COMBINATION OF GEODETIC NETWORKS

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PREFACE

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COMBINATION OF GEODETIC NETWORKS

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PREFACE

This report is an unaltered version of the author's doctoral thesis entitled "A Study of the Combination of Terrestrial and Satellite Geodetic Networks". A copy of this report was submitted to the Geodetic Surveys of Canada in partial fulfillment of a Research Contract.

The thesis advisor and supervisor of the research carried out was Dr. E. J. Krakiwsky. Acknowledgement of the assistance rendered by others is given in the ACKNOWLEDGEMENTS.

ABSTRACT

The rigorous combination of terrestrial and satellite geodetic networks is not easily accomplished. There are many factors to be considered. The more important are how to deal with terrestrial networks that are separated into horizontal and vertical components which are not usually coincident; the relation of each component to a different datum; and the existence of unmodeled systematic errors in terrestrial observables. Satellite networks are inherently three-dimensional and are relatively free of systematic errors. In view of these facts, and with present practical considerations in mind, fourteen alternate mathematical models for the combination of terrestrial and satellite geodetic networks are investigated, catalogued and categorized in this report.

To understand the reasoning behind the formulation of the models presented and the interpretation of the results obtained, some basic definitions and properties of datums, and satellite and terrestrial networks are presented.

Based on previous investigations and the author's interpretation of the problem of combining geodetic networks, the models under study are split into two major groups. The first group treats datum

transformation parameters as known, while the second includes them as unknowns to be estimated in the combination procedure. Each model is investigated in terms of its dimensionality, unknown parameters to be estimated, observables, and the estimation procedure utilized.

The group of three-dimensional models that treat the datum transformation parameters as unknowns to be estimated are themselves separated into two parts. The Bursa, Molodensky, and Veis models contain only one set of rotation parameters each, while the Hotine, Krakiwsky-Thomson, and Vanicek-Wells models each contain two sets of unknown rotations. For the combination of terrestrial and satellite networks, the latter three models represent physical reality.

The models that are not three-dimensional do not take advantage of the inherent tri-dimensionality of satellite networks. Thus, when the satellite network data is split into horizontal and vertical components for combination with terrestrial data, the covariance between the components is omitted. Even though the use of two and one-dimensional combination models are required at present due to the sparseness of adequate terrestrial data and the need for the solution of practical problems, it is not recommended for the future.

The Bursa model is recommended for the combination of two or more satellite networks. However, when combining terrestrial and satellite networks, when datum transformation parameters are unknown, none of the Bursa, Molodensky, or Veis models are adequate. In this case, the Hotine or Krakiwsky-Thomson model which parameterize the lower order systematic errors in the terrestrial network, should be used. To combine several terrestrial datums and one satellite datum and determine the orientation of these with respect to the Average Terrestrial system, the Vanicek-Wells model should be used.

A combination of the Krakiwsky-Thomson and Vanicek-Wells models is seen to be the best, from a theoretical point of view, for the combination of a satellite and several terrestrial networks. Such a solution will yield the datum transformation parameters between each of the datums involved, the orientation of each datum with respect to the Average Terrestrial coordinate system, and parameters representing the overall systematic orientation and scale errors of each terrestrial network.

No substantiative conclusions could be given based on the numerical testing carried out. A sparseness of adequate data prevented this. The numerical testing has not been wasted, however. The type and quality of data required for several models has been demonstrated. Further, the available data was utilized to substantiate the fact that the proposed solution of the Krakiwsky-Thomson model is possible.

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0. INTRODUCTION

0.1 Background

In the broad spectrum of activities covered by geodesy, one of the primary tasks is the establishment of geodetic networks. These networks, which may be of a local or regional nature, or even of global extent, have a variety of uses in the realms of both scientific and applied geodesy. The establishment of geodetic networks and the datums to which they are referred are massive tasks. Using only classical terrestrial observables, the problems encountered are numerous. The 1927 North American Datum and associated horizontal networks are an example of the results obtained using limited terrestrial observables and limited computing facilities. Three-dimensional satellite geodetic networks are a new tool that can be used by geodesists in the establishment of terrestrial geodetic datums and networks.

The notion of combining terrestrial and satellite geodetic networks, for the purpose of solving some of the problems associated with terrestrial datums and networks, began with the establishment of the first geodetic satellite networks. Some of the first mathematical models for the combination process were applied in the investigation of the position and orientation of the terrestrial network datum with respect to that of the satellite network [Bursa, 1962; Bursa, 1967; Lambeck, 1971]. In many instances combination models were derived utilized in conjunction with other geodetic investigations [Veis, 1960; Molodensky et al, 1962; Badekas, 1969]. As the accuracy and density of satellite determined geodetic networks increased, the mathematical models used to combine them with their terrestrial counterparts have become more varied and sophisticated. For example, there are those that parameterize both the position and orientation of the terrestrial network datum with respect to the satellite network datum and the unknown systematic errors in the terrestrial networks [Hotine, 1969; Krakiwsky and Thomson, 1974]. Another is useful in the combination of several terrestrial networks with one satellite network in which the orientation of their datums with respect to the Average Terrestrial coordinate system is determined [Wells and Vanicek, 1975]. In the future, as satellite methods yield coordinates of centimetre accuracy, it is envisaged that rigorously combined satellite and terrestrial networks, along with the results of new technology such as VLBI, will be important tools in the establishment of three-dimensional, time-varying, geodetic coordinates that are necessary for geophysical and geodynamic purposes.

0.2 Objective and Methodology

What mathematical model should be used for the rigorous combination of satellite and terrestrial geodetic networks? There is no simple answer to this. There are many models available and the use of one or another of them is dependent on several factors, not the least of which are the ultimate objectives of the user. To be able to choose a particular model to solve a certain problem, the user should be aware of the implications of the choice. The objective of this study is to outline how to choose an appropriate mathematical model for the combination of satellite and terrestrial geodetic networks and why certain models should be used in different circumstances.

The main method used in this report is to catalogue, categorize, analyse, and test several models. This approach of covering the broad spectrum of the combination of satellite and terrestrial geodetic networks as opposed to intense investigation of one or two mathematical models was arrived at because of certain circumstances. After some preliminary research, it was found that various opinions existed regarding the foundation, formulation, use, and interpretation of several models. These points required clarification. Further, it was found that there was insufficient data to adequately investigate any one model in which the final analysis and conclusions would be based on the solution of an actual combination of some existing satellite and terrestrial networks. These facts lead to the present format of the study.

To accomplish the stated objective using the aforementioned methodology, several tasks have to be completed. One of these is to substantiate why satellite geodetic networks should be combined with their terrestrial counterparts and to set out the concepts upon which the combination mathematical models are based. This involves the definition, in terms of current geodetic thought, of geodetic datums and the parameters that are used to define them. Chapter 1 is devoted to this. In Chaper 2, geodetic networks (terrestrial and satellite), their relationships with their respective datums and their inherent properties, are described. Section I is concluded by Chapter 3 in which the rationale for combining geodetic networks is given, along with a classification and listing of several models.

Another task is to examine several combination models in detail in order that a logical scheme of categorization can be produced. This involves the study of a priori assumptions, dimensionalities of models, treatment of unknown parameters such as datum transformation elements and systematic errors in the terrestrial networks,and the interpretation of results. These are the underlying considerations in Section II and Section III. In the former, three separate chapters deal with the combination of terrestrial and satellite networks when the transformation parameters between their respective datums are considered known, while in the latter, two chapters deal with models in which datum transformation parameters are treated as unknowns to be solved for in the combination procedure. Section IV, TEST RESULTS, presents some numerical tests using data from North American terrestrial and satellite geodetic networks.

The fulfillment of the objective of this study is attained in Chapter 9. Here, the various models are presented in three tables. These outline the proper application and interpretation of the various models considered for the combination of terrestrial and satellite geodetic networks.

0.3 Scope

Several events in geodesy prompted this research. By the late 1960's, the coordinates of terrestrial points were being determined with 5 m (1 σ) accuracy using satellite methods. Today, 1 m standard deviations are commonplace and decimetre accuracy is predicted for the near future. The problems inherent in the North American terrestrial geodetic network and the decision to redefine it was another contributing factor. It was recognized that the three-dimensional satellite networks could contribute invaluably in the positioning and orientating of a new geodetic datum and in the definition of a more accurate and homogeneous terrestrial geodetic network. Finally, there was the fact that the geodetic record, of which geodetic networks are an integral part, was being utilized more frequently in the solution of related scientific and practical problems. In order that this contribution be more valuable, geodesists must move towards the so-called four-dimensional system - a rigorous system of three-dimensional, time-varying coordinates.

As explained previously, the aim of this study is to cover the broad spectrum of the combination of geodetic networks. However, it was carried out within a certain framework. To have immediate practical value it was decided to devote a major portion of the study to combination models in which the ultimate objective was the positioning and orienting of a geodetic datum and the definition of a more accurate and homogeneous terrestrial geodetic network. Further, models in which presently available geodetic network data could be realistically used were given priority. Due to the inherently three-dimensional nature of satellite networks and geodetic datums, a concentrated effort was made on the three-dimensional models.

The aforementioned constraints have not inhibited this study. Combination models and procedures utilized in other studies are included. Several new models and variations of older ones are presented. Detailed explanations of previously used models, lacking in some studies, are given. The estimation techniques required to obtain solutions for all models are explained. Test results, and their interpretation, are given for several solutions.

0.4 Contributions

This study has resulted in several contributions to the subject of the combination of satellite and terrestrial geodetic networks. Nine of these, considered to be the most significant are:

(i) a comprehensive description of classical and contemporary geodetic network datums and their positioning and orienting in the earth body;

(ii) an enumeration of the sources of systematic errors in the observables used to define terrestrial networks;

(iii) the discovery of some shortcomings in several of the presently used combination models;

(iv) an explanation of the differences amongst presently used three-dimensional combination models;

(v) an alternative derivation of the Hotine combination model;

(vi) the development of a new model for the combination of terrestrial and satellite networks in which the lower order effects of systematic errors in the terrestrial network are modeled by three rotations and a scale difference parameter;

(vii) the generation of numerical results from the solution of several models using the same data and an explanation of the differences obtained;

(viii) the cataloguing and categorizing of fourteen models for the combination of terrestrial and satellite geodetic networks;

(ix) the construction of tables that can be used to choose a correct model, under a given set of conditions, for the combination of satellite and terrestrial geodetic networks.

SECTION I

GEODETIC DATUMS, GEODETIC NETWORKS, AND COMBINATION PROCEDURES

1. GEODETIC DATUMS

The word "datum" is defined as "a real or assumed thing used as a basis for calculations" [Webster's, 1951]. The definition of a geodetic datum is then that "thing" to which geodetic computations are referred. In the past there has been much confusion regarding geodetic datums, particularly with regards to their relationship with geodetic networks [e.g. Jones, 1973]. This was brought on, in part, by the mixing of the datum with its position and orientation within the earth body, and with the mixing of the networks themselves and the datum to which they were referred.

The aim here is to first define the geodetic datums presently used in North America for terrestrial geodetic networks (horizontal and vertical), to point out the relationships between them, and to show the connection between present geodetic coordinates and those of a unified three-dimensional coordinate system. Second, the classical means of positioning and orienting a horizontal geodetic datum in the earth body is presented. The classical parameters are related to those of contemporary three dimensional methods. The vertical datum, and its position, is of no less importance. However, due to the complexities of the problems related to it [e.g. Bomford, 1971], a detailed discussion was deemed beyond the scope of this work.

Finally, the modern concept of a datum required for three dimensional geodesy is presented. The means of establishing datums for a satellite networks are given, with reference to two specific examples. The position and orientation of the two datums with respect to each other are discussed in detail.

1.1 Classical Geodetic Datums

During the late seventeenth and early eighteenth centuries, after having been considered in the past a plane, a convex disk, and a sphere, the earth was determined to be ellipsoidal in shape. Further work in the nineteenth and early twentieth centuries by mathematicians and geodesists showed that the earth's shape is best represented by one of its equipotential surfaces, the geoid.

The historical developments in the determination of the size and shape of the earth, and numerous other physical problems, have led to the traditional splitting of the triplet of coordinates used to describe the positions of terrain points into horizontal and vertical components. Further, due to the inherent differences between the respective terrestrial observables used in the different mathematical models for horizontal and vertical networks, two separate geodetic datums must be defined.

The geodetic datum used for classical horizontal terrestrial networks is a rotational ellipsoid whose size and shape are traditionally given by the lengths of its semi-major and semi-minor axes, a and b

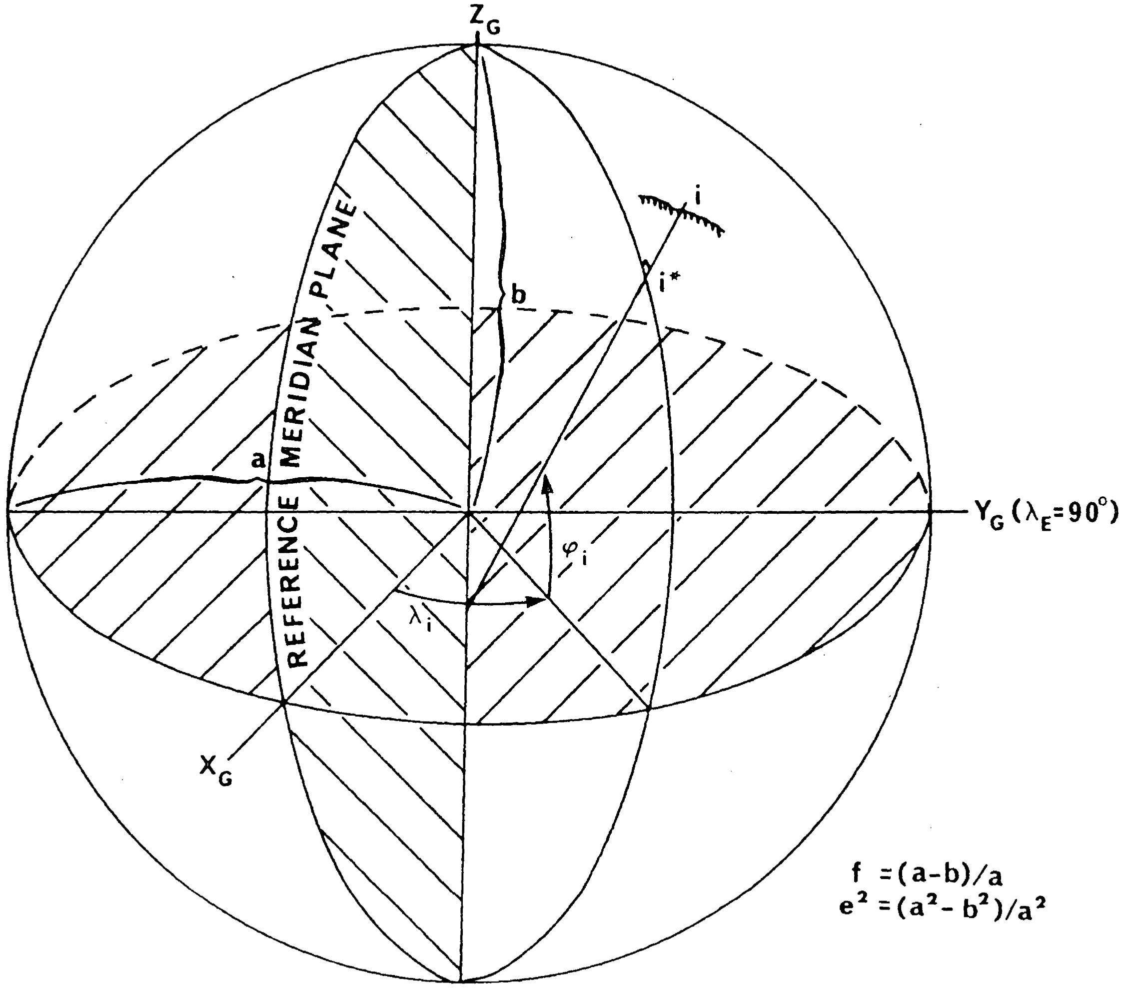

Figure 1-1

The Reference Ellipsoid, Geodetic Latitude and Longitude, and the Geodetic Coordinate System respectively (Figure 1-1) or its semi-major axis and flattening, f. Horizontal network computations are carried out on the surface of this reference ellipsoid. Its size and shape, and position in the earth body is generally such that it is a "best fitting" ellipsoid which approximates the geoid most closely. This may refer to the whole earth or a particular region of it [Heiskanen and Moritz, 1967]. The determination of the dimensions of the reference ellipsoid is a complex problem in its own right and is not covered in this report.

For this datum to be used for terrestrial network computations, its position and orientation with respect to some earth fixed coordinate system must be given. This may be accomplished via some observations and adherence to certain conditions at a terrestrial network initial point (1.3). Thus, a classical horizontal geodetic network datum is completely defined by the size and shape of the reference ellipsoid and its position and orientation.

The geodetic datum used for vertical networks in North America is nominally the geoid. The geoid is defined as that equipotential surface of the earth which "most nearly coincides with the undisturbed mean sea level" [Mueller and Rockie, 1966]. The delimitation of the geoid, as a base for vertical networks, is resolved via the determination of mean sea level at tide-gauge stations [Bomford, 1971; Ku, 1970; Lennon, 1974]

The distance between the reference ellipsoid and geoid at any point, the geoidal height N (Figure 1-2), is the "connecting link" between classical horizontal and vertical geodetic network

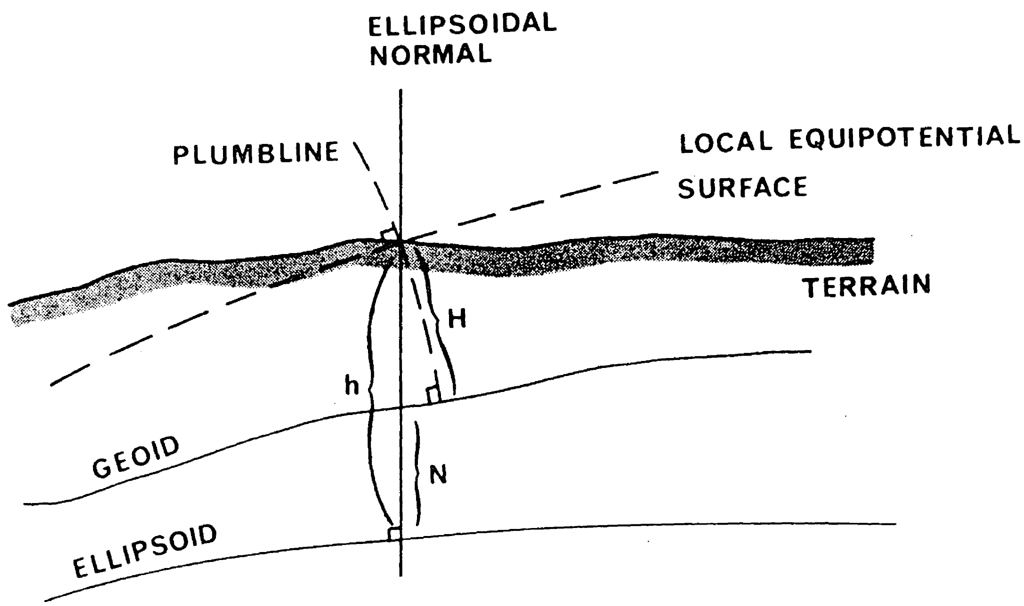

Figure 1-2

Reference Surfaces and Heights

coordinates. That is, to express the position of a terrain point relative to one datum, the aforementioned quantity must be known. In the next chapter (1.2), the use of the geoidal height in relating ellipsoidal geodetic coordinates and orthometric height to Cartesian coordinates is given.

1.2 Ellipsoidal Geodetic Coordinates, Orthometric Height, and the Geodetic Coordinate System

The relationships amongst "classical" geodetic coordinates, and between them and Cartesian coordinates are important to the discussions of the positioning and orienting of a classical horizontal network datum and the relationship of this with three-dimensional geodetic concepts.

The horizontal position of a terrain point i is given on the surface of the reference ellipsoid as a set of curvilinear coordinates, the geodetic latitude (ϕ_i) and longitude (λ_i) [Krakiwsky and Wells, 1971] (Figure 1-1). With respect to the Geodetic Coordinate system, the point can be expressed as a triplet of Cartesian coordinates (x_i^*, y_i^*, z_i^*) in terms of the ellipsoid curvilinear coordinates by [Heiskanen and Moritz, 1967]

$$\begin{bmatrix} x_i^* \\ y_i^* \\ z_i^* \end{bmatrix} = \begin{bmatrix} N_i^* \cos \phi_i \cos \lambda_i \\ N_i^* \cos \phi_i \sin \lambda_i \\ N_i^* (1-e^2) \sin \phi_i \end{bmatrix} \tag{1-1}$$

where N_i^* is the prime vertical radius of curvature and e^2 is the square of the first eccentricity of the ellipsoid.

The vertical coordinate of a terrain point is given by its orthometric height, H [e.g. Vanicek, 1972]. In order to relate this quantity to the horizontal network datum, the geoidal height, N, must be known. The two quantities are added together to yield the ellipsoidal height, h (Figure 1-2). This simple addition procedure neglects the curvature of the actual plumbline and introduces an error of less than one millimetre in the ellipsoidal height [Heiskanen and Moritz, 1967].

The triplet (ϕ_i, λ_i, h_i) describes the position of a terrain point with respect to one datum. In terms of Cartesian coordinates, one obtains [Heiskanen and Moritz, 1967]

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} (N_i^*+h_i) \cos \phi_i \cos \lambda_i \\ (N_i^*+h_i) \cos \phi_i \sin \lambda_i \\ (N_i^*(1-e^2)+h_i) \sin \phi_i \end{bmatrix} . \qquad (1\text{-}2)$$

The Cartesian system to which the triplet (x_i, y_i, z_i) refers is called the Geodetic coordinate system (Figure 1-1). It is a right-handed coordinate system whose origin is coincident with the origin of the reference ellipsoid. The Z_G axis is directed along the minor axis of the ellipsoid, and the X_GZ_G plane is in the plane of the reference geodetic longitude. The Y_GZ_G plane is 90° east of the X_GZ_G plane, and the X_GY_G plane is coincident with the equatorial plane of the ellipsoid. The orientation and position of this system is discussed in 1.3.

It is easily seen that via the geoidal height N, and equation (1-2), one has the relationships between coordinates expressed on classical datums (ϕ_i, λ_i, H_i), and those expressed in terms of a three dimensional coordinate system (ϕ_i, λ_i, h_i) or (x_i, y_i, z_i). The latter are used extensively in this report in a contemporary and clear definition of a geodetic datum (1.4) and the formulation of several mathematical models for the combination of terrestrial and satellite geodetic networks (Section III).

1.3 Positioning and Orienting of a Terrestrial Horizontal Network Datum

A body in three-dimensional space has six degrees of freedom with respect to some fixed reference. An ellipsoid of rotation, used as a terrestrial horizontal network datum, is located in the earth body by six parameters with respect to some physical properties of the earth represented by the Average Terrestrial (AT) coordinate system (Figure 1-3). The Average Terrestrial system, conventionally right handed, is defined as having its origin at the earth's centre of gravity, its third (Z_{AT}) axis oriented towards the Conventional International Origin (CIO) defined by the International Polar Motion Service, and its first (X_{AT}) axis oriented towards the Greenwich Mean Astronomical Meridian as defined by the Bureau International de l'Heure [e.g. Mueller, 1969]. The condition to be fulfilled when positioning and orienting the reference ellipsoid is

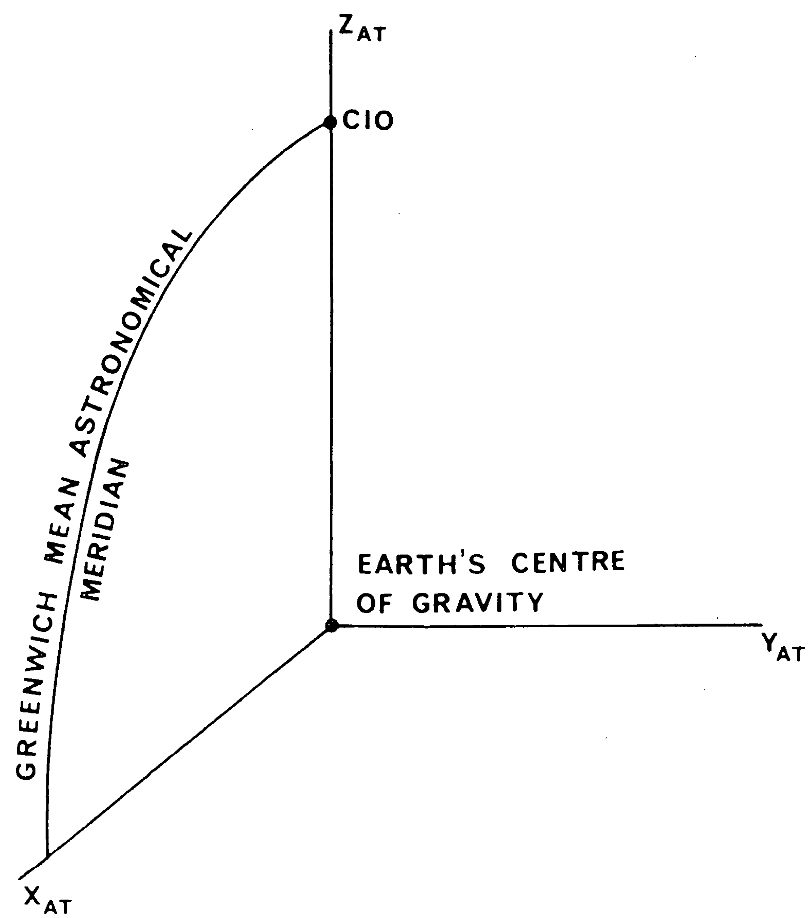

Figure 1-3

The Average Terrestrial Coordinate System that its axes (Geodetic Coordinate system) be parallel with those of the earth fixed system (Average Terrestrial system). The fulfillment of this condition is convenient since it tends to simplify several geodetic equations.

Before proceeding further with the accepted procedure of positioning and orienting a horizontal terrestrial network reference ellipsoid, it is necessary to distinguish between geodetic and geocentric datums. Strictly speaking, a geocentric datum is one whose origin is coincident with the earth's centre of gravity, and whose third (Z) axis is coincident with the earth's polar axis of inertia [e.g. Vanicek, 1975]. Such a system can be attained by developing a set of data such as gravity anomalies, geoidal heights, or deflections of the vertical, into an infinite series of spherical harmonics and dropping out the first degree terms [Heiskanen and Moritz, 1967]. Another possibility lies in the use of dynamic satellite geodetic observations in the establishment of satellite geodetic networks [Anderle and Tanenbaum, 1974].

Ideally, three translation components and three rotations are the simplest elements with which to express the position and orientation of the reference ellipsoid with respect to the Average Terrestrial coordinate system (1.4). The traditional approach does not do this directly. First a terrestrial point is chosen and designated as the network "initial point" (k). At this point, a set of computed (via terrestrial geodetic observables) and defined quantities are combined to yield the required six orienting and positioning parameters.

Using astronomical observations, one determines the astronomic latitude (Φ_k) and longitude (Λ_k) of the initial point, and the astronomic azimuth of at least one emanating line (A_{ki}). The standard deviations of these quantities will be of the order 0.1 to 0.3 arc seconds for Φ_k and Λ_k and 0.2 to 0.4 arc seconds for A_{ki} [Mueller, 1969]. The orthometric height of the initial point, H_k, can be determined via geodetic leveling. Its standard deviation, σ_{H_k}, will be a function of several factors affecting geodetic levelling. For example, using a recommended formula for approximating standard deviations of levelling network points in North America ($E = 1.8\ K^{2/3}$ where K is the distance from the reference stations(s)* in km) [NASA, 1973], the uncertainty of the orthometric height at Meade's Ranch, Kansas (the initial point of the present North American geodetic networks) can be estimated to be 0.2 m to 0.3 m. Finally, one is able to measure the zenith distance (Z_{ki}) in the Local Astronomic coordinate system [e.g. Krakiwsky and Wells, 1971] on any emanating line with a minimum standard deviation of the order of 1 arc second [Heiskanen and Moritz, 1967]. The observed values are used in the positioning and orienting of the network datum.

The quantities that are required in order to start a terrestrial geodetic network are the geodetic coordinates of the initial point (ϕ_k, λ_k, h_k), and the geodetic azimuths (α_{k_i}) and zenith distances (Z_{k_i}) of two emanating lines. Note that the determination of the above quantities must be carried out in such a way as to adhere to the condition of parallelity of axes. Further, it may be required that the ellipsoid be "best-fitting" in terms of

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* Mareograph Station(s)

geoid-ellipsoid separation over a certain region. The latter problem is not considered here.

One approach to datum establishment is to assign some "errorless" geodetic coordinates to the initial point ($\sigma_{\phi_k} = \sigma_{\lambda_k} = \sigma_{h_k} = 0$) such that the differences $(\Phi_k - \phi_k)$ and $(\Lambda_k - \lambda_k)$ are sufficiently small in order that their second powers can be neglected [Bomford, 1971; Heiskanen and Moritz, 1967; Vanicek and Wells, 1974]. As a consequence the components of the astrogeodetic deflections of the vertical and geoidal height at the initial point are expressed by [Heiskanen and Moritz, 1967]

$$\xi_k = \Phi_k - \phi_k \ , \qquad (1\text{-}3)$$

$$\eta_k = (\Lambda_k - \lambda_k) \cos \phi_k \ , \qquad (1\text{-}4)$$

$$N_k = h_k - H_k \ . \qquad (1\text{-}5)$$

These expressions are valid if the conditions for parallelity of axes, expressed via [Hotine, 1969; Vanicek and Wells, 1974]

$$z_{ki} = Z_{ki} + (\Lambda_k - \lambda_k) \cos \phi_k \sin A_{ki} + (\Phi_k - \phi_k) \cos A_{ki} , \qquad (1\text{-}6)$$

$$\alpha_{ki} = A_{ki} - (\Lambda_k - \lambda_k) \sin \phi_k - [(\Phi_k - \phi_k) \sin A_{ki} - (\Lambda_k - \lambda_k) \cos A_{ki} \cos \phi_k] \cot Z_{ki} \qquad (1\text{-}7)$$

are fulfilled.

Using this approach, the values of ξ_k, η_k, N_k, z_{ki}, and α_{ki} each have standard deviations as a result of error propagation in the respective equations. This implies that while the origin of the ellipsoid is fixed in space via the assigned geodetic coordinates, the orientation of its axes is not. Hence, due to errors in the geodetic observations at the initial point, the axes of the ellipsoid may not be parallel with those of the Average Terrestrial coordinate system.

An alternative procedure is to define the values of the deflections of the vertical (ξ_k, η_k) and geoidal height (N_k) at the terrestrial initial point [Heiskanen and Moritz, 1967] ($\sigma_{\xi_k} = \sigma_{\eta_k} = \sigma_{N_k} = 0$). In this case, the initial point geodetic coordinates will have some standard deviations as a result of error propagation in (1-3) through (1-5), and z_{ki} and α_{ki} as a result of error propagation in (1-6) and (1-7) respectively. Again, the uncertainties are due to the errors in the original geodetic measurements used to determine Φ_k, Λ_k, A_{ki}, H_k and Z_{ki}. The result again is the possible misalignment of the ellipsoidal axes with those of the Average Terrestrial coordinate system. A further complication of this approach is that the geodetic coordinates of the initial point can not be considered as fixed quantities since they do have some uncertainty.

The six parameters that are traditionally chosen to represent the position and orientation of the reference ellipsoid are (ϕ_k, λ_k, N_k, ξ_k, η_k, α_{ki}) [Vanicek, 1975; Krakiwsky and Wells, 1971; Mueller, 1974(a)]. However, these parameters are regarded as fixed quantities with no regard for the errors previously outlined. Only one of the parallelism conditions is applied, namely 1-7, which is known as the Laplace equation. Further, since an attempt is usually made to keep $z_{ki} \simeq 90°$, a truncated version of the Laplace equation is used [Mueller, 1974(a)],

$$A_{ki} - \alpha_{ki} = \eta_k \tan \phi_k \ . \tag{1-8}$$

Horizontal network computations are then initiated assuming parallelity of axes has been achieved by the truncated version of the Laplace equation. The truncated version of the Laplace equation is applied intermitently throughout the network with the hope of maintaining the parallelity of the reference ellipsoid and Average Terrestrial system axes. If one views the geodetic network as a separate but intricately connected entity from the datum and its position and orientation, the aforementioned claim is impossible. In this case, the astronomic azimuths, introduced via the Laplace equation, serve only to control the orientation of the geodetic network. On the other hand, if one assumes that the datum is represented by the geodetic network itself [Mueller, 1974(a), Vanicek, 1975], the claim may have some validity. To the author's knowledge, this has never been proven theoretically and this doubt is also upheld in [e.g. Hotine, 1969; Mueller, 1969].

The neglect of the second parallelism condition (1-6), the application of a truncated Laplace equation, and the assumption of fixed parameters at the initial point, were perhaps adequate in the past. Such datum establishment procedures may still suffice for the initial iteration of the several required for the establishment of a terrestrial horizontal network datum and related geodetic network (2.1). However, with the advent of more precise measurements, greatly increased data gathering and computing facilities, and the establishment of three-dimensional geodetic networks, a geodetic datum must be better defined. The non-parallelity of axes can create problems for three-dimensional terrestrial networks [Hotine, 1969; Vanicek and Wells, 1974]. The rigorous combination of terrestrial networks with those networks defined by satellite observations and inertial positioning systems requires that the transformation parameters between respective datums be known.

1.4 Contemporary Concepts

The intrinsic three-dimensional character of the problem of describing the position of a terrestrial point, and the recently acquired ability to directly determine the Cartesian coordinates of any point using observations of artificial earth satellites, have given rise to the necessity of an expanded concept of a geodetic datum, its positioning and orienting. The geodetic coordinates of any terrain point (ϕ_i, λ_i, h_i) are equivalent to the Cartesian triplet $(x_i, y_i, z_i)_G$. The Geodetic Coordinate system is then the reference frame, or datum, of a homogeneous three-dimensional terrestrial network. The reference frame used for a satellite geodetic network is a set of Cartesian axes. Using dynamic satellite procedures, an earth centred coordinate system is implied [Anderle, 1974(a)], while using geometric methods, the position of the origin of the reference frame is chosen arbitrarily [Schmid, 1974].

The datum for a geodetic network can be defined as a particular reference coordinate system. For example, the Geodetic Coordinate system is the datum of a unified terrestrial geodetic network, and its six position and orientation parameters are expressed with respect to the Average Terrestrial system. These are the three components (x_o, y_o, z_o) of the translation vector, $\bar{r}_o$, and three rotations (ε_x, ε_y, ε_z) (Figure 1-4). In order that the axes be parallel, it is obviously necessary that $\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$. The translation components may assume any value, but will be dependent on other specified criteria. The position and orientation elements (x_o, y_o, z_o, ε_x, ε_y, ε_z) may or may not have associated standard deviations, depending on how they are determined.

The relationship between the Geodetic and Average Terrestrial coordinates of any terrain point i is given by (Figure 1-4)

$$(\vec{R}_i)_{AT} = (\vec{r}_o)_{AT} + R_1(\varepsilon_x)R_2(\varepsilon_y)R_3(\varepsilon_z)\ (\vec{r}_i)_G \tag{1-9}$$

or

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_{AT} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_{AT} + R_1(\varepsilon_x)\ R_2(\varepsilon_y)\ R_3(\varepsilon_z) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}_G \tag{1-10}$$

The quantities $R_1(\varepsilon_x)$, $R_2(\varepsilon_y)$, $R_3(\varepsilon_z)$ are the well-known rotation matrices

$$R_1(\varepsilon_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_x & \sin\varepsilon_x \\ 0 & -\sin\varepsilon_x & \cos\varepsilon_x \end{bmatrix} \tag{1-11}$$

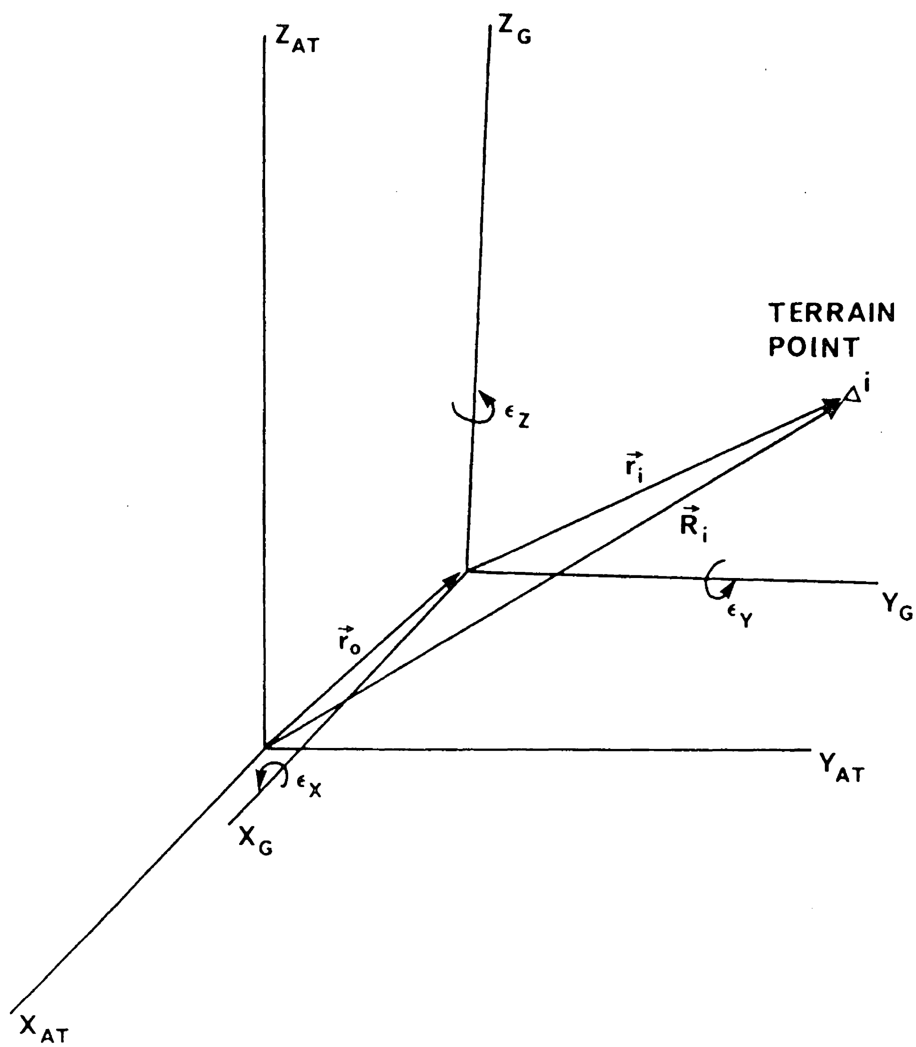

Figure 1-4

Relationship of the Average Terrestrial and Geodetic Coordinate Systems

$$R_2(\varepsilon_y) = \begin{bmatrix} \cos\varepsilon_y & 0 & -\sin\varepsilon_y \\ 0 & 1 & 0 \\ \sin\varepsilon_y & 0 & \cos\varepsilon_y \end{bmatrix}, \qquad (1\text{-}12)$$

$$R_3(\varepsilon_z) = \begin{bmatrix} \cos\varepsilon_z & \sin\varepsilon_z & 0 \\ -\sin\varepsilon_z & \cos\varepsilon_z & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (1\text{-}13)$$

The main problem with the aforementioned approach is the estimation of the Average Terrestrial coordinate system. One way of doing this is through the definition of a satellite network using dynamic procedures, combined with terrestrial determinations of the longitude origin and CIO [Anderle, 1974(a)]. Through the coordinates of the network points, the system is then recoverable. It is estimated that the geocentre may be estimated with a standard deviation of 1 m, the orientation of the spin axis with respect to the CIO pole to 5 m [Anderle, 1974(b)]. It should be noted that this coordinate system is "dynamic." However, it can be used for present geodetic networks since the time-variations are below the level of position errors [Anderle and Tanenbaum, 1974].

The use of the coordinates of geodetic network points for the recovery of datum parameters appears to be in contradiction with previous concepts (1.1, 1.3). This is not so. As before, the datum, or reference coordinate system, and associated network, are separate but intricately related. The coordinates of network points are used to recover the origin, position, and orientation of the datum. They are not used to define the datum. This approach is completely consistent with that presented previously. In addition, it is a substantial part of the foundation of several mathematical models used for the combination of terrestrial and satellite geodetic networks (Section III).

The adoption of a Cartesian coordinate system as a geodetic network datum, and the use of the parameters (x_o, y_o, z_o, ε_x, ε_y, ε_z) for position and orientation, are consistent with contemporary geodetic goals. The next obvious step is to move to a time-varying three-dimensional system required for geodynamics [Mather, 1974], although this step is beyond the scope of this report and not treated herein.

2. TERRESTRIAL AND SATELLITE GEODETIC NETWORKS

A geodetic network can be said to be a geometric object in which the various network points are uniquely defined by their coordinates. The coordinates are not directly observable but are derived via some observables amongst various network points. Using appropriate functional relationships, the observables are used to compute a homogeneous set of coordinates of the network points. Geodetic networks are intricately tied to their geodetic datums, thus the coordinates of the network points can, under certain conditions, be utilized to recover the parameters used to determine the datum position and orientation.

Geodetic networks may be regional or global in extent. The set of precisely coordinated terrestrial points can be used for geophysical studies or the tracking of artificial satellites, the location of national or international boundaries, the making of maps or the exploration for natural resources, and numerous other tasks. Thus, they must satisfy the requirements of both scientific investigators and the geodetic engineers.

The precision and homogeneity of a set of network point coordinates are basically dependent on the observables and the completeness of the mathematical models employed in subsequent network computations. One of the fundamental problems with terrestrial geodetic networks is the accumulation of unaccounted for systematic errors. Satellite networks are not as susceptible to this type of problem. Used in combination with their terrestrial counterparts, they offer a means of recovery and control of accumulated systematic errors in the terrestrial networks.

The aim of this chapter is to describe the natures of terrestrial and satellite geodetic networks with particular reference to the national and continental networks in North America. This includes a summary of the observations that are made and the functional relationships and computing techniques that are used to obtain rigorous solutions. The connections of the networks with their respective datums, the expected precision of network coordinates, and examples of the sources of systematic errors are enumerated.

2.1 Terrestrial Geodetic Networks

Traditionally, the triplet of coordinates used to describe the position of a terrain point has been split into horizontal and vertical components. The result of this is the development of separate horizontal and vertical geodetic networks. In general, the reasons for such a practice may be classed as psychological, historical, physical, and mathematical [Krakiwsky, 1972; Marussi, 1974]. The continuation of this custom is now based on practical issues. While new networks could be established in a three-dimensional mode, the required data to transform older networks is not presently available.

In North America, adherence to this classical geodetic custom has led to the present horizontal and vertical networks [McLellan, 1974; Baker, 1974; Villasana, 1974]. Horizontal networks are obtained by projecting the actual geodetic network to a mathematical surface, the reference ellipsoid, while vertical networks are nominally treated in the natural environment of the earth's gravity field without reference to any ficticious surface. Although some experimental work towards the establishment of three-dimensional terrestrial geodetic networks has been carried out in North America [Fubara, 1972; Hradilek, 1972; Vincenty, 1973], the redefined North American networks will be separated into horizontal and vertical components.

2.1.1 Horizontal Geodetic Networks

The datum used for horizontal geodetic networks is a reference ellipsoid (1.1). The network initial point, where datum position and orientation parameters are determined (1.3), is the starting point for network computations. The networks have been, and are presently, established by triangulation, trilateration, and traversing. The observables are horizontal directions, distances, and zenith distances between various network points. At certain intervals, astronomic observations are made for the determination of astronomic latitude, longitude, and azimuth. Each of the aforementioned is subject to some estimable errors and unknown systematic errors. The former are accounted for in the network adjustment in the variance-covariance matrix of observations, the latter tend to propagate systematically through the network causing some distortions.

Horizontal directions contain unaccounted for errors due to lateral refraction which can amount to 2 arc-seconds in extreme circumstances [Kukkamaki, 1961; Kukkamaki, 1949; Bomford, 1971]. Distances measured with electronic and electro-optical instruments are subject to unknown systematic errors due to inadequate atmospheric data and incomplete refraction models. This type of error has been determined to amount to 4 ppm in some instances [Jones, 1971]. Zenith distances, used to compute height differences in horizontal networks, may yield accuracies of 2 cm under experimentally controlled conditions [Hradilek, 1972]. In normal circumstances, zenith distance measurements have standard deviations of the order of 1 to 5 arc-seconds, yielding accuracies in heights much larger than that quoted above [Heiskanen and Moritz, 1967]. The astronomic quantities (Φ, Λ, α) can be determined with standard deviation of the order of 0.3 arc-seconds. In the North American geodetic networks, standard deviations of astronomic latitude and longitude are not expected to be below 0.5 arc-seconds, while in many determinations unknown systematic errors due to timing and the use of various star catalogues are estimated to be 1.5 and 0.4 arc-seconds respectively [Merry, 1975]. Astronomic azimuths, used in the Laplace equation (1-7) to control orientation of the network, are subject to unknown errors due to a lack of knowledge of plumbline curvature, atmospheric refraction, and the use of various star catalogues.

The observed quantities are projected, using a series of reductions [Mueller, 1974(a)], to the reference ellipsoid for network computations. To do this, the orthometric height (H), geoidal height (N), and components of the astrogeodetic deflections of the vertical (ξ, η) are required at each network point. These quantities are subject to errors, random and systematic, which propagate into the reduced measurements. The problems with trigonometric heights have already been given above. When vertical network points are coincident with horizontal network points, the more precise heights are utilized. However, they too are subject to error, albeit of a lower magnitude (2.1.2). N, ξ, and η are subject to errors due to those in the observations used to compute them. Further, the data required for direct computation of N, ξ, and η at each network point are not generally available, thus alternate procedures must be used. For example, using a least-squares surface fitting technique [Vanicek and Merry, 1973], errors in geoidal heights are estimated to be 2 m or greater [Merry, 1975].

The rigorous computation of an extensive terrestrial horizontal geodetic network is a complex problem [Thomson and Chamberlain, 1975]. The distortions created due to non-rigorous adjustment techniques, the lack of proper reduction procedures, and so on, in the present North American horizontal geodetic network have been studied [Thomson, 1970; Merry and Vanicek, 1973; Thomson et al., 1974]. In any new adjustment using only terrestrial data it is expected that a rigorous solution would be computed. The resulting coordinates (ϕ, λ), and associated variance-covariance matrix, would be free of errors of the magnitude in the present North American framework.

However, such a solution would still contain unknown systematic errors. While these may not be detectable over limited regions, they would be evident in a continental context. The reasons for the existence of these errors are four-fold:

1) Unknown and unmodelled errors in direction, distance and zenith distance measurements;
2) Unknown errors in the astronomic azimuths used for the control of network orientation;
3) Errors in H, N, ξ, η propagate into the reduced observations;
4) The initial discordant geodetic coordinate system results in the errors of 3).

Using only terrestrial data, there appears to be no way to completely eliminate, or at least model, all of the above. It is expected that the standard deviation of any network point coordinates, $\sigma_{\phi,\ \lambda}$ (m), with respect to the initial point, resulting from a rigorous computation process will be equal to or less than [NASA, 1973]

$$\sigma_{\phi,\ \lambda} = 0.020\ K^{2/3} \qquad (2\text{-}1)$$

where K is the distance, in kilometres, of the point in question from the initial point. The high precision geodimeter traverse surveys being carried out in the United States of America [Meade, 1967] are expected to yield more precise results. However, even these are expected to contain some residual systematic errors due to unknown errors in distances and astronomic azimuths [NASA, 1973].

2.1.2 Vertical Geodetic Networks

In Canada and the United States of America, the aim is to express the coordinates of points comprising the vertical geodetic networks as orthometric heights. The datum for the networks is the geoid. The position of the datum is determined via the monitoring of mean sea-level at several mareograph stations [McLellan, 1974]. Such networks should be established using precise spirit levelling and measured gravity [Krakiwsky and Mueller, 1966; Vanicek et al., 1972; Heiskanen and Moritz, 1967]. The orthometric height differences, obtained through appropriate computation procedures [Vanicek, 1972], are utilized in a suitable adjustment model to yield a homogeneous set of vertical coordinates (H_i) and associated variance-covariance matrix (Σ_H). As with horizontal terrestrial networks, the establishment of a vertical network is fraught with problems. Neglecting the current state of affairs in the present North American vertical networks in which, amongst other things, normal gravity is used in place of measured gravity [Nassar and Vanicek, 1975], there are several sources of errors which are much more difficult to isolate and model.

The use of mean sea-level, as determined from tide-gauge observations at various coastal locations, is a source of error. Mean sea level is not completely coincident with the geoid, departing from that equipotential surface by 1 to 2 metres under various conditions

[Lisitzin and Pattulo, 1961; Lennon, 1974]. Further, there are problems with the tide gauges themselves [Ku, 1970; Lennon, 1974]. All of this leads to the question of the stability of the definition of the datum position, and the propagation of unknown errors into the vertical network coordinates.

The orthometric height differences used in an adjustment process are subject to several sources of error. The observed spirit-levelled height differences are subject to errors due to thermal effects on the level and the effects of atmospheric refraction [Entin, 1959]. In reducing the measured height differences, the astronomic effect, which can amount to 0.1 mm per kilometre, should be accounted for [Holdahl, 1974].

Some of the above errors can be entirely eliminated. Reliable estimates for other errors can be obtained and accounted for in the rigorous computation of a vertical network. There will be residual errors, however, that can not be removed, such as the unknown refraction effects. These unknown errors will tend to affect the orientation of the network with respect to its datum. While the resulting network distortion (tilt) may be concealed over a small area, the effects may be detectable and significant when working with a continental network.

2.1.3 Three-Dimensional Terrestrial Geodetic Networks

Although it may presently be impractical to subject the entire North American geodetic framework to a three-dimensional computation procedure, the advantages of such a system should not be ignored. The major benefit to geodesists, geophysicists, and many other users of geodetic network data, is the complete definition in space of each network point by the triplet $(x, y, z)_G$ or (ϕ, λ, h) and the associated variance-covariance matrix.

In establishing a three-dimensional network, all terrestrial observations of horizontal directions, slope distances, and zenith distances or spirit-levelled height differences, plus the astronomic latitude, longitude, and azimuth are used in the network adjustment. The mathematical models required for this are available [Hotine, 1969; Heiskanen and Moritz, 1967] and have been tested [Fubara, 1972; Vincenty, 1973]. The advantages of such an approach are that observed quantities do not have to be reduced to a reference ellipsoid, fewer astronomic observations are required, the degrees of freedom of the solution is increased by combining horizontal and vertical adjustments, and the method as a whole is more rigorous and straightforward [Chovitz, 1974; Vincenty, 1973; Fubara, 1972].

Opponents of three-dimensional terrestrial networks invariably point to the inpracticality of the required spirit-levelling and the problem of vertical refraction. As pointed out by Vincenty [1973], using zenith distance measurements "the vertical component of spatial positions can be determined with the same accuracy as the horizontal components, provided that we use suitable observational procedures and a sound theoretical approach". Evidence of this is given by results in which the vertical coordinates, determined in a three-dimensional adjustment, agreed with spirit levelled heights to within 2.2 cm [Hradilek, 1972].

The basic problem in North America is that the networks have not been designed or observed with an eventual three-dimensional system in mind. Only 2% of the horizontal network points are coincident with vertical network points and only 10% of the former have measured zenith distances, many of which are of questionable accuracy [Chovitz, 1974]. An exception to this are the precise geodimeter traverses in the United States of America [Meade, 1967] in which the observations required for a three-dimensional network adjustment are available.

In order to make full use of the three-dimensional satellite networks in North America (2.2), a set of homogeneous three-dimensional terrestrial network coordinates for points coincident with satellite network points is desirable. This can be easily achieved by combining readjusted horizontal and vertical network coordinates and geoidal heights. The ellipsoidal heights are obtained by the addition of orthometric and geoidal heights and the variance-covariance matrix by

$$\Sigma_h = \Sigma_H + \Sigma_N , \qquad (2\text{-}2)$$

where Σ_H and Σ_N are the variance-covariance matrices of the readjusted orthometric heights and recomputed geoidal heights respectively. The result of this is the set of coordinates (ϕ_i, λ_i, h_i), with a variance-covariance matrix $\Sigma_{\phi\lambda h}$ in which there is no correlation between the horizontal and vertical components. When Cartesian coordinates $(x_i, y_i, z_i)_G$ are required, they are computed using (1-2). The associated variance-covariance matrix, $\Sigma_{x\,y\,z}$, is computed using the covariance law by

$$\Sigma_{x\,y\,z} = C\,\Sigma_{\phi\,\lambda\,h}\,C^T . \tag{2-3}$$

The transformation matrix C is composed of 3x3 submatrices, C_i, of the form

$$C_i = \begin{bmatrix} \frac{\partial x_i}{\partial \phi_i} & \frac{\partial x_i}{\partial \lambda_i} & \frac{\partial x_i}{\partial h_i} \\ \frac{\partial y_i}{\partial \phi_i} & \frac{\partial y_i}{\partial \lambda_i} & \frac{\partial y_i}{\partial h_i} \\ \frac{\partial z_i}{\partial \phi_i} & \frac{\partial z_i}{\partial \lambda_i} & \frac{\partial z_i}{\partial h_i} \end{bmatrix} , \tag{2-4}$$

$$C_i = \begin{bmatrix} -(M_i+h_i)\sin\phi_i\cos\lambda_i & -(N_i^*+h_i)\cos\phi_i\sin\lambda_i & \cos\phi_i\cos\lambda_i \\ -(M_i+h_i)\sin\phi_i\sin\lambda_i & (N_i^*+h_i)\cos\phi_i\cos\lambda_i & \cos\phi_i\sin\lambda_i \\ (M_i+h_i)\cos\phi_i & 0 & \sin\phi_i \end{bmatrix} , \tag{2-5}$$

where M_i is the meridian radius of curvature of the reference ellipsoid.

The aforementioned approach is not intended to replace an eventual rigorous three-dimensional terrestrial geodetic network adjustment. It is, however, a rigorous procedure to obtain three-dimensional terrestrial coordinates, although the model is incomplete in that the statistical covariance between horizontal and vertical components is not present.

Three-dimensional networks are subject to many of the same unknown errors as those in the classical geodetic networks. The only ones that are eliminated in a three-dimensional adjustment are those attributable to the reduction of observations to their respective datums. If the procedure outlined above is used, all of the errors previously described (2.1.1, 2.1.2) will be present to cause unknown orientation and scale errors in the network.

2.2 Satellite Geodetic Networks

The methods of analysis of observations of artificial earth satellites, for the purpose of computing terrestrial positions, can be placed in two general categories: geometric and dynamic.

In a geometric analysis, the satellite is used strictly as a high elevation active or passive target. A satellite position at any instant of time is treated as an unknown set of parameters, independent of all other positions, to be determined on the basis of observations made at that instant. Computations for this approach are not subject to errors in the adopted force field such as uncertainties in the earth's gravity field, atmospheric drag, radiation pressure, and tidal effects. Tracking station coordinates, computed using the geometric method, are subject to errors due to uncertainties in the effects of tides, crustal motion, and polar motion. The origin of the datum of the coordinated terrestrial points is dependent on definition from external sources.

The orientation of the Cartesian axes, and the scale of the system, are dependent on the observing techniques employed. For example, an optical network has to be given scale from some external source while the orientation of the datum axes is inherent in the observed spatial directions measured with respect to a star background. On the other hand, a range network provides no information on the orientation of the reference frame. The observations generally used in geometric solutions are those of simultaneous spatial directions obtained by photographing the satellite against the background of the stars, and simultaneous satellite ranges using electronic range and laser range equipment.

In a dynamic analysis, the satellite is considered subject to the forces affecting its motion, thus successive satellite positions are functionally related. Dynamic methods are considered to be statistically stronger than geometric methods because of the vast increase in the number of degrees of freedom in the former arising from the reduction of the number of unknowns required to define satellite positions over a certain time span. This procedure is, however, also subject to errors due to uncertainties in the effects of tides, crustal movement and polar motion. The origin of the datum of the resulting three-dimensional satellite network is the earth's centre of gravity. This is achieved by setting the first degree gravity field coefficients, used in orbit computations, to zero. The direction of the X-axis is defined using external information. The orientation of the Z-axis may be defined in the dynamic analysis.

However, in most solutions, this is usually carried out using a combination of satellite determined and terrestrial data. The source of scale for dynamic solutions is primarily the earth's gravitational constant. However, in the case of electronic range and range-difference and laser range observing systems, the adopted value of the velocity of light and the earth's gravitational constant are used to introduce scale.

In addition there are a variety of methods that are derived directly from the aforementioned such as quasi-geometric, semi-dynamic, short-arc, and translocation. These are sometimes used to gain some benefits from the general methods. In many instances, a combination of several techniques are used in a simultaneous solution for terrestrial station positions and other geodetic parameters.

There have been several tens of satellite geodetic networks established throughout the world to serve various functions. In recent years, several satellite solutions for terrestrial station coordinates have been completed. Amongst these are the geometric WN-12 and WN-14 solutions [Mueller, 1974(b)], the Doppler dynamic solution NWL-9D [Anderle, 1974(b)], and the World Geometric Satellite Triangulation (BC-4) Network [Schmid, 1974]. Of greater importance to the present situation in North America are the Canadian and American Doppler networks and the North American densification of the BC-4 global network. The stated role of these latter networks is the support of the redefinition of the North American datum and terrestrial geodetic networks [Schmid, 1970; McLellan, 1974; Strange et al., 1975]. For this reason, the establishment of these networks are analysed in more detail (2.2.1, 2.2.2).

2.2.1 North American Densification of the World Geometric Satellite Triangulation (BC-4) Network

The results of the completed World Satellite Triangulation (BC-4) Network were published in late 1974 [Schmid, 1974]. The mean positional error of the forty-five stations is 4.5 m (1σ). The twenty-three station North American Densification (Figure 2-1) of the global network was completed early in 1975 [Pope, 1975]. The solution of the densification network was carried out independent of the world net solution, although there are six common stations between the two of them. The reduction of observations and the adjustment of the latter network were done in the same manner as for the world network [Pope, 1975].

The datum of the North American Densification Network is a set of near-geocentric Cartesian axes. The Z-axis was made parallel to the mean rotation axis of the earth for a certain epoch (CIO) by virtue of the orientation of the interpolated satellite directions. The orientation of the X-axis (longitude origin) and position of the origin were determined by assigning near-geocentric coordinates to one station in the network. A comparison of the World BC-4 Network reference frame with that of the WN-14 Global Satellite Results [Mueller, 1974(b)] indicates that they are separated by a 14 m translation vector, and that the two systems are rotated by 0".11 in longitude with respect to each other. In a similar test, this time with respect to the Doppler NWL-9D system, the translation vector BC-4 to NWL9-D was found to be 30 m and the difference in X-axis orientation 0".61 [Schmid, 1974].

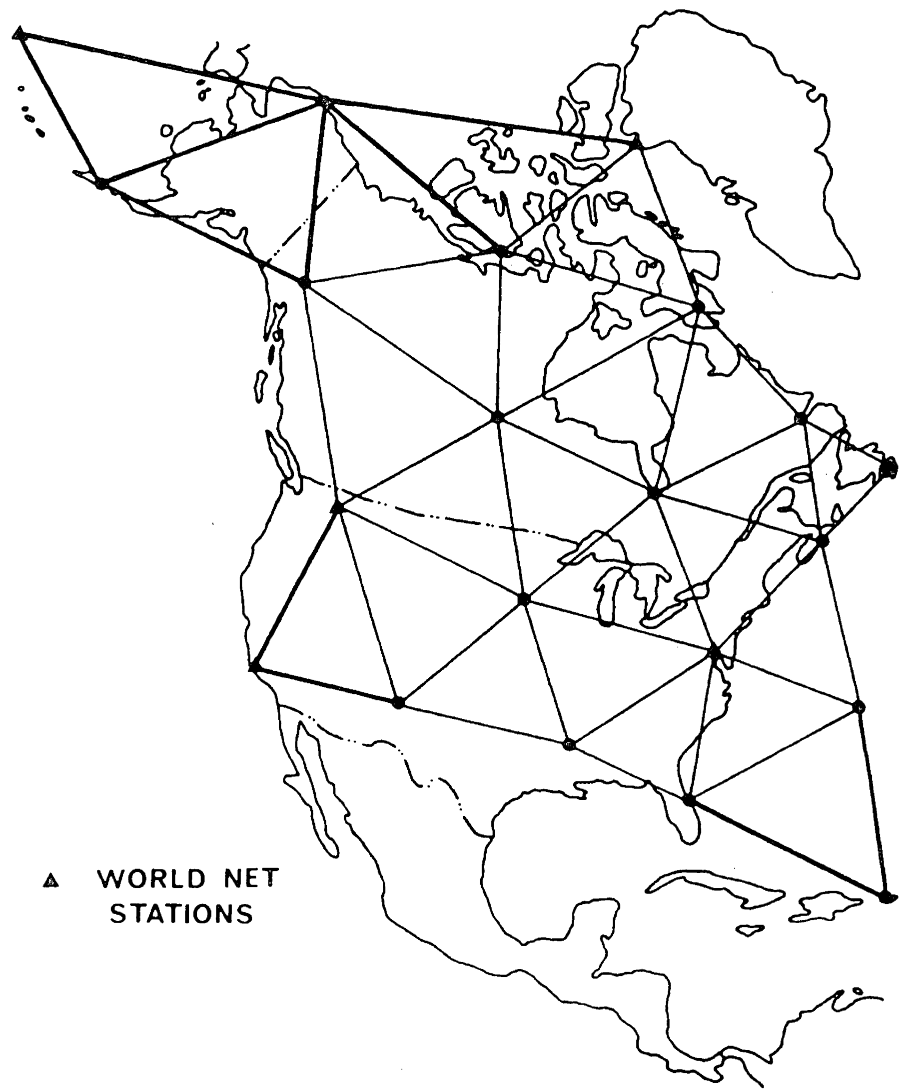

Figure 2-1

North American Densification of the World Geometric Satellite Triangulation (BC-4) Network

Now, the sources of error in the BC-4 network are examined. The principle of satellite triangulation is to combine spatial directions to satellites and one or more spatial distance measurements in a three-dimensional triangulation adjustment. In the network being discussed here, the directions, expressed in terms of right-ascension and declination were obtained, via a complex procedure, from photographs of satellites against a star background. The required spatial distances were determined from precise terrestrial traverses. The standard deviation of such a spatial direction is estimated to be 0.24 arc-seconds [Schmid, 1972; Schmid, 1974], while the two North American base lines have standard deviations of 3.53 m and 1.59 m over distances of 3.5×10^6 m and 1.4×10^6 m respectively [Schmid, 1974].

There are several sources of errors in the observations. The terrestrially measured base lines are subject to the errors found in any terrestrial network (2.1). The direction measurements, which make up the greatest percentage of observations, have error sources that may be summarized as being dependent on [Schmid, 1965]

(i) the comparator measurements of stars and satellite images;

(ii) the star catalogue data;

(iii) the time determination associated with star and satellite image exposures;

(iv) atmospheric scintillation;

(v) emulsion distortion occurring during development.

The positional accuracy resulting from rigorous satellite triangulation is indepdendent of station location [Schmid, 1965].

Internal and external consistency checks have been carried out with the global BC-4 results. In one internal test, the computed spatial distances from the adjusted network (one fixed point, one constrained base line) were compared to precisely determined terrestrial spatial distances. The total distance difference for the six lines was 9.16 m in a total distance of 17.5×10^6m, or 1.9 ppm [Schmid, 1974]. External checks showed that the network is scaled 2 ppm smaller than the Doppler NWL-9D solution [Schmid, 1974], and 2.3 ppm smaller than the WN-14 solution [Mueller, 1974(b)].

Although there are possibilities of unknown systematic errors in the global BC-4 network, Schmid [1974] stated "error theoretical investigations indicate that the result, derived in principle by interpolation into the astronomical right ascension-declination system, is essentially free of systematic errors". Since the North American Densification Network was established by the same procedures, it is logical to assume that the aforementioned statement applies to this network as well. It was expected that the three-dimensional positions of the Densification could be determined with an accuracy of 3 m to 4 m in all components [Schmid, 1970]. The final results yielded mean standard deviations of 3.4 m, 4.2 m, and 4.7 m in the X, Y and Z components respectively. Computed spatial distances, azimuths, and vertical angles, on lines $.83 \times 10^6$ m and greater in length, have standard deviations of the order of 5.5 m, 0.8 arc-seconds, and 1.0 arc-seconds respectively [Pope, 1975].

2.2.2 North American Doppler Networks

The geodetic Doppler networks in Canada and the United States are expected to contain a total of 350 points upon completion. In Canada, Doppler points have been established at intervals of 250 km to 500 km, and by the end of 1974 had numbered 76 (Figure 2-2). These networks are to be used in the redefinition of the North American horizontal terrestrial geodetic networks [McLellan, 1974; Strange et al., 1975].

The nominal reference frame for the Doppler network is the Average Terrestrial Coordinate System. Studies have indicated that the origin is within 1 m of the geocentre and that the primary pole (Z-axis) is oriented such that it is within 5 m, or 0.15 arc-seconds, of the CIO [Anderle, 1974(b)]. The errors in position and orientation are largely due to unknown errors in the coefficients used to represent the earth's gravity field [Anderle, 1974(b)]. The correct orientation of the longitude origin (X-axis) has not been resolved. Comparisons with other satellite network reference frames have yielded relative longitude rotations of up to 1.1 arc seconds (34 m on the equator) [Mueller, 1974(b)].

The observations being used in the establishment of these networks are Doppler shift measurements of signals emitted by satellites of the United States Navy Navigation system. Using geodetic receivers and a precise ephemeris, recent studies have shown that three-dimensional coordinates of points can be determined with an accuracy (1σ) of 1 m or less in all three components [Kouba, 1975;

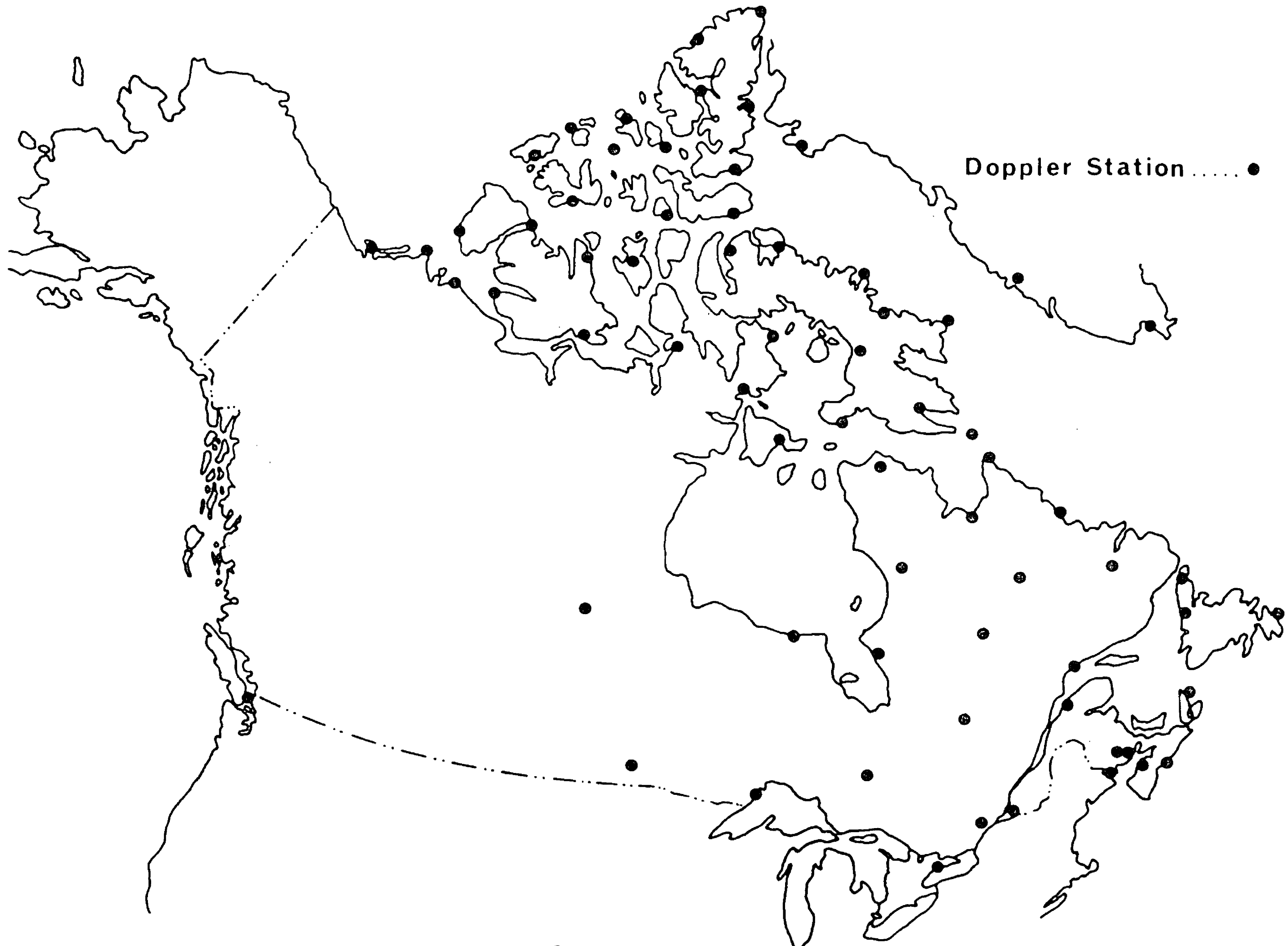

Figure 2-2

1974 Status of Canadian Doppler Network

Hothem, 1975]. The Doppler measurements are subject to several sources of errors. Typically, the random error in the measurement of a range difference is in the order of 10 cm [Anderle, 1974(b)]. The main sources of observation errors are due to instrument noise, timing, and ionospheric and tropospheric refraction. In the computation of coordinates, several effects must be accounted for in order to produce precise positions. These include orbit determination uncertainties, data rejection criteria, and the direction of satellite motion [Kouba, 1975; Hothem, 1975]. The methods used for position computation are such that several of the unknown errors (orbital biases, timing, unknown tropospheric refraction) are modelled and solved for simultaneously with the position determination [e.g. Brown, 1970; Kouba and Boal, 1975; Wells, 1974].

The internal and external consistency of the Doppler networks have been investigated. For example, a Doppler network in Atlantic Canada was found to have an RMS standard deviation of 0.9 m. This is compared to 1.2 m and 6.7 m for readjusted terrestrial and preliminary Geometric Satellite Triangulation in the same area [Wells et al., 1974]. The scale of the Doppler network in the United States has been tested against some external standards. When Doppler network distances were compared to four VLBI distances (0.8×10^6 m to 5.0×10^6 m in length), the maximum difference (Doppler minus VLBI) was 5.1 m. After scaling down the Doppler distances by 1 ppm, the maximum difference was 2.1 m [Strange et al., 1975].

As with Satellite Triangulation, the accuracy of a Doppler network point is not dependent on terrestrial position. Although many solutions yield only point positions with no correlation between network points, a multistation solution, with a full covariance matrix, is possible [Kouba and Boal, 1975]. This type of solution yields a homogeneous Doppler network in which coordinate standard deviations are in the order of 1 m or less.

3. GENERAL CONCEPTS REGARDING THE COMBINATION OF GEODETIC NETWORKS

The problem of the redefinition of the North American geodetic networks and the existence of several satellite networks on the continent requires that some guidance be available to indicate how to use all available data to solve the aforementioned problems. In view of this, a general flow chart for the study of the problem of network combinations has been devised (Figure 3-1) [Krakiwsky and Thomson, 1974]. Investigations involving each of the elements and completion of the flow is felt to constitute a logical approach to this study.

The reasons for the combination of terrestrial and satellite networks are given in 3.1 in the context of how the satellite network data can be used to supplement classical terrestrial data in the solution of problems related to the latter networks. A set of parameters by which to classify the mathematical models, in the general sense, is given in 3.2. The classifications presented are used as the basis for the separation of the various combination models given in this report.

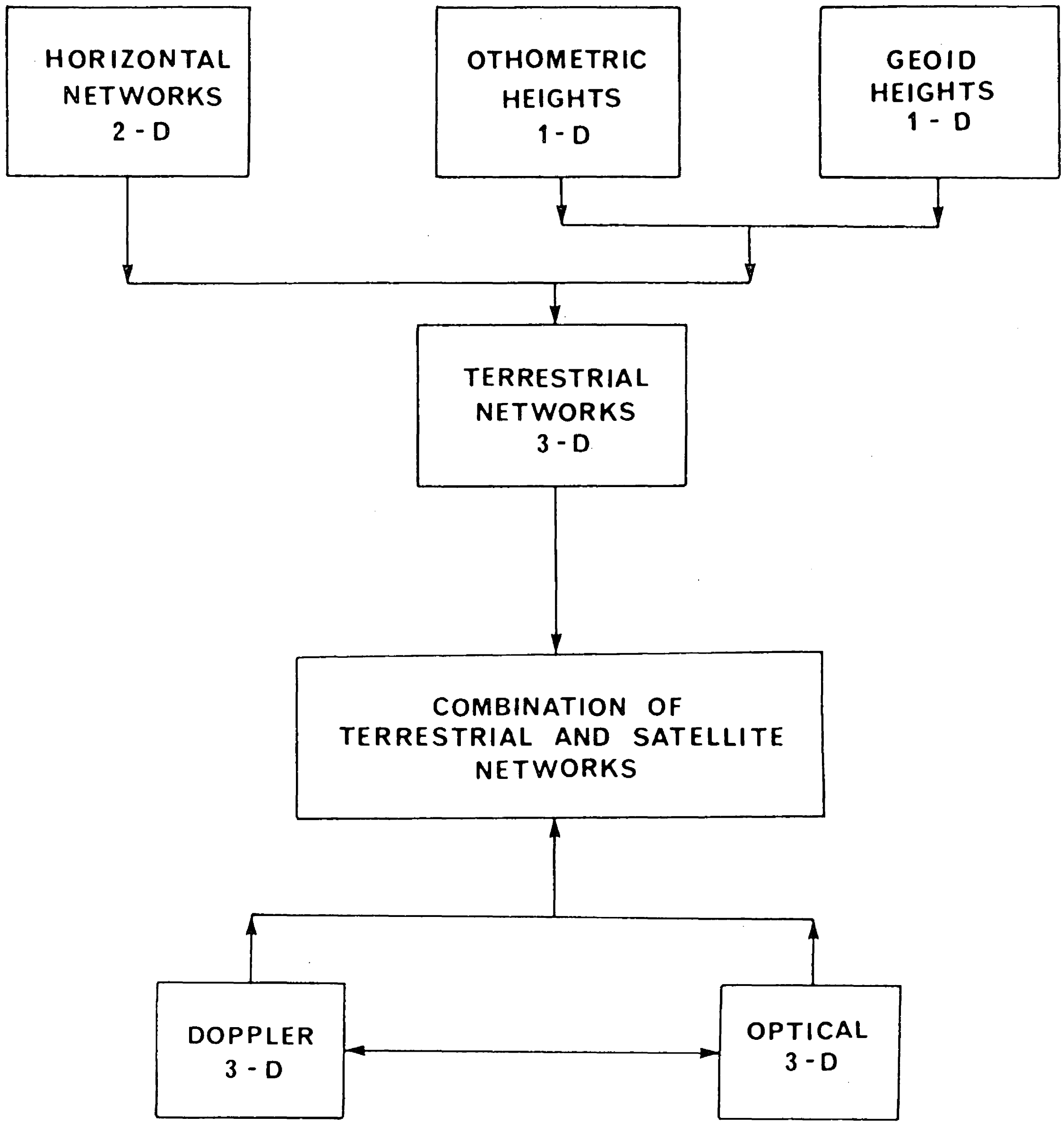

Figure 3-1

Combination of Geodetic Networks

3.1 Rationale

Satellite networks provide new and independent sources of data. Whether or not this data will in fact yield a significant contribution to the solution of terrestrial network problems is dependent on factors such as the accuracy and homogeneity of the satellite network data and the mathematical model employed in the combination procedure.

Geodetic datums for terrestrial networks are established independent of their networks (1.1). The terrestrial networks are intricately tied to their respective datums, but are in fact separate entities (2.1). The datum of satellite networks are implied via various phenomena such as the physics of the model used, the observations used, and the estimation process (2.2). In this case, the coordinates of network points can be used to recover the position and orientation of the datum with respect to some other reference frame. Due to differences in the establishment of satellite and terrestrial network datums, their origins will generally not be coincident nor will their reference frame axes be parallel. In order to be able to use data from one network as observables in the other, one must use only that data which is invariant of the coordinate system (spatial angles and spatial distances) or the datum differences must be modelled.

Datum transformation parameters can be solved for in the combination of satellite and terrestrial networks. In cases where the satellite network is the result of dynamic analyses of satellite observations, the data can be used to assist in the positioning and orienting of the terrestrial network datum with respect to the Average Terrestrial coordinate system. Further, if the satellite network is one of global extent, it can be used to relate several regional terrestrial datums.

Satellite networks are inherently three-dimensional. They are comparatively free of systematic errors and the relative accuracy of station coordinates are not dependent on station separation (2.2). Due to the sequential nature of the establishment of terrestrial geodetic networks, the relative standard errors of coordinates increases with interstation distance. There are many unresolved errors in terrestrial networks as a result of misoriented datums, and scale and orientation problems in the networks. Satellite data can serve to strengthen terrestrial networks.

Good quality satellite network data can be used in a terrestrial network in place of further terrestrial observables. For example, the older horizontal networks in North America had only limited numbers of observed base lines and astronomic azimuths for scale and control of orientation of the network. In addition to adding scale and orientation control, the use of the satellite network data in a terrestrial network adjustment will constrain the usual build-up of random and systematic errors. Intuitively, the result should be a more internally consistent terrestrial network.

Combination models in which terrestrial network systematic errors in scale and orientation are modelled are available. These types of mathematical models do not treat individual systematic errors of terrestrial observables. Their function is to model, in a mean sense, the overall effects in a few parameters.

In summary, the main reasons for combining satellite and terrestrial geodetic networks can be given as:

(i) independent source of reliable data;

(ii) strengthening of classical terrestrial networks;

(iii) control of or modelling and removal of systematic errors in terrestrial networks;

(iv) relating various terrestrial and satellite datums.

3.2 Classification of Mathematical Models

The alternative procedures for the combination of satellite and terrestrial geodetic networks can be divided into two broad groups:

(i) those in which the datum transformation parameters are considered known;

(ii) those which treat the datum transformation parameters as unknowns to be solved for in the combination solution.

Within each of the aforementioned groups, there are four classification parameters used. They are the dimensionality of the models, the type of observables required, the parameterization involved in any particular model, and the estimation procedure used to solve the model.

Due to the traditional splitting of terrestrial geodetic networks, combination models are available that reflect the division. This means that the three-dimensional satellite data has to be split also. There are three common types of models - one, two, and three-dimensional.

Table 3-1 is structured around the aforementioned divisions.

Several types of observables are used when combining satellite and terrestrial geodetic networks. One may use all of the original observables from both networks, or quasi-observables from one and the original observations from the other (Bursa; Table 3-1). Still other procedures require quasi-observables from two or more satellite and terrestrial networks.

The unknown parameters in any combination model are important in its use and the interpretation of results. Some models contain no additional unknown parameters due to the combination process (Anderle, Meade; Table 3-1). Others contain unknown datum transformation parameters (Bursa, Wells and Vanicek, Tscherning, Mueller, Lambeck; Table 3-1), and still others contain these plus parameters to model unknown errors in one network (Hotine, Krakiwsky and Thomson; Table 3-1).

The estimation procedures vary from one mathematical model to the other. In many cases the estimation technique used could be replaced by another method to yield the same results. Some investigations require only certain results, thus very simple estimation techniques are employed.

Many studies of the combination of terrestrial and satellite geodetic networks have been completed. Each of these have employed only one specific mathematical model. In most instances, little attention has been paid to whether or not certain models should have been used. As will be shown later in this report, some of the accepted combination procedures have been improperly used.

Investigator(s)	Datum Transformation Parameters	Dimensionality	Observables	No. of Unknown Parameters	Estimation Procedure	Remarks
Bursa [1967]	Unknown	3	Quasi-observables from terrestrial networks	3 rotations	Least-Squares Parametric Adjustment	Terrestrial network direction cosines computed from geodetic coordinates; those from satellite network obtained from optical observations directly.
Anderle [1974c]	Known	2	Quasi-observables from terrestrial network; weighted Doppler coordinates	0	Least-Squares Parametric Adjustment	Terrestrial network measurements computed from precise geodimeter traverses.
Wells and Vanicek [1975]	Unknown	3	Quasi-observables from both satellite and terrestrial networks	3 translation components 4 rotations 1 scale difference	Least-Squares Parametric Adjustment	Network-coordinates used as observables; data from several datums is used in a solution.

Classification of Mathematical Models

Table 3-1.

Investigator(s)	Datum Transformation Parameters	Dimensionality	Observables	No. of Unknown Parameters	Estimation Procedure	Remarks
Tscherning [1975]	Unknown	3	Quasi-observables from both satellite and terrestrial networks plus some terrestrial observables	3 translation components	Least-Squares Collocation	Terrestrial observables included free-air gravity anomalies, deflections of the vertical and height anomalies.
Mueller et al [1970]	Unknown	3	Quasi-observables (coordinates) from both satellite and terrestrial networks	3 rotations 1 scale difference	Least-Squares Parametric Adjustment	Translation components were considered to be known.
Merry [1975]	Known	1	Quasi-observables from both satellite and terrestrial networks	1 geoidal height	Least-Squares Estimation	Geoidal height determined from satellite ellipsoidal height and terrestrial orthometric height used as constraints in geoid determination.
Lambeck [1971]	Unknown	3	Quasi-observables from both satellite and terrestrial networks	3 translation components 3 rotations 1 scale difference	Least-Squares Combined Adjustment	Model yields adjusted coordinates of common network points.

Table 3-1 (cont'd)

Investigator(s)	Datum Transformation Parameters	Dimensionality	Observables	No. of Unknown Parameters	Estimation Procedure	Remarks
Meade [1974]	Known	2	Quasi-observables from satellite network	0	Least-Squares Parametric Adjustment	Distances and azimuths computed from Doppler network coordinates.
Hotine [1969]	Unknown	3	Quasi-observables from both satellite and terrestrial networks	3 translation components 3 rotations 2 orientation parameters 1 scale diff.	Least-Squares Combined Adjustment	2 rotations and scale difference are used to model overall systematic errors in terrestrial network.
Krakiwsky and Thomson [1974]	Unknown	3	Quasi-observables from both satellite and terrestrial networks	3 translation components 6 rotations 1 scale difference	Least-Squares Combined Adjustment	3 rotations and scale difference are used to model overall systematic errors in terrestrial network.

Table 3-1 (cont'd)

SECTION II

COMBINATION PROCEDURES WHEN DATUM TRANSFORMATION PARAMETERS ARE KNOWN

4. THREE-DIMENSIONAL MODELS

The natural approach to utilizing satellite data is in a three-dimensional model. With respect to present terrestrial geodetic networks in North America, this is not possible. However, with a view towards the future design of terrestrial networks, it is important to investigate how satellite network data can be used in this mode.

There are several approaches to entering the satellite data into the solution of a terrestrial network. One can use the original observations - directions, ranges, range differences - or some quasi-observable such as direction cosines derived from this data. Some investigators have opted for this approach in their investigations [Bursa, 1967]. However, the easiest approach is to take the results of the completed satellite network - coordinates and associated variance-covariance matrix - and use this information, or some quasi-observables derived from it, in the combination solution with terrestrial network data.

Since the datum transformation parameters are assumed to be known in this instance, the satellite network data can be used in several ways. For example, one can use the data in a simple model to parameterize unknown systematic errors in a terrestrial network. The data can be added to a network adjustment to supplement the terrestrial observables and to control the build-up of random and systematic errors.

Several estimation procedures may be employed. For reasons of expediency, efficiency, and ease of application, the method of least squares is suggested here. Of course, different situations may require either a batch or stepwise approach. Within these bounds, one is free to choose a parametric, combined, collocation, or any other well known least squares procedure.

Obviously, many satellite-terrestrial network combination models within the stated limitations are possible (known datum transformation parameters, three-dimensional models). Given here are several models which are considered to be practically feasible at the present time.

4.1 A Parameterization of Scale and Orientation Errors in a Terrestrial Network

The data required for the model are the Cartesian coordinates of both the satellite (X_i, Y_i, Z_i) and terrestrial (x_i, y_i, z_i) networks. These coordinates are assumed to be the results of independent network adjustments so that each set of coordinates has associated variance-covariance matrices Σ_{XYZ} and Σ_{xyz}. The satellite coordinates and their variance-covariance matrix are transformed to the terrestrial network coordinate system using the known datum transformation parameters (x_o, y_o, z_o, ε_x, ε_y, ε_z) using the relationship (1-9).

The combination model is given by (Figure 4-1)

$$\vec{F} = (\vec{r}_k)_G + (1+\kappa)\ R(\vec{r}_{ki})_G - (\vec{\rho}_i)_G = 0 \qquad (4\text{-}1)$$

where $\vec{\rho}_i$ is the transformed position vector $(X_i, Y_i, Z_i)^T_G$ of the satellite point, $\vec{r}_k$ $(= \vec{\rho}_k)$ is the position vector of the terrestrial network initial point $(x_k, y_k, z_k)^T_G$, $\vec{r}_{ki}$ is the terrestrial network position vector with respect to the initial point $(x_{ki}, y_{ki}, z_{ki})^T_G$, κ is an unknown scale difference to be estimated, and R represents a product of three rotation matrices containing unknown network orientation parameters to be estimated. The matrix R is given by

$$R = R_1(\psi_x),\ R_2(\psi_y),\ R_3(\psi_x) \qquad (4\text{-}2)$$

where R_1, R_2, R_3 are the well known rotation matrices ((1-11), (1-12), (1-13)), and ψ_x, ψ_y, ψ_z are small rotations about the x, y, and z axes respectively of a local geodetic coordinate system at the initial point k, whose axes are parallel to those of the Geodetic system (Figure 4-1). Assuming differentially small values for ψ_x, ψ_y, ψ_z, and neglecting higher than first order terms, (4-2) reduces to

$$R = \begin{bmatrix} 1 & \psi_z & -\psi_y \\ -\psi_z & 1 & \psi_x \\ \psi_y & -\psi_x & 1 \end{bmatrix}. \qquad (4\text{-}3)$$

The reasoning for this model is as follows. First, it is assumed that after transformation, the differences in the coordinates of common network points are statistically significant. Second, it is presumed that the incompatability is caused by the presence of unknown systematic errors in the terrestrial network. Finally,

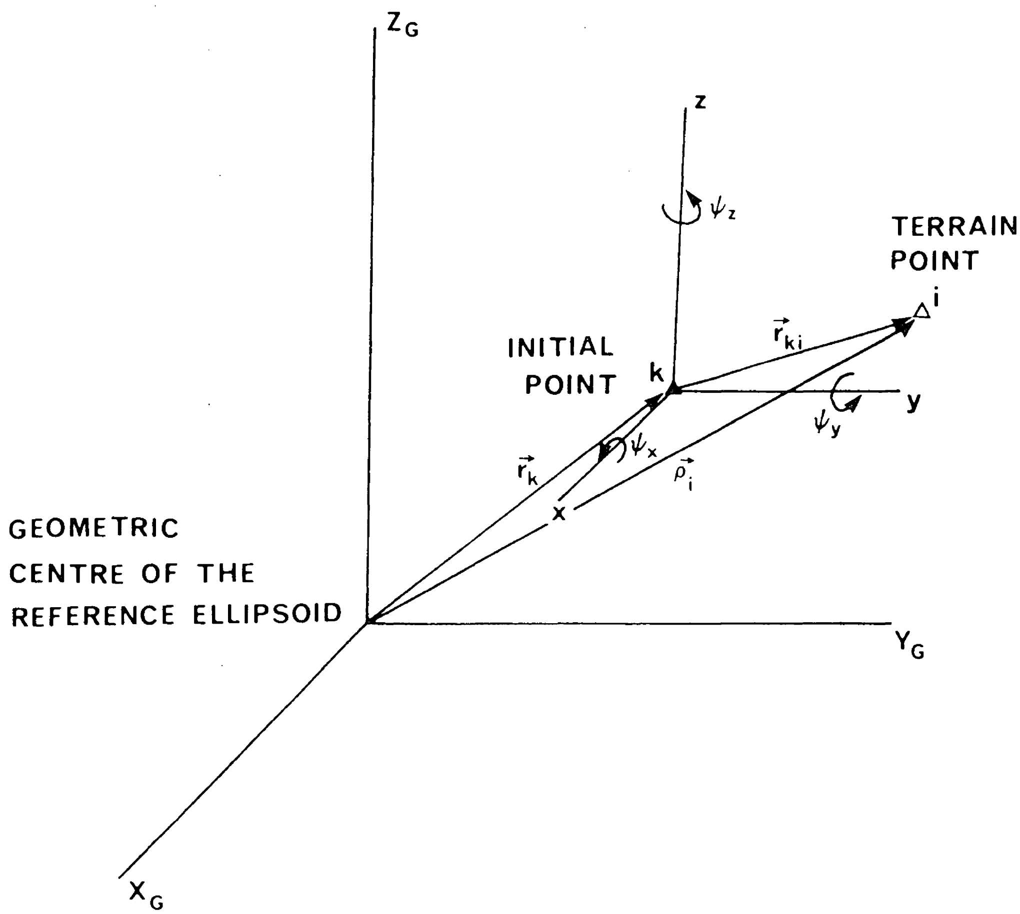

Figure 4-1

A Parameterization of Terrestrial Network Systematic Errors the unknown errors are parameterized by four parameters - one scale difference and three rotations.

The solution to this model can be obtained using a combined method least squares estimation procedure, expressed functionally as

$$F(\bar{X}, \bar{L}) = 0, \qquad (4\text{-}4)$$

where $\bar{X}$ represents the unknown parameters (scale and orientation) and $\bar{L}$ the observables (coordinates of common network points). A linearization of the non-linear model (4-1) yields the matrix expression

$$A\hat{X} + B\hat{V} + W^{O} = 0 \;, \qquad (4\text{-}5)$$

in which A and B are design matrices, $\hat{X}$ is an estimate of the unknown parameters, $\hat{V}$ the residuals of the observables, and W^{O} the misclosure vector.

Expansion of (4-1) gives

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}_G + (1+\kappa) \begin{bmatrix} 1 & \psi_z & -\psi_y \\ -\psi_z & 1 & \psi_x \\ \psi_y & -\psi_x & 1 \end{bmatrix} \begin{bmatrix} x_{ki} \\ y_{ki} \\ z_{ki} \end{bmatrix}_G - \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_G . \qquad (4\text{-}6)$$

Now, assuming that the initial values of the unknown parameters (X^{O}) are zero, the design matrices, A and B, and misclosure vector, W^{O}, are given by

$$A = \frac{\partial \bar{F}}{\partial X} = \left[\begin{array}{ccc|c} 0 & -z_{ki} & y_{ki} & x_{ki} \\ z_{ki} & 0 & -x_{ki} & y_{ki} \\ -y_{ki} & x_{ki} & 0 & z_{ki} \end{array}\right] , \qquad (4\text{-}7)$$

$$B = \frac{\partial \bar{F}}{\partial L} = \begin{bmatrix} 1 & 0 & 0 & | & -1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & -1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & -1 \end{bmatrix} , \quad (4\text{-}8)$$

$$W = F(X^o, L) = \begin{bmatrix} x_k + x_{ki} - X_i \\ y_k + y_{ki} - Y_i \\ z_k + z_{ki} - Z_i \end{bmatrix} , \quad (4\text{-}9)$$

where L are the observed values of the observables. The solution to (4-5) is given by [Kouba, 1970; Krakiwsky, 1975]

$$\hat{X} = -(A^T(B\Sigma_L B^T)^{-1} A)^{-1}(B\Sigma_L B^T)^{-1} W^o , \quad (4\text{-}10)$$

$$\hat{V} = \Sigma_L B^T K \quad , \quad (4\text{-}11)$$

$$Q_{\hat{X}} = (A^T (B\Sigma_L B^T)^{-1} A)^{-1} \quad , \quad (4\text{-}12)$$

$$Q_{\hat{L}} = (\Sigma_L - \Sigma_L B^T((B\Sigma_L B^T)^{-1} - (B\Sigma_L B^T)^{-1} A Q_{\hat{X}} A^T (B\Sigma_L B^T)^{-1}) B\Sigma_L) , \quad (4\text{-}13)$$

where the correlate vector, K is given by

$$K = -(B\Sigma_L B^T)^{-1} (A\hat{X} + W^o) . \quad (4\text{-}14)$$

In the above equations, Σ_L is the variance-covariance matrix of the observables, the coordinates (satellite Σ_{XYZ} and terrestrial Σ_{xyz}) of the common network points. $Q_{\hat{X}}$ and $Q_{\hat{L}}$ are the weight coefficient matrices of the estimated parameters and adjusted observables respectively. The associated variance-covariance matrices are given by

$$\Sigma_{\hat{X}} = \sigma_o^2 \, Q_{\hat{X}} , \quad (4\text{-}15)$$

and

$$\Sigma_{\hat{L}} = \sigma_o^2 \, Q_{\hat{L}} , \quad (4\text{-}16)$$

where σ_o^2 is the a priori variance factor. The least squares estimate of the unknown parameters is given by

$$\hat{\hat{X}} = X^o + \hat{X} \ , \qquad (4-17)$$

and the adjusted observables (coordinates) by

$$\hat{\hat{L}} = L + \hat{V} \ . \qquad (4-18)$$

This model, and its proposed solution, is similar to one in which the datum transformation parameters are considered to be unknown (7.2). Variations on this theme are possible. If the terrestrial coordinates are expressed as ellipsoidal (ϕ, λ, h), one may wish to express the orientation unknowns in terms of an azimuth rotation (dA), and prime vertical ($d\mu$) and meridian ($d\nu$) tilts about the Local Geodetic coordinate system at k. As shown in (7.3), the results are equivalent to using ψ_x, ψ_y, and ψ_z as has been presented here.

4.2 Satellite Coordinates as Weighted Parameters

This is the most straightforward and simple method to combine geodetic networks. The satellite network data is used directly in a three-dimensional terrestrial network adjustment, such as that proposed by Vincenty [1973]. This model requires the adjusted satellite network coordinates and associated variance-covariance matrix transformed to the desired coordinate system, and the usual terrestrial observables.

Assuming that a least squares parametric estimation procedure is used for the network computation, given by

$$F(\bar{X}) = \bar{L} \,, \qquad (4\text{-}19)$$

then the satellite network coordinates are used as initial approximate coordinates in the formation of the design matrix A and misclosure vector W. The usual matrix expression for the solution vector is given by

$$\hat{X} = -(A^T \Sigma_L^{-1} A)^{-1} \, A^T \Sigma_L^{-1} W \,, \qquad (4\text{-}20)$$

where Σ_L^{-1} is the weight matrix of the observables. However, in this case there is information regarding the a priori coordinates, which is characterised by its variance-covariance matrix Σ_{XYZ}. The solution vector is now given by

$$\hat{X} = -(\Sigma_{XYZ}^{-1} + A^T \Sigma_L^{-1} A)^{-1} \, A^T \Sigma_L^{-1} W \,. \qquad (4\text{-}21)$$

The final network coordinates are given by

$$\hat{\hat{X}} = X^o + \hat{X} \,, \qquad (4\text{-}22)$$

and the adjusted terrestrial observables by

$$\hat{\hat{L}} = L + \hat{V} \qquad (4\text{-}23)$$

where

$$\hat{V} = A\hat{X} + W \,. \qquad (4\text{-}24)$$

The variance-covariance matrix of the adjusted network coordinates is computed as

$$\Sigma_{\overline{XYZ}} = \sigma_o^2 (\Sigma_{XYZ}^{-1} + A^T \Sigma_L^{-1} A)^{-1} \,. \qquad (4\text{-}25)$$

Through equations (4-21) and (4-25) it is easily seen how the contribution of the satellite network coordinates are entered into the solution of the problem. It should be noted that this effect must also be accounted for in the estimated variance factor $(\hat{\sigma}_o^2)$ by

$$\hat{\sigma}_o^2 = \frac{X^T \Sigma_{XYZ}^{-1} X + V^T \Sigma_L^{-1} V}{n - u - n_X} , \qquad (4-26)$$

where n is the number of unknown parameters in the network (coordinates of every point, refraction coefficient, etc.), u represents the number of observation equations, and n_X the number of known satellite determined coordinates in the network.

This model has several effects on the computation of the network and its results. First, the knowledge of the coordinates reduces the number of unknowns in the estimation procedure. If the standard deviations of the satellite determined network coordinates are less than those of the terrestrial network as computed using only terrestrial data, the combination of satellite and terrestrial data will constrain the solution to conform with the weighted satellite coordinate data. The effect of the constraints is to prevent the build-up of random and systematic errors in the network . The problem is that if the errors are large, they will propagate into the residuals of the terrestrial observables. Existence of problems like this will show up in a statistical analysis of the results of the network adjustment.

Such an analysis may lead to an attempt to model suspected systematic errors in the terrestrial observables. This may include one or more unknown parameters in scale and orientation. However, since this type of modeling is not dependent on the use of satellite network data, it is not covered herein.

When using this model for combining satellite and terrestrial networks, one must take care with the "fixing" of a terrestrial initial point. It is necessary to realize that the variance-covariance matrix of the satellite network points gives an estimate of the accuracy of the network coordinates with respect to the datum origin. If one satellite point ($\vec{r}_k$) is chosen to be fixed, the variance-covariance matrix Σ_{xyz} must be altered to reflect the fact that the satellite coordinates will have variances, and covariance amongst them, with respect to the fixed point. This is done using the equation

$$\vec{r}_{ik} = \vec{r}_i - \vec{r}_k \tag{4-27}$$

and the covariance law to yield a matrix $\Sigma_{(\Delta x\ \Delta y\ \Delta z)_{ik}}$ by

$$\Sigma_{(\Delta x\ \Delta y\ \Delta z)_{ik}} \quad G\Sigma_{(xyz)_{ik}} G^T \tag{4-28}$$

in which

$$G = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right], \tag{4-29}$$

and

$$\Sigma_{(xyz)_{ik}} = \left[\begin{array}{c|c} \Sigma_{(xyz)_{ik}} & COV_{(xyz)_{ik}} \\ COV_{(xyz)_{ki}} & \Sigma_{(xyz)_{ik}} \end{array}\right]. \tag{4-29a}$$

This procedure propagates the errors of the fixed satellite point into all others rather than simply, and incorrectly, discarding it.

An alternative to the above is to accept the satellite network datum as the one to be used and let the weighted satellite network coordinates define it in the network adjustment process. In this case, no terrestrial points can be held fixed. This is considered to be a logical solution due to the nature and relationship of and between satellite networks and their datums.

4.3 Satellite Coordinate Differences as Observables

As in the case of using satellite network coordinates as weighted parameters, the satellite coordinates are first transformed, using the known datum transformation parameters, to the desired coordinate system. Then, observation equations of the form

$$\begin{aligned} X_{ij} &= X_j - X_i \\ Y_{ij} &= Y_j - Y_i \\ Z_{ij} &= Z_j - Z_i \end{aligned} \tag{4-30}$$

or equivalently

$$\begin{aligned} \phi_{ij} &= \phi_j - \phi_i \\ \lambda_{ij} &= \lambda_j - \lambda_i \\ h_{ij} &= h_j - h_i \end{aligned} \tag{4-31}$$

are formed. These equations are used to compute elements of the design matrix (A) and misclosure vector (W^{ρ}). The associated variance-covariance matrix of the observations is given by the covariance law as

$$\Sigma_{\Delta} = G \Sigma_{XYZ} G^T , \tag{4-32}$$

where the elements of G are given by

$$G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] . \tag{4-33}$$

The matrix Σ_{Δ} then becomes part of the variance-covariance matrix of the observables. The network can be solved using a simple least squares parametric estimation.

The major drawback with this approach as compared to that presented in 4.2 is the extra computational effort required to enter the same amount of satellite data into a terrestrial network adjustment. In order to get all information into the solution, it is necessary to write observation equations of the form (4-30) or (4-31) amongst all satellite points. Then, the associated variance-covariance matrix must be generated using (4-32). The comments given in 4.2 regarding the constraints imposed on the terrestrial network apply here as well. Another disadvantage of this procedure is that the satellite network coordinates can not be used to define a datum.

4.4 Computed Spatial Distances, Azimuths, and Vertical Angles as Observables

This model requires that the spatial distances, azimuths, and vertical angles computed from the transformed satellite network coordinates be entered as observables in the three-dimensional terrestrial network computation procedure. The spatial distances are given by

$$S_{ij} = ((X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2)^{1/2} \qquad (4\text{-}34)$$

The azimuth (a_{ij}) and vertical angle (V_{ij}), expressed in the Local Geodetic coordinate system at i, are given by [Krakiwsky and Thomson, 1974]

$$a_{ij} = \tan^{-1} \left(\frac{Y_{ij}}{X_{ij}}\right) , \qquad (4\text{-}35)$$

$$V_{ij} = \sin^{-1} \left(\frac{Z_{ij}}{S_{ij}}\right) . \tag{4-36}$$

In (4-35) and (4-36), the coordinate differences (X_{ij}, Y_{ij}, Z_{ij}) are expressed in the Local Geodetic Coordinate system. For generation of the associated variance-covariance matrix, which requires the use of coordinate differences in the chosen Geodetic coordinate system, the equation

$$\begin{bmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{bmatrix}_{LG} = \begin{bmatrix} -\sin\phi_i \cos\lambda_i & -\sin\phi_i \sin\lambda_i & \cos\phi_i \\ -\sin\lambda_i & \cos\lambda_i & 0 \\ \cos\phi_i \cos\lambda_i & \cos\phi_i \sin\lambda_i & \sin\phi_i \end{bmatrix} \begin{bmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{bmatrix}_G \tag{4-37}$$

must be used. This leads to the transformation matrix, G_{SaV}, to be used in the covariance law (4-32), given by

$$G_{SaV} = \left[\begin{array}{ccc|ccc} \frac{\partial S_{ij}}{\partial X_i} & \frac{\partial S_{ij}}{\partial Y_i} & \frac{\partial S_{ij}}{\partial Z_i} & \frac{\partial S_{ij}}{\partial X_j} & \frac{\partial S_{ij}}{\partial Y_j} & \frac{\partial S_{ij}}{\partial Z_j} \\ \frac{\partial a_{ij}}{\partial X_i} & \frac{\partial a_{ij}}{\partial Y_i} & \frac{\partial a_{ij}}{\partial Z_i} & \frac{\partial a_{ij}}{\partial X_j} & \frac{\partial a_{ij}}{\partial Y_j} & \frac{\partial a_{ij}}{\partial Z_j} \\ \frac{\partial V_{ij}}{\partial X_i} & \frac{\partial V_{ij}}{\partial Y_i} & \frac{\partial V_{ij}}{\partial Z_i} & \frac{\partial V_{ij}}{\partial X_j} & \frac{\partial V_{ij}}{\partial Y_j} & \frac{\partial V_{ij}}{\partial Z_j} \end{array}\right] . \tag{4-38}$$

Obviously, this model inputs an equivalent amount of information as compared to the previous two procedures (4.2 and 4.3) if distances, azimuths, and vertical angles between <u>all</u> satellite network points are used. Again, the major problem is the extra effort to be expended to get the same results.

Anderle [1974(c)] used a combination of this model and that presented in (4.2) in combining portions of the United States Doppler network and geodimeter traverses. The Doppler coordinates were used as weighted parameters, and distances, azimuths and vertical angles were computed from geodimeter traverse data. The recommendations as a result of this test were as follows:

> "The North American Datum readjustment should be based upon simultaneous adjustment of Doppler satellite and terrestrial data ..... The vertical adjustment should be made prior to or simultaneously with the horizontal adjustment in order that the strength of the Doppler determinations of absolute height can be used to prevent distortion of the vertical datum at the edges or along spurs which would transfer errors into the horizontal adjustment."

It should be noted that a variation of this procedure is possible in which knowledge of the datum transformation parameters is not needed. Spatial distances and spatial angles amongst three-dimensional network points are independent of any coordinate system. Satellite network spatial distances are computed via (4-34), while the spatial angle between any two satellite network interstation vectors $\vec{\rho}_{ij}$ and $\vec{\rho}_{i\ell}$ is given by

$$\theta_{ji\ell} = \cos^{-1}\left(\frac{\vec{\rho}_{ij} \cdot \vec{\rho}_{i\ell}}{|\vec{\rho}_{ij}|\;|\vec{\rho}_{i\ell}|}\right) \tag{4-39}$$

In the presently used three-dimensional terrestrial network adjustments [Vincenty, 1973; Fubara, 1972] the spatial angle data can not be utilized. If changes were made to accommodate the use as observables of both distances and spatial angles derived from the satellite network, the effects on the terrestrial network would be similar.

5. TWO-DIMENSIONAL MODELS

While any two-dimensional approach to the combination of satellite and terrestrial networks is not as desirable as the three-dimensional procedures, there are some practical considerations involved. In North America a three-dimensional approach is not possible at present due to the distribution of terrestrial data. It is therefore necessary to eliminate the height component and associated variances and covariances from the satellite network data to use the latter in a horizontal terrestrial network adjustment.

The basic drawback of the two-dimensional procedures is the partial loss of satellite network information. Covariance amongst the horizontal (ϕ, λ) and vertical (h) components of each network point, and covariance between the horizontal coordinates of one point and the height components of all other points, is not taken into account.

Three alternative two-dimensional procedures are presented. They have been used in this and other studies to generate test results and are considered to be feasible two-dimensional combination approaches. The assumptions are that proper datum transformations are to be applied to work in the chosen Geodetic coordinate system, and that the terrestrial network computations are to be carried out on the surface of a reference ellipsoid.

5.1 Satellite Coordinates as Weighted Parameters

The satellite network coordinates, coincident with the horizontal terrestrial network points, are used as weighted parameters in a least-squares parametric adjustment of the latter network. Assuming that the computations will be carried out in the chosen Geodetic coordinate system, the satellite network coordinates and associated variance-covariance matrix must be transformed to the former system using the known transformation parameters.

The procedure to prepare the satellite network data is as follows. First, compute (ϕ_i, λ_i, h_i) from (X_i, Y_i, Z_i) (if required) by the iterative procedure for h_i and ϕ_i [Heiskanen and Moritz, 1967]

$$P = (X_i^2 + Y_i^2)^{1/2} , \tag{5-1}$$

$$\tan \phi_i = \frac{Z_i}{P} \left(1 - e^2 \frac{N_i^*}{N_i^* + h_i}\right)^{-1} , \tag{5-2}$$

$$h_i = \frac{P}{\cos \phi_i} - N_i^* . \tag{5-3}$$

Note that in the first iteration, one sets $h = 0$ so that

$$\tan \phi_i = \frac{Z_i}{P} (1-e^2)^{-1} . \tag{5-4}$$

The longitude is given directly by

$$\lambda_i = \tan^{-1}(Y_i/X_i). \tag{5-5}$$

Alternately, one may use a closed form solution for the above as suggested by M.K. Paul [1973]. After this, $\Sigma_{\phi,\lambda,h}$ is computed using the covariance law by

$$\Sigma_{\phi\lambda h} = G\,\Sigma_{XYZ}\,G^T , \tag{5-6}$$

where for each point i

$$G = \begin{bmatrix} -\dfrac{\sin\phi_i\cos\lambda_i}{M_i+h_i} & -\dfrac{\sin\phi_i\sin\lambda_i}{M_i+h_i} & \dfrac{\cos\phi_i}{M_i+h_i} \\[2ex] -\dfrac{\sin\lambda_i}{(N_i^*+h_i)\cos\phi_i} & \dfrac{\cos\lambda_i}{(N_i^*+h_i)\cos\phi_i} & 0 \\[2ex] \cos\phi_i\cos\lambda_i & \cos\phi_i\sin\lambda_i & \sin\phi_i \end{bmatrix} . \tag{5-7}$$

The result of (5-6) is a fully populated variance-covariance matrix $\Sigma_{\phi\lambda h}$ whose elements, for each set of points i and j, are

$$\Sigma_{\phi\lambda h} = \left[\begin{array}{ccc|ccc} \sigma^2_{\phi_i} & \sigma_{\phi_i\lambda_i} & \sigma_{\phi_i h_i} & \sigma_{\phi_i\phi_j} & \sigma_{\phi_i\lambda_j} & \sigma_{\phi_i h_j} \\ \sigma_{\phi_i\lambda_i} & \sigma^2_{\lambda_i} & \sigma_{\lambda_i h_i} & \sigma_{\lambda_i\phi_j} & \sigma_{\lambda_i\lambda_j} & \sigma_{\lambda_i h_j} \\ \sigma_{\phi_i h_i} & \sigma_{\lambda_i h_i} & \sigma^2_{h_i} & \sigma_{h_i\phi_j} & \sigma_{h_i\lambda_j} & \sigma_{h_i h_j} \\ \hline & & & \sigma^2_{\phi_j} & \sigma_{\phi_j\lambda_j} & \sigma_{\phi_j h_j} \\ & \mathrm{cov}_{ji} & & \sigma_{\phi_j\lambda_j} & \sigma^2_{\lambda_j} & \sigma_{\lambda_j h_j} \\ & & & \sigma_{\phi_j h_j} & \sigma_{\lambda_j h_j} & \sigma^2_{h_j} \end{array}\right] \tag{5-8}$$

This matrix, $\Sigma_{\phi\lambda h}$ (5-8), is inverted to yield a weight matrix, $\Sigma^{-1}_{\phi\lambda h}$, pertaining to the satellite network coordinates. To be used in the two dimensional terrestrial network adjustment, the rows and columns pertaining to the ellipsoidal height are rigorously eliminated during the inversion process.

The idea of this model is to use satellite coordinates that have an accuracy that will tend to constrain the usual build-up of random and systematic errors in a terrestrial network. These errors will now overflow into the residual and solution vectors. Unfortunately if the residuals increase too much, the adjustment may not be accepted as a result of some statistical testing (such as an analysis of variance).

The two-dimensional model has been used in some studies of the combination of North American horizontal terrestrial and Doppler geodetic networks. In one United States study [Dracup, 1975], five weighted* Doppler positions were used in a 1566 station network. The results of several adjustments, with and without the Doppler network data, were compared to see what the effects were on the network. In his conclusions, Dracup [1975] stated

> " ... geodesists have a powerful and accurate tool, in the form of Doppler positions, for strengthening existing networks and for establishing the fundamental framework in those areas which are now devoid of control, to which conventional geodetic networks may be fitted. Although the inclusion of Doppler positions cannot eliminate observational problems, there are occasions where it might be possible to uncover poor observations or network geometry not previously suspected".

In a similar study [Chamberlain et al., 1976] weighted Doppler positions helped to determine network scale differences of 2.3 ppm

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* No information regarding the computation of $\Sigma_{\phi\lambda}^{-1}$ was available.

when using geodimeter or tellurometer measurements, and an azimuth orientation discrepancy of ± 0.5 arc-seconds. Further discussion regarding some test results when using this procedure is given in 10.1.

As with the three-dimensional model of this type, variations of it can be implemented. For example, the chosen terrestrial geodetic coordinate system may be that of the satellite network thus eliminating the need to transform the coordinates and their variance-covariance matrix. Again, the argument of whether or not to "fix" one point as a terrestrial network initial point enters the situation. The steps to be taken, and the implications, are the same as those given in 4.2.

5.2 Satellite Coordinate Differences as Observables

The steps outlined in 5.1 regarding the application of transformation parameters to the satellite network coordinates and their variance-covariance matrix must be carried out. Observation equations of the form

$$\phi_{ij} = \phi_j - \phi_i \quad , \qquad (5\text{-}9)$$

$$\lambda_{ij} = \lambda_j - \lambda_i \quad ,$$

are computed. Weights of the quasi-observables are computed via the covariance law in the same manner as described in 4.3. These weights are added, in appropriate locations, to the weight matrix of observables,

Σ_L, to be used in the least-squares parametric adjustment of the terrestrial network.

This procedure adds equivalent information to the terrestrial network as the method of weighted parameters as long as coordinate differences amongst all network points are included. As with the three-dimensional model of this type, the added quasi-observables will constrain and strengthen the network. This can help to determine weak points in the terrestrial network, and add some scale and orientation information.

The major drawback with this approach is the extra computational effort required to obtain the same results as those of 5.1.

5.3 Computed Distances and Azimuths as Observables

The major problem here is to compute the required ellipsoidal distances and azimuths and their variance-covariance matrix. The most rigorous approach is to compute the ellipsoidal coordinates of the satellite network points (ϕ_i, λ_i) and their variance-covariance matrix $\Sigma_{\phi\lambda}$. Then, using rigorous ellipsoidal formulae [e.g. Bomford, 1971; Krakiwsky and Thomson, 1974], compute the geodetic distances S_{ij} and azimuths α_{ij} amongst all satellite network points. It should be noted that the α_{ij} are geodetic, and not astronomic azimuths. This means that they are not subject to computation via the Laplace equation (1-7). The quantities are then used in the formulation of distance and azimuth observation equations for the least-squares parametric terrestrial network adjustment.

The associated variance-covariance matrix of the quasi-observables, $\Sigma_{S\alpha}$, is obtained via

$$\Sigma_{S\alpha} = G_{S\alpha} \Sigma_{\phi\lambda} G^{T}_{S\alpha} \quad , \tag{5-11}$$

in which $\Sigma_{\phi\lambda}$ is obtained simply by eliminating from $\Sigma_{\phi\lambda h}$ (5-8) those elements that pertain to the ellipsoidal heights. The elements of $G_{S\alpha}$ are given adequately by using spherical approximations for S_{ij} and α_{ij}, namely

$$S_{ij} = R\theta \tag{5-12}$$

$$\alpha_{ij} = \cot^{-1}\beta \tag{5-13}$$

in which

$$R = (R_i + R_j)/2 \quad , \tag{5-14}$$

where R_i and R_j are the Euler radii of curvature, and

$$\theta = \cos^{-1} (\sin \phi_i \sin \phi_j + \cos \phi_i \cos \phi_j \cos (\lambda_j - \lambda_i)), \tag{5-15}$$

$$\beta = \frac{\tan \phi_j \cos \phi_i - \sin \phi_i \cos (\lambda_j - \lambda_i)}{\sin (\lambda_j - \lambda_i)} \quad . \tag{5-16}$$

$G_{S\alpha}$ is computed by

$$G_{S\alpha} = \left[\begin{array}{cc|cc} \frac{\partial S_{ij}}{\partial \phi_i} & \frac{\partial S_{ij}}{\partial \lambda_i} & \frac{\partial S_{ij}}{\partial \phi_j} & \frac{\partial S_{ij}}{\partial \lambda_j} \\ & & & \\ \frac{\partial \alpha_{ij}}{\partial \phi_i} & \frac{\partial \alpha_{ij}}{\partial \lambda_i} & \frac{\partial \alpha_{ij}}{\partial \phi_j} & \frac{\partial \alpha_{ij}}{\partial \lambda_j} \end{array} \right] \quad , \tag{5-17}$$

where

$$\frac{\partial S_{ij}}{\partial \phi_i} = \frac{-R}{\sin\theta}(\cos\phi_i \sin\phi_j - \sin\phi_i \cos\phi_j \cos(\lambda_j-\lambda_i)), \quad (5\text{-}18)$$

$$\frac{\partial S_{ij}}{\partial \lambda_i} = \frac{-R}{\sin\theta}(\cos\phi_i \cos\phi_j \sin(\lambda_j - \lambda_i)), \quad (5\text{-}19)$$

$$\frac{\partial S_{ij}}{\partial \phi_j} = \frac{-R}{\sin\theta}(\sin\phi_i \cos\phi_j - \cos\phi_i \sin\phi_j \cos(\lambda_j-\lambda_i)), \quad (5\text{-}20)$$

$$\frac{\partial S_{ij}}{\partial \lambda_j} = \frac{-R}{\sin\theta}(-\cos\phi_i \cos\phi_j \sin(\lambda_j - \lambda_i)), \quad (5\text{-}21)$$

$$\frac{\partial \alpha_{ij}}{\partial \phi_i} = \sin^2\alpha_{ij} \frac{\tan\phi_j \sin\phi_i \quad \cos\phi_i \cos(\lambda_j-\lambda_i)}{\sin(\lambda_j-\lambda_i)}), \quad (5\text{-}22)$$

$$\frac{\partial \alpha_{ij}}{\partial \lambda_i} = -\sin^2\alpha_{12}(\frac{\sin(\lambda_j-\lambda_i)\sin\phi_i}{(\sin(\lambda_j-\lambda_i))^2} \quad (5\text{-}23)$$

$$\frac{+\cos(\lambda_j-\lambda_i)(\tan\phi_i \cos\phi_i - \sin\phi_i \cos(\lambda_j-\lambda_i)}{}), \quad (5\text{-}23)$$

$$\frac{\partial \alpha_{ij}}{\partial \phi_j} = -\sin^2\alpha_{12}(\frac{\sec^2\phi_{ij}\cos\phi_i}{\sin(\lambda_j-\lambda_i)}), \quad (5\text{-}24)$$

$$\frac{\partial \alpha_{ij}}{\partial \lambda j} = -\sin^2\alpha_{12}(\frac{\sin^2(\lambda_j-\lambda_i)\ \sin\phi_j\ -}{(\sin(\lambda_j-\lambda_i))^2}$$

$$\frac{-\cos(\lambda_j-\lambda_i)(\tan\phi_j \cos\phi_i - \sin\phi_i \cos(\lambda_j-\lambda_i))}{}). \quad (5\text{-}25)$$

An alternative approach is to compute the spatial distances and azimuths, in the geodetic coordinate system, amongst all network points using (4-34) and (4-35) respectively. These computed quantities are then projected to the reference ellipsoid using rigorous methods [e.g. Bomford, 1971; Krakiwsky and Thomson, 1974], noting that the azimuths need no correction for the deflection of the vertical. The transformation matrix $G^*_{S\alpha}$ required for error propagation is generated via equations (4-34) and (4-35) to yield

$$G^*_{S\alpha} = \begin{bmatrix} -\frac{X_{ij}}{S_{ij}} & -\frac{Y_{ij}}{S_{ij}} & -\frac{Z_{ij}}{S_{ij}} & \frac{X_{ij}}{S_{ij}} & \frac{Y_{ij}}{S_{ij}} & \frac{Z_{ij}}{S_{ij}} \\ \frac{\sin\lambda_i X_{ij} - \cos\phi_i \cos\lambda_i Y_{ij}}{X^2_{ij} + Y^2_{ij}} & \frac{\cos\lambda_i X_{ij} - \sin\phi_i \sin\lambda_i Y_{ij}}{X^2_{ij} + Y^2_{ij}} & \frac{\cos\phi_i Y_{ij}}{X^2_{ij}+Y^2_{ij}} & -\frac{\sin\lambda_i X_{ij} + \sin\phi_i \cos\lambda_i Y_{ij}}{X^2_{ij} + Y^2_{ij}} & -\frac{\cos\lambda_i X_{ij} + \sin\phi_i \sin\lambda_i Y_{ij}}{X^2_{ij} + Y^2_{ij}} & -\frac{\cos\phi_i Y_{ij}}{X^2_{ij}+Y^2_{ij}} \end{bmatrix} \qquad (5\text{-}26)$$

in which the coordinate differences (X_{ij}, Y_{ij}, Z_{ij}) are expressed in the chosen Geodetic coordinate system. The variance-covariance matrix is then given by

$$\Sigma^*_{S\alpha} = G^*_{S\alpha}\, \Sigma_{XYZ}\, G^{*T}_{S\alpha} \quad . \qquad (5\text{-}27)$$

This approach is based on the assumption that the projections of S_{ij} and α_{ij} are carried out without introducing any errors, and that the distances and azimuths so deduced will not differ significantly from the quantities computed using ellipsoidal coordinates in rigorous ellipsoidal geodetic formulae. In some test computations (10.2) in which satellite station separations of up to 1000 km exist, maximum differences in the distances and azimuths computed via the two aforementioned procedures were found to be 0.29 m (0.3 ppm) and -0.842 arc-seconds respectively. The maximum difference in standard deviations of the quasi-observables, computed using $G_{S\alpha}$ and $G^{*}_{S\alpha}$ were found to be -0.01 m for distances and -0.204 arc-seconds for azimuths.

The effects of this model in a terrestrial horizontal geodetic network are the same as those of 5.1 and 5.2, as long as all data is utilized. The extra computational effort over either of the previous two alternatives is obvious.

Meade [1974] used this two-dimensional model in a combination of portions of the United States terrestrial and Doppler geodetic networks. In two separate network adjustments, one controlled by conventional terrestrial base lines and astronomic azimuths and the other by azimuths and distances computed from four Doppler positions, he found mean differences of 10 ppm in scale and 0.45 arc-seconds in azimuth. Further investigations, using the aforementioned comparison of network adjustment results, brought to light an azimuth discrepancy of 3 arc-seconds in a portion of the terrestrial network [Dracup, 1975].

6. ONE DIMENSIONAL MODELS

As pointed out in 5, horizontal coordinates (ϕ, λ) and their variance-covariance matrix can be easily extracted from the three-dimensional Cartesian coordinates and associated variance-covariance matrix of a satellite network. Similarly, the height components (h) and their variance-covariance matrix can be split from the original data. The objective here is to examine briefly the possibilities of utilizing this height data in combination with terrestrially determined height information.

Again, the loss of covariance can be detrimental to this type of approach. Whether or not the discarded covariance between height and horizontal coordinates is significant is dependent on the degree of correlation between them. This could only be determined through some numerical testing in which the same data is used for both a three-dimensional combination and then separate two-dimensional and one-dimensional combinations.

6.1 Vertical Networks and Geoidal Heights

The information used in the combination of satellite and terrestrial height networks comes from the equation

$$h = H + N \ . \tag{6-1}$$

The ellipsoidal height (h) is obtained directly from satellite networks. Using known datum transformation parameters in equation (1-9), the satellite network coordinates are transformed into the desired geodetic coordinate system. Then, using the procedures outlined in 5.1, the satellite network coordinates (ϕ, λ, h) and associated variance-covariance matrix are computed. From these, the ellipsoidal heights and variance-covariance matrix are extracted. In satellite networks already completed, such as the North American Densification of the World Geodetic Satellite Triangulation, the mean standard deviation of the ellipsoidal height is of the order of 7 m [Pope, 1975]. The results in Doppler networks are much better, yielding standard deviations for h of 0.6 m to 1.6 m [Kouba, 1976(b)]. In a recent paper, Kouba [1976(a)]. has described a procedure termed "Doppler Levelling" by which height differences between stations 50 km apart can be determined to 0.4 m (1 σ). Further, he proposes that with improvements in instrumentation and error modelling Doppler Levelling will yield height differences with standard deviations of 0.2 m or less for the previously mentioned station separation.

The orthometric heights, H, of points in the North American networks are determined as outlined in 2.1.2. Using the rule of thumb given in 1.3, the standard deviation of the orthometric height difference between stations 50 km apart would be of the order of 0.02 m. This is one order of magnitude better than the predicted results for ellipsoidal height differences using Doppler Levelling.

There are several approaches to the computation of the geoid. The results of some recent astrogravimetric geoid computations at the University of New Brunswick indicate that the geoidal height difference can be determined with a standard deviation of the order of 0.5 m for stations 50 km apart [Merry, 1975]. This accuracy is of the same order as those presently attainable for ellipsoidal height differences by Doppler Levelling.

In combining terrestrial and satellite vertical networks there are basically two possibilites to be considered. First, using terrestrially determined orthometric heights and satellite determined ellipsoidal heights (transformed to the desired geodetic coordinates system via known transformation parameters) (6-1) yields geoidal heights. These geoidal heights, with $\sigma_N \approx 0.5$ m, can then be used as constraints in geoid computations [Merry, 1975]. The second approach is to use geoidal heights combined with satellite network ellipsoidal heights to yield orthometric heights. The resulting orthometric heights, with standard deviations of the order of 0.5 m, could be used for the reduction of observables in terrestrial horizontal networks. They would also be of sufficient quality to be used as part of an orthometric height network. For example, such heights would have sufficient accuracy to be used in a 1:50000 mapping program.

The conclusion here is that the combination of terrestrial and satellite vertical networks are best utilized in supplying constraints for geoid computations. The accuracies of satellite determined ellipsoidal heights and geoid computations have not reached the stage where the resulting orthometric heights can be used for more than "lower" order vertical information.

SECTION III: COMBINATION PROCEDURES WHEN DATUM TRANSFORMATION PARAMETERS ARE UNKNOWN

7. STANDARD MODELS

The determination of datum transformation parameters is a three-dimensional problem. The rigorous combination of terrestrial and satellite geodetic networks will yield the transformation parameters between their respective datums. Satellite networks are immediately ready to be combined, while classical terrestrial networks must first be made three-dimensional (1.4) before they can be utilized.

Several models have been developed which describe the functional relationships between pairs of three-dimensional coordinates. In each, the network Cartesian coordinates are used as quasi-observables and thus receive corrections as a result of the combination estimation procedure.

Three mathematical models, noted as "standard" due to their extensive use over a number of years, are given herein.* The models differ from each other in several ways, including a priori conditions, the type of coordinates used, and the interpretation of results. These differences, and others, are examined fully in 7.4.

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* The names given to these three models - Bursa, Molodensky, Veis - have already been used by several authors. This practice is followed here and is not meant to indicate who may have been responsible for the original derivation of the models.

7.1 Bursa

The Bursa model [Bursa, 1962; Wolf, 1963; Badekas, 1969; Lambeck, 1971] expresses the relationship between two coordinate systems by three translations (x_o, y_o, z_o), three rotations (ε_x, ε_y, ε_z), and a scale change (κ). The two sets of network coordinates for any terrain point i are used as observables in the model given by (Figure 7-1)

$$\vec{F}_i = (\vec{r}_o)_1 + (1+\kappa)\ R_\varepsilon\ (\vec{r}_i)_2 - (\vec{\rho}_i)_1 = 0\ . \qquad (7\text{-}1)$$

In the above, $\vec{r}_o$ is the translation vector between the origin of coordinate systems 1 and 2, R_ε is the matrix given by

$$R_\varepsilon = R_1(\varepsilon_x),\ R_2(\varepsilon_y),\ R_3(\varepsilon_z)\ , \qquad (7\text{-}2)$$

where R_1, R_2, and R_3 are the rotation matrices given by equations (1-11), (1-12) and (1-13), and $\vec{\rho}_i$ and $\vec{r}_i$ are the position vectors of the terrain point i in coordinate systems 1 and 2 respectively.

Expanding (7-2) as in 4.1, and substituting in (7-1) yields

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_1 + (1+\kappa) \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}_2 - \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}_1 = 0\ . \qquad (7\text{-}3)$$

This model is solved easily using a combined case least squares estimation procedure (4.1). The elements of the design matrix A are given by

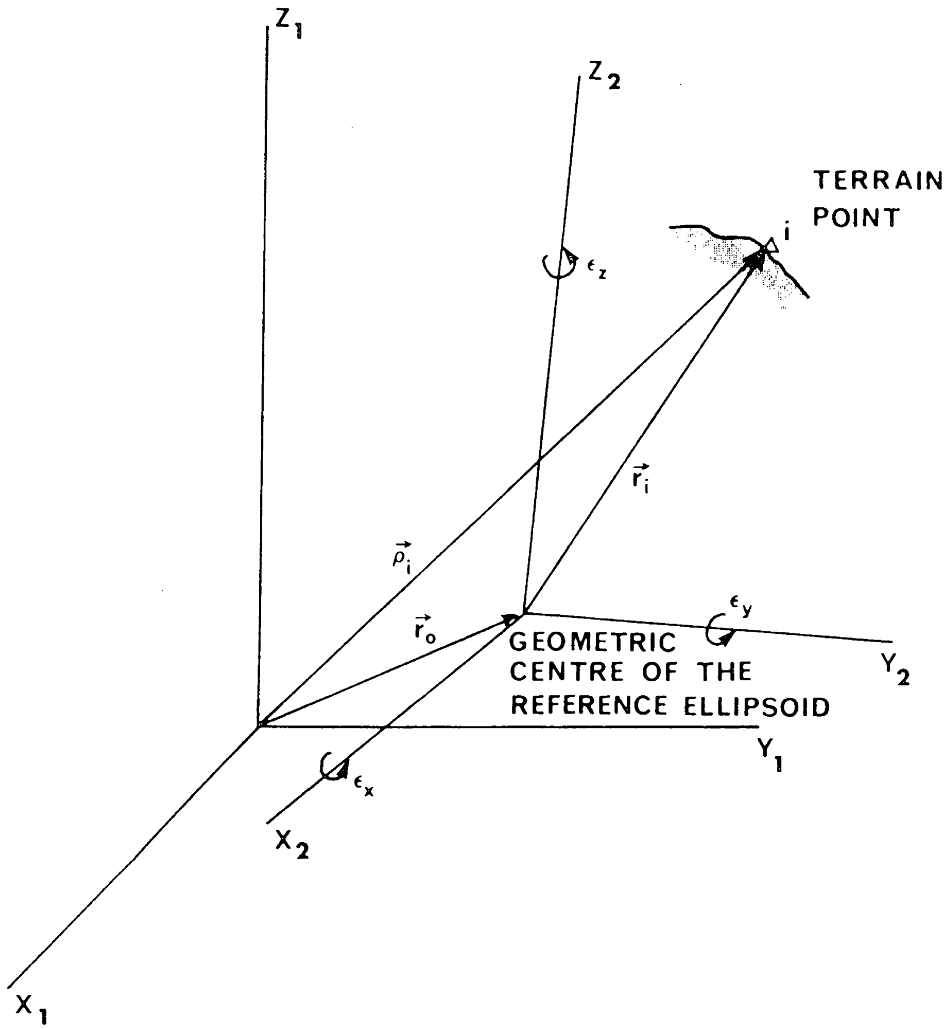

Figure 7-1

Bursa Model

$$A_i = \frac{\partial \vec{F}_i}{\partial X}\bigg|_{X,L} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -(z_i+\kappa^o z_i) & (y_i+\kappa^o y_i) & (x_i+\varepsilon_z^o y_i-\varepsilon_y^o z_i) \\ 0 & 1 & 0 & (z_i+\kappa^o z_i) & 0 & -(x_i+\kappa^o x_i) & (y_i+\varepsilon_x^o z_i-\varepsilon_z^o x_i) \\ 0 & 0 & 1 & -(y_i+\kappa^o y_i) & (x_i+\kappa^o x_i) & 0 & (z_i+\varepsilon_y^o x_i-\varepsilon_x^o y_i) \end{array}\right] , \tag{7-4}$$

where the superscript ^{o} indicates initial approximate values of the unknown parameters, X. Similarly, the design matrix B is given by

$$B_i = \frac{\partial \vec{F}_i}{\partial L_i}\bigg|_{X,L} = \left[\begin{array}{ccc|ccc} (1+\kappa^o) & (\varepsilon_z^o+\kappa^o\varepsilon_z^o) & (\varepsilon_y^o+\kappa^o\varepsilon_y^o) & -1 & 0 & 0 \\ -(\varepsilon_z^o+\kappa^o\varepsilon_z^o) & (1+\kappa^o) & -(\varepsilon_x^o+\kappa^o\varepsilon_x^o) & 0 & -1 & 0 \\ -(\varepsilon_z^o+\kappa^o\varepsilon_y^o & (\varepsilon_x^o+\kappa^o\varepsilon_x^o) & (1+\kappa^o) & 0 & 0 & -1 \end{array}\right] . \tag{7-5}$$

If the point of expansion is taken as

$$(X^o)^T = (x_o^o,\ y_o^o,\ z_o^o,\ \varepsilon_x^o,\ \varepsilon_y^o,\ \varepsilon_z^o,\ \kappa^o) = 0 , \tag{7-6}$$

then A and B reduce to

$$A_i = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -z_i & y_i & x_i \\ 0 & 1 & 0 & z_i & 0 & -x_i & y_i \\ 0 & 0 & 1 & -y_i & x_i & 0 & z_i \end{array}\right] , \tag{7-7}$$

$$B_i = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] . \tag{7-8}$$

The misclosure vector is given by

$$W_i^o = F(X^o,\ L) , \tag{7-9}$$

which in this case is

$$W_i^o = \begin{bmatrix} x_i - X_i \\ y_i - Y_i \\ z_i - Z_i \end{bmatrix} . \qquad (7\text{-}10)$$

The results of the estimation procedure are the solution vector, $\hat{X}$, the residual vector, $\hat{V}$, and their associated variance-covariance matrices, $\Sigma_{\hat{X}}$ and $\Sigma_{\hat{V}}$. The final results are the transformation components, $\hat{\bar{X}}$, given by

$$\hat{\bar{X}} = X^o + \hat{X} \quad , \qquad (7\text{-}11)$$

and the adjusted common network coordinates,

$$\hat{\bar{L}} = L + \hat{V} \ . \qquad (7\text{-}12)$$

Their respective variance-covariance matrices are given by (4-15) and (4-16).

The use of the coordinates of common network points as observables is important in the analysis of this model. This is valid when the networks are relatively free of systematic errors. If, however, one of the networks contains systematic errors, these unknown errors will be confused with the datum transformation parameters. Furthermore, this approach assumes that the network coordinates can be used to recover a datum. As explained earlier (3.1), this is true for satellite networks and their datums. The conclusion drawn here, as a result of a study of the model and the generation of several sets of test results, is that the Bursa model is adequate for the combination of two satellite networks, but not for a terrestrial and a satellite geodetic network.

Variations of the Bursa model can be achieved by solving for fewer unknown parameters. For example, if it is suspected that the two coordinate systems involved have a scale difference, and are translated parallel to each other, one need only solve for these four parameters. Other combinations of unknowns are possible and in many instances tests should be carried out so as not to try and solve for more unknown transformation parameters than are really present.

7.2 Molodensky

This model is given by [Molodensky et al., 1962; Badekas, 1969; Mueller et al., 1970]

$$\vec{F}_i = (\vec{r}_o)_1 + (\vec{r}_k)_2 + (1+\kappa)\; R_\psi (\vec{r}_{ki})_2 - (\vec{\rho}_i)_1 = 0 \; . \tag{7-13}$$

The newly introduced vector $\vec{r}_k$ is that of the "initial point" of the second network. The quantity R_ψ is a combined rotation matrix

$$R_\psi = R_1(\psi_x)\; R_2(\psi_y)\; R_3(\psi_z) \; , \tag{7-14}$$

where R_1, R_2 and R_3 are the rotation matrices given previously (4.1). The vector $\vec{r}_{ki}$ represents the position vector differences of the second network (Figure 7-2). In an expanded form, (7-13) becomes

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_1 + \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}_2 + (1+\kappa) \begin{bmatrix} 1 & \psi_z & -\psi_y \\ -\psi_z & 1 & \psi_x \\ \psi_y & -\psi_x & 1 \end{bmatrix} \begin{bmatrix} x_i - x_k \\ y_i - y_k \\ z_i - z_k \end{bmatrix}_2 - \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_1 = 0 \; . \tag{7-15}$$

The solution of this model is easily obtained using a combined case least squares estimation procedure (4.1). The solution

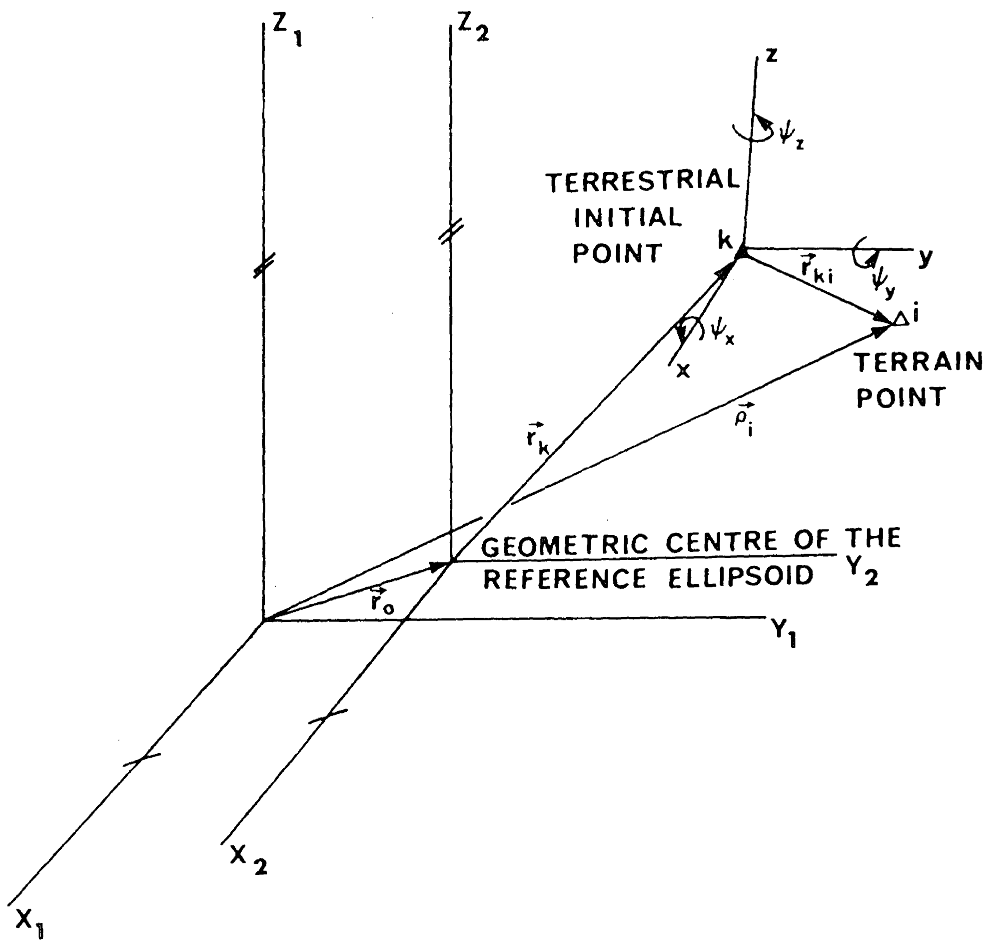

Figure 7-2

Molodensky Model

vector yields up to seven parameters (three translations, three rotations, and a scale difference), and a residual vector whose elements are corrections to the "observed" coordinates and coordinate differences. The design matrix A is given by elements

$$A_i = \frac{\partial \vec{F}_i}{\partial X}\Big|_{X^o, L} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -z_{ki} & y_{ki} & x_{ki} \\ 0 & 1 & 0 & z_{ki} & 0 & -x_{ki} & y_{ki} \\ 0 & 0 & 1 & -y_{ki} & x_{ki} & 0 & z_{ki} \end{array}\right], \tag{7-16}$$

where

$$(X^o)^T = (x_o^o, y_o^o, z_o^o, \psi_x^o, \psi_y^o, \psi_z^o, \kappa^o) = 0. \tag{7-17}$$

Similarly, B is given by

$$B_i = \frac{\partial \vec{F}_i}{\partial L_i}\Big|_{X^o, L} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right], \tag{7-18}$$

and the misclosure vector W^o by

$$W_i^o = F_i(X^o, L_i) = \begin{bmatrix} x_k + x_{ki} - X_i \\ y_k + y_{ki} - Y_i \\ z_k + z_{ki} - Z_i \end{bmatrix}. \tag{7-19}$$

The model requires the position vector of the initial point of the second network and coordinate data for at least three other common points.

The rotations and scale difference in this model refer to the geodetic network. This fact is obvious upon examination of the design matrix A, whose elements are the coordinate difference vectors, $\vec{r}_{ki}$. The main problem is that coordinate systems one and two are assumed to be parallel. This means that all rotation errors are attributed to the network difference vectors $\vec{r}_{ki}$. As with the Bursa model, this type of approach leads to the confusion of two sets of rotations - those between the coordinate systems and those associated with the second network. Also, it should be emphasized that the scale difference in this model is a network scale difference (note the last column of the design matrix A_i (7-16) which refers to the network position vectors $\vec{r}_{ki}$).

This model is not used to combine two satellite networks since in most cases these networks have no network initial points. Furthermore, in the transformation between two satellite networks, no a priori assumption regarding parallelity of axes is made.

If the Molodensky model is used to combine a terrestrial and a satellite geodetic network, the a priori assumption of parallelity of datum axes may be erroneous. Further, the rotations referring to the misaligned coordinate axes and those referring to the misoriented terrestrial network will be confused.

7.3 Veis

The Veis model [Veis, 1960; Badekas, 1969] is mathematically equivalent to the Molodensky model. The rotations, denoted dA, $d\mu$, $d\nu$, are referred to the Local Geodetic coordinate system at the initial point k (Figure 7-2). A rotation about the Z_{LG} axis, dA, corresponds to a rotation in azimuth, about the Y_{LG} axis, $d\mu$, a tilt in the meridian plane, and about the X_{LG} axis,

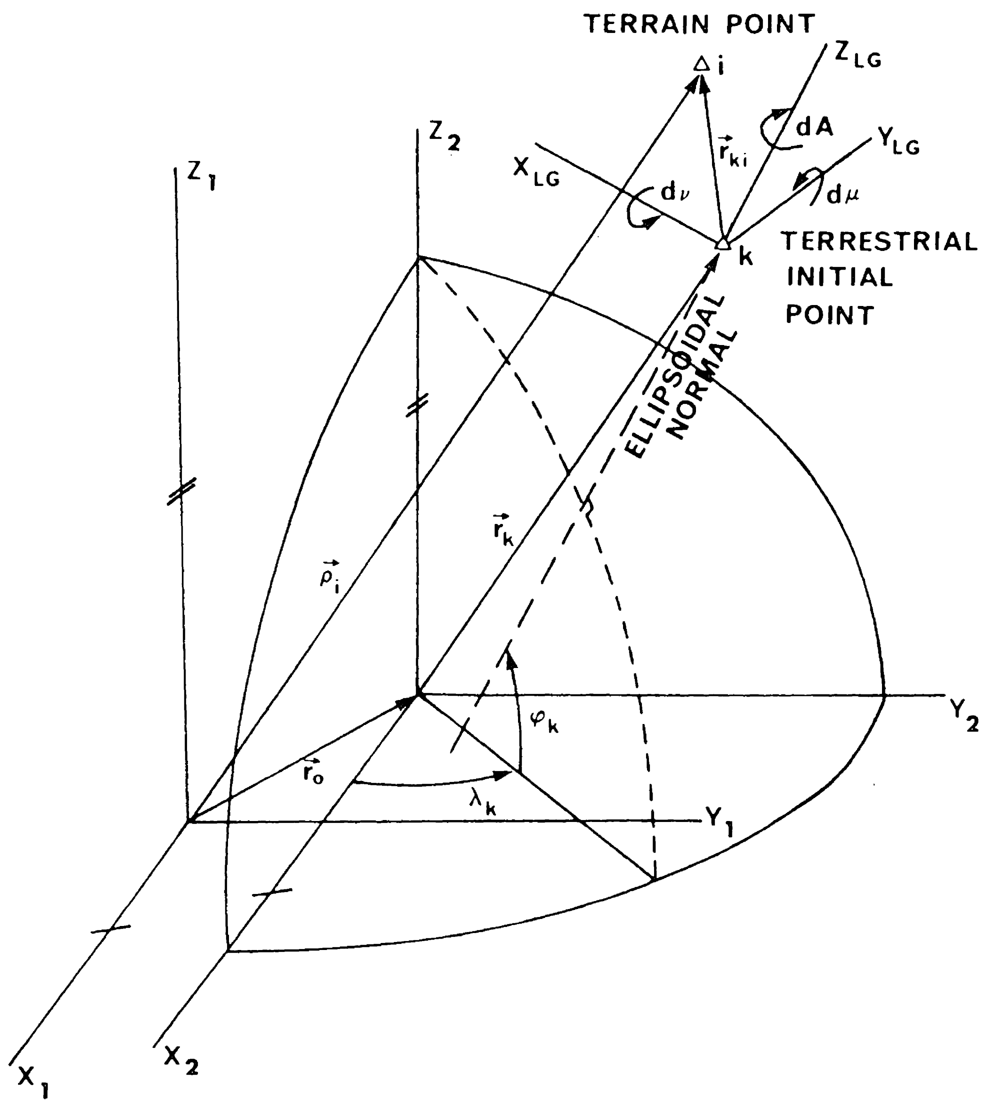

Figure 7 - 3

Veis Model

$d\nu$, a tilt in the prime vertical plane.

The model is expressed as

$$\vec{F}_i = (\vec{r}_o)_1 + (\vec{r}_k)_2 + (1+\kappa)\ R_V\ (\vec{r}_{ki})_2 - (\vec{\rho}_i)_1 = 0\ , \qquad (7\text{-}20)$$

where

$$R_V = R_3(180-\lambda_k)\ R_2(90-\phi_k)\ P_2R_1(d\nu)\ R_2(d\mu)\ R_3(dA)\ P_2R_2(\phi_k-90)R_3(\lambda_k-180)\,. \qquad (7\text{-}21)$$

In (7-21), the new quantities (ϕ_k, λ_k) are the geodetic coordinates of the initial point, and P_2 is a reflection matrix given by

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (7\text{-}22)$$

The first set of rotations, R_2 (ϕ_k-90) $R_3(\lambda_k$-180) and the reflection P_2, are required to transform the difference vector $\vec{r}_{ki}$ to the Local Geodetic system at k from the coordinate system (2) in which they are formulated. The rotations $R_1(d\nu)$ $R_2(d\mu)$ $R_3(dA)$, in which dA, $d\mu$, $d\nu$ are the unknown rotation parameters to be determined, yield a matrix

$$R_1(d\nu)\ R_2(d\mu)\ R_3(dA) = \begin{bmatrix} 1 & dA & -d\mu \\ -dA & 1 & d\nu \\ d\mu & -d\nu & 1 \end{bmatrix}\ . \qquad (7\text{-}23)$$

The final set of orthogonal transformation matrices in (7-21) are required to rotate the transformed difference vectors $\vec{r}_{ki}$ back to the second coordinate system. The final expanded result of (7-21) is given by

$$R_V = \begin{bmatrix} 1 & -\sin\phi_k dA - \cos\phi_k d\nu & \sin\lambda_k \cos\phi_k dA + \cos\lambda_k d\mu - \sin\phi_k \sin\lambda_k d\nu \\ \sin\phi_k dA + \cos\phi_k d\nu & 1 & -\cos\phi_k \cos\lambda_k dA + \sin\lambda_k d\mu + \sin\phi_k \cos\lambda_k d\nu \\ -\cos\phi_k \sin\lambda_k dA - \cos\lambda_k d\mu + \sin\phi_k \sin\lambda_k d\nu & \cos\phi_k \cos\lambda_k dA - \sin\lambda_k d\mu - \sin\phi_k \cos\lambda_k d\nu & 1 \end{bmatrix} . \quad (7\text{-}24)$$

The solution of the Veis model is the same as for the Molodensky model (combined case least-squares estimation). The design matrix B_i and misclosure vector W_i^o are equivalent to (7-18) and (7-19), respectively. The design matrix A_i has the form

$$A_i = \frac{\partial \vec{F}}{\partial \bar{X}}\Big|_{X^o, L} = \begin{bmatrix} 1 & 0 & 0 & -\sin\phi_k y_{ki} + \sin\lambda_k \cos\phi_k z_{ki} & \cos\lambda_k z_{ki} & -\cos\phi_k y_{ki} - \sin\phi_k \sin\lambda_k z_{ki} & x_{ki} \\ 0 & 1 & 0 & \sin\phi_k x_{ki} - \cos\phi_k \cos\lambda_k z_{ki} & \sin\lambda_k z_{ki} & \cos\phi_k x_{ki} + \sin\phi_k \cos\lambda_k z_{ki} & y_{ki} \\ 0 & 0 & 1 & -\cos\phi_k \sin\lambda_k x_{ki} + \cos\phi_k \cos\lambda_k y_{ki} & -\cos\lambda_k x_{ki} - \sin\lambda_k y_{ki} & \sin\phi_k \sin\lambda_k x_{ki} - \sin\phi_k \cos\lambda_k y_{ki} & z_{ki} \end{bmatrix}, \quad (7\text{-}25)$$

where

$$X^{oT} = [x_o, y_o, z_o, dA, d\mu, d\nu, \kappa] = 0 . \quad (7\text{-}26)$$

The interpretation of the results are the same as for the Molodensky model. The Molodensky rotations can be derived from the

Veis model. The functional relationships are

$$\psi_x = -\cos\phi_k \cos\lambda_k dA + \sin\lambda_k d\mu + \sin\phi_k \cos\lambda_k d\nu \ , \qquad (7\text{-}27)$$

$$\psi_y = -\cos\phi_k \sin\lambda_k dA - \cos\lambda_k d\mu + \sin\phi_k \sin\lambda_k d\nu \ , \qquad (7\text{-}28)$$

$$\psi_z = -\sin\phi_k dA - \cos\phi_k d\nu \ . \qquad (7\text{-}29)$$

The inverse of the above yields the Veis rotations in terms of those of Molodensky, namely

$$dA = -\cos\lambda_k \cos\phi_k \psi_x - \sin\lambda_k \cos\phi_k \psi_y - \sin\phi_k \psi_z \ , \qquad (7\text{-}30)$$

$$d\mu = \sin\lambda_k \psi_x - \cos\lambda_k \psi_y \ , \qquad (7\text{-}31)$$

$$d\nu = -\cos\lambda_k \sin\phi_k \psi_x + \sin\lambda_k \sin\phi_k \psi_y - \cos\phi_k \psi_z \ . \qquad (7\text{-}32)$$

The problems encountered when using the Veis model for the combination of terrestrial and satellite networks are the same as those of the Molodensky model. This model should not be used for the combination of satellite networks.

7.4 Comparison of Bursa, Molodensky and Veis Models

Based on the comparisons of the models, and the knowledge of the properties of terrestrial and satellite networks, conclusions can be drawn on their respective advantages in the combination of terrestrial and satellite geodetic networks.

All three models contain a maximum of seven unknown transformation parameters. A solution for each of the models is easily obtained via a combined case least squares estimation procedure. The Molodensky and Veis models are based on the same a priori assumption of parallelity of coordinate system axes, and contain explicit provisions for a network initial point for the second network. These latter two models use the same observables - $\vec{r}_{ki}$ and $\vec{\rho}_i$. As was shown in 7.3, the Molodensky and Veis rotations are easily related to one another.

There are several significant differences in the three models, particularly between the Bursa model, and the Molodensky and Veis models. The Bursa rotations and scale difference refer directly to the coordinate systems as is indicated by the elements of the A matrix (7-4). This interpretation is based on the fact that the two networks involved can be used to recover the origin and orientation of their respective datum axes. As argued previously (3.1), this is only possible in the case of satellite networks. On the other hand, the Molodensky and Veis models use only one network in this fashion (the first). The second network observables are network coordinate differences, and the rotations and scale difference apply to these quantities. The Bursa model has no provision for an initial point of the second network at which there are no second network coordinates. In contrast to the Molodensky and Veis models, all network points in the Bursa model are treated equivalently. Between the Molodensky and Veis models, the only difference is the orientation of the coordinate system at the initial point k. This fact has no bearing on the solution or interpretation of results.

Comparing the Bursa and Molodensky models mathematically, one obtains the difference

$$\vec{r}_{o_M} - \vec{r}_{o_B} + (Q_M - Q_B)\,\Delta\vec{r}_{ik} - Q_B\vec{r}_k + (\kappa_M - \kappa_B)\,\Delta\vec{r} - \kappa_B\vec{r}_k = 0, \qquad (7\text{-}33)$$

in which

$$Q_M = R_\psi - I, \qquad (7\text{-}34)$$

$$Q_B = R_\varepsilon - I, \qquad (7\text{-}35)$$

and the subscripts B and M refer to the Bursa and Molodensky models. If one assumes the two solutions will be the same (i.e. $\vec{r}_{o_M} = \vec{r}_{o_B}$; $Q_M = Q_B$; $\kappa_M = \kappa_B$), then one obtains

$$(Q_B + \kappa_B I)\,\vec{r}_k = 0 \qquad (7\text{-}36)$$

This expression (7-36) is satisfied if and only if $Q_B = 0$ and $\kappa_B = 0$. This is further proof of the differences between the Bursa and Molodensky (or Veis) models.

The main problem with all three models is the assumption of only one set of rotation parameters. As has been discussed previously, it is unlikely that two coordinate systems, either satellite and terrestrial datums, or two satellite or terrestrial datums, would be parallel. Thus, in any combination procedure, some or all of the Bursa rotations would be present. When a terrestrial network is involved, it is known that it would contain errors which may be modelled by either the Molodensky or Veis rotations and a scale difference.

It should be noted that when combining a terrestrial and a satellite network using the Bursa and Molodensky models, identical numerical values for the rotations are obtained (see 11.0 and 11.2). However, the translation components will differ. The reason for this is that the scale difference parameter, κ, is applied only to the coordinate differences in the Molodensky model (7-13), while in the Bursa model (7-1), it is applied to all position vectors, including that of the terrestrial initial point.

The conclusions to be drawn are first that two satellite networks, relatively free of systematic errors with respect to each other, can be combined using a Bursa model. This has been done by several investigators [Anderle, 1974(b); Mueller, 1974(b)]. The Bursa model is not adequate for the combination of a satellite and a terrestrial network, mainly because the one set of rotations does not adequately model the true situation. The Molodensky and Veis models can not be used to combine two satellite networks due to the a priori assumption of parallelity of datum axes and the necessity of having a network initial point for one of the networks. The latter models may be used to combine terrestrial and satellite networks since an initial point would be available for the terrestrial network, and the rotations and scale difference can model the errors in this latter network. However, the assumption of axes parallelity again makes the models unacceptable.

Basically, all three models - Bursa, Molodensky and Veis - are inadequate for the combination of terrestrial and satellite networks for one fundamental reason: they do not contain sufficient transformation unknowns to adequately describe the relationship between the two networks and their datums. This inadequacy led to the study and development of the more complex models of 8.1, 8.2 and 8.3.

8. RECENT MODELS

In 1.3 the problems related to the positioning and orienting of a terrestrial datum were enumerated and in 2.1 the existence of systematic errors in the terrestrial networks themselves were pointed out. When terrestrial and satellite networks are combined and the datum transformation parameters are known (SECTION II), the datum orientation and position problems are not involved, and systematic errors are assumed to have been removed in some acceptable manner. However, when datum position and orientation parameters are unknown, the combination of networks becomes complex. Further, if one attempts to model the systematic errors in the terrestrial network or to investigate the parallelism of Geodetic and satellite coordinate systems with respect to the Average Terrestrial system, other complications arise.

The Bursa, Molodensky and Veis models (7) are not adequate for the treatment of the above mentioned problems. The predominant inadequacy of the aforementioned models is the existence of only one set of rotations in each. The Hotine (8.1), Krakiwsky-Thomson (8.2), and Vanicek-Wells (8.3) models do not have this inadequacy as the models contain two sets of rotations. The first two models are formulated such that in the network combination process the systematic errors in the terrestrial network are modelled by up to three orientation parameters and a scale difference. The third model (Vanicek-Wells) is concerned partly with the lack of parallelism of Geodetic and satellite system axes with those of the Average Terrestrial system.

The data used in each of these models are the adjusted coordinates of three-dimensional terrestrial and satellite networks. It may be argued that the Geodetic coordinate system axes can be made "nearly" parallel to those of the Average Terrestrial system using classical methods, and that the systematic errors in a terrestrial network can be effectively modelled and removed during the estimation process. However, due to the many problems involved, it is unlikely that the above will be the case and it is the common belief amongst several geodesists that residual model errors will remain in the datum transformation parameters and the orientation and scale of the terrestrial network. The models presented herein are proposed as possible solutions to the removal of the above mentioned residual errors.

8.1 Hotine

In his monograph Mathematical Geodesy, Hotine [1969] argues that there should be two sets of rotations in a model intended for the combination of geodetic networks. His argument for this is stated as:

> "In addition to the initial choice of a discordant system of geodetic coordinates, the network itself may have systematic errors of scale and orientation for which an

> allowance should be made before we adjust the network to adjacent work or into a fixed system of a worldwide triangulation."

He goes on to state that if there is only one set of rotations in the combination model, namely those pertaining to the discordant geodetic coordinate system,

> "...the effect of a systematic orientation error in the network could be concealed by evaluating false values of the rotation parameters ω_1, ω_2, ω_3."*

Thus, he makes a separation of rotation parameters (ω_1, ω_2, ω_3) for the discordant geodetic coordinate system from the orientation parameters for the systematic errors in the terrestrial network. The two parameters are $d\alpha$, a change in azimuth, and $d\beta$, a change in zenith distance. The azimuth parameter, $d\alpha$ is a rotation about the z-axis of the local geodetic coordinate system at the terrestrial initial point k. The zenith distance parameter, $d\beta$, is a constant applied to all lines radiating from the terrestrial initial point, k. In addition, network scale error is accounted for by a scale difference parameter, κ. Counting the datum translation parameters, this gives a total of nine unknown parameters in a combination model expressed as (Figure 8-1)

$$\vec{F}_i = (\vec{r}_o)_1 + R_\varepsilon\{(\vec{r}_k)_2 + (1+\kappa)\ R_3(180-\lambda_k)\ R_2(90-\phi_k)\ P_2\ R_H \quad P_2\ R_2(\phi_k-90)\ R_3(\lambda_k-180)\ (\vec{r}_{ki})_2\} - (\vec{\rho}_i)_1 = 0 \quad . \tag{8-1}$$

$(\vec{r}_o)_1$, R_ε, $(\vec{r}_k)_2$, and $(\vec{\rho}_i)_1$ are defined in 7.1. $(\vec{r}_{ki})$, the position vector of i with respect to the initial point k, is defined in terms of the vector length $|\vec{r}_{ki}|$, and the azimuth (α_{ki}) and zenith distance (β_{ki}) in the local geodetic coordinate system at k. The matrix R_H contains the unknown parameters κ, $d\alpha$, $d\beta$, and is derived below.

The Hotine model as given by (8-1) is expanded as follows:

First, $\vec{r}_{ki}$ is expressed by

*Hotine's rotations ω_1, ω_2, ω_3 are equivalent to the values ε_1, ε_2, and ε_3 used in this work.

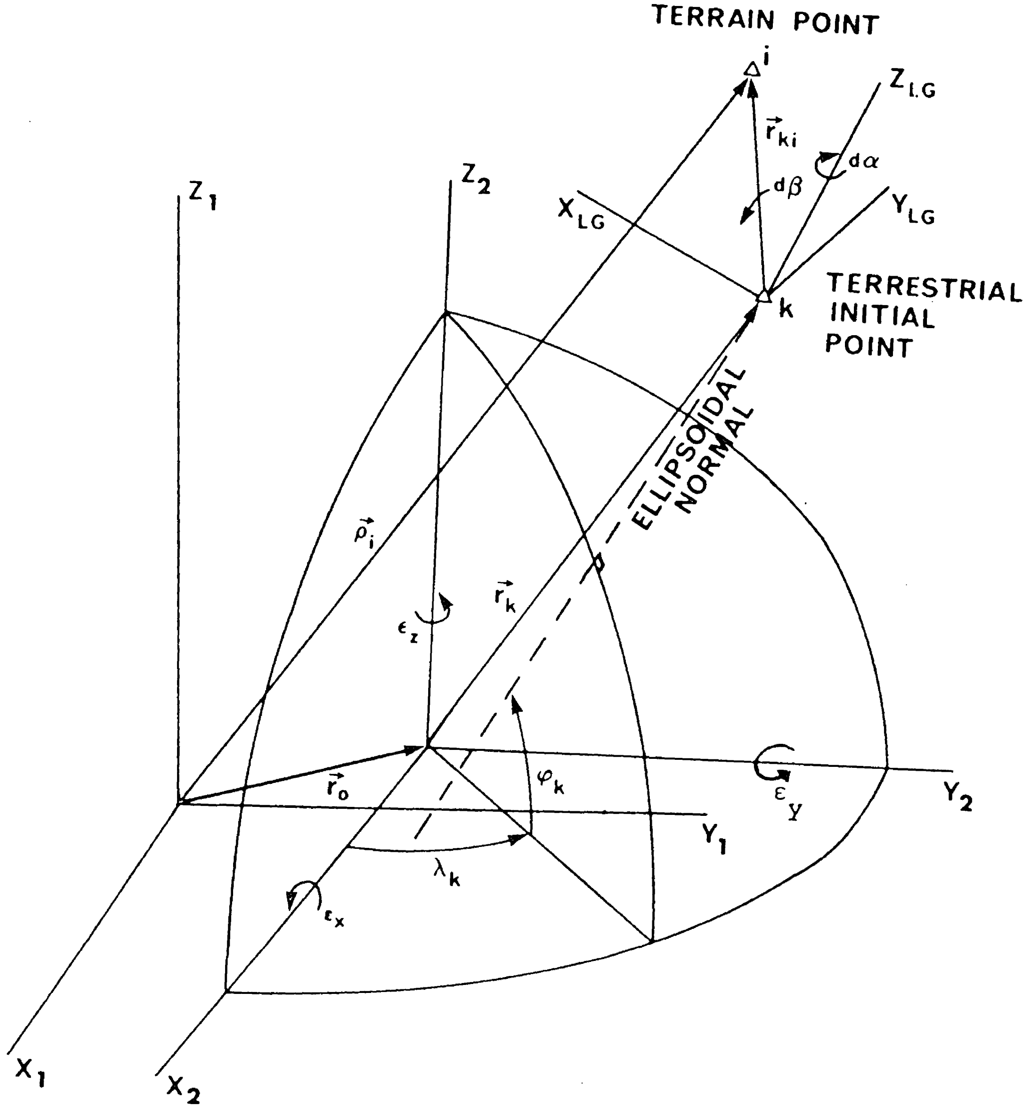

Figure 8 - 1

Hotine Model

$$\vec{r}_{ki} = \begin{bmatrix} x_{ki} \\ y_{ki} \\ z_{ki} \end{bmatrix} = \begin{bmatrix} |\vec{r}_{ki}| \sin\beta_{ki} \cos\alpha_{ki} \\ |\vec{r}_{ki}| \sin\beta_{ki} \sin\alpha_{ki} \\ |\vec{r}_{ki}| \cos\beta_{ki} \end{bmatrix} . \qquad (8\text{-}2)$$

After evaluating the partial derivatives $\partial\vec{r}_{ki}/\partial|\vec{r}_{ki}|$, $\partial\vec{r}_{ki}/\partial\alpha_{ki}$, $\partial\vec{r}_{ki}/\partial\beta_{ki}$, the change in coordinates due to the changes in $|\vec{r}_{ki}|$, α_{ki}, and β_{ki} can be expressed by

$$\begin{bmatrix} dx_{ki} \\ dy_{ki} \\ dz_{ki} \end{bmatrix} = \left[\begin{array}{c|c|c} \sin\beta_{ki} \cos\alpha_{ki} & -|\vec{r}_{ki}|\sin\beta_{ki}\sin\alpha_{ki} & |\vec{r}_{ki}| \cos\beta_{ki}\cos\alpha_{ki} \\ \sin\beta_{ki} \sin\alpha_{ki} & |\vec{r}_{ki}|\sin\beta_{ki}\cos\alpha_{ki} & |\vec{r}_{ki}| \cos\beta_{ki}\sin\alpha_{ki} \\ \cos\beta_{ki} & 0 & -|\vec{r}_{ki}| \sin\beta_{ki} \end{array}\right] \begin{bmatrix} dr_{ki} \\ d\alpha_{ki} \\ d\beta_{ki} \end{bmatrix} . \qquad (8\text{-}3)$$

Using the expressions

$$\begin{aligned} x'_{ki} &= x_{ki} + dx_{ki} , \\ y'_{ki} &= y_{ki} + dy_{ki} , \\ z'_{ki} &= z_{ki} + dz_{ki} , \end{aligned} \qquad (8\text{-}4)$$

where the prime (') indicates the changed values of coordinates due to dr_{ki}, $d\alpha_{ki}$, and $d\beta_{ki}$, and setting

$$\kappa = \frac{dr_{ki}}{|\vec{r}_{ki}|} , . \qquad (8\text{-}5)$$

the matrix R_H is

$$R_H = \left[\begin{array}{c|c|c} (1+\kappa) & -d\alpha & \cos\alpha_{ki} d\beta \\ d\alpha & (1+\kappa) & \sin\alpha_{ki} d\beta \\ -\dfrac{d\beta}{\cos\alpha_{ki}} & 0 & (1+\kappa) \end{array}\right] . \qquad (8\text{-}6)$$

The expanded model (8-1) is now written as

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_1 + \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y & x_k \\ -\varepsilon_z & 1 & \varepsilon_x & y_k \\ \varepsilon_y & -\varepsilon_x & 1 & z_k \end{bmatrix} + \begin{bmatrix} (1+\kappa) & -d\alpha & \cos\alpha_{ki}\, d\beta \\ d\alpha & (1+\kappa) & \sin\alpha_{ki}\, d\beta \\ -\frac{d\beta}{\cos\alpha_{ki}} & 0 & (1+\kappa) \end{bmatrix} \begin{bmatrix} x_{ki} \\ y_{ki} \\ z_{ki} \end{bmatrix}_2 - \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_1 = 0. \tag{8-7}$$

Hotine [1969] also mentions the possibility of including two more parameters in his model - da and de which are changes in the size and shape of the reference geodetic ellipsoid. These have not been included herein for two reasons. The inclusion of da and de may be used if one were working with the problem of determining the "Figure of the Earth"(defining the Geodetic coordinate system which best fits the Astronomic system which is beyond the scope of this work). Further, the inclusion of da would tend to eliminate the systematic error in scale of the terrestrial network since the ellipsoid would be rescaled to absorb scale errors. As stated by Hotine [1969]

> "this procedure would vitiate the height dimension and would result in some inaccuracy even in a two-dimensional adjustment which ignores geodetic heights ....".

Since there are only two networks involved in this combination procedure the estimation procedure for its solution is not easily formulated. Hotine did not propose any estimation procedure for the solution of his combination model. Regarding the model and its solution, he stated [Hotine, 1969]:

> "This procedure assumes that the parameters are independent and that second-order effects can be either neglected or removed by some process of iteration, although in some cases, the parameters, especially the rotations, will be strongly correlated".

The estimation procedure proposed for the Krakiwsky-Thomson model can be used for the solution of the Hotine model. The design matrices and misclosure vectors are the same as those given in 8.2 except for A_{22_i} which pertains to the unknown orientation and scale parameters in the second (terrestrial) network. For the Hotine model, A_{22_i} is given by

$$A_{22_i} = \frac{\partial F}{\partial \bar{X}_2 | X_2^o, L_2} = \begin{bmatrix} -y_{ki} & \cos \alpha_{ki} z_{ki} & x_{ki} \\ x_{ki} & \sin \alpha_{ki} z_{ki} & y_{ki} \\ 0 & -x_{ki}/\cos\alpha_{ki} & z_{ki} \end{bmatrix} , \qquad (8\text{-}8)$$

where

$$X_2^o = (d\alpha,\ d\beta,\ \kappa)^T = 0 \ . \qquad (8\text{-}9)$$

The solution is obtained using the matrix equations given in 8.2.

The Hotine model can be used to combine a terrestrial and a satellite geodetic network. The solution yields the datum transformation parameters $\hat{\bar{X}}_1$, adjusted network coordinates $\hat{\bar{L}}_1$ and $\hat{\bar{L}}_2$, three parameters $(\bar{X}_2)$ which represent the systematic errors in the terrestrial (second) network, and their associated variance-covariance matrices. The one drawback of this model is the parameterization of the network orientation errors. The parameters $d\alpha$ and $d\beta$ can not be split to give either the Molodensky (ψ_1, ψ_2, ψ_3) or Veis $(d\mathbf{A}, d\mu, d\nu)$ types of representation. The latter are particularly desirable if one is going to compute the changes in the deflection components of the vertical at the initial point that have occurred due to the combination process.

8.2 Krakiwsky-Thomson

Like the Hotine model, this one contains two sets of rotations - a first set (ε_x, ε_y, ε_z) for the misorientation of the second coordinate system with respect to the first and a second set (Ψ_x, Ψ_y, ψ_z) or (dA, dμ, dν) for the misoriented network. When first published [Krakiwsky and Thomson, 1974], it was given as

$$\vec{F}_i = (\vec{r}_o)_1 + (1+\kappa)\ R_\varepsilon\{(\vec{r}_k)_2 + R_\psi(\vec{r}_{ki})_2\} - \vec{\rho}_1 = 0. \qquad (8\text{-}10)$$

The scale difference in the above is interpreted as a system scale difference. Thus, (1+κ), along with the system rotations in R_ε (7-2), yields a totally redefined initial point ($\vec{r}_k$). Also, the systematic scale error in the terrestrial network, represented by the vectors ($\vec{r}_{ki}$), will be absorbed in the system scale difference. To have the scale difference parameter apply to the second network and obtain a parameterization of network orientation errors in terms of azimuth and tilts in the prime vertical and meridian planes, the Krakiwsky-Thomson model is written as [Thomson and Krakiwsky, 1975] (Figure 8-2)

$$\vec{F}_i = (\vec{r}_o)_1 + R_\varepsilon\{(\vec{r}_k)_2 + (1+\kappa)\ R_V(\vec{r}_{ki})_2\} - \vec{\rho}_1 = 0. \qquad (8\text{-}11)$$

Expanding (8-11) with quantities previously defined yields

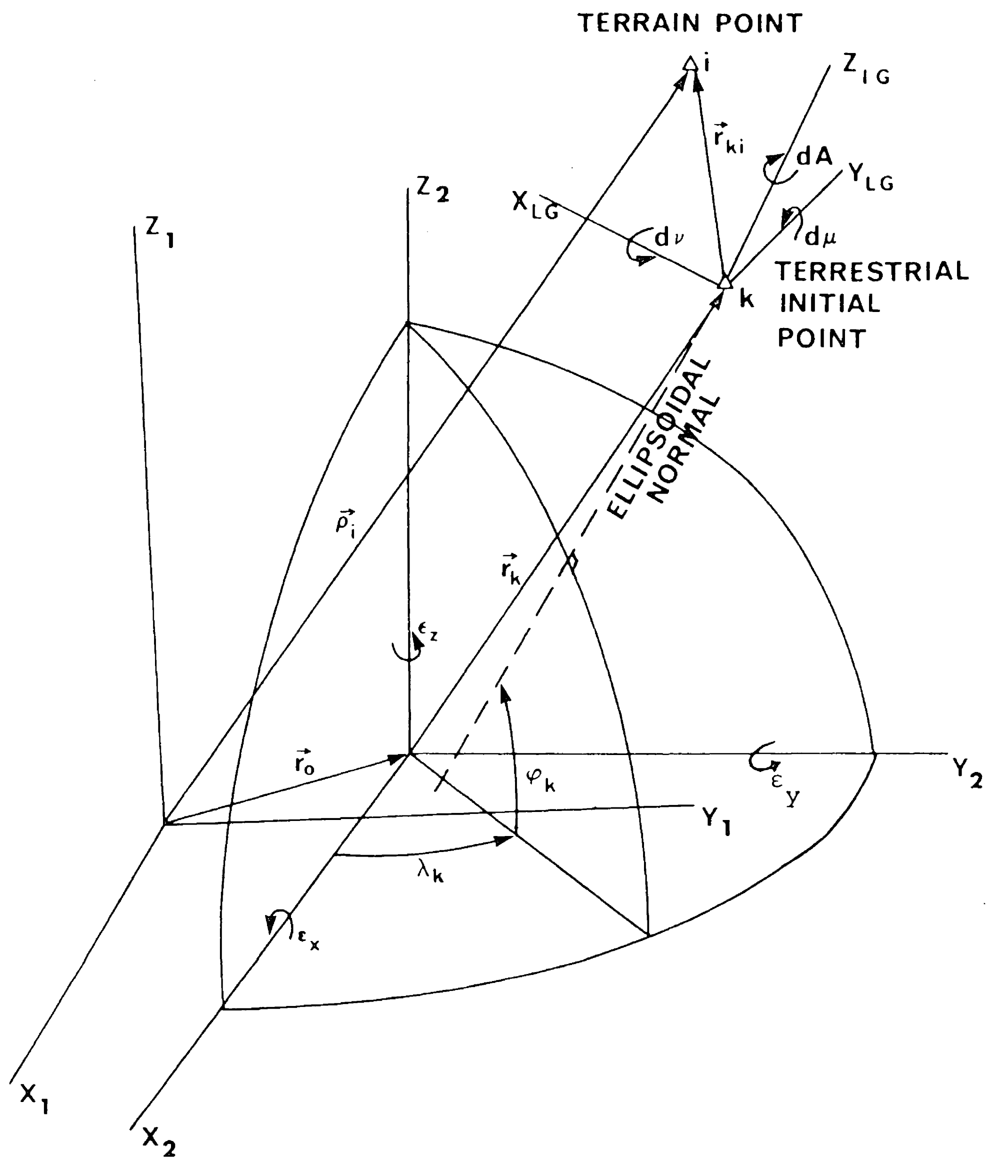

Figure 8-2

Krakiwsky-Thomson Model

$$
\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_1 + \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \left\{ \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}_2 + (1+\kappa) \begin{bmatrix} 1 & -\sin\phi_k dA - \cos\phi_k d\nu & \sin\lambda_k \cos\phi_k dA + \cos\lambda_k d\mu - \sin\phi_k \sin\lambda_k d\nu \\ \sin\phi_k dA + \cos\phi_k d\nu & 1 & -\cos\phi_k \cos\lambda_k dA + \sin\lambda_k d\mu + \sin\phi_k \cos\lambda_k d\nu \\ -\cos\phi_k \sin\lambda_k dA - \cos\lambda_k d\mu + \sin\phi_k \sin\lambda_k d\nu & \cos\phi_k \cos\lambda_k dA - \sin\lambda_k d\mu - \sin\phi_k \cos\lambda_k d\nu & 1 \end{bmatrix} \begin{bmatrix} x_{ki} \\ y_{ki} \\ z_{ki} \end{bmatrix}_2 \right\} - \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_1 = 0 \,. \qquad (8\text{-}12)
$$

This model contains ten unknown parameters, of which six are rotations. In order to compute a solution, there must be at least four network points for which Cartesian coordinates are available in the two coordinate systems.

Since there are two sets of unknown rotations and only two networks (one satellite, one terrestrial) are being combined, a special least-squares estimation procedure is required to obtain a solution. The estimation model, in functional form, is

$$F_1(\bar{X}_1, \bar{L}_1) = 0 , \qquad (8\text{-}13)$$

$$F_2(\bar{X}_1, \bar{X}_2, \bar{L}_2) = 0 , \qquad (8\text{-}14)$$

where $\bar{X}_1$ are the coordinate system transformation parameters (x_o, y_o, z_o, ε_x, ε_y, ε_z);

$\bar{X}_2$ are the rotation and scale differences parameters pertaining to the second network (κ, dA, $d\mu$, $d\nu$);

$\bar{L}_1$ are the observables (coordinate differences $(x_{ik}, y_{ik}, z_{ik})_2$ and coordinates $(X_i, Y_i, Z_i)_1$ of the "inner zone")(Figure 8-3);

$\bar{L}_2$ are the observables (coordinate differences $(x_{ik}, y_{ik}, z_{ik})_2$ and coordinates $(X_i, Y_i, Z_i)_1$ of the outer zone)(Figure 8-3).

The reason for the splitting of the coordinates of common points into inner and outer zones is to make the solution possible for the two sets of rotation parameters in the Krakiwsky-Thomson model. The inner zone contains sufficient observables (L_1) to solve for the unknown parameters ($\hat{X}_1$). The common network points of the inner zone should be sufficiently close to the terrestrial initial point so that the observables (coordinates) of the second (terrestrial) network will not contain significant systematic errors. The outer zone then contains the remaining common network stations.

The observables L_1 and L_2 are correlated. Thus, the variance-covariance matrix of the observables is given by

$$\Sigma_L^* = \left[\begin{array}{c|c} \Sigma_{L_1} & COV_{L_1L_2} \\ \hline COV_{L_2L_1} & \Sigma_{L_2} \end{array}\right] . \qquad (8\text{-}15)$$

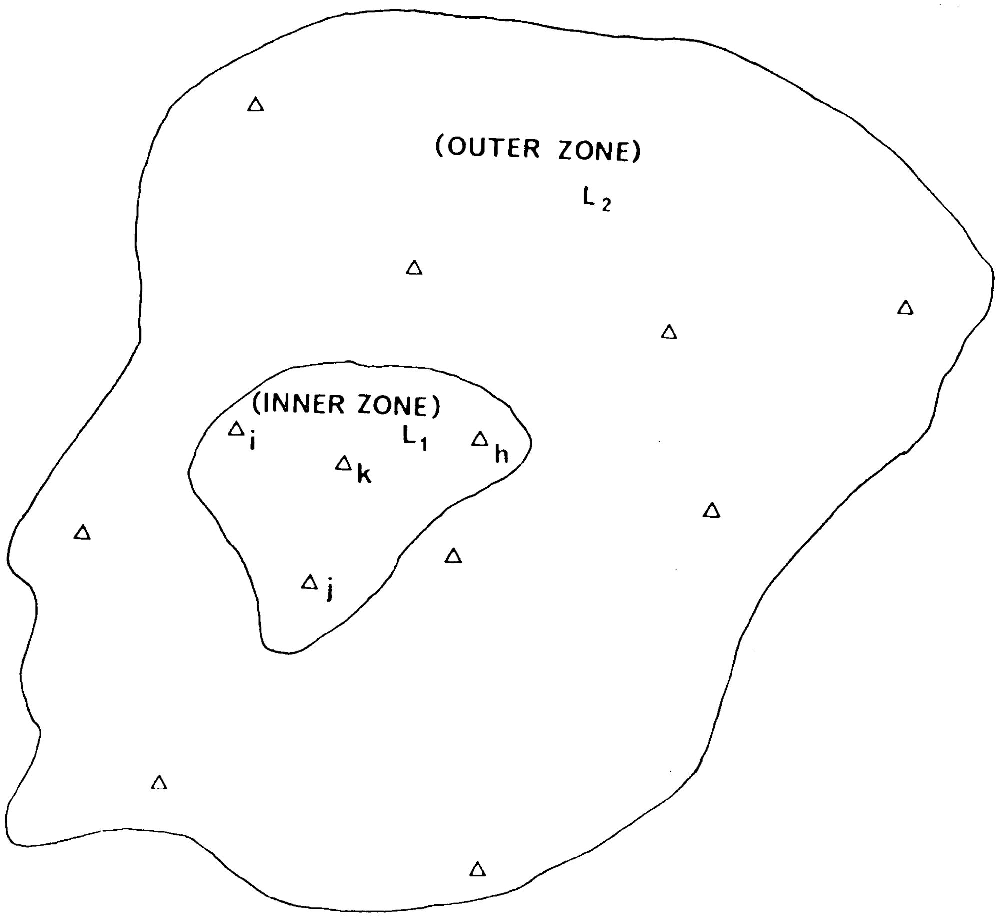

Figure 8 - 3

Estimation Procedure for the Krakiwsky-Thomson Model

Linear Taylor series expansions of F_1 (8-13) and F_2 (8-14) yields the matrix equations

$$A_{11}\hat{X}_1 + B_{11}\hat{V}_1 + W_1^o = 0 \ , \tag{8-16}$$

$$A_{21}\hat{X}_1 + A_{22}\hat{X}_2 + B_{22}\hat{V}_2 + W_2^o = 0 \ , \tag{8-17}$$

where

$$A_{11} = \frac{\partial F_1}{\partial \bar{X}_1 | X_1^o,\ L_1} \ , \quad B_{11} = \frac{\partial F_1}{\partial \bar{L}\ | X^o,\ L_1} \ ,$$

$$A_{21} = \frac{\partial F_2}{\partial \bar{X}_1 | X_1^o,\ L_2} \ , \ A_{22} = \frac{\partial F_2}{\partial \bar{X}_2 | X_2^o,\ L_2} \ , \ B_{22} = \frac{\partial F_2}{\partial \bar{L}\ | X_2^o, L_2} \ . \tag{8-18}$$

The least squares normal equations relating the unknown quantities ($\hat{X}_1$, $\hat{X}_2$, $\hat{V}_1$, $\hat{V}_2$) to the known quantities (A_{11}, B_{11}, W_1, A_{21}, A_{22}, B_{22}, W_2^o, Σ_L^*) are normally obtained from the variation function [Krakiwsky, 1975]

$$\phi = [\hat{V}_1^T \quad \hat{V}_2^T] \begin{bmatrix} \Sigma_{L_1} & COV_{L_1 L_2} \\ COV_{L_2 L_1} & \Sigma_{L_2} \end{bmatrix}^{-1} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix}$$

$$-2\hat{K}_1^T(A_{11}\hat{X}_1 + B_{11}\hat{V}_1 + W_1) - 2K_2^T(A_{12}\hat{X}_1 + A_{22}\hat{X}_2 + B_{22}\hat{V}_2) \ . \tag{8-19}$$

However, due to the form of the variance-covariance matrix Σ_L^* and the need to invert it for the development of the normal equation system, an alternate approach has been chosen.

The development given here of the matrix equations required for the solution of the proposed least-squares estimation model is similar to that used to derive the stepwise collocation equations [Moritz, 1973; Krakiwsky, 1975]. A new, and simplified, model

$$F(\bar{X}^*, \bar{L}^*) = 0 \tag{8-20}$$

in which $\bar{X}^*$ contains $\bar{X}_1$ and $\bar{X}_2$, and $\bar{L}^*$ contains $\bar{L}_1$ and $\bar{L}_2$ as will be shown shortly, is used to develop the well known combined-case least-squares matrix equations (see equations (4-10) to (4-14) inclusive in Chapter 4)

$$\hat{X}^* = -(A^{*T}(B^*\Sigma_L^* B^{*T})^{-1}A^*)^{-1} A^{*T}(B^*\Sigma_L^* B^{*T})^{-1} W^* , \tag{8-21}$$

$$\hat{V}^* = \Sigma_L^* B^{*T} K^* , \tag{8-22}$$

$$\hat{K} = -(B^*\Sigma_L^* B^{*T})^{-1} (A^*\hat{X}^* + W^*) , \tag{8-23}$$

$$Q_{\hat{X}} = (A^{*T}(B^*\Sigma_L^* B^{*T})^{-1} A^*)^{-1} , \tag{8-24}$$

$$Q_{\hat{L}} = (\Sigma_L^* - \Sigma_L^* B^{*T}((B^*\Sigma_L^* B^T)^{-1} - (B^*\Sigma_L^* B^{*T})^{-1} A^* Q_{\hat{X}} A^{*T}(B^*\Sigma_L^* B^{*T}))B^*\Sigma_L^*) . \tag{8-25}$$

Now the matrices, denoted by * above, are split to take into account the stepwise procedure required for the solution while maintaining the covariance between the inner and outer zone observables. The unknowns and residuals are given by

$$\hat{X}^* = \begin{bmatrix} \hat{X}_1 \\ \hline \hat{X}_2 \end{bmatrix} \quad \text{and} \quad \hat{V}^* = \begin{bmatrix} \hat{V}_1 \\ \hline \hat{V}_2 \end{bmatrix} . \tag{8-26}$$

The weight coefficient matrices are given by

$$Q_{\hat{X}} = \left[\begin{array}{c|c} Q_{\hat{X}_1} & COV_{\hat{X}_1\hat{X}_2} \\ \hline COV_{\hat{X}_2\hat{X}_1} & Q_{\hat{X}_2} \end{array}\right] \tag{8-27}$$

and

$$Q_{\hat{L}} = \left[\begin{array}{c|c} Q_{\hat{L}_1} & COV_{\hat{L}_1\hat{L}_2} \\ \hline COV_{\hat{L}_2\hat{L}_1} & Q_{\hat{L}_2} \end{array}\right] , \tag{8-28}$$

and the final variance-covariance matrices by

$$\Sigma_{\hat{X}*} = \sigma_o^2 \; Q_{\hat{X}_1} , \tag{8-29}$$

$$\Sigma_{\hat{L}*} = \sigma_o^2 \; Q_{\hat{L}} . \tag{8-30}$$

The misclosure and correlate vectors are

$$W* = \left[\begin{array}{c} W_1 \\ \hline W_2 \end{array}\right] \quad \text{and} \quad \hat{K}* = \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right] . \tag{8-31}$$

The hyper-design matrix A* has the form

$$A* = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right] . \tag{8-32}$$

When the initial approximate values, X_1^o and X_2^o are set to zero, then

$$A_{11_i} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -(z_k+z_{ki}) & (y_k+y_{ki}) \\ 0 & 1 & 0 & (z_k+z_{ki}) & 0 & -(x_k+x_{ki}) \\ 0 & 0 & 1 & -(y_k+y_{ki}) & (x_k+x_{ki}) & 0 \end{array}\right] , \tag{8-33}$$

for the r_1 observations and u_1 unknowns of the inner zone, the elements of A_{21} are computed just as those of A_{11} for the r_2 observations and u_1 unknowns of the outer zone, and

$$A_{22_i} = \left[\begin{array}{ccc|c} -\sin\phi_k y_{ki} + \sin\lambda_k \cos\phi_k z_{ki} & \cos\lambda_k z_{ki} & -\cos\phi_k y_{ki} - \sin\phi_k \sin\lambda_k z_{ki} & x_{ki} \\ \sin\phi_k x_{ki} - \cos\phi_k \cos\lambda_k z_{ki} & \sin\lambda_k z_{ki} & \cos\phi_k x_{ki} + \sin\phi_k \cos\lambda_k z_{ki} & y_{ki} \\ -\cos\phi_k \sin\lambda_k x_{ki} + \cos\phi_k \cos\lambda_k y_{ki} & -\cos\lambda_k x_{ki} - \sin\lambda_k y_{ki} & \sin\phi_k \sin\lambda_k x_{ki} - \sin\phi_k \cos\lambda_k y_{ki} & z_{ki} \end{array}\right], \tag{8-34}$$

for the r_2 observations and u_2 unknowns of the outer zone. The hyper-design matrix B* is given by

$$B^* = \left[\begin{array}{c|c} B_{11} & 0 \\ \hline 0 & B_{22} \end{array}\right], \tag{8-35}$$

where

$$B_{11_i} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] . \tag{8-36}$$

The three sections of B_{11} pertain to the initial point $(\vec{r}_k)$, the coordinate differences $(\vec{r}_{ki})$, and the coordinates $(\vec{\rho}_i)$. If there are coordinates $(\vec{\rho}_k)$ at the initial point, then

$$B_{11_i} = [{}_3I_3 \,\vdots\, -{}_3I_3] \tag{8-37}$$

since $\vec{r}_{kk} = 0$. The full design matrix, which has dimensions $(r_1 x 2r_1)$ is given by

$$
B_{11} = \left[\begin{array}{c|cccccc}
{}_3I_3 & -{}_3I_3 & 0 & 0 & 0 & 0 & \cdots \\
{}_3I_3 & 0 & {}_3I_3 & -{}_3I_3 & 0 & 0 & \cdots \\
{}_3I_3 & 0 & 0 & 0 & {}_3I_3 & -{}_3I_3 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}\right] . \qquad (8\text{-}38)
$$

B_{22} is given by

$$
B_{22_i} = [{}_3I_3 \,\vdots\, -{}_3I_3] , \qquad (8\text{-}39)
$$

and the full matrix is

$$
B_{22} = \begin{bmatrix}
{}_3I_3 & -{}_3I_3 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & {}_3I_3 & -{}_3I_3 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & {}_3I_3 & -{}_3I_3 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{bmatrix} . \qquad (8\text{-}40)
$$

The misclosure vector, for inner zone points, is

$$
W^o_{1_i} = \begin{bmatrix} x_k + x_{ki} - X_i \\ y_k + y_{ki} - Y_i \\ z_k + z_{ki} - Z_i \end{bmatrix} , \qquad (8\text{-}41)
$$

and the elements of W^o_2 are evaluated for the outer zone points using the same expression.

The solution is now generated using the expanded matrices in equations (8-21) through (8-25) inclusively. For example, (8-21) is now

$$\left[\begin{array}{c}\hat{X}_1\\ \hline \hat{X}_2\end{array}\right] = -\left\{\left[\begin{array}{c|c}A_{11} & 0\\ \hline A_{21} & A_{22}\end{array}\right]^T \left(\left[\begin{array}{c|c}B_{11} & 0\\ \hline 0 & B_{22}\end{array}\right]\left[\begin{array}{c|c}\Sigma_{L_1} & COV_{L_1L_2}\\ \hline COV_{L_2L_1} & \Sigma_{L_2}\end{array}\right]\left[\begin{array}{c|c}B_{11} & 0\\ \hline 0 & B_{22}\end{array}\right]^T\right)^{-1}\right.$$

$$\left.\left[\begin{array}{c|c}A_{11} & 0\\ \hline A_{22} & A_{22}\end{array}\right]\right\}^{-1}\left[\begin{array}{c|c}A_{11} & 0\\ \hline A_{21} & A_{22}\end{array}\right]^T\left(\left[\begin{array}{c|c}B_{11} & 0\\ \hline 0 & B_{22}\end{array}\right]\left[\begin{array}{c|c}\Sigma_{L_1} & COV_{L_1L_2}\\ \hline COV_{L_2L_1} & \Sigma_{L_2}\end{array}\right]\left[\begin{array}{c|c}B_{11} & 0\\ \hline 0 & B_{22}\end{array}\right]^T\right)^{-1}\left[\begin{array}{c}W_1\\ \hline W_2\end{array}\right]. \qquad (8\text{-}42)$$

This expression for the unknown parameters is equivalent to a set of sequential expressions in which $\hat{X}_1$ and $\hat{X}_2$ are solved for in explicitly separate matrix expressions.

The Krakiwsky-Thomson model as developed here, or expanded using Molodensky type rotations to express the misorientation of a network, can be used to combine a satellite and a terrestrial geodetic network. If the quantities dA, dμ, dν are computed, then ψ_1, ψ_2, ψ_3 can be computed using (7-25) to (7-25), or vice-versa via (7-28) to (7-30).

Test results of the combination of a terrestrial and a satellite network have been generated (12) using the Krakiwsky-Thomson model as developed herein.

8.3 Vanicek-Wells

The stated objective of this model is to enable one "to examine numerically the parallelism of geodetic systems (based on terrestrial observations) and satellite systems (based on satellite observations) and the average terrestrial system" [Wells and Vanicek, 1975]. The model is given by [Wells and Vanicek, 1975] (Figure 8-4)

$$\bar{F}_i = \vec{r}_G + R_\Lambda \kappa(\vec{r}_k + \vec{r}_{ki}) - \vec{r}_S + R_{\omega\psi\varepsilon}\vec{\rho}_i = 0 , \qquad (8\text{-}43)$$

where $\vec{r}_G$ is the translation vector between the Average Terrestrial and Geodetic systems, $\vec{r}_S$ is the translation vector between the Average Terrestrial and the satellite systems, κ is the scale difference in the Geodetic system, R_Λ and $R_{\omega\psi\varepsilon}$ are rotation matrices (see below), and $\vec{r}_k$, $\vec{r}_{ki}$, and $\vec{\rho}_i$ are as previously defined.

The rotation matrix $R_{\omega\psi\varepsilon}$, which contains the three rotations of the three satellite system axes with respect to the Average Terrestrial axes is given by

$$R_{\omega\psi\varepsilon} = \begin{bmatrix} 1 & \varepsilon & -\psi \\ -\varepsilon & 1 & \omega \\ \psi & -\omega & 1 \end{bmatrix} . \qquad (8\text{-}44)$$

The originators of this model have proven that under certain conditions only four datum position and orientation parameters exist (three translations, one azimuth rotation) [Vanicek and Wells, 1974]. This occurs when the orientation and position of the datum is defined at a terrestrial initial point, at which point equations (1-3) and (1-4) are accepted by definition. Under this condition, equation (1-6) is satisfied. This then leaves only the azimuth orientation condition (1-7) to be satisfied. Thus, the rotation matrix, R_Λ, pertaining to the Geodetic coordinate system contains only the azimuth orientation unknown Δ and is given by [Wells and Vanicek, 1975].

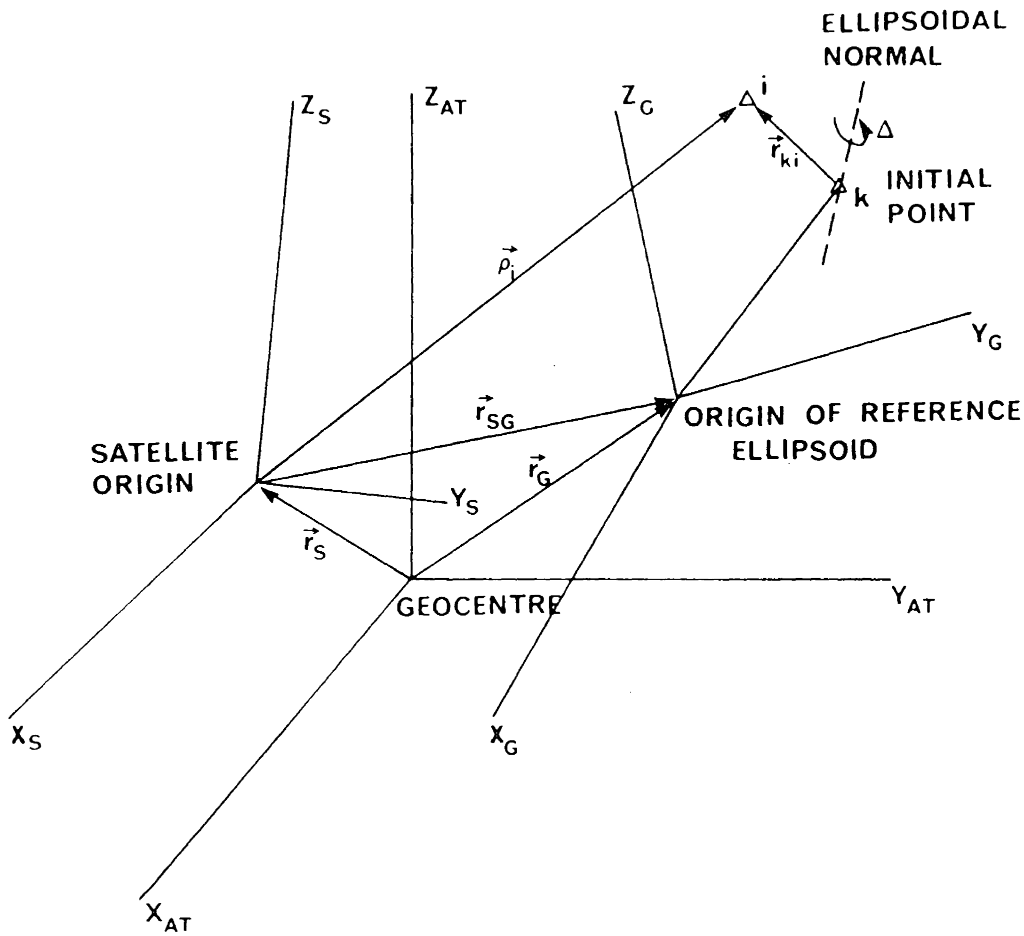

Figure 8-4

Vaníček - Wells Model

$$R_\Delta = \begin{bmatrix} 1 & \Delta \sin \phi_k & -\Delta \cos \phi_k \sin \lambda_k \\ -\Delta \sin \phi_k & 1 & \Delta \cos \phi_k \cos \lambda_k \\ \Delta \cos \phi_k \sin \lambda_k & -\Delta \cos \phi_k \cos \lambda_k & 1 \end{bmatrix} . \qquad (8\text{-}45)$$

There are eight unknowns to be solved for using this model - ω, ψ, ε, Λ, κ, $\bar{r}_{SG}$. The last quantity, the difference vector $\bar{r}_{SG}$, replaces $\bar{r}_S$ and $\bar{r}_G$ since the centre of gravity (origin of the Average Terrestrial system) is unknown. As with the previous two models presented (Hotine and Krakiwsky-Thomson), this one contains two unknown sets of rotation parameters. One satellite network and several geodetic networks, having common points, are combined in one parametric least-squares solution. Details of the development of the model and its solution are given in [Wells and Vanicek, 1975].

In some test computations, Wells and Vanicek [1975] have shown how this model is used to combine one satellite network with up to five geodetic. Their conclusions include the comment that care should be taken in the selection of both data points and datums used in the application of this model.

8.4 Comparison of the Hotine, Krakiwsky-Thomson, and Vanicek-Wells Models

All three models contain two sets of unknown rotation parameters. The observables used in each of them are the Cartesian coordinates of points common to the satellite and terrestrial networks involved. The estimation procedure used in each case is the method of least-squares, although different cases are used for the Hotine and Krakiwsky-Thomson models from the Vanicek-Wells model. Each model yields the three components of the position vector $\bar{r}_o$ ($\bar{r}_{SG}$ in Vanicek-Wells) - the translation vector between the origins of the satellite and Geodetic coordinate systems.

In the Hotine and Krakiwsky-Thomson models, the first set of rotations (ε_x, ε_y, ε_z) expresses the alignment of the Geodetic system axes with respect to those of the satellite system. The Vanicek-Wells rotations (ω, ψ, ε) express the alignment of the satellite axes with respect to those of the Average Terrestrial system. The second Vanicek-Wells rotation, Λ, is that which relates the Geodetic system with the Average Terrestrial, whereas the Hotine ($d\alpha$, $d\beta$) and Krakiwsky-Thomson ($d\alpha$, $d\mu$, $d\nu$) parameters are a parameterization of the systematic errors in the terrestrial network.

The scale difference parameter, κ, is a Geodetic coordinate system scale factor in the Vanicek-Wells model. In the Hotine model, it (κ) is an expression of the systematic scale error in the terrestrial network. The Krakiwsky-Thomson model is such that κ can be entered to express either of the above. In the development of the model herein (8.2), it is treated as an expression of systematic scale distortion in the terrestrial network.

The Hotine and Krakiwsky-Thomson models have been developed for the combination of one satellite and one terrestrial network, thus only two coordinate systems are involved. Both of these can be easily expanded to accommodate more satellite networks in the combination procedure [Krakiwsky and Thomson, 1974]. There are several coordinate systems involved in the Vanicek-Wells model - the Average Terrestrial, one satellite and several Geodetic.

All of the above mentioned similarities and differences stem from the different objectives of the models. The Hotine and Krakiwsky-Thomson are concerned with the combination of a satellite and a terrestrial network, the orientation and position of the Geodetic system with respect to the satellite system, and a parameterization of systematic errors in the terrestrial network. The objective of the Vanicek-Wells model is the determination of the orientation of a set of satellite system axes, and that of several Geodetic systems, with respect to the Average Terrestrial system. Also included are parameters that yield the position and scale of the Geodetic system with respect to that of the satellite network.

8.5 Comparison of Hotine, Krakiwsky-Thomson, Vanicek-Wells Models with those of Bursa, Molodensky, and Veis

This comparison of models is made on the basis of combining satellite and terrestrial geodetic networks. This does not preclude the fact that none of the Bursa, Veis, or Molodensky models are considered to be adequate for the task (7.4).

All of the models covered in 7 and 8 are three-dimensional. Datum transformation parameters are treated as unknowns. The observables utilized in each model are the Cartesian coordinates of points common to the satellite and terrestrial networks under consideration. The solutions are obtained using a least squares estimation procedure.

In each model, the translation vector, $\bar{r}_o$ ($\bar{r}_{SG}$ in the Vanicek-Wells model), between the satellite and terrestrial network datums appears explicitly. The Bursa, Hotine, and Krakiwsky-Thomson models have an explicit set of unknown rotation parameters between the satellite and geodetic coordinate systems, while the Vanicek-Wells models solves for rotations between these two systems and the Average Terrestrial coordinate system. The Molodensky and Veis models have no provision for a discordant geodetic coordinate system.

The parameterization of systematic errors in the terrestrial network is accomplished via three rotations ($d\alpha$, $d\mu$, $d\nu$ or ψ_x, ψ_y, ψ_z) and a scale difference in the Molodensky, Veis, and Krakiwsky-Thomson models. The Bursa and Vanicek-Wells models do not have this provision.

In the Bursa and Vanicek-Wells models, the terrestrial initial point is totally redefined, that is, scaled and rotated. The terrestrial network initial point is rotated but not scaled in the presented versions of the Hotine and Krakiwsky-Thomson models. As pointed out in 8.1 and 8.2, the inclusion of (da, de) in the Hotine model, or having κ as a system scale in the Krakiwsky-Thomson model, would cause a total redefinition of the terrestrial network initial point.

The major differences in the models are that the first set - Bursa, Molodensky, Veis - contains only one set of unknown rotation parameters, while the second set - Hotine, Thomson-Krakiwsky, Vanicek-Wells - contains two sets of unknown rotation parameters. For this reason, more complex least squares estimation procedures are required for two of the latter set (Hotine, Krakiwsy-Thomson) while the third (Vanicek-Wells) involves several terrestrial geodetic datums.

Finally, it should be noted that the latter three models are far more flexible than the Bursa, Molodensky, and Veis models, and will thus more easily reflect physical reality.

9. SUMMARY

Fourteen mathematical models for the combination of terrestrial and satellite networks have been examined. The examination of each model has been carried out under the terms spelled out in 3.2. Where necessary, a complete development of the models has been presented. The purpose here is to summarize, in three Tables (9-1, 9-2, 9-3), the characteristics and recommended uses of each of the models.

The Tables are split into two major segments. The first (Table 9-1) covers seven of the eight models studied in which the datum transformation parameters (satellite-terrestrial) must be known before the models can be used. The second segment (Tables 9-2 and 9-3) includes all six models investigated in which some or all of the datum transformation parameters are solved for during the combination process.

For the six models given in Table 9-1, there are four major features outlined for each model. These are Dimensionality, Unknown Parameters, Observables, and Estimation Procedure. The use of each model is the combination of a satellite and a terrestrial network. The implementation is given by the name of each model and the estimation procedure recommended. The major drawbacks are given in the Remarks column of Table 9-1. In the case of the two-dimensional models, one obvious negative factor is the loss of one dimension of the three-dimensional satellite network information.

The one-dimensional combination procedures covered in Chapter 6 are not given in Table 9-1. The reason for this is that the coverage herein is recognized as being superficial. Details regarding the use of satellite network data in combination with terrestrial vertical network data are covered adequately in other reports [e.g. Merry, 1975; John, 1976].

Tables 9-2 and 9-3 summarize the characteristics and uses of six three-dimensional models contained in this report. These six models are those in which some or all of the datum transformation parameters (satellite-terrestrial) are treated as unknown parameters. In addition to indicating which datum transformation parameters are treated as unknown parameters in the combination procedures, Table 9-2 also points out in which models the overall systematic errors in the terrestrial network are parameterized, and by how many parameters. The estimation procedure utilized for each model is also given.

Table 9-3 deals specifically with the recommended uses of the models covered in Table 9-2. The recommended uses given are the result of the in depth analysis of these models for this study. The recommended uses of several of the models were tested for this report (Chapters 11 and 12), while others have been tested elsewhere [e.g. Wells and Vanicek, 1975].

One combined use of the Hotine and Vanicek-Wells or Krakiwsky-Thomson and Vanicek-Wells models has not been given elsewhere. This proposed combination of two models (Table 9-2) is seen to be beneficial in several ways. First, several terrestrial datums will be tied together rigorously via one global satellite network. Second, the orientation of the satellite and terrestrial datum axes with respect to those of the Average Terrestrial coordinate system will be solved for. Finally, the overall effects of the systematic errors in scale and orientation in each of the terrestrial networks will be parameterized. Thus the combination procedure yields a system of globally connected terrestrial networks whose datums will be correctly oriented and whose systematic orientation and scale errors will have been modelled. Of course, such a solution is not practical at present, but it will be worth serious consideration in the future.

Model	Dimensionality	Unknown Parameters	Observables	Estimation Procedure	Remarks
(4.1) Parameterization of Terrestrial Network Scale and Orientation	3	3 rotation parameters 1 scale difference	Quasi-observables (coordinates and coordinate differences) from both satellite and terrestrial networks	Combined Case Least-Squares	Parameterization of overall effects of scale and orientation systematic errors in the terrestrial network.
(4.2) Satellite Coordinates as Weighted Parameters	3	Coordinates of the terrestrial network points	Quasi-observables (coordinates) from satellite network; original terrestrial observables	Parametric Case Least-Squares with Weighted Parameters	No parameterization of systematic errors in the terrestrial network. Any errors present overflow into the solution and residual vectors during the estimation.
(4.3) Satellite Coordinate Differences as Observables	3	Coordinates of the terrestrial network points	Quasi-observables (coordinate differences) from satellite network; original terrestrial observables	Parametric Case Least-Squares	No parameterization of systematic errors in the terrestrial network. Any errors present overflow into the solution and residual vectors during the estimation procedure. Same results as (4.2) with more computational effort.

Table 9-1

Datum Transformation Parameters Known - General Characteristics

of the Combination Models Studied.

Model	Dimensionality	Unknown Parameters	Observables	Estimation Procedure	Remarks
(4.4) Satellite Distances, Azimuths, and Zenith Distances as Observables	3	Coordinates of terrestrial network points	Quasi-observables (computed distances, azimuths, zenith distances) from the satellite network; original terrestrial observables	Parametric Case Least-Squares	No parameterization of systematic errors in the terrestrial network. Any errors present overflow into the solution and residual vectors during the estimation procedure. Same results as (4.2) and (4.3) with more computational effort.
(5.1) Satellite Coordinates as Weighted Parameters	2	Coordinates of the terrestrial network points	Quasi-observables (coordinates) from the satellite network; original observables from the horizontal terrestrial network	Parametric Case Least-Squares with Weighted Parameters	No parameterization of systematic errors in the terrestrial network.
(5.2) Satellite Coordinate Differences as Observables	2	Coordinates of the terrestrial network points	Quasi-observables (coordinate differences) from the satellite network; original observables from the horizontal terrestrial network	Parametric Case Least-Squares	No parameterization of systematic errors in the terrestrial network. Same results as (5.1) with more computational effort.

Table 9-1 (Cont'd)

Model	Dimensionality	Unknown Parameters	Observables	Estimation Procedure	Remarks
(5.3) Satellite Distances and Azimuths as Observables	2	Coordinates of the terrestrial network points	Quasi-observables (distances and azimuths) from the satellite network; original observables from the horizontal terrestrial network	Parametric Case Least-Squares	No parameterization of systematic errors in the terrestrial network. Same result as (5.1) and (5.2) with more computational effort

Table 9-1 (Cont'd)

3-D Model	Datum Transformation Parameters (Terr.-Sat.)			Parameterization of Systematic Errors in the Terrestrial Network		Orientation of Datums w.r.t. A.T. System		Estimation Procedure
	Translations	Rotations	Scale Difference	Rotations	Scale Difference	Satellite	Terrestrial	
(7.1) Bursa	3	3	1					Combined Case Least-Squares
(7.2) Molodensky	3			3	1			Combined Case Least Squares
(7.3) Veis	3			3	1			Combined Case Least Squares
(8.1) Hotine	3	3		1 azimuth rotation 1 zenith dist. parameter	1			Stepwise Least Squares
(8.2) Krakiwsky-Thomson	3	3		3	1			Stepwise Least Squares
(8.3) Vanicek-Wells	3		1			3	1	Parametric Case Least Squares

Table 9-2.

Datum Transformation Parameters Unknown - General Characteristics of the Combination Models Studied.

MODEL / USE	Bursa	Molodensky	Veis	Hotine	Krakiwsky-Thomson	Vanicek-Wells
Combination of two or more satellite networks	X					
Combination of a terrestrial and one or more satellite networks				X	X	
Combination of several terrestrial datums and a satellite datum via network coordinates						X
Combination of several terrestrial and satellite network (overall systematic terrestrial network errors parameterized)				X and	X and X	X
Combination of a terrestrial and a satellite network - geodetic and satellite coordinate system axes are parallel		X or	X			

Table 9-3.

Summary of the Uses of the Three-Dimensional

(Unknown Datum Transformation Parameters) Models Studied.

SECTION IV

TEST RESULTS

10. MODELS FOR WHICH DATUM TRANSFORMATION PARAMETERS ARE KNOWN

Eight procedures for the combination of satellite and terrestrial networks, when datum transformation parameters are considered known, have been presented. A lack of adequate terrestrial data has prevented the testing of the three-dimensional models. The testing of the one-dimensional procedures are being carried out and reported by other investigators [e.g. Kouba, 1976(a); John, 1976].

The combination of a horizontal terrestrial network and the equivalent components of a satellite network can be accomplished by any one of three procedures (5.1, 5.2, 5.3). The preferred approach, that of using satellite coordinates as weighted parameters in a terrestrial network adjustment, was tested for this report (10.1). The preparation of computed satellite network azimuths and distances for use as observables in a terrestrial network adjustment is also included (10.2).

The effects of satellite network distances on the adjustment of a terrestrial network have been reported previously [Thomson and Krakiwsky, 1975]. In the aforementioned computation, 28 distances from a Doppler network and 3 distances from the North American Densification of the Worldwide Geometric Satellite Triangulation

(BC-4) network were added as observables to the rigorous adjustment of a portion of the Canadian geodetic network in Eastern Canada. When compared to results generated without these extra distances, it was found that the satellite network distances caused an increase in the scale of the network of 2.3 ppm, and a mean rotation of $-0''.10$ arc-seconds.

10.1 Satellite Coordinates as Weighted Parameters

Parts of the Canadian horizontal terrestrial geodetic and Doppler networks were used in this test (Figure 10-1). There are 53 terrestrial network stations, of which five have Doppler determined coordinates (Figure 10-1). The networks cover an area of approximately 28110 km^2. The terrestrial observables used were 249 direction measurements, 94 distances (geodimeter and tellurometer measurements), and 2 astronomic azimuths. The observations, and their variances, were supplied by the Geodetic Survey of Canada [McLellan, 1973]. The Doppler coordinates are given in Table 10-1. The distances between Doppler points vary from 60 km (UNB-PLEASANT) to 280 km (UNB-BIO). The full variance-covariance matrix, Σ_{XYZ}, for the five sets of Doppler coordinates was obtained from the Geodetic Survey of Canada [Kouba, 1976(b)].

The Doppler data was treated as follows. First, it was scaled down by 1 ppm. This has been advocated by several investigators [e.g. Strange et al., 1975; Meade, 1974], and is compatible with the results of tests carried out by the author (11

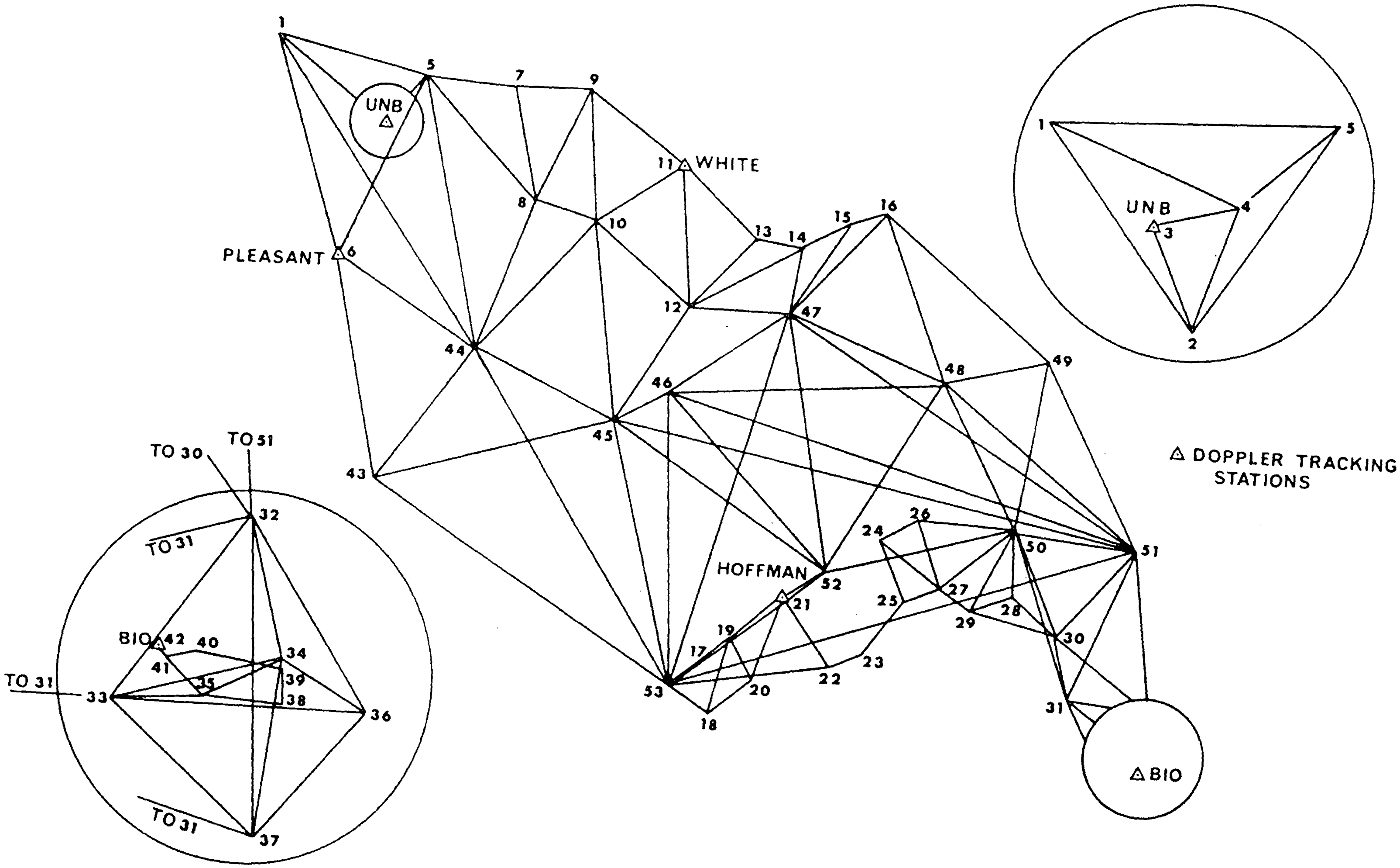

Figure 10 - 1

Doppler Coordinates as Weighted Parameters in a Horizontal Network Adjustment

Sta. Name & No.	X (m)	Y (m)	Z (m)	σ_x (m)	σ_y (m)	σ_z (m)
3 UNB	1761276.50	-4078247.17	4561415.33	1.34	1.35	1.26
6 PLEASANT	1765429.45	-4121681.36	4521317.48	1.35	1.32	1.18
11 WHITE	1848533.19	-4046217.11	4555689.58	1.36	1.21	1.13
21 HOFFMAN	1908996.09	-4093297.58	4488666.55	1.32	1.20	1.15
42 BIO	2018844.69	-4069146.31	4462376.69	1.28	1.20	1.14

TABLE 10-1.

DOPPLER COORDINATES USED AS WEIGHTED PARAMETERS and 12). The variance-covariance matrix, Σ_{XYZ}, was augmented by an external variance-covariance matrix, Σ_{EXT} to yield

$$\Sigma^*_{XYZ} = \Sigma_{XYZ} + \Sigma_{EXT} \qquad (10\text{-}1)$$

which expresses the accuracy of the Doppler coordinates with respect to their coordinate system. Σ_{EXT} is given by

$$\Sigma_{EXT} = \begin{bmatrix} \Sigma_b & \Sigma_b & \Sigma_b & \cdots \\ \Sigma_b & \Sigma_b & \Sigma_b & \cdots \\ \Sigma_b & \Sigma_b & \Sigma_b & \cdots \\ \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdots & \end{bmatrix} \qquad (10\text{-}2)$$

in which, for this test, Σ_b are 3x3 diagonal submatrices with elements 1 m^2 [Kouba, 1975]. The Doppler data was then transformed to the desired Geodetic coordinate system using the inverse of equation (1-9), with $x_o = -25.4$ m, $y_o = 152.4$ m, $z_o = 177.7$ m, and $\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$ [Boal, 1975]. The ellipsoidal coordinates for the Doppler stations, and their associated variance-covariance matrix, $\Sigma_{\phi\lambda}$, were generated using the procedures outlined in 5.1.

To test the overall effects of the Doppler network information, two parametric least-squares adjustments were performed. The adjustment procedure and software used are described in [Thomson and Chamberlain, 1975]. The first adjustment contained only the terrestrial observables. One point was fixed (transformed Doppler coordinates of 3 UNB), and the datum was the Clarke 1866 ellipsoid with axes oriented parallel to those of the Doppler datum. The position of the reference ellipsoid relative to the Doppler datum origin are given above. The second adjustment contained the Doppler network coordinates as weighted parameters. The Clarke 1866 ellipsoid was the reference surface. However, the final position and orientation of the datum were determined by the Doppler coordinates and their associated variance-covariance matrix.

The results of the adjustment of the terrestrial observables are given in Table 10-2. The χ^2 analysis of variance at 95% was not rejected. As a result of a normal test of the individual residuals, only two direction observations were rejected.

The results of the adjustment in which the Doppler coordinates were treated as weighted parameters are given in Table 10-3. The χ^2 analysis of variance at 95% was not rejected. Based on a normal distribution test of individual residuals, six directions were rejected. Two of the rejections were the same as those rejected when only terrestrial observables were used. The remaining four are all associated with the short lines used to tie 3 UNB (Doppler tracking station) to the main framework (Figure 10-1).

The overall effect of the Doppler network data is depicted in Figure 10-2. The network did not undergo changes in scale and orientation. The effect of the weighted Doppler coordinates was to shift the entire network in a northeasterly direction by approximately 1.3 m.

Table 10-4 shows what changes occurred in the Doppler coordinates and their standard deviations as a result of the adjustment. As can be seen, there was virtually no change in the standard

Network Points:		53
Observations:	Directions:	249
	Distances:	94
	Azimuths:	2

Fixed Points: 1 (3 UNB)

$\sigma_o^2 = 1.00$ $\hat{V}^T P \hat{V} = 190.00$ df = 170 $\hat{\sigma}_o^2 = 1.12$

χ^2 ANOVA (95%): $0.91 \leq 1.00 \leq 1.40$ (not rejected)

Residual Rejection: $P_r(-C\sigma_i < V_i < C\sigma_i) = 95\%$*

2 rejected observations (2 directions)

* C: from standard normal distribution

σ_i: standard deviation of the observable

V_i: residual

TABLE 10-2

RESULTS OF LEAST-SQUARES ADJUSTMENT USING TERRESTRIAL OBSERVABLES

Network Points: 53

Observations: Directions: 249

Distances: 94

Azimuths: 2

Doppler Coordinates: 5 points (3 UNB, 6 PLEASANT, 11 WHITE, 21 HOFFMAN, 42 BIO)

$\sigma_o^2 = 1.00 \quad \hat{V}^T P \hat{V} = 197.25 \quad X^T P_X X = 6.15 \quad df = 178 \quad \hat{\sigma}_o^2 = 1.14$

χ^2 ANOVA (95%): $0.94 \leq 1.00 \leq 1.42$ (not rejected)

Residual Rejection: $P_r\,(-C\sigma_i < V_i < C\sigma_i) = 95\%$*

6 rejected observations (6 directions)

* C: from standard normal distribution

σ_i: standard deviation of observable

V_i: residual

TABLE 10-3

RESULTS OF LEAST-SQUARES ADJUSTMENT USING TERRESTRIAL OBSERVABLES AND DOPPLER COORDINATES AS WEIGHTED PARAMETERS

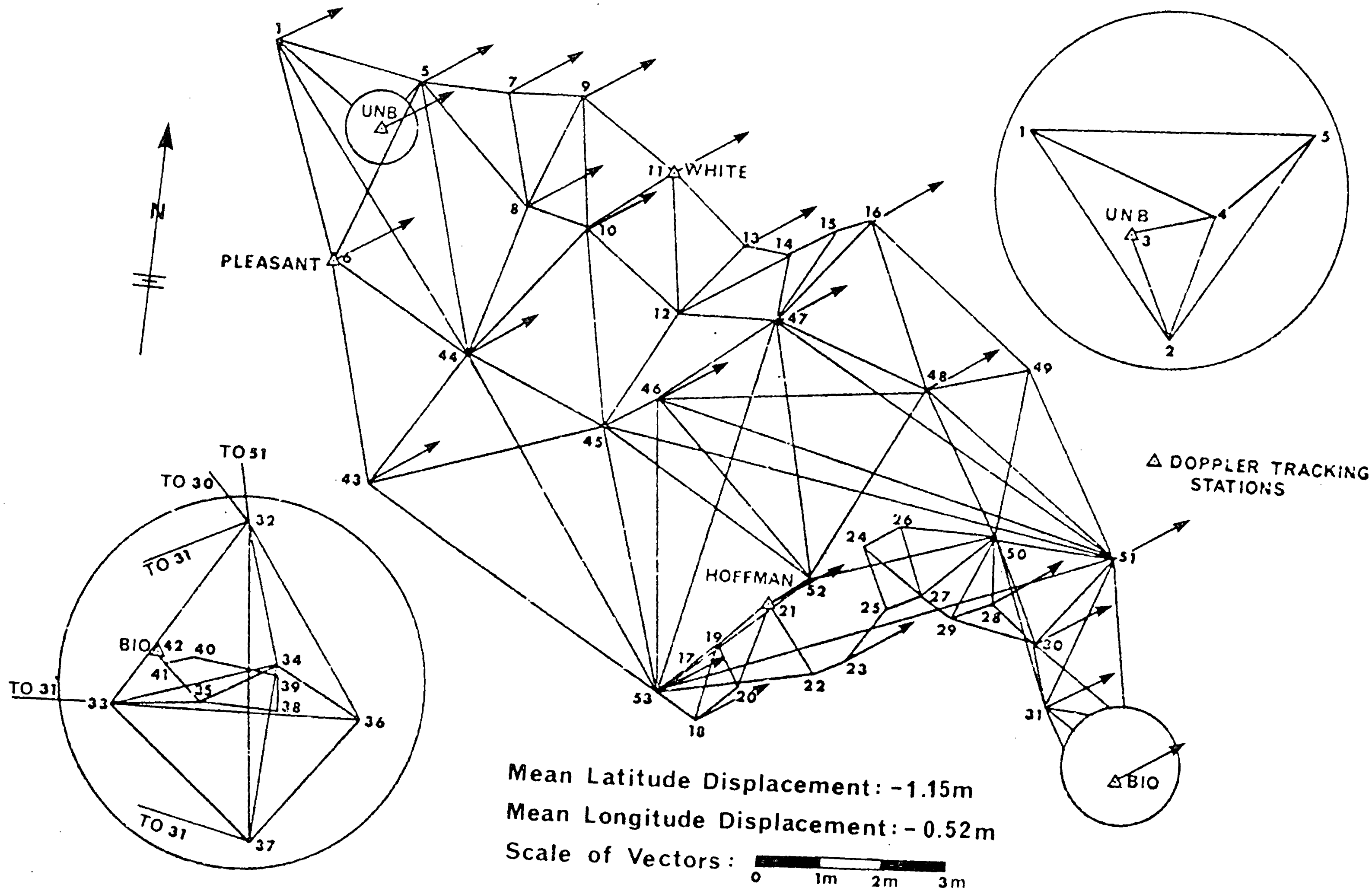

Figure 10-2
Displacement Vectors: Adjustment with Terrestrial Observables Only Minus Adjustment with Doppler Coordinates as Weighted Parameters

Station	ϕ	λ	σ_ϕ (arc-secs)	σ_λ (arc-secs)
3 UNB	45°57'00".825	-66°38'32".647	0.042	0.064
6 PLEASANT	45°25'50".858	-66°48'49".306	0.039	0.066
11 WHITE	45°52'28".526	-65°26'49".255	0.038	0.064
24 HOFFMAN	45°00'53".930	-64°59'51".252	0.038	0.062
42 BIO	44°40'59".210	-63°36'46".690	0.038	0.060

Doppler Coordinates Prior to the Adjustment

Station	ϕ	λ	σ_ϕ (arc-secs)	σ_λ (arc-secs)
3 UNB	45°57'00".861	-66°38'32".626	0.037	0.042
6 PLEASANT	45°25'50".853	-66°48'49".302	0.037	0.041
11 WHITE	45°52'28".542	-65°26'49".271	0.036	0.041
24 HOFFMAN	45°00'53".927	-64°59'51".208	0.036	0.039
42 BIO	44°40'59".220	-63°36'46".725	0.037	0.039

Coordinates of Doppler Points After Adjustment

TABLE 10-4

EFFECTS ON DOPPLER COORDINATES AS A RESULT OF COMBINATION WITH

TERRESTRIAL DATA

deviations of the latitudes. However, the terrestrial data caused a significant improvement in the standard deviations of the longitudes. The standard deviations of all coordinates resulting from this adjustment were in the ranges $0''.036$ to $0''.038$ for latitudes and $0''.039$ to $0''.043$ for longitudes. When no Doppler coordinates were used as weighted parameters, the standard deviations varied from $0''.00$ for the latitude and longitude of the fixed point to $0''.046$ and $0''.041$ in latitude and longitude respectively for other points. Thus, by allowing the Doppler coordinates and their associated variance-covariance matrix to define the coordinate system and constrain the terrestrial observables, the result is a more uniform variance-covariance matrix of adjusted coordinates.

The conclusion reached as a result of this test is that the two sets of data (terrestrial observables and transformed Doppler coordinates) are compatible. The shift of the network was due entirely to the fact that the weighted Doppler coordinates defined a new position of the datum from that given by the one fixed point in the first adjustment. The rejection of the extra direction observations in the second adjustment was the only sign of incompatibility of data. It is likely that the Doppler coordinates of 3 UNB are the problem since these direction observables were not rejected in the first adjustment.

A third adjustment of the test network was performed to investigate the effects of neglecting the covariance amongst and between the weighted Doppler coordinates. The correlation coefficients amongst the latitude components varied from 0.7 to 0.8, and amongst longitude components from 0.5 to 0.6. The correlation coefficients between the latitudes and longitudes were very small, never exceeding an absolute value of 0.03. Correlation between the latitude and longitude elements before the addition of Σ_{EXT} is much greater than the value given above.

The results of the adjustment, using only the diagonal elements of the weight matrix, P_X, are given in Table 10-4. The χ^2 analysis of variance at 95% was not rejected. Only two observables were rejected, and these were the same two directions as were rejected when no weighted Doppler coordinates were used.

The total effect on the network was to shift it approximately 1.0 m in a northeasterly direction. The mean shifts in latitude and longitude were -0.9 m and -0.6 m respectively (adjustment with terrestrial observables minus the adjustment with Doppler coordinates as weighted parameters). These values are somewhat less than the shifts that took place when the full weight matrix was used (Figure 10-2). There were no changes in the scale and orientation of the network.

The neglect of the covariance also appears to mask problems of compatibility between the two sets of data. This is indicated by the fact that in this adjustment, the direction observables rejected when a full weight matrix was used are not rejected. Overall, it appears as if the use of Doppler coordinates as weighted parameters in the adjustment of terrestrial data is a viable approach to the combination of the horizontal components of the networks when the required transformation parameters are known. It is recommended

Network Points:		53
Observations:	Directions:	249
	Distances:	94
	Azimuths:	2

Doppler Coordinates: 5 points (3 UNB, 6 PLEASANT, 11 WHITE, 21 HOFFMAN, 42 BIO)

$\sigma_o^2 = 1.00$ $\hat{V}^T P \hat{V} = 190.93$ $\hat{X}^T P_X \hat{X} = 5.01$ df = 178 $\hat{\sigma}_o^2 = 1.10$

χ^2 ANOVA (95%): $0.90 \leq 1.00 \leq 1.37$ (not rejected)

Residual Rejection: $P_r\ (-C\sigma_i < V_i < C\sigma_i) = 95\%$*

2 rejected observables (2 directions)

* C: from standard normal distribution

σ_i: standard deviation of observable

V_i: residual

TABLE 10-5

RESULTS OF LEAST-SQUARES ADJUSTMENT USING TERRESTRIAL OBSERVABLES AND DOPPLER COORDINATES AS WEIGHTED PARAMETERS (DIAGONAL ELEMENTS ONLY OF P_X)

that a full weight matrix (P_X) be used to avoid masking any problems that may exist in either of the two sets of data.

10.2 Computation of Satellite Network Distances, Azimuths and Associated Variance-Covariance Matrix

In 5.3, it was shown how satellite network distances and azimuths can be utilized as observables in a terrestrial network adjustment. While this procedure requires more computational effort to input an equivalent amount of satellite network data as when the satellite network coordinates are used as weighted parameters, there may be cases where the use of computed distances and azimuths is desirable [Anderle, 1974(c); Meade, 1974].

To treat the observables (computed distances and azimuths) rigorously in a two-dimensional network adjustment, it is necessary to have their full variance-covariance matrix, $\Sigma_{S\alpha}$. Two procedures for computing $\Sigma_{S\alpha}$ are given in 5.3. The first, given by equation (5-11), is rigorous. The second, computed via (5-27), is an approximate method. Computationally, the latter procedure has some advantages. However, these advantages can be overridden by the introduction of significant errors in the computed distances, azimuths, and associated variance-covariance matrix.

A portion of the Canadian Doppler network in eastern Canada [Kouba, 1976(b)] has been used to demonstrate the magnitude of the differences, of both the computed observables and their standard deviations, that can be expected when using the approximate method instead of the rigorous one. Table 10-6 gives the Doppler coordinates, and their standard deviations, that were used. Table 10-7 lists the computed ellipsoidal distances, azimuths, and associated standard deviations as computed via the rigorous ellipsoidal procedure. Values for the same quantities, computed via the Cartesian coordinates and then projected to the ellipsoid, are given in Table 10-8. The comparison of the two sets of results is given in Table 10-9.

The largest distance difference (S^*-S, Table 10-9) determined was 0.29 m for a line of length 1111.5 km, which is 0.3 ppm. For the two shortest lines, 326.5 km and 276.1 km, the distance differences were zero. For five of the distances, the distance differences (S^*-S, Table 10-9) were greater in magnitude than 10% of the values of the associated standard deviations of those distances (σ_S, Table 10-7). Such errors propagate systematically in a terrestrial network. The differences in the standard deviations of all distances were insignificant ($\sigma_S^*-\sigma_S$, Table 10-9). Thus, it can be concluded that for this practical example, the effects of the approximations introduced by using (5-27) are insignificant with respect to the satellite network distances.

The errors introduced into the azimuths are significant. Eight of the azimuth differences ($\alpha^*-\alpha$, Table 10-9) are greater by a factor of 2 (approximately) than the standard deviations of the rigorously computed azimuths (σ_α, Table 10-7). Only the azimuths of the two shortest lines (326.5 km and 276.1 km) have differences that are less than 10% of their standard deviations. All of the standard deviation differences ($\sigma_\alpha^*-\sigma_\alpha$, Table 10-9) are large relative to their

STA. NAME & NO.	X (m)	Y (m)	Z (m)	σ_x (m)	σ_y (m)	σ_z (m)
1 GOOSE BAY	1888555.65	-3319617.94	5091144.81	1.34	1.22	1.16
2 ST JOHNS	2612796.34	-3429075.79	4684923.87	1.44	1.34	1.21
3 BIOANT	2018845.72	-4069145.92	4462375.70	1.42	1.24	1.17
4 MATANE	1606493.42	-3888716.94	4777519.73	1.40	1.34	1.26
5 UNB	1761273.74	-4078249.66	4561416.97	1.40	1.25	1.20

TABLE 10-6

COORDINATES OF DOPPLER TRACKING STATIONS

FROM	TO	ELLIPSOIDAL DISTANCE (m)	σ_s (m)	GEODETIC AZIMUTH	σ_α (")
1 GOOSE BAY	2 ST JOHNS	838154.45	1.13	136°29'36".604	0.318
1 GOOSE BAY	3 BIOANT	987950.88	0.83	195°09'46".699	0.306
1 GOOSE BAY	4 MATANE	708724.62	1.12	228°06'59".046	0.393
1 GOOSE BAY	5 UNB	934811.57	0.97	211°25'57".142	0.312
2 ST JOHNS	3 BIOANT	901842.76	1.34	253°10'15".412	0.338
2 ST JOHNS	4 MATANE	1111552.61	1.38	282°40'51".811	0.275
2 ST JOHNS	5 UNB	1079118.45	1.39	265°32'21".281	0.287
3 BIOANT	4 MATANE	549620.96	1.10	328°12'33".489	0.473
3 BIOANT	5 UNB	276126.24	1.27	301°43'53".562	0.593
4 MATANE	5 UNB	326496.30	1.02	167°29'27".602	0.834

TABLE 10-7

RIGOROUSLY COMPUTED DOPPLER NETWORK ELLIPSOIDAL DISTANCES AND AZIMUTHS.

FROM	TO	REDUCED SPATIAL DISTANCE S* (m)	σ^*_s (m)	REDUCED SPATIAL AZIMUTH α*	σ^*_α (m)
1 GOOSE BAY	2 ST JOHNS	838154.28	1.13	136°29'35".840	0.284
1 GOOSE BAY	3 BIOANT	987950.82	0.82	195°09'47".233	0.273
1 GOOSE BAY	4 MATANE	708724.53	1.12	228°06'59".588	0.321
1 GOOSE BAY	5 UNB	934811.40	0.96	211°25'57".988	0.266
2 ST JOHNS	3 BIOANT	901842.55	1.33	253°10'16".080	0.237
2 ST JOHNS	4 MATANE	1111552.90	1.37	282°40'51".268	0.201
2 ST JOHNS	5 UNB	1079118.29	1.38	265°32'21".657	0.197
3 BIOANT	4 MATANE	549620.98	1.09	328°12'33".107	0.450
3 BIOANT	5 UNB	276126.24	1.27	301°43'53".465	0.748
4 MATANE	5 UNB	326496.30	1.02	167°29'27".545	0.801

TABLE 10-8

DOPPLER NETWORK SPATIAL DISTANCES AND AZIMUTHS REDUCED TO THE REFERENCE ELLIPSOID

FROM	TO	S* - S (m)	S* - S (ppm)	$\sigma_s^* - \sigma_s$ (m)	$\alpha^* - \alpha$ (")	$\sigma_\alpha^* - \sigma_\alpha$ (")
1 GOOSE BAY	2 ST JOHNS	-0.17	-0.2	0.00	0.758	-0.034
1 GOOSE BAY	3 BIOANT	-0.05	0.0	-0.01	-0.533	-0.034
1 GOOSE BAY	4 MATANE	-0.10	-0.1	0.00	-0.535	-0.072
1 GOOSE BAY	5 UNB	-0.18	-0.2	-0.01	-0.842	-0.045
2 ST JOHNS	3 BIOANT	-0.20	-0.2	-0.01	-0.666	-0.101
2 ST JOHNS	4 MATANE	0.29	0.3	-0.01	0.541	-0.075
2 ST JOHNS	5 UNB	-0.17	-0.2	-0.01	-0.375	-0.090
3 BIOANT	4 MATANE	0.02	0.0	-0.01	0.377	-0.023
3 BIOANT	5 UNB	0.00	0.0	0.00	0.093	-0.204
4 MATANE	5 UNB	0.00	0.0	0.00	0.055	-0.032

TABLE 10-9

DIFFERENCES IN DISTANCES, AZIMUTHS, AND ASSOCIATED STANDARD DEVIATIONS standard deviations (σ_α, Table 10-7). At present, there is no satisfactory explanation for the two extremely large standard deviation differences of -0".101 and -0".204 ($\sigma^*_\alpha - \sigma_\alpha$, Table 10-9). Differences in azimuths and their standard deviations of the magnitudes found in this example are not acceptable.

11. STANDARD THREE-DIMENSIONAL MODELS - UNKNOWN DATUM TRANSFORMATION PARAMETERS

The numerical testing carried out using the Bursa, Molodensky, and Veis models has been done for two major reasons. First, a proper use of the Bursa model is indicated. Second, the different results obtained using the Bursa or Molodensky and Veis models, when the same data is used in each, is pointed out.

Unfortunately, a lack of sufficient data hindered the solution of any major network combinations. However, the type of data required for rigorous three-dimensional procedures is shown.

11.1 Bursa Model

As pointed out in 7.1, the Bursa model should be used for the combination of two satellite networks. In this instance, the networks define the coordinate systems involved. The rotations solved for (ε_x, ε_y, ε_z) can not be confused with any systematic errors in either network and thus represent the orientation of one coordinate system with respect to the other. Further, the scale difference parameter, κ, is a system scale factor that can not be confused with either a change in the size of a reference ellipsoid or systematic scale errors in one satellite network or the other. In this case, κ is the difference in scale due to the different approaches used to scale the networks.

The Bursa model has been used by several investigators. Anderle [1974(b)] combined 37 stations of the World Satellite Triangulation Network (BC-4) with a network determined dynamically using Doppler measurements to Transit satellites. Schmid [1974] carried out a similar combination computation.

In order not to duplicate this type of solution, and since sufficient data (coordinates and variance-covariance matrices) for the United States or Canadian Doppler networks and North American Densification of the World Satellite Triangulation (BC-4) Network was not readily available, an example of another use of the Bursa model is given in the following.

The Geodetic Survey of Canada computes several sets of Cartesian coordinates for each point in a Doppler network. For example, one set may be the results using the broadcast ephemeris of the Transit satellites, while others are the results using the precise ephemeris from one or more satellites [Kouba, 1975]. In order that the full benefit of all data can be obtained, or to study the differences between the broadcast and precise ephemeris coordinate systems, the Bursa model is used to combine the two networks. An example, for five stations in Atlantic Canada, is given (Figure 11-1).

The data for this test - Cartesian coordinates (Table 11-1) and full variance-covariance matrix - were supplied by the Geodetic

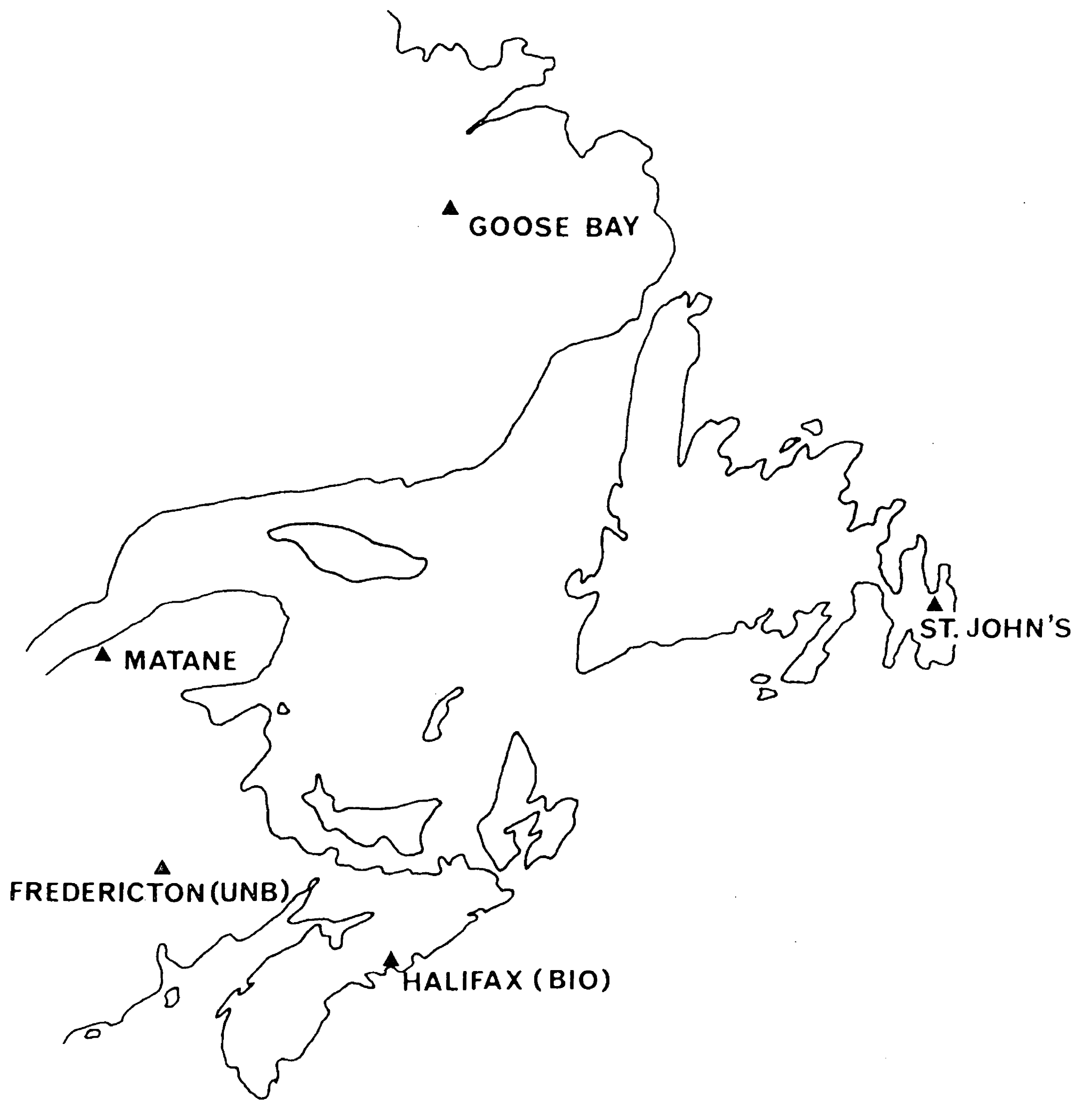

Figure 11-1

Five Doppler Tracking Stations in Atlantic Canada

STA. NAME & NO.	X (m)	Y (m)	Z (m)	σ^*_x (m)	σ^*_y (m)	σ^*_z (m)
1 GOOSE BAY	1888563.50	-3319612.36	5091145.80	4.57	4.67	4.54
2 ST. JOHNS	2612805.27	-3429070.10	4684925.21	4.52	4.71	4.56
3 BIOANT	2018852.49	-4069145.29	4462379.60	4.49	4.63	4.53
4 MATANE	1606501.10	-3888717.43	4777524.76	4.51	4.63	4.52
5 UNB	1761283.05	-4078248.72	4561419.76	4.53	4.61	4.56

DOPPLER COORDINATES - BROADCAST EPHEMERIS OF SATELLITES

12 & 13 FOR DAYS 147 - 151/1974.

STA. NAME & NO.	X (m)	Y (m)	Z (m)	σ^*_x (m)	σ^*_y (m)	σ^*_z (m)
1 GOOSE BAY	1888555.65	-3319617.94	5091144.81	1.34	1.22	1.16
2 ST. JOHNS	2612796.34	-3429075.79	4684923.87	1.44	1.34	1.21
3 BIOANT	2018845.72	-4069145.92	4462375.70	1.42	1.24	1.17
4 MATANE	1606493.42	-3888716.94	4777519.73	1.40	1.34	1.26
5 UNB	1761273.74	-4078249.66	4561416.97	1.40	1.25	1.20

DOPPLER COORDINATES - PRECISE EPHEMERIS OF SATELLITE 14

DAYS 148 - 151/1973.

* σ_x, σ_y, σ_z with respect to the coordinate system.

TABLE 11 - 1.

COMBINATION OF TWO SETS OF DOPPLER COORDINATES

Survey of Canada [Kouba, 1976(b)]. The variance-covariance matrices (Σ_{XYZ}) for the broadcast and precise ephemeris results represent the accuracy of the solutions. The accuracy of the coordinates with respect to their respective coordinate systems is given by

$$\Sigma^{*}_{XYZ} = \Sigma_{XYZ} + \Sigma_{EXT} \tag{11-1}$$

where Σ_{EXT} has the form [Kouba, 1975]

$$\Sigma_{ext} = \begin{bmatrix} \Sigma_b & \Sigma_b & \Sigma_b & \cdots\cdots \\ \Sigma_b & \Sigma_b & \Sigma_b & \cdots\cdots \\ \Sigma_b & \Sigma_b & \Sigma_b & \cdots\cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix} \tag{11-2}$$

where Σ_b is a 3x3 submatrix. At present, only the diagonal elements of Σ_b are available, and they are 1 m^2 for precise ephemeris results and 16 m^2 for broadcast ephemeris results [Kouba, 1975]. This results in incorrect correlation amongst and between the coordinates. Since Σ^{*-1}_{XYZ} is used as the weight matrix of observations in the estimation procedure of the Bursa model, this lack of information could have an effect on the results.

The Doppler tracking stations are separated by between 276 km (FREDERICTON-U.N.B. to HALIFAX-BIO) and 1078 km (FREDERICTON-U.N.B. to ST. JOHN'S). Four tests were carried out in which the precise ephemeris data was treated as referring to coordinate system 1 and the broadcast data to system 2 of the Bursa model (7-1). In each computation, different numbers and combinations of unknown datum transformation parameters were solved for (Table 11-2).

Of the four tests, only two solutions (Tests #1 and #2, Table 11-2) were not rejected based on a χ^2 analysis of variance at 95% probability. Test #1 had seven unknown parameters (x_o, y_o, z_o, ε_x, ε_y, ε_z, κ) while Test #2 had only six (x_o, y_o, z_o, ε_x, ε_y, ε_z). The elimination of the scale difference parameter, which in the results of Test #1 was significant (κ = 2.0 ppm, σ_κ = 1.0 ppm), caused a marked change in the translation components of Test #2. The rotations were not affected by the elimination of κ. In both solutions, the estimates of ε_y were found to be equal to or less than their respective standard deviations. Although neither solution was rejected, $\hat{V}^T P \hat{V}$ of Test #2 (15.80) was much greater than that of Test #1 (11.85) while the increase in df was only 1. Thus, in addition to the changes in the translation components caused by the elimination of κ from the Test #2 solution, there was also an overall increase in the magnitude of the residuals. However, in both Test #1 and Test #2, the residuals for this small sample were found to be normally distributed, and no residuals were outside the range

$$- c\sigma_{L_i} < \hat{V}_i < c\sigma_{L_i} \quad , \qquad (11\text{-}3)$$

where σ_{L_i} is the a priori standard deviation of the observable and c is the value of the standard normal distribution at 95% probability.

The adjusted coordinates of the five points (Test #1) are given in Table 11-3. As can be observed by comparing these values (Table 11-3) with those before the adjustment (Table 11-1), changes have taken place as a result of the combination process. The maximum coordinate changes for the precise ephemeris data was -1.11 m

Test #1

$x_o = 14.8$ m	$\sigma_{x_o} = 10.7$ m	$\varepsilon_x = -0\overset{''}{.}90$	$\sigma_{\varepsilon_x} = 0\overset{''}{.}25$	
$y_o = 16.7$ m	$\sigma_{y_o} = 8.8$ m	$\varepsilon_y = 0\overset{''}{.}26$	$\sigma_{\varepsilon_y} = 0\overset{''}{.}26$	$\kappa = -2.0$ ppm $\sigma_\kappa = 1.0$ ppm
$z_o = 20.1$ m	$\sigma_{z_o} = 9.6$ m	$\varepsilon_z = 0\overset{''}{.}70$	$\sigma_{\varepsilon_z} = 0\overset{''}{.}30$	

$\sigma_o^2 = 1.00$ df = 8 $\hat{\sigma}_o^2 = 1.48$

Test #2

$x_o = 10.3$ m	$\sigma_{x_o} = 10.4$ m	$\varepsilon_x = -0\overset{''}{.}88$	$\sigma_{\varepsilon_x} = 0\overset{''}{.}25$	
$y_o = 24.0$ m	$\sigma_{y_o} = 8.0$ m	$\varepsilon_y = 0\overset{''}{.}23$	$\sigma_{\varepsilon_y} = 0\overset{''}{.}26$	κ: eliminated from the solution
$z_o = 10.3$ m	$\sigma_{z_o} = 7.1$ m	$\varepsilon_z = 0\overset{''}{.}71$	$\sigma_{\varepsilon_z} = 0\overset{''}{.}30$	

$\sigma_o^2 = 1.00$ df = 9 $\hat{\sigma}_o^2 = 1.76$

Test #3

$x_o = -4.8$ m	$\sigma_{x_o} = 4.0$ m	ε_x	
$y_o = -9.0$ m	$\sigma_{y_o} = 6.0$ m	ε_y eliminated from the solution	$\kappa = -1.9$ ppm $\sigma_\kappa = 1.0$ ppm
$z_o = 5.9$ m	$\sigma_{z_o} = 6.7$ m	ε_z	

$\sigma_o^2 = 1.00$ df = 11 $\hat{\sigma}_o^2 = 2.65$

Test #4

$x_o = 9.6$ m	$\sigma_{x_o} = 4.6$ m	ε_x	
$y_o = -2.0$ m	$\sigma_{y_o} = 4.7$ m	ε_y eliminated from the solution	κ: eliminated from the solution
$z_o = -3.0$ m	$\sigma_{z_o} = 4.6$ m	ε_z	

$\sigma_o^2 = 1.00$ df = 12 $\hat{\sigma}_o^2 = 2.71$

Table 11-2

BURSA MODEL TESTS

COMBINATION OF TWO SETS OF DOPPLER DATA

Sta. Name & No.	X (m)	Y (m)	Z (m)	σ_x (m)	σ_y (m)	σ_z (m)
1 GOOSE BAY	1888564.64	-3319610.72	5091146.87	4.52	4.64	4.51
2 ST JOHNS	2612804.61	-3429069.93	4684925.44	4.49	4.67	4.55
3 BIOANT	2018853.32	-4069144.70	4462380.00	4.46	4.60	4.52
4 MATANE	1606501.21	-3888716.19	4777524.81	4.47	4.61	4.51
5 UNB	1761282.39	-4078248.54	4561420.96	4.48	4.59	4.53

ADJUSTED DOPPLER COORDINATES - BROADCAST EPHEMERIS

OF SATELLITES 12 & 13 FOR DAYS 147-151/1973

Sta. Name & No.	X (m)	Y (m)	Z (m)	σ_x	σ_y	σ_z
1 GOOSE BAY	1888556.06	-3319617.61	5091144.63	1.28	1.20	1.14
2 ST JOHNS	2612796.60	-3429075.63	4684924.11	1.36	1.30	1.20
3 BIOANT	2018844.61	-4069146.12	4462375.92	1.29	1.19	1.14
4 MATANE	1606493.55	-3888717.95	4477520.35	1.28	1.24	1.18
5 UNB	1761274.29	-4078249.51	4561416.31	1.28	1.19	1.15

ADJUSTED DOPPLER COORDINATES - PRECISE EPHEMERIS OF

SATELLITE 14 FOR DAYS 148-151/1973

TABLE 11-3

ADJUSTED COORDINATES AS A RESULT OF THE COMBINATION OF TWO

SETS OF DOPPLER COORDINATES

in X (BIOANT), -1.01 m in Y (MATANE), and -0.66 m in Z (UNB). The maximum coordinate changes for the broadcast ephemeris data was 1.14 m in X and 1.64 m in Y, both of which occurred at the point GOOSEBAY, and 1.20 m in Z (UNB). In all instances, there was no significant change in the standard deviations of the coordinate values.

A second test was made using the Bursa model for the combination of a satellite and a terrestrial network. This model is <u>not recommended</u> for use in this way. The test was run solely for the sake of comparing results with those of the Molodensky and Veis models. The results are presented to illustrate numerically the fact that the Bursa and Molodensky or Veis models are not equivalent.

The data used is given in Tables 11-4 and 11-5. These are the coordinates - preliminary terrestrial results for the Transcontinental Traverse and Doppler - for twenty-one stations in the United States of America (Figure 11-2). This data was supplied by the National Geodetic Survey of the United States [Meade, 1975; Strange, 1975].

Of the twenty-one triplets of Doppler coordinates, only fifteen had recorded standard deviations resulting from the solutions for these coordinates. The remaining six points were given standard deviations equal to the means of the values (σ_ϕ, σ_λ, σ_h) of the other fifteen. No covariance was available, either amongst station coordinates or between various stations. No estimate of external accuracy was given. Values of 0.504 m^2, 0.518 m^2, 0.533 m^2

STA. NAME & NO.	LATITUDE ϕ	LONGITUDE λ	h (m)	σ^*_ϕ (m)	σ^*_λ (m)	σ^*_h (m)
MDS. RCH. 10006	39°13'26".642	261°27'27".477	566.60	0.12	0.17	0.11
BLTSVLE 53002	39°01'39:288	283°10'27".262	1.00	0.12	0.20	0.12
NEWTON 51025	30°54'24".714	266°23'56".765	44.90	0.17	0.28	0.16
IRAAN 51039	30°52'15".370	258°03'58".284	866.97	0.15	0.25	0.15
ARTHUR 51041	41°38'26".726	258°24'01".345	1151.98	0.13	0.19	0.12
LOVELL 51043	44°48'01".457	251°39'16".310	1193.35	0.12	0.20	0.12
HORSE 51044	41°36'44".678	252°12'53".888	2208.72	0.11	0.20	0.11
ALBQUE 51048	34°56'43".490	253°32'23".305	1800.36	0.15	0.25	0.15
TERBON 51066	44°23'31".282	238°42'12".208	861.28	0.14	0.23	0.14
MINWEL 51067	32°57'44".997	261°54'35".104	323.84	0.15	0.23	0.14
YOLEE 51068	30°41'46".311	278°15'59".114	-16.24	0.12	0.19	0.12
ASHEPO 51069	32°45'31".674	279°26'36".774	-38.68	0.13	0.20	0.12
RIOVIST 51089	38°08'31".754	238°16'33".529	10.23	0.16	0.24	0.15
DACOUNT 51103	32°04'19".495	253°31'03".740	1239.34	0.15	0.25	0.15
OPELOUS 51121	30°37'55".231	267°50'02".412	-15.99	0.15	0.25	0.15
FT DAVIS 51123	30°40'16".420	255°58'36".396	2307.58	0.15	0.25	0.15

Table 11-4

STA. NAME & NO.	LATITUDE ϕ	LONGITUDE λ	h (m)	σ^*_ϕ (m)	σ^*_λ (m)	σ^*_h (m)
PILLPT 10055	37°29'53".123	237°30'04".985	13.51	0.15	0.25	0.15
CASH 10021	37°33'06".952	273°55'09".742	229.40	0.19	0.29	0.17
UKAMISS 10022	34°47'15".796	271°45'29".375	211.70	0.26	0.41	0.22
TERMISS 10023	33°33'54".992	270°50'03".480	103.60	0.20	0.30	0.18
OXALIS 53063	36°54'50".743	239°26'44".530	-1.44	0.14	0.22	0.13

a = 6378145.0 1/f = 298.25

NOTE: σ_ϕ, σ_λ, σ_h were not given for stations 4, 18, 15, 16, 17. The values given here are the means of the values σ_ϕ, σ_λ, σ_h of all other 15 stations.

* σ_ϕ, σ_λ, σ_h represent the internal accuracy of the solution.

TABLE 11-4 (cont'd)

DOPPLER COORDINATES COINCIDENT WITH TRANSCONTINENTAL TRAVERSE POINTS

STA. NAME & NO.	LATITUDE ϕ	LONGITUDE λ	$\sigma_\phi = \sigma_\lambda$ (m)	H (m)	σ_H (m)	N (m)	σ_H (m)
MDS. RCH. 10006	39°13'26".686	261°27'29".494	0.0	599.40	0.0	0.0	0.0
BLTSVLE 53002	39°01'39".261	283°10'26".899	3.0	42.80	0.3	-2.7	5.9
NEWTON 51025	30°54'24".080	266°23'57".134	2.0	85.70	0.2	2.1	3.9
IRAAN 51039	30°52'14".847	258°04'00".426	2.0	898.90	0.2	0.6	3.8
ARTHUR 51041	41°38'27".013	258°24'03".844	1.0	1179.40	0.1	4.6	2.2
LOVELL 51043	44°48'02".028	251°39'19".733	2.0	1213.55	0.2	6.8	3.9
HORSE 51044	41°36'56".136	252°12'57".024	1.8	2231.30	0.2	5.2	3.4
TERBON 51066	44°23'32".167	238°42'17".070	3.1	891.30	0.3	-16.4	6.1
ALBQUE 51048	34°56'43".351	253°32'25".929	1.8	1829.60	0.2	0.5	3.5
MINWEL 51067	32°57'44".602	261°54'36".968	1.6	359.60	0.2	-2.0	3.1
YULEE 51068	30°41'45".626	278°15'59".246	3.0	17.00	0.3	4.4	5.7
ASHEPO 51069	32°45'31".194	279°26'36".789	2.9	2.20	0.3	-0.3	5.6
RIOUIST 51089	38°08'32".353	238°16'37".829	3.2	51.80	0.3	-30.7	6.2
DACOUNT 51103	32°04'19".097	253°31'06".299	2.1	1271.60	0.2	-3.5	4.0
OPELOUS 51121	30°37'54".595	267°50'05".405	2.2	19.50	0.2	3.1	4.1
FT. DAVIS 51123	30°40'15".872	255°58'44:120	2.1	2065.60	0.2	1.1	4.0

Table 11-5

STA. NAME & NO.	LATITUDE ϕ	LONGITUDE λ	σ_{ϕ} ϕ_{λ} (m)	H (m)	σ_H (m)	N (m)	σ_H (m)
PILLPT 10056	37°29'53".441	237°30'09".749	3.3	53.82	0.3	-32.8	6.4
CASH 10021	37°33'06".818	273°55'10".341	2.1	267.00	0.2	-1.4	4.1
UKAMISS 10022	34°47'15".482	271°45'30".184	2.0	247.10	0.2	3.3	4.0
TERMISS 10023	33°33'54".598	270°50'04".408	2.1	138.60	0.2		4.0
OXALIS 53063	36°54'51".184	239°26'48".612	3.1	40.08	0.3	-30.4	6.0

a = 6378206.4   1/f = 294.98525

*STN. MDS RCH. (10006) IS THE TERRESTRIAL NETWORK INITIAL POINT

TABLE 11-5 (cont'd)

U.S.A. TRANSCONTINENTAL TRAVERSE COORDINATES

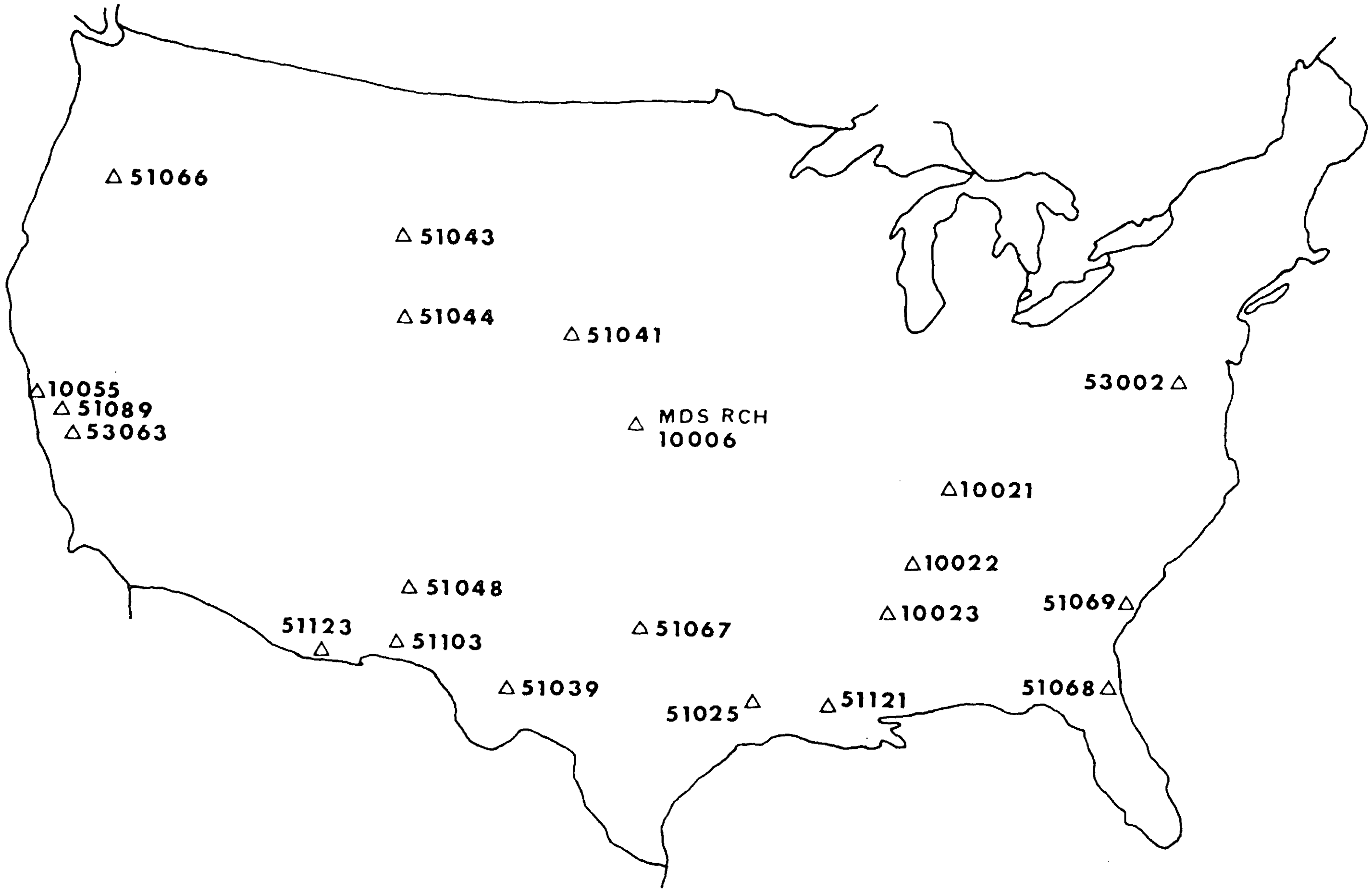

Figure 11-2

Twenty-one Doppler Tracking Stations in the United States of America

[Beuglass, 1974] were added to each of σ_ϕ^2, σ_λ^2, σ_h^2 for each station to yield realistic variances for the Doppler coordinates with respect to their coordinate system.

No estimates of the accuracy of the Transcontinental Traverse coordinates (ϕ, λ) were given since they represent preliminary results [Strange, 1975]. In lieu of this, the rule of thumb [NASA, 1973]

$$\sigma_\phi = \sigma_\lambda = 0.020\ K^{2/3} m \tag{11-4}$$

was used, where K is the distance of (ϕ, λ) from the terrestrial initial point. In this test, (σ_{ϕ_k}, σ_{λ_k}) have been assumed to be equal to zero, although as explained previously (1.3), this need not be the case.

Similarly, no accuracy estimates were given for either H or N. Again, some rules of thumb were adopted to have some standard deviations. Assuming that all heights were determined by spirit levelling, the relationship [NASA, 1973]

$$\sigma_H = 1.8\ K^{2/3} \cdot 10^{-3}\ m \tag{11-5}$$

was used. For N, assuming that astrogeodetic methods had been used, σ_N was approximated by [Badekas, 1969]

$$\sigma_N = 0.035\ K^{1/2} + 0.075\ K^{1/2} + 3.88 \cdot 10^{-6} \cdot K^{5/3} m\ . \tag{11-6}$$

In this test, the ellipsoidal height ($h_k = H_k + N_k$) at the terrestrial network initial point has been assumed to have $\sigma_h = 0$. Again, this will most likely not be the case in practice (1.3).

The Doppler network coordinates are referred to system 1 in the Bursa model (7.1), while the Transcontinental Traverse is related to system 2. Since the Doppler coordinates were given in terms of geodetic coordinates (ϕ, λ, h), they, and their variance-covariance matrix, were transformed to a Cartesian coordinate system using (1-2) and (2-3) respectively. The Transcontinental Traverse (terrestrial) coordinates were expressed in terms of (ϕ, λ, H, N), and they too were transformed to a Geodetic Cartesian system using the procedure outlined in 2.1.3. The semi-major axes and flattenings of the two ellipsoids involved are given in Tables 11-4 and 11-5.

Four test runs were made using the data described. The resulting transformation parameters are given in Table 11-6. In all cases, the χ^2 analysis of variance failed at the 95% confidence level. Although no indepth analysis of the reasons for the failures could be carried out, it is the author's opinion that the approximate method by which variances were generated for the terrestrial coordinates was the major problem. This is particularly evident for Test #3 (Table 11-6) in which all transformation parameters were significant. The other reason, which is of greater consequence, is that the model (Bursa) is not suitable for the combination of terrestrial and satellite networks.

A brief examination of the results of Tests #1 and #2 (Table 11-6) shows that the rotation parameters generated are all insignificant. In all tests, the translation components are significant, and Tests #1 and #3 show the scale difference to be of significant magnitude.

In comparing the results reported here with those of Strange et al. [1975], it was found that the translation components

Test #1			
$x_o = -22.6$ m	$\sigma_{x_o} = 6.3$ m	$\varepsilon_x = 0''.10$	$\sigma_{\varepsilon_x} = 0''.29$
$y_o = 154.4$ m	$\sigma_{y_o} = 5.8$ m	$\varepsilon_y = 0''.02$	$\sigma_{\varepsilon_y} = 0''.16$
$z_o = 172.6$ m	$\sigma_{z_o} = 7.3$ m	$\varepsilon_z = 0''.10$	$\sigma_{\varepsilon_z} = 0''.17$ $\kappa=1.1$ ppm $\sigma_\kappa=0.4$ ppm
$\sigma_o^2 = 1.00$	df = 56		$\hat{\sigma}_o^2$ 0.54
Test #2			
$x_o = -22.7$ m	$\sigma_{x_o} = 6.3$ m	$\varepsilon_x = 0''.07$	$\sigma_{\varepsilon_x} = 0''.29$
$y_o = 150.0$ m	$\sigma_{y_o} = 5.5$ m	$\varepsilon_y = 0''.03$	$\sigma_{\varepsilon_y} = 0''.16$
$z_o = 177.6$ m	$\sigma_{z_o} = 7.0$ m	$\varepsilon_z = 0''.12$	$\sigma_{\varepsilon_z} = 0''.17$ κ: eliminated from the solution.
$\sigma_o^2 = 1.00$	df = 57		$\hat{\sigma}_o^2 = 0.64$
Test #3			
$x_o = -25.3$ m	$\sigma_{x_o} = 0.8$ m	ε_x:	
$y_o = 156.6$ m	$\sigma_{y_o} = 2.1$ m	ε_y: eliminated from the solution	$\kappa = 1.1$ ppm $\sigma_\kappa = 0.4$ ppm
$z_o = 174.8$ m	$\sigma_{z_o} = 1.9$ m	ε_z:	
$\sigma_o^2 = 1.00$	df = 59		$\hat{\sigma}_o^2 = 0.52$
Test #4			
$x_o = -26.0$ m	$\sigma_{x_o} = 0.7$ m	ε_x:	
$y_o = 151.7$ m	$\sigma_{y_o} = 0.7$ m	ε_y: eliminated from the solution	κ: eliminated from the solution
$z_o = 179.1$ m	$\sigma_{z_o} = 0.7$ m	ε_z:	
$\sigma_o^2 = 1.00$	df = 60		$\hat{\sigma}_o^2 = 0.62$

TABLE 11-6

BURSA MODEL TESTS

COMBINATION OF A DOPPLER AND A TERRESTRIAL NETWORK

(Test #1, Table 11-6) agreed to within 2 m in all three components. The scale difference of 1.1 ppm compares favourably with the 1.0 ppm. However, Strange et al. [1975] found significant rotations ε_y (-0".10, σ = 0".04) and ε_z (0".19, σ = 0".04). This latter difference is most likely due to the somewhat different data used and the different variances attributed to the terrestrial coordinates.

For each test run, the adjusted coordinates of each network and their variance-covarinace matrices were computed. In all cases, there was a significant decrease in the standard deviations of the terrestrial coordinates. For example, before the network combination, the standard deviations of the Cartesian coordinates of point 51068 (YULEE) were 3.06 m, 5.11 m, and 3.89 m in the x, y, and z components respectively. After the combination they were 0.69 m, 0.36 m, and 0.36 m. There was little change in the standard deviations of the Doppler coordinates.

There were significant changes in the terrestrial coordinate values in all tests. For example, the adjusted terrestrial coordinates for Test #4 experienced changes of -2.77 m to 2.76 m in x, -2.54 m to 4.01 m in y, and -3.70 m to 0.39 m in z. There were no changes greater than $\pm$ 0.2 m (Test #4) in any Doppler coordinates.

Comparisons of the Bursa model results with those of the Veis and Molodensky are given in 11.3.

11.2 Veis and Molodensky Models

These models have been used in some studies in which the combination of terrestrial and satellite networks were carried out [e.g. Badekas, 1969; Mueller and Kumar, 1975]. The test computations reported herein have been carried out solely for the purposes of comparison with results generated via the Bursa model.

The Doppler and terrestrial data used is given in Tables 11-4 and 11-5 respectively. Five test runs were made, each of which contained different sets of unknown parameters. The resulting translation components and network scale and orientation parameters are given in Table 11-7.

None of the test runs were acceptable on the basis of a χ^2 analysis of variance at 95%. One of the reasons for this, as stated regarding the Bursa model test, is most probably the approximate methods by which terrestrial coordinate variances were generated. The fact can not be discounted, however, that the failure of the χ^2 analysis of variance is an indication that the Veis (or Molodensky) model is not adequate for the combination of terrestrial and satellite networks.

In all five test computations, only the translation components and scale difference were found to be significant. As with the Bursa model tests, there was little change in the Doppler coordinates or their associated variances. Significant changes in terrestrial coordinates and their variances occurred as a result of the Veis (or Molodensky) combination procedure.

Comparisons of the results generated for this report could

Test #1

$x_o = -26.2$ m $\sigma_{x_o} = 0.7$ m $dA = -0.04$ $\sigma_{dA} = 0''.09$

$y_o = 151.2$ m $\sigma_{y_o} = 0.7$ m $d\mu = -0''.09$ $\sigma_{d\mu} = 0''.29$

$z_o = 179.3$ m $\sigma_{z_o} = 0.7$ m $d\nu = -0''.10$ $\sigma_{d\nu} = 0''.20$ $\kappa = 1.1$ ppm $\sigma_\kappa = 0.4$ ppm

Molodensky Rotations:

$\psi_x = 0''.10$ $\psi_y = 0''.02$ $\psi_z = 0''.10$

$\sigma_o^2 = 1.00$ df = 56 $\hat{\sigma}_o^2 = 0.54$

Test #2

$x_o = -26.1$ m $\sigma_{x_o} = 0.7$ m $dA = -0''.04$ $\sigma_{dA} = 0''.09$

$y_o = 151.8$ m $\sigma_{y_o} = 0.7$ m $d\mu = -0''.06$ $\sigma_{d\mu} = 0''.29$

$z_o = 179.2$ m $\sigma_{z_o} = 0.7$ m $d\nu = -0''.12$ $\sigma_{d\nu} = 0''.20$ κ: eliminated from the solution

Molodensky Rotations:

$\psi_x = 0''.07$ $\psi_y = 0''.03$ $\psi_z = 0''.12$

$\sigma_o^2 = 1.00$ df = 57 $\hat{\sigma}_o^2 = 0.64$

Test #3

$x_o = -26.1$ m $\sigma_{x_o} = 0.7$ m dA: eliminated

$y_o = 151.2$ m $\sigma_{y_o} = 0.7$ m dμ: from the solution

$z_o = 179.3$ m $\sigma_{z_o} = 0.7$ m dν: $\kappa = 1.1$ ppm $\sigma_\kappa = 0.4$ ppm

$\sigma_o^2 = 1.00$ df = 59 $\hat{\sigma}_o^2 = 0.52$

Table 11-7

Test #4

x_o = -26.0 m	σ_{x_o} = 0.7 m	dA:	eliminated from the solution
y_o = 151.7 m	σ_{y_o} = 0.7 m	dμ:	
z_o = 179.1 m	σ_{z_o} = 0.7 m	dν:	κ: eliminated from the solution
σ_o^2 = 1.00	df = 60		$\hat{\sigma}_o^2$ = 0.62

Test #5

x_o = -26.1 m	σ_{x_o} = 0.7 m	dA = -0"04 σ_{dA} = 0"09	
y_o = 151.2 m	σ_{y_o} = 0.7 m	dμ: eliminated from the solution	
z_o = 179.3 m	σ_{z_o} = 0.7 m	dν:	κ=1.1 ppm σ_κ=0.4 ppm

Molodensky Rotations:

ψ_x = -0"02	ψ_y = -0"03	ψ_z = 0"02
σ_o^2 = 1.00	df = 58	$\hat{\sigma}_o^2$ = 0.53

TABLE 11-7 (cont'd)

VEIS AND MOLODENSKY MODEL TESTS

COMBINATION OF A TERRESTRIAL AND A DOPPLER NETWORK not be compared with those of other investigators. The reason for this is that no outside investigators have attempted to combine the U.S. Transcontinental Traverse and Doppler networks using either a Veis or a Molodensky model.

11.3 Comparison of Results

In 7.4, it was shown that the Bursa and Veis (or Molodensky) models are not mathematically equivalent. This was brought to light in a comparison of the treatment of a terrestrial initial point in the formulation of each of the models, and in a comparison of the elements of the design matrices A_i in the estimation procedure. As has been shown in other investigations, the models can be made to be equivalent under certain conditions concerning the terrestrial initial point and the formulation of the mathematical models [Krakiwsky and Thomson, 1974; Mueller and Kumar, 1975].

A comparison of the test results generated using the same data in each of the models helps to further illustrate the differences in the models. In each of Tests #1 and #2 (Tables 11-6 and 11-7), the translation components of the Bursa model solution are different from those of the Veis (or Molodensky) model solution, while the scale difference parameters and rotation parameters are equivalent numerically. The latter is easily understood. Since there is only one set of rotation parameters and one scale difference parameter in each model all rotation and scale errors, whether they are the result of discordant, mis-scaled coordinate systems or misoriented mis-scaled networks, are represented by these parameters. The results of Tests #3 (Tables 11-6 and 11-7), from which all orientation parameters were eliminated, show that the two sets of translation components (Bursa and Veis or Molodensky) are still significantly different - 0.8 m in x_o, 5.4 m in y_o, and -4.5 m in z_o. The scale difference parameters are, however, equivalent. Only in Test #4 of each model, in which all rotation and scale parameters have been eliminated from each solution, are the results (translation components) equivalent.

12. RECENT THREE-DIMENSIONAL MODELS - UNKNOWN DATUM TRANSFORMATION PARAMETERS

Each of the three models in this category - Hotine, Krakiwsky-Thomson, and Vanicek-Wells - contain two sets of unknown rotation parameters. The separation of the two sets of rotations in the former two models is achieved via a specific least-squares estimation procedure, while the latter involves several terrestrial networks in a single parametric least-squares solution. In all three instances, final conclusions regarding the validity and usefulness of the models will only be possible when the proper data is available for numerical testing. Proper data is considered to be two or more sets of terrestrial and satellite network coordinates that are the result of rigorous but separate network computation procedures. Unfortunately, this type of data is not readily available at present. Thus, the tests carried out merely indicate that the models proposed, and their methods of solution, are feasible.

Contained herein are test results of the combination of a Doppler and a terrestrial network using the Krakiwsky-Thomson model.

The Vanicek-Wells model was not tested for this report. Test results for this model have been reported elsewhere [Wells and Vanicek, 1975].

12.1 Krakiwsky-Thomson Model

The data given in 11.1 (Tables 11-4 and 11-5) was used to generate some test results for the combination of a Doppler and a terrestrial network. The estimation procedure outlined in 8.2 was utilized.

Three sets of results are given in Table 12-1. Based on a χ^2 analysis of variance at 95%, none of the results were accepted. Several assumptions may be made regarding this, none of which can be investigated at present. The data used was not complete. There was no covariance information amongst and between the Doppler coordinates. The variances of the terrestrial coordinates were generated using some rules of thumb (11.1), and therefore there was no covariance amongst and between coordinates. Of course, there is the possibility that the model and/or the estimation procedure may be inadequate. However, with the data used, no conclusive evidence of this could be deduced.

As required for the estimation procedure, the observables (network coordinates) were split into two parts (inner and outer zones). The inner zone consisted of 5 points that were all with 1000 km of the terrestrial network initial point. The remaining observables (Doppler

Test #1

Datum Transformation Parameters

$x_o = -25.1$ m $\sigma_{x_o} = 6.5$ m $\varepsilon_x = -0''.01$ $\sigma_{\varepsilon_x} = 0''.53$

$y_o = 152.2$ m $\sigma_{y_o} = 10.3$ m $\varepsilon_y = 0''.29$ $\sigma_{\varepsilon_y} = 0''.22$

$z_o = 180.6$ m $\sigma_{z_o} = 12.8$ m $\varepsilon_z = -0''.19$ $\sigma_{\varepsilon_z} = 0''.21$

Network Parameters

$dA = -0''.25$ $\sigma_{dA} = 0''.28$

$d\mu = -0''.18$ $\sigma_{d\mu} = 0''.64$

$d\nu = -0''.30$ $\sigma_{d\nu} = 0''.23$ $\kappa = 1.1$ ppm $\sigma_\kappa = -.5$ ppm

Molodensky Network Rotations

$\psi_x = 0''.18$ $\psi_y = -0''.03$ $\psi_z = 0''.39$

$\sigma_o^2 = 1.00$ df = 53 $\hat{\sigma}_o^2 = 0.46$

Test #2

Datum Transformation Parameters

$x_o = -30.6$ m $\sigma_{x_o} = 5.0$ m ε_x: eliminated from the solution

$y_o = 152.0$ m $\sigma_{y_o} = 1.0$ m ε_y:

$z_o = 179.3$ m $\sigma_{z_o} = 0.7$ m $\varepsilon_z = -0''.18$ $\sigma_{\varepsilon_z} = 0''.21$

Network Parameters

$dA = 0''.01$ $\sigma_{dA} = 0''.21$

$d\mu = -0''.21$ $\sigma_{d\mu} = 0''.35$

$d\nu = -0''.34$ $\sigma_{d\nu} = 0''.23$ $\kappa = 1.1$ ppm $\sigma_\kappa = 0.5$ ppm

Molodensky Network Rotations

$\psi_x = 0''.24$ $\psi_y = 0''.18$ $\psi_z = 0''.26$

$\sigma_o^2 = 1.00$ df = 55 $\hat{\sigma}_o^2 = 0.47$

TABLE 12-1

Test #3

Datum Transformation Parameters

x_o = -26.2 m	σ_{x_o} = 0.7 m	ε_x eliminated from the solution
y_o = 151.3 m	σ_{y_o} = 0.7 m	ε_y
z_o = 179.3 m	σ_{z_o} = 0.7 m	ε_z

Network Parameters

dA = -0".12 σ_{dA} = 0".10

dμ eliminated from the solution

dν κ = 1.2 ppm σ_κ = 0.5 ppm

Molodensky Network Rotations

ψ_x = -0".01	ψ_y = -0".09	ψ_z = 0".07
σ_o^2 = 1.00	df = 58	$\hat{\sigma}_0^2$ = 0.50

TABLE 12-1 (cont'd)

TEST RESULTS OF THE KRAKIWSKY-THOMSON MODEL and terrestrial coordinates of 16 points) were in the outer zone. This split was purely arbitrary since, due to a lack of sufficient data, no conclusive testing could be carried out regarding the optimal separation of observables.

In all three sets of test results (Table 12-1), the system and network rotation parameters were insignificant. Attempts to isolate significant rotations, such as longitudinal (ε_z) system rotation (Table 12-1, Test #2) and an azimuthal (dA) network rotation (Table 12-1, Test #3) did not improve the solution. Only the datum translation components (x_o, y_o, z_o) and the network scale difference (κ) proved to be significant.

It should be noted that the network rotations determined are expressed in three ways. The Krakiwsky-Thomson model developed herein is in terms of Veis-type network rotations. As pointed out in 7.3, Molodensky-type network rotations are easily computed from these.

As expected, the variances of the terrestrial network coordinates improved (decreased) as a result of the combination procedure. Changes in the terrestrial coordinates of up to 2.8 in x, 3.3 m in y, and 3.5 m in z took place. All Doppler coordinates changed by 0.2 m or less.

No further analysis was carried out. As indicated earlier, the lack of data was a major problem and hindered adequate testing.

SECTION V

CONCLUSIONS AND RECOMMENDATIONS

13. CONCLUSIONS AND RECOMMENDATIONS

As a result of the research and analyses carried out for this study, the following conclusions have been arrived at:

1. No matter what methodology is used to establish a datum for terrestrial geodetic networks, the quantities and procedures used must be stated clearly and explicitly. Assuming that the practice of using a reference ellipsoid as a horizontal network datum will be continued, it should be remembered that it is a three-dimensional object and must be positioned and oriented as such in the earth body. This means that the conditions for parallelity of axes must be applied at the terrestrial network initial point. Further, the manner by which the initial point geodetic coordinates are obtained must be clearly defined. Only then will the rigorous propagation of errors, into appropriate quantities, be possible.

All of the information mentioned above is essential for a total understanding of how a particular datum has been chosen, positioned, and oriented. This will be particularly crucial, for example, for an in depth analysis of the preliminary results of the redefined Canadian geodetic networks.

2. Precise and homogeneous satellite networks, such as the Canadian Doppler network, contain errors that are of a lower order of magnitude than those that could be expected in a terrestrial network of similar extent. Such networks can provide essential information for:
   (i) the establishment of a terrestrial datum;
   (ii) the modelling of systematic errors in terrestrial networks;
   (iii) the establishment of a homogeneous, three-dimensional, terrestrial geodetic network.
3. To utilize a maximum amount of satellite network data with the least amount of computational effort in the adjustment of terrestrial network, the use of satellite network coordinates as weighted parameters is the most practical. The basic disadvantages of this type of approach are that:
   (i) the datum transformation parameters must be known;
   (ii) unmodelled systematic errors will affect the solution and residual vectors.
4. Three dimensional models for the combination of satellite and terrestrial geodetic networks are preferable. The advantages are:
   (i) no loss of covariance information such as when the data is split into horizontal (ϕ, λ) and vertical (h) components;
   (ii) datum transformation parameters can be solved for rigorously;
   (iii) the overall effects of systematic errors in the terrestrial network can be modelled;
   (iv) given sufficient terrestrial data, the orientation of the satellite and terrestrial datums with respect to Average Terrestrial axes can be determined.

Arguments regarding the insignificance of correlation between (ϕ, λ) and (h) components in networks such as the Canadian Doppler can not be assumed, particularly on a continental basis.

5. The Bursa and Veis (or Molodensky) models are not mathematically equivalent. Under certain conditions, they give equivalent numerical results. None of these three models are considered to adequately model the unknowns in the combination of terrestrial and satellite networks.
6. The Bursa model is adequate for the combination of two or more satellite networks.
7. The Krakiwsky-Thomson and Hotine models reflect more adequately than the other models the real physical situation that exists when combining a satellite and a terrestrial network.
8. The Vanicek-Wells model is the only model that relates the orientation of the axes of a satellite datum and those of several terrestrial datums to the Average Terrestrial coordinate system.
9. Theoretically, a combination of the Krakiwsky-Thomson and Vanicek-Wells models can be used to combine several terrestrial datums and their related networks and a satellite datum and its network. Such a procedure will yield the orientations of each datum with respect to one another and the Average Terrestrial coordinate system, the position of each datum with respect to the other, and a parameterization of the overall effects of systematic scale and orientation errors in each terrestrial network.

This study has led to four <u>recommendations</u> that, in the author's opinion, are important in the Redefinition of the North American geodetic networks.

1. The North American Densification of the Worldwide Satellite Triangulation (BC-4) Network should be investigated, once sufficient terrestrial survey ties become available, with respect to the Canadian and U.S. Doppler Networks. Such an investigation should lead to the eventual use of this data in the redefinition of the North American geodetic networks.
2. The Krakiwsky-Thomson model should be fully tested when adequate data becomes available. The data should consist of:
   (i) the complete definitions of the terrestrial and satellite datums;
   (ii) the coordinates and associated variance-covariance matrix of a homogeneous satellite network, such as that of the completed Canadian Doppler network;
   (iii) the coordinates and associated variance-covariance matrix of a readjusted, homogeneous, three-dimensional terrestrial network with several hundred network points common to those in (ii) above;

   or

   The readjusted horizontal (ϕ, λ) and vertical (H) network coordinates and associated variance-covariance matrices, plus the geoidal heights (N) and associated variance-covariance matrix with several hundred network points common to those in (ii) above.
3. Serious consideration should be given to the three-dimensional

combination of the satellite (Doppler, BC-4) and readjusted terrestrial networks in the redefinition of the North American geodetic networks. The several hundred network points involved would form the basis of future three-dimensional networks.

4. There should be efforts made towards the rigorous establishment of a three-dimensional terrestrial network in North America. This will permit, in the future, an unadulterated use of inherently three-dimensional information, such as that obtained via satellite and inertial positioning.

REFERENCES

Anderle, Richard J., and Mark G. Tanenbaum (1974). Practical Realization of a Reference System for Earth Dynamics by Satellite Methods. NWL Technical Report, No. TR-3151. U.S. Naval Weapons Laboratory, Dahlgren, Virginia.

Anderle, Richard J. (1974(a)). Role of Artificial Satellites in Redefinition of the North American Datum. The Canadian Surveyor, Vol. 28, No. 5.

Anderle, Richard J. (1974(b)). Transformation of Terrestrial Survey Data to Doppler Satellite Datum. Journal of Geophysical Research, Vol. 79, No. 35.

Anderle, R.J. (1974(c)). Simultaneous Adjustment of Terrestrial Geodimeter and Satellite Doppler Observations for Geodetic Datum Definitions. NWL Technical Report TR-3129. U.S. Naval Weapons Laboratory.

Badekas, John (1969). Investigations Related to the Establishment of a World Geodetic System. Reports of the Department of Geodetic Science Report No. 124, The Ohio State University, Columbus.

Baker, Captain Leonard S. (1974). Geodetic Networks in the United States. The Canadian Surveyor, Vol. 28, No. 5.

Beauglass, L.K.(1974). Positions for 1973 Based on Doppler Satellite Observations. NWL Technical Report, No. TR-3181.

Boal, J.D. (1975). Private Communication with the author.

Bomford, G. (3rd. ed. 1971). Geodesy. Oxford University Press, London.

Brown, D.C. (1970) Near Term Prospects for Positional Accuracies of 0.1 to 1.0 Metres from Satellite Geodesy. DBA Systems, Incorporated, Melbourne, Florida.

Bursa, M. (1962). The Theory of the Determination of the Non parallelism of the Minor Axis of the Reference Ellipsoid, Polar Axis of Inertia of the Earth, and Initial Astronomical and Geodetic Meridians from Observations of Artificial Earth Satellites. Translation from Geophysica et Geodetica, No. 6.

Bursa, Milan (1967). On the Possibility of Determining the Rotating Elements of Geodetic Reference System on the Basis of Satellite Observations. Studia Geophysica et Geodaetica, No. 11

Chamberlain, C.C., D.B. Thomson and E.J. Krakiwsky (1976). A Study of Systematic Errors in Horizontal Geodetic Networks. Department of Surveying Engineering Contract Report, Geodetic Survey of Canada Contract No. OSU5-0119, University of New Brunswick, Fredericton.

Chovitz, Bernard (1974). Three-Dimensional Model Based on Hotine's "Mathematical Geodesy". The Canadian Surveyor, Vol. 28, No. 5.

Dracup, J.F. (1975). Use of Doppler Positions to Control Classical Geodetic Networks. Presented to the International Association of Geodesy, International Union of Geodesy and Geophysics, XVI General Assembly, Grenoble, France.

Enin, I.I. (1959). Main Systematic Errors in Precise Levelling. Bulletin Geodesique, No. 52.

Fubara, D.M.J. (1972). Three-Dimensional Adjustment of Terrestrial Geodetic Networks. The Canadian Surveyor, Vol. 26, No. 4.

Heiskanen, Weikko A., and Helmut Moritz (1967). Physical Geodesy. W.H. Freeman and Company, San Francisco.

Holdahl, S. (1974) Time and Heights. The Canadian Surveyor, Vol. 28, No. 5.

Hothem, Larry D. (1975). Evaluation of Precision and Error Sources Associated with Doppler Positioning. Presented at the 1975 Spring Annual Meeting of the American Geophysical Union, Washington, D.C.

Hotine, M. (1969). Mathematical Geodesy. ESSA Monograph No. 2. U.S. Department of Commerce, Washington, D.C.

Hradilek, L. (1972). Refraction in Trigonometric and Three-Dimensional Terrestrial Networks. The Canadian Surveyor, Vol. 26, No. 1.

John, S. (1976). The Determination of the Astrogravimetric Geoid Using Doppler Satellite Geoidal Heights as Constraints. Department of Surveying Engineering, M.Sc. Thesis (in prep.). University of New Brunswick

Jones, Harold E. (1971). Systematic Errors in Tellurometer and Geodimeter measurements. The Canadian Surveyor, Vol. 25, No. 4.

Jones, Harold E. (1973). Geodetic Datums in Canada. The Canadian Surveyor, Vol. 27, No. 3.

Kouba, J. (1970). Generalized Least Squares Expressions and MATLAN Programming. Department of Surveying Engineering, M.Sc. Thesis, University of New Brunswick, Fredericton.

Kouba, J. (1976(a)). Doppler Satellite Levelling. The Canadian Surveyor (to be published).

Kouba, J. (1975). Doppler Satellite Control in Establishing Geodetic Control Network. Contributions of the Earth Physics Branch, No. 34.

Kouba, J. and J.D. Boal (1975). Program GEODOP. Geodetic Survey of Canada, Ottawa.

Kouba, J. (1976(b)). Private communication with the author.

Krakiwsky, E.J. and I.I. Mueller (1966). Proposed Establishment of a First-Order Height System in the U.S.A. Presented at the 47th Annual Meeting of the American Geophysical Union, Washington, April 19-22.

Krakiwsky, E.J. and D.E. Wells (L971). Coordinate Systems in Geodesy. Department of Surveying Engineering, Lecture Notes No. 16, University of New Brunswick, Fredericton.

Krakiwsky, E.J. (1972). Three Dimensional Coordinates in Surveying. An Invited Paper to the Second Colloquium on Surveying and Mapping for the Petroleum Industry, Banff, May 24 to 26.

Krakiwsky, E.J. and D.B. Thomson (1974(a)). Geodetic Position Computations. Department of Surveying Engineering, Lecture Notes No. 39, University of New Brunswick, Fredericton.

Krakiwsky, E.J. and D.B. Thomson (1974(a)). Mathematical Models for the Combination of Terrestrial and Satellite Networks. The Canadian Surveyor, Vol. 28, No. 5.

Krakiwsky, E.J. (1975). A Synthesis of Recent Advances in the Least Squares Method. Department of Surveying Engineering Lecture Notes No. 42, University of New Brunswick, Fredericton.

Ku, L.F. (1970). The Evaluation of the Performance of Water Level Recording Instruments. Symposium on Coastal Geodesy, Munich, July.

Kukkamaki, T.J. (1949). On Lateral Refraction in Triangulation. Bulletin Geodesique, No. 11.

Kukkamaki, T.J. (1961). Lateral Refraction on the Sideward Slope. Annales Academiae Scientiarum Fennicae, Series A, III Geologica - Geographica, 61.

Lambeck, K. (1971). The Relation of Some Geodetic Datums to a Global Geocentric Reference System. Bulletin Geodesique No. 99.

Lennon, G.W. (1974). Mean Sea Level as a Reference for Geodetic Levelling. The Canadian Surveyor, Vol. 28, No. 5.

Lisitzin, E. and J.G. Pattulo (1961). The Principal Factors Influencing the Seasonal Oscillation of Sea-Level. Journal of Geophysical Research, Vol. 66, No. 3.

Marussi, Prof. Antonio (1974). Geodetic Networks in Space. The Canadian Surveyor, Vol. 28, No. 5.

Mather, R.S. (1974). Geodetic Coordinates in Four Dimensions. The Canadian Surveyor, Vol. 28, No. 5.

McLellan, C.D. (1974). Geodetic Networks in Canada. The Canadian Surveyor, Vol. 28, No. 5.

McLellan, D. (1973). Private communication with the author.

Meade, Buford K. (1967). High-Precision Geodimeter Traverse Surveys in the United States. Presented to the XIV General Assembly of IUGG, International Association of Geodesy, Lucern, Switzerland.

Meade, B.K. (1974). Doppler Data Versus Results from High Precision Traverse. The Canadian Surveyor, Vol. 28, No. 5.

Meade, B.K. (1975). Private communication with the author.

Merry, C.L. and P. Vanicek (1973). Horizontal Control and the Geoid in Canada. The Canadian Surveyor, Vol. 27, No. 1.

Merry, Charles L. (1975). Studies Towards an Astrogravimetric Geoid for Canada. Department of Surveying Engineering, Technical Report No. 31, University of New Brunswick, Fredericton.

Molodensky, M., V. Yeremeyev, M. Yurkina (1962). Methods for Study of the External Gravitational Field and Figure of the Earth. Israel Program for Scientific Translations, Jerusalem.

Moritz, Helmut (1973). Stepwise and Sequential Collocation. Reports of the Department of Geodetic Science, Report No. 203, The Ohio State University, Columbus, Ohio.

Mueller, Ivan I. and John D. Rockie (1966). Gravimetric and Celestial Geodesy - A Glossary of Terms. Frederick Unger Publishing Co., New York.

Mueller, Ivan I. (1969). Spherical and Practical Astronomy as Applied to Geodesy. Frederick Unger Publishing Co., New York.

Mueller, I.I., C.R. Schwarz and J.P. Reilly (1970). Analysis of Geodetic Satellite (GEOS I) Observations in North America. Bulletin Geodesique No. 96.

Mueller, Ivan I. (1974(a)). Review of Problems Associated with Conventional Geodetic Datums. The Canadian Surveyor, Vol. 28, No. 5.

Mueller, I.I. (1974(b)). Global Satellite Triangulation and Trilateration Results. Journal of Geophysical Research, Vol. 29, No. 35.

Mueller, I.I. and M. Kumar (1975). The OSU 275 System of Satellite Tracking Station Coordinates. Reports of the Department of Geodetic Science, Report No. 228. The Ohio State University, Columbus, Ohio.

NASA (3rd ed., 1973). NASA Directory of Observation Station Locations, Volumes 1 and 2. Prepared by Computer Sciences Corporation, Falls Church, Virginia.

Nassar, M.M. and P. Vanicek (1975). Levelling and Gravity. Department of Surveying Engineering, Technical Report No. 33, University of New Brunswick, Fredericton.

Paul, M.K. (1973). A Note on Computation of Geodetic Coordinates from Geocentric (Cartesian) Coordinates. Bulletin Geodesique No. 108.

Pope, A. (1975). The North American Densification Satellite Triangulation Network. Personal communication with the author.

Schmid, Hellmut H. (1965). Precision and Accuracy Consideration for the Execution of Geodetic Satellite Triangulation. The Use of Artificial Satellites for Geodesy, Volume II. Publication of the National Technical University, Athens, Greece.

Schmid, Hellmut H. (1970). A World Survey Control System and Its Implications for National Control Networks. Papers of the 1970 Annual Meeting, Canadian Institute of Surveying.

Schmid, Hellmut H. (1972). Status of Data Reduction and Analysis Methods for the Worldwide Geometric Satellite Triangulation Program. Geophysical Monograph 15, American Geophysical Union, Washington.

Schmid, Hellmut H. (1974). World Wide Geometric Satellite Triangulation. Journal of Geophysical Research, Vol. 79, No. 35.

Strange, W.E., L.D. Hothem, M.B. White (1975). The Satellite Doppler Station Network in the United States. Presented to the International Union of Geodesy and Geophysics, International Association of Geodesy, XVI General Assembly, Grenoble, France.

Strange, W. (1975). Private communication with the author.

Thomson, D.B. (1970). The Readjustment of Geodetic Networks in Eastern Canada. M.Sc. Thesis, Department of Surveying Engineering, University of New Brunswick, Fredericton.

Thomson, D.B., M.M. Nassar, and C.L. Merry (1974). Distortions of Canadian Geodetic Networks due to the Neglect of Deflections of the Vertical and Geoidal Heights. The Canadian Surveyor, Vol. 28, No. 5.

Thomson, D.B. and E.J. Krakiwsky (1975). Alternative Solutions to the Combination of Terrestrial and Satellite Geodetic Networks. Contributions of the Earth Physics Branch, No. 45, Ottawa.

Thomson, D.B. and C. Chamberlain (1975). UNBSA - A Rigorous Computation Procedure for Extensive Horizontal Geodetic Networks. Presented at the 68th Annual Meeting of the Canadian Institute of Surveying, Fredericton.

Tscherning, C.C. (1975). Determination of Datum-Shift Parameters Using Least Squares Collocation. Presented at the 6th Symposium on Mathematical Geodesy, Siena, Italy.

Vanicek, Petr (1972). Physical Geodesy II. Department of Surveying Engineering, Lecture Notes No. 24, University of New Brunswick, Fredericton.

Vanicek, P., J.D. Boal, and T.A. Porter (1972). Proposals for a More Modern System of Heights for Canada. Department of Energy, Mines and Resources, Surveys and Mapping Branch, Technical Report No. 72-3.

Vanicek, P. and C.L. Merry (1973). Determination of the Geoid from Deflections of the Vertical Using a Least-Squares Surface Fitting Technique. Bulletin Geodesique, No. 109.

Vanicek, P. and D.E. Wells (1974). Positioning of Geodetic Datums. The Canadian Surveyor, Vol. 28, No. 5.

Vanicek, P. (1975). Report on Geocentric and Geodetic Datums. Department of Surveying Engineering, Technical Report No. 32, University of New Brunswick, Fredericton.

Veis, G. (1960). Geodetic Uses of Artificial Satellites. Smithsonian Contributions to Astrophysics, Volume 3, Number 9. Smithsonian Institution, Washington.

Villasana, J. Alberto (1974). Geodetic Networks in Mexico. The Canadian Surveyor, Vol. 28, No. 5.

Vincenty, T. (1973). Three-Dimensional Adjustment of Geodetic Networks. DMAAC Geodetic Survey Squadron, F.E. Warren AFB, Wyoming.

Webster's New World Dictionary (1951). Nelson, Foster, & Scott Ltd., Toronto.

Wells, D.E. (1974). Doppler Satellite Control. Department of Surveying Engineering, Technical Report No. 29, University of New Brunswick, Fredericton.

Wells, D.E., E.J. Krakiwsky and D.B. Thomson (1974). Internal and External Consistency of Doppler, Satellite Triangulation, and Terrestrial Networks. The Canadian Surveyor, Vol. 28, No. 5.

Wells, D.E. and P. Vanicek (1975). Alignment of Geodetic and Satellite Coordinate Systems to the Average Terrestrial System. Bulletin Geodesique No. 117.

Wolf, Helmut (1963). Geometric Connection and Re-Orientation of Three Dimensional Triangulation Nets. Bulletin Geodesique, No. 68.