V-11}$$

Substituting for GC^{D} from equation (V-4), the Helmert gravity correction becomes:

$$GC^{H}_{ij} = -\frac{\bar{h}_{ij}}{G} [\Delta\Delta g^{F}_{ij} + \Delta\delta\gamma_{o,ij} - 0.2238\ \Delta h_{ij}] , \tag{V-12}$$

which is identical to equation (4-27).

Now the Vignal correction VC_{ij} (based on actual gravity) is given [e.g. Vaníček, 1972] by:

$$VC_{ij} = DC_{ij} + \frac{\bar{\gamma}'^{V}_{i}-G}{G} h_i - \frac{\bar{\gamma}'^{V}_{j}-G}{G} h_j , \tag{V-13}$$

where:

$$\bar{\gamma}'^{V}_{i} = \gamma_{o,i} - 0.1543\ h_i . \tag{V-14}$$

The corresponding expression for $V\tilde{C}_{ij}$, based on normal gravity will be:

$$V\tilde{C}_{ij} = D\tilde{C}_{ij} + \frac{\bar{\gamma}'^{*}_{i}-G}{G} h_i - \frac{\bar{\gamma}'^{*}_{j}-G}{G} h_j , \tag{V-15}$$

in which $\bar{\gamma}'^{*}_{i}$ is given by equation (V-8). Consequently, the Vignal gravity correction can be expressed as:

$$GC^{V}_{ij} = GC^{D}_{ij} + \frac{1}{G} [(\bar{\gamma}'^{V}_{i} - \bar{\gamma}'^{*}_{i})h_i - (\bar{\gamma}'^{V}_{j} - \bar{\gamma}'^{*}_{j})\ h_j)] . \tag{V-16}$$

From equations (V-8) and (V-14), we can write:

$$(\bar{\gamma}'^{V}_{i} - \bar{\gamma}'^{*}_{i}) = \delta\gamma_{o,i} . \tag{V-17}$$

Using the relations between h_i, h_j and $\bar{h}_{ij}$, as mentioned earlier, and substituting in (V-16), we obtain:

$$GC^{V}_{ij} = GC^{D}_{ij} + \frac{1}{G} [-\bar{h}_{ij}\ \Delta\delta\gamma_{o,ij} - \Delta h_{ij}\ \delta\bar{\gamma}_{o,ij}] . \tag{V-18}$$

Using again equation (V-4), the Vignal gravity correction becomes:

$$GC^{V}_{ij} = \frac{1}{G} [\Delta h_{ij} \, \overline{\Delta g}^{F}_{ij} - \bar{h}_{ij} \, \Delta\delta\gamma_{o,ij}] \, , \qquad \text{(V-19)}$$

which is identical to equation (4-36).