ROBUSTNESS ANALYSIS OF GEODETIC NETWORKS

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July 2006

GEODESY AND GEOMATICS ENGINEERING
UNB

TECHNICAL REPORT NO. 242

ROBUSTNESS ANALYSIS OF GEODETIC NETWORKS

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July 2006

PREFACE

This technical report is a reproduction of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Geodesy and Geomatics Engineering, July 2006. The research was supervised by Dr. Petr Vaníček and Dr. Peter Dare.

Berber, Mustafa (2006). *Robustness Analysis of Geodetic Networks*. Ph.D. dissertation, Department of Geodesy and Geomatics Engineering, Technical Report No. 242, University of New Brunswick, Fredericton, New Brunswick, Canada, 121 pp.

ABSTRACT

After geodetic networks (e.g., horizontal control, levelling, GPS etc.) are monumented, relevant measurements are made and point coordinates for the control points are estimated by the method of least squares and the 'goodness' of the network is measured by a precision analysis making use of the covariance matrix of the estimated parameters. When such a network is designed, traditionally this again uses measures derived from the covariance matrix of the estimated parameters. This traditional approach is based upon propagation of random errors.

In addition to this precision analysis, reliability (the detection of outliers/gross errors/blunders among the observations) has been measured using a technique pioneered by the geodesist Baarda. In Baarda's method a statistical test (data-snooping) is used to detect outliers. What happens if one or more observations are burdened with an error? It is clear that these errors will affect the observations and may produce incorrect estimates of the parameters. If the errors are detected by the statistical test then those observations are removed, the network is readjusted, and we obtain the final results.

Here the consequences of what happens when errors are not detected by Baarda's test are considered. This may happen for two reasons: (i) the observation is not sufficiently checked by other independent observations; and, (ii) the test does not recognize the gross error. By how much can these undetected errors influence the network? If the influence of the undetected errors is small the network is called robust, if it is not it is called a weak network.

In the approach described in this dissertation, traditional reliability analysis (Baarda's approach) has been augmented with geometrical strength analysis using strain in a technique called robustness analysis. Robustness analysis is a natural merger of reliability and strain and is defined as the ability to resist deformations induced by the maximum undetectable errors as determined from internal reliability analysis.

To measure robustness of a network, the deformation of individual points of the network is portrayed by strain. The strain technique reflects only the network geometry and accuracy of the observations. However, to be able to calculate displacements caused by the maximum undetectable errors, the initial conditions have to be determined. Furthermore, threshold values are needed to evaluate the networks. These threshold values are going to enable us to assess the robustness of the network. If the displacements of individual points of the network are worse than the threshold values, we must redesign the network by changing the configuration or improving the measurements until we obtain a network of acceptable robustness.

The measure of robustness should be independent of the choice of a datum so that the analysis of a network using a different datum will give the same answer. Robustness should be defined in terms of invariants rather than the primitives (the descriptors for deformation, e.g., dilation, differential rotation and shear) since a datum change will change the strain matrix and therefore the primitive values. Since dilation, differential rotation and total shear are invariants in 2D, whatever the choice of the datum is the results for dilation, differential rotation and total shear will be identical. Moreover this should be the case for 3D robustness analysis.

Robustness of a network is affected by the design of the network and accuracy of the observations. Therefore the points that lack robustness in the network may be remedied either by increasing the quality of observations and/or by increasing the number of observations in the network. A remedial strategy is likely to be different for different networks since they have different geometry and different observations. There might not be a solution fitting all networks but in this thesis a general strategy is given.

In this dissertation first the initial conditions for 2D networks have been formulated then the threshold values for 2D networks have been developed. Application of robustness analysis to 1D networks has been investigated and the limitation of robustness analysis for 1D networks is addressed. The initial condition for 1D networks has also been formulated. Application of robustness analysis to 3D networks has been researched. Moreover, the initial conditions have been formulated. To evaluate 3D networks, the threshold values have been developed. Strain invariants in 3D have been researched. It is proven that dilation and differential rotation are invariants in 3D. It has been discovered that total shear is not invariant in 3D Euclidean space. Therefore the maximum shear strain in eigenspace has been extended into a 3D formulation. The relation between 3D and 2D in terms of invariants has been shown. For the networks which need to be improved, a remedial strategy has been described.

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. Petr Vaníček since he accepted me as a PhD student and gave me the opportunity to do research with him. His guidance and support are invaluable. Here I also would like to express my appreciation to Mrs. Vaníček, rest in peace, for her delicious Thanksgiving and Christmas dinners.

I would like to thank to my co-supervisor, Chair of the department, Dr. Peter Dare for his endless support, encouragement and patience.

I am grateful to Dr. Mike Craymer for his guidance and support without his help this research would not be possible. Whenever I needed him he was always on the other side of the e-mail.

I would like express my gratitude to my supervisory committee members Dr. Anna Szostak-Chrzanowski and Dr. Marcelo Santos for their help during my PhD studies.

I would like to thank to Dr. Marwan Hassan for his time and reference support.

I owe special thanks to Dr. Susan Nichols, Director of Graduate Studies, for keeping me in Canada. Without her help I would not even have been able to start my studies at UNB.

I am deeply thankful to Rick Wilkins for his time and software support.

My thanks also go to Dr. Hakan Kutoglu and Dr. Robert Tenzer.

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List of Symbols

Latin alphabet

A	design matrix
C	multiplication factor
d_i	displacement
d_{ij}	distance in kilometres between points i and j
e	strain
E	strain matrix
I_1, I_2, I_3	invariants
Δl_i	maximum undetectable error
M	maximum shear strain
O	origin of the coordinate system
O_x, O_y, O_z	coordinate axes
P	weight matrix
R	rotation matrix
r_i	redundancy number of the i^{th} observation
r_{ij}	semi-major axis of the 95% confidence region
u	displacement in the x direction
v	displacement in the y direction
w	displacement in the z direction
Δx	estimate for the displacements

X_0, Y_0, Z_0	initial conditions
X_i, Y_i, Zi	coordinates of the points

Greek alphabet

γ	total shear
Γ	total shear in 3D
δ_i	threshold value
$\delta_{r_{ij}}$	relative displacement between points i and j
$\sqrt{\lambda_0}$	the value of the shift of the postulated distribution in the alternative hypothesis
σ	dilation
Σ	dilation in 3D
σ_a	semi-major axis of the ellipse
σ_b	semi-minor axis of the ellipse
σ_{l_i}	a priori value of standard deviation of the i^{th} observation
$\sigma^2_{x_i}$	variance of the X coordinate
$\sigma^2_{y_i}$	variance of the Y coordinate
σ_{xy_i}	covariance of X and Y coordinate
τ	pure shear
υ	simple shear

ω differential rotation

Ω differential rotation in 3D

CHAPTER ONE

INTRODUCTION

This chapter introduces the developments and problems in robustness analysis. In section 1.1 the developments which have been made about robustness analysis so far are briefly mentioned and further developments are addressed. Section 1.2 provides a literature review. The contribution of this research is outlined in section 1.3 and in section 1.4 an outline of this dissertation is given.

1.1 Research Motivation

After geodetic networks (e.g., horizontal control, levelling, GPS etc.) are established, relevant measurements are made and point coordinates are estimated by the method of least squares. However, the method of least squares does not give any information about the robustness of networks. To measure robustness of a network the degree of the deformation of individual points of the network is measured by strain.

In statistical literature robustness means insensitivity to gross errors or outliers in the data. In the approach described in this dissertation, traditional reliability analysis (Baarda's approach) has been augmented with geometrical strength analysis using strain in a technique called robustness analysis. This is outlined in Vaníček et al. [2001] for 2D networks. In the developments which are addressed in this thesis displacements (‘potential deformation’ that could be introduced by the undetected errors in the observations) are quite different than what is described in Vaníček et al. [2001]. So how can these displacements be obtained in 2D networks? In order to be able to do that, first the initial conditions (coordinates which are obtained minimizing the norm of the displacement vectors at all points in the network) need to be determined. Having obtained the initial conditions the displacements can be computed for 2D networks. However some threshold values (the values which displacements are compared against which ideally should not be exceeded) are needed to evaluate the networks. These threshold values are going to enable us to assess the robustness of networks. Furthermore, how can robustness analysis be applied to 1D networks? Can it be applied directly or is there a limitation? Are the applications of robustness analysis to 3D networks the same as the application of robustness analysis to 2D networks?

In Vaníček et al. [2001] a complete and detailed description of the potential network deformation in terms of three independent measures representing robustness in scale, in orientation and in configuration are given (these are also called ‘robustness primitives’ or ‘robustness measures’) for 2D networks. However robustness should be defined in terms of invariants (unaffected by a designated operation. In the case described in this dissertation, unaffected by translation, scale and rotation of a coordinate system) rather than the primitives. Hence we use primitives with the property of invariants. Therefore the primitives which are not invariant will not be employed. Vaníček et al. [2001] shows that the primitives; dilation, differential rotation and total shear are invariants in 2D. It means that no matter what the choice of the datum is the results for dilation, differential rotation and total shear will be identical. Moreover this should be also the case for 3D robustness analysis. So what are the invariants in 3D?

Furthermore can the invariants in 2D be obtained from the 3D solution? How can this transformation be achieved? In geodetic networks some points might lack robustness. If there is lack of robustness according to the robustness analysis of a network, what should be the remedial strategy?

In this thesis, application of robustness analysis to 3D and 1D networks is going to be investigated. Threshold values for 2D and 3D networks are going to be determined. Then, to be able to talk about robustness analysis of geodetic networks in 3D, the research is going to be expanded to determine the invariants in 3D.

1.2 Literature Review

Love [1944] gives a historical background for the mathematical theory of elasticity and investigates the general theory of strain and analysis of strain. Information about analysis of strain may also be obtained in Sokolnikoff [1956] and Timoshenko and Goodier [1970]. These authors are either mathematicians or from the field of engineering mechanics.

Terada and Miyabe [1929] used strain to describe deformation of the earth surface caused by earthquakes. According to Pope [1966], in a series of papers in the Bulletin of the Institute for Earthquake Research of the University of Tokyo, Terada, Miyabe, Tsuboi and others described these techniques and applied them to various areas in Japan and Taiwan. The next scientist interested in strain analysis was Kasahara. In Kasahara [1957], [1958a], [1958b] and [1964], the work of Terada, Miyabe and Tsuboi were referenced and the earlier analysis was extended. Burford [1965] computed the components of strain for an arc of triangulation in Southern California. Independently, Frank [1966] derived methods for computation of strain components and pointed out their advantages and disadvantages. The above authors are geologists or geophysicists [Pope, 1966]. Pope (the first known geodesist to deal with strain analysis) also used this technique for application to repeated geodetic surveys to determine crustal movements.

The use of strain to analyze the strength of a geodetic network was first attempted at University of New Brunswick, this was performed by Thapa [1980]. In this study, the impact of incompatible observations in horizontal geodetic networks was investigated using strain analysis. Vaníček et al. [1981] elaborated on this approach. In Dare and Vaníček [1982] a new method for strain analysis of horizontal geodetic networks based on the measurement of the network deformation was presented. Dare [1983] developed a method for the strength analysis of geodetic networks using strain; also studied was the effect of scale change, twist or shear. In Craymer et al. [1987] a program package called NETAN for the interactive covariance, strain and strength analysis of networks was introduced. Vaníček et al. [1991] combined the reliability technique introduced by Baarda and the geometrical strength analysis method into one technique called ‘robustness analysis’. Vaníček and Ong [1992] investigated the datum independence problem in robustness analysis. In Krakiwsky et al. [1993] further developments of robustness analysis were given, such as singularities in robustness, precision of robustness measures and interpretation of robustness measures.

Szabo et al. [1993] described robustness analysis of horizontal geodetic networks. Craymer et al. [1993a], [1993b] and [1995] presented further findings about robustness analysis. Robustness analysis of horizontal geodetic networks was also studied by Ong [1993] and Amouzgar [1994]. Vaníček et al. [1996] describe a more economical algorithm for searching for the most influential observations in large networks, investigated alternative methods of defining the local neighborhood for which strain measures are computed for each point, and proposed a method of network classification that takes into account both precision (random errors) and accuracy (systematic biases) of point positions. Vaníček et al. [2001] (with amendments in Craymer and Vaníček [2002]) summarized the findings about robustness analysis and gave an explicit proof for the robustness datum independence. In this dissertation, the above references have been used and further developments are addressed.

1.3 Contribution of this Research

- Application of robustness analysis to 2D networks is outlined in Vaníček et al. [2001] and Craymer and Vaníček [2002]. However to be able to calculate displacements caused by maximum undetectable errors, the initial conditions have to be determined. In this dissertation the initial conditions for 2D networks have been formulated.
- In order to be able to assess the goodness of geodetic networks some threshold values are needed. In this thesis the threshold values for 2D networks have been developed.

- Application of robustness analysis to 1D networks has been investigated and the limitation of robustness analysis for 1D networks is addressed. The initial condition for 1D networks has also been formulated.
- Application of robustness analysis to 3D networks has been researched. Moreover, the initial conditions have been formulated.
- To evaluate 3D networks, the threshold values have been developed.
- Strain invariants in 3D have been researched. It is proven that dilation and differential rotation are invariants in 3D. It has been discovered that total shear is not invariant in 3D Euclidean space. Therefore the maximum shear strain in eigenspace has been adopted in 3D formulation.
- The relation between 3D and 2D in terms of invariants has been shown.
- If a network is not robust (i.e., there are some points that lack robustness) at a required level it needs to be improved. In this dissertation a remedial strategy to overcome the lack of robustness has been described.

1.4 Outline of Dissertation

The logical progression of thought is presented as concise as possible. In order not to interrupt the flow of the arguments, all the derivations are provided in the appendices.

Chapter 2 gives a review of robustness analysis and shows the application of robustness analysis to 3D, 2D and 1D geodetic networks.

Chapter 3 explains how displacements at networks points are computed and it addresses how threshold values for 2D and 3D networks are determined.

Chapter 4 presents the numerical examples for 2D and 3D networks. If the networks are not all robust at the required level, they need to be remedied. In Chapter 4 the remedial strategies are described. And it also opens a discussion about how geodetic networks should be assessed.

Chapter 5 analyzes the invariants in 3D and provides the numerical results for 3D invariants. Since it is shown that total shear is not invariant in 3D, maximum shear strain is extended to 3D. It also outlines the relation between 3D and 2D in terms of invariants.

Chapter 6 concludes the thesis and lists the recommendations for further research.

Appendix-I shows how strain matrix is estimated for 1D, 2D and 3D cases.

Appendix-II explains how initial conditions are determined.

Appendix-III outlines the use of principal strains in 3D.

Appendix-IV gives the proofs for rotational invariance in 3D.

Finally a complete reference list and the author’s vita are presented.

CHAPTER TWO

APPLICATION OF ROBUSTNESS ANALYSIS

This chapter outlines the application of robustness analysis to different dimensions. In section 2.1 a review of robustness analysis is given. In section 2.2 application of robustness analysis to 3D networks, in section 2.3 application of robustness analysis to 2D networks and in section 2.4 application of robustness analysis to 1D networks are addressed.

2.1 A Review of Robustness Analysis

After geodetic networks (e.g., horizontal control, levelling, GPS etc.) are established, they are measured and point coordinates are estimated by the method of least squares. What happens if one or more observations are burdened with errors? It is clear that these errors will affect the observations and may produce incorrect estimates of the parameters. Therefore they normally should be detected and corrected. Generally in practice they are removed and the network is readjusted. To detect the errors among the observations Baarda's method of statistical testing (data-snooping) is often used. What happens if these errors are not detected by the test? This may happen for two reasons: (i) the observation is not sufficiently checked by other independent observations; and, (ii) the test does not recognize the gross error. These situations were first investigated by Baarda [1968].

An element of the maximum undetectable error vector $\mathbf{\Delta l}$ among the observations which would not be detected by a statistical test is given by Baarda [1968] as

$$\Delta l_i = \sqrt{\lambda_0}\,\frac{\sigma_{l_i}}{\sqrt{r_i}} \tag{2.1}$$

where $\sqrt{\lambda_0}$ is the value of the shift (non-centrality parameter) of the postulated distribution in the alternative hypothesis. σ_{l_i} is the a priori value of standard deviation of the i^{th} observation and r_i is the redundancy number of the i^{th} observation. Redundancy number is a number between 0 and 1 and it gives the 'controllability' of the observation. If it is 1, it is said that the observation is very well controlled. If it is 0, it means that only the minimum number of observations are connecting the point to the rest of the network. The estimate for the displacements $\mathbf{\Delta x}$ caused by the maximum undetectable errors $\mathbf{\Delta l}$ in the observations is given by

$$\mathbf{\Delta x} = \left(\mathbf{A}^T\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{P}\mathbf{\Delta l} \tag{2.2}$$

where **A** is the design matrix and **P** is the weight matrix. The entire procedure of computing $\mathbf{\Delta x}$ is explained amply both in Vaníček et al. [1991] and [2001]. In this dissertation the correlations among the observations are not considered. In case of correlations the reliability measures must be reformulated. For more information about the correlated cases, interested readers are referred to Chen and Wang [1996] and Schaffrin [1997].

The non-centrality parameter $\sqrt{\lambda_0}$ in eq. (2.1) is function of both the probability of committing a Type-I error (α) and the probability of committing a Type-II error (β). The values to use for α and β are not easy to define. The problems associated with this issue are discussed in Krakiwsky et al. [1993]. Krakiwsky et al. [1993] state that Baarda used 0.1% for α but did not realize that this value corresponds to α/n for in-context testing (for more information about in-context testing interested readers are referred to Krakiwsky et al. [1994]). Furthermore, in order to obtain a reasonable α (e.g., 5%) β had to take on a high value of 20%. But they also articulated that 20% is too large to use in geodetic practice so instead they recommend a more conservative β value of 5%. Therefore throughout this thesis 5% for α and 5% for β are assumed. On the other hand one might think that by simply changing the value of β the displacements can be tailored to meet any specified value. Note that higher confidence (1-α) and power levels (1-β) would lead to a larger non-centrality parameter and thus require better network design and redundancy to achieve the same accuracy. This in turn leads to higher costs that may render the survey uneconomical.

One may be interested in measuring the deformations using strain on a local scale such as engineering applications or regional and global scale such as crustal dynamics analysis. Let us start by saying that within the context of geodetic network coordinate estimation robustness analysis can be applied wherever the Least Squares Method is applied. However one must not get the false impression that when using robustness analysis one is dealing with an actual deformation (physical movement of points) due to external forces such as crustal motions. With robustness analysis one is dealing with potential deformation that could be introduced into the coordinates by undetected errors in the observations. So throughout this dissertation the term ‘displacements’ means ‘potential deformation’ that could be introduced by the undetected errors in the observations and this will be considered as a kind of potential ‘displacements’. On the other hand strain analysis has been utilized in crustal dynamics analysis for many years; interested readers are referred to Pietrantino and Riguzzi [2004] for examples and details.

Nonetheless, the problem with the displacements in eq. (2.2) is that their estimates are datum dependent. This means that these estimates depend not only on the geometry of the network and the accuracy of the observations but also on the selection of constraints for the adjustment; this has nothing to do with the network deformation. Robustness of a network should depend only on the network geometry and accuracy of the observations. Therefore the strain technique is used as it is independent of adjustment constraints and reflects only the network geometry and accuracy of the observations. To measure robustness of a network the degree of deformation at individual points of the network is measured by strain. Traditional reliability analysis (Baarda's approach) has been augmented with the strain technique and termed robustness analysis; this is outlined in the following sections.

2.2 Robustness Analysis of 3D Networks

Vaníček et al. [1991] formulated the expressions for the various deformation primitives in 3D. However the problem was that *geodetic* networks are inherently only 2D in nature since they lie on the surface of the Earth whose variations in height are much smaller than those in the horizontal dimension. In other words the configuration of geodetic networks are usually nearly two-dimensional, i.e., superficial, as the heights may not differ too much from each other. Then the strain in the dimension perpendicular to the surface may become ill-defined.

Coordinates can be curvilinear or cartesian. The problem with the curvilinear coordinates is that when the points have the same height, the height difference between points relative to each other is zero. In this case it is thought that instead of curvilinear coordinates cartesian coordinates can be used. In this thesis two coordinate systems are used: Conventional Terrestrial (CT) system and Local Geodetic (LG) system (for more information about the coordinate systems see Vaníček and Krakiwsky [1986]). Naturally most satellite positioning work is done in CT system and most traditional horizontal positions are given in LG system. Since the magnitude (length) of the displacement vector is independent of the coordinate system, networks can be assessed in any coordinate system.

If 1st, 2nd, 3rd order etc. geodetic networks are taken into account, generally horizontal distances between/among these points are of the order of a few kms whereas the heights between/among these points do not vary much especially in urban areas. In this dissertation these types of networks are analyzed whereas special networks such as high precision networks for engineering projects or open pit mines are not considered.

Let us denote the displacement of a point P_i by

$$\mathbf{\Delta x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{2.3}$$

where u is the displacement in the x direction, v is the displacement in the y direction and w is the displacement in the z direction. Then the strain matrix is [Love, 1944; Sokolnikoff, 1956 and Timoshenko & Goodier, 1970]

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} & \frac{\partial u_i}{\partial z} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} & \frac{\partial v_i}{\partial z} \\ \frac{\partial w_i}{\partial x} & \frac{\partial w_i}{\partial y} & \frac{\partial w_i}{\partial z} \end{bmatrix}. \tag{2.4}$$

This may be applied to geodetic networks as follows:

$$a_i + \frac{\partial u_i}{\partial x}\left(X_j - X_i\right) + \frac{\partial u_i}{\partial y}\left(Y_j - Y_i\right) + \frac{\partial u_i}{\partial z}\left(Z_j - Z_i\right) = u_j \tag{2.5}$$

$$b_i + \frac{\partial v_i}{\partial x}\left(X_j - X_i\right) + \frac{\partial v_i}{\partial y}\left(Y_j - Y_i\right) + \frac{\partial v_i}{\partial z}\left(Z_j - Z_i\right) = v_j \tag{2.6}$$

$$c_i + \frac{\partial w_i}{\partial x}\left(X_j - X_i\right) + \frac{\partial w_i}{\partial y}\left(Y_j - Y_i\right) + \frac{\partial w_i}{\partial z}\left(Z_j - Z_i\right) = w_j \tag{2.7}$$

where all the partial derivatives as well as the absolute terms a_i, b_i, c_i and the coordinates X_i, Y_i and Z_i refer to point P_i and point P_j is connected (by an observation) to the point of interest, point P_i. The estimation of **E** may be seen in Appendix I. The strain matrix can be decomposed into two as follows

$$\mathbf{E} = \frac{1}{2}(\mathbf{E} + \mathbf{E}^T) + \frac{1}{2}(\mathbf{E} - \mathbf{E}^T) \tag{2.8}$$

$$\mathbf{E} = \mathbf{S} + \mathbf{A} \tag{2.9}$$

where the matrix **S** describes symmetrical differential deformation and the matrix **A** describes anti-symmetrical differential deformation at a point as

$$\mathbf{S}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{1}{2}\left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_i}{\partial x} + \frac{\partial u_i}{\partial y}\right) & \frac{\partial v_i}{\partial y} & \frac{1}{2}\left(\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w_i}{\partial x} + \frac{\partial u_i}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w_i}{\partial y} + \frac{\partial v_i}{\partial z}\right) & \frac{\partial w_i}{\partial z} \end{bmatrix} \tag{2.10}$$

and

$$\mathbf{A}_i = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_i}{\partial z} - \frac{\partial w_i}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w_i}{\partial x} - \frac{\partial u_i}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w_i}{\partial y} - \frac{\partial v_i}{\partial z}\right) & 0 \end{bmatrix}. \tag{2.11}$$

Thus robustness primitives in 3D can be calculated for each point in the network accordingly, for example, Sokolnikoff [1956], Ramsay [1967], Vaníček et al. [1991] show that:

Dilation
$$\sigma_i = \frac{1}{3}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\right) \tag{2.12}$$

Pure shear (xy)
$$\tau_{xy_i} = -\tau_{yx_i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y}\right) \tag{2.13}$$

Pure shear (xz)
$$\tau_{xz_i} = -\tau_{zx_i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x} - \frac{\partial w_i}{\partial z}\right) \tag{2.14}$$

Pure shear (yz)
$$\tau_{yz_i} = -\tau_{zy_i} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y} - \frac{\partial w_i}{\partial z}\right) \tag{2.15}$$

Simple shear (xy)
$$\upsilon_{xy_i} = -\upsilon_{yx_i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right) \tag{2.16}$$

Simple shear (xz)
$$\upsilon_{xz_i} = -\upsilon_{zx_i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x}\right) \tag{2.17}$$

Simple shear (yz) $$\upsilon_{yz_i} = -\upsilon_{zy_i} = \frac{1}{2}\left(\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y}\right) \tag{2.18}$$

Differential rotation (xy) $$\omega_{xy_i} = -\omega_{yx_i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}\right) \tag{2.19}$$

Differential rotation (xz) $$\omega_{xz_i} = -\omega_{zx_i} = \frac{1}{2}\left(\frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial x}\right) \tag{2.20}$$

Differential rotation (yz) $$\omega_{yz_i} = -\omega_{zy_i} = \frac{1}{2}\left(\frac{\partial v_i}{\partial z} - \frac{\partial w_i}{\partial y}\right) \tag{2.21}$$

Total shear (xy) $$\gamma_{xy_i} = \sqrt{\tau_{xy_i}^2 + \upsilon_{xy_i}^2} \tag{2.22}$$

Total shear (xz) $$\gamma_{xz_i} = \sqrt{\tau_{xz_i}^2 + \upsilon_{xz_i}^2} \tag{2.23}$$

Total shear (yz) $$\gamma_{yz_i} = \sqrt{\tau_{yz_i}^2 + \upsilon_{yz_i}^2} \tag{2.24}$$

2.3 Robustness Analysis of 2D Networks

Robustness analysis of 2D networks may be thought as a special case of 3D analysis. In Vaníček et al. [2001] a displacement of a point P_i is denoted as

$$\mathbf{\Delta x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{2.25}$$

where u is the displacement in the x direction and v is the displacement in the y direction. Then the tensor gradient with respect to position is

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} \end{bmatrix}. \tag{2.26}$$

The estimation of **E** may be seen in Appendix I. Using eq. (2.8) the strain matrix **E** can be decomposed into its symmetric **S** and anti-symmetric **A** parts; i.e.,

$$\mathbf{E} = \mathbf{S} + \mathbf{A} \tag{2.27}$$

where

$$\mathbf{S}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{1}{2}(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}) \\ \frac{1}{2}(\frac{\partial v_i}{\partial x} + \frac{\partial u_i}{\partial y}) & \frac{\partial v_i}{\partial y} \end{bmatrix},\ \mathbf{A}_i = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}) \\ \frac{1}{2}(\frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}) & 0 \end{bmatrix} \tag{2.28}$$

and the primitives can be formulated as follows, for example, see Frank [1966], Ramsay [1967], Vaníček et al. [2001].

Dilation $$\sigma_i = \frac{1}{2}(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}) \tag{2.29}$$

Pure shear $$\tau_i = \frac{1}{2}(\frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y}) \tag{2.30}$$

Simple shear $$\upsilon_i = \frac{1}{2}(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}) \tag{2.31}$$

Differential rotation $$\omega_i = \frac{1}{2}(\frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}) \tag{2.32}$$

Total shear $$\gamma_i = \sqrt{\tau_i^2 + \upsilon_i^2} \tag{2.33}$$

2.4 Robustness Analysis of 1D Networks

Classical Geodetic height networks are measured utilizing Trigonometric Heighting or Differential Levelling techniques. 'Measurements' are the height differences between/among the points. Thus heights are determined well whereas horizontal coordinates are only approximately known. Therefore in this type of network, one concentrates on the displacements in the vertical direction. In sections 2.2 and 2.3 it was shown how the strain technique is applied to 3D and 2D networks but how can it be applied to 1D networks?

Let us denote the displacement of a point P_i by

$$\mathbf{\Delta x}_i = [\Delta z_i] = [w_i] \tag{2.34}$$

where w is the displacement in the z direction. Then the tensor gradient with respect to position is

$$\mathbf{E}_i = \left[\frac{\partial w_i}{\partial z}\right]. \tag{2.35}$$

The estimation of **E** may be seen in Appendix I. However since there is only one component in the strain matrix not all the primitives (e.g., dilation, pure shear, simple shear, differential rotation and total shear) can be defined. Only expansion may be defined which is dilation.

With eq. (2.35) one must not get the false impression that the application of robustness analysis to the vertical dimension views the problem as purely one dimensional. This formulation does not assume that all points in the network lie on a vertical line located at a single horizontal position. As one may envisage the control points have their horizontal coordinates as well. However as it is indicated above, in 1D networks, heights are determined well whereas horizontal coordinates are only approximately known. Obviously if horizontal coordinates are only approximately known it means that the displacements in 2D cannot be determined properly. That is why in this type of network one only concentrates on the displacements in the vertical direction.

On the other hand in 1D networks there are some issues which need a closer look. Strain is calculated as

$$e = \frac{l' - l}{l} \tag{2.36}$$

where l' is the deformed length and l is the original length of the object. This is the definition in the theory of elasticity, whereas in Geodesy distances (in 1D networks the height differences between the points) can be thought of as beams of rigid length. Then in 1D networks one might run into a problem. When two points have nearly the same height (a common occurrence) the strain with respect to height may become extremely large and mislead the results.

To overcome this problem finite strain could be used since the strain with respect to height might become very large. In eq. (2.36), if the ratio of numerator to denominator becomes large, it is said that the strain is no longer infinitesimal strain; under these circumstances it becomes finite strain. The difference between finite strain and infinitesimal strain is that in finite strain the higher-order differentials of displacements are considered (Love [1944]). Furthermore if the two points have exactly the same height finite strain would not help because as can be seen in eq. (2.37), the result would be infinity since

$$e = \frac{\Delta l}{l} = \frac{\Delta l}{0} = \infty . \tag{2.37}$$

Moreover it was investigated that the height from a surface such as ellipsoid or geoid could be defined but then the physical changes from that surface to the point of interest would not be explained; see Fig. 2.1.

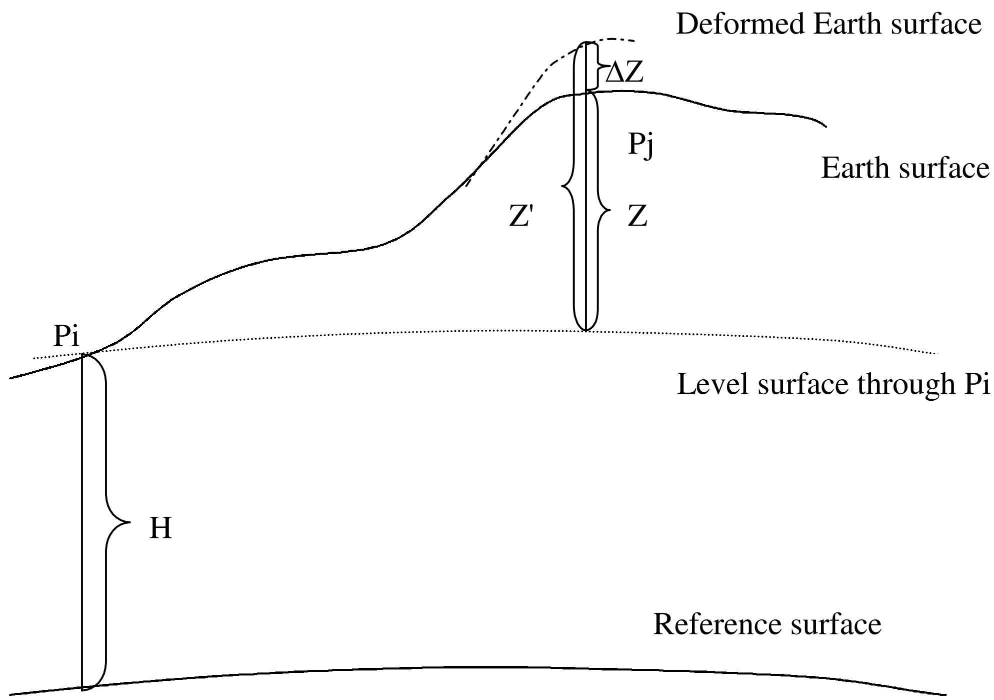

Fig. 2.1 Height from a surface.

If the strain is calculated by taking into account the height from a reference surface, we write

$$e = \frac{H + Z' - (H + Z)}{H + Z} = \frac{\Delta Z}{H + Z} \tag{2.38}$$

where H is the height of the point of interest (Pi) (at which the strain is sought) from a reference surface, Z is the height difference between the point of interest and a connected point (Pj), Z' is the deformed height and ΔZ is the displacement. Nevertheless there is not an immediate answer for application of robustness analysis to 1D networks. Computation of displacements and threshold values are given in Chapter 3.

CHAPTER THREE

THRESHOLD VALUES FOR GEODETIC NETWORKS

This chapter addresses how threshold values are determined for 2D and 3D networks. To accomplish this in section 3.1 computations of displacements are shown. In sections 3.2 and 3.3 determination of threshold values for 2D and 3D geodetic networks are addressed.

3.1 Computation of Displacements

In robustness analysis the deformation caused by the maximum undetectable errors is estimated through strain analysis. In other words by using strain analysis one moves from displacement field to strain field. However, here some threshold values for the robustness primitives are sought. This means that one has to return from strain field to displacement field to determine how much the displacements caused by maximum undetectable errors a network could “handle”. To be able to compute the threshold values for networks firstly initial conditions for networks must be determined. Only then can the threshold values for robustness primitives be calculated.

Application of robustness analysis to geodetic networks has been outlined in Chapter 2. But how can the displacements be calculated? It is postulated that displacements in 3D are computed as follows

$$\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} & \frac{\partial u_i}{\partial z} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} & \frac{\partial v_i}{\partial z} \\ \frac{\partial w_i}{\partial x} & \frac{\partial w_i}{\partial y} & \frac{\partial w_i}{\partial z} \end{bmatrix} \begin{bmatrix} X_i - X_0 \\ Y_i - Y_0 \\ Z_i - Z_0 \end{bmatrix} \tag{3.1}$$

where u is the displacement in the x direction and v is displacement in the y direction and w is displacement in the z direction. In 2D, displacements are calculated as

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} \end{bmatrix} \begin{bmatrix} X_i - X_0 \\ Y_i - Y_0 \end{bmatrix}. \tag{3.2}$$

In eqs. (3.1) and (3.2), the coordinates X_i, Y_i and Z_i refer to point P_i, which is the point of interest and X_0, Y_0 and Z_0 are the initial conditions.

Equations (3.1) and (3.2) are a system of first order differential equations. In order to solve them, they should be integrated; the integration process is explained in Vaníček and Kwimbere [1988]. Here the idea is that to solve the system of differential equations, one ordinarily needs the boundary values. However in our case we have no idea what the boundary values should be. Thus instead the boundary values are replaced by a condition that the norm of all displacement vectors at all points in the network should be a minimum. This is similar to a generalized inverse solution. It turns out that this minimization yields a set of useful ‘initial conditions’, the condition that the displacement of the centroid be equal to 0.

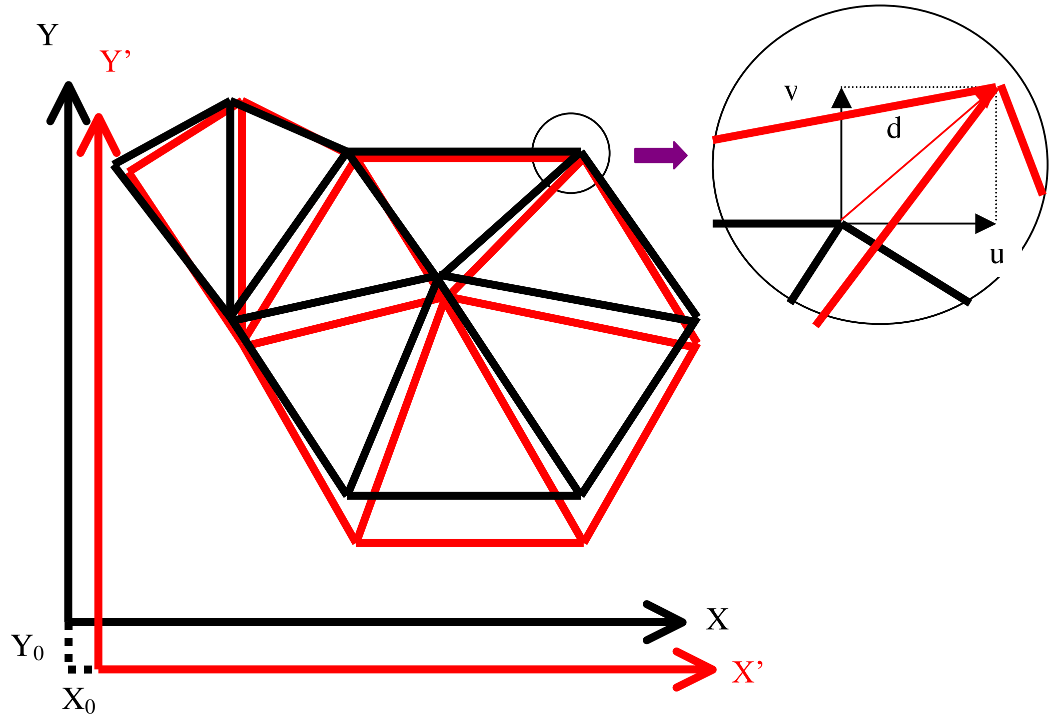

Fig. 3.1 Representation of initial conditions

In Fig. 3.1, the network in black was the network before the deformation. After the deformation, it is deformed and became the network in red. In the circle window one of the points in this network is zoomed in. In this window d is the total displacement at this point. To determine the initial conditions the norm of these displacement vectors at all points in the network is minimized. If the norm of the displacement vectors at all points in the network is minimized, the initial conditions are attained. Essentially these initial conditions tell us where the network was before the deformation. The calculation of initial conditions may be seen in Appendix II. One thing should be emphasized here that without determining the initial conditions this research would not be possible. After computing the displacements for each point in the network, the total displacement at each point in 3D is calculated from

$$d_i = \sqrt{u_i^2 + v_i^2 + w_i^2} \tag{3.3}$$

and accordingly in 2D it can be calculated from

$$d_i = \sqrt{u_i^2 + v_i^2}\ . \tag{3.4}$$

3.2 Threshold Values for 2D Networks

As it is expressed in section 2.1, the displacements computed using eq. (2.2) are datum dependent. This is the reason that strain analysis is introduced to robustness analysis since the strain technique is independent of adjustment constraints and reflects only the network geometry and accuracy of the observations. On the other hand to assess the displacements some threshold values are needed. Vaníček et al. [1991] state that the relative confidence region represents the relative accuracy between the two stations. It is not datum dependent and is most often used to define the accuracy of a network. This is the main reason that the following specifications are used to determine the threshold values.

The specifications given by the Canadian Geodetic Survey Division (GSD) are used to compute threshold values. The GSD specifications are given in GSD [1978].

A survey point of a network is classified according to whether the semi-major axis of the 95% confidence region, with respect to other points of the network, is less than or equal to:

$$r_{ij} = C\ (d_{ij} + 0.2) \tag{3.5}$$

where r_{ij} is in centimetres, d_{ij} is the distance in kilometres between points P_i and P_j and C is a dimensionless factor assigned according to the order of survey as given in Table 3.1.

Table 3.1 The values for factor C for different orders of geodetic networks.

Order	Average Length	Factor C
1st	20 km	2
2nd	15 km	5
3rd	10 km	12
4th	5 km	30

In robustness analysis the effect of the maximum undetectable errors which would not be detected by the statistical test on the network are searched for. Although the GSD specifications are given for random errors it means that by definition some certain amount of error is accepted in the GSD networks. Here a tacit acceptance has been made of certain values as ‘acceptable’, random or systematic. In this thesis these specifications are used as an example but they will likely vary from country to country. As the GSD specifications relate to pairs of points, in this dissertation the following formula is implemented:

$$\delta_{r_{ij}} = \sqrt{(u_j - u_i)^2 + (v_j - v_i)^2} \tag{3.6}$$

where i and j are the points in question and $\delta_{r_{ij}}$ is the relative displacement between points P_i and P_j.

Here the relative displacements are calculated using eq. (3.6) to be able to compare them with the specifications used by GSD. The absolute displacements are also calculated at network points using eq. (3.4) for help in the interpretation and graphical display.

$\forall i, j$; i,j=1,2,…,n, $\delta_{r_{ij}}$ is calculated and compared with r_{ij}. If for $\forall i, j$: $\delta_{r_{ij}} < r_{ij}$ then the network is robust at the required level of probability. If for some i,j: $\delta_{r_{ij}} > r_{ij}$, the network is weak, i.e., some of the points do not meet the required level of robustness (at the required level of probability). Numerical examples are shown in Chapter 4.

While working on 2D networks, having in mind that the configuration of geodetic networks are usually nearly two-dimensional (i.e., superficially, the heights may not differ too much from each other) the application of robustness analysis to 3D networks was a very distant possibility. So the GSD specifications which are given in GSD [1978] are used to compute threshold values in 2D. As time went by it seemed that the application of robustness analysis to 3D networks might be possible. To accomplish this, the threshold values for 3D networks needed to be determined. This is explained in the following section.

3.3 Threshold Values for 3D Networks

The "Accuracy Standards for Positioning" GSD [1996] are used to compute threshold values. The standard ellipse representing the one-sigma network accuracy of the adjusted horizontal coordinates at point P_i, is defined by its major (a) and minor (b) semi-axes. Using the elements of the covariance matrix of the parameters, they are computed as can be seen in eqs. (3.7) and (3.8). Derivations of these equations may be found in Mikhail and Gracie [1981], page 227, equation numbers 8-56 and 8-57. Since the accuracy standards for control points in geodetic networks are sought, eigenvalues of the covariance matrix are calculated as

$$\sigma_{a_i} = \left[\left(\sigma^2_{x_i} + \sigma^2_{y_i}\right)/2 + q_i\right]^{1/2} \tag{3.7}$$

$$\sigma_{b_i} = \left[\left(\sigma_{x_i}^2 + \sigma_{y_i}^2\right)/2 - q_i\right]^{1/2} \tag{3.8}$$

where q_i is an intermediate result calculated as follows

$$q_i = \left[\left(\sigma_{x_i}^2 - \sigma_{y_i}^2\right)^2/4 + \sigma_{xy_i}^2\right]^{1/2} \tag{3.9}$$

and where

$\sigma_{x_i}^2$ is the variance of X coordinate (m^2)

$\sigma_{y_i}^2$ is the variance of Y coordinate (m^2)

σ_{xy_i} is the covariance of X and Y coordinates (m^2)

To obtain semi-axes of the 95% confidence ellipse, we write

$$\sigma_{a_{95_i}} = 2.45\,\sigma_{a_i} \tag{3.10}$$

$$\sigma_{b_{95_i}} = 2.45\,\sigma_{b_i}\,. \tag{3.11}$$

The 95% confidence interval representing the network accuracy of the ellipsoidal height is obtained by multiplying σ_{h_i} (in units of metres), which is extracted from the covariance matrix, by the expansion factor 1.96 for a single variate probability distribution. Therefore in this dissertation the following formula is implemented:

$$\delta_i = \sqrt{\sigma_{a_{95_i}}^2 + \sigma_{b_{95_i}}^2 + \sigma_{h_{95_i}}^2} \tag{3.12}$$

where $\sigma_{a_{95}}$ is semi-major axis of the 95% confidence ellipse, $\sigma_{b_{95}}$ is semi-minor axis of the 95% confidence ellipse, $\sigma_{h_{95}}$ is 95% confidence interval of height component and δ_i is the threshold value which the displacements are compared against. In this equation the horizontal semi-axes must be scaled by (2.795/2.447) and the vertical interval by

(2.795/1.960). So first one needs to replace the 2D expansion factor and the 1D expansion factor with the 3D expansion factor before forming the 3D limit which is an approximation of the 3D confidence ellipsoid.

The confidence ellipsoid could be used to determine the accuracy of adjusted coordinates at network points for GPS networks since one may have the full covariance matrix for GPS observations. However, with the traditional approach of classical (terrestrial) three dimensional networks, horizontal and vertical coordinates are obtained separately so generally a full variance covariance matrix is not available. Therefore in this thesis the general case which has already been suggested by GSD of Canada is implemented.

Here the displacements are calculated using eq. (3.3) to be able to compare them with the threshold values. The threshold values for each point in the network are computed using eq. (3.12). Since the magnitudes (lengths) of displacements are invariant from coordinate systems, comparison can be made in any coordinate system.

d_i is calculated and compared with δ_i. If for $\forall i = 1,2,\ldots,n : Pi :$ $d_i < \delta_i$ it means that the network is robust at the required level of probability. If for some i: $d_i > \delta_i$ the network is weak at Pi, i.e., some of the points do not meet the required level of robustness (at the required level of probability). Numerical examples are shown in Chapter 4.

CHAPTER FOUR

NUMERICAL RESULTS

In this chapter the robustness analysis technique is applied to simulated, terrestrial and GPS networks. In section 4.1 numerical examples for 2D networks and in section 4.2 numerical examples for 3D networks are presented. If there is lack of robustness according to the robustness analysis of a network, it needs to be improved and this is explained in section 4.3. Section 4.4 opens a discussion about the application of robustness analysis to geodetic networks.

4.1 Numerical Examples for 2D Networks

The following examples are based on a datum specified by minimal constraints. If more than minimal constraints are used, one faces a very different problem which is an overconstrained solution. An overconstrained solution shows a deformation different from that of a minimally constrained solution. The additional deformation due to introduced constraints over and above the minimal ones is not considered in this dissertation.

To be able to show the power of the technique three different networks are examined. The first network is the HOACS2D network. It is a synthetic horizontal network, shown in Fig. 4.1. The network consists of 11 points, one of which (point 1) is fixed, 38 directions, 19 distances and 1 azimuth from point 6 to point 5. The distances are assigned a realistic standard deviation of 3 mm + 2 ppm while the directions are assigned a standard deviation of 0.5". The datum orientation is defined by the azimuth with a standard deviation of 1".

In this network all directions and distances are measured. On the other hand, as can be seen from Fig. 4.1, the geometry of the network is not good. The redundancy numbers of the observations at the edge of the network are low compared to the other observations in the network. Hence bigger displacements are obtained at edge points.

From the detailed analysis of the original observations it is found out that since the distances between the points 8-10, 8-11, 9-10 and 10-11 are longer than the other distances in the network, their standard deviations are larger compared to the other distance observations in the network. Hence bigger displacements are obtained at these points.

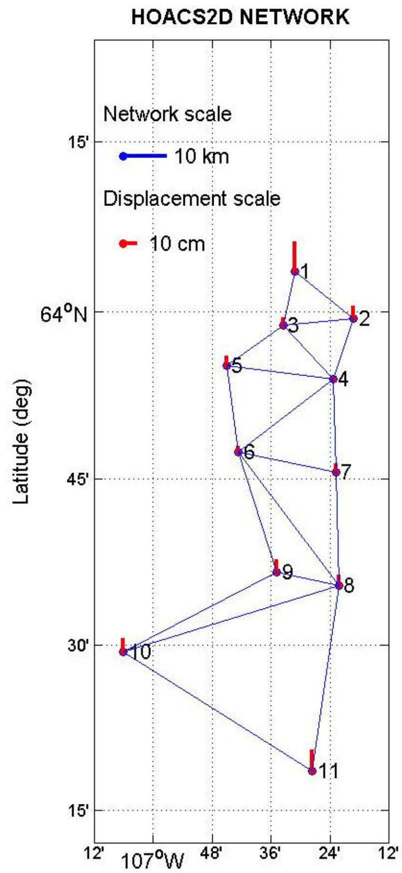

Fig. 4.1 Displacements in HOACS2D network.

The displacements are computed using eq. (3.4) and plotted in Fig. 4.1. Then relative displacements are calculated using eq. (3.6) and compared with the specifications from eq. (3.5); the comparisons are given in Table 4.1. In this network for $\forall ij : 1,2,...,n : (P_i P_j) :\ \delta_{r_{ij}} < r_{ij}$ so it is a totally robust network.

Table 4.1 Relative displacements and threshold values for HOACS2D network (m).

Points	δr_{ij}	r_{ij}
1-2	0.40	1.59
1-3	0.35	2.82
2-3	0.07	1.43
2-4	0.13	1.34
3-4	0.08	1.51
3-5	0.10	1.45
4-5	0.09	0.88
4-6	0.06	0.99
4-7	0.12	0.79
5-6	0.04	1.82
6-7	0.18	0.84
6-8	0.20	0.57
6-9	0.22	0.42
7-8	0.03	0.97
8-9	0.02	1.33
8-10	0.08	0.76
8-11	0.13	0.62
9-10	0.08	0.59
10-11	0.10	0.76

The second network is called Realnet and is shown in Fig. 4.2. It is an example of a real horizontal terrestrial network in southern Quebec consisting of 58 points, one of which (point 1) is fixed, 307 directions, 125 distances and 1 azimuth observation from point 1 to point 3. The range of the standard deviations for the direction observations are 0".6-2".0 for the distance observations 1-34 cm and the standard deviation for the azimuth observation is 1". Note that the robustness of 4 points (#19, 23, 25, 55) is undefined. This is due to a singularity at these points which are linked to the rest of the network by only one observation. Singularity cases have been addressed by Vaníček et al. [2001].

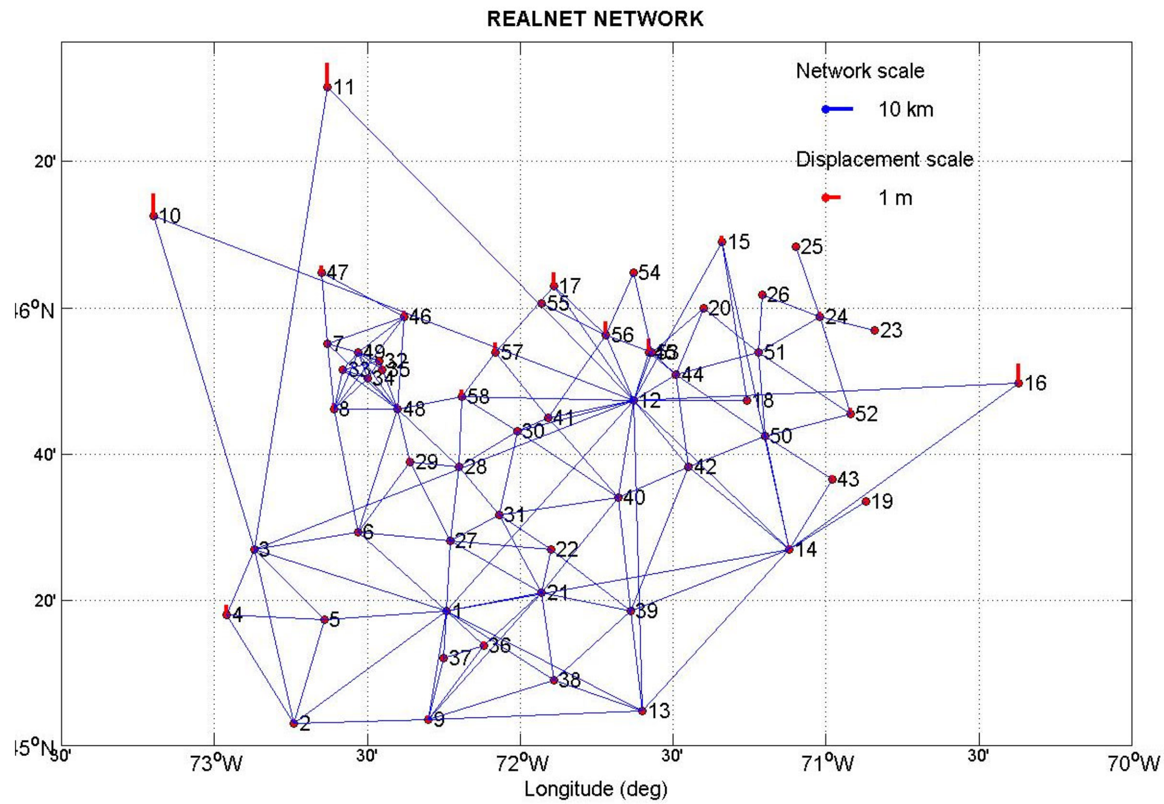

Fig. 4.2 Displacements in Realnet network.

As can be seen from Fig. 4.2, the geometry of the network is not good there are some very low controlled points such as points 10, 11 and 16. Moreover, most of the distances in the network were not measured. Therefore the redundancy numbers of these observations are rather low. Hence very big displacements are obtained at these points. Therefore in this network for some pair of points $\delta_{r_{ij}} > r_{ij}$; these pairs of points are identified in Table 4.2 by an asterisk.

The displacements are computed using eq. (3.4) and plotted in Fig. 4.2 and then relative displacements are calculated using eq. (3.6) to be able to compare them with the specifications; the comparisons are given in Table 4.2.

Table 4.2 Relative displacements and threshold values for Realnet network (m).

Points	δr_{ij}	r_{ij}	Points	δr_{ij}	r_{ij}	Points	δr_{ij}	r_{ij}	Points	δr_{ij}	r_{ij}
1-27	0.20	0.88	8-34	0.47	1.39	*17-57	0.66	0.45	32-34	0.09	1.59
1-12	0.34	1.43	*8-48	1.23	0.81	20-51	0.39	0.93	32-33	0.30	2.93
1-14	0.42	1.79	9-37	0.63	0.81	20-44	0.28	0.91	33-49	0.25	2.11
1-21	0.33	0.50	*9-36	0.64	0.48	20-53	0.22	0.88	33-34	0.29	2.04
1-13	0.11	1.13	9-21	0.44	0.86	21-22	0.19	1.50	33-48	0.56	0.84
1-38	0.23	0.66	9-38	0.47	0.68	21-40	0.36	0.64	34-49	0.30	2.33
1-36	0.33	1.60	9-13	0.19	1.10	21-14	0.51	1.29	34-48	0.77	1.28
1-37	0.34	1.49	*10-12	2.75	2.60	21-39	0.38	0.46	35-49	0.29	2.32
1-9	0.30	0.57	*11-12	2.99	2.20	21-38	0.20	0.44	36-37	0.17	1.30
1-2	0.45	0.98	*12-17	1.38	0.71	21-27	0.20	0.55	38-39	0.28	0.52
1-5	0.31	0.63	*12-56	1.20	0.89	21-31	0.22	0.47	39-40	0.06	0.59
1-3	0.21	1.03	*12-45	1.72	1.56	22-31	0.09	0.81	39-42	0.18	0.81
1-6	0.18	0.60	12-53	0.44	1.61	22-39	0.19	0.52	40-42	0.19	1.00
*2-4	0.76	0.65	12-15	0.47	0.92	22-27	0.14	0.53	*40-41	0.57	0.55
2-3	0.65	0.92	12-44	0.27	1.61	24-26	0.48	0.76	*41-57	1.13	0.42
2-5	0.14	0.57	12-16	1.77	1.97	24-51	0.69	0.92	*41-56	1.02	0.51
2-9	0.19	0.70	12-18	0.21	0.59	*24-52	1.12	0.53	42-44	0.06	0.48
*3-10	2.24	1.78	12-14	0.11	1.11	26-51	0.41	1.79	42-50	0.19	0.41
*3-11	2.54	2.38	12-42	0.27	0.45	27-28	0.25	0.95	43-50	0.40	0.43
3-12	0.54	2.08	12-13	0.23	1.58	27-31	0.07	1.71	44-45	1.60	2.61
3-6	0.33	0.55	12-40	0.40	0.50	27-29	0.37	0.45	44-53	0.38	2.56
*3-5	0.52	0.51	12-28	0.59	0.95	28-58	0.58	0.92	44-51	0.14	0.43
*3-4	1.39	0.91	12-30	0.31	0.61	28-30	0.43	0.89	44-50	0.17	0.55
*4-5	0.88	0.50	12-41	0.35	0.44	28-31	0.31	0.78	46-47	0.21	0.48
*6-8	0.68	0.64	*12-58	1.11	0.87	28-3	0.54	1.13	46-49	0.81	1.68
6-48	0.56	0.67	13-38	0.31	0.48	28-29	0.54	1.51	46-48	0.19	0.46
6-29	0.43	0.45	13-39	1.12	0.51	*28-48	0.53	0.45	48-58	0.31	0.84
6-27	0.07	0.47	13-14	0.31	1.11	29-48	0.28	1.78	48-49	0.81	0.88
7-47	0.92	0.92	14-15	0.39	1.59	*30-58	0.84	0.83	50-15	0.69	0.99
*7-46	0.80	0.42	14-50	0.53	0.61	30-41	0.47	2.62	50-51	0.04	0.42
7-49	0.04	2.59	14-43	0.27	0.42	30-40	0.10	0.62	*50-52	0.55	0.46
7-48	0.79	0.48	*14-16	1.66	1.45	30-31	0.13	0.45	*51-52	0.56	0.56
7-8	0.50	0.84	14-39	0.42	0.88	31-40	0.15	0.63	53-54	0.37	0.41
8-33	0.69	3.04	14-42	0.37	0.67	32-49	0.24	1.85	53-56	1.19	1.50
8-49	0.51	0.82	15-18	0.66	0.80	32-35	0.23	0.57	*54-56	1.23	0.89
*8-46	1.28	0.58	*17-56	2.35	0.92	32-48	0.83	1.49	57-58	1.70	1.74

As can be seen in Fig. 4.2, the identified pairs of points have a big displacement at least at one of the points. The reasons for these displacements are, first, the distances are not measured (discovered from detailed analysis of the original observations) and, second, the points are not well controlled.

However, if the redundancy number of the observations increases the displacements get smaller. For example, although the distances from point 17 to the connected points were not measured, the redundancy number of the observations is higher at point 17 compared to points 10, 11 and 16. This causes the displacement at point 17 to be smaller than the displacements at points 10, 11 and 16 and similarly for points 24, 56 and 57.

The third network is called Northwest Territories Network; it is shown in Fig. 4.3. It is an example of a real GPS network. It consists of 33 points, one of which (point 1) is fixed, and 402 coordinate differences. The range of the baseline component standard deviations are 8-774 mm since it is a rather old GPS network. Note that although GPS networks are intrinsically 3D, only the horizontal (2D) component of the network is analyzed here.

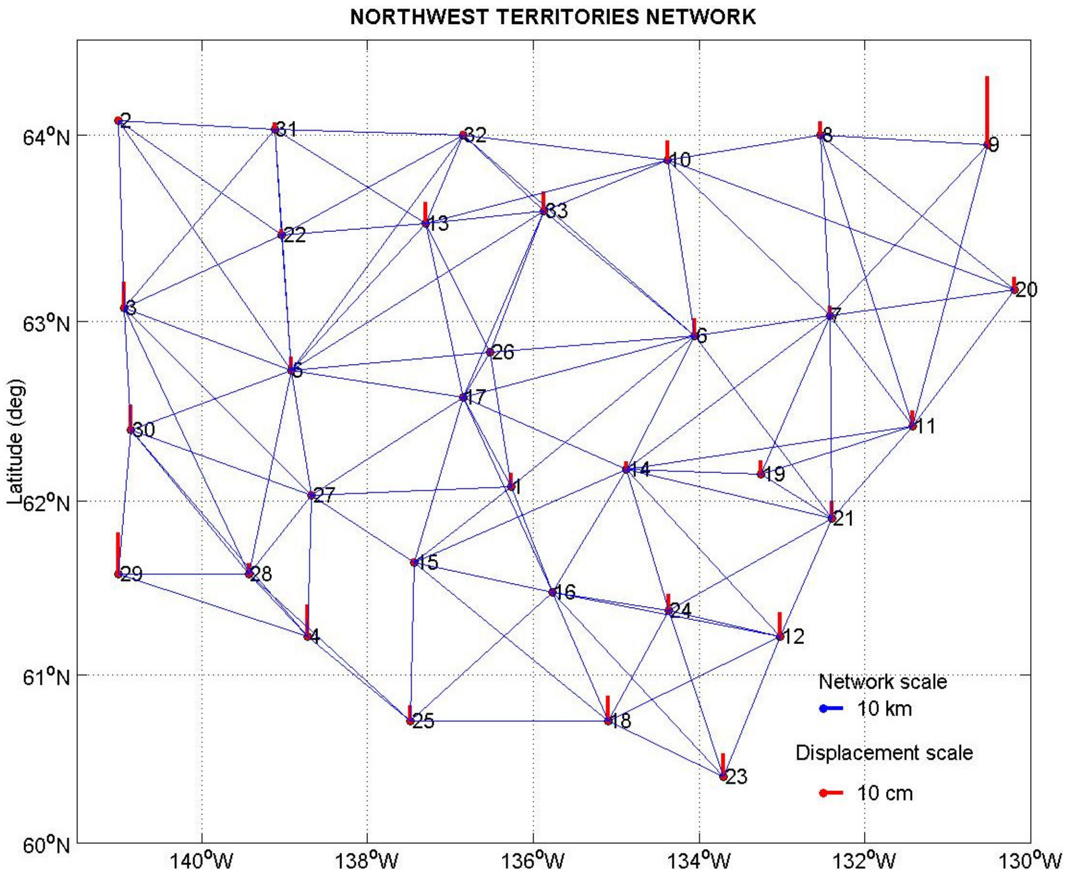

Fig. 4.3 Displacements in Northwest Territories network.

As can be seen from Fig. 4.3 generally the displacements are bigger at the edge of the network since the redundancy number of the observations is rather small compared to the other points in the network. However, as soon as the redundancy number of the observations increases the displacements get smaller. For example, at point 9 there are 3 connections whereas at point 20 there are 4 connections and the displacement is smaller at point 20 than at point 9. Nevertheless, points 8, 10, 13 and 33 have some observations which have large standard deviations (see Table 4.3). Therefore the displacements at these points are larger compared to the other points in the network. However, at point 2, standard deviations of the observations are smaller compared to the maximum standard deviations at points 8, 10, 13 and 33. Similar situations also occur at points 31 and 32.

Table 4.3 Standard deviations of some of the observations in NWT network (m).

Points	$\sigma_{\Delta X}$	$\sigma_{\Delta Y}$	Points	$\sigma_{\Delta X}$	$\sigma_{\Delta Y}$
2-5	0.083	0.063	13-10	0.111	0.078
2-3	0.060	0.042	13-5	0.121	0.067
2-31	0.051	0.018	33-32	0.147	0.100
2-22	0.034	0.027	33-13	0.095	0.046
8-20	0.028	0.018	33-6	0.125	0.091
8-11	0.152	0.072	33-10	0.036	0.041
8-10	0.068	0.064	33-26	0.072	0.069
8-9	0.083	0.054	33-17	0.060	0.077
8-7	0.024	0.012	33-13	0.093	0.053
13-31	0.038	0.025	10-32	0.110	0.097
13-26	0.079	0.073	10-33	0.036	0.041
13-33	0.093	0.053	10-6	0.088	0.103
13-32	0.073	0.053	10-7	0.123	0.099
13-22	0.050	0.044	10-8	0.068	0.064
13-17	0.054	0.067	10-20	0.104	0.094

In Table 4.3 $\sigma_{\Delta X}$ is the standard deviation for the coordinate difference in the x direction and $\sigma_{\Delta Y}$ is the standard deviation for the coordinate difference in the y direction. The displacements are calculated using eq. (3.4) and plotted in Fig. 4.3 and then relative displacements are calculated using eq. (3.6) to able to compare them with the specifications; the comparisons are given in Table 4.4. In this network for $\forall ij : 1,2,...,n : (P_i P_j) :\ \delta_{r_{ij}} < r_{ij}$ so it is a totally robust network.

Table 4.4 Relative displacements and threshold values for NWT Network (m).

Points	δr_{ij}	r_{ij}	Points	δr_{ij}	r_{ij}	Points	δr_{ij}	r_{ij}	Points	δr_{ij}	r_{ij}
1-15	0.05	1.55	5-32	0.07	3.51	10-20	0.11	4.46	15-25	0.10	2.06
1-16	0.06	1.44	5-27	0.06	1.58	10-13	0.13	2.97	15-27	0.02	1.56
1-17	0.07	1.28	5-28	0.05	2.61	11-14	0.08	3.62	16-23	0.12	3.30
1-26	0.08	1.68	5-30	0.07	2.13	11-19	0.14	1.99	16-24	0.09	1.54
1-27	0.07	2.52	6-14	0.05	1.85	11-21	0.13	1.56	16-17	0.01	2.71
1-6	0.09	2.95	6-7	0.04	1.68	11-20	0.05	2.10	16-18	0.13	1.84
2-22	0.03	2.37	6-10	0.08	2.14	12-23	0.03	1.94	16-25	0.09	2.49
2-3	0.12	2.28	6-21	0.05	2.86	12-24	0.04	1.48	17-26	0.01	0.63
2-31	0.03	1.84	6-17	0.09	2.96	12-18	0.13	2.49	17-27	0.01	2.27
2-5	0.06	3.67	6-26	0.09	2.53	12-16	0.14	3.01	17-33	0.10	2.48
3-31	0.14	2.84	6-33	0.02	2.39	12-14	0.12	2.93	18-23	0.15	1.70
3-5	0.07	2.20	6-32	0.08	3.69	12-21	0.09	1.67	18-24	0.10	1.62
3-22	0.14	2.13	7-11	0.06	1.72	13-26	0.12	1.76	18-25	0.07	2.61
3-27	0.13	3.28	7-14	0.03	3.16	13-33	0.02	1.43	19-21	0.10	1.06
3-28	0.09	3.64	7-19	0.10	2.14	13-31	0.13	2.12	21-24	0.08	2.39
3-30	0.04	1.48	7-20	0.02	2.27	13-32	0.11	1.13	22-31	0.01	1.27
4-25	0.24	1.71	7-8	0.05	2.14	13-17	0.11	2.16	22-32	0.05	2.46
4-27	0.16	1.84	7-9	0.34	2.79	13-22	0.13	1.74	23-24	0.05	2.26
4-28	0.11	1.15	7-21	0.07	2.55	14-15	0.05	2.94	25-28	0.14	2.84
4-29	0.06	2.59	7-10	0.09	2.70	14-16	0.04	1.83	26-33	0.10	1.86
4-30	0.05	3.49	8-11	0.01	3.69	14-17	0.04	2.25	27-28	0.06	1.28
5-13	0.05	2.43	8-20	0.05	2.97	14-19	0.08	1.69	27-30	0.13	2.42
5-17	0.07	2.14	8-10	0.15	1.85	14-21	0.05	2.66	28-29	0.17	1.65
5-26	0.07	2.45	8-9	0.29	1.97	14-24	0.08	1.91	28-30	0.07	2.35
5-33	0.04	3.64	9-11	0.29	3.56	15-17	0.01	2.17	29-30	0.11	1.81
5-22	0.08	1.67	10-32	0.08	2.44	15-16	0.01	1.80	31-32	0.06	2.22
5-31	0.08	2.93	10-33	0.11	1.58	15-18	0.14	3.25	32-33	0.09	1.30

In Table 4.4, r_{ij} values have been obtained using eq. (3.5). Clearly in this equation threshold values depend on the distance and order of the network. Since Northwest Territories network is a very large network, accordingly r_{ij} values are large. It seems that threshold values are much larger compared to relative displacements and one might think about the economics and question the quality of the measurements. However, note that this is a first order network and with threshold values we are talking about threshold values for any geodetic network. In this case threshold values for 2D networks.

4.2 Numerical Examples for 3D Networks

Two GPS networks have been examined. The first network called Simple GPS network is shown in Fig. 4.4. It is a small real GPS network consisting of 7 points, one of which (point 1) is fixed, and 42 coordinate differences. The range of the baseline component standard deviations are 0.7-3.1 mm.

As can be seen from Fig. 4.4 some of the displacements are bigger at the edge of the network. However, as soon as the redundancy number of the observations increases the displacements get smaller. For example, at point 5 there are 5 connections whereas at point 1 there are 3 connections and the displacement is smaller at point 5 than at point 1. Standard deviations in this network are quite small therefore the displacements are quite small.

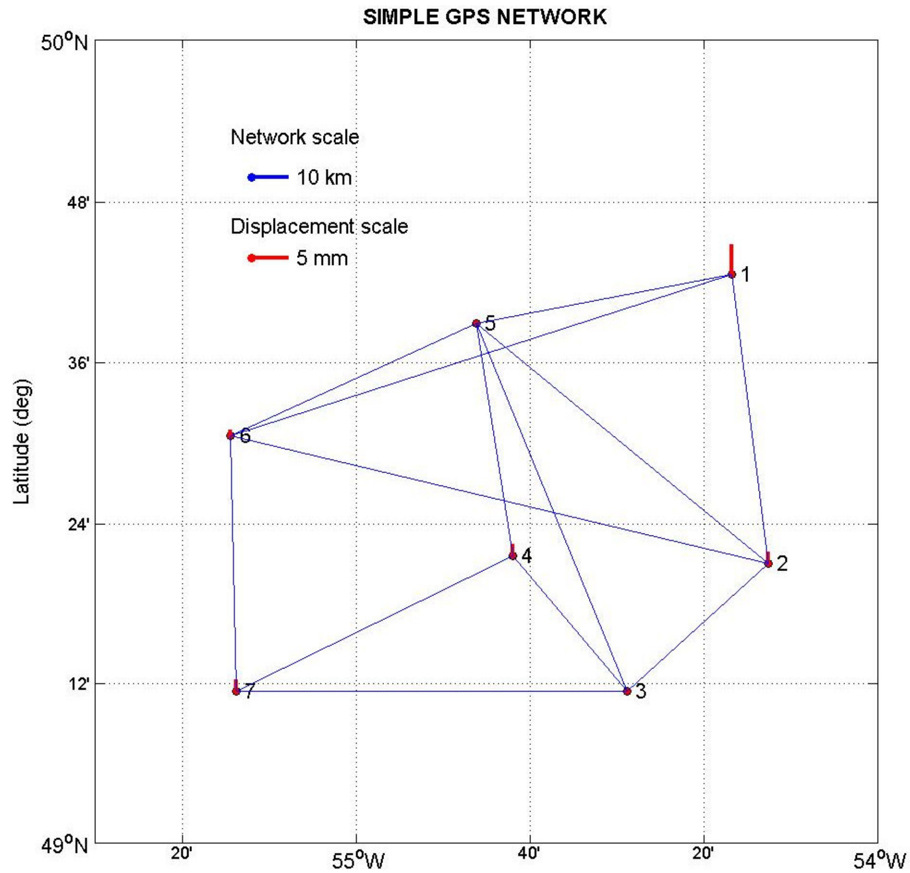

Fig. 4.4 Displacements in Simple GPS network.

The displacements are calculated using eq. (3.3) and plotted in Fig. 4.4 and the threshold values are computed using eq. (3.12) and then the displacements are compared against the threshold values. The comparisons are given in Table 4.5. Since for $\forall i = 1,2,...,n : Pi :$ in the network the displacements are less than the threshold values this is a totally robust network.

Table 4.5 Displacements and threshold values for Simple GPS network (m).

Points	d_i	δ_i
2	0.002	0.005
3	0.000	0.005
4	0.002	0.006
5	0.000	0.005
6	0.001	0.005
7	0.002	0.007

The second network is the same Northwest Territories Network used in section 4.1; it is shown in Fig. 4.5. It consists of 33 points, one of which (point 1) is fixed, and 402 coordinate differences. The range of the baseline component standard deviations are 8-774 mm; these values look quite large since it is a rather old GPS network.

As can be seen from Fig. 4.5 generally the displacements are bigger at the edge of the network since the redundancy number of the observations is rather small compared to the other points in the network. However, as soon as the redundancy number of the observations increases the displacements get smaller. For example, at point 9 there are 3 connections whereas at point 20 there are 4 connections and the displacement is smaller at point 20 than at point 9.

It is seen from detailed analysis of the original observations that points 5 and 22 have some observations which have large standard deviations. These less precise observations affect the points in their vicinity; therefore we get a large displacement at point 22. And also points 12, 21 and 24 have some observations which have large standard deviations. Consequently, these observations affect that area of the network and we get some significant displacements at these points. At point 29 not only is the redundancy number of the observations low but also the observations have large standard deviations therefore we obtain the largest displacement of the whole network at this point. Most of the other points are in the middle of the network therefore their redundancy numbers are quite high compared to the other points at the edge of the network, thus we do not get large displacements at these points.

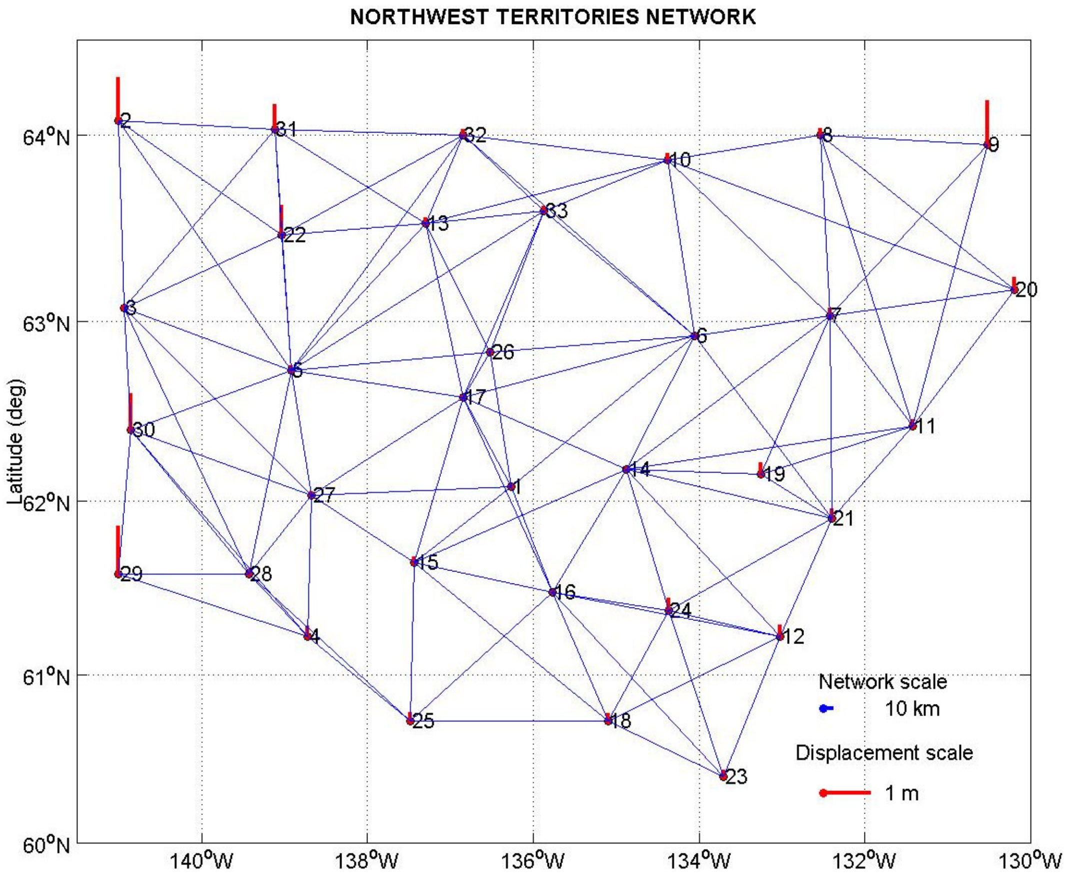

Fig. 4.5 Displacements in Northwest Territories network.

The displacements are computed using eq. (3.3) and plotted in Fig. 4.5 and the threshold values are calculated using eq. (3.12) and then the displacements are compared with the threshold values the comparisons are given in Table 4.6. For some points in this network displacements are larger than the threshold values so it is not a robust network. Therefore the network is not robust at all the points at the required level and so it needs to be improved. These points are identified in Table 4.6 by an asterisk.

Table 4.6 Displacements and threshold values for NWT network (m).

Points	d_i	δ_i	Points	d_i	δ_i
*2	0.91	0.14	*18	0.16	0.13
3	0.07	0.14	*19	0.25	0.19
*4	0.23	0.14	*20	0.28	0.15
5	0.09	0.12	*21	0.20	0.13
6	0.05	0.15	*22	0.63	0.12
*7	0.17	0.15	23	0.13	0.23
8	0.14	0.15	*24	0.28	0.23
*9	0.93	0.25	*25	0.18	0.15
10	0.14	0.14	26	0.07	0.08
11	0.14	0.25	27	0.08	0.14
*12	0.25	0.17	28	0.11	0.13
13	0.12	0.12	*29	1.02	0.27
14	0.02	0.10	*30	0.76	0.13
*15	0.12	0.11	*31	0.54	0.12
16	0.03	0.08	32	0.13	0.15
17	0.01	0.06	33	0.09	0.13

In Table 4.6, δ_i values have been obtained using eq. (3.12). In this equation threshold values depend on the semi-axes of the 95% confidence ellipse and the 95% confidence interval of height component which are obtained using the variances and covariances of the coordinates. Therefore δ_i values are quite close to the d_i values. In this 3D case it seems that threshold values are more realistic. Therefore some of the points do not meet the robustness requirements. One thing that should be emphasized here is that the threshold values we are talking about are threshold values for any geodetic network.

This is the same network which is used in section 4.1 for 2D analysis. Nonetheless the results seem very different. The results are different because in the 2D analysis the relative displacements are calculated and compared with the threshold values which are developed for 2D. For 3D analysis the absolute displacements are calculated and compared with the threshold values which are developed for 3D. As it is pointed out above, the formulation of threshold values in 2D and 3D are quite different. Due to the theory of robustness analysis (in 2D and 3D cases; design matrix **A**, weight matrix **P**, redundancy numbers and therefore the maximum undetectable errors are different) the results for the strain matrix in 2D and 3D are different. Accordingly, the invariants in 2D and 3D are different, therefore the relation between 2D and 3D in terms of invariants needs to be investigated; this is outlined in section 5.7.

In sections 4.1 and 4.2 geodetic networks have been analyzed in 2D and 3D. It turns out that some of the networks are not all robust at the required level and they need to be remedied. The remedial strategies are described in the following section.

4.3 Remedial Strategy

Robustness of a network is affected by the design of the network and accuracy of the observations. Therefore the points that lack robustness in the network may be remedied either by increasing the quality of observations and/or by increasing the number of observations (controllability) in the network.

A remedial strategy is likely to be different for different networks since they have different geometry and different observations. There might not be a solution fitting all networks but here a general strategy is given.

Increasing the number of observations increases the redundancy of the network, therefore we get lower primitive values. Adding an observation should never deteriorate the solution because the current network will already be there. In fact by adding an observation the current network is only improved since the redundancy in the network is improved.

If we add an observation and its quality is not good enough, this observation might not contribute to the solution at all. Let us say we have an observation with an infinite standard deviation then its weight will be zero therefore it will not affect the solution.

Increasing the quality of the observations or increasing the number of observations in a network generally affects the points the observations are connected to. Increasing the quality of direction observations at network points affects the points around these points since the decreased standard deviation affects the surrounding points.

If there is lack of robustness in terms of scale (scale is represented by dilation) at a network point distance observations should be added to the network. If there is lack of robustness in terms of differential rotation at a network point adding direction observations or some distance observations should improve the robustness. Adding these observations is going to tighten the network for that area. In addition because GPS observations provide both scale and direction information and are cheaper to perform over long distances they may be incorporated as well.

Some things still need to be investigated, for example if we get a big value of, let us say, pure shear. How can this problem be cured? What is the source of this problem?

4.4 Discussion

In robustness analysis, the deformation caused by the maximum undetectable errors is measured through strain analysis. In other words by using strain analysis one moves from displacement field to strain field. First the threshold values are determined in the strain field using the relations among the primitives. Preliminary thoughts are addressed in Berber et al. [2003]. However at that time we could not quite succeed in this venture. Therefore we needed to return from strain field to displacement field to define the threshold values. To be able to compute the threshold values for networks, firstly initial conditions must be determined. Only after that the threshold values can be determined for robustness primitives. That is why the specifications from GSD are utilized to determine threshold values. By definition these specifications give the amount of error which is accepted in the GSD networks.

After expanding robustness analysis to 3D, another problem which is needed to overcome was the determination of the invariants in 3D. In the examples in sections 4.1 and 4.2 one datum is selected and the results are produced. However, the robustness measure should not depend on the choice of a datum. Robustness should be defined in terms of invariants rather than the primitives since a datum change will change the strain matrix and therefore the primitive values.

Vaníček et al. [2001] shows that dilation, differential rotation and total shear are invariants in 2D. This means that no matter what the choice of the datum is the results for dilation, differential rotation and total shear will be the same for each solution. This should also be the case for 3D robustness analysis. So the search for invariants in 3D is crucial. These issues are addressed in Chapter 5.

CHAPTER FIVE

INVESTIGATIONS ON INVARIANTS

In this chapter invariants in 3D are investigated. First, in section 5.1 strain invariants are introduced. In section 5.2 the investigations which have been made on invariants in 3D are addressed. In section 5.3 the proofs about invariants in 3D are given. In section 5.4 the proofs are supported by numerical results. Further developments about the shear invariant are addressed in section 5.5. In section 5.6 numerical results for principal strains in 3D are given. In section 5.7 the relation between 3D and 2D in terms of invariants is shown.

5.1 Strain Invariants

The three principal axes of a strain ellipsoid which are the eigenvalues of the following equation (see Appendix III for their derivation) are obtained from

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0 \tag{5.1}$$

where

$$
\begin{aligned}
I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\
I_2 &= \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{zz} - \sigma_{yy}\sigma_{zz} \qquad (5.2) \\
I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2
\end{aligned}
$$

where $\sigma_{xx}, \sigma_{xy}, ..., \sigma_{zz}$ are the components of the symmetrical part of the strain matrix. I_1, I_2

and I_3 are invariants of strain and must be the same for all choices of coordinate axes x,y,z. In fact invariant theory originates from linear algebra as outlined in linear algebra text books, for example, Paige and Swift [1961], Shields [1964], Kaplansky [1969] and Kaye and Wilson [1998].

Derivation of invariants is based on the assumption that there exist three mutually perpendicular planes at a point O (origin) on which the shear strain vanish. The remaining normal strain components on these three planes are called principal strains. Correspondingly, the three mutually perpendicular axes that are normal to the three planes are called principal axes. Under a deformation of a body any infinitesimal sphere in the body is deformed into an infinitesimal ellipsoid called the strain ellipsoid. The principal axes of the strain ellipsoid have the directions of the principal axes of strain. So using only the principal values, the invariants are derived from eq. (5.1) [Boresi et al. 1993]. Finding the three roots of eq. (5.1), the three principal axes of the strain ellipsoid (which are the eigenvalues of this equation) can be defined.

5.2 Investigations on Invariants in 3D

The aim of the following sections is to show the useful strain invariants in 3D, i.e., those that are useful for geodetic analysis. Since linear combinations of invariants are also invariant, one can obtain an infinite number of invariants. However since we use primitives in robustness analysis here we concentrate on dilation, differential rotation and shear invariants. The invariants are summarized for 2D in Table 5.1. On top of these “useful” invariants, there are some others listed, for instance, by Love [1944].

Table 5.1 Invariants in 2D.

Robustness primitives	2D
Dilation	invariant
Pure shear	not invariant
Simple shear	not invariant
Differential rotation	invariant
Total shear	invariant
Maximum shear strain	invariant

5.2.1 Dilation

The ‘dilation invariant’ is the only linear invariant. It is given by (see Appendix IV for more information)

$$\forall n = 1, 2, 3: \quad {}^{n}\sigma = \frac{1}{n}\sum_{i=1}^{n}\sigma_i \tag{5.3}$$

where n is the dimensionality of the problem. The relation between the two and three dimensional cases is

$$\Sigma = \frac{1}{3}\left(2\sigma + \sigma_3\right) \tag{5.4}$$

where the two dimensional dilation invariant is denoted by ${}^{2}\sigma$ or simply by σ. σ_3 represents the dilation in third dimension. Σ denotes the dilation invariant in 3D.

Dilation invariant is equal to the first invariant which is given in eq. (5.2). In other words $\Sigma = I_1$ and it is an invariant in 3D. If an invariant is multiplied by a value, the result is again an invariant. Thus the fraction in eq. (5.4) should not confuse the readers. Conventions about strain tensor notation are addressed in Sokolnikoff [1956].

5.2.2 Differential Rotation

The ‘differential rotation magnitude invariant’ is a quadratic invariant. It is postulated to be given by the following expression in 3D

$$\Omega = \sqrt{\omega_{xy}^2 + \omega_{xz}^2 + \omega_{yz}^2}\ . \tag{5.5}$$

While in 2D the invariant ω^2 is simply a scalar quantity [Vaníček et al. 2001], this 3D invariant Ω can be interpreted as the square of the length of the ‘differential rotation vector’, where the components are equal to the 2D scalar invariants ω_{xy}, ω_{xz} and ω_{yz} in the coordinate planes xy, xz and yz respectively. It is known from the investigation of the 2D invariants that these invariants are nothing else but magnitudes of differential rotation vectors ω_i i=1,2,3 that are perpendicular to the coordinate planes xy, xz and yz respectively. In other words $\omega_1 = \omega_{xy}$, $\omega_2 = \omega_{xz}$ and $\omega_3 = \omega_{yz}$.

5.2.3 Shear

The total shear invariant in 2D is defined as follows

$$\gamma_{xy} = \sqrt{\tau_{xy}^2 + \upsilon_{xy}^2} \tag{5.6}$$

Total shear in 3D is postulated to be given by

$$\Gamma = \sqrt{\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2} \tag{5.7}$$

where the formulae for γ_{xy}, γ_{xz} and γ_{yz} are given in eqs. (2.22), (2.23) and (2.24) respectively. Since pure shear (τ) and simple shear (ν) are normally thought of as scalar quantities, it is not immediately obvious how they should be interpreted geometrically. Neither is it obvious how the 2D scalar invariants are to be shown geometrically. But the fact is that they are invariant under a rotation of the 2D coordinate system in the corresponding manifold [Vaníček et al. 2001].

5.3 Proof of Invariance in 3D

In Vaníček et al. [2001] the invariants in 2D are investigated. But what happens if we expand our research from 2D to 3D? In 3D, as it was the case in 2D, since translations do not affect strain, the solutions based on datums with different origins will be the same. In 3D, the solutions for scale change should be identical to each other since scale change has only a second order effect. However the solution for orientation change will be different. Consider two analyses of a network, the first using coordinate system (x, y, z) where

$$\Delta\mathbf{x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{5.8}$$

and the strain matrix is as follows

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} & \frac{\partial u_i}{\partial z} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} & \frac{\partial v_i}{\partial z} \\ \frac{\partial w_i}{\partial x} & \frac{\partial w_i}{\partial y} & \frac{\partial w_i}{\partial z} \end{bmatrix}. \tag{5.9}$$

The second analysis uses coordinate system (x^*, y^*, z^*) which is rotated clockwise about Ox (along x axis) by an angle α and then subsequently rotated clockwise about Oy (along y axis) by an angle β, next after the α and β rotations the axes are rotated clockwise about Oz (along z axis) by an angle γ. Then

$$\Delta\mathbf{x}_i^* = \begin{bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{bmatrix} = \mathbf{R} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{5.10}$$

where **R** is the rotation matrix and the strain matrix is

$$\mathbf{E}_i^* = \begin{bmatrix} \frac{\partial u_i^*}{\partial x^*} & \frac{\partial u_i^*}{\partial y^*} & \frac{\partial u_i^*}{\partial z^*} \\ \frac{\partial v_i^*}{\partial x^*} & \frac{\partial v_i^*}{\partial y^*} & \frac{\partial v_i^*}{\partial z^*} \\ \frac{\partial w_i^*}{\partial x^*} & \frac{\partial w_i^*}{\partial y^*} & \frac{\partial w_i^*}{\partial z^*} \end{bmatrix}. \tag{5.11}$$

From now on, for simplicity, subscript i is going to be omitted. The proof for dilation invariant may be shown as

$$\Sigma^* = \frac{1}{3}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} \right) \tag{5.12}$$

where the equivalents of $\frac{\partial u^*}{\partial x^*}$, $\frac{\partial v^*}{\partial y^*}$ and $\frac{\partial w^*}{\partial z^*}$ are given in Appendix IV. After replacing the partials in eq. (5.12) and performing the necessary algebraic manipulations we see that

$$\Sigma^* = \frac{1}{3}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right) = \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \Sigma . \tag{5.13}$$

It means that Σ which is equal to I_1 is invariant under a rotation of a coordinate system.

Under an orientation change if the partials are expressed using tensor analysis summation convention we write (see Appendix IV)

$$\frac{\partial u^{*r}}{\partial x^{*s}} = \frac{\partial u^{*r}}{\partial u^{\alpha}} \frac{\partial u^{\alpha}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x^{*s}} \tag{5.14}$$

where Greek indices imply summation over the values (1,2,3). r and s are the free indices (information on indices' role can be found in Boresi and Chong [2000]). Since our jacobian matrix (a jacobian matrix is a matrix containing partial derivatives) is the rotation matrix we can write

$$\mathbf{E}^* = \mathbf{R}\,\mathbf{E}\,\mathbf{R}^{\mathrm{T}} \tag{5.15}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathrm{Cos}\beta\mathrm{Cos}\gamma & \mathrm{Cos}\beta\mathrm{Sin}\gamma & -\mathrm{Sin}\beta \\ \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma - \mathrm{Cos}\alpha\mathrm{Sin}\gamma & \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma + \mathrm{Cos}\alpha\mathrm{Cos}\gamma & \mathrm{Sin}\alpha\mathrm{Cos}\beta \\ \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma + \mathrm{Sin}\alpha\mathrm{Sin}\gamma & \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma - \mathrm{Sin}\alpha\mathrm{Cos}\gamma & \mathrm{Cos}\alpha\mathrm{Cos}\beta \end{bmatrix} \tag{5.16}$$

It means that using eq. (5.15) we can transform the strain matrix from first coordinate system (x, y, z) to second coordinate system (x^*, y^*, z^*). Using eq. (2.8) the strain matrix can be decomposed into its symmetric **S** and anti-symmetric **A** parts. If we do that we can write

$$\mathbf{E}^* = \mathbf{R}\,\mathbf{S}\,\mathbf{R}^{\mathrm{T}} + \mathbf{R}\,\mathbf{A}\,\mathbf{R}^{\mathrm{T}} \tag{5.17}$$

This means that the transformation from first coordinate system to second coordinate system is commutative. The proof may be seen in Appendix IV.

Now whether Ω is invariant in a rotation of a coordinate system in 3D is investigated. For invariance we require that

$$\Omega = \Omega^* \tag{5.18}$$

which is equivalent to

$$\omega_{xy}^2 + \omega_{xz}^2 + \omega_{yz}^2 = \omega_{xy}^{*2} + \omega_{xz}^{*2} + \omega_{yz}^{*2} \tag{5.19}$$

If eqs. (2.19), (2.20) and (2.21) are substituted in eq. (5.19) we get

$$\Omega^* = \left(\frac{\partial u}{\partial y}\right)^2 - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 - 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \left(\frac{\partial w}{\partial y}\right)^2 \tag{5.20}$$

Equation (5.20) is equal to Ω which is given in eq. (5.5). It means that Ω is invariant under a rotation of a coordinate system. Intermediate derivations may be seen in Appendix IV. Numerical results are given in section 5.4.

According to investigations which have been carried out regarding total shear Γ, it is not invariant in a rotation of a coordinate system in Euclidean space. The proof may be seen in Appendix IV. In eq. (IV.69) it is hypothesized that the total shear in the rotated system would be different than the one in the original system. This means that eq. (IV.102) must not be equal to zero. It seems that eq. (IV.102) might be equal to zero for two reasons: (i) if all the differentials in this equation are zero - this means that there is no deformation; and, (ii) if the variable α is zero - this means that there is no rotation. However these are not the case with the robustness analysis of geodetic networks. On the other hand despite extensive efforts, it has not been possible to prove mathematically that eq. (IV.102) will not be zero for all other possible cases. Instead it has been shown numerically that the total shear in the rotated system is different than the one in the original system - the numerical results may be seen in Table 5.2.

One might think that the expression for 3D total shear in eq. (IV.75) should reduce to the 2D expression by setting to zero all differentials involving w and z. However this argument is valid from eq. (IV.70) to eq. (IV.72). Indeed in these equations if w and z (which represent the third dimension) are set to zero then 3D total shear reduces to 2D total shear. Nevertheless if the equivalents of simple and pure shears from (2.13) to (2.18) are substituted in eq. (IV.72) this argument is no longer valid because in eq. (IV.73) if all the differentials involving w and z are set to zero, 3D total shear will not reduce to 2D total shear - some strain matrix components will remain in the equation. So it means that from now on we do not have any control over the total shear in 3D. Besides it is not known how simple and pure shear 'behave'. Therefore in the evaluation of 3D total shear, the role that pure and simple shears play is worth investigating in order to be able to understand total shear in 3D. That is why this issue is addressed in section 6.2 as a recommendation for further research. On the other hand this argument is valid on differential rotation. Because if all the differentials involving w and z in the equations from (IV.63) to (IV.65) are set to zero, differential rotation in 3D always reduces to differential rotation in 2D however this is not the case with total shear.

As can be seen from the results in Tables 5.2, total shear is not invariant under a rotation of a coordinate system in 3D. This agrees with the hypothesis made in eq. (IV.69). Since it can be assumed now that total shear is not invariant in 3D, it will not used as a measure in robustness analysis. Therefore maximum shear strain is recommended instead. This is explained in section 5.5.

5.4 Numerical Results for 3D Invariants

In order to be able to show the invariants in 3D numerically, Northwest Territories network in Fig. 4.5 is used. First in CT system dilation, differential rotation and total shear are calculated then using eq. (5.15) strain matrix is transformed to LG system and then once again dilation, differential rotation and total shear are computed. The results are given in Table 5.2.

Table 5.2 Values for Northwest Territories Network.

	Σ		Ω		Γ	
	CT	LG	CT	LG	CT	LG
1	17.1E-06	17.1E-06	13.2E-05	13.2E-05	1.46E-04	1.37E-04
2	9.6E-06	9.6E-06	68.5E-06	68.5E-06	7.43E-05	7.15E-05
3	1.2E-06	1.2E-06	11.9E-06	11.9E-06	1.37E-05	1.21E-05
4	3.3E-06	3.3E-06	24.2E-06	24.2E-06	2.95E-05	2.56E-05
5	1.4E-06	1.4E-06	12.6E-06	12.6E-06	1.45E-05	1.28E-05
6	-0.1E-06	-0.1E-06	10.3E-06	10.3E-06	1.11E-05	1.04E-05
7	-5.2E-06	-5.2E-06	27.5E-06	27.5E-06	3.02E-05	2.97E-05
8	-6.4E-06	-6.4E-06	15.1E-06	15.1E-06	1.87E-05	2.00E-05
9	-6.7E-06	-6.7E-06	36.0E-06	36.0E-06	4.24E-05	3.77E-05
10	2.0E-06	2.0E-06	12.5E-06	12.5E-06	1.35E-05	1.32E-05
11	-4.6E-06	-4.6E-06	33.5E-06	33.5E-06	3.63E-05	3.49E-05
12	-11.7E-06	-11.7E-06	80.1E-06	80.1E-06	8.83E-05	8.36E-05
13	-2.3E-06	-2.3E-06	20.1E-06	20.1E-06	2.32E-05	2.08E-05
14	-4.2E-06	-4.2E-06	19.4E-06	19.4E-06	2.25E-05	2.11E-05
15	3.5E-06	3.5E-06	12.1E-06	12.1E-06	1.67E-05	1.47E-05
16	6.7E-06	6.7E-06	20.2E-06	20.2E-06	2.39E-05	2.47E-05
17	4.6E-06	4.6E-06	21.4E-06	21.4E-06	2.39E-05	2.37E-05
18	17.4E-06	17.4E-06	48.6E-06	48.6E-06	5.87E-05	6.06E-05
19	-23.0E-06	-23.0E-06	83.4E-06	83.4E-06	9.52E-05	9.61E-05
20	-6.7E-06	-6.7E-06	18.1E-06	18.1E-06	2.16E-05	2.31E-05
21	-18.8E-06	-18.8E-06	74.2E-06	74.2E-06	8.38E-05	8.37E-05
22	16.3E-06	16.3E-06	42.2E-06	42.2E-06	5.22E-05	5.47E-05
23	16.2E-06	16.2E-06	54.3E-06	54.3E-06	6.28E-05	6.37E-05
24	-11.7E-06	-11.7E-06	80.1E-06	80.1E-06	8.83E-05	8.36E-05
25	-17.7E-06	-17.7E-06	53.0E-06	53.0E-06	6.32E-05	6.49E-05
26	3.4E-06	3.4E-06	23.7E-06	23.7E-06	2.72E-05	2.46E-05
27	-1.0E-06	-1.0E-06	21.7E-06	21.7E-06	2.38E-05	2.19E-05
28	4.9E-06	4.9E-06	8.1E-06	8.1E-06	1.44E-05	1.36E-05
29	1.9E-06	1.9E-06	10.8E-06	10.8E-06	1.22E-05	1.28E-05
30	-8.0E-06	-8.0E-06	68.2E-06	68.2E-06	7.53E-05	7.02E-05
31	16.3E-06	16.2E-06	42.2E-06	42.2E-06	5.22E-05	5.47E-05
32	-1.5E-06	-1.5E-06	6.7E-06	6.7E-06	8.80E-06	7.63E-06
33	5.0E-06	5.0E-06	28.3E-06	28.3E-06	3.13E-05	3.06E-05

As can be seen from the results only dilation Σ and differential rotation Ω are invariant under a rotation of a coordinate system in 3D.

5.5 Shear in terms of Principal Strains in 3D

In section 5.3 it is indicated that total shear Γ is not invariant in a rotation of a coordinate system in 3D Euclidean space. However in order to be able to see the displacements caused by maximum undetectable errors in terms of shear, we need a measure which is going to represent the shear and be invariant in a rotation of a coordinate system. Grafarend and Voosoghi [2003] showed that in eigenspace the maximum shear strain is invariant in 2D. Here maximum shear strain is extended to 3D.

Maximum shear strain is defined as the difference between maximum and minimum eigenvalue. Therefore the eigenvalues in 3D should be computed. The three principal axes of strain ellipsoid which are the eigenvalues of eq. (5.1) are calculated. Grafarend and Voosoghi [2003] uses Γ as a symbol to represent the maximum shear strain in 2D, however, here this symbol has already been used to represent total shear in 3D. Hence the symbol M is going to be used to represent the maximum shear strain in 3D.

$$M = \sigma_b - \sigma_s \tag{5.21}$$

where σ_b is the maximum eigenvalue and σ_s is the minimum eigenvalue. Since eigenvalues are computed using the symmetrical part of the strain matrix, in eq. (5.21) the values for σ_b and σ_s will be always real.

If we think about an infinitesimal sphere at a network point in the undeformed state, in the deformed state this infinitesimal sphere becomes an infinitesimal ellipsoid. It means that the infinitesimal sphere is deformed along the principal axis therefore the direction of the maximum shear strain follows the direction of the maximum eigenvalue.

5.6 Numerical Results for Principal Strains in 3D

For numerical tests the Northwest Territories network in Fig. 4.5 is used. First in CT system invariants and principal strains are calculated then using eq. (5.15) the strain matrix is transformed into the LG system and then once again invariants and principal strains are computed. The results are given in Tables 5.3 and 5.4.

Table 5.3 Results for Northwest Territories Network in CT system.

	I_1	I_2	I_3	σ_1	σ_2	σ_3
1	5.13E-05	1.75E-08	5.38E-15	1.61E-04	-1.09E-04	-3.07E-07
2	2.87E-05	4.69E-09	3.53E-16	8.44E-05	-5.56E-05	-7.53E-08
3	3.74E-06	1.40E-10	-2.03E-17	1.38E-05	-1.02E-05	1.44E-07
4	1.00E-05	6.01E-10	1.59E-16	3.01E-05	-1.99E-05	-2.65E-07
5	4.17E-06	1.58E-10	-4.82E-17	1.47E-05	-1.08E-05	3.03E-07
6	-4.17E-07	1.07E-10	-3.68E-17	9.98E-06	-1.07E-05	3.43E-07
7	-1.57E-05	7.52E-10	-1.65E-16	2.05E-05	-3.64E-05	2.21E-07
8	-1.91E-05	2.14E-10	4.20E-17	8.05E-06	-2.70E-05	-1.93E-07
9	-2.00E-05	1.25E-09	2.40E-16	2.69E-05	-4.67E-05	-1.91E-07
10	6.02E-06	1.54E-10	-1.50E-17	1.57E-05	-9.82E-06	9.70E-08
11	-1.38E-05	1.12E-09	3.11E-17	2.72E-05	-4.10E-05	-2.79E-08
12	-3.52E-05	6.41E-09	-2.18E-15	6.42E-05	-9.97E-05	3.41E-07
13	-7.01E-06	4.10E-10	-3.17E-17	1.70E-05	-2.41E-05	7.75E-08
14	-1.27E-05	3.69E-10	1.76E-16	1.42E-05	-2.64E-05	-4.70E-07
15	1.06E-05	1.56E-10	5.11E-17	1.90E-05	-8.01E-06	-3.36E-07
16	2.01E-05	4.07E-10	1.04E-16	3.27E-05	-1.23E-05	-2.57E-07
17	1.37E-05	4.63E-10	-6.43E-17	2.94E-05	-1.58E-05	1.38E-07
18	5.22E-05	2.33E-09	3.20E-17	8.10E-05	-2.87E-05	-1.38E-08
19	-6.89E-05	6.93E-09	-4.04E-15	5.52E-05	-1.25E-04	5.86E-07
20	-2.00E-05	3.18E-10	3.93E-17	1.06E-05	-3.04E-05	-1.23E-07
21	-5.65E-05	5.49E-09	3.79E-16	5.11E-05	-1.08E-04	-6.89E-08
22	4.88E-05	1.80E-09	6.84E-17	7.33E-05	-2.45E-05	-3.81E-08
23	4.85E-05	2.93E-09	-3.98E-16	8.35E-05	-3.52E-05	1.36E-07
24	-3.52E-05	6.41E-09	-2.18E-15	6.42E-05	-9.97E-05	3.41E-07
25	-5.32E-05	2.85E-09	1.06E-15	3.33E-05	-8.61E-05	-3.69E-07
26	1.02E-05	5.59E-10	-7.90E-17	2.92E-05	-1.92E-05	1.41E-07
27	-2.98E-06	4.73E-10	-9.66E-17	2.02E-05	-2.34E-05	2.05E-07
28	1.46E-05	7.52E-11	2.97E-17	1.87E-05	-3.68E-06	-4.32E-07
29	5.63E-06	1.35E-10	-6.80E-18	1.47E-05	-9.16E-06	5.03E-08
30	-2.41E-05	4.65E-09	1.86E-15	5.74E-05	-8.12E-05	-3.98E-07
31	4.88E-05	1.80E-09	6.84E-17	7.33E-05	-2.45E-05	-3.81E-08
32	-4.43E-06	4.86E-11	-2.92E-18	5.06E-06	-9.55E-06	6.05E-08
33	1.49E-05	8.13E-10	3.94E-17	3.70E-05	-2.20E-05	-4.86E-08

If the results in Table 5.3 are compared with the results in Table 5.4, it is seen that the results for invariants and principal strains are virtually identical (since these are very small numbers, there will be some slight differences due to rounding errors) for each solution. This is based on the eigenvalue theorem which is introduced in section 5.1.

Table 5.4 Results for Northwest Territories Network in LG system.

	I_1	I_2	I_3	σ_1	σ_2	σ_3
1	5.13E-05	1.75E-08	5.37E-15	1.61E-04	-1.09E-04	-3.06E-07
2	2.87E-05	4.69E-09	3.52E-16	8.44E-05	-5.56E-05	-7.51E-08
3	3.74E-06	1.40E-10	-2.03E-17	1.38E-05	-1.02E-05	1.44E-07
4	1.00E-05	6.01E-10	1.58E-16	3.01E-05	-1.99E-05	-2.65E-07
5	4.17E-06	1.58E-10	-4.82E-17	1.47E-05	-1.08E-05	3.03E-07
6	-4.15E-07	1.07E-10	-3.68E-17	9.98E-06	-1.07E-05	3.44E-07
7	-1.57E-05	7.52E-10	-1.65E-16	2.05E-05	-3.64E-05	2.21E-07
8	-1.91E-05	2.14E-10	4.20E-17	8.06E-06	-2.70E-05	-1.93E-07
9	-2.00E-05	1.25E-09	2.40E-16	2.69E-05	-4.67E-05	-1.91E-07
10	6.02E-06	1.54E-10	-1.50E-17	1.57E-05	-9.82E-06	9.70E-08
11	-1.38E-05	1.12E-09	3.11E-17	2.72E-05	-4.10E-05	-2.78E-08
12	-3.52E-05	6.41E-09	-2.18E-15	6.42E-05	-9.97E-05	3.41E-07
13	-7.01E-06	4.10E-10	-3.17E-17	1.70E-05	-2.41E-05	7.75E-08
14	-1.27E-05	3.69E-10	1.76E-16	1.42E-05	-2.64E-05	-4.70E-07
15	1.06E-05	1.56E-10	5.10E-17	1.90E-05	-8.01E-06	-3.35E-07
16	2.01E-05	4.07E-10	1.03E-16	3.27E-05	-1.23E-05	-2.57E-07
17	1.37E-05	4.63E-10	-6.44E-17	2.94E-05	-1.58E-05	1.39E-07
18	5.22E-05	2.33E-09	3.20E-17	8.10E-05	-2.87E-05	-1.38E-08
19	-6.89E-05	6.93E-09	-4.04E-15	5.52E-05	-1.25E-04	5.86E-07
20	-2.00E-05	3.18E-10	3.93E-17	1.06E-05	-3.04E-05	-1.23E-07
21	-5.65E-05	5.49E-09	3.74E-16	5.11E-05	-1.08E-04	-6.81E-08
22	4.88E-05	1.80E-09	6.81E-17	7.33E-05	-2.45E-05	-3.79E-08
23	4.85E-05	2.93E-09	-3.98E-16	8.35E-05	-3.52E-05	1.36E-07
24	-3.52E-05	6.41E-09	-2.18E-15	6.42E-05	-9.97E-05	3.41E-07
25	-5.32E-05	2.85E-09	1.06E-15	3.33E-05	-8.61E-05	-3.69E-07
26	1.02E-05	5.59E-10	-7.93E-17	2.92E-05	-1.92E-05	1.42E-07
27	-2.98E-06	4.73E-10	-9.66E-17	2.02E-05	-2.34E-05	2.05E-07
28	1.46E-05	7.52E-11	2.97E-17	1.87E-05	-3.68E-06	-4.31E-07
29	5.64E-06	1.35E-10	-6.78E-18	1.47E-05	-9.16E-06	5.02E-08
30	-2.41E-05	4.65E-09	1.86E-15	5.74E-05	-8.12E-05	-3.98E-07
31	4.88E-05	1.80E-09	6.81E-17	7.33E-05	-2.45E-05	-3.79E-08
32	-4.43E-06	4.86E-11	-2.92E-18	5.06E-06	-9.54E-06	6.04E-08
33	1.49E-05	8.12E-10	3.90E-17	3.69E-05	-2.20E-05	-4.80E-08

If the maximum shear strain values are computed in both CT and LG systems, it would be possible to see if maximum shear strain is invariant in 3D. The results are given in the following table.

Table 5.5 Maximum shear strain values in 3D.

	CT system	LG system
1	1.61E-04	1.61E-04
2	8.43E-05	8.43E-05
3	1.37E-05	1.37E-05
4	2.98E-05	2.98E-05
5	1.44E-05	1.44E-05
6	1.04E-05	1.04E-05
7	3.62E-05	3.62E-05
8	2.68E-05	2.68E-05
9	4.65E-05	4.65E-05
10	1.56E-05	1.56E-05
11	4.10E-05	4.10E-05
12	9.94E-05	9.94E-05
13	2.40E-05	2.40E-05
14	2.59E-05	2.59E-05
15	1.87E-05	1.87E-05
16	3.24E-05	3.24E-05
17	2.93E-05	2.93E-05
18	8.10E-05	8.10E-05
19	1.24E-04	1.24E-04
20	3.03E-05	3.03E-05
21	1.08E-04	1.08E-04
22	7.33E-05	7.33E-05
23	8.34E-05	8.34E-05
24	9.94E-05	9.94E-05
25	8.57E-05	8.57E-05
26	2.91E-05	2.91E-05
27	2.32E-05	2.32E-05
28	1.83E-05	1.83E-05
29	1.46E-05	1.46E-05
30	8.08E-05	8.08E-05
31	7.33E-05	7.33E-05
32	9.49E-06	9.48E-06
33	3.70E-05	3.69E-05

As can be seen from the results, ignoring small computational rounding errors, maximum shear strain is invariant under a rotation of a coordinate system in 3D. Now we can summarize the invariants both in 2D and 3D as shown in Table 5.6.

Table 5.6 Invariants in 2D and 3D.

Robustness primitives	2D	3D
Dilation	invariant	invariant
Pure shear	not invariant	not invariant
Simple shear	not invariant	not invariant
Differential rotation	invariant	invariant
Total shear	invariant	not invariant
Maximum shear strain	invariant	invariant

5.7 Relation between 3D and 2D in terms of Invariants

In the previous sections the invariants in 3D have been shown. Now the question is: how are the 3D invariants related to the 2D invariants? This question is crucial if we want to do the strain analysis of geodetic networks in 3D and obtain the results in 2D. Or if we want to compare the invariants in 2D with their 3D counterparts. These may be predicated for verification purposes, for example, having the 3D solution at hand and the 2D solution may be required, or the older equivalent 2D network can be analyzed.

The relation between 3D and 2D in terms of dilation has been developed in eq. (5.4). However, how can the relation between 3D and 2D in terms of differential rotation be shown? At the outset of the research the 3D solution (which is coming from the CT system) was to be compared with the solution on the 2D manifold (xy plane) in the LG system.

Several approaches have been tested, nonetheless the problem has not been overcome. Eventually it is realized that this transformation was not possible due to a dimensionality problem. The dimensionality problem is that the solution coming from CT system is in 3D whereas the solution on the 2D manifold in LG system is in 2D. From eqs. (2.19) and (2.32) it is known that the formula for differential rotation in 3D in xy plane and the formula for differential rotation in 2D are the same. Thus the results coming from 3D solution and 2D solution must be the same. Actually since measurements are taken in the Local Astronomic (LA) system (distances are independent of the system) and later on transformed to the LG system, this comparison should have been done in the LG system. If the results coming from the 3D solution and 2D solution are compared in the LG system, it is seen in section 5.8 that they are the same.

The differential rotation in 3D is given in section 5.2.2. However the differential rotation in 2D may be understood as the length of rotation vector $\vec{\omega}$ that stands out of the 2D manifold. The displacement $\mathbf{\Delta x} = (u, v)^T$ of a 2D position $\mathbf{x} = (x, y)^T$ due to the differential rotation ω is given by

$$\mathbf{\Delta x} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \omega y \\ -\omega x \end{bmatrix} = \omega \begin{bmatrix} y \\ -x \end{bmatrix} \tag{5.22}$$

This equation can be written equivalently using the three dimensional symbolism as

$$\mathbf{\Delta x} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \omega y \\ -\omega x \\ 0 \end{bmatrix} = \omega \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} \tag{5.23}$$

Another, equivalent way of writing this equation is

$$\boldsymbol{\Delta x} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \omega y \\ -\omega x \\ 0 \end{bmatrix} \tag{5.24}$$

where $\times$ denotes the vector product.

In the last equation, the differential rotation vector $[0, 0, \omega]^T$ is a vector of magnitude ω, perpendicular to the xy plane, or $\boldsymbol{\omega}_1$ in our alternative notation (see section 5.2.2), so that

$$\boldsymbol{\Delta x} = -\boldsymbol{\omega}_1 \times \mathbf{x} \tag{5.25}$$

Similarly, we may show that $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ are perpendicular to planes xz and yz respectively. This shows that the 2D matrix multiplication is equivalent to a vector multiplication by the 'rotation vector' that is perpendicular to the 2D manifold.

For clarity the alternative notation which was introduced section 5.2.2 is used. Using that we write

$$\boldsymbol{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} \omega_{2,3} \\ \omega_{1,3} \\ \omega_{1,2} \end{bmatrix} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3 \tag{5.26}$$

which is a vector fixed in space implied by the strain matrix, but its magnitude is invariant in any coordinate transformation. This can be seen by realizing that under any rotation of the original coordinate system, neither the magnitudes, nor the configuration or the triad ($\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$) change.

The simplest way of computing the projection of $\boldsymbol{\Omega}$ onto the normal to the 2D manifold (to get the value of $\omega = \omega_{1,2}$ that is sought) is to first rotate the 3D coordinate system into a position where the xy plane coincides with the 2D manifold. The rotation can be done by the standard formula:

$$\boldsymbol{\Omega}^* = \mathbf{R}\,\boldsymbol{\Omega} \tag{5.27}$$

where **R** is the rotation matrix (see section 5.3) that transforms coordinates from the original 3D coordinate system into the coordinate system which is denoted by an asterisk in which the xy plane is identical to the 2D. The z coordinate of $\boldsymbol{\Omega}^*$ is the scalar ω that is sought.

Having completed the solution for differential rotation it is time to move onto shear. Then the question is: how can the relation between 3D and 2D in terms of shear be shown? At the beginning it is thought that the relation could be achieved simply using projection methods. Nevertheless it turned out that the answer was not that easy. Afterwards it is found out that total shear is not invariant in 3D (see section 5.3). Therefore maximum shear strain is employed in eigenspace.

The 3D system of principal axes is not related to the 2D system of principal axes in any obvious way hence the relation has to be worked out in a step-by-step manner. The maximum shear strain in 3D and the maximum shear strain in 2D are two scalars but different in themselves. This is why for comparison the maximum shear strain in 3D must be transformed to 2D. If the vector of strain eigenvalues $(\Lambda_1, \Lambda_2, \Lambda_3)^T = \boldsymbol{\Lambda}$ from the standard formula are computed we get

$$\det(\boldsymbol{\Lambda}\mathbf{I} - \mathbf{S}) = \Lambda^3 - I_1\Lambda^2 - I_2\Lambda - I_3 \tag{5.28}$$

where I_1, I_2, I_3 are the invariants in 3D. The solution for cubic equations is outlined in Dickson [1914]. If we set

$$\Lambda = y + \frac{I_1}{3} \tag{5.29}$$

we get a reduced cubic equation

$$y^3 + py + q = 0 \tag{5.30}$$

where

$$p = -I_2 - \frac{I_1^2}{3}, \qquad q = -I_3 - \frac{I_1 I_2}{3} - \frac{2I_1^3}{27} \tag{5.31}$$

Using Cardan's formula [see Dickson, 1914; MacDuffee, 1954] the three roots of eq. (5.30) are obtained as

$$\begin{aligned}
y_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}} \\
y_2 &= \omega \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \omega^2 \sqrt[3]{-\frac{q}{2} - \sqrt{R}} \\
y_3 &= \omega^2 \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \omega \sqrt[3]{-\frac{q}{2} - \sqrt{R}}
\end{aligned} \tag{5.32}$$

where

$$R = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \tag{5.33}$$

and

$$\omega = \frac{-1 + \sqrt{-3}}{2}, \qquad \omega^2 = \frac{-1 - \sqrt{-3}}{2} \tag{5.34}$$

However, if the symmetric part of the strain matrix (**S**) is used, the three roots are always real [Boresi et al., 1993]. In this case the above equations are simplified to the following equations [MacDuffee, 1954].

$$y_1 = 2\sqrt[3]{-\frac{q}{2} + \sqrt{R}}$$

$$y_2 = -\sqrt[3]{-\frac{q}{2}+\sqrt{R}} - \sqrt[3]{-\frac{q}{2}-\sqrt{R}}\,\sqrt{3} \tag{5.35}$$

$$y_3 = -\sqrt[3]{-\frac{q}{2}+\sqrt{R}} + \sqrt[3]{-\frac{q}{2}-\sqrt{R}}\,\sqrt{3}$$

Then the roots of eq. (5.28) are obtained as follows

$$\Lambda_1 = y_1 + \frac{I_1}{3}$$

$$\Lambda_2 = y_2 + \frac{I_1}{3} \tag{5.36}$$

$$\Lambda_3 = y_3 + \frac{I_1}{3}$$

If we look at the formulae in eq. (5.36) closely, because $y_1 > y_2 > y_3$ we see that the biggest root is Λ_1 and the smallest root is Λ_3. Since the maximum shear strain is defined as the difference between the biggest and the smallest eigenvalue, to compute the maximum shear in 3D we write

$$M = \Lambda_1 - \Lambda_3 \tag{5.37}$$

If Λ_1 and Λ_3 are substituted in eq. (5.37) we reach

$$M = 3\sqrt[3]{-\frac{q}{2}+\sqrt{R}} - \sqrt{3}\,\sqrt[3]{-\frac{q}{2}-\sqrt{R}} \tag{5.38}$$

In the 2D case, if the vector of strain eigenvalues $(\lambda_1, \lambda_2)^T = \boldsymbol{\lambda}$ from the standard formula are computed we get

$$\det(\boldsymbol{\lambda}\mathbf{J} - \mathbf{S}) = \lambda^2 - J_1\lambda - J_2 \tag{5.39}$$

where J_1 and J_2 are the invariants in 2D. The roots of eq. (5.39) are obtained from

$$\lambda_1 = \frac{J_1 + \sqrt{J_1^2 + 4J_2}}{2} \qquad\qquad \lambda_2 = \frac{J_1 - \sqrt{J_1^2 + 4J_2}}{2} \tag{5.40}$$

It is clear that $\lambda_1 > \lambda_2$. So to compute the maximum shear in 2D, we write

$$\mu = \lambda_1 - \lambda_2 \tag{5.41}$$

If we substitute λ_1 and λ_2 in eq. (5.41) we arrive at

$$\mu = \sqrt{J_1^2 + 4J_2} \tag{5.42}$$

We want to move from the maximum shear in 3D to the maximum shear in 2D. As can be seen, eqs. (5.38) and (5.42) only depend on the invariants. However the invariants in 3D and 2D are different. Therefore we should find a relation between the invariants in 3D and invariants in 2D. If we look at eq. (5.2) we see that the following relation is valid

$$I_1 = J_1 + \sigma_{zz} \tag{5.43}$$

where σ_{zz} is one of the components of the symmetrical part of the strain matrix. So we need to get I_1 out of eq. (5.38). Let us start with

$$M^3 = \left(3\sqrt[3]{-\frac{q}{2} + \sqrt{R}} - \sqrt{3}\sqrt[3]{-\frac{q}{2} - \sqrt{R}} \right)^3 \tag{5.44}$$

If the necessary algebraic manipulations are performed we obtain I_1 as

$$I_1 = -3\sqrt[3]{\sqrt{R} - \frac{M^3 - K}{27} + \frac{I_3}{2} + \frac{I_1 I_2}{6}} \tag{5.45}$$

where

$$K = -3\left(3\sqrt[3]{-\frac{q}{2} + \sqrt{R}} \right)^2 \left(\sqrt{3}\sqrt[3]{-\frac{q}{2} - \sqrt{R}} \right) + 3\left(3\sqrt[3]{-\frac{q}{2} + \sqrt{R}} \right)\left(\sqrt{3}\sqrt[3]{-\frac{q}{2} - \sqrt{R}} \right)^2$$

$$-\left(\sqrt{3}\sqrt[3]{-\frac{q}{2}-\sqrt{R}}\right)^3 \quad (5.46)$$

Using eq. (5.43) we can write

$$J_1 = -3\sqrt[3]{\sqrt{R}-\frac{M^3-K}{27}+\frac{I_3}{2}+\frac{I_1 I_2}{6}}-\sigma_{zz} \quad (5.47)$$

If eq. (5.47) is substituted in eq. (5.42) we get

$$\mu = \sqrt{\left(-3\sqrt[3]{\sqrt{R}-\frac{M^3-K}{27}+\frac{I_3}{2}+\frac{I_1 I_2}{6}}-\sigma_{zz}\right)^2+4J_2} \quad (5.48)$$

This is the equation which has been sought. However, as can be seen, computing the maximum shear strain in 2D from the maximum shear strain in 3D is very cumbersome operation. That is why instead of computing the maximum shear strain in 2D from the maximum shear strain in 3D, the maximum shear strain in 2D should be calculated separately.

5.8 Numerical Results for 3D and 2D in terms of Invariants

Let us start with a simulated strain matrix in the CT system

$$\mathbf{E}_{CT} = \begin{bmatrix} 3 & 10 & 7 \\ 5 & 1 & 6 \\ 13 & 0 & 9 \end{bmatrix} \quad (5.49)$$

Then using the eqs. (2.19), (2.20) and (2.21) the differential rotation vector in 3D is obtained as

$$\mathbf{\Omega}_{CT} = \begin{bmatrix} \omega_{2,3} \\ \omega_{1,3} \\ \omega_{1,2} \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 2.5 \end{bmatrix} \tag{5.50}$$

So we can calculate the differential rotation in 3D as

$$\Omega_{CT} = \sqrt{\omega_{2,3}^2 + \omega_{1,3}^2 + \omega_{1,2}^2} = 24.25 \tag{5.51}$$

If we transform this strain matrix in (5.49) utilizing the rotation of the strain matrix in eq. (5.15) we write

$$\mathbf{E}_{LG} = \mathbf{R}\,\mathbf{E}_{CT}\mathbf{R}^{T} \tag{5.52}$$

and we get the values in $\mathbf{E}_{LG}$ as follows

$$\mathbf{E}_{LG} = \begin{bmatrix} 17.1132 & 2.5822 & 4.9083 \\ 3.0308 & -5.5367 & -1.0377 \\ 5.0977 & 8.7989 & 1.4235 \end{bmatrix} \tag{5.53}$$

where

$$\mathbf{R} = \begin{bmatrix} 0.6387 & 0.6108 & 0.4680 \\ 0.6911 & -0.7227 & 0 \\ -0.3383 & -0.3235 & 0.8837 \end{bmatrix} \tag{5.54}$$

This is the rotation matrix which is used to transform the strain matrix from CT system to LG system. Since we want to compare the 3D solution coming from CT system with the one on the 2D manifold (xy plane) in LG system, the differential rotation coming from the top-left side of this strain matrix should be computed. This is because the top-left side of this matrix corresponds the xy plane in LG system. If we do that we get

$$\mathbf{E}_{LG} = \begin{bmatrix} 17.1132 & 2.5822 \\ 3.0308 & -5.5367 \end{bmatrix} \tag{5.55}$$

The symmetrical and anti-symmetrical part of this matrix are as follows

$$\mathbf{S}_{LG} = \begin{bmatrix} 17.1132 & 2.8065 \\ 2.8065 & -5.5367 \end{bmatrix} \tag{5.56}$$

$$\mathbf{A}_{LG} = \begin{bmatrix} 0 & -0.2243 \\ 0.2243 & 0 \end{bmatrix} \tag{5.57}$$

So the differential rotation in xy plane in LG system is

$$\omega_{xy} = -0.2243 \tag{5.58}$$

As can be seen, the value of ω_{xy} does not match with $\omega_{1,2}$ in (5.50). This is due to the dimensionality problem. If we calculate the differential rotation vector in LG system in 3D we get

$$\mathbf{\Omega}_{LG} = \begin{bmatrix} \omega_{2,3} \\ \omega_{1,3} \\ \omega_{1,2} \end{bmatrix} = \begin{bmatrix} -4.9183 \\ -0.0947 \\ -0.2243 \end{bmatrix} \tag{5.59}$$

The differential rotation vector in 3D in LG system is

$$\Omega_{LG} = \sqrt{\omega_{2,3}^2 + \omega_{1,3}^2 + \omega_{1,2}^2} = 24.25 \tag{5.60}$$

This proves that the magnitude of the differential rotation vector is invariant under a rotation of a coordinate system.

From eqs. (2.19) and (2.32) we know that as long as the 2D manifold consists of the xy plane in 3D, the formula for differential rotation in 3D in xy plane and the formula for differential rotation in 2D are the same. Thus the results coming from 3D solution and 2D solution must be the same. Actually since measurements are taken in LA system and later on transformed to LG system, this comparison should have been done in LG system. We already have the results for differential rotation in 3D in LG system. So what we need to do is calculate the differential rotation in 2D in LG system. Let us extract the 2D strain matrix from 3D strain matrix in LG system using top-left side of the strain matrix. If we do that we get

$$\mathbf{E}_{LG} = \begin{bmatrix} 17.1132 & 2.5822 \\ 3.0308 & -5.5367 \end{bmatrix} \tag{5.61}$$

Using eq. (2.8) we can calculate the symmetrical and anti-symmetrical part of the strain matrix as follows

$$\mathbf{S}_{LG} = \begin{bmatrix} 17.1132 & 2.8065 \\ 2.8065 & -5.5367 \end{bmatrix} \tag{5.62}$$

$$\mathbf{A}_{LG} = \begin{bmatrix} 0 & -0.2243 \\ 0.2243 & 0 \end{bmatrix} \tag{5.63}$$

From the anti-symmetrical part of the strain matrix it is clear that the differential rotation in 3D xy plane and the differential rotation in 2D in LG system (5.57) are the same. Since (5.63) gives the differential rotation in xy plane in 3D we can rotate the 3D coordinate system using eq. (5.27) into a position where the xy plane coincides with the 2D manifold.

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS

This chapter concludes the developments which have been made throughout this thesis. In section 6.1 the conclusions are addressed and in section 6.2 recommendations for further research are outlined.

6.1 Conclusions

Application of robustness analysis to 2D networks is known. However, in order to be able to calculate the displacements in 2D networks the initial conditions must be computed. In this dissertation the initial conditions for 2D networks have been formulated. Furthermore the threshold values are needed to evaluate the networks. These threshold values enable us to assess the robustness of networks. In this thesis the specifications given by Geodetic Survey Division are used to compute the threshold values. The results prove that this approach works well.

Robustness analysis is a very powerful technique capable of providing a picture of the analyzed network. If a network has some defects, the robustness analysis technique reveals them and portrays them.

When the redundancy number of the observations is lower we obtain bigger displacements. If a network has some deficiency (such as if the distances are not measured) we can determine the weakness of the network for these points. This lack of measurements also lowers the redundancy number of the observations in the network. However, if the redundancy number of the observations increases the displacements get lower. The observations with large standard deviations cause bigger displacements at the connected points.

In 1D networks since there is only one component in the strain matrix not all the primitives can be defined. Only dilation may be defined. Moreover when two points have very nearly the same height (a common occurrence) the strain with respect to height might become extremely large and mislead the results. Therefore in 1D networks there are some issues which one should be aware of.

As well as 1D networks the robustness analysis technique cannot be directly applied to 3D networks. Because if two points have very nearly the same height, the strain with respect to height might become extremely large. In this case it is thought that instead of curvilinear coordinates cartesian coordinates can be used. Since magnitudes (length) of displacements are independent from the coordinate system, networks can be assessed in any coordinate system.

Since displacements are computed from a system of first order differential equations, to solve them, these equations should be integrated. Therefore the initial conditions have to be determined. In this thesis the initial conditions for 3D networks have been developed. Furthermore the threshold values are needed to evaluate 3D networks. These threshold values enable us to evaluate the robustness of 3D networks.

In this dissertation accuracy standards for positioning given by GSD are used to compute threshold values for 3D networks. The confidence ellipsoid could be used to determine accuracy of adjusted coordinates at network points for GPS networks since one may have the full covariance matrix for GPS observations. However, with the traditional approach, horizontal and vertical coordinates are obtained separately so generally a full variance covariance matrix is not available for the points in the classical (terrestrial) three dimensional networks. Therefore in this thesis the combination of 2D and 1D networks is implemented following the suggestion made by Geodetic Survey Division of Canada.

Robustness of a network should be defined in terms of invariants rather than the primitives since a datum change will change the strain matrix therefore the primitive values. So in this thesis invariants in 3D are investigated. According to the results, in 3D Euclidean space, dilation and differential rotation are invariants and maximum shear strain is invariant in eigenspace. This means that no matter what the choice of the datum is the results for these invariants will be the same for each solution. It has been shown that total shear is not invariant in 3D.

In this dissertation it is shown how the 3D invariants are related to the 2D invariants. This is crucial if we want to do the strain analysis of geodetic networks in 3D and obtain the results in 2D. Or if we want to compare the invariants in 2D with their 3D counterparts. It has been shown that maximum shear strain should be calculated in 2D rather than being determined via its 3D counterpart.

Robustness of a network is affected by the design of the network and accuracy of the observations. Therefore the points that lack robustness in the network may be remedied either by increasing the quality of observations and/or by increasing the number of observations in the network. A remedial strategy is likely to be different for different networks since they have different geometry and different observations. There might not be a solution fitting all networks but in this thesis a general strategy has been developed.

6.2 Recommendations for Further Research

In this dissertation the norm of the displacement vectors at all points in the network is minimized. However there is an infinite number of solutions to obtain the displacements from strain; for example, minimizing the mean displacement for the network, or the largest displacement, or the sum of absolute displacements or the median displacement etc. The mentioned solutions might be developed and tested.

When two points have very nearly the same height (a common occurrence) the strain with respect to height might become extremely large and mislead the results. Therefore a technique which is going to handle the problem of application of robustness analysis to 1D networks should be developed.

According to the investigations which have been carried out about total shear, in 3D, total shear is not invariant in a rotation of a coordinate system in Euclidean space however in this thesis it is shown that in 3D, maximum shear strain is invariant in eigenspace. Therefore the relation between total shear and maximum shear strain needs to be investigated.

In the evaluation of 3D total shear, the role that the pure and simple shears play is worth investigating in order to be able to understand total shear in 3D.

Some things still need to be investigated for example if we get a big value of, let us say pure shear. How can this problem be cured? What is the source of this problem? In order to be able to answer these questions a synthetic network should be simulated and all the observations in that network should be built one by one.

In this thesis the correlations among the observations are not considered. However to obtain rigorously correct results, the correlations among the observations may be considered.

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APPENDIX I: ESTIMATION OF STRAIN MATRIX

I.1 Estimation of Strain Matrix in 2D

In Vaníček et al. [2001] a displacement of a point Pi is denoted as

$$\mathbf{\Delta x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{I.1}$$

where u is the displacement in the x direction and v is the displacement in the y direction. Then the tensor gradient with respect to position is

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} \end{bmatrix}. \tag{I.2}$$

For $\forall j = 0,1,...,t$ (t is the number of connection) the displacements u and v can be calculated as follows (see also Craymer and Vaníček [2002]).

$$a_i + \left(\frac{\partial u_i}{\partial x}\right)\left(X_j - X_i\right) + \left(\frac{\partial u_i}{\partial y}\right)\left(Y_j - Y_i\right) = u_j \tag{I.3}$$

$$b_i + \left(\frac{\partial v_i}{\partial x}\right)\left(X_j - X_i\right) + \left(\frac{\partial v_i}{\partial y}\right)\left(Y_j - Y_i\right) = v_j \tag{I.4}$$

where all the partial derivatives as well as the absolute terms a_i, b_i and the coordinates X_i, Y_i refer to point P_i. In matrix form:

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} a_i \\ \dfrac{\partial u_i}{\partial x} \\ \dfrac{\partial u_i}{\partial y} \end{bmatrix} = \mathbf{u_i} \tag{I.5}$$

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} b_i \\ \dfrac{\partial v_i}{\partial x} \\ \dfrac{\partial v_i}{\partial y} \end{bmatrix} = \mathbf{v_i} \tag{I.6}$$

If these equations are solved using the method of least squares, we get

$$\forall i \text{ in the network} \quad \begin{bmatrix} a_i \\ \dfrac{\partial u_i}{\partial x} \\ \dfrac{\partial u_i}{\partial y} \end{bmatrix} = \left(\mathbf{K}_i^T \mathbf{K}_i\right)^{-1} \mathbf{K}_i^T \mathbf{u}_i = \mathbf{Q}_i \mathbf{u}_i \tag{I.7}$$

$$\forall i \text{ in the network} \quad \begin{bmatrix} b_i \\ \dfrac{\partial v_i}{\partial x} \\ \dfrac{\partial v_i}{\partial y} \end{bmatrix} = \left(\mathbf{K}_i^T \mathbf{K}_i\right)^{-1} \mathbf{K}_i^T \mathbf{v}_i = \mathbf{Q}_i \mathbf{v}_i \tag{I.8}$$

Assembling (I.7) and (I.8) into a hypermatrix, we reach

$$\forall i \text{ in the network} \quad \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_i & 0 \\ 0 & \mathbf{Q}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \tag{I.9}$$

Since we are looking for the relation between the displacement vector and the strain matrix, absolute terms are no interest to us. So we can eliminate the first row of the $\mathbf{Q}_i$ matrix. If we show the reduced matrix with $\mathbf{T}$ and substitute the eqs. (I.1) and (I.2) in eq. (I.9), we get

$$\forall i \text{ in the network} \quad \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i \boldsymbol{\Delta}\mathbf{x}_i \tag{I.10}$$

Using eq. (2.2) in eq. (I.10), we obtain

$$\forall i \text{ in the network} \quad \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{P} \Delta \mathbf{l} \tag{I.11}$$

I.2 Estimation of Strain Matrix in 1D

Let us denote the displacement of a point Pi by

$$\boldsymbol{\Delta}\mathbf{x}_i = [\Delta z_i] = [w_i] \tag{I.12}$$

where w is the displacement in the z direction. Then the tensor gradient with respect to position is

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial w_i}{\partial z} \end{bmatrix}. \tag{I.13}$$

For all j = 0,1,...,t the displacement w can be calculated as follows

$$c_i + \frac{\partial w_i}{\partial z}\left(Z_j - Z_i\right) = w_j \tag{I.14}$$

where Z_i is the height of the point of interest, Z_j is the height of the connected point. It means that there will be one equation for each connection. In matrix form:

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} c_i \\ \frac{\partial w_i}{\partial z} \end{bmatrix} = \mathbf{w}_i \tag{I.15}$$

If this equation is solved using LSE as it is shown in section I.1, the following equation is obtained.

$$\forall i \text{ in the network} \quad \begin{bmatrix} c_i \\ \frac{\partial w_i}{\partial z} \end{bmatrix} = \left(\mathbf{K}_i^T \mathbf{K}_i\right)^{-1} \mathbf{K}_i^T \mathbf{w}_i = \mathbf{Q}_i \mathbf{w}_i \tag{I.16}$$

Using the same reasoning in section I.2 and substituting from eqs. (I.12) and (I.13), we get

$$\forall i \text{ in the network} \quad \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i \boldsymbol{\Delta}\mathbf{x}_i \tag{I.17}$$

Using eq. (2.2) in eq. (I.17), we obtain

$$\forall i \text{ in the network} \quad \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i \left(\mathbf{A}^T \mathbf{P} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{P} \Delta \mathbf{l} \tag{I.18}$$

I.3 Estimation of Strain Matrix in 3D

Let us denote the displacement of a point P_i by

$$\mathbf{\Delta x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{I.19}$$

then the tensor gradient with respect to position is [Love, 1944; Sokolnikoff, 1956 and Timoshenko and Goodier, 1970]

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} & \frac{\partial u_i}{\partial z} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} & \frac{\partial v_i}{\partial z} \\ \frac{\partial w_i}{\partial x} & \frac{\partial w_i}{\partial y} & \frac{\partial w_i}{\partial z} \end{bmatrix}. \tag{I.20}$$

This may be applied to geodetic networks as follows where u_i, v_i and w_i are the displacements of point Pi relative to point Pj:

$$a_i + \frac{\partial u_i}{\partial x}\left(X_j - X_i\right) + \frac{\partial u_i}{\partial y}\left(Y_j - Y_i\right) + \frac{\partial u_i}{\partial z}\left(Z_j - Z_i\right) = u_j \tag{I.21}$$

$$b_i + \frac{\partial v_i}{\partial x}\left(X_j - X_i\right) + \frac{\partial v_i}{\partial y}\left(Y_j - Y_i\right) + \frac{\partial v_i}{\partial z}\left(Z_j - Z_i\right) = v_j \tag{I.22}$$

$$c_i + \frac{\partial w_i}{\partial x}\left(X_j - X_i\right) + \frac{\partial w_i}{\partial y}\left(Y_j - Y_i\right) + \frac{\partial w_i}{\partial z}\left(Z_j - Z_i\right) = w_j \tag{I.23}$$

where all the partial derivatives as well as the absolute terms a_i, b_i c_i and the coordinates X_i, Y_i and Z_i refer to point Pi and point Pj is connected (by an observation) to the point of interest, point Pi. In matrix form:

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ \frac{\partial u_i}{\partial z} \end{bmatrix} = \mathbf{u}_i \tag{I.24}$$

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \\ \frac{\partial v_i}{\partial z} \end{bmatrix} = \mathbf{v}_i \tag{I.25}$$

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} c_i \\ \frac{\partial w_i}{\partial x} \\ \frac{\partial w_i}{\partial y} \\ \frac{\partial w_i}{\partial z} \end{bmatrix} = \mathbf{w}_i \tag{I.26}$$

Using LSE, we obtain

$$\forall i \text{ in the network} \quad \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ \frac{\partial u_i}{\partial z} \end{bmatrix} = \left(\mathbf{K}_i^T \mathbf{K}_i\right)^{-1} \mathbf{K}_i^T \mathbf{u}_i = \mathbf{Q}_i \mathbf{u}_i \tag{I.27}$$

$$\forall i \text{ in the network} \quad \begin{bmatrix} b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \\ \frac{\partial v_i}{\partial z} \end{bmatrix} = \left(\mathbf{K}_i^T \mathbf{K}_i\right)^{-1} \mathbf{K}_i^T \mathbf{v}_i = \mathbf{Q}_i \mathbf{v}_i \tag{I.28}$$

$$\forall i \text{ in the network} \quad \begin{bmatrix} c_i \\ \frac{\partial w_i}{\partial x} \\ \frac{\partial w_i}{\partial y} \\ \frac{\partial w_i}{\partial z} \end{bmatrix} = \left(\mathbf{K}_i^T \mathbf{K}_i\right)^{-1} \mathbf{K}_i^T \mathbf{w}_i = \mathbf{Q}_i \mathbf{w}_i \tag{I.29}$$

Assembling (I.27), (I.28) and (I.29) into a hypermatrix, we get

$$\forall i \text{ in the network} \quad \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ \frac{\partial u_i}{\partial z} \\ b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \\ \frac{\partial v_i}{\partial z} \\ c_i \\ \frac{\partial w_i}{\partial x} \\ \frac{\partial w_i}{\partial y} \\ \frac{\partial w_i}{\partial z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_i & 0 & 0 \\ 0 & \mathbf{Q}_i & 0 \\ 0 & 0 & \mathbf{Q}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \\ \mathbf{w}_i \end{bmatrix} \tag{I.30}$$

Using the same reasoning in section I.2 and substituting from eqs. (I.19) and (I.20), we obtain

$$\forall i \text{ in the network} \quad \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i \Delta \mathbf{x}_i \tag{I.31}$$

Substituting eq. (2.2) in eq. (I.31), we get

$$\forall i \text{ in the network} \quad \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{P} \Delta \mathbf{l} \tag{I.32}$$

APPENDIX II: DETERMINATION OF INITIAL CONDITIONS

II.1 Determination of Initial Conditions for 2D Networks

It is clear that eq. (3.2) is a system of first order differential equations. In order to solve the system, it should be integrated. Therefore the initial conditions have to be determined. In order to be able to calculate the initial conditions, the displacements caused by maximum undetectable errors in network points should be minimized. This means that the norm of the displacement vectors for all points in the network should be minimum. i.e.,

$$\min_{(X_0, Y_0 \in R)} \sum_{i=1}^{n} \|\Delta\vec{r}\|_i = \min_{(X_0, Y_0 \in R)} \sum_{i=1}^{n} (u_i^2 + v_i^2) \tag{II.1}$$

Here we are looking for the relation between the initial conditions and the strain parameters, therefore the absolute terms a_i and b_i are no interest to us. So if the absolute terms are removed from eqs. (I.3) and (I.4) and then if these reduced equations are employed, we get

$$= \min_{(X_0, Y_0 \in R)} \sum_{i=1}^{n} \left[\left(\frac{\partial u_i}{\partial x}(X_i - X_0) + \frac{\partial u_i}{\partial y}(Y_i - Y_0) \right)^2 + \left(\frac{\partial v_i}{\partial x}(X_i - X_0) + \frac{\partial v_i}{\partial y}(Y_i - Y_0) \right)^2 \right] \tag{II.2}$$

If this equation is differentiated with respect to X_0, we can write

$$\frac{\partial \sum_{i=1}^{n} \|\Delta\vec{r}\|_i}{\partial X_0} = 0 \tag{II.3}$$

We get

$$\sum_{i=1}^{n}\left(-2\left(\frac{\partial u_i}{\partial x}(X_i - X_0)+\frac{\partial u_i}{\partial y}(Y_i - Y_0)\right)\frac{\partial u_i}{\partial x}-2\left(\frac{\partial v_i}{\partial x}(X_i - X_0)+\frac{\partial v_i}{\partial y}(Y_i - Y_0)\right)\frac{\partial v_i}{\partial x}\right)=0 \tag{II.4}$$

Using the same reasoning above, the constant terms are removed. So we write

$$\sum_{i=1}^{n}\left(\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x}(X_i - X_0)+\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x}(Y_i - Y_0)\right)+\left(\frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial x}(X_i - X_0)+\frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x}(Y_i - Y_0)\right)\right)=0 \tag{II.5}$$

If they are taken into parentheses, we can write

$$\sum_{i=1}^{n}\left(\left(-\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x}-\frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial x}\right)X_0+\left(-\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x}-\frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x}\right)Y_0+\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial x}\right)X_i+\left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x}\right)Y_i\right)=0 \tag{II.6}$$

If this equation is expressed in the following form,

$$\sum_{i=1}^{n}(a_1 X_0 + b_1 Y_0 + c_1) = 0 \tag{II.7}$$

We can write that

$$a_1 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial x}\right)^2+\left(\frac{\partial v_i}{\partial x}\right)^2\right] \tag{II.8}$$

$$b_1 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x}\right) \tag{II.9}$$

$$c_1 = \sum_{i=1}^{n}\left[\left(\left(\frac{\partial u_i}{\partial x}\right)^2+\left(\frac{\partial v_i}{\partial x}\right)^2\right)X_i+\left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x}\right)Y_i\right] \tag{II.10}$$

If the same equation is differentiated with respect to Y_0, we can write

$$\frac{\partial \sum_{i=1}^{n} \|\Delta \vec{r}\|_i}{\partial Y_0} = 0 \qquad \text{(II.11)}$$

We get

$$\sum_{i=1}^{n}\left(\left(-\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y}\right)X_0 + \left(-\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial y}\right)Y_0 + \left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y}\right)X_i + \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial y}\right)Y_i\right) = 0 \qquad \text{(II.12)}$$

If this equation is expressed in the following form,

$$\sum_{i=1}^{n}\left(a_2 X_0 + b_2 Y_0 + c_2\right) = 0 \qquad \text{(II.13)}$$

We can write

$$a_2 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y}\right) \qquad \text{(II.14)}$$

$$b_2 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial y}\right)^2 + \left(\frac{\partial v_i}{\partial y}\right)^2\right] \qquad \text{(II.15)}$$

$$c_2 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y}\right)X_i + \left(\left(\frac{\partial u_i}{\partial y}\right)^2 + \left(\frac{\partial v_i}{\partial y}\right)^2\right)Y_i\right] \qquad \text{(II.16)}$$

If these system of linear equations (II.7) and (II.13) are solved with the compact form, we obtain the initial conditions X_0 and Y_0 as follows

$$\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{(II.17)}$$

The initial conditions X_0, Y_0 are substituted in eq. (3.2) to calculate the displacements u and v for each point in the network.

II.2 Determination of Initial Condition for 1D Networks

In geodetic height networks the height differences between/among the points are the measurements. Therefore heights are determined well whereas horizontal coordinates are only approximately known. So in 1D networks one concentrates on the displacements in z direction. Then we start with

$$\min_{(Z_0 \in R)} \sum_{i=1}^{n} \|\Delta\vec{r}\|_i = \min_{(Z_0 \in R)} \sum_{i=1}^{n} w_i^2 \tag{II.18}$$

Using the similar approach which is expressed in section II.1, if eq. (I.14) is employed, we get

$$= \min_{(Z_0 \in R)} \sum_{i=1}^{n} \left[\frac{\partial w_i}{\partial z} (Z_i - Z_0) \right]^2 \tag{II.19}$$

If this equation is differentiated with respect to Z_0 we can write

$$\frac{\partial \sum_{i=1}^{n} \|\Delta\vec{r}\|_i}{\partial Z_0} = 0 \tag{II.20}$$

We get

$$\sum_{i=1}^{n} \left[-2 \frac{\partial w_i}{\partial z} \left(\frac{\partial w_i}{\partial z} Z_i - \frac{\partial w_i}{\partial z} Z_0 \right) \right] = 0 \tag{II.21}$$

Since constants are no interest to us we write

$$Z_0 = \frac{\sum_{i=1}^{n} \left(\frac{\partial w_i}{\partial z} \right)^2 Z_i}{\sum_{i=1}^{n} \left(\frac{\partial w_i}{\partial z} \right)^2} \tag{II.22}$$

The initial condition Z_0 is substituted in the following equation to calculate the displacement w for each point in the network

$$w_i = \frac{\partial w_i}{\partial z} (Z_i - Z_0). \tag{II.23}$$

II.3 Determination of Initial Conditions for 3D Networks

Equation (3.1) is a system of first order differential equations. In order to solve the system, it should be integrated. Therefore the initial conditions (X_0, Y_0, Z_0) have to be determined. In order to be able to calculate the initial conditions, the displacements caused by maximum undetectable errors in network points should be minimized. This means that the norm of the displacement vectors for all points in the network should be minimum.

$$\min_{(X_0, Y_0, Z_0 \in R)} \sum_{i=1}^{n} \left\| \Delta\vec{r} \right\|_i = \min_{(X_0, Y_0, Z_0 \in R)} \sum_{i=1}^{n} \left(u_i^2 + v_i^2 + w_i^2 \right) \tag{II.24}$$

Using the similar approach which is expressed in section II.1, if eqs. (I.21), (I.22) and (I.23) are employed, we obtain

$$
= \min_{(X_0, Y_0, Z_0 \in R)} \sum_{i=1}^{n} \left[\left(\frac{\partial u_i}{\partial x}(X_i - X_0) + \frac{\partial u_i}{\partial y}(Y_i - Y_0) + \frac{\partial u_i}{\partial z}(Z_i - Z_0) \right)^2 + \right.
$$

$$
\left(\frac{\partial v_i}{\partial x}(X_i - X_0) + \frac{\partial v_i}{\partial y}(Y_i - Y_0) + \frac{\partial v_i}{\partial z}(Z_i - Z_0) \right)^2 + \tag{II.25}
$$

$$
\left. \left(\frac{\partial w_i}{\partial x}(X_i - X_0) + \frac{\partial w_i}{\partial y}(Y_i - Y_0) + \frac{\partial w_i}{\partial z}(Z_i - Z_0) \right)^2 \right]
$$

If this equation is differentiated with respect to X_0, we can write

$$
\frac{\partial \sum_{i=1}^{n} \|\Delta \vec{r}\|_i}{\partial X_0} = 0 \tag{II.26}
$$

We get

$$
\sum_{i=1}^{n} \left[-2\frac{\partial u_i}{\partial x} \left(\frac{\partial u_i}{\partial x} X_i - \frac{\partial u_i}{\partial x} X_0 + \frac{\partial u_i}{\partial y} Y_i - \frac{\partial u_i}{\partial y} Y_0 + \frac{\partial u_i}{\partial z} Z_i - \frac{\partial u_i}{\partial z} Z_0 \right) \right.
$$

$$
-2\frac{\partial v_i}{\partial x} \left(\frac{\partial v_i}{\partial x} X_i - \frac{\partial v_i}{\partial x} X_0 + \frac{\partial v_i}{\partial y} Y_i - \frac{\partial v_i}{\partial y} Y_0 + \frac{\partial v_i}{\partial z} Z_i - \frac{\partial v_i}{\partial z} Z_0 \right) \tag{II.27}
$$

$$
\left. -2\frac{\partial w_i}{\partial x} \left(\frac{\partial w_i}{\partial x} X_i - \frac{\partial w_i}{\partial x} X_0 + \frac{\partial w_i}{\partial y} Y_i - \frac{\partial w_i}{\partial y} Y_0 + \frac{\partial w_i}{\partial z} Z_i - \frac{\partial w_i}{\partial z} Z_0 \right) \right] = 0
$$

Since constants are no interest to us we write

$$
\sum_{i=1}^{n} \left[\left(-\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial x} - \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial x} \right) X_0 + \left(-\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial y} \right) Y_0 + \right.
$$

$$
\left(-\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial z} - \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial z} - \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial z} \right) Z_0 + \left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial x} \right) X_i +
$$

$$
\left. \left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial y} \right) Y_i + \left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial z} \right) Z_i \right] = 0
$$

(II.28)

To simplify this equation it can be expressed in the following form,

$$\sum_{i=1}^{n}\left(a_1X_0 + b_1Y_0 + c_1Z_0 + d_1\right) = 0 \tag{II.29}$$

We can write

$$a_1 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial v_i}{\partial x}\right)^2 + \left(\frac{\partial w_i}{\partial x}\right)^2\right] \tag{II.30}$$

$$b_1 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial y}\right) \tag{II.31}$$

$$c_1 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial z}\right) \tag{II.32}$$

$$d_1 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial x}\right)X_i + \left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial y}\right)Y_i + \left(\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial x}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial z}\right)Z_i\right] \tag{II.33}$$

If the same equation is differentiated with respect to Y_0, we can write

$$\frac{\partial \sum_{i=1}^{n}\left\|\Delta\vec{r}\right\|_i}{\partial Y_0} = 0 \tag{II.34}$$

We get

$$\sum_{i=1}^{n}\left[\left(-\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x} - \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial x}\right)X_0 + \left(-\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial y} - \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial y}\right)Y_0 + \left(-\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial z} - \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial z} - \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial z}\right)Z_0 + \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial x}\right)X_i + \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial y}\right)Y_i + \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial z}\right)Z_i\right] = 0 \tag{II.35}$$

To simplify this equation it can be expressed in the following form,

$$\sum_{i=1}^{n}\left(a_2X_0 + b_2Y_0 + c_2Z_0 + d_2\right) = 0 \tag{II.36}$$

We can write

$$a_2 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial x}\right) \tag{II.37}$$

$$b_2 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial y}\right)^2 + \left(\frac{\partial v_i}{\partial y}\right)^2 + \left(\frac{\partial w_i}{\partial y}\right)^2\right] \tag{II.38}$$

$$c_2 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial z}\right) \tag{II.39}$$

$$\begin{aligned} d_2 = \sum_{i=1}^{n} \Bigg[& \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial x}\right)X_i + \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial y}\right)Y_i \\ & \left(\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial y}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y}\frac{\partial w_i}{\partial z}\right)Z_i \Bigg] \end{aligned} \tag{II.40}$$

If the same equation is differentiated with respect to Z_0, we can write

$$\frac{\partial \sum_{i=1}^{n}\left\|\Delta\vec{r}\right\|_i}{\partial Z_0} = 0 \tag{II.41}$$

We get

$$\begin{aligned} \sum_{i=1}^{n} \Bigg[& \left(-\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial x} - \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial x}\right)X_0 + \left(-\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial y} - \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial y}\right)Y_0 + \\ & \left(-\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial z} - \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial z} - \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial z}\right)Z_0 + \left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial x}\right)X_i + \\ & \left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial y}\right)Y_i + \left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial z}\right)Z_i \Bigg] = 0 \end{aligned} \tag{II.42}$$

To simplify this equation it can be expressed in the following form,

$$\sum_{i=1}^{n}\left(a_3 X_0 + b_3 Y_0 + c_3 Z_0 + d_3\right) = 0 \tag{II.43}$$

We can write

$$a_3 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial x}\right) \tag{II.44}$$

$$b_3 = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial y}\right) \tag{II.45}$$

$$c_3 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial z}\right)^2 + \left(\frac{\partial v_i}{\partial z}\right)^2 + \left(\frac{\partial w_i}{\partial z}\right)^2\right] \tag{II.46}$$

$$d_3 = \sum_{i=1}^{n}\left[\left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial x} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial x}\right)X_i + \left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial y}\right)Y_i + \left(\frac{\partial u_i}{\partial z}\frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial z}\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial z}\frac{\partial w_i}{\partial z}\right)Z_i\right] \tag{II.47}$$

If these equations (II.29), (II.36) and (II.43) are solved with the compact form, we obtain the initial conditions X_0, Y_0 and Z_0 as follows

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{II.48}$$

The initial conditions X_0, Y_0 and Z_0 are substituted in eq. (3.1) to calculate the displacements u, v and w for each point in the network.

APPENDIX III: PRINCIPAL STRAINS IN 3D

If we change our notation to distinguish between the symmetrical and anti-symmetrical part of the strain matrix we write

$$e_{ij} = \sigma_{ij} + \omega_{ij} \tag{III.1}$$

where e_{ij} are the components of strain matrix, σ_{ij} are the components of symmetrical part of the strain matrix and ω_{ij} are the components of anti-symmetrical part of the strain matrix. The strain vectors $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$ and $\boldsymbol{\sigma}_z$ are written as follows

$$\begin{aligned}
\boldsymbol{\sigma}_x &= \sigma_{xx}\mathbf{i} + \sigma_{xy}\mathbf{j} + \sigma_{xz}\mathbf{k} \\
\boldsymbol{\sigma}_y &= \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k} \\
\boldsymbol{\sigma}_z &= \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}
\end{aligned} \tag{III.2}$$

where **i**, **j** and **k** are the unit vectors. Now consider the strain vector $\boldsymbol{\sigma}_P$ on an arbitrary oblique plan P through point O see Fig. III.1.

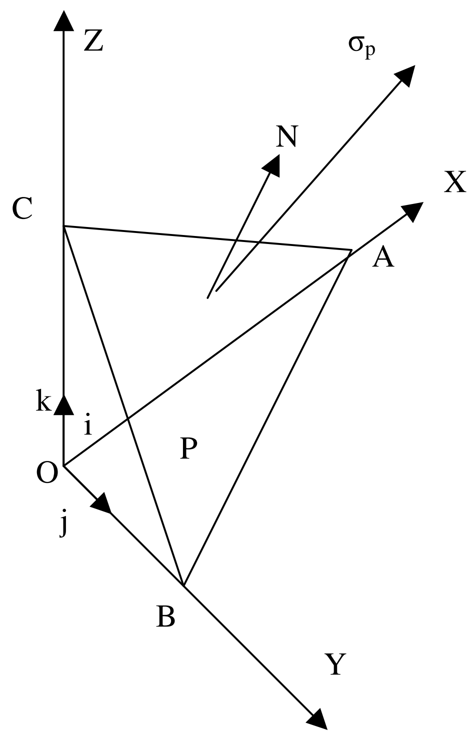

Fig III.1 Strain vector on an oblique plane having a normal N.

The unit normal vector to plane P is

$$\mathbf{N} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k} \tag{III.3}$$

where l, m and n are the direction cosines of unit vector **N**. The ratios of areas OBC, OAC and OBA to area ABC equal to l, m and n respectively. Therefore we can write

$$\boldsymbol{\sigma}_P = l\boldsymbol{\sigma}_x + m\boldsymbol{\sigma}_y + n\boldsymbol{\sigma}_z \tag{III.4}$$

$\boldsymbol{\sigma}_P$ can also be written in terms of its projections as follows:

$$\boldsymbol{\sigma}_P = \sigma_{P_x}\mathbf{i} + \sigma_{P_y}\mathbf{j} + \sigma_{P_z}\mathbf{k} \tag{III.5}$$

Using eq. (III.4) we can write

$$\sigma_{P_x} = l\sigma_{xx} + m\sigma_{yx} + n\sigma_{zx}$$

$$\sigma_{P_y} = l\sigma_{xy} + m\sigma_{yy} + n\sigma_{zy} \tag{III.6}$$

$$\sigma_{P_z} = l\sigma_{xz} + m\sigma_{yz} + n\sigma_{zz}$$

The normal strain σ_{PN} on the plane P is the projection of the vector $\boldsymbol{\sigma}_P$ in the direction of **N** that is

$$\sigma_{PN} = \boldsymbol{\sigma}_P \ \mathbf{N} \qquad \text{(III.7)}$$

Substituting eqs. (III.3), (III.5) and (III.6) in eq. (III.7) we get

$$\sigma_{PN} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + mn(\sigma_{yz} + \sigma_{zy}) + nl(\sigma_{xz} + \sigma_{zx}) + lm(\sigma_{xy} + \sigma_{yx})$$

$$\sigma_{PN} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2mn\sigma_{yz} + 2\ln\sigma_{xz} + 2lm\sigma_{xy} \qquad \text{(III.8)}$$

Let x,y,z and X,Y,Z denote two rectangular coordinate systems with a common origin. The cosines of the angles between the coordinate axes are given in the table below.

Table III.1 Direction cosines between the coordinate axes

	x	y	z
X	l_1	m_1	n_1
Y	l_2	m_2	n_2
Z	l_3	m_3	n_3

The strain components σ_{XX}, σ_{YY}, σ_{ZZ},… are defined with reference to X,Y,Z axes in the same manner as σ_{xx}, σ_{yy}, σ_{zz},…are defined relative to the axes x,y,z. σ_{XX} is the normal strain component on a plane perpendicular to axis X and σ_{XY} and σ_{XZ} are shear strain components on this plane. Hence using eq. (III.8) we can write

$$\sigma_{XX} = l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + n_1^2\sigma_{zz} + 2m_1n_1\sigma_{yz} + 2n_1l_1\sigma_{zx} + 2l_1m_1\sigma_{xy} \qquad \text{(III.9)}$$

$$\sigma_{YY} = l_2^2\sigma_{xx} + m_2^2\sigma_{yy} + n_2^2\sigma_{zz} + 2m_2n_2\sigma_{yz} + 2n_2l_2\sigma_{zx} + 2l_2m_2\sigma_{xy} \qquad \text{(III.10)}$$

$$\sigma_{ZZ} = l_3^2\sigma_{xx} + m_3^2\sigma_{yy} + n_3^2\sigma_{zz} + 2m_3n_3\sigma_{yz} + 2n_3l_3\sigma_{zx} + 2l_3m_3\sigma_{xy} \qquad \text{(III.11)}$$

σ_{XY} is the Y component of the strain vector $\boldsymbol{\sigma}_X$ acting on the plane perpendicular to the X axis. Thus σ_{XY} may be evaluated by forming the scalar product of the vector $\boldsymbol{\sigma}_X$ with a unit vector parallel to the Y axis which is

$$\mathbf{N}_2 = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k} \tag{III.12}$$

By eqs. (III.5), (III.6) and (III.12) σ_{XY} is determined as

$$\sigma_{XY} = \boldsymbol{\sigma}_X \mathbf{N}_2 = \boldsymbol{\sigma}_Y \mathbf{N}_1$$

$$\begin{aligned}\sigma_{XY} &= l_1 l_2 \sigma_{xx} + m_1 m_2 \sigma_{yy} + n_1 n_2 \sigma_{zz} + (m_1 n_2 + m_2 n_1)\sigma_{yz} \\ &\quad + (l_1 n_2 + l_2 n_1)\sigma_{zx} + (l_1 m_2 + l_2 m_1)\sigma_{xy}\end{aligned} \tag{III.13}$$

Using similar procedures

$$\begin{aligned}\sigma_{XZ} = \boldsymbol{\sigma}_X \mathbf{N}_3 &= l_1 l_3 \sigma_{xx} + m_1 m_3 \sigma_{yy} + n_1 n_3 \sigma_{zz} + (m_1 n_3 + m_3 n_1)\sigma_{yz} \\ &\quad + (l_1 n_3 + l_3 n_1)\sigma_{zx} + (l_1 m_3 + l_3 m_1)\sigma_{xy}\end{aligned} \tag{III.14}$$

$$\begin{aligned}\sigma_{YZ} = \boldsymbol{\sigma}_Y \mathbf{N}_3 &= l_2 l_3 \sigma_{xx} + m_2 m_3 \sigma_{yy} + n_2 n_3 \sigma_{zz} + (m_2 n_3 + m_3 n_2)\sigma_{yz} \\ &\quad + (l_2 n_3 + l_3 n_2)\sigma_{zx} + (l_2 m_3 + l_3 m_2)\sigma_{xy}\end{aligned} \tag{III.15}$$

There exist three mutually perpendicular planes at point O on which the shear strain vanish. The remaining normal strain components on these three planes are called principal planes. The strain vector on principal planes is given by

$$\boldsymbol{\sigma}_P = \sigma\,\mathbf{N} \tag{III.16}$$

where σ is the magnitude of the strain vector $\boldsymbol{\sigma}_P$ and $\mathbf{N}$ the unit normal to a principal plane. Projections of $\boldsymbol{\sigma}_P$ along x,y,z, axes are

$$\sigma_{P_x} = \sigma\, l$$

$$\sigma_{P_y} = \sigma\, m \tag{III.17}$$

$$\sigma_{P_z} = \sigma\, n$$

Hence by eq. (III.6) we obtain

$$l(\sigma_{xx} - \sigma) + m\sigma_{xy} + n\sigma_{xz} = 0$$

$$l\sigma_{xy} + m(\sigma_{yy} - \sigma) + n\sigma_{yz} = 0 \tag{III.18}$$

$$l\sigma_{xz} + m\sigma_{yz} + n(\sigma_{zz} - \sigma) = 0$$

For eqs. (III.18) to posses nontrivial solutions, the determinant of the coefficients of l, m and n must vanish. Thus

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma \end{vmatrix} = 0 \tag{III.19}$$

Expanding the determinant we get

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0 \tag{III.20}$$

where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{zz} - \sigma_{yy}\sigma_{zz} \tag{III.21}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2$$

I_1, I_2 and I_3 are invariants of strain and must be the same for all choices of coordinate axes x,y,z [Boresi et al. 1993].

APPENDIX IV: ROTATIONAL INVARIANCE IN 3D

IV.1 Derivation of Partials in the Rotated System

Consider two analysis of a network, the first using coordinate system (x, y, z) where

$$\Delta \mathbf{x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \qquad \text{(IV.1)}$$

then the strain matrix is

$$\mathbf{E}_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} & \frac{\partial u_i}{\partial z} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} & \frac{\partial v_i}{\partial z} \\ \frac{\partial w_i}{\partial x} & \frac{\partial w_i}{\partial y} & \frac{\partial w_i}{\partial z} \end{bmatrix}. \qquad \text{(IV.2)}$$

The second analysis uses coordinate system (x^*, y^*, z^*) which is rotated clockwise about Ox (along x axis) by an angle α and then subsequently rotated clockwise about Oy (along y axis) by an angle β next after the α and β rotations the axes are rotated clockwise about Oz (along z axis) by an angle γ. Thus

$$\Delta \mathbf{x}_i^* = \begin{bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{bmatrix} = \mathbf{R} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \qquad \text{(IV.3)}$$

and the strain matrix is

$$\mathbf{E}_i^* = \begin{bmatrix} \frac{\partial u_i^*}{\partial x^*} & \frac{\partial u_i^*}{\partial y^*} & \frac{\partial u_i^*}{\partial z^*} \\ \frac{\partial v_i^*}{\partial x^*} & \frac{\partial v_i^*}{\partial y^*} & \frac{\partial v_i^*}{\partial z^*} \\ \frac{\partial w_i^*}{\partial x^*} & \frac{\partial w_i^*}{\partial y^*} & \frac{\partial w_i^*}{\partial z^*} \end{bmatrix}. \tag{IV.4}$$

For simplicity subscript i is from now on going to be omitted. Where

$$\mathbf{R} = \begin{bmatrix} \mathrm{Cos}\beta\mathrm{Cos}\gamma & \mathrm{Cos}\beta\mathrm{Sin}\gamma & -\mathrm{Sin}\beta \\ \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma - \mathrm{Cos}\alpha\mathrm{Sin}\gamma & \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma + \mathrm{Cos}\alpha\mathrm{Cos}\gamma & \mathrm{Sin}\alpha\mathrm{Cos}\beta \\ \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma + \mathrm{Sin}\alpha\mathrm{Sin}\gamma & \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma - \mathrm{Sin}\alpha\mathrm{Cos}\gamma & \mathrm{Cos}\alpha\mathrm{Cos}\beta \end{bmatrix} \tag{IV.5}$$

The displacement vector in the second system is

$$\Delta\mathbf{x}^* = \mathbf{R}\,\Delta\mathbf{x} \tag{IV.6}$$

then we can write

$$\Delta\mathbf{x}^* = \begin{bmatrix} \mathrm{Cos}\beta\mathrm{Cos}\gamma u + \mathrm{Cos}\beta\mathrm{Sin}\gamma v - \mathrm{Sin}\beta w \\ (\mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma - \mathrm{Cos}\alpha\mathrm{Sin}\gamma)u + (\mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma + \mathrm{Cos}\alpha\mathrm{Cos}\gamma)v + \mathrm{Sin}\alpha\mathrm{Cos}\beta w \\ (\mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma + \mathrm{Sin}\alpha\mathrm{Sin}\gamma)u + (\mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma - \mathrm{Sin}\alpha\mathrm{Cos}\gamma)v + \mathrm{Cos}\alpha\mathrm{Cos}\beta w \end{bmatrix} \tag{IV.7}$$

Since the rotation matrices are orthogonal the displacement vector in the first system is

$$\Delta\mathbf{x} = \mathbf{R}^T\,\Delta\mathbf{x}^* \tag{IV.8}$$

where

$$\mathbf{R}^T = \begin{bmatrix} \mathrm{Cos}\beta\mathrm{Cos}\gamma & \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma - \mathrm{Cos}\alpha\mathrm{Sin}\gamma & \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma + \mathrm{Sin}\alpha\mathrm{Sin}\gamma \\ \mathrm{Cos}\beta\mathrm{Sin}\gamma & \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma + \mathrm{Cos}\alpha\mathrm{Cos}\gamma & \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma - \mathrm{Sin}\alpha\mathrm{Cos}\gamma \\ -\mathrm{Sin}\beta & \mathrm{Sin}\alpha\mathrm{Cos}\beta & \mathrm{Cos}\alpha\mathrm{Cos}\beta \end{bmatrix} \tag{IV.9}$$

Then we can write

$$\Delta\mathbf{x} = \begin{bmatrix} Cos\beta Cos\gamma\, u^* + (Sin\alpha Sin\beta Cos\gamma - Cos\alpha Sin\gamma)\, v^* + (Cos\alpha Sin\beta Cos\gamma + Sin\alpha Sin\gamma) w^* \\ Cos\beta Sin\gamma\, u^* + (Sin\alpha Sin\beta Sin\gamma + Cos\alpha Cos\gamma)\, v^* + (Cos\alpha Sin\beta Sin\gamma - Sin\alpha Cos\gamma)\, w^* \\ -Sin\beta\, u^* + Sin\alpha Cos\beta\, v^* + Cos\alpha Cos\beta\, w^* \end{bmatrix}$$

(IV.10)

The partials with respect to first coordinate system are

$$\frac{\partial u^*}{\partial u} = Cos\beta Cos\gamma \tag{IV.11}$$

$$\frac{\partial u^*}{\partial v} = Cos\beta Sin\gamma \tag{IV.12}$$

$$\frac{\partial u^*}{\partial w} = -Sin\beta \tag{IV.13}$$

$$\frac{\partial v^*}{\partial u} = Sin\alpha Sin\beta Cos\gamma - Cos\alpha Sin\gamma \tag{IV.14}$$

$$\frac{\partial v^*}{\partial v} = Sin\alpha Sin\beta Sin\gamma + Cos\alpha Cos\gamma \tag{IV.15}$$

$$\frac{\partial v^*}{\partial w} = Sin\alpha Cos\beta \tag{IV.16}$$

$$\frac{\partial w^*}{\partial u} = Cos\alpha Sin\beta Cos\gamma + Sin\alpha Sin\gamma \tag{IV.17}$$

$$\frac{\partial w^*}{\partial v} = Cos\alpha Sin\beta Sin\gamma - Sin\alpha Cos\gamma \tag{IV.18}$$

$$\frac{\partial w^*}{\partial w} = Cos\alpha Cos\beta \tag{IV.19}$$

The partials with respect to second coordinate system are

$$\frac{\partial x}{\partial x^*} = \mathrm{Cos}\beta\mathrm{Cos}\gamma \tag{IV.20}$$

$$\frac{\partial x}{\partial y^*} = \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma - \mathrm{Cos}\alpha\mathrm{Sin}\gamma \tag{IV.21}$$

$$\frac{\partial x}{\partial z^*} = \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Cos}\gamma + \mathrm{Sin}\alpha\mathrm{Sin}\gamma \tag{IV.22}$$

$$\frac{\partial y}{\partial x^*} = \mathrm{Cos}\beta\mathrm{Sin}\gamma \tag{IV.23}$$

$$\frac{\partial y}{\partial y^*} = \mathrm{Sin}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma + \mathrm{Cos}\alpha\mathrm{Cos}\gamma \tag{IV.24}$$

$$\frac{\partial y}{\partial z^*} = \mathrm{Cos}\alpha\mathrm{Sin}\beta\mathrm{Sin}\gamma - \mathrm{Sin}\alpha\mathrm{Cos}\gamma \tag{IV.25}$$

$$\frac{\partial z}{\partial x^*} = -\mathrm{Sin}\beta \tag{IV.26}$$

$$\frac{\partial z}{\partial y^*} = \mathrm{Sin}\alpha\mathrm{Cos}\beta \tag{IV.27}$$

$$\frac{\partial z}{\partial z^*} = \mathrm{Cos}\alpha\mathrm{Cos}\beta \tag{IV.28}$$

Using the chain rule for differentiation

$$\frac{\partial u^*}{\partial x} = \frac{\partial u^*}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial u^*}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u^*}{\partial w}\frac{\partial w}{\partial x} \tag{IV.29}$$

$$\frac{\partial u^*}{\partial y} = \frac{\partial u^*}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial u^*}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u^*}{\partial w}\frac{\partial w}{\partial y} \tag{IV.30}$$

$$\frac{\partial u^*}{\partial z} = \frac{\partial u^*}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial u^*}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u^*}{\partial w}\frac{\partial w}{\partial z} \tag{IV.31}$$

$$\frac{\partial v^*}{\partial x} = \frac{\partial v^*}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial v^*}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial v^*}{\partial w}\frac{\partial w}{\partial x} \quad \text{(IV.32)}$$

$$\frac{\partial v^*}{\partial y} = \frac{\partial v^*}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial v^*}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial v^*}{\partial w}\frac{\partial w}{\partial y} \quad \text{(IV.33)}$$

$$\frac{\partial v^*}{\partial z} = \frac{\partial v^*}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial v^*}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial v^*}{\partial w}\frac{\partial w}{\partial z} \quad \text{(IV.34)}$$

$$\frac{\partial w^*}{\partial x} = \frac{\partial w^*}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w^*}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w^*}{\partial w}\frac{\partial w}{\partial x} \quad \text{(IV.35)}$$

$$\frac{\partial w^*}{\partial y} = \frac{\partial w^*}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w^*}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial w^*}{\partial w}\frac{\partial w}{\partial y} \quad \text{(IV.36)}$$

$$\frac{\partial w^*}{\partial z} = \frac{\partial w^*}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w^*}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial w^*}{\partial w}\frac{\partial w}{\partial z} \quad \text{(IV.37)}$$

Using the chain rule for differentiation

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial u^*}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial u^*}{\partial y}\frac{\partial y}{\partial x^*} + \frac{\partial u^*}{\partial z}\frac{\partial z}{\partial x^*} \quad \text{(IV.38)}$$

$$\frac{\partial u^*}{\partial y^*} = \frac{\partial u^*}{\partial x}\frac{\partial x}{\partial y^*} + \frac{\partial u^*}{\partial y}\frac{\partial y}{\partial y^*} + \frac{\partial u^*}{\partial z}\frac{\partial z}{\partial y^*} \quad \text{(IV.39)}$$

$$\frac{\partial u^*}{\partial z^*} = \frac{\partial u^*}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial u^*}{\partial y}\frac{\partial y}{\partial z^*} + \frac{\partial u^*}{\partial z}\frac{\partial z}{\partial z^*} \quad \text{(IV.40)}$$

$$\frac{\partial v^*}{\partial x^*} = \frac{\partial v^*}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial v^*}{\partial y}\frac{\partial y}{\partial x^*} + \frac{\partial v^*}{\partial z}\frac{\partial z}{\partial x^*} \quad \text{(IV.41)}$$

$$\frac{\partial v^*}{\partial y^*} = \frac{\partial v^*}{\partial x}\frac{\partial x}{\partial y^*} + \frac{\partial v^*}{\partial y}\frac{\partial y}{\partial y^*} + \frac{\partial v^*}{\partial z}\frac{\partial z}{\partial y^*} \quad \text{(IV.42)}$$

$$\frac{\partial v^*}{\partial z^*} = \frac{\partial v^*}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial v^*}{\partial y}\frac{\partial y}{\partial z^*} + \frac{\partial v^*}{\partial z}\frac{\partial z}{\partial z^*} \tag{IV.43}$$

$$\frac{\partial w^*}{\partial x^*} = \frac{\partial w^*}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial w^*}{\partial y}\frac{\partial y}{\partial x^*} + \frac{\partial w^*}{\partial z}\frac{\partial z}{\partial x^*} \tag{IV.44}$$

$$\frac{\partial w^*}{\partial y^*} = \frac{\partial w^*}{\partial x}\frac{\partial x}{\partial y^*} + \frac{\partial w^*}{\partial y}\frac{\partial y}{\partial y^*} + \frac{\partial w^*}{\partial z}\frac{\partial z}{\partial y^*} \tag{IV.45}$$

$$\frac{\partial w^*}{\partial z^*} = \frac{\partial w^*}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial w^*}{\partial y}\frac{\partial y}{\partial z^*} + \frac{\partial w^*}{\partial z}\frac{\partial z}{\partial z^*} \tag{IV.46}$$

Using tensor analysis summation convention the eqs. from (IV.38) to (IV.46) can be expressed as

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*} \tag{IV.47}$$

$$\frac{\partial u^*}{\partial y^*} = \frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*} \tag{IV.48}$$

$$\frac{\partial u^*}{\partial z^*} = \frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*} \tag{IV.49}$$

$$\frac{\partial v^*}{\partial x^*} = \frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*} \tag{IV.50}$$

$$\frac{\partial v^*}{\partial y^*} = \frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*} \tag{IV.51}$$

$$\frac{\partial v^*}{\partial z^*} = \frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*} \tag{IV.52}$$

$$\frac{\partial w^*}{\partial x^*} = \frac{\partial w^*}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial x^*} \tag{IV.53}$$

$$\frac{\partial w^*}{\partial y^*} = \frac{\partial w^*}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^*} \tag{IV.54}$$

$$\frac{\partial w^*}{\partial z^*} = \frac{\partial w^*}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial z^*} \tag{IV.55}$$

Using tensor analysis summation convention, the eqs. from (IV.47) to (IV.55) can be expressed as

$$\frac{\partial u^{*r}}{\partial x^{*s}} = \frac{\partial u^{*r}}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial x^{*s}} \tag{IV.56}$$

where Greek indices imply summation over the values (1,2,3). r and s are the free indices. Indices' role can be found in Boresi and Chong [2000].

IV.2 Invariance of Dilation

Dilation is given in 3D as follows

$$\Sigma^* = \frac{1}{3}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} \right) \tag{IV.57}$$

From the eqs. (IV.38), (IV.42) and (IV.46), we write

$$\frac{\partial u^*}{\partial x^*} = (\mathrm{Cos}\beta \mathrm{Cos}\gamma \frac{\partial u}{\partial x} + \mathrm{Cos}\beta \mathrm{Sin}\gamma \frac{\partial v}{\partial x} - \mathrm{Sin}\beta \frac{\partial w}{\partial x})\mathrm{Cos}\beta \mathrm{Cos}\gamma$$
$$+ (\mathrm{Cos}\beta \mathrm{Cos}\gamma \frac{\partial u}{\partial y} + \mathrm{Cos}\beta \mathrm{Sin}\gamma \frac{\partial v}{\partial y} - \mathrm{Sin}\beta \frac{\partial w}{\partial y})\mathrm{Cos}\beta \mathrm{Sin}\gamma \qquad \text{(IV.58)}$$
$$+ (\mathrm{Cos}\beta \mathrm{Cos}\gamma \frac{\partial u}{\partial z} + \mathrm{Cos}\beta \mathrm{Sin}\gamma \frac{\partial v}{\partial z} - \mathrm{Sin}\beta \frac{\partial w}{\partial z})(-\mathrm{Sin}\beta)$$

and

$$\frac{\partial v^*}{\partial y^*} = ((\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma - \mathrm{Cos}\alpha \mathrm{Sin}\gamma)\frac{\partial u}{\partial x} + (\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma + \mathrm{Cos}\alpha \mathrm{Cos}\gamma)\frac{\partial v}{\partial x} + \mathrm{Sin}\alpha \mathrm{Cos}\beta \frac{\partial w}{\partial x})$$
$$(\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma - \mathrm{Cos}\alpha \mathrm{Sin}\gamma) + ((\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma - \mathrm{Cos}\alpha \mathrm{Sin}\gamma)\frac{\partial u}{\partial y} + (\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma +$$
$$\mathrm{Cos}\alpha \mathrm{Cos}\gamma)\frac{\partial v}{\partial y} + \mathrm{Sin}\alpha \mathrm{Cos}\beta \frac{\partial w}{\partial y})(\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma + \mathrm{Cos}\alpha \mathrm{Cos}\gamma) + ((\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma -$$
$$\mathrm{Cos}\alpha \mathrm{Sin}\gamma)\frac{\partial u}{\partial z} + (\mathrm{Sin}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma + \mathrm{Cos}\alpha \mathrm{Cos}\gamma)\frac{\partial v}{\partial z} + \mathrm{Sin}\alpha \mathrm{Cos}\beta \frac{\partial w}{\partial z})\mathrm{Sin}\alpha \mathrm{Cos}\beta$$
$$\text{(IV.59)}$$

and then

$$\frac{\partial w^*}{\partial z^*} = ((\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma + \mathrm{Sin}\alpha \mathrm{Sin}\gamma)\frac{\partial u}{\partial x} + (\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma - \mathrm{Sin}\alpha \mathrm{Cos}\gamma)\frac{\partial v}{\partial x} +$$
$$\mathrm{Cos}\alpha \mathrm{Cos}\beta \frac{\partial w}{\partial x})(\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma + \mathrm{Sin}\alpha \mathrm{Sin}\gamma) + ((\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma + \mathrm{Sin}\alpha \mathrm{Sin}\gamma)\frac{\partial u}{\partial y} +$$
$$(\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma - \mathrm{Sin}\alpha \mathrm{Cos}\gamma)\frac{\partial v}{\partial y} + \mathrm{Cos}\alpha \mathrm{Cos}\beta \frac{\partial w}{\partial y})(\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma - \mathrm{Sin}\alpha \mathrm{Cos}\gamma) +$$
$$((\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Cos}\gamma + \mathrm{Sin}\alpha \mathrm{Sin}\gamma)\frac{\partial u}{\partial z} + (\mathrm{Cos}\alpha \mathrm{Sin}\beta \mathrm{Sin}\gamma - \mathrm{Sin}\alpha \mathrm{Cos}\gamma)\frac{\partial v}{\partial z} + \mathrm{Cos}\alpha \mathrm{Cos}\beta \frac{\partial w}{\partial z})$$
$$\mathrm{Cos}\alpha \mathrm{Cos}\beta$$
$$\text{(IV.60)}$$

If $\frac{\partial u^*}{\partial x^*}$, $\frac{\partial v^*}{\partial y^*}$ and $\frac{\partial w^*}{\partial z^*}$ are replaced in eq. (IV.57) and the necessary algebraic manipulations are performed we see that

$$\Sigma^* = \frac{1}{3}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right) = \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \Sigma \tag{IV.61}$$

It means that Σ is invariant under a rotation of a coordinate system.

IV.3 Invariance of Differential Rotation

Since whether Ω is invariant in a rotation of a coordinate system in 3D is investigated, we require

$$\Omega = \Omega^* \tag{IV.62}$$

which is equal to

$$\omega_{xy}^2 + \omega_{xz}^2 + \omega_{yz}^2 = \omega_{xy}^{*2} + \omega_{xz}^{*2} + \omega_{yz}^{*2} \tag{IV.63}$$

where

$$\left(\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\right)^2 = \left(\frac{1}{2}\left(\frac{\partial u^*}{\partial y^*} - \frac{\partial v^*}{\partial x^*}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial u^*}{\partial z^*} - \frac{\partial w^*}{\partial x^*}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial v^*}{\partial z^*} - \frac{\partial w^*}{\partial y^*}\right)\right)^2 \tag{IV.64}$$

After eliminating the constants we can write

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial u^*}{\partial y^*} - \frac{\partial v^*}{\partial x^*}\right)^2 + \left(\frac{\partial u^*}{\partial z^*} - \frac{\partial w^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial z^*} - \frac{\partial w^*}{\partial y^*}\right)^2 \tag{IV.65}$$

If we open the parentheses on the right hand side, we get

$$= \frac{\partial u^*}{\partial y^*}\frac{\partial u^*}{\partial y^*} - 2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*} + \frac{\partial v^*}{\partial x^*}\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial z^*}\frac{\partial u^*}{\partial z^*} - 2\frac{\partial u^*}{\partial z^*}\frac{\partial w^*}{\partial x^*} + \frac{\partial w^*}{\partial x^*}\frac{\partial w^*}{\partial x^*} + \frac{\partial v^*}{\partial z^*}\frac{\partial v^*}{\partial z^*}$$

$$- 2\frac{\partial v^*}{\partial z^*}\frac{\partial w^*}{\partial y^*} + \frac{\partial w^*}{\partial y^*}\frac{\partial w^*}{\partial y^*} \qquad \text{(IV.66)}$$

If the eqs. from (IV.47) to (IV.55) are employed

$$\begin{aligned}
&= \frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*}\frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*} - 2\frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*}\frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*} + \frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*}\frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*} \\
&+ \frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*}\frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*} - 2\frac{\partial u^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*}\frac{\partial w^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*} + \frac{\partial w^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*}\frac{\partial w^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^*} \\
&+ \frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*}\frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*} - 2\frac{\partial v^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^*}\frac{\partial w^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*} + \frac{\partial w^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*}\frac{\partial w^*}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^*}
\end{aligned}$$

$$\text{(IV.67)}$$

which is equal to

$$\Omega^* = \left(\frac{\partial u}{\partial y}\right)^2 - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 - 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \left(\frac{\partial w}{\partial y}\right)^2 = \Omega$$

$$\text{(IV.68)}$$

Equation (IV.68) is evaluated using MAPLE software. Numerical results are given in section 5.4.

IV.4 Total Shear

It is postulated here that total shear Γ is invariant in a rotation of a 3D coordinate system in Euclidean space. So we postulate that

$$\Gamma - \Gamma^* = 0 \tag{IV.69}$$

From eq. (5.7) we know that

$$\Gamma = \sqrt{\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2} \tag{IV.70}$$

If we substitute eqs. (2.22), (2.23) and (2.24) in eq. (IV.70), we get

$$\Gamma^2 = \left(\sqrt{\tau_{xy}^2 + \upsilon_{xy}^2}\right)^2 + \left(\sqrt{\tau_{xz}^2 + \upsilon_{xz}^2}\right)^2 + \left(\sqrt{\tau_{yz}^2 + \upsilon_{yz}^2}\right)^2 \tag{IV.71}$$

which is equal to

$$\Gamma^2 = \tau_{xy}^2 + \upsilon_{xy}^2 + \tau_{xz}^2 + \upsilon_{xz}^2 + \tau_{yz}^2 + \upsilon_{yz}^2 \tag{IV.72}$$

If we substitute eqs. from (2.13) to (2.18) in eq. (IV.72), we obtain

$$\Gamma^2 = \left(\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right)^2$$
$$+ \left(\frac{1}{2}\left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)\right)^2 + \left(\frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right)^2 \tag{IV.73}$$

We know that an invariant remains invariant after it is multiplied by a constant so we can write

$$\Gamma^2 = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \tag{IV.74}$$

If we open the parentheses we get

$$\Gamma^2 = \left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial z} + \left(\frac{\partial w}{\partial z}\right)^2 +$$

$$\left(\frac{\partial u}{\partial z}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}+\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2-2\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}+\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}+\left(\frac{\partial w}{\partial y}\right)^2$$

(IV.75)

For total shear in the rotated system we write

$$\Gamma^* = \sqrt{\gamma_{xy}^{*2}+\gamma_{xz}^{*2}+\gamma_{yz}^{*2}} \tag{IV.76}$$

If we substitute eqs. (2.22), (2.23) and (2.24) in eq. (IV.76), we get

$$\Gamma^{*2} = \left(\sqrt{\tau_{xy}^{*2}+\upsilon_{xy}^{*2}}\right)^2+\left(\sqrt{\tau_{xz}^{*2}+\upsilon_{xz}^{*2}}\right)^2+\left(\sqrt{\tau_{yz}^{*2}+\upsilon_{yz}^{*2}}\right)^2 \tag{IV.77}$$

which is equal to

$$\Gamma^{*2} = \tau_{xy}^{*2}+\upsilon_{xy}^{*2}+\tau_{xz}^{*2}+\upsilon_{xz}^{*2}+\tau_{yz}^{*2}+\upsilon_{yz}^{*2} \tag{IV.78}$$

If we substitute eqs. from (2.13) to (2.18) in eq. (IV.78), we obtain

$$\Gamma^{*2} = \left(\frac{1}{2}\left(\frac{\partial u^*}{\partial x^*}-\frac{\partial v^*}{\partial y^*}\right)\right)^2+\left(\frac{1}{2}\left(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*}\right)\right)^2+\left(\frac{1}{2}\left(\frac{\partial u^*}{\partial x^*}-\frac{\partial w^*}{\partial z^*}\right)\right)^2+\left(\frac{1}{2}\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial x^*}\right)\right)^2$$
$$+\left(\frac{1}{2}\left(\frac{\partial v^*}{\partial y^*}-\frac{\partial w^*}{\partial z^*}\right)\right)^2+\left(\frac{1}{2}\left(\frac{\partial v^*}{\partial z^*}+\frac{\partial w^*}{\partial y^*}\right)\right)^2 \tag{IV.79}$$

Using the same reasoning about invariants we can write

$$\Gamma^{*2} = \left(\frac{\partial u^*}{\partial x^*}-\frac{\partial v^*}{\partial y^*}\right)^2+\left(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*}\right)^2+\left(\frac{\partial u^*}{\partial x^*}-\frac{\partial w^*}{\partial z^*}\right)^2+\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial x^*}\right)^2+\left(\frac{\partial v^*}{\partial y^*}-\frac{\partial w^*}{\partial z^*}\right)^2+\left(\frac{\partial v^*}{\partial z^*}+\frac{\partial w^*}{\partial y^*}\right)^2 \tag{IV.80}$$

If we open the parentheses we get

$$=\frac{\partial u^*}{\partial x^*}\frac{\partial u^*}{\partial x^*}-2\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*}+\frac{\partial v^*}{\partial y^*}\frac{\partial v^*}{\partial y^*}+\frac{\partial u^*}{\partial y^*}\frac{\partial u^*}{\partial y^*}+2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}+\frac{\partial v^*}{\partial x^*}\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial x^*}\frac{\partial u^*}{\partial x^*}-2\frac{\partial u^*}{\partial x^*}\frac{\partial w^*}{\partial z^*}$$

$$+\frac{\partial w^*}{\partial z^*}\frac{\partial w^*}{\partial z^*}+\frac{\partial u^*}{\partial z^*}\frac{\partial u^*}{\partial z^*}+2\frac{\partial u^*}{\partial z^*}\frac{\partial w^*}{\partial x^*}+\frac{\partial w^*}{\partial x^*}\frac{\partial w^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\frac{\partial v^*}{\partial y^*}-2\frac{\partial v^*}{\partial y^*}\frac{\partial w^*}{\partial z^*}+\frac{\partial w^*}{\partial z^*}\frac{\partial w^*}{\partial z^*}$$

$$+\frac{\partial v^*}{\partial z^*}\frac{\partial v^*}{\partial z^*}+2\frac{\partial v^*}{\partial z^*}\frac{\partial w^*}{\partial y^*}+\frac{\partial w^*}{\partial y^*}\frac{\partial w^*}{\partial y^*} \tag{IV.81}$$

An invariant must be invariant for any rotation changes therefore in the rotation matrix in (IV.5) only α is chosen as an arbitrary angle and β and γ are set to zero. Essentially this rotation change has also been worked out considering all three angles α, β, γ as variables. However this just makes the results more complicated. In fact choosing α as a variable and setting β and γ are to zero simplifies the following differentiations:

$$\frac{\partial u^*}{\partial u}=1 \tag{IV.82}$$

$$\frac{\partial u^*}{\partial v}=0 \tag{IV.83}$$

$$\frac{\partial u^*}{\partial w}=0 \tag{IV.84}$$

$$\frac{\partial v^*}{\partial u}=0 \tag{IV.85}$$

$$\frac{\partial v^*}{\partial v}=\mathrm{Cos}\alpha \tag{IV.86}$$

$$\frac{\partial v^*}{\partial w}=\mathrm{Sin}\alpha \tag{IV.87}$$

$$\frac{\partial w^*}{\partial u}=0 \tag{IV.88}$$

$$\frac{\partial w^*}{\partial v}=-\mathrm{Sin}\alpha \tag{IV.89}$$

$$\frac{\partial w^*}{\partial w} = \mathrm{Cos}\alpha \tag{IV.90}$$

and

$$\frac{\partial x}{\partial x^*} = 1 \tag{IV.91}$$

$$\frac{\partial x}{\partial y^*} = 0 \tag{IV.92}$$

$$\frac{\partial x}{\partial z^*} = 0 \tag{IV.93}$$

$$\frac{\partial y}{\partial x^*} = 0 \tag{IV.94}$$

$$\frac{\partial y}{\partial y^*} = \mathrm{Cos}\alpha \tag{IV.95}$$

$$\frac{\partial y}{\partial z^*} = -\mathrm{Sin}\alpha \tag{IV.96}$$

$$\frac{\partial z}{\partial x^*} = 0 \tag{IV.97}$$

$$\frac{\partial z}{\partial y^*} = \mathrm{Sin}\alpha \tag{IV.98}$$

$$\frac{\partial z}{\partial z^*} = \mathrm{Cos}\alpha \tag{IV.99}$$

Substituting these differentials in the differentials from (IV.29) to (IV.46) and using MAPLE software, eq. (IV.81) yields

$$\Gamma^{*^2} = 4\mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha\left(\frac{\partial w}{\partial y}\right)^2 + 4\mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha\left(\frac{\partial v}{\partial z}\right)^2 + 2\mathrm{Sin}^2\alpha\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\mathrm{Cos}^2\alpha\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}$$

$$+2\sin^2\alpha\cos^2\alpha\left(\frac{\partial w}{\partial z}\right)^2+2\sin^2\alpha\cos^2\alpha\left(\frac{\partial v}{\partial y}\right)^2-2\sin^2\alpha\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+2\cos^4\alpha\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}$$

$$-2\cos^2\alpha\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}+2\cos^2\alpha\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+2\cos^3\alpha\sin\alpha\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}+2\cos^3\alpha\sin\alpha\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}$$

$$+8\sin^2\alpha\cos^2\alpha\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}+2\sin^3\alpha\cos\alpha\frac{\partial w}{\partial z}\frac{\partial w}{\partial y}-2\cos^4\alpha\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-2\sin^2\alpha\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}$$

$$+2\left(\frac{\partial u}{\partial x}\right)^2+2\sin^4\alpha\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}-2\sin^4\alpha\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}+2\sin^2\alpha\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}-2\cos^2\alpha\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

$$+\cos^2\alpha\left(\frac{\partial w}{\partial x}\right)^2+\cos^2\alpha\left(\frac{\partial u}{\partial z}\right)^2+\sin^2\alpha\left(\frac{\partial w}{\partial x}\right)^2+2\cos^4\alpha\left(\frac{\partial v}{\partial y}\right)^2+2\sin^4\alpha\left(\frac{\partial w}{\partial z}\right)^2$$

$$+\cos^2\alpha\left(\frac{\partial u}{\partial y}\right)^2+\sin^2\alpha\left(\frac{\partial u}{\partial z}\right)^2+\cos^2\alpha\left(\frac{\partial v}{\partial x}\right)^2+2\cos^4\alpha\left(\frac{\partial w}{\partial z}\right)^2+2\sin^4\alpha\left(\frac{\partial v}{\partial y}\right)^2$$

$$+\sin^2\alpha\left(\frac{\partial u}{\partial y}\right)^2+\sin^2\alpha\left(\frac{\partial v}{\partial x}\right)^2+\cos^4\alpha\left(\frac{\partial v}{\partial z}\right)^2+\sin^4\alpha\left(\frac{\partial w}{\partial y}\right)^2+2\cos\alpha\sin^3\alpha\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}$$

$$-2\sin\alpha\cos^3\alpha\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}-2\sin\alpha\cos^3\alpha\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}-2\sin^3\alpha\cos\alpha\frac{\partial v}{\partial z}\frac{\partial v}{\partial y}-2\sin^3\alpha\cos\alpha\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}$$

$$+\cos^4\alpha\left(\frac{\partial w}{\partial y}\right)^2+\sin^4\alpha\left(\frac{\partial v}{\partial z}\right)^2 \qquad \text{(IV.100)}$$

If the necessary algebraic manipulations are performed eq. (IV.100) is simplified to

$$\Gamma^{*2}=2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\left(1+2\sin^2\alpha\cos^2\alpha\right)+$$

$$\left(\frac{\partial w}{\partial y}\right)^2\left(1+2\sin^2\alpha\cos^2\alpha\right)+\left(\frac{\partial v}{\partial y}\right)^2\left(2\sin^2\alpha\cos^2\alpha+2\cos^4\alpha+2\sin^4\alpha\right)+$$

$$\left(\frac{\partial w}{\partial z}\right)^2\left(2\sin^2\alpha\cos^2\alpha+2\cos^4\alpha+2\sin^4\alpha\right)+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}-2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-2\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}+$$

$$\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\left(2\cos^4\alpha+2\sin^4\alpha\right)-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\left(2\cos^4\alpha+2\sin^4\alpha\right)+\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\left(2\cos^3\alpha\sin\alpha-2\sin^3\alpha\cos\alpha\right)$$

$$+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\left(2\cos^3\alpha\sin\alpha-2\sin^3\alpha\cos\alpha\right)+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\left(2\sin^3\alpha\cos\alpha-2\sin\alpha\cos^3\alpha\right)+$$

$$\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\left(2\mathrm{Sin}^3\alpha\mathrm{Cos}\alpha - 2\mathrm{Sin}\alpha\mathrm{Cos}^3\alpha\right) + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\left(8\mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha\right) \qquad \text{(IV.101)}$$

So we see that

$$\Gamma^{*2} - \Gamma^2 = 2\left(\frac{\partial v}{\partial z}\right)^2 \mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha + 2\left(\frac{\partial w}{\partial y}\right)^2 \mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha + 2\left(\frac{\partial v}{\partial y}\right)^2 \left(\mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha + \mathrm{Cos}^4\alpha\right.$$

$$\left. + \mathrm{Sin}^4\alpha - 1\right) + 2\left(\frac{\partial w}{\partial z}\right)^2 \left(\mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha + \mathrm{Cos}^4\alpha + \mathrm{Sin}^4\alpha - 1\right) + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\left(\mathrm{Cos}^4\alpha + \mathrm{Sin}^4\alpha - 1\right)$$

$$- 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\left(\mathrm{Cos}^4\alpha + \mathrm{Sin}^4\alpha - 1\right) + 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\left(\mathrm{Cos}^3\alpha\mathrm{Sin}\alpha - \mathrm{Sin}^3\alpha\mathrm{Cos}\alpha\right) +$$

$$2\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\left(\mathrm{Cos}^3\alpha\mathrm{Sin}\alpha - \mathrm{Sin}^3\alpha\mathrm{Cos}\alpha\right) + 2\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\left(\mathrm{Sin}^3\alpha\mathrm{Cos}\alpha - \mathrm{Sin}\alpha\mathrm{Cos}^3\alpha\right) +$$

$$2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\left(\mathrm{Sin}^3\alpha\mathrm{Cos}\alpha - \mathrm{Sin}\alpha\mathrm{Cos}^3\alpha\right) + 8\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\mathrm{Sin}^2\alpha\mathrm{Cos}^2\alpha \qquad \text{(IV.102)}$$

If it can be shown that eq. (IV.102) is not equal to zero, that would prove that total shear is *not* invariant in a rotation of a coordinate system in Euclidean space. This is discussed in section 5.3.

IV.5 Commutativeness of Strain Matrix

The strain matrix can be decomposed into two as follows

$$\mathbf{E} = \frac{1}{2}(\mathbf{E} + \mathbf{E}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{E} - \mathbf{E}^{\mathrm{T}}) \qquad \text{(IV.103)}$$

$$\mathbf{E} = \mathbf{S} + \mathbf{A} \qquad \text{(IV.104)}$$

The strain matrix in the rotated system is computed as

$$\mathbf{E}^* = \mathbf{R}\ \mathbf{E}\ \mathbf{R}^T \tag{IV.105}$$

Substituting eq. (IV.104) in eq. (IV.105), we get

$$\mathbf{E}^* = \mathbf{R}\ (\mathbf{S} + \mathbf{A})\ \mathbf{R}^T \tag{IV.106}$$

$$\mathbf{E}^* = (\mathbf{R}\ \mathbf{S} + \mathbf{R}\ \mathbf{A})\ \mathbf{R}^T \tag{IV.107}$$

$$\mathbf{E}^* = \mathbf{R}\ \mathbf{S}\ \mathbf{R}^T + \mathbf{R}\ \mathbf{A}\ \mathbf{R}^T \tag{IV.108}$$

This is the equation which is given in eq. (5.17) and this proves that the transformation from first coordinate system to second coordinate system is commutative.

IV.6 Dilation in 2D and 3D

In section 2.2, dilation in 3D is given as

$$\sigma = \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \tag{IV.109}$$

If $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ are represented by σ_1, σ_2 and σ_3 respectively, and the dilation in 3D is shown by ${}^3\sigma$, we can write

$${}^3\sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{IV.110}$$

In fact this is the result produced by eq. (5.3) for the 3D case. For the 2D case, eq. (5.3) produces the following

$$^{2}\sigma = \frac{1}{2}(\sigma_1 + \sigma_2). \tag{IV.111}$$

If $^{2}\sigma$ is shown by σ we can write

$$\sigma = \frac{1}{2}(\sigma_1 + \sigma_2) \tag{IV.112}$$

$$2\sigma = \sigma_1 + \sigma_2 \tag{IV.113}$$

and $^{3}\sigma$ is shown by Σ, using eq. (IV.110) we can write

$$\Sigma = \frac{1}{3}(2\sigma + \sigma_3). \tag{IV.114}$$

This is the equation which is given in eq. (5.4).

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