HONORING THE ACADEMIC LIFE OF PETR VANICEK

Edited by
MARCELO SANTOS

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HONORING THE ACADEMIC LIFE OF PETR VANÍČEK

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Petr Vaníček, born in Suzice, Czech Republic in July 1935, received his B.Sc in 1959 (Geodetic Engineer—Czech Technical University, Prague), Honours in the highest category of the state professional examination in surveying in 1963, Ph.D. in 1968 (Mathematical Physics—Czechoslovak Academy of Science, Prague), and Dr.Sc. in 1993 from the Czech Academy of Science, Prague.

Petr joined the Department of Surveying Engineering at the University of New Brunswick (UNB) in Canada in 1971, after working in various institutions in Czechoslovakia, England, and Canada. In 1976, he became a full professor at UNB, and later Adjunct Professor in four University of Toronto departments: Survey Science, Geology, Physics, and Civil Engineering.

Numerous invitations as a visiting scientist and professor have included universities in Brazil, Germany, Sweden, the U.S.A., South Africa, Iran, and Mexico. More than 60 students have completed their studies and specialized in Geodesy at the M.Sc.E. and Ph.D. levels under his supervision. Dr. Vaníček is author of over 300 publications and co-author of several prestigious books on geodesy, including *Geodesy, the Concepts*, which became a textbook used around the world.

Dr. Vaníček has been the recipient of various prestigious awards such as the Senior Distinguished Scientist - Humboldt Foundation (FRG) award in 1989, and the J. Tuzo Wilson medal for outstanding contributions to Canadian geophysics in 1996, by the Canadian Geophysical Union. He is one of few to receive the U.S. Academy of Science/National Research Council "Visiting Senior Scientist Award." He actively collaborated with several national and international institutions. Among them, the International Association of Geodesy, in which he served as member and president of several Special Study Groups and Commissions, and as Editor-in-Chief of the journals *Manuscripta Geodaetica* and *Bulletin Géodésique*. He has served as president and past president of the Canadian Geophysical Union from 1987 to 1991. In 1999 he became an Honorary Research Professor at UNB. Dr. Vaníček officially retired from UNB in 1999, but he still carries on his research as Professor Emeritus (since 2001) and within the Geodetic Research Laboratory.

Introduction

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This publication honoring the academic life of Petr Vaníček is being published by the Department to mark the occasion when Petr became Professor Emeritus and his 31 plus years of contribution to the field of geodesy as a member of our Department. Over these years, Petr has influenced a whole generation of geodesists, either directly (as supervisor or as teacher) or indirectly (by means of his publications).

I had the opportunity to meet Petr in 1987, during a week-long course he gave at the University of São Paulo. A few years later I became his graduate student. I feel that I can speak on behalf of his graduate students. If I could highlight some of the lessons from the many we learned from him (and in my case, I still do as a colleague now) I would choose two of them. First, treat and teach geodesy as a scientific subject, with rigour. The second is to never play numerical tricks to solve a problem. Instead, we should understand the problem, respect the physics behind it to have it properly formulated, and after that apply the appropriate mathematical tools.

This publication is a rather modest collection of reminiscences, technical papers, and personal statements by contributors in countries other than just Canada – Australia, Brazil, Germany, Iran, Slovak Republic, Sweden, and the United States.

Some of the papers in this report highlight contributions by Petr to geodesy but not all of them. Petr has been involved in many research projects and has come up with many innovations, inventions, novel theories, and techniques. Here is a (still incomplete) list:

- Least-squares spectral analysis (LSSA)
- Diagrammatic approach to least-squares estimation
- Theory of motion of horizontal pendulums
- New technique for geoid determination
- Precise gravimetric geoid determination technique
- Inverse gravimetric problem

- Tidal corrections to geodetic quantities
- Sequential tidal analysis
- GPS orbit determination
- New method for determining vertical crustal movements
- Analysis of sea-level variations
- A technique for the determination of the sea-surface topography
- Horizontal crustal motion determination
- Four-dimensional positioning
- Theory of horizontal datum positioning
- Effect of geodetic datum misalignment
- Rigorous densification of geodetic networks
- Analysis and maintenance of geodetic networks
- Geometrical strength analysis of horizontal geodetic networks
- Systematic effects in levelling
- Correlation of levelled height differences
- GPS positioning
- GPS surveys validation and specification
- Continental slope foot determination
- New navigation algorithm
- Generalization of cartographic features
- Errors in boundaries
- Theory of orthometric height

I would like to thank our compilation team, Robert Tenzer and Wendy Wells, for their precious help in organizing the publication.

Congratulations, Petr!

Marcelo Santos
editor

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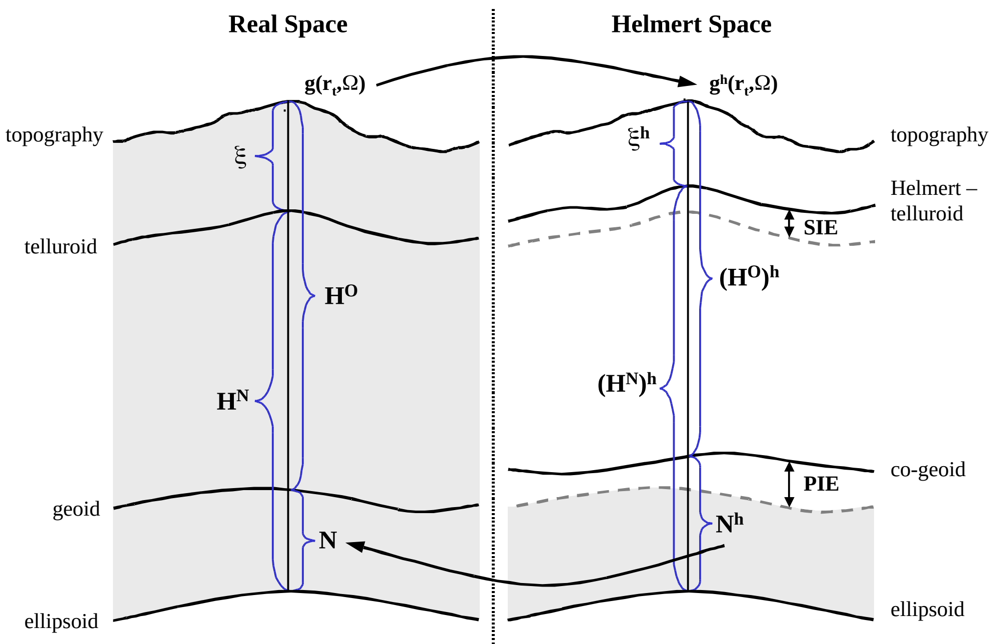


The Helmert-Stokes approach diagram.

International Union of Geodesy and Geophysics (IUGG)

XVIII General Assembly

Hamburg, FRG, August 15-27, 1983

PROCEEDINGS OF THE
INTERNATIONAL ASSOCIATION OF GEODESY (IAG)
SYMPOSIA

THE FUTURE OF GEODETIC NETWORKS

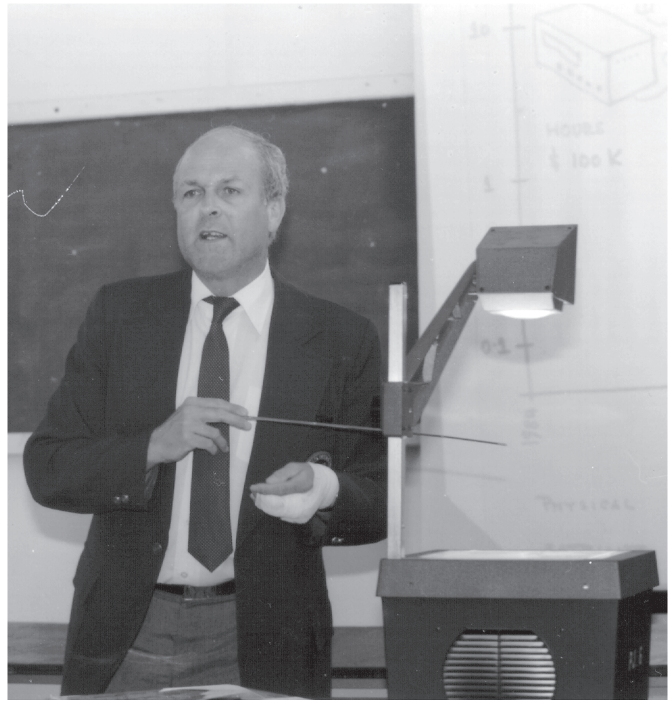
PETR VANÍČEK, DAVID E. WELLS, ADAM CHRZANOWSKI,
ANGUS C. HAMILTON, RICHARD B. LANGLEY,
JOHN D. MCLAUGHLIN, BRADFORD G. NICKERSON

In 1983, a prediction:
"… postulate the limit of this evolution: a cheap "wrist locator" giving instantaneous positions to an accuracy of 1 mm."
"One day, perhaps 100 years from now, the wrist locator will exist"

In 1977, Petr spent three weeks in Brazil lecturing at the Instituto Militar de Engenharia in Rio de Janeiro on positioning of geodetic datums, Earth tides, and satellite geodesy.

Left to right: Denizar Blitzkow, Gérard Lachapelle, Petr Vanícek, Ed Krakiwsky. Taken on the occasion of the Colloquium on Surveying and Mapping Education which was held at UNB on 12-14 June 1985. Photographer unknown.

In 1980, Petr addressed the second international symposium on problems related to the redefinition of North-American vertical geodetic networks, held in Ottawa.

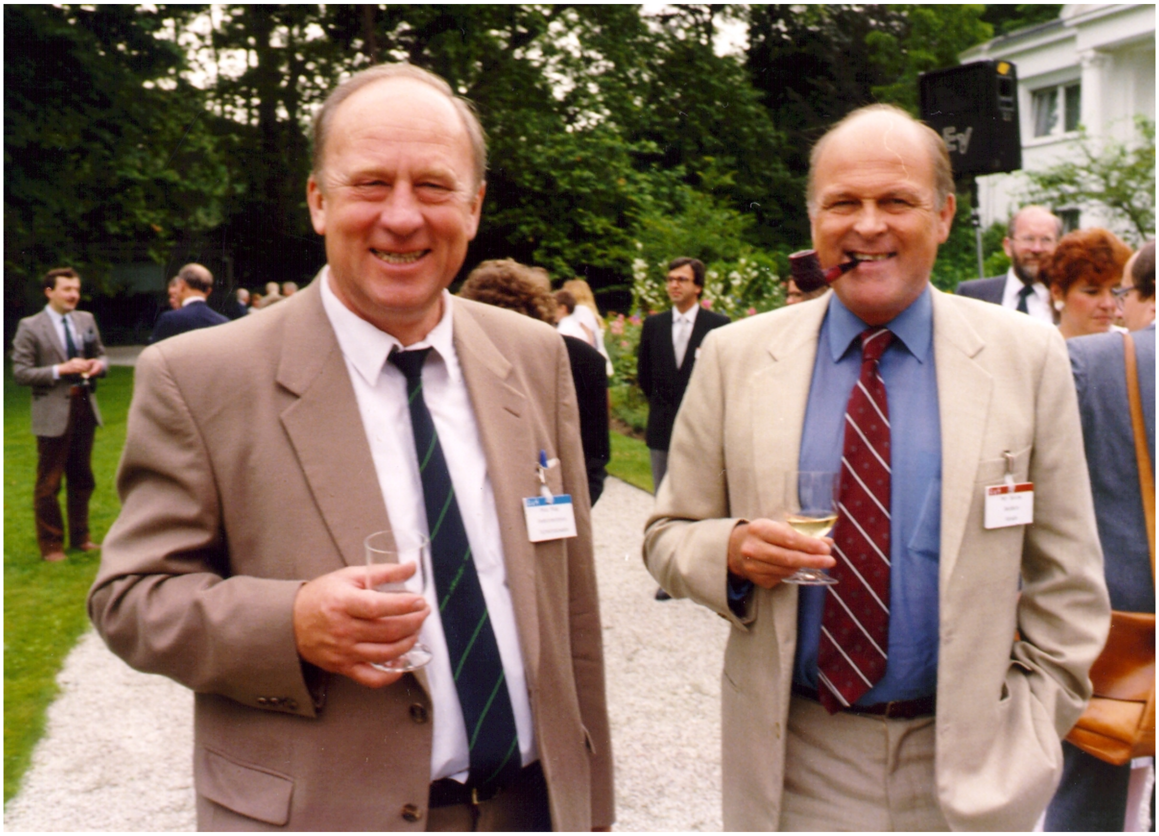


Petr toured South Africa in 1980. He visited the Department of Surveying and Mapping, University of Natal, Durban, where he lectured on levelling and crustal movements.

In 1984, Petr visited the Institute of Photogrammetry, National Cheng Kung University, Taiwan, and lectured on geodetic research in Canadian universities.

In 1986 Petr gave a three-week training course on geodetic networks and earth's gravity field at the University of Nairobi, Kenya.

Petr visited the Geodetic Institute of the University of Stuttgart in 1990. He lectured on robustness of geodetic networks.



In 1990, Petr, a recipient of the Humboldt Foundation Award, is shown here at the recipient's party.

Petr receiving a Dr.Sc. from the Czech Academy of Science, Prague, in 1994.

The Canadian Geophysical Union awarded the J. Tuzo Wilson medal to Petr in 1996 for outstanding contributions to Canadian geophysics.

In 1998, Petr received a plaque from fellow members of the Committee on Geodetic Aspects of the Law of the Sea in recognition of his contributions to GALOS.

April 1999, Petr arrives for his last class (above) to be greeted by balloons, a cake, and a plaque (below) presented by Michael Sutherland, President of the Geodesy and Geomatics Graduate Association.

Petr explained once more what geodesy is (in 4 million words or less) at the department's 40th Anniversary reunion workshop in 2001.

At the 17 May 2001 Encaenia ceremony the honorary rank of Professor Emeritus was conferred on Petr. To his right are N.B. Premier Bernard Lord and UNB President Liz Parr-Johnson.

Petr Vaníček: Colleague and Friend

[Adapted from notes prepared for Petr's retirement party in July 1999.]

Angus Hamilton
Professor Emeritus, Geodesy and Geomatics Engineering
University of New Brunswick, Fredericton, NB, E3B 5A3 Canada

They used to say, long ago, that old fire-horses missed the sound of the fire alarm. Well, old professors miss the chance to have a captive audience so I was very pleased when Dave [Wells] suggested that I could have a few minutes tonight to say something about Petr to Petr and to all of you. You will have to tolerate a bit of history.

It's not often that I get a chance to talk about my geodetic life. Yes, I did have one. In fact at one time I thought I was a geodesist. Please bear with me, I'm going to go back more than 40 years. In 1957, the IUGG – the *International Union of Geodesy and Geophysics* – met in Toronto and the Dominion Geodesist arranged that four of us from the Geodetic Survey of Canada could attend for one week and four for the other week. Up until that time the Dominion Geodesist was the only member of the Survey who had been allowed to go to an IUGG meeting. It was an eye-opener. Here was the *International Association of Geodesy* [one of the Associations in the IUGG] – meeting in Canada and there was not one research paper from Canada. There was just our National Report where we bragged about all the field work we had done. The technical/scientific presentations were dominated by Europeans with just a couple of American contributions.

Fast forward a decade. In 1968, at a meeting of the *Commission on Recent Crustal Movements* in Leningrad, a British scientist by the name of Lenon asked me if there were any opportunities in Canada for a young Czeck geodesist who had been working with him on earth tides. He explained that this young man would prefer to return to Czecko-Slovakia but that he was so unhappy with its communist government that he was looking for some place else to work until his homeland got free of its communist dictatorship. I noted the name and address and when I got back to my job – I then had the grand title of "Coordinator, Research and Training," in the Surveys and Mapping Branch of the Department of Energy, Mines and Resources [now Natural Resources Canada] and I was trying to break the Branch's image of being just a production shop for surveys and maps. I did the paper work to establish a Post-Doctorate Fellow position, the first ever in the Branch, and succeeded in getting Petr appointed to it. Then, early in 1971, both Petr and I were invited to join the Department of Surveying Engineering at the University of New Brunswick and to use that trite old phrase: "The rest is history!"

However, I want to relate one incident that happened a few months after Petr joined the S&M Branch. Without delay, Petr had set to work on a couple of papers and, from time to time would bring me a draft of one of them for discussion. Even if I couldn't help much with the technical content I assumed I could help with the English. One day I changed a word that I thought I knew the meaning of and Petr very courteously suggested I should look it up in the dictionary. I did, and, of course, I was wrong!

In the few minutes that I've been given I can't begin to even list Petr's accomplishments but I would be remiss if I didn't mention a few of them:

- 337 publications: Books, text books, lecture notes, refereed papers, research reports, invited papers, presented papers, reviews. What many of us consider to be his *magnum opus*, "Geodesy: The Concepts," 690 pages, published in 1982 with Ed Krakiwsky as co-author, reprinted several times, a revised edition in 1986 with translations into Chinese and Spanish, reprinted in China and Iran, paperback edition, etc. and etc.

- More than $2.25M dollars in research funding: Earth tides, crustal movements, redefinition of the North American datum, redefinition of the vertical reference system, geoid determination, time varying positions, applications of NAVSTAR, applications of GPS, sea surface topography, marine geodesy, ocean mapping, to name some of them.

- More than 35 graduate students have survived his stringent demands and earned their degrees. He has flooded the world with PhDs in geodesy, at least 15 of them!

- Editor. On the editorial boards of most of the prestigious geodetic journals in the world.

- Awards and honours: Ph D University of Prague in 1968, DSc Czeck Academy of Sciences in 1993. Visiting professor, visiting scientist, visiting research fellow, positions for which there is much competition at the University of Parana and at the University of Sao Paulo in Brazil, U.S. Academy of Science award at the National Geodetic Survey in Washington, at the University of Stuttgart, at the Royal Institute of Technology in Stockholm, at the South African Council of Scientific and Industrial Research, and a very prestigious one: the Humboldt Foundation award.

- Associations. Petr has been a member and a contributor to nearly all the commissions and the study groups in the IAG - the International Association of Geodesy. He was a founding member and has always been a very active member of the Canadian Geophysical Union. He has served on countless committees and was president 1987-89. His service was recognized in 1996 when he was presented with the Tuzo Wilson medal for outstanding contributions to Canadian geophysics.

I started with the depressing scene I witnessed at the 1957 IUGG meeting and though I haven't been to an IUGG meeting for many years I know that if I went I would no longer be embarrassed by the lack of Canadian input. I know that there would be a very respectable Canadian showing and I know that much of it would be the result of Petr's efforts to build a strong geodetic culture in Canada.

As a happily defrocked geodesist I have an excellent answer to the question, "What was your major contribution to geodesy in Canada?" My answer: "My major contribution to geodesy in Canada was in arranging for Petr Vaníček to come to Canada."

Petr Vaníček

Ed Krakiwsky
Professor Emeritus, University of Calgary
Calgary, Alberta, Canada

It is with great pleasure that I write this communication to help celebrate Petr's retirement and appointment as Professor Emeritus at UNB.

I was told that since we co-authored the book – *Geodesy: The Concepts* – I would have some inside information on Petr to share with our colleagues. We did spend some five years writing the book and during that period we put in several hundred hours toiling side-by-side and debating face-to-face. I got to know Petr very well indeed.

"Was Not Afraid of a Challenge"

Shortly after I joined UNB in 1968 (fresh from my doctoral studies at Ohio State University) Petr was interviewed at UNB for a professorship. During the interview, he witnessed that I was the only geodesist (idealistic, young and inexperienced) on the Department's faculty, while there were about six photogrammetry, and surveying and mapping personnel. I told him that we badly needed assistance in developing the geodetic science dimension of the Department and we were looking for a serious person to take on that challenge. Petr could clearly see the "geodetic vacuum" that existed at that time, but was not at all intimidated by it. He quickly accepted UNB's offer and challenge, and as they say, the rest is history.

Through years of consistent contributions on the teaching, research and professional service fronts, he was one of the key individuals who helped take UNB from its humble beginnings to one of the top universities in the world in our field. To this day, students from all over the world leave their families and jobs to come to UNB to study in the Department.

"Played a Key Role in Establishing Geodetic Research Funding for Canadian Universities"

Circa 1971, I recall Petr and I sitting in his office and discussing research funding for geodesy in Canada. After we did some complaining to each other, Petr said why don't we do something about it and help the Geodetic Survey, EMR, and the Gravity Survey, EPB, get some research funding from the central budget of the federal government in Ottawa. The EMR and EPB branches were the only organizations in Canada at that time overseeing and performing geodetic operations. At that time not a single dollar came to UNB for geodetic research from the federal government, however, the New Brunswick provincial government was beginning to fund control surveys applied research, which helped begin the establishment of the geodetic research culture at UNB.

Petr came up with the idea that we should get a national initiative going that would define geodetic research initiatives for all of Canada. During a year-long period, a series of meetings were held in Ottawa with our geodetic colleagues and shortly thereafter a conference was staged to define priority geodetic research initiatives. This led to the setting of federal budgets for geodetic research in Canada; a practice that persists to the present day.

UNB then began to receive moneys for geodetic research, the first of which was for Navy Navigation Satellite System (NNSS) Doppler Satellite Surveys, and shortly thereafter, moneys began to flow to UNB and other universities for research into the redefinition of geodetic networks, and geoid determination.

Clearly, Petr had a strong sense of duty to work hard at fixing a problem that was thwarting the development of geodesy in Canada, and thereby made a valuable contribution to establishing the first and continued funding for geodetic research in Canada. As a direct result, today, young researchers have a leg up in getting started in their teaching and research careers.

"Lecture Notes and More Lecture Notes, and then Two Books"

One of the best things a Professor can do for their students is to communicate and document his/her thoughts and knowledge so that students have material to examine, ponder over, and critically exam on their own time. Petr unselfishly wrote countless sets of lecture notes for his students, time that he could have spent on publishing more papers than he did in order to help bolster his annual merit increments. This meant spending many late nights poring over manuscripts. He even prepared sample problems for them to work on to test their knowledge. This helped them become accountable for the knowledge they were attempting to learn from him.

The question that always came to my mind was: How could one person write so many lecture notes on such a diverse range of topics, such as tensors, gravity field, data analysis, geoid determination, earth tides, and others (which I have already forgotten about)? The answer came one day when Petr shared with me that after his Dipl. Ing. degree he spent several years rethinking and restudying all of his mathematics and physics at Charles University, Prague. This sojourn culminated in his doctorate, which indeed served him well in publishing several hundred pages of such complete and sophisticated sets of lecture notes, over 100 scientific papers, and two text books.

"The Birth of Our Book"

It was circa 1974, when Petr and I decided to write *Geodesy: The Concepts*. That decision profoundly changed our lives, in that the priorities in our respective personal lives would be significantly affected, and furthermore, the two of us would be inextricably "welded together" for a five-year period. A day never passed when we weren't working on the book, whether talking together, writing in solitude, debating whether materials should be in or out, researching citations, or involving our students in test computations and, of course, proofreading the manuscript.

The beginnings of the book occurred in the summer of 1974, when I left Fredericton for a one-year sabbatical at the French Space Center, Toulouse. I thought that I would have one years worth of peace and sampling wines while in France, but not so. Petr called me in Toulouse from Canada and said that we better get started on the book and develop a detailed outline that we could follow while being separated. Before I knew it he flew into Toulouse and the next day we were off to Rayol Sur Mer on the Cote d'Azur to begin work. In Rayol, he moved in with his vacationing Aunt and Uncle (who was a Professor at the Sorbonne), and I got an apartment for my family.

Every morning during that week in August 1974, we would wade over to a tiny island and work on the outline for the book. Our families thought we were rather eccentric. We conceived the book to have several sections each corresponding to a main area of geodesy (e.g., positioning, gravity field, etc.), and each section would have several chapters devoted to its respective subjects (e.g., 1-D, 2-D, 3-D positioning), and finally the chapters would be continuously numbered from 1 to a total of some 27 (actual). This format was developed on the tiny island and then used as our road map. We only made minor deviations from the "Rayol Map" during the five years of writing.

"The Book: A Labour of Love"

More books are started than finished, and we both were cognizant of this fact. Petr, being the determined individual that he truly is, at no time ever indicated to me that he would relent or would entertain the idea of us quitting. So, we by definition (as he would say) were to complete the task and that was it. For five years he pulled more than his weight, and I am willing to concede that he lead us through very difficult circumstances to eventually complete the book, but of course not on time nor on budget; that would be asking too much. The contract with our publisher began in 1974 and called for a 350-page book to be completed in 3 years. The book ended up being about 700 pages and took 5 years to complete (6 years to publication date). The publishing house didn't even flinch, because they knew we were bleeding all the way and "producing in one book as much as any two authors could be expected to do" (a paraphrase from Prof. Erik Grafarend, Germany, in his review of our book).

I recall us discussing how we were spending many late nights (that was Petr) and many early mornings (that was me) writing the book, and finding that we were beginning to labour under the stress. Petr would work through the night and then come straight to the University; I would get up at 4AM, go to a truck stop for a three-egg breakfast, and then write up until my first lecture. Petr said we needed a change of venue and quickly got us separate NRC Exchange Scientist Visitations to Brazil. We spent several one-month stints in Brazil writing the book in the mornings and evenings and lecturing (for our keep) in the afternoons at the university in Curitiba. The visitations to Brazil were very productive periods in our writing, and we will always remember the meals of 50-shrimp per plate servings and black beans that gave us our sustenance for our 15-hour days.

Another innovation of Petr's was, that after each of us would write a given chapter from scratch, the other would critique it in detail. The golden rule was that this had to be done as quickly as possible. Petr would always finish his critique in record time (he would work through the night), setting an example for the younger author. This is the way Petr would keep us moving along the path of albeit, ever slowing progress as we began to labour towards the end of completing the series of 27 chapters.

Someone had to look after the figures and illustrations. Petr jumped in and would redraw every figure before it went to the Draftsperson. I watched him with continued amazement on how he could switch from formulating an equation to drafting a figure with artistic flare. Not once did he complain about the increasing workload he was taking on in order to get the book done.

All the effort proved to be worth it, for the book got very favorable reviews from several international geodesists. The then President Gerard Lachapelle of the Canadian Institute of Surveying gave the book a "Citation for Excellence". Petr spearheaded the writing of the second edition in 1986, a third reprint in paperback in 1996, and he is managing the planned third edition that is coming out sometime in 2003.

"The Melting Of the Weld"

The ultimate test for Petr's resilience came in 1979, four years into the writing of the book, when I accepted the Chairmanship of the new program of Geomatics Engineering at the University of Calgary. This meant that he would be "left holding the book in Fredericton", while I breezed on to greener pastures in Calgary. I still had four chapters of the book to write and so did Petr. My departure did not take Petr's focus away from completing the book; he established efficient communications with me in Calgary and carried the book to the finish line.

Looking back, the amazing thing is that Petr and I didn't even have a single argument that led to bad feelings. We, however, did have countless, incredibly intense debates about the material, correctness of equations and the like, but we always searched for the truth in what we put down on paper. What you see in the book is the absolute best that we could have done. Doing one's best is a trademark Petr has earned during his career.

The weld that kept us together while writing the book has only melted in the sense that the book had been completed. The weld is, however, still there.

"Petr: You Deserve to Go and Enjoy Yourself"

I will always remember Petr as an incredibly talented and unique individual, courageous and not afraid of any challenge, a prolific writer, a relentless worker, a person who wouldn't and couldn't tolerate fools, and a person who has contributed immensely to his university and profession. If his golf game can ascend to the level of his professional accomplishments, he will soon be playing in the PGA!

With Best Regards,
November 11, 2002

Statement of Appreciation

Khagendra Thapa
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Phone 231 591 2660

Thank you for providing me the opportunity to write about Prof. Vaníček. I was a graduate student at the University of New Brunswick from 1978/80. I thoroughly enjoyed my time at UNB. It is a great place for learning. Dr. Vaníček was my adviser and I did my research in a topic suggested by him.

It was a great pleasure working under Dr. Vaníček. He is a very kind, honest, and smart man. He encouraged me to get involved and participate in conferences. After I was done with my research and completed my degree, he helped me stay at the Geodetic Survey of Canada in Ottawa where I implemented the software developed by me. The software used strain tensor to detect inconsistent observations and constraints in Geodetic Networks.

We also published an article in *Manuscripta Geodetica* based on my research at UNB under Dr. Vaníček. I believe after I left UNB, a number of people did masters and Ph.D. degrees about the topic I started.

Prof. Vaníček has made a great difference in my life. He is one of the smartest individuals I ever met. I have a lot of respect for him. I wish him best of luck and happy, healthy, and peaceful retired life. However, I feel that he is the kind of man who will find it difficult to completely retire.

With best wishes
Khagendra Thapa

Ocean Tide Loading

Spiros Pagiatakis
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It was in the early 1980s when Petr introduced to me the exciting phenomenon of the ocean tide loading, that is, the periodic change of the size, shape and gravity field of the earth due to rising and falling of the ocean tides. At that time Geodesy was able to achieve 10^{-9} – 10^{-10} precision in baseline determination using Very Long Baseline Inteferometry (VLBI), while the then under development Global Positioning System (GPS) was promising similar repeatability for continental-scale baselines when in full constellation. Gravity field observations were no exception, as absolute and superconducting gravimetry could achieve something like 10^{-9} and 10^{-12} precision, respectively. Apparently, the ocean load effect that amounts to about the same level (i.e., 10^{-9} for both displacement and gravity) must be taken into consideration when positions and gravity observations need to be determined that precisely.

Initially, the task of evaluating the ocean load effects was relatively simple, as I had to modify existing elastic Green's functions (*Farrell*, 1972) and subsequently convolve them with an ocean tide model to numerically evaluate 3-D displacement, gravity and tilt. Exciting times then, as Schwiderski's global ocean tide model was starting to make its real debut in the scientific community. What was less exciting was the use of punch cards and card readers for the development of convolution software. To make things even more challenging, the IBM 3090-180 VF main frame system would take over one hour of CPU for a single station determination and the jobs had to run overnight in deferred mode… not particularly productive when trying to debug the program!

Petr was emphatically interested in applying the ocean load corrections primarily to VLBI observables and later on to tilt and gravity observations that the UNB tidal station had been producing. I, on the other hand, had some long term plans regarding the ocean tide load effect; modifying existing Green's functions on a simple elastic Earth model was not after all a really exciting proposition, yet it was the first step and Petr was very adamant about it… you crawl before you walk!

At that time, Richard Langley joined the Department and as a VLBI expert gave a good boost in the modification of the Canadian VLBI software package to accept ocean load corrections. Very encouraging results sprang out of this effort and a few papers (e.g., *Pagiatakis et al*., 1982; *Pagiatakis and Langley*, 1985; *Pagiatakis and Vaníček*, 1985) publicized the first interesting and promising results. Then NASA became interested in the convolution software for VLBI data processing and analysis. It was the time when my MSc thesis (*Pagiatakis*, 1983) saw the light of day after many revisions; surely, Petr knew (and knows) how to use heavily and effectively his red pen… not particularly pleasant for his pupils but certainly much appreciated later on!

Ocean load modelling became a real esoteric matter after 1983 as I started making the first steps towards the development of an ocean tide load model on a realistic Earth that included, among others, compressibility, anisotropy, viscoelasticity and rotation. Petr's graduate course *"Gravimetric Satellite Geodesy"* introduced the concepts of Lagrangean mechanics and proved to be the catalyst for the development of the new model along with the unforgettable Thursday night faculty club meetings where every concept, idea and equation was meticulously examined, tested and grilled before the next step was attempted.

The efforts for the development of a realistic ocean tide load model came to fruition in 1988 with the completion of my PhD thesis (*Pagiatakis*, 1998), which was also published in the Geophysical Journal International (*Pagiatakis*, 1990) and elsewhere (e.g., *Pagiatakis*, 1991). Petr definitely shares the success of the model as his supervision and guidance were effective and invaluable. In these latter publications the model is described in great detail and an excellent summary of the ocean tide load effect is given by *Jentzsch* (1997), who also gives appropriate credit to my work. In this contribution, I will emphasise a few aspects of the model that in my view characterise it uniquely.

Lateral (transverse) anisotropy in the Earth is an important feature of the model that was taken into consideration via the strain energy function introduced by Love in 1927. The compressional and shear seismic wave velocities of the Preliminary Reference Earth Model – PREM (*Dziewonski and Anderson*, 1981) were used to numerically evaluate this function and assess the effect of the anisotropy of the Earth on the load numbers. Results showed that lateral anisotropy in the Earth (upper layers up to 225 km in depth) may affect the load numbers by as much as 2.5 percent, depending on the load extent (*Pagiatakis*, 1990; Fig. 1).

The departure of the Earth's rheology from perfect elasticity was described by a standard–linear–type solid. The grain-boundary relaxation model for the dissipation mechanism within the Earth was adopted and the thermodynamic state of the Earth was accounted for, through its absolute temperature, Gibbs free activation energy, viscosity and Q profiles. The viscosity profile of the Earth was synthesised from various models. Characteristically, the following viscosities were accepted and used: 2.5_10^{21} Pa s for the lower mantle, 10^{21} Pa s for the transition zone and 10^{16} Pa s for the low velocity zone. Viscoelasticity introduces an increase in the absolute value of the load numbers. For semidiurnal tides and all degrees of expansion the load numbers were systematically larger than on an elastic Earth but never exceeded 0.2 percent. However, for fortnightly periods and degree of expansion n=100 I found that the load numbers increase by about one percent with respect to an elastic Earth.

Load numbers on a viscoelastic Earth are complex quantities that introduce a phase shift between the applied load and resulting effect. Results showed that complex Green's functions for the tangential load displacements exhibit the largest phase shift reaching nearly 6º. This is explained by the fact that energy dissipation in shear is dominant.

Calculation of load numbers within the Earth (as functions of depth) was a by-product of the study and showed that after a depth of about 1.2 times the wavelength of the load, load numbers approach asymptotically to zero. I came up with a rule-of-thumb, which states that the deformations induced by surface loads penetrate the Earth to a depth that is about twice the extent of the load. As a consequence, a load of $n \geq 500$ takes place only in the lithosphere (*Pagiatakis*, 1990; Fig. 3).

Rotation of the Earth was considered at semidiurnal frequencies only and the results indicated that load numbers can be affected by as much as three percent (3rd load number *l*_) (*Pagiatakis,* 1990; Fig. 2). I found a weak latitude dependence of the load numbers for n≤4.

Ocean loading effect calculations are achieved through convolution program LOADSDP (*Pagiatakis*, 1992) that is now in v. 5.0, and is used by other research groups as well (e.g., *Andersen et al*., 1993; *Sovers et al*., 1993; *Kouba et al*., 1994; *Andersen*, 1995; *Dragert and Hyndman*, 1995; *Mireault et al*., 1996; *Agnew*, 1996; *Yang et al*., 1996). Through the years there has been a considerable effort to improve the ocean load calculations by augmenting the global ocean tide models with local/regional ones, especially close to the Canadian coasts (*Pagiatakis*, 1992, *Lambert et al*., 1991; *Lambert et al*., 1998). Efforts to improve the model and LOADSDP software will continue as Geodesy consistently achieves higher accuracy in positioning and in gravity measurements.

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An "Outsider's" Perspective on Petr Vaníček's Approaches to Regional Geoid Determination

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ABSTRACT

This paper overviews Petr Vaníček's philosophies and approaches to regional geoid determination that have been published in the open literature over the last three decades. This includes the formulation and application of the generalised Stokes scheme, mitigation of the truncation error through a deterministic kernel modification, and the determination of appropriate boundary values, including downward continuation, on the geoid.

1. Introduction

Petr Vaníček's interest in geoid determination appears to have begun in the mid-1970s [e.g., Merry and Vaníček, 1973; 1974; Vaníček and Merry, 1974]. There was then a hiatus in his activity in this subject, only as measured by publications in the open literature, until the late-1980s (excepting Vaníček and John [1983]). Arguably, this marks the 'turning point' in Petr's interest in the determination of the geoid. It is when he first conceived what was later to be called the generalised Stokes scheme [Vaníček and Sjöberg, 1989; 1991], coupled with the use of a deterministically modified integration kernel [Vaníček and Kleusberg, 1987; Vaníček et al., 1987].

The generalised Stokes scheme with Petr's deterministically modified kernel has been used to compute regional gravimetric geoid models of Canada [Vaníček et al., 1987; 1990; 1995; Vaníček and Kleusberg, 1987], among other areas [e.g., Kadir et al., 1999; Featherstone et al., 2002 submitted]. This is a fundamentally different approach to the so-called remove-compute-restore technique, using the (unmodified) spherical Stokes kernel, which seems to have gained a wider acceptance. However, based on citations, most the users of the remove-compute-restore technique appear to be unaware of Petr's approach. I shall not try to speculate upon the reasons for this.

Over the past decade, Petr has turned his attention to the so-called Stokes-Helmert technique [e.g., Vaníček and Martinec, 1994, Vaníček et al., 1999], where terrestrial gravity anomalies are evaluated on the geoid boundary, essentially by using refinements to Helmert's second method of condensation that account for additional terms required to compute the 'centimetre' geoid. Importantly, this includes the downward continuation of gravity anomalies from the topographic surface to the (Helmert co-) geoid, followed by the computation of the corresponding primary indirect topographical effects.

Petr's approach to gravimetric geoid determination mirrors his general philosophy towards geodetic science in general. Essentially, he takes nothing published in the literature for granted and, instead, prefers to re-examine each problem from first principles. For some strange reason, this approach seems to have attracted criticism from some authors. Take the following example in favour of Petr's approach: the approximations used to derive the relevant formulae may have been valid at the time, but these cannot be assumed to be valid in perpetuity. This rationale is particularly valid in the geodetic sciences, where improved measurement techniques often render older (while permissible at the time of their use) approximations invalid when using modern data.

This paper attempts to present an "outsider's" view of Petr's approaches to regional gravimetric geoid determination. Admittedly, this is not a truly independent perspective because I have collaborated with Petr for several years. Nevertheless, I have tried to be as objective as possible. Our collaboration began in 1988 while I was a graduate student at the University of Oxford and Petr emailed me with some computer software. I first met Petr at the 1989 IAG General Assembly in Edinburgh, where he provided me with preprints of his papers on the generalised Stokes scheme and modified kernels [Vaníček and Sjöberg, 1989; 1991].

At Petr's invitation, I spent a two-month sabbatical at UNB in 1996, where we worked on the effect of three different modifications of Stokes's kernel in the combined solution for the geoid [Vaníček and Featherstone, 1998] and the determination of boundary values on the geoid [Vaníček et al., 1999]. In 1998, Petr visited Australia, funded by a prestigious C. Y. O'Connor Fellowship from Curtin University of Technology, when we worked on horizontal datum transformations [Featherstone and Vaníček, 1999] and the optimal degree of expansion of a global geopotential model and optimal integration radius in geoid determination; work that I have yet to write up for journal publication (sorry Petr!). Petr will again come to Australia in 2003, when we intend to work on height systems and vertical datums.

2. The Generalised Stokes Scheme

2.1 The reference spheroid (global geopotential model)

Petr's acceptance of the need to use low-frequency geoid information from a satellite-only global geopotential model is unquestioned. He has always correctly maintained that this is the best source of long-wavelength geoid information. Therefore, this forms the primary basis of his generalised Stokes scheme [Vaníček and Sjöberg, 1991; Vaníček et al., 1996;

Martinec and Vaníček, 1996], where the low-degree reference spheroid is used instead of the reference ellipsoid. To this day, Petr remains vehement about the spherical harmonic degree of satellite-only global geopotential model that should be used in this scheme.

The use of a satellite-only derived global geopotential model is advocated by Petr, and some other authors, for the simple reason that it is independent of the terrestrial gravity data that are use to compute a regional gravimetric geoid model. Recall that combined global geopotential models include terrestrial gravity data that are subsequently used again to compute the regional geoid model. Petr's objection to using a combined global geopotential model is because of the *unknown* correlations of errors between such a global model and the terrestrial gravity data. Instead, a satellite-only global geopotential model is used as the reference spheroid so that these correlations can correctly be assumed to be zero [cf. Najafi et al., 1999].

In addition, most satellite-only global geopotential models (excepting those derived from the current and planned dedicated satellite gravimetry missions) are computed to a spherical harmonic degree and order that is generally higher than the measurements allow for. For instance, satellite-only global geopotential models are limited in precision due to a combination of: the power-decay of the gravitational field with altitude; the inability to track complete satellite orbits using ground-based stations; atmospheric refraction; imprecise modelling of atmospheric drag, non-gravitational and third-body perturbations; and incomplete sampling of the global gravity field due to the limited number of satellite orbital inclinations available.

In his co-authored textbook [Vaníček and Krakiwsky, 1982; 1986], Petr gives a rationale for the maximum feasible degree and order of a satellite-only global geopotential model based on the ranging accuracy to and altitude of geodetic satellites, coupled with Kaula's [allegedly unpublished] rule-of-thumb. For terrestrially tracked geodetic satellites, this yields degree 20 [e.g., Vaníček et al., 1987; 1990; 1995; 1996], which can be confirmed by simple inspection of the signal-to-noise spectra of satellite-only global geopotential models. However, this cut-off value is likely to increase with the release of satellite-only global geopotential models based on the dedicated satellite gravity field missions, as argued by Featherstone [2002 submitted].

2.2 The GBVP for the generalised Stokes scheme

The above arguments led to Petr's unerring preference for the so-called generalised Stokes scheme for regional gravimetric geoid determination, where an adapted [Martinec and Vaníček, 1996; Vaníček et al., 1996] low-degree satellite-only global geopotential model is used as the reference spheroid, as opposed to the reference ellipsoid. The so-called spheroidal Stokes kernel [Vaníček and Kleusberg, 1987; Vaníček and Sjöberg, 1989; 1991] then becomes the kernel implicit in this scheme, which can be simply computed from the spherical

Stokes kernel by removing the low-degree Legendre polynomials up to an including the degree and order of the reference spheroid used. This is analogous with the Wong and Gore [1969] modification, but only under certain conditions [e.g., Featherstone, 2002 submitted].

The high-frequency (residual) terrestrial gravity anomalies are computed by subtracting the gravity anomalies implied by the same degrees and orders of the same satellite-only global geopotential used to define the reference spheroid. This is analogous with the remove-compute-restore technique, and is thus very easy to implement in practice. Therefore, it is curious that only a few investigators have chosen to implement, or at the very least trial, this technique. Another often-omitted consideration is the downward-continuation of the satellite-only geopotential coefficients through the topography to the geoid, which must be included [cf. Vaníček et al., 1996; Martinec and Vaníček, 1996].

Petr's earlier application of the generalised Stokes scheme appears to have assumed that Stokes's spherical solution of the geodetic boundary-value problem (GBVP) was appropriate for this application [e.g., Vaníček et al., 1986; 1990]. To validate this, albeit retrospectively, Martinec and Vaníček [1996] and Vaníček et al. [1996] returned to first principles and reformulated the GBVP for the reference spheroid so as to demonstrate its validity. This essentially completed Petr's work on the generalised Stokes scheme (only as measured by publications in the open literature).

Finally, another less well-acknowledged, though important, benefit of the generalised Stokes scheme is its implicit reduction of spherical approximation error associated with above formulations [cf. Vaníček and Sjöberg, 1991]. Quite recently, there has been renewed interest in the ellipsoidal correction to the spherical solution of the GBVP, with numerous papers recently published in or submitted to the *Journal of Geodesy*. However, their relation to the generalised Stokes scheme remains to be investigated fully, though the ellipsoidal corrections to the gravity anomaly [Vaníček et al., 1999] and the reference spheroid [Vaníček et al., 1996] may suffice.

3. Mitigation of the Truncation Error

When a regional gravimetric geoid model is computed using only a limited spatial coverage of terrestrial gravity anomaly data, the so-called truncation error results. The use of a low-degree (degree-20) satellite-only global geopotential model in the generalised Stokes scheme leads to a larger truncation error than the remove-compute-restore technique, which routinely uses the complete expansion (degree-360) of a combined global geopotential model. The penalty of the remove-compute-restore technique is that there is less control over the techniques that were used to compute the combined global geopotential model, as well as the less preferable filtering properties of the spherical Stokes kernel [cf. Vaníček and Featherstone, 1998]. However, Petr's approach includes explicit evaluation of a truncation bias term, which reduces the truncation error.

3.1 Petr's deterministic kernel modification

In the mid-1980s, Petr realised that the issue of the larger truncation error associated with the generalised Stokes scheme needed addressing through an appropriate modification to the spheroidal Stokes kernel. While several options for kernel modification were available by then, Petr chose to approach the problem largely from first principles and to use deterministic considerations. The rationale for using a deterministic modification, as opposed to stochastic (statistical) modification is that the variances the gravity field data are not accurately known at all frequencies. Importantly, Petr's formulation is for the generalised Stokes scheme, whereas previous kernel modifications were only for the spherical Stokes formula.

Vaníček and Kleusberg [1987; also see Vaníček and Sjöberg, 1989; 1991] adapted Molodensky's theory, which had been formulated for the truncated spherical Stokes formula, to the generalised Stokes scheme thus yielding a new deterministic kernel modification. This modified kernel has been used for all the gravimetric geoid models of Canada computed at UNB [Vaníček et al., 1987; 1990; 1995] and has recently been used elsewhere (described later). The primary motivation of this deterministic kernel modification is to reduce the upper bound of the truncation error [Vaníček and Sjöberg, 1991]. Featherstone et al. [1998] refined this approach to take advantage of an increased rate of convergence of the truncation error at the zero crossing points of Petr's kernel, which was used to compute the most recent Australian geoid model [Featherstone et al., 2001].

3.2 Separate evaluation of the truncation bias

Unlike many kernel modifications that aim to reduce the truncation error to the point at which it can be neglected (e.g., under the data-driven criteria at the time), Petr chose to explicitly compute a truncation bias (i.e., to further reduce the truncation error) using a combined global geopotential model [e.g., Vaníček et al., 1987; 1990; 1995]. Therefore, Petr's approach to regional geoid determination, unlike others, includes *three* primary contributions: those from the reference spheroid (satellite-only global geopotential model), terrestrial gravity anomalies in a spherical cap, and the explicit computation of the truncation bias. Novàk et al. [2000] give a self-consistent summary of this data combination.

Recalling Petr's objection to using a combined global geopotential model because of correlated errors, his deterministic kernel modification reduces the truncation bias (as well as the truncation error that must be neglected) in the generalised Stokes scheme such that the geoid solution is less sensitive to these correlations. Martinec [1993] demonstrated this empirically, while working with Petr at UNB, where the truncation bias need only be computed (from degree 21) to degree ~120. The same study showed that the difference between truncation biases computed from two different global geopotential models, while in their own right appearing to contribute less than 1 cm to the geoid for degrees greater than 120, differed by over 5 cm.

To summarise, the Vaníček and Kleusberg [1987] kernel modification (also see Vaníček and Sjöberg [1991]) aims to reduce the truncation error associated with the generalised Stokes scheme, but a component is also computed (herein called the truncation bias) as a separate contribution to the geoid. Novàk et al. [2000], as well as summarising the approach, vindicate the use of this scheme using a synthetic (i.e., simulated) regional gravity field model, which shows that these techniques, as well as the computer software used to implement them at UNB, can deliver a regional geoid model to an accuracy of slightly less than 1 cm (ignoring data errors).

4. Determination of Boundary Values on the Geoid

In the early-1990s, Petr turned his attention more to the determination of appropriate gravity anomalies at the geoid as the boundary values required for the proper solution of the GBVP by Stokesian methods. Before this time, many authors had relied upon the assumptions and linear approximations given by Moritz [1968], which relates more to Molodensky's theory. Indeed, many still continue to do so today, including me [Featherstone et al., 2001]. Petr again chose to re-examine this issue from first principles, and his work in this area appears to continue to date. Therefore, the review material below should not be interpreted as being complete.

4.1 The Stokes-Helmert technique

By way of a 'refresher', the proper solution of the GBVP by Stokes's method requires gravity anomalies on the boundary surface of the geoid. Moreover, these quantities should be harmonic (i.e., satisfy Laplace's equation), which requires mathematical removal of the topographic masses and their subsequent restoration on or below the geoid so as to preserve the Earth's mass. Petr elected to use Helmert's second method of condensation, where the topographic masses are removed and subsequently restored as a surface layer on the Helmert geoid to yield a harmonic gravity field outside the Helmert geoid [Vaníček and Martinec, 1994; Martinec et al., 1993]. This type of remove-restore technique should not be confused with the geoid remove-compute-restore technique described earlier.

Petr and colleagues have addressed the development of the so-called Stokes-Helmert technique in a series of papers [Martinec et al., 1993; Vaníček and Martinec, 1994; Martinec and Vaníček, 1994a; 1994b; Vaníček et al., 1996], which should be read in conjunction with one another so as to gain a full understanding of the problem, as well the refinements made over the years. A useful summary of these studies, with some additional theories and results, is given in Vaníček et al. [1999]. Essentially, the Stokes-Helmert approach uses terrestrial Helmert gravity anomalies at the Helmert co-geoid as input to the generalised Stokes scheme to compute the Helmert co-geoid, which must then be transformed to the geoid by a consistent application of the primary indirect topographical effect.

The direct topographical effect of the masses above the geoid is evaluated at the topographic surface and applied to the second-order free-air gravity anomalies Vaníček and Martinec [1994]. The equations for this are given in Vaníček and Kleusberg, [1987], Martinec et al. [1993], Vaníček and Martinec [1994] and Martinec et al. [1996]. The Vaníček and Kleusberg [1987] paper led to the 'famous' debate with Wang and Rapp [1990] on the role of the topographical correction in geoid studies. Martinec et al. [1993] show that their solution is equivalent to the Wang and Rapp [1990] solution only under the Pellinen assumption of a linear correlation between the gravity anomaly and the topographic height, which may not hold in all areas.

Vaníček and Martinec [1994] gave an embryonic formulation and solution of the geodetic boundary-value problem for the Stokes-Helmert scheme. Here, the direct topographical effect (DTE) was introduced, which requires the use of an anisotropic integration kernel that is a function of the topographic height. As such it is not suited to efficient FFT evaluation, but Martinec et al. [1996] show that this kernel must be used to avoid numerical instabilities associated with the planar Moritz kernel. Vaníček and Martinec [1994] also introduced the primary indirect topographical effect (PITE) and primary indirect topographical effect (SITE) for the Stokes-Helmert scheme.

Martinec and Vaníček [1994a] expand on the DTE by formulating it for a spherical Earth, instead of the widely used planar approximation. This formulation also allows for the inclusion of lateral density variations, which were later implemented by Martinec et al. [1996] and Huang et al. [2001]. Martinec and Vaníček [1994a] also introduced the concept of the spherical Bouguer shell (as opposed to the well-known Bouguer plate), which reduces to the planar formula. Martinec and Vaníček [1994b] then expanded on the PITE that is compatible with the above DTE (i.e., for a spherical model). This includes a terrain roughness term and is evaluated at the geoid.

Associated with the Stokes-Helmert technique is the role of a higher-than-second-degree reference spheroid for use in the generalised Stokes scheme (described earlier). This was described in Vaníček et al. [1996] and Martinec and Vaníček [1996] and effectively applies the DTE and PITE to the reference spheroid, as well as downward continuation, to the reference spheroid because it lies within the topographic masses. Ellipsoidal corrections are also considered here, in addition to the ellipsoidal corrections applied to the terrestrial gravity anomalies [Vaníček et al., 1999].

4.2 Downward continuation of Helmert gravity anomalies

Given that the Helmert gravity anomalies are defined on the topographic surface [cf. Martinec et al., 1993; Vaníček and Martinec, 1994], which is now becoming more widely acknowledged, it is necessary to downward-continue these to the geoid boundary before solution of the GBVP. However, downward continuation of any potential field data suffers from the restriction that it is very sensitive to the effects of high-frequency noise. Essentially, this can become amplified during the downward continuation process, and has to be reduced by some appropriate regularisation. This is because downward continuation is a non-linear problem [e.g., Vaníček et al., 1999].

Petr's work on downward continuation began in parallel with his work on the Stokes-Helmert technique [Vaníček and Martinec, 1994] and has continued [e.g., Sun and Vaníček, 1996; 1998; Vaníček et al., 1999]. First however, it is informative to review the debate generated by Wang and Rapp [1990]. This pointed out the difference between evaluating the terrain correction at the topographic surface [Vaníček and Kleusberg, 1987] versus Moritz's [1968] evaluation at the geoid. Under the so-called Pelinnen assumption of a linear correlation between the gravity anomaly and elevation, the Moritzian terrain correction implicitly includes an approximate downward continuation [Martinec et al., 1993].

Petr elected not to use the Moritzian approach because of this Pellinen assumption, which does not always hold due to topographic mass density variations. Instead, he has retained his evaluation of the DTE at the topographic surface, then downward-continues the mean Helmert anomalies to the Helmert co-geoid. The regularisation applied is through the use of 5 arc-minute mean Helmert anomalies [cf. Sun and Vaníček, 1998], though other regularisation techniques would have to be used for higher resolution grids.

Vaníček et al. [1999] review the above approaches, as well as including some refinements to the previously proposed theories, as follows. The secondary indirect topographical effect (SITE) is now applied at the topographic surface instead of on the geoid, as was the case in earlier papers. Two ellipsoidal correction terms are now considered; one to the Helmert gravity disturbance at the topography and the other for the spherical Stokes assumption. These ellipsoidal corrections can be applied iteratively using an *a priori* estimate of the Helmert disturbing potential, or included in the kernel (i.e., boundary operator).

Vaníček et al. [1999] also propose some new 'correction' terms, as follows. The correction for the orthometric height is applied to the computation of the free-air gravity anomaly so as to account for the geoid-ellipsoid separation in the second-order fee-air correction [Vaníček and Martinec, 1994]. Atmospheric effects are also considered in more detail than previously. A new term, called the condensed terrain effect (CTE) is introduced which effectively combines the topographical roughness term [Vaníček and Martinec, 1994] and the

DTE. If all the above terms are considered, then the best possible mean Helmert gravity anomalies at the geoid should ensue.

4.3 The role of topographical mass-density variations

The determination of the geoid by Stokes's theory (arguably as opposed to Molodensky's theory) requires the determination of equipotential surfaces inside the Earth's gravitating masses. However, the practical solution of geodetic boundary-value problems requires harmonic gravity field quantities (i.e., those satisfying Laplace's and not Poisson's equation). While the Stokes-Helmert technique generates a harmonic field that is less sensitive to lateral topographical density variations, these are still required to be accurate to approximately five-percent to realise a one-centimetre geoid [Vaníček and Martinec, 1994; also see Martinec et al., 1996].

Presumably due to the lack of topographical mass density data, most authors have - until relatively recently - neglected the role of such data in gravimetric geoid computations [cf. Tziavos and Featherstone, 2001]. As such, most should strictly be classified as quasi-geoid models. In the mid-1990s, Petr began to investigate the use of topographical mass density data on gravimetric geoid determination. In these studies, the test area of the Canadian Rocky Mountains was used, which gives a good estimate of how large these effects can be [Vaníček et al., 1996; Martinec et al., 1995; Huang et al., 2001]. Likewise, Martinec et al., [1996] choose Lake Superior to show that the lateral density contrast between water and rocks can cause an effect of ~10 cm on the geoid.

Huang et al. [2001] investigate the role of topographical mass density variations, derived from geological maps using a geographical information system, on the gravimetrically computed geoid. In this contribution, several new terms are introduced that can be seen as adaptations of the earlier DTE, PITE and SITE. The new terms are the so-called direct density effect (DDE), the primary indirect density effect (PIDE), and the secondary indirect density effect (SIDE). These are applied to the same points as their topographical counterparts. This shows that density variations must be considered to yield the centimetre geoid.

4.4 Spherical Bouguer gravity anomalies and terrain corrections

Following the introduction of the spherical equations for the DTE and the PITE [Martinec and Vaníček, 1994a; 1994b], Vaníček et al. [2001] and Novàk et al. [2001] revisit the role of the spherical Bouguer correction to the gravity anomaly. Bouguer corrections play a role in geoid determination through the DTE and PITE, as well as usually producing a smoother gravity anomaly field that can reduce the detrimental effect of aliasing during gravity interpolation and prediction. Vaníček et al. [2001] argue that the Bouguer anomaly has an unclear physical meaning, which may impact upon geophysical interpretations of this quantity. Novàk et al. [2001] subsequently introduce the spherical terrain correction to the above spherical Bouguer gravity anomaly, as well as including remote zone effects.

5. Practical Implementation of Petr's Techniques

As well as concentrating on developing appropriate theories for regional geoid determination, Petr and his colleagues have investigated the practical computation of regional geoid models of the whole or parts of Canada. Such an approach is always useful to empirically verify and validate any new theory, but of course is plagued by the problems of working with observational data. As such, Petr has concentrated on developing the theoretical basis that allows computation of a geoid one order of magnitude more precise than the Canadian data are likely to allow (i.e., 1 cm versus 1 dm).

5.1 Canadian geoid models computed at UNB

Notwithstanding his earlier studies [e.g., Merry and Vaníček, 1974; Vaníček and John, 1983], Petr's work on practically implementing his own techniques and theories began in the mid-1980s with the computation of regional gravimetric geoid models over Canada [Vaníček et al., 1987; Vaníček and Kleusberg, 1987]. To the best of my knowledge, this UNB87 geoid model was the first nation-wide gravimetric geoid model of Canada. This early study used the generalised Stokes technique in conjunction with the deterministically modified kernel (for degree 20), but omitted the complete Stokes-Helmert scheme, considerations of proper downward continuation, and the many other 'correction' terms derived since [e.g., Vaníček et al., 1999].

Petr's subsequent geoid models of Canada [Vaníček et al., 1990] or parts thereof [Vaníč ek et al., 1995] have sequentially added Petr's refinements to the determination of boundary values, as well as refinements to his suites of computer software (described later). These geoid models still use the degree-20 generalised Stokes scheme and the same deterministically modified kernel, but use different global geopotential models and terrestrial datasets for their evaluation. Due to time constraints, the UNB95 geoid model [Vaníček et al., 1995] only covered a limited area of the Canadian Rocky Mountains. I assume that Petr's work on the practical computation of the Canadian geoid is ongoing.

5.2 Australian geoid models computed using the generalised Stokes scheme

Featherstone et al. [2002 submitted] have implemented the generalised Stokes scheme [Vaníček and Sjöberg, 1991] over Australia, in conjunction with the Vaníček and Kleusberg [1987] deterministically modified kernel and separate computation of the truncation bias term [cf. Novàk et al., 2000]. This uses the degree-20 expansion of the EGM96S satellite-only global geopotential model [Lemoine et al., 1998] as the reference spheroid and the

EGM96 combined global geopotential model [*ibid*.] to compute the truncation bias term to degree and order 120 [cf. Martinec, 1993].

However, this study did not use Petr's approaches for the determination of boundary values at the geoid [cf. Vaníček et al., 1999]. Instead, it used the same 2' by 2' grid of mean Faye gravity anomalies (as approximations of mean Helmert anomalies under the so-called Pellinen assumption; Martinec et al. [1993]) that were also used for AUSGeoid98 [Featherstone et al., 2001]. This approach is probably permitted over Australia when considering that no nation-wide density model is available, and given the smaller and generally smooth topographic elevations in Australia (maximum orthometric height ~2228m). The Stokesian integration was performed over a spherical cap of six degrees radius from the computation point using the one-dimensional FFT technique [cf. Novàk et al., 2000].

A slight variation of the 'standard' remove-compute-restore technique was also used so as to give a comparison of the two techniques using the same terrestrial data. Here, the degree-360 EGM96 combined global geopotential model was used and no truncation bias term computed. Instead, the truncation error had to be assumed negligible beyond degree 360, which may or may not be a valid assumption. To be consistent with the UNB approach, the (unmodified) Stokesian integration in the remove-compute-restore technique was also performed over a six-degree cap using the one-dimensional FFT technique.

Table 1 gives a summary of the fit of the various geoid models to 1013 GPS-levelling points in Australia. Firstly, it is acknowledged that such comparisons are equivocal because of the error budget associated with the GPS and levelling data used. The GPS data come from a variety of vintages, and their accuracy can thus be questioned. The Australian Height Datum (AHD) is now widely acknowledged to contain distortions of over 1 m. In addition, the AHD uses an ambiguously defined normal-orthometric height system that is strictly incompatible with gravimetric (quasi-)geoid heights [e.g., Featherstone, 1998].

Table 1. Descriptive statistics of the differences between the 1013 GPS-AHD heights and (quasi-) geoid heights computed from various techniques (units in metres).

(quasi-)geoid model	*max.*	*min.*	*mean*	*st.dev*
EGM96S to degree 20	6.970	-6.513	1.530	2.505
AUSGeoid_UNB	3.580	-2.932	0.094	0.495
EGM96 to degree 360	5.524	-3.018	0.251	0.441
AUSGeoid_RCR	3.580	-2.932	0.094	0.791
AUSGeoid98 (Featherstone et al., 2001)	3.558	-2.572	-0.002	0.314

From Table 1, it is interesting to observe that EGM96 to degree 360 gives a better fit to the GPS-AHD data than Petr's approach. Given the error budget of the various data sources, this difference is not statistically significant, but there are numerous reasons that could account for this observation, notably systematic errors in the Australian gravity anomalies. Nevertheless, Petr's technique does significantly improve upon EGM96S. Conversely, the remove-compute restore (RCR) technique (though with a six-degree cap) does not improve upon EGM96. Again, this is probably due to systematic errors in the Australian data.

Most importantly, the UNB technique clearly improves upon the RCR technique in Australia. The results for AUSGeoid98 (Table 1) are the best, but this is because further optimisation of the integration kernel [Featherstone et al., 1998] and cap radius (one-degree) was performed in relation to a majority of the GPS-AHD data [Featherstone et al., 2001]. Accordingly, such a comparison of this with the UNB and RCR techniques is unfair and should not necessarily be interpreted as an improved technique. Further work is needed to optimise the integration radius and degree of global geopotential model, which is the work that I am yet to write up from Petr's visit to Perth in 1998.

6. Petr's Geoid-related Studies

In addition to the numerous theoretical derivations and practical computations of regional geoid models, Petr's geoid-related studies have extended to the – mainly geophysical – applications of the geoid and improvements in numerical integration.

6.1 Numerical integration

The Canadian geoid models computed at UNB have routinely used quadrature-based numerical integration of the generalised Stokes formula [Vaníček et al., 1987; 1990; 1995; Vaníček and Kleusberg, 1987]. Due to the limited computer power available at those times, the integration domain was divided among different element sizes, with small as possible elements close to the computation point, where the generalised Stokes kernel varies rapidly, and increasingly larger elements towards the boundary of the integration cap. Nevertheless, the computational burden remained large.

Petr has always appeared to be sceptical about the fast Fourier transform (FFT) technique to regional geoid computation, presumably because of his background in spectral analysis, preferring to continue using quadrature-based numerical integration. The primary argument used by most authors in favour of the FFT technique is computational speed. However, this may be at the overriding expense of numerical accuracy. It is interesting to note that the evolution of the FFT technique in geoid determination has partly reverted back to quadrature-based numerical integration, where the 1D-FFT [Haagmans et al., 1993] is used only along the parallels where it is exact and numerical integration is used elsewhere.

Assuming that computation time was the only factor in the widespread preference for the FFT, Huang et al. [2001] implemented some numerical 'tricks' in the UNB quadrature-based numerical integration that exploit symmetries in the isotropic Stokesian kernel to make this technique nearly as fast as the FFT. Importantly, this gives an exact result and does not rely on any kernel approximation, as was the case with the earlier FFT techniques. As such, arguments based solely on computational efficiency should not be used to choose the geoid computation technique; accuracy should always be the overriding factor.

6.2 Geo-scientific applications of the geoid

Petr's interest in the geophysical applications of the geoid began (as measured only by publications) in the late-1980s [Christou et al., 1989]. This used the image-processed UNB87 geoid model to reveal geological structures over Canada that correlate well with previously mapped structures. Petr then co-edited a book on the *Geoid and its Geophysical Interpretations* [Vaníček and Christou, 1994]. This involved invited contributions from several authors and, as well as covering some of the theory of the determination of the geoid, looked to its applications in the Earth sciences. Most interestingly, the chapter on regional gravimetric geoid computation did not include a complete description of Petr's techniques.

With on of his ex-graduate students, Peter Vajda, Petr has also investigated the inversion of the geoid to deduce point-mass distributions inside the Earth [Vajda and Vaníček, 1997; 1998a; 1998b; 1999a; 1999b]. These studies recognise that a regional geoid results from a truncated integration, and thus treats the inversion accordingly. This work contributes to IAG special study group 3.177 on *Synthetic Modelling of the Earth's Gravity Field*, of which Petr is a full member. This group aims to produce a simulated Earth gravity field for, among other things, the validation of geoid computation theories and has been used to trail Petr's techniques [cf. Novàk et al., 2000].

7. Summary and Concluding Remarks

This paper has attempted to give an "outsider's" perspective of Petr Vaníček's approaches to regional gravimetric geoid determination. It is acknowledged that is not a truly independent or external review, because Petr and I have collaborated and jointly published our research (see the reference list). Nevertheless, I have tried to be as objective as possible.

Petr's work in the mid-1980s on what was later to be called the generalised Stokes scheme and deterministically modified kernels is still used today, and is now being trialled in other parts of the world with some quite promising results. His work from the early 1990s to present has focussed on the determination of appropriate boundary values at the geoid via the so-called Stokes-Helmert technique. These techniques have been applied to a series of geoid models for Canada, and are now gaining acceptance in some other parts of the world.

The question then arises as to what else is there for Petr to work on concerning the geoid? The answer to this should be simple: enjoy his retirement! However, I suspect that Petr's interest in the geoid, and in geodesy in general, will not wane with something so trivial as retirement. The repetition of 'this is beyond the scope of the current paper' - or similar - is probably testament to this.

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Least-Squares Spectral Analysis – LSSA

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ABSTRACT

To realize the full potential of increasingly more accurate measurements, scientists are now faced with the task of modelling ever smaller effects on their observations to improve their results. The problem, however, is that there is often little understanding of the cause and effect relation between these so-called systematic effects and the measurements. Spectra and autocorrelation functions can be used to help diagnose and improve the modelling of these systematic effects in measurements. However, standard techniques for computing spectra and autocorrelation functions require the data to be evenly spaced, which is often not satisfied in practice.

The approach taken here is to develop a general technique for determining autocorrelation functions for data which are unevenly spaced. This is an indirect method whereby the systematic effects, represented by the residuals from an incomplete a priori deterministic model, are transformed into a power spectrum and then into an autocorrelation function. To accommodate unevenly spaced data, a general least squares transform and its inverse are developed. The inverse transform is used to obtain the autocorrelation function from the least squares spectrum originally developed by Vaníček [1971]. This formulation can accommodate unequally spaced data, random observation errors, arbitrary frequency selection, arbitrarily weighted and correlated observations, as well as the presence of any a priori deterministic model. The conventional Fourier transform and spectrum are shown to be just special cases of this more general least squares formulation. It is also shown how the individual spectral components in the least squares spectrum and inverse transform can be estimated either independently of or simultaneously with each other.

The advantages and limitations of the least squares transforms and spectra are illustrated through tests with simulated data. The technique of using autocorrelation functions to model systematic effects is also illustrated with two real applications; one based on the precise measurement of the extension of a baseline spanning the San Andreas fault in California, and another based on the measurement of ellipsoidal heights using a GPS receiver under the influence of the effects of Selective Availability. These tests show that the use of fully populated weight matrices generally results in an increase in the value of the standard deviations of the estimated model parameters, thereby providing more realistic estimates of the uncertainties. On the other hand, the effect of correlations among the observations on the least squares estimates of model parameters was found not to be very significant.

Fashionable spectral estimation techniques have been related almost entirely to fast Fourier transform (FFT) algorithms for the determination of the power spectrum. The FFT approach is computationally efficient and produces reasonable results for a large class of signal processes (Kay and Marple, 1981). However, FFT methods are not a panacea in spectral analysis. There are many inherent limitations, the most prominent being the requirement that the data be equally spaced and equally weighted (e.g., Press et al., 1992). Pre-processing of the data is inevitable in these cases, it is unsatisfactory and it performs poorly (Press and Teukolsky, 1988).

Least-squares spectral analysis (LSSA) was developed by Vaníček (1969; 1971) as an alternative to the classical Fourier methods. LSSA bypasses all inherent limitations of Fourier techniques, such as the requirement that the data be equally spaced, equally weighted, with no gaps and datum shifts and provides the following advantages: ***a)*** systematic noise (coloured or other) can be rigorously suppressed without producing any shift of the existing spectral peaks (Taylor and Hamilton, 1972), ***b)*** time series with unequally spaced values can be analysed without pre-processing (Maul and Yanaway, 1978; Press et al., 1992), ***c)*** time series with an associated covariance matrix can be analysed (Steeves, 1981a); and ***d)*** statistical testing on the significance of spectral peaks can be performed (Steeves, 1981a; Pagiatakis, 1999).

Promising and powerful as it may sound, least-squares spectral analysis has received relatively little attention. It has been applied successfully in its original (Vaníček, 1971) or alternate forms, by a number of researchers in the field of observational astronomy (e.g., Lomb, 1976; Scargle, 1982; Press et al., 1992), or in the field of geodetic science (e.g., Maul and Yanaway, 1978; Merry and Vaníček, 1981; Steeves, 1981b; Delikaraoglou, 1984; Pagiatakis and Vaníček, 1986; Pagiatakis, 2000). Alternate forms of LSSA developed by Lomb (1976) and later by Scargle (1982) were based on Vaníček's original formulation. However, they did not appropriately recognise the original source and now many researchers refer to Lomb's and Scargle's algorithms. That is so unfortunate!

The first time I came across LSSA was in 1981 when I was working as a graduate student on a term paper for the *"Approximation and Time Series"* course given by Petr. My topic dealt with the various spectral analysis techniques including LSSA. I'm not sure if I ever understood the method then as I could hardly follow Petr's original papers that used intricate functional analysis theory. I remember that I strategically tried to avoid the details and present just the basics, which of course did not work very well! I couldn't tell exactly the difference between FFT and LSSA. Nevertheless, that was my initiation to the method which later on I used for the analysis of tidal records. Petr's original code had already been pepped up into powerful software (Wells and Vaníček, 1978). I call that version v. 1.0 as it was the first to be documented and become available to users. A newer version, v. 2.0, was produced in 1995 (Wells et al., 1995).

My involvement with the method became more intimate as the years went by and I dared to modify the software even further to include new base functions (other than linear trend and datum shifts) that led to v. 3.0. V. 4.0 included a few statistical tests on the residual series and the significance of peaks. By that time I had distributed the software to a few eager users who gave me invaluable feed back for further improvements. Meanwhile, based on Robin's findings (Steeves, 1981a), I worked to further develop the statistical tests on the significance of peaks, work that culminated in 1999 with a publication in the *Journal of Geodesy* (Pagiatakis, 1999). The findings of that research were immediately implemented into the software along with other changes (autoregressive process, random walk, testing of the significance of the estimated parameters, etc.) thus, reaching v. 5.0. This version included the calculation of power spectral density directly from the least-squares spectrum, something analogous to the result from FFT algorithm. This least-squares power spectral density (LS-PSD) comes in two different forms: LS-PSD in units of decibel (dB) and in units of variance normalized by the frequency. Both forms can be used effectively to evaluate the noise characteristics of the residual series at any frequency band and of course to communicate the LSSA results to traditional FFTers who come second as special cases of the LS-PSD, when the series are equally spaced!

About the same time period, Michael Craymer was considering spectra and autocorrelation functions to diagnose and improve the modelling of the systematic effects in measurements. Once again the standard techniques for computing spectra (cf. FFT) and autocorrelation functions were inadequate for most problems, as evenly-spaced data are generally required. Resorting to the general idea of the least-squares spectral analysis, Michael developed an indirect method whereby the systematic effects in geodetic and geophysical observations, represented by the residuals from an incomplete a priori deterministic model, were transformed into a power spectrum and then into an autocorrelation function. The inverse transform was used to obtain the autocorrelation function from the least squares spectrum. This formulation can accommodate unequally spaced data, random observation errors, arbitrary frequency selection, arbitrarily weighted and correlated observations, as well as the presence of any a priori deterministic model. Michael also showed that the conventional Fourier transform and spectrum are just special cases of this more general least squares formulation (Craymer, 1998), a fact that I also showed independently one year later (Pagiatakis, 1999).

I'm now looking back at this extraordinary method of analysis of time series and thinking of those who still use the FFT… they are struggling to find optimum ways to fill in the gaps, or make their inherently unequally-spaced series evenly-spaced, while they are scratching their heads to find ways to account for the fact the time series values are not equally weighted. Can such an analysis be done via FFT? Of course, but at the expense of altering significantly the content of the original series! Will the results be reliable? Certainly not, but it is being done anyway as for some there is no other method…

In my new capacity as professor at the Department of Earth and Atmospheric Science I was thrilled to teach the “*Time Series and Spectral Analysis*,” for two consecutive years. I have modified the course content to include LSSA and I consider the students at York University privileged to learn this method. I spend many lectures on the method and its advantages and my students now use the software in various projects outside the course work. It takes time to break the status-quo but it’s coming along well. There is a never ending effort to improve the software and v. 6.0 is now under development to meet the needs of new research, such as the analysis of superconducting gravimeter data (e.g., Pagiatakis, 2000) and VLBI baseline residual series and VLBI nutation residuals for discovering new periodicities, verifying theoretical geophysical models and understanding better the physics of our planet. V. 6.0 will be accepting complex input series, it will be capable of performing cross-spectral analysis of two input time series, and it will be performing response analysis by estimating gain and phase spectra. I am certain that another version may do wavelet analysis vial least squares. New statistical tests must be developed to check the significance of cross-spectral peaks as well.

I am certain that other colleagues who grew up under Petr’s guidance may have similar experiences to share. My short exposé presented herein is by no means complete and exhaustive, or it is perhaps inaccurate at times, and I am sure I have excluded, unintentionally of course, the work of others. My sincere apologies! I just wanted to share my own experience on such an important contribution by Petr. Petr, rest assured that your method is being applied daily and your original code has been and is being improved to meet the needs of new research!

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The Geoid Involvement in my Scientific Activities

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In August 01, 1977 I was officially admitted as researcher and teacher at the Institute of Astronomy and Geophysics, University of São Paulo (IAG-USP). As a young scientist and with a challenge to start a program of research and activities in geodesy in the most famous and important university of Brazil, I addressed my attention to gravimetry as a first step. A research sponsorship application to FAPESP (Fundação de Pesquisa do Estado de São Paulo) received a positive answer and a La Coste&Romberg gravity meter was purchased. A long history of gravity surveys started and extended for many years in different parts of Brazil in a first step. In the 1990s other countries in South America have also been involved in surveys under my coordination.

An important fact that drove an attention to the geoid and addressed the efforts in the direction of this subject, was a visit of Petr Vaníček to the Federal University of Paraná with a short stay in São Paulo visiting IAG/USP, in 1978. The possibility of a six months stay in the University of New Brunswick, at that time, Department of Surveying Engineering, was discussed and a work plan established. A request for a fellowship was submitted to CNPq/CIDA, approved a few months later.

On January 1980 I initiated a stay at the New Brunswick University on the advise of Petr Vaníček. His clear advise, great interest on the subject, extreme care on the concepts and kind attention and encouragement during my stay, increased more and more my excitement on the geoid theory and software developments. That was the beginning of my long involvement with the geoid in Brazil, extended to South America in the mid of 1990s.

In 1986, I obtained my PhD degree at Institute of Astronomy and Geophysics at the University of São Paulo (IAG/USP) with a thesis on the combination of different kinds of data for geoid computation.

I transferred to Polytechnic School at the University of São Paulo (EPUSP) in January of 1990. The new position at one of the most active and dynamic institutes of the University of São Paulo, addressed a real increase in my activities on gravimetry, on the geoid and also on the GPS. A project of cooperation between EPUSP, the University of Leeds and IBGE (Instituto Brasileiro de Geografia e Estatística) was the beginning of a consistent and a continuous effort to infill gravity gaps in Brazil. In the mid of 1990s the efforts were extended to other countries in South America: Argentina, Chile, Paraguay and Ecuador. Different versions of a geoid model have already been estimated for South America using different software suites and a continuous improvement on data quality and distribution has been successfully accomplished.

I never lost my contact with Petr Vaníček. We have been talking in very many different meetings in symposium and congresses, discussing points about the geoid, theory, data manage, kernel modification and others.

I conclude saying that Petr Vaníček had a very strong and positive influence on my scientific life and on the geoid activities in South America.

The GPS Wrist Locator

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The first Navstar Global Positioning System test satellite was launched on 22 February 1978. By December of that year, there were four test or Block I satellites in orbit. Outside of military circles, there were very few research centres doing any preliminary investigations on the use of GPS. One of those centres of GPS research was the University of New Brunswick. The first UNB report on the potential use of GPS – *Navstar Performance Analysis* – was published in 1980 by David Wells who had just joined the department's professoriate and Demitris Delikaraoglou, Petr Vaníček's research assistant who had a freshly minted UNB Ph.D. degree on his wall. That first report was quickly followed by *Application of Navstar/GPS to Geodesy in Canada: Pilot Study*, authored by all three researchers. So Petr took an early interest in GPS, forseeing its usefulness in geodesy in addition to its many other potential applications.

Over the next few years, Petr authored or co-authored several papers and reports on GPS. One of the most prescient publications was a paper Petr gave at the International Union of Geodesy and Geophysics quadrennial meeting in Hamburg in 1983. "The Future of Geodetic Networks," written by a team of UNB researchers and subsequently published in the proceedings of the Inernational Association of Geodesy symposia, correctly forecast the demise of conventional geodetic networks and predicted that anyone with a GPS receiver would be able to establish the coordinates of virtually any point of interest without direct recourse to conventional geodetic control. The paper predicted that GPS receivers (or positioning devices based on some future technology) would become so accurate, so inexpensive, and so small that finding one's position would become as common as telling the time. In fact, the paper predicted that we would wear these devices on our wrists just like wristwatches.

The prediction of the wrist locator came true in 1999 when Casio introduced the first GPS watch.

To celebrate Petr's successful GPS-crystal-ball gazing, I am happy to contribute the following reprint of an article I wrote for my column *Innovation* in *GPS World* magazine. The article overviews the evolution of the GPS receiver culminating in the GPS wrist locator predicted by Petr some 20 years ago.

Smaller and Smaller: The Evolution of the GPS Receiver

Richard B. Langley University of New Brunswick

We have reached another GPS milestone. Just a few months ago, GPS World *celebrated its 10th anniversary. The first issue of the magazine (a bimonthly in its first year of publication) appeared in January/February 1990. The "Innovation" column has appeared in every regular issue of* GPS World, *and this month's column is number 100.*

Throughout the column's 10-year history, we have examined many innovative developments in the GPS world, including improvements in precise positioning, velocity determination, and the transfer of time; in applications such as real-time dredge positioning, monitoring the deformation of the Earth's crust, the Earth's rotation, and the state of its ionosphere; and the use of GPS on various plaftforms such as submersible vehicles, aircraft, and satellites.

Many of these developments were possible because of advances in GPS receiver technology. The technology has resulted in GPS receivers becoming smaller and more convenient to use and recently permitted receivers so small that they can be incorporated into cellular telephones and other devices. On the occasion of the 100th "Innovation" column, what better time to review the progress of GPS receiver technology throughout the past 20 years and to take a peek into its future.

Just like Grant Williams in the classic science fiction film, *The Incredible Shrinking Man*, the GPS receiver keeps getting smaller and smaller. This trend is shared by portable electronic equipment of all kinds, fueled by consumer demand for products that are smaller, lighter, faster, and cheaper. Cellular telephones, pagers, and personal digital assistants, for example, all have become smaller during the past few years while the number of features they offer has increased.

The desire to embed GPS units in other products has driven the miniaturization trend in particular, with manufacturers continually striving to reduce the number of components needed to build a receiver. In fact, a basic GPS receiver can now be fabricated with just two or three major integrated circuits. The increasing use of digital technology in GPS receivers has aided these advances, allowing a reduction in power consumption and size. Manufacturers have been able to shrink receiver components to the point that one company recently put a "Dick Tracy"–style wristwatch GPS receiver on the market.

But size is not all that matters. We have witnessed many advances in GPS receiver technology during the past 20 years. At the high end, receivers for engineering and scientific applications have become more powerful with all-satellites-in-view tracking capabilities, low-noise measurements, and real-time-kinematic on-the-fly ambiguity resolution, to mention only a few currently available enhancements. At the low end, features can be traded for cost, with basic handheld receivers now offered for less than $100. But receiver miniaturization is perhaps the most significant and far-reaching of all the developments.

To better understand this miniaturization trend, we shall examine in this article some of the technological advances that have reduced the size of a GPS receiver from two racks of equipment to that of a slightly bulky wristwatch. But first, let's briefly review the basic GPS receiver building blocks that are common to all units, regardless of size.

ESSENTIAL ELEMENTS

A GPS receiver consists of a number of basic components: an antenna with optional preamplifier, a radio-frequency and intermediate-frequency (RF/IF) "front end" section, a signal tracker/correlator section, and a microprocessor that controls the receiver, processes the signals, and computes the receiver's coordinates. The receiver will also include a power supply and memory devices for storing instructions and data.

Antenna. The antenna's job is to convert the energy in the electromagnetic waves arriving from the satellites into an electrical current that can be handled by the electronics in the receiver. The antenna's size and shape are very important, as these characteristics govern, in part, the antenna's ability to pick up and pass on to the receiver the very weak GPS signals. Typical antennas for GPS use include microstrip patches and quadrifilar helices. An *active* antenna is one that, besides the antenna element, includes a low-noise preamplifier (with one or more associated filters) to boost the weak signals. This type of unit is used if the antenna and receiver are separated by some distance.

A Front End. A GPS receiver's RF/IF section serves to translate the frequency of the signals arriving at the antenna to a lower frequency, called an intermediate frequency or IF, which is more easily managed by the rest of the receiver. A receiver component known as a local oscillator performs this task by combining the incoming RF signal with a pure sinusoidal signal or tone. Most GPS units use precision quartz crystal oscillators, enhanced versions of the regulators commonly found in wristwatches. Some geodetic-quality devices provide for the option of obtaining the local oscillator signal from an external source such as an atomic frequency standard (rubidium vapor, cesium beam, or hydrogen maser), which has a higher frequency stability. Receivers supplied with such a signal can produce carrier-phase measurements with less clock noise.

The IF signal contains all of the modulation that is present in the transmitted signal; only the carrier has been shifted in frequency. The shifted carrier's frequency is simply the difference between the original received carrier frequency and that of the local oscillator. It is often called a beat frequency in analogy to the beat note heard when two musical tones very close together are played simultaneously. Some receivers employ multiple IF stages, reducing the carrier frequency in steps. Filters are used at each IF stage to suppress out-of-band interference and other undesired signals. The final IF signal passes to the signal trackers or correlators.

Correlators. The omnidirectional antenna of a GPS receiver simultaneously intercepts signals from all satellites above the antenna's horizon. The receiver must be able to isolate the signals from each particular satellite to measure the code pseudorange and the phase of the carrier. Isolation is achieved by using a number of channels and assigning each signal to a particular channel. This is not difficult to accomplish, because the unique C/A-code or portion of the P-code transmitted by each satellite enables easy discrimination of the different signals.

Until the early 1990s, receivers with single channels were cheaper to build than multichannel units.

The channels in a GPS receiver may be implemented in one of two basic ways. A receiver may have dedicated channels that continuously track particular satellites. A minimum of four such channels tracking the L1 signals of four satellites with good geometry would be required to determine three coordinates of position and the receiver clock offset. Additional channels permit tracking of more satellites including, perhaps, GLONASS or SBAS (satellite-based augmentation system) satellites, or the L2 signals for ionospheric delay correction or both.

The other channelization concept uses one or more sequencing channels. A sequencing channel "listens" to a particular satellite for a period of time, making measurements on that satellite's signal before it switches to another satellite. A single channel receiver must sequence through four satellites to obtain a three-dimensional position fix. Before a first fix can be obtained, however, the receiver has to dwell on each satellite's signal for at least 30 seconds to acquire sufficient data from the satellite's broadcast message. The time to first fix and the time between position updates can be reduced by having a pair of sequencing channels.

A variation of the sequencing channel concept is the multiplexing channel. With a multiplexing channel, a receiver sequences through the satellites at a fast rate so that all of the broadcast messages from the individual satellites are acquired essentially simultaneously. For a multiplexing receiver, the time to first fix can be 30 seconds or less, the same as for a receiver with dedicated multiple channels.

Until the early 1990s, receivers with single channels were cheaper to build than multichannel units but, because of their slowness, were restricted to low-speed applications. Most modern civil GPS receivers have 8–12 dedicated channels. Some have 24 or even 40 channels for multisystem and dual-frequency processing. Receivers with dedicated channels have greater sensitivity, because they can make measurements on the signals more often.

Microprocessor and Memory. A microprocessor controls a GPS receiver's operation. As these devices have become more and more powerful, they have taken on more of the signal processing tasks in the receiver. The microprocessor's software, that is, the instructions for running the receiver, is embedded in memory either within the microprocessor itself or in auxiliary integrated circuits within the receiver.

The microprocessor works with digital samples of pseudorange and carrier phase, acquired as a result of analog-to-digital conversion at some point in the signal flow through the receiver and the cross-correlation process. The receiver uses these data samples to establish its position, velocity, and time. The microprocessor may also run routines to filter this raw data in order to reduce the effect of noise or to get more reliable positions and velocities when the receiver is in motion.

In addition, the microprocessor may be required to carry out computations for waypoint navigation, to convert coordinates from the standard World Geodetic System of 1984 (WGS 84) geodetic datum to a regional one, and to combine differential corrections from ground-based or satellite-based augmentation systems. It also manages the input of commands from the user, the information display, and the flow of data through its communication port, if it has one.

Power Supply. Most GPS receivers have internal DC power supplies, often in the form of rechargeable batteries. Some even use "penlight" batteries. The latest receivers have been designed to draw as little current as possible to extend the operating time between battery charges or replacement. The receiver may also include a small onboard lithium battery for keeping memory circuits "alive" while the receiver is not operating.

These basic building blocks have appeared in essentially all GPS receivers ever built. But the size, power requirements, and capabilities of these blocks have evolved over the past 20 years or so. Before taking a look at the functioning aspects of these blocks in a modern miniature GPS receiver, it will be instructive to look at how far we have come, which will also help us appreciate the progress of technology during the past two decades.

RECEIVER RUNDOWN

The first GPS receivers, built in the 1970s, were used for the initial testing of the GPS concept. Mostly developed under contracts with the Department of Defense (DoD), these units were prototypes for various branches of the military. For our purpose, however, we will primarily focus on the technological advances of receivers developed for the civil market, with occasional reference to military receivers.

The first commercially available GPS receiver was the STI-5010 built by Stanford Telecommunications, Inc. It was a dual-frequency, C/A- and P-code, slow-sequencing receiver. Cycling through four satellites took about five minutes, and the receiver unit alone required about 30 centimeters of rack space. External counters, also requiring rack space, made pseudorange measurements. An external computer controlled the receiver and computed positions. A variant of the STI-5010 was developed for the GPS control segment tracking stations.

The Macrometer. Researchers at the Massachusetts Institute of Technology initially introduced another early civil GPS receiver, the Macrometer V-1000, in 1982, and Litton Aero Service subsequently commercialized it. Users could transport the unit and successfully deployed it using both small vehicles and helicopters, despite its size of 58 × 56 × 64 centimeters and weight of 73 kilograms, exclusive of the 18-kilogram antenna. A far cry from a wristwatch!

These units used codeless squaring to make carrier-phase measurements, which meant they made no pseudorange measurements nor did they decode any broadcast ephemeris data. Real-time operations were thus ruled out. Furthermore, clock synchronization using the GPS signals was not possible. As a result of these limitations, users needed to employ at least two receivers simultaneously. Plus, before being deployed, they had to be synchronized to each other as well as to Coordinated Universal Time (UTC). Auxiliary equipment permitted UTC synchronization, using time signals such as those from one of the Geostationary Operational Environmental Satellites.

The Macrometer recorded its data in nonvolatile bubble memory in the receiver, and users transferred it to a small data cartridge after the recording session. Data could then be processed after the mission using ephemerides obtained from other sources. This particular receiver was known for its well-designed, although large, antenna which had exceptional phase-center stability. The Macrometer V-1000 and its successor, the dual-frequency Macrometer II, were used extensively for a number of years.

The TI 4100. Also introduced in 1982 was the first relatively compact civil GPS receiver, the Texas Instruments TI 4100, also known as the Navstar Navigator. This receiver could make both C/A- and P-code measurements along with carrier-phase measurements on both L1 and L2 frequencies. Its single hardware channel could track four satellites simultaneously through a multiplexing arrangement. The $37 \times 45 \times 21$–centimeter receiver/processor had a handheld control and display unit and an optional dual-cassette data recorder for saving measurements for postprocessing. The unit, although portable, weighed 25 kilograms and consumed 110 watts of power (the receiver doubled as a hand warmer). Field operation required a supply of automobile batteries.

During their heyday, the TI 4100 and the Macrometer were used around the world for high-precision surveying and the establishment of geodetic networks. In fact, surveyors and geodesists were the primary GPS users while the GPS constellation consisted of only the prototype Block I satellites. The concepts of single- and double-difference observations were pioneered with these early survey-quality receivers.

Beginning around the mid-1980s, a number of companies entered the market and started producing receivers for surveying, navigation, and time transfer. Ever since, a continuous evolution of GPS receivers has taken place with receiver capabilities being enhanced while size and power consumption have been reduced.

Here Come the Handhelds. GPS receiver technology experienced a major evolution in 1988. In that year, the Collins Division of Rockwell International demonstrated a prototype of the first handheld receiver. It was about the same size as a large-size package of cigarettes and therefore affectionately known as the "Virginia Slim" after a popular brand. Researchers designed and built the prototype receiver under contract from the DoD's Defense Advanced Research Projects Agency (DARPA). It was a two-channel, dual-frequency, P-code unit, employing custom-designed integrated circuits based, in part, on gallium arsenide (GaAs) semiconductor technology. The use of GaAs circuitry in a GPS receiver was, and is, unusual as most receivers are based on conventional silicon technology (see the "Semiconductor Basics" sidebar).

In addition to its novel use of GaAs circuitry, the receiver featured the first hybrid analog/digital microwave monolithic integrated circuit (IC) chip. Also the first all-digital receiver, the unit performed code-correlation using digital rather than analog signals. Achievement of all-digital GPS processing opened the door for developers to exploit very-large-scale integration (VLSI) in GPS receivers.

The DARPA receiver efforts sparked development of handheld receivers by other companies. Magellan introduced the first commercial handheld GPS unit in 1988, named the NAV 1000. In a $19 \times 8.9 \times 5.3$–centimeter, 850-gram package, its single sequencing channel could track four satellites. During that year, Trimble introduced its Trimpak. It was a three-channel sequencing

SEMICONDUCTOR BASICS

The miniaturization of the GPS receiver was made possible by the phenomenal developments in semiconductor technology during the past 20 years. But just what is a semiconductor? A semiconductor is a material such as silicon (Si) or germanium (Ge) in which the electrical conductivity lies between that of conductors and insulators. However, the resistivity of pure silicon, for example, is so high (about 10^{11} times that of copper) that it is virtually an insulator and therefore not much use in electronic circuits. But by adding minute amounts of specific "impurity" or replacement atoms, a process called *doping*, the resistivity can be reduced by a factor of 10^6 or more. Replacement atoms include antimony, phosphorus, arsenic, and aluminum. The doped semiconductor conducts current either by negative or positive carriers depending on the dopant. Actually, the positive carrier is virtual. The electron still carries the charge, but it is the physical absence of the electron that moves, and this virtual carrier is called a *hole.*

By combining semiconducting material that has an abundance of electrons (negative or n-type) with material that has an abundance of holes (positive or p-type), electronic circuit devices can be created in which the current flow through the device can be externally controlled, operating as a rectifier, a basic switch, or an amplifier. Examples of semiconductor devices include diodes and transistors. To build a complete electronic circuit out of these basic components may require just one semiconductor, such as a "minimalist" radio using only a single diode (a "crystal set"), to many millions as found in computers.

To significantly reduce the parts count necessary to build complex pieces of electronic equipment, such as GPS receivers, transistors and other electronic components, including capacitors and resistors, are crafted into a single semiconducting chip, forming an integrated circuit. A chip may contain just a few transistors or more than a million. The Power PC 620 microprocessor chip, for example, incorporates almost seven million transistors, along with many other electronic components. The number of transistors that semiconductor manufacturers can put on a chip doubles about every 18 months.

Some integrated circuits use simple bipolar transistors. *Bipolar* refers to the type of transistor construction that has either two p-regions, one on either side of an n-region (a pnp transistor) or two n-regions and one p-region (an npn transistor). This transistor "sandwich" is called a junction transistor. Most GPS receiver front-end chips use bipolar silicon technology.

However, the transistors in some integrated circuits are complementary metal oxide semiconductors (CMOS). Whereas bipolar devices use junction transistors, CMOS devices use a particular kind of field effect transistor, the MOSFET. Similar to bipolar transistors, MOSFETs have p- and n-type regions, but the central region is made up of a substrate of silicon semiconductor material on top of which is an insulating layer of silicon dioxide (glass), and on top of the oxide layer is a metal "gate."

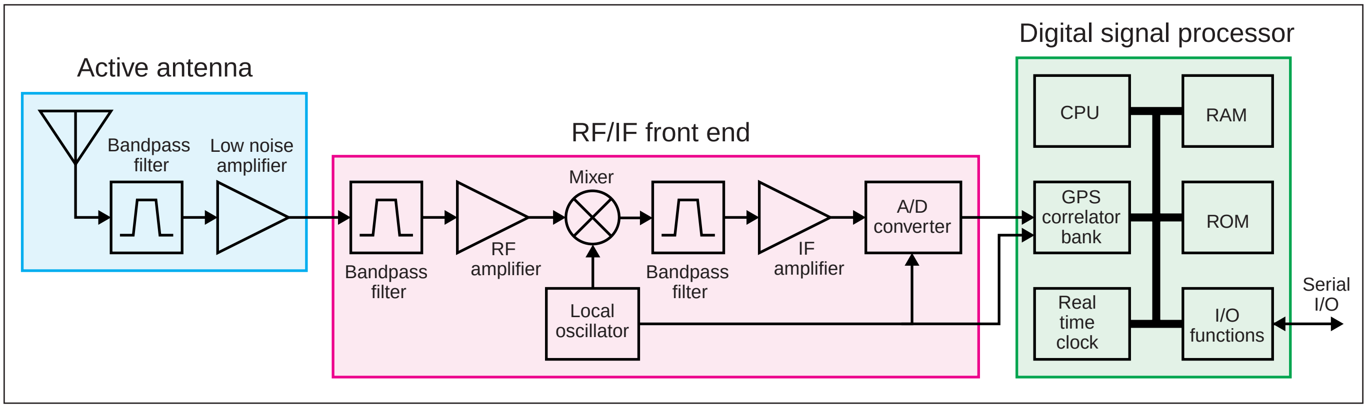


Figure 1. A GPS receiver can be constructed from just two major integrated circuits: an RF/IF front end ("radio chip") and a digital signal processor.

When a voltage is applied to that metal gate, an electric field is established that penetrates through the insulator into the substrate, allowing current to flow in a channel along the top of the substrate. CMOS combines both p- and n- channel devices in the same substrate to achieve high noise immunity and low power consumption — a negligible amount during standby. This accounts for the widespread use of CMOS in battery-operated equipment. Furthermore, by 1996, individual MOSFETs could be made extremely small, less than 500 nanometers in length (which is approximately the wavelength of yellow light), allowing large numbers of them to be packed into a single integrated circuit.

Silicon is not the only semiconductor used to make ICs for GPS receivers. ICs made from gallium arsenide (GaAs) and silicon germanium (SiGe) are also available. Gallium arsenide has greater electron mobility and a correspondingly higher maximum operating frequency than typical silicon devices. Its low-noise characteristics make it an appropriate choice for discrete low-noise amplifiers and for mixers operating at high frequencies. The key advantages of silicon-germanium technology are high-frequency performance at lower power consumption and extremely low phase noise with apparently lower processing cost compared to GaAs-based devices.

receiver and could track eight satellites. Although slightly larger than the NAV 1000, measuring 19.8 × 6.4 × 22.4 centimeters and weighing 1.5 kilograms, it could still be held in one hand. Coalition forces in the 1991 Gulf War widely used both military and civil versions of the NAV 1000 and the Trimpak.

The next significant development in the miniaturization of the GPS receiver occurred in 1993. In that year, the Mayo Foundation and Motorola demonstrated the first GPS multichip module (MCM) receiver. MCM construction is similar to conventional printed circuit boards (PCBs) but uses smaller physical geometries and a special substrate, the material on which a circuit is constructed or fabricated. They took an existing six-channel GPS receiver design that used single-chip packaging and conventional PCB technology and miniaturized it to a double-sided MCM measuring just 3.5 × 3.6 × 1.0 centimeters fully assembled.

The MCM contained both surface-mount components and bare die (chip circuitry) on both sides of an eight-layer substrate. The unit dissipated about 1.3 watts. In combining the very low-level analog RF signals with high-level digital processing of these signals on the same substrate, the developers had to take special care to prevent the digital signals from interfering with the RF signals by carefully placing parts.

Several companies now offer small handheld GPS receivers built with a small number of integrated circuits. Some of these ICs are generic ones used in different kinds of electronic devices, but some of them have been specially developed for GPS use. These application-specific integrated circuits (ASICs) have significantly reduced the number of components needed to build GPS receivers and hence their size and power consumption. In fact, the chip manufacturers offer the ASICs to other manufacturers for building standalone GPS receivers or for embedding a receiver into another product.

THE WORKINGS OF A CHIPSET

It is common now for the RF and IF functions of a GPS receiver to be combined on a single ASIC, supplying the IF signal to another ASIC: a digital signal processing chip, which typically includes an eight- or 12-channel correlator array. Although a GPS receiver's microprocessor may be a separate IC, ASICs have been developed that combine the correlators and support functions (such as serial ports) together with the microprocessor in a single chip. Today, the chip can even include onboard read-only memory (ROM) and random-access memory (RAM), making a complete two-chip GPS receiver.

Figure 1 presents a block diagram of a GPS receiver built using only two major integrated circuits. The signal from the antenna feeds into the RF/IF front-end chip. Depending on the antenna cabling's length and front-end input signal level requirements, an active antenna (with a preamplifier) rather than a passive antenna may be required. In the RF/IF chip, the signal is first bandpass filtered and amplified before being mixed with the signal from the local oscillator to generate the IF. The bandpass filtering helps to protect the GPS signal from interference, for example, from cellular phone signals.

Some chip designs have external local oscillators that supply timing signals to both the RF/IF chip and the signal processing chip. Some designs use more than one IF stage, reducing the intermediate frequency in steps with filtering and amplification at each step. The filters and the crystal for the oscillator are typically outside the chip package. Some front-end designs use larger encapsulated-module packaging and include the filters and

FURTHER READING

For the first paper to discuss the possibility and potential for a GPS "wrist locator," see

■ "The Future of Geodetic Networks," by P. Vaníček, D.E. Wells, A. Chrzanowski, A.C. Hamilton, R.B. Langley, J.D. McLaughlin, and B.G. Nickerson, published in the *Proceedings of the International Association of Geodesy Symposia*, International Union of Geodesy and Geophysics, XVIII General Assembly, Hamburg, Germany, August 15–27, 1983, Vol. 2, pp. 372–379.

For more details on some of the early GPS receivers, see

■ *Guide to GPS Positioning,* edited by D. Wells, originally published in 1986 by Canadian GPS Associates and recently reprinted by the University of New Brunswick, Fredericton, Canada, as Department of Geodesy and Geomatics Engineering Lecture Note No. 58.

For further details about modern GPS receiver mechanization, see

■ "GPS Receivers," by A.J. Van Dierendonck, Chapter 8 in *Global Positioning System: Theory and Applications,* Vol. I, edited by B.W. Parkinson and J.J. Spilker Jr., American Institute of Aeronautics and Astronautics, Inc., Washington D.C. 1996.

■ "GPS Receiver Architectures and Measurements," by M.S. Braasch and A.J. Van Dierendonck in *Proceedings of the IEEE,* Vol. 87, No. 1, pp. 48–64, 1999.

For a review of the past, present, and future of semiconductor technology, see

■ *The Solid-state Century,* Scientific American Special Issue, Vol. 8, No. 1, 1997.

even the local oscillator crystal.

Typically, the RF/IF chip includes an onboard analog-to-digital converter that samples the final IF signal with one- or two-bit quantization before passing it on to the signal processing chip. A synchronizing clock signal is also passed. Once the signal is sampled, all further signal processing is done digitally.

Processing the Digital Signal. The digital signal processing (DSP) chip incorporates a central processing unit (CPU) microprocessor, which is the workhorse of a GPS receiver. Some DSP designs do all of the signal processing in a powerful CPU, but it is more common to use hardware correlators onboard the chip. Typically an eight- or 12-channel bank of correlators is provided. The DSP chip may include onboard ROM and RAM, or the memory may be provided by separate ICs. Some GPS chips run their software from flash memory that can be upgraded by the user. The DSP chip would also include interface and power-management circuitry.

The power for both the RF/IF and DSP chips is usually provided by a 5- or 3.3-volt DC supply. Power consumption depends on chip usage, but typically the RF/IF chip consumes 100–200 milliwatts and the DSP chip, 300–500 milliwatts.

WRIST-MOUNT GPS

Multichip modules that implement multiple unpackaged chips on a single substrate have enabled manufacturers to complete receivers as small as 30 × 30 × 6 millimeters, small enough to put in a wristwatch. In fact, researchers at the University of New Brunswick (UNB) first predicted exactly that — a "wrist locator" — in a paper presented at the International Union of Geodesy and Geophysics quadrennial meeting in Hamburg, Germany, in 1983. In that paper, they said "With the miniaturization and cost reduction being experienced continually, it is surely safe to postulate the limit of this evolution: a cheap 'wrist locator' giving instantaneous positions to an accuracy of 1 [millimeter]." Elsewhere in the paper, they suggested a price for this technological wonder of $10, and that it would be available sometime in the twenty-first century.

Costing about $400 and giving Standard Positioning Service accuracies, the first "wrist locator" came on the market last year — before the twenty-first century began. While the UNB researchers may have been a bit overly optimistic in the capabilities and cost of the "wrist locator," the basic prediction came true earlier than expected. And there's room for further development.

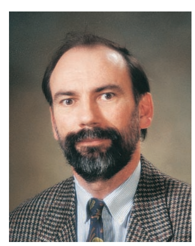
ANYTHING BUT DISAPPEARING

The demand for smaller, cheaper, power-thrifty GPS receivers will continue. Receiver designers have already reduced the number of ICs needed to just two. Is a one-chip GPS receiver possible, and how small can the receiver be made? A complete GPS system on a single chip should be possible, but cross-talk between the digital signals and the weak analog signals on the chip will need to be addressed. New chip packaging technologies also will help in reducing the size. Integration of the receiver into a wearable computer built into clothes may even be possible one day.

While these technologies may seem far fetched, scientists are already working on such developments. A limiting factor could be the size of the receiver's antenna. The smaller the antenna, the less sensitive it is. But GPS antennas smaller than 2.5 × 2.5 centimeters are available today. The GPS signals are fairly weak, but with more sensitive front ends and more sophisticated signal processing, even smaller antennas should be possible.

Researchers have built a solid-state molecular switch that is electronically configurable. This work could lead to the development of chemically-assembled ICs with molecular-sized devices that would need a miniscule amount of power. Thus, as each of these elements improve, so too do the opportunities for new, even smaller GPS receivers.

In *The Incredible Shrinking Man*, Grant Williams gets smaller and smaller until he disappears in a puff of wind. The GPS receiver will not get that small, but it could conceivably become small enough to be implanted under the skin and, coupled with other sensors and a transmitter, used to track people and their vital health signs. As hard as it may be to believe, at least one company is already working on a prototype for just such a device. ■



"Innovation" is a regular column featuring discussions about recent advances in GPS technology and its applications as well as the fundamentals of GPS positioning. The column is coordinated by Richard Langley of the Department of Geodesy and Geomatics Engineering at the University of New Brunswick, who appreciates receiving your comments as well as topic suggestions for future columns. To contact him, see the "Columnists" section on page 4 of this issue.

Use of Deformation Monitoring Measurements in Solving Geomechanical Problems

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Note: this paper has been prepared on the occasion of the retirement of Prof. Dr. Petr Vaníček, a close friend of the authors. Prof. Dr. Petr Vaníček in his scientific career has made significant contributions to the field of deformation analysis. The authors hope that this paper is complementary to his work.

ABSTRACT

The role of deformation surveys is much broader than just the conventional determination of the geometrical status of the deformable object. By comparing the observed deformation quantities with the designed deformations obtained from the finite element analysis (deterministic model) one can verify the designed geomechanical parameters of the material and one may also determine the actual deformation mechanism. In this presentation, determination of effects of hydrological changes on ground subsidence in a potash mine in Canada is discussed as an example.

Key words: finite element method, deformation analysis, ground subsidence

1. Introduction

Analysis of deformations of any type of a deformable body includes geometrical analysis and physical interpretation. Geometrical analysis describes the change in shape and dimensions of the monitored object, as well as its rigid body movements (translations and rotations). The ultimate goal of the geometrical analysis is to determine in the whole deformable object the displacement and strain fields in the space and time domains. In the generalised method of deformation analysis (Chen, 1983; Chrzanowski et al., 1983; Chrzanowski et al., 1986), the displacement field is obtained by fitting an appropriate displacement function to the measured deformation quantities (e.g., absolute or relative displacements, strain, tilts, changes in distances or angles). The strain field is determined from the displacement function by using the well-known strain-displacement relationships (Timoshenko and Goodier, 1970; Shames, 1979; Chrzanowski et al., 1983)

Physical interpretation is to establish the relationship between the causative factors (loads) and the deformations (Chen and Chrzanowski, 1986). This can be determined either by

- a statistical method, which analyses the correlation between the observed deformations and loads, or

- a deterministic method, which utilizes information on the loads, properties of the material, and physical laws governing the stress-strain relationship.

The deterministic modelling requires solving differential equations for which closed form solutions may be difficult or impossible to obtain. Therefore, numerical methods, such as the Finite Element Method (FEM) are used. In case of rock and soil materials, the in-situ geomechanical properties may significantly differ from the laboratory values. This must be taken under consideration when performing deterministic modelling of deformation.

By comparing the geometrical model of deformations, derived from the observed deformation quantities, with the designed deformations obtained from FEM, one can verify the designed geomechanical parameters (Szostak-Chrzanowski et al., 2002a). In addition, with properly designed monitoring surveys (Kuang and Chrzanowski, 1994), one may also determine the actual deformation mechanism (Chrzanowski and Szostak-Chrzanowski, 1993; Chrzanowski and Szostak-Chrzanowski, 1995). Thus, the role of deformation monitoring surveys becomes much broader than just the conventional determination of the geometrical status of the deformable object.

The concept of using the results of monitoring surveys in solving geomechanical problems has been implemented in many projects, for example in:

- determination of the deformation mechanism in a slope stability study in Sparwood, B.C. (Chrzanowski et al., 1991),

- determination of the deformation mechanism at hydro-electric power station in Mactaquac N.B. (Chrzanowski and Szostak-Chrzanowski, 1995),

- determination of the stress changes in the rock mass due to underground copper mining in Poland (Chrzanowski et al., 2000),

- verification of design geotechnical parameters at the large earth dams of the Diamond Valley Lake reservoir in California (Szostak-Chrzanowski et al., 2002b).

In this paper, after a short introduction to the behaviour and modelling of rock and soil material, determination of effects of hydrological changes on ground subsidence in a potash mine in New Brunswick, Canada is discussed as an example.

2. Rock Behaviour

2.1 Behaviour of Brittle Rock

The rock mass behavior in some range of compressive stresses may be assumed as linear elastic. The relation between stress and strain is expressed as:

$$\sigma = \mathbf{D}\,\varepsilon \tag{1}$$

where σ is the stress vector, **D** is a constitutive matrix , and ε is the strain vector.

The constitutive matrix for isotropic material in case of plane strain is expressed as:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1+2\nu}{2} \end{bmatrix} \tag{2}$$

where E is Young modulus, ν is Poisson ratio.

The behaviour of rock mass subjected to tensional stresses may be assumed as behaviour of ‘no-tension’ material (Zienkiewicz et al., 1968). In case of the presence of tensional stress in only one direction, the constitutive matrix includes directional Young modulus E_a, and directional Poisson ratio ν_a and, for case of plane strain, has the form:

$$\mathbf{D}_a = \frac{E_a}{(1+\nu)(1-\nu-2n\nu_a^2)}\begin{bmatrix} n(1-\nu_a^2) & n\nu_a(1+\nu) & 0 \\ n\nu_a(1+\nu) & 1-\nu^2 & 0 \\ 0 & 0 & m(1+\nu)(1-\nu-2n\nu_a^2) \end{bmatrix} \tag{3}$$

where $\nu = E / E_a$, $m = G_a / E_a$, $G_a = E_a / (2\,(1+\nu_a))$, and $\nu_a = (3/2 - \nu) / (1+ E / E_a)$.

After reaching the critical values of stress (compressive or tensional) the rock mass behaves as a non-linear material. The problem of non-linearity of the material is solved through sequential linear elastic solutions with iterative updating of the constitutive matrix through changes of the modulus of elasticity, E.

2.2 Behaviour of Salt Rock

The behaviour of salt is time dependent (visco-elastic) and the strain may be expressed as the sum of two components:

$$\varepsilon = \varepsilon_e + \varepsilon_c \qquad (4)$$

where ε_e is the elastic strain and ε_c is the creep strain. It is difficult to obtain the long term characteristics of the creep model. Generally it requires several years of in-situ creep measurements to arrive at the proper model.

The salt rock may also be considered as a non-Newtonian liquid with high and not constant viscosity (Mraz et al., 1987, Physics Today, 2000). As a non-Newtonian liquid, the intact salt rock deposits are under isotropic lithostatic stress conditions. Therefore, the shear stress in intact salt rock is equal to zero. Development of shearing stresses due to mining activity causes the flow of the salt mass into the excavated areas until new equilibrium state of stresses is achieved. The 'flow-in' zone is determined by the areas in which maximum shearing stresses are developed.

3. Modelling of Rock and Soil Deformation Using FEM

3.1 Basic Definitions in Modelling of Deformations

Using the well known principles of the finite element displacement approach, the global equilibrium equation for the investigated object maybe written as:

$$\mathbf{K}\,\delta = \mathbf{r} + \mathbf{f}^{b} + \mathbf{f}^{\sigma_o} + \mathbf{f}^{\varepsilon_o} \qquad (7)$$

where δ is the vector of nodal displacements (in a two-dimensional analysis), $\mathbf{K}$ is the stiffness matrix of the material, $\mathbf{f}^b$ is the loading vector of body forces, $\mathbf{f}^{\sigma o}$ is the loading vector from initial stresses, $\mathbf{f}^{\varepsilon o}$ is the loading vector from initial strains, and $\mathbf{r}$ is the vector of external forces concentrated at nodal points. The global matrices and vectors are calculated through a superimposition of local (at each element or at each node of the FEM mesh) matrices $\mathbf{K}_e$ and vectors $\mathbf{f}_e^{b}$, $\mathbf{f}^{\varepsilon o}$, and $\mathbf{f}^{\sigma o}$.

The stiffness matrix is given:

$$\mathbf{K}_e = \iint \mathbf{B}_e^T \mathbf{D} \mathbf{B}_e t dxdy \quad (8)$$

where $\mathbf{B}_e$ is the matrix relating strains in the element to its nodal displacements, **D** is the constitutive matrix of the material t is the unit thickness of the elements (two-dimensional analysis). Since the stiffness matrix **K** is singular, boundary conditions must be applied in order to solve equation (7) for the displacements.

3.2 Modelling of Deformation due to Mining in Brittle and Salt Rocks

A method of modelling the deformation and stress change in rock due to mining activity has been developed at UNB (Szostak-Chrzanowski and Chrzanowski, 1991, Chrzanowski et al., 1998a) based on FEM. The method is known as the S-C method. The most important concept of the S-C method is an introduction of the ‘weak zone’ in the qualitative model of ground subsidence. The ‘weak zone’ is delineated by the FEM elements in which the maximum shearing stresses develop at the boundary between the zone of rocks subjected to tensional stresses above the underground opening and the surrounding rocks subjected to compressive stresses. The boundary surface of the maximum shearing stresses has been identified by Kratzsch (1983) as a slippage surface of rocks reduces transferring of the tensional stresses beyond the boundary.

Once the weak zone is introduced into the FEM model, the basic task is to change sequentially the values of E in individual FEM elements according to criteria based on the critical tensional stresses. In the case of elements in which both principal stresses (in two-dimensional analysis) exceed the critical value, the value of E = 0 is placed in the given elements. In cases of only one principal stress being tensional, the material is assumed to be anizotropic with E = 0 in the direction of the critical principal stress and the constitutive matrix is changed to reflect the transversely isotropic material. The FEM analysis is repeated until no more elements are identified as having developed critical tensional stresses.

Modelling of salt rocks in salt and potash mines is performed in two phases:

- modelling of the salt rock response of the load and calculation of the expected deformation at the top of the salt formation (at the cap rock); In the model the final maximum subsidence at the top of the salt formation, the original value of E in the 'flow' zone is decreased to give the same volume of the subsidence basin (under the cap rock) as is the volume of mining openings (minus the volume of the compacted backfill) which will be filled up by the salt rock convergence, and

- modelling of the response of the overburden rocks (above the cap rock) which are treated as brittle and non-tension material.

4. Modelling of Ground Subsidence and Hydrological Changes in PCS Potash Mine

4.1 Monitoring of Ground Subsidence

Mining of a large deposit of high-grade sylvinite in New Brunswick has been carried out by Potash Corporation of Saskatchewan (PCS) since the mid 1980s. Potash and salt mining at PCS takes place at depths between 400 m to 700 m within a 25 km long dome-shaped salt pillow in which the potash is preserved in steeply dipping flanks (Figure 1). A strong, arch shaped, caprock provides an excellent natural support for the overlain brittle rocks. Potash is mined by using a mechanized cut-and-fill method with up to 100% extraction in the 1000 m long and about 150 m high stopes. Unsupported openings are up to 25 m wide. The potash deposit is structurally complex with a variable dip and width. Salt mining is by multi-level room-and-pillar method. Trans Canada Highway runs along the longitudinal axis of the mine and is affected by ground subsidence.

Annual monitoring of ground subsidence over the PCS mining operation near Sussex, N.B., has been carried out by UNB since 1989. Figure 2 shows the layout of the mining workings and the network of monitoring surveys consisting of precision levelling, traversing with total stations, and GPS surveys. In 1995, a finite element analysis using S-C method was performed to model the maximum expected subsidence along a selected cross-section (line A-A in Figure 2). A summary of the results was presented in (Chrzanowski et al. 1998 b). The expected subsidence profile was to follow a regular shape with its maximum subsidence located above the room-and-pillar salt extraction (approximately above the centre of the salt dome). Figure 3 shows the observed and FEM modelled subsidence in 1995.

Since 1997, a significant increase in water inflow to the mine was noticed at lower levels of potash extraction near the investigated cross-section and a secondary subsidence basin started occurring on the surface at the north end of the investigated cross-section (Figure 4). In 2001, a new FEM analysis of ground subsidence was undertaken using the S-C method to explain whether the water inflow from an unknown aquifer could cause the development of the secondary subsidence basin.

4.2 Behaviour of Aquifer Material Due to Change of Piezometric Head

Deformation of the soil layer of the aquifer is mainly due to water head change. The deformation may be calculated using the theory of consolidation. For clayey or sandy soil the deformation is given by Shueng Shu-guang et al. (1984). Subsidence model derived by Bravo et al. (1991) uses the principle of relationship of elastic compaction of soil and ground-water piezometric head. For example, Riley (1984) shows that a 24 m change of water level gives 0.076 m subsidence on the top of the aquifer.

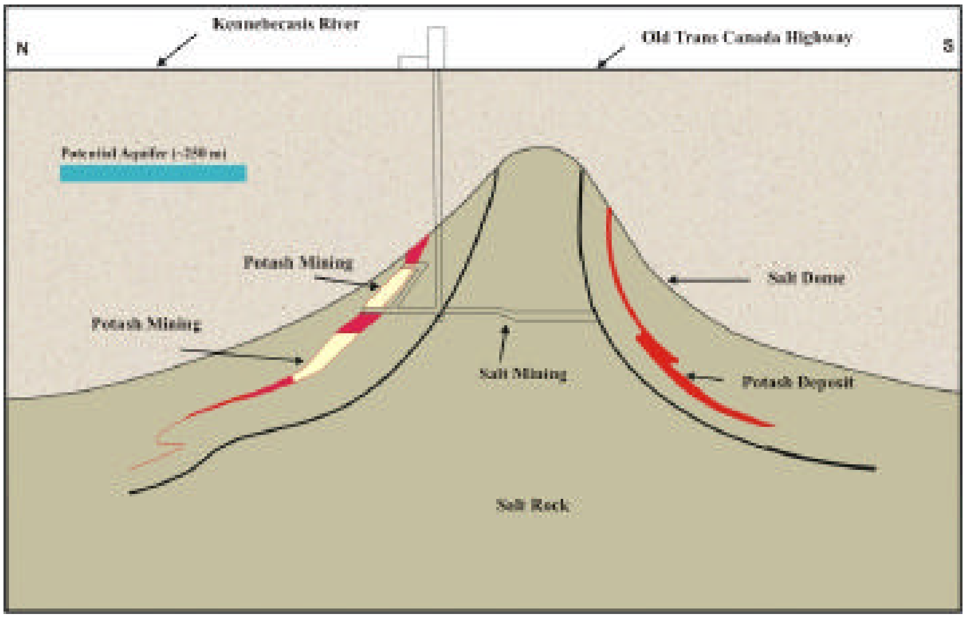

Figure 1. Cross-section of the PCS potash and salt mine in New Brunswick.

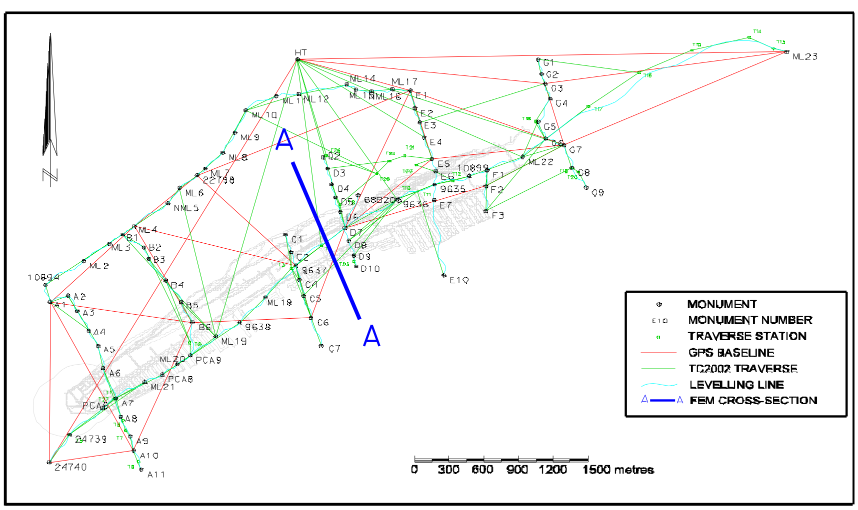

Figure 2. Mine layout and the monitoring scheme.

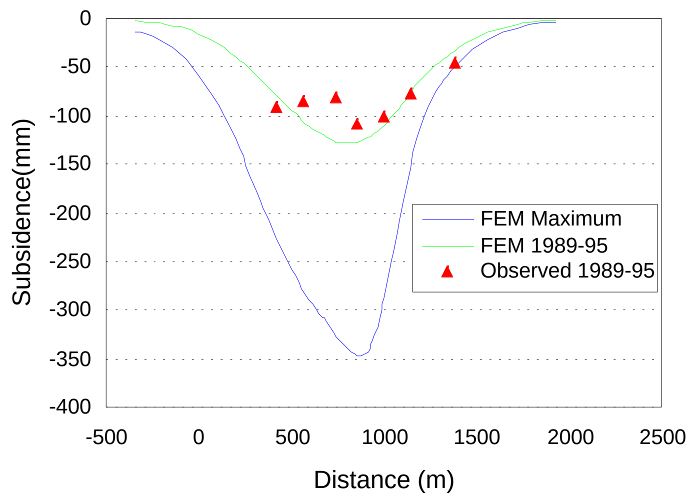

Figure 3. Observed and FEM modelled subsidence in 1995.

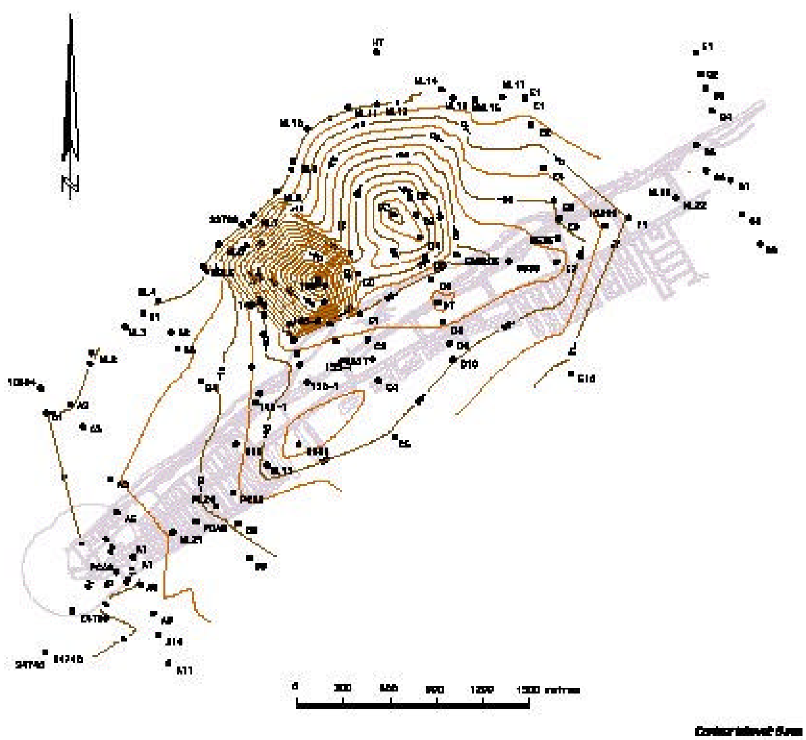

Figure 4. Subsidence isolines.

The following two basic models have been analysed using FEM and the S-C method:

- analysis of ground subsidence as caused only by extraction of potash and salt, and
- analysis of ground subsidence as above, with an addition of possible effects of hydrological changes in a hypothetical aquifer.

Due to a lack of information on the time dependent effects of mineral extraction and hydrological changes in the aquifer, the identification of the best model had to be based only on a qualitative analysis by comparing the shape of the FEM calculated subsidence profile with the observed subsidence curve.

In the first model, the Young modulus in the “weak zones” over salt and potash excavations was selected in the FEM analysis to give the calculated maximum subsidence the same as the observed maximum subsidence for the observation period 1995-2001. Figure 5 shows the FEM calculated profile of the surface subsidence in comparison with the observed subsidence. The irregular observed subsidence couldn’t be caused by the mineral extraction alone. Therefore, the second model with an effect of hydrological changes in the assumed aquifer (Figure 1) was analyzed.

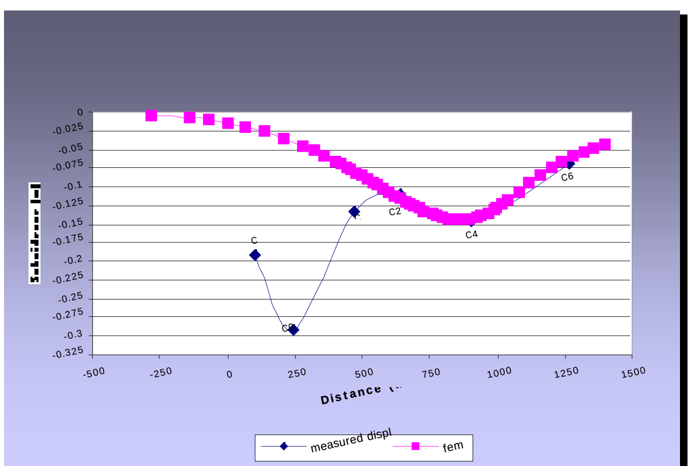

Figure 5. Measured subsidence along A-A Line and calculated (FEM) subsidence due to only mineral extraction.

Three analyses were performed for three assumed depths of the aquifer: 350 m, 250 m, and 150 m. In each analysis, it was assumed that the centre of the aquifer is located under the point of the maximum subsidence of the secondary subsidence basin and that the compressibility value in the aquifer is such that the effect on the surface subsidence is approximately equal to the observed maximum subsidence of 0.3 m. The dimensions of the aquifer were arbitrarily taken as having the width of 330 m and thickness of 40 m. The analysis of the aquifer at the depth of 150 m gave the best agreement between the observed and modeled subsidence curve (Figure 6).

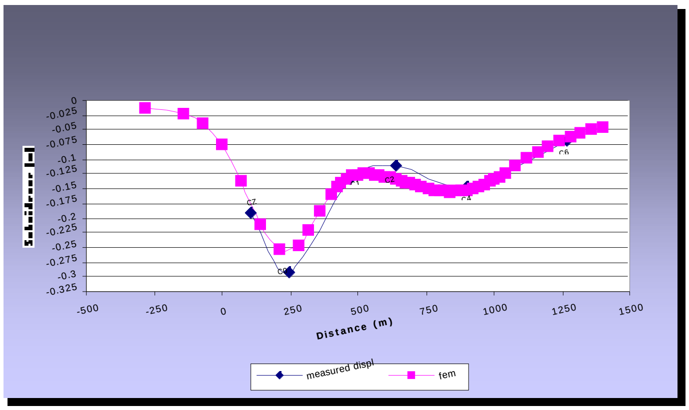

Figure 6. Measured subsidence along A-A Line and calculated (FEM) subsidence with aquifer at 150 m depth.

This preliminary study has illustrated the potential for modelling hydrological changes by comparing the results of the FEM analysis with the observed surface subsidence.

5. Conclusion

The presented case study demonstrates the usefulness of the monitoring surveys in solving geomechanical problems. This broadens the role of the monitoring surveys. In order to take a full adventure of geodetic monitoring surveys, geodetic engineers must learn basics of deterministic modelling and physical interpretation of the surveys.

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Generalizations of Least-Squares Spectral Analysis

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ABSTRACT

Spectral analysis is often needed in the analysis of geodetic measurements, and more generally, in time series applications. The Least-Squares Spectral Analysis (LSSA) of Vaníček [1969, 1971] has extended the classical Fourier analysis to include trend components and irregularly spaced observations. These nonstationarity considerations can be somewhat generalized to include quasi-stationary situations where windowed Fourier transforms in the sense of Gabor transforms are applicable. In more general contexts, the Fourier basis functions have to be generalized to wavelets for multiresolution analysis and synthesis. Furthermore, geodetic applications often imply spherical computations where Fourier transforms become spherical harmonic transforms and wavelet transforms become spherical wavelet transforms. Simple numerical experimentation shows the advantages in multiresolution analysis of nonstationary data sequences. Concluding remarks with some practical recommendations are included.

1. Introduction

The classical methods of spectral and harmonic analysis are based on stationarity assumptions and Fourier analysis. However in practice, nonstationarity in time and nonhomogeneity in space characterize most physical data sequences. In some cases, it is simply a matter of removing trend components before treating the residual space or time series as stationary but the situation is more complex in general. The least-squares approach of Vaníček [1969, 1971] was a major step in generalizing the Fourier approach but with Gabor and wavelet transforms, the field has advanced greatly over the past decades. Equispaced (in time or space) data sequences will be assumed in the following discussion, with stationarity generally referring to space or time depending on the context. Nonequispaced data sequences are also the object of current research, including the implications of breaks and gaps in geophysical observations (e.g., Palmer and Smylie, [2002]).

The multifrequency analysis of Fourier representations has led to multiscalar, multispectral and multitemporal analysis and synthesis with wavelet transforms. Since most observational data contain different information at different scales, frequencies and times, the need for multiresolution analysis and synthesis is very common in the geosciences and elsewhere. Among the best known applications are data compression, nonlinear estimation, pattern identification and feature extraction.

The principal objective of spectral analysis is most often the localization of features of interest in the spectrum of a time series. In other words, in classical spectral analysis, the frequencies where the signal exhibits maximum strength or energy are rendered in the power spectrum or the spectral density function. In multi-resolution signal analysis, localization of features of interest is possible not only in frequency windows but also in spatial and/or temporal windows, assuming available observational data. Furthermore, the spatial and/or spectral and/or temporal windows are adaptive with wavelets, unlike the fixed-width windows in Gabor analysis.

Harmonic analysis which generalizes spectral analysis in connection with boundary value problems of the spherical Laplace equation is also very important in geodesy. Spherical harmonic transforms correspond to Fourier transforms on the sphere and multiresolution analysis and synthesis also generalize LSSA in spherical applications. In view of the non commutativity of convolutions on the sphere, special considerations are required when implementing Gabor and wavelet transforms in spherical applications. Non Abelian harmonic analysis and synthesis still have many challenging research problems. In practice, more research and development are definitely warranted for gravity field and related applications.

2. Classical Fourier Analysis

The use of sinusoidal basis functions for analyzing signals goes back to Fourier's studies of heat diffusion which led to the 1807 claim that any periodic function can be represented as a series of harmonically related sinusoids. This idea had a profound impact in mathematical analysis, physics and engineering. However, it took over one and a half centuries to fully understand the convergence of Fourier series and develop the theory of Fourier integrals.

Given a (square integrable) function $h(t)$, t $\mathbb{R}$, i.e., the real line, its continuous Fourier transform is given by

$$F_c h(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \quad \text{or} \quad F_c h(\tau) = \int_{-\infty}^{\infty} h(t) e^{-i2\pi\tau t} dt$$

and the corresponding inverse is simply

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_c h(\omega) e^{+i\omega t}\, d\omega \qquad \text{or} \qquad h(t) = \int_{-\infty}^{\infty} F_c h(\tau) e^{+i2\pi\tau t}\, d\tau$$

with frequency $\omega = 2\pi\tau$, in terms of the normalized frequency τ. The Fourier transform and the periodogram $|F_c h(\tau)|^2$ have been standard tools in signal analysis for decades.

With discrete data values $\{h(0), h(\Delta t), \ldots, h((N-1)\Delta t)\}$, $N > 1$, the corresponding discrete Fourier transform is given by

$$F_D h(m\Delta\tau) = \Delta t \sum_{N=0}^{N-1} h(n\Delta t) e^{-i2\pi m\Delta\tau n\Delta t} = \Delta t \sum_{N=0}^{N-1} h(n\Delta t) e^{-i2\pi m n/N}$$

with the inverse discrete Fourier transform

$$h(n\Delta t) = \Delta\tau \sum_{m=0}^{N-1} F_D h(m\Delta\tau) e^{+i2\pi m\Delta\tau n\Delta t} = \Delta\tau \sum_{m=0}^{N-1} F_D h(m\Delta\tau) e^{+i2\pi m n/N}$$

with $N\,\Delta t\,\Delta\tau = 1$ and discrete frequencies in terms of $\Delta\tau$, within the Nyquist limits, i.e., $-(2\Delta t)^{-1} \le m\Delta\tau \le (2\Delta t)^{-1}$. The periodogram $|F_D h(m\Delta\tau)|^2$ is well known to provide, with proper smoothing, excellent approximations for the power spectrum of the original data sequence.

If one writes the discrete Fourier transform $\{F_D h(0), F_D h(\Delta\tau), \ldots, F_D h((N-1)\Delta\tau)\}$ in matrix form using the notation $W = \exp[-i2\pi/N]$, i.e.,

$$\Delta t \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W & \cdots & W^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & W^{N-1} & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} h(0) \\ h(\Delta t) \\ \cdots \\ h((N-1)\Delta t) \end{bmatrix}$$

it is clear, especially when N is a power of 2, that the symmetries in the Fourier matrix can be exploited to optimize the computations. Fast Fourier Transforms (FFTs) have revolutionized these computations with only O(NlogN) operations instead of the $O(N^2)$ operations required with the discrete Fourier transforms.

3. Gabor Harmonic Analysis

Gabor analysis goes back to his fundamental 1946 paper on a "Theory of Communication" in which he introduced a two-dimensional time-frequency analysis approach for a one-dimensional time data sequence. A windowed sinusoid leads to a "quantum of information," which is characterized by time and frequency. Gabor proposed the use of a Gaussian window, which is to be translated along the time axis of the signal. Other windows have since been used to achieve similar objectives [Feichtinger and Strohmer, 1998].

In 1948, Shannon also published "A Mathematical Theory of Communication" in which the foundations of information theory were proposed. Both Gabor and Shannon attempted to represent the time-frequency information contents in quantitative terms for communication purposes. One major difference between the two approaches is that Gabor did not consider any additive noise in his analysis. These engineering problems are related to Heisenberg uncertainty and phase space density, and Shannon's impact was certainly most significant in communications with his famous Capacity Theorem, and the estimation of an unknown signal of some known type disturbed by additive noise was solved by Kolmogorov and Weiner (e.g., Pierce, [1980]; Blais, [1988]).

In quasi-stationary contexts, the previous Fourier transform approach is naturally extended to short-time or windowed Fourier (or Gabor) transforms with continuous form

$$G_c h(t,\tau) = \int_{-\infty}^{\infty} h(s)\overline{w(t-s)}\, e^{-i2\pi s\tau} ds$$

assuming some (normalized) window function w(.), with corresponding inverse transform

$$h(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_c h(s,\tau) w(t-s) e^{+i2\pi s\tau} ds\, d\tau$$

with frequency τ. These integrals are understood in the mean square sense. The windowed Fourier (or Gabor) transform and the spectogram $|G_c h(t,\tau)|^2$ have become standard tools in signal analysis.

With discrete data values $\{h(0), h(\Delta t), \ldots, h((N\text{-}1)\Delta t)\}$, $N > 1$, the corresponding discrete short-time or windowed Fourier (or Gabor) transform is given by

$$G_D h(n\Delta t, m\Delta\tau) = \Delta t \sum_{k=0}^{N-1} h(k\Delta t)\overline{w((n-k)\Delta t)}\; e^{-i2\pi k\Delta t m\Delta\tau}$$

with the inverse discrete short-time or windowed Fourier (or Gabor) transform

$$h(n\Delta t) = \Delta\tau \sum_{k=0}^{M-1} \sum_{r=0}^{N-1} G_D h(k\Delta t, r\Delta\tau) w((n-k)\Delta t) e^{+i2\pi k\Delta t r\Delta\tau}$$

with $N \Delta t \Delta\tau = 1$ and discrete frequencies in terms of $\Delta\tau$, within the Nyquist limits, i.e. $-(2\Delta t)^{-1} \leq m\Delta\tau \leq (2\Delta t)^{-1}$. Notice that in practice, the number of "voices" M is not necessarily equal to the number N of data values and discrete frequencies. An example will illustrate the situation in section 6.

4. Multiresolution Wavelet Analysis

Wavelet theory is a generalization of Fourier analysis and such generalization can be viewed from different mathematical perspectives. At least three different approaches are common in the technical literature: functional analysis, digital signal analysis and algebraic group theory [Jaffard et al., 2001; Blais, 2002]. The functional analysis approach will be used in the following introductory discussion.

Consider sequences of real data $\{x_1, x_2, x_3, \ldots, x_N\}$ in some subspace S^N in $\mathbb{R}^N$, to be decomposed into complementary sequences $\{u_1, u_2, u_3, \ldots, u_{N/2}\}$ and $\{v_1, v_2, v_3, \ldots, v_{N/2}\}$ by means of a smoothing or scaling transformation ϕ with decimation

$$\phi : \{x_1, x_2, x_3, \ldots, x_N\} \rightarrow \{u_1, u_2, u_3, \ldots, u_{N/2}\},$$

and a detail or wavelet transformation ψ with decimation

$$\psi : \{x_1, x_2, x_3, \ldots, x_N\} \rightarrow \{v_1, v_2, v_3, \ldots, v_{N/2}\}.$$

Dyadic scaling is used here for simplicity, with N assumed to be a power of 2. With appropriate definition of the transformations ϕ and ψ, the corresponding synthesis of the original sequence is readily obtained:

$$\phi^*\{u_1, u_2, u_3, \ldots, u_{N/2}\} + \psi^*\{v_1, v_2, v_3, \ldots, v_{N/2}\} = \{x_1, x_2, x_3, \ldots, x_N\},$$

in which the superscript * indicates adjoint or dual. In terms of spaces, the decomposition

$$\phi : S^N \rightarrow U^{N/2} \qquad \text{and} \qquad \psi : S^N \rightarrow V^{N/2}$$

which implies the relationship

$$S^N = U^{N/2} \qquad V^{N/2}$$

with some appropriate summation   of the subspaces $U^{N/2}$ and $V^{N/2}$. Repeating the preceding binary decomposition leads to

$$\begin{array}{ccccccccccccc}
S^N & \rightarrow & U^{N/2} & \rightarrow & U^{N/4} & \rightarrow & \dots & \rightarrow & U^2 & \rightarrow & U^1 & \rightarrow & U^0 \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
 & & V^{N/2} & & V^{N/4} & & \dots & & V^2 & & V^1 & & V^0
\end{array}$$

and hence the complete decomposition

$$S^N = U^0 \qquad V^0 \qquad V^1 \qquad V^2 \qquad \dots \qquad V^{N/4} \qquad V^{N/2}.$$

This decomposition can be interpreted in terms of some "mean" value and detail information at different resolutions. Depending on the choice of basis functions (or equivalently, scaling and wavelet transformations), the multiresolution decomposition can be very different and hence, designed to exhibit the detail information of interest [Mallat, 1998; Teolis, 1998].

The scaling or smoothing transformation $\phi(t)$ is often associated with lowpass filtering with decimation while the detail or wavelet transformation $\psi(t)$ is associated with highpass filtering with decimation. The filtering operation is usually carried out as a convolution, which is a simple product in the Fourier domain. The decimation part cannot be interpreted as a filtering operation since it cannot be implemented as a convolution. Hence, one must be careful in the practical implementations of these multiresolution decompositions for analysis and reconstructions for synthesis.

For illustration purposes, consider the decomposition of a short data sequence $S_o := \{0, 1, 1, 2, 3, 5, 8, 13\}$ using a two-coefficient Haar filter ($\alpha_1 = \alpha_2 = 1/\sqrt{2}$). The Haar basis is really the simplest boxlike function basis. The three-level decomposition goes as follows:

$S_1 = L_\alpha \, S_o = \{0.707, 2.121, 5.657, 14.849\}$
$R_1 = H_\alpha \, S_o = \{-0.707, -0.707, -1.414, -3.536\}$
$S_2 = L_\alpha \, S_1 = \{2, 14.5\}$
$R_2 = H_\alpha \, S_1 = \{-1, -6.5\}$
$S_3 = L_\alpha \, S_2 = \{11.667\}$
$R_3 = H_\alpha \, S_2 = \{-8.839\}$

giving the wavelet transform of S_o as

$WT[S_o] = \{11.667, -8.839, -1, -6.5, -0.707, -0.707, -1.414, -3.536\}$,

where L_α and H_α correspond to the lowpass and highpass filters with taps $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1, -\alpha_2\}$, respectively, followed by decimation. Using the inverse filters, the original sequence can be recovered from the wavelet transform (computational details and other examples can be found in [Blais, 1996]). These special paraunitary filtering operations are exactly information preserving or reversible as Fourier transforms.

5. Spherical Analysis and Synthesis

Two-dimensional Fourier analysis corresponds to a periodic planar domain $[0, 2\pi) \times [0, 2\pi)$ and equivalently on a torus because of the double periodicity. The spherical surface is topologically different and requires different basis functions: sinusoidal functions in longitude and Legendre functions in latitude.

The orthogonal or Fourier expansion of a function $f(\theta, \lambda)$ on the sphere $\mathbf{S}^2$ is given by

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{|m|\le n} f_{n,m} Y_n^m(\theta,\lambda)$$

using colatitude θ and longitude λ, where the basis functions $Y_n^m(\theta,\lambda)$ are called the spherical harmonics satisfying the (spherical) Laplace equation $\Delta_S Y_n^m(\theta,\lambda) = 0$, for all $|m| \le n$ and $n = 0, 1, 2, \ldots$. This is an orthogonal decomposition in the Hilbert space $\mathbf{L}^2(\mathbf{S}^2)$ of functions square integrable with respect to the standard rotation invariant measure $d\sigma = \sin\theta \, d\theta \, d\lambda$ on $\mathbf{S}^2$. In particular, the Fourier or spherical harmonic coefficients appearing in the preceding expansion are obtained as inner products

$$f_{n,m} = \int_{S^2} f(\theta,\lambda) \overline{Y}_n^m(\theta,\lambda) d\sigma$$

in terms of the associated Legendre functions

$$\overline{Y}_n^m(\theta,\lambda) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_{nm}(\cos\theta) e^{im\lambda}$$

with the overbar denoting the complex conjugate. In most practical applications, the functions $f(\theta,\lambda)$ are band-limited in the sense that only a finite number of those coefficients are nonzero, i.e., $f_{n,m} \equiv 0$ for all $n \geq N$.

The usual geodetic spherical harmonic formulation is slightly different with

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda\right] \bar{P}_{nm}(\cos\theta)$$

where

$$\bar{C}_{nm} = \frac{1}{4\pi} \int_{S^2} f(\theta,\lambda) \cos m\lambda \, \bar{P}_{nm}(\cos\theta) \, d\sigma$$

$$\bar{S}_{nm} = \frac{1}{4\pi} \int_{S^2} f(\theta,\lambda) \sin m\lambda \, \bar{P}_{nm}(\cos\theta) \, d\sigma$$

and

$$\bar{P}_{nm}(\cos\theta) = \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}} \, P_{nm}(\cos\theta)$$

$$\bar{P}_{n}(\cos\theta) = \sqrt{2n+1} \, P_{n}(\cos\theta)$$

In this geodetic formulation, the overbars refer to the assumed normalization.

A spherical harmonic expansion satisfies all the conditions for a (harmonic) Shannon wavelet expansion. Denoting the n-th degree harmonic expansion of a function w W by w_n W_n and the n-th degree harmonic term by h_n H_n, one has the decomposition

$$\begin{array}{ccccccc} W_n & \rightarrow & W_{n-1} & \rightarrow & W_{n-2} & \rightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & H_{n-1} & & H_{n-2} & & \cdots \end{array}$$

where $W_n = W_{n-1}$ H_{n-1} and dimensionwise, $(n+1)^2 = n^2 + (2n+1)$, with corresponding (variable) scale factor $s = n / (n-1) > 1$ for all $n > 1$. For dyadic scaling, assuming n to be a power of 2, one would write

$$\begin{array}{ccccccc} W_n & \rightarrow & W_{n/2} & \rightarrow & W_{n/4} & \rightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & H'_{n/2} & & H'_{n/4} & & \cdots \end{array}$$

where $H'_{n/2}$ denotes the detail space such that $W_n = W_{n/2} \quad H'_{n/2}$ with corresponding dimensions $(n + 1)^2$, as expected (see Blais and Provins [2002] for further details).

However, as the Shannon scaling function is a simple box function in the spectral domain, oscillations can be expected in the spatial domain. This is completely analogous to the Haar scaling function implying oscillations in the spectral domain. In case of stationary processes, spherical harmonic transforms are quite adequate but with only quasi-stationarity, windowing or Gabor transform strategies are more appropriate as in planar contexts. With general nonstationary processes, the harmonic basis functions have to be replaced with more specialized wavelet functions or paraunitary filters for multiresolution analysis and synthesis. The selection of optimal bases in spherical wavelet transforms is an open problem for most applications. More research on optimal adapted wavelets for specific application areas is obviously warranted.

6. Simple Numerical Experimentation

For illustration purposes, let us consider two simulated data sequences in terms of different sinusoids:

$$T_1 := \{\cos(0.2n) + 3\sin(0.3n)\cos(0.4n) + 2\cos(0.9n) \mid n = 0, 1023\}$$

and

$$T_2 := \{\cos(0.2n) \mid n = 0, 255\} \quad \{3\sin(0.3n)\cos(0.4n) \mid n = 256, 767\}$$
$$\{2\cos(0.9n) \mid n = 768, 1023\}.$$

These are shown in Figure 1 (a) and (b), respectively. It is easy to see that the first sequence T_1 is stationary while the second T_2 is not. The situation corresponds to observations of signals with components that are not always present for one reason or another.

The periodograms for T_1 and T_2 are shown in Figure 2 (a) and (b), respectively, using the conventional logarithmic scale. The spectral peaks are properly displayed in both cases although the second periodogram displays much interference that would very difficult to analyze without additional information about the observations. The spectral power exhibited in the peaks is also quite different in the second periodogram as expected.

Using Malvar wavelet transforms, which are essentially windowed cosine transforms, the spectrograms of both data sequences are shown in Figure 3 (a) and (b), respectively. The first spectrogram shows spectral ridges at the proper frequencies (corresponding to the peaks in the periodogram), which are essentially unchanging in time. In the second spectrogram, the spectral ridges correspond very well to the situation in the three segments of T_2, and provide very useful information for the analysis of the observations in real world applications. Actually, the spectrogram is much clearer and more explicit than the corresponding periodogram for any spectral analysis of the data sequence T_2.

7. Concluding Remarks

Nonstationarity is common in geodesy and other geosciences. As the classical Fourier approach relies on sinusoidal basis functions, some generalization is required for numerous applications. The windowing strategy in Gabor transforms is perhaps the simplest modification of Fourier transforms in cases where estimating and removing trends is not really feasible. In other words, in numerous cases, the underlying process is not stationary and hence, some other strategies have to be investigated.

Windowing Fourier transforms is very powerful and readily applicable in numerous cases. However, when sinusoidal basis functions are not appropriate even in the local sense, then other (affine) wavelet basis functions have to be considered. These wavelet functions usually vanish outside some compact support. Decompositions for analysis and reconstructions for synthesis are achieved by scaling and translating these wavelet functions thus allowing multiresolution analysis and synthesis under severe nonstationarity conditions.

In geodetic applications, spherical harmonic transforms generally replace the Fourier transforms and similar strategies apply. As the spherical situation is intrinsically different from the planar cases, some computational complications are unavoidable. Furthermore, numerous geodetic applications involve voluminous datasets of scalar, vectorial and tensorial quantities with spatial and temporal dimensions. More research and development investigations are clearly warranted for computational efficiency and reliability in general nonstationary geodetic applications.

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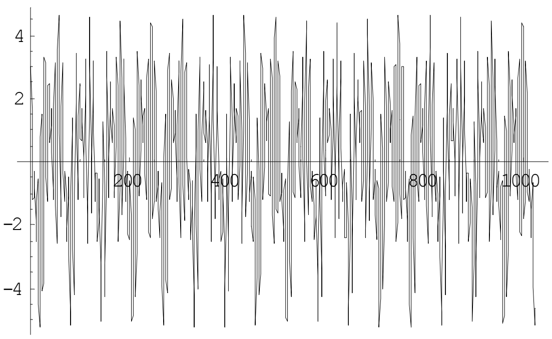

Figure 1. a) First Data Sequence T_1.

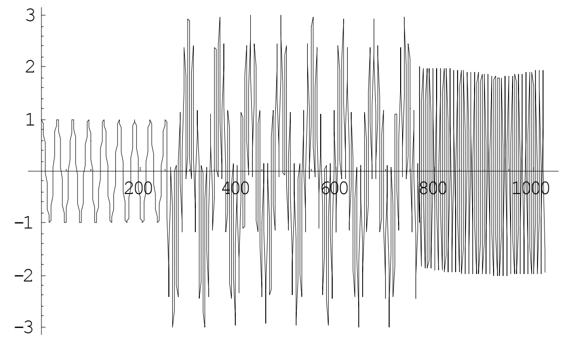

Figure 1. b) Second Data Sequence T_2.

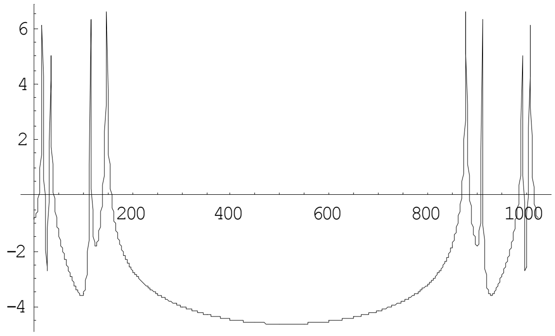

Fig. 2. a) Periodogram (log scale) of T_1.

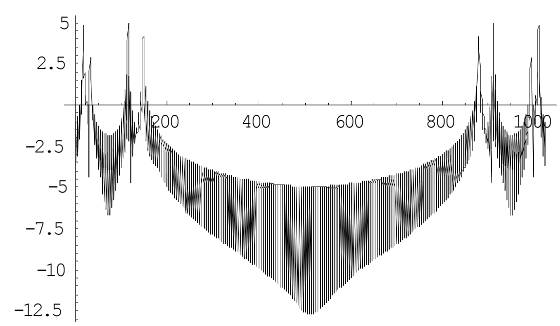

Fig. 2. b) Periodogram (log scale) of T_2.

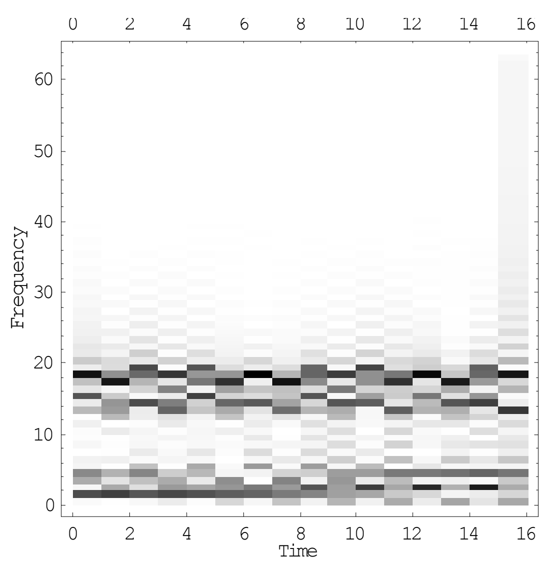

Fig. 3. a) Spectrogram Using Malvar Wavelet Transform of T_1.

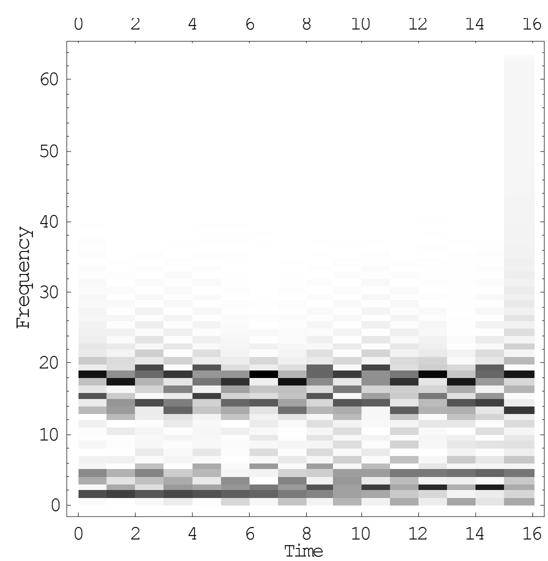

Fig. 3. b) Spectrogram Using Malvar Wavelet Transform of T_2.

A Closed Form Representation Of Somigliana- Pizzetti Gravity

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ABSTRACT

Somigliana-Pizzetti (SOM-PI) gravity, namely the modulus of the gradient of SOM-PI gravity potential of a *Level Ellipsoid* („International Reference Ellipsoid“), is computed in a closed form to the *subNanoGal level*. Such a representation of ellipsoidal gravity needed for the computation of *ellipsoidal vertical deflections* as well *as ellipsoidal geoidal / quasi-geoidal undulations* extends standard representations in the Geodesist's Handbook by *H. Moritz* and *E. Groten* . An introduction into SOM-PI gravity functionals is *P. Vaníček and E. Krakiwsky*: Geodesy: The concept, North Holland Publishing Company, Amsterdam 1982.

Introduction

The *International Gravity Formula* based upon the *Somigliana Pizzetti* reference potential of a level ellipsoid (International Reference Ellipsoid) has been always of focal geodetic interest, in particular of *P. Vaníček and E. Krakiwsky* (*1982*). The main argument to use the *Somigliana-Pizzetti* theory of the gravity field of a rotational symmetric level ellipsoid originates from the close relation to an equilibrium figure of the Earth like the *MacLaurin Equilibrium Figure*. Such an argument has dominated the textbook of *P. Vaníček and E. Krakiwsky* (*1982*) as well as of *H.Moritz* (*1990*). For instance, if we define the disturbing potential with respect to the SOM-PI reference potential we have to exclude the ellipsoidal harmonic terms of degree/order (0,0), (1,1), (1,0), (1,1), (2,0) as well as the centrifugal potential from the spectral bandlimited representation of the gravity field, namely its disturbing potential and disturbing gravity. While the SOM-PI reference potential is *not* a harmonic function - thanks to the centrifugal potential – the disturbing potential with respect to the SOM-PI reference potential *is* a harmonic function. Accordingly the horizontal as well as the vertical derivative of the disturbance potential which is related to the vertical deflections and the gravity disturbance have to be defined with respect to an ellipsoidal figure of equilibrium. A geophysical interpretation accordingly is directly possible, namely the departure of an ellipsoidal figure of equilibrium.

The drawback of SOM-PI gravity is its complicated *series representation* we may call ugly (*H. Moritz* (*2000*), *E. Groten* (*2000*)). Therefore we aim at a new closed, simple form representation of SOM-PI, accurate to the sub-nanoGal level. Indeed we hope that with our closed form representation we gain more interest in the *Geodetic Community* for equilibrium figures as a reference, namely for ellipsoidal coordinates in *Geometric Geodesy* (GPS: Global Problem Solver) as well as in Physical Geodesy (ellipsoidal harmonics) just honoring those great geodesists who finished by their spectacular expeditions for Peru and Lapland the dark Medievals when the *Plane* and the *Sphere* ruled the geodetic view of the world. Of course, we in particular hope to have brought in a particular present to *P. Vaníček*, my mentor for many, many years.

Ellipsoidal reference gravity intensity of type Somigliana-Pizzetti

Here we aim at computing the modulus of reference gravity $g(l, f, u) =: \|\mathrm{grad}\, w(l, f, u)\|$ with respect to the *ellipsoidal reference potential* of type *Somigliana-Pizzetti*. The detailed computation of $\|\mathrm{grad}\, w(l, f, u)\|$ is presented by means of *Table* 1, two lemmas and two corollaries.

Since the reference potential of type *Somigliana-Pizzetti* $w(f, u)$ depends only on spheroidal latitude f and spheroidal height u, the modulus of the reference gravity vector (1) is a nonlinear operator based upon the *lateral derivative* $\mathrm{D}_f w$ and the *vertical derivative* $\mathrm{D}_u w$. As soon as we depart from the standard representation of the gradient operator in orthogonal coordinates, namely $\mathrm{grad}\, w = \mathbf{e}_l (g_{ll})^{1/2} \mathrm{D}_l w + \mathbf{e}_f (g_{ff})^{1/2} \mathrm{D}_f w + \mathbf{e}_u (g_{uu})^{1/2} \mathrm{D}_u w$ we arrive at the standard form of $\|\mathrm{grad}\, w\|$ of type (1). Here for the *near field computation* we shall assume $x := (u^2 + e^2)^{-1} (\mathrm{D}_f w / \mathrm{D}_u w)^2 = 1$. Accordingly by means of (2) we expand $\sqrt{1+x}$ in binomial series and are led to the first order approximation of $g =: \|\mathrm{grad}\, w\|$ by (3). Obviously up to $O(2)$ it is sufficient to compute the vertical derivative $|\mathrm{D}_u w|$. An explicit version of $\mathrm{D}_u w$ is given by (4) and (5), in the form of ellipsoidal base functions by (6).

Table 1: Reference gravity intensity of type Somigliana-Pizzetti
International Gravity Formula

$$
\begin{aligned}
g &= \|\mathrm{grad}\, w(f, u)\| \\
&= \sqrt{\langle \mathrm{grad}\, w(f, u) \,|\, \mathrm{grad}\, w(f, u) \rangle} \\
&= \sqrt{\frac{1}{u^2 + e^2 \sin^2 f} (\mathrm{D}_f w)^2 + \frac{u^2 + e^2}{u^2 + e^2 \sin^2 f} (\mathrm{D}_u w)^2} \qquad (1) \\
&= \sqrt{\frac{u^2 + e^2}{u^2 + e^2 \sin^2 f}} \sqrt{(\mathrm{D}_u w)^2 + \frac{1}{u^2 + e^2} (\mathrm{D}_f w)^2}
\end{aligned}
$$

$$g = \sqrt{\frac{u^2 + e^2}{u^2 + e^2 \sin^2 f}}\,|\mathrm{D}_u\, w|\sqrt{1 + \frac{1}{u^2 + e^2}\left(\frac{\mathrm{D}_f\, w}{\mathrm{D}_u\, w}\right)^2} \qquad (2)$$

$$x := (u^2 + e^2)^{-1}(\mathrm{D}_f\, w / \mathrm{D}_u\, w)^2$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathrm{O}(x^3) \quad " \, |x| < 1$$

if $x = (u^2 + e^2)^{-1}(\mathrm{D}_f\, w / \mathrm{D}_u\, w)^2 < 1$, then

$$g = \sqrt{\frac{u^2 + e^2}{u^2 + e^2 \sin^2 f}}\,|\mathrm{D}_u\, w| + \mathrm{O}(2) \qquad (3)$$

"the first version of $\mathrm{D}_u\, w$"

$$\begin{aligned} \mathrm{D}_u w = &-\frac{GM}{u^2 + e^2} + \frac{1}{6}\mathrm{W}^2 a^2 (3\sin^2 f - 1) \times \\ &\times \frac{6u/e^2 \operatorname{arc\,cot}(u/e) - (3u^2/e^2 + 1)e/(e^2 + u^2) - 3/e}{(3b^2/e^2 + 1)\operatorname{arc\,cot}(b/e) - 3b/e} \\ &+ \mathrm{W}^2 u \cos^2 f \end{aligned} \qquad (4)$$

$$\begin{aligned} \mathrm{D}_u w = &- \frac{GM}{u^2 + e^2} \\ &- \frac{1}{6}\mathrm{W}^2 a^2 \frac{1}{u^2 + e^2} \frac{6(u^2/e^2 + 1)u \operatorname{arccot}(u/e) - 2e(3u^2/e^2 + 2)}{(3b^2/e^2 + 1)\operatorname{arccot} b/e - 3b/e} \\ &+ \frac{1}{2}\mathrm{W}^2 a^2 \frac{\sin^2 f}{u^2 + e^2} \frac{6(u^2/e^2 + 1)u \operatorname{arccot}(u/e) - 2e(3u^2/e^2 + 2)}{(3b^2/e^2 + 1)\operatorname{arccot} b/e - 3b/e} \\ &+ \mathrm{W}^2 u \cos^2 f \end{aligned} \qquad (5)$$

"the second version of $\mathrm{D}_u\, w$"

$$D_u w = \frac{GM}{e}\left[Q_{00}^*\left(\frac{u}{e}\right)\right]' + \frac{\sqrt{5}}{15} W^2 a^2 P_{20}^*(\sin f)\frac{[Q_{20}^*(u/e)]'}{Q_{20}^*(b/e)} \qquad (6)$$

A more useful closed-form representation of the reference gravity intensity of type Somigliana-Pizzetti will be given by two lemmas and two corollaries. *First* if we collect the coefficients of $\{1, \cos^2 f, \sin^2 f\}$ by $\{g_0, g_c, g_s\}$ we are led to the representation of g by (7) - (10) elegantly expressed in the *first lemma*. *Second* if we take advantage of the (a,b) representation of $\sqrt{u^2 + e^2} / \sqrt{u^2 + e^2 \sin^2 f}$ and decompose g_0 according to $g_0(\sin^2 f + \cos^2 f)$ we are led to the alternative elegant representation of g by (11) - (18) presented in the *second lemma*.

Lemma (Reference gravity intensity of type Somigliana-Pizzetti,
International Gravity Formula):

If $x := (u^2 + e^2)^{-1}(\mathrm{D}_f w / \mathrm{D}_u w)^2 = 1$ holds where $w = w(f, u)$ is the reference potential field of *type Somigliana-Pizzetti*, then its gravity field intensity $g(f, u)$ can be represented up to the order $\mathrm{O}(2)$ by

$$g = \sqrt{\frac{u^2 + e^2}{u^2 + e^2 \sin^2 f}} \left| g_0 + g_c \cos^2 f + g_s \sin^2 f \right| \quad (7)$$

subject to

$$\begin{aligned} g_0 := &-\frac{GM}{a^2 + (u+b)(u-b)} - \frac{1}{3} \mathrm{W}^2 a^2 \frac{1}{a^2 + (u+b)(u-b)} \times \\ &\times \frac{3(u^2/e^2 + 1)u \operatorname{arccot}(u/e) - e(3u^2/e^2 + 2)}{(3b^2/e^2 + 1)\operatorname{arccot}(\mathrm{b/e}) - 3\mathrm{b/e}} \end{aligned} \quad (8)$$

$$g_c := \mathrm{W}^2 \mathrm{u} \quad (9)$$

$$\begin{aligned} g_s := &\mathrm{W}^2 a^2 \frac{1}{a^2 + (u+b)(u-b)} \times \\ &\times \frac{3(u^2/e^2 + 1)u \operatorname{arccot}(u/e) - e(3u^2/e^2 + 2)}{(3b^2/e^2 + 1)\operatorname{arccot}(\mathrm{b/e}) - 3\mathrm{b/e}} \end{aligned} \quad (10)$$

Lemma (Reference gravity intensity of type Somigliana-Pizzetti,
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$$\sqrt{\frac{u^2+e^2}{u^2+e^2\sin^2 f}} = \frac{\sqrt{a^2+(u+b)(u-b)}}{\sqrt{a^2\sin^2 f + b^2\cos^2 f + (u+b)(u-b)}} \quad (11)$$

$$g_0 + g_c \cos^2 f + g_s \sin^2 f = g_a \cos^2 f + g_b \sin^2 f \quad (12)$$

$$g_a := g_0 + g_c \quad (13)$$

$$g_b := g_0 + g_s \quad (14)$$

$$\mathrm{G}_a := (g_0 + g_c)\sqrt{1 + e^2/u^2} \quad (15)$$

$$\mathrm{G}_b := g_b = g_0 + g_s \quad (16)$$

$$g(u,f) = \sqrt{a^2+(u+b)(u-b)}\frac{\left|g_b \sin^2 f + g_a \cos^2 f\right|}{\sqrt{a^2\sin^2 f + b^2\cos^2 f + (u+b)(u-b)}} \quad (17)$$

$$g(u,f) = \frac{\left|\sqrt{a^2+(u+b)(u-b)}\ \mathrm{G}_b \sin^2 f + u\ \mathrm{G}_a \cos^2 f\right|}{\sqrt{a^2\sin^2 f + b^2\cos^2 f + (u+b)(u-b)}} \quad (18)$$

Two special cases of the modulus of the reference gravity vector of *type Somigliana-Pizzetti* will be finally presented. By means of the *first corollary* we specialize to $u = b$, namely assuming a location of points on the *level ellipsoid* $\mathrm{E}^2_{a,b}$, the *International Reference Ellipsoid*. In contrast, in the *second corollary* we present to you results of a computation of the *International Gravity Formula* at the equator $f = 0$ and the poles $f = \pm \frac{p}{2}$. On the one side we find the characteristic formulae (19) - (21) for $u = b$, on the other side (22) - (25) for $f = \pm \frac{p}{2}, u\ ^1\ b$ *and* $u = b$.

Corollary (Reference gravity intensity of type Somigliana-Pizzetti,
special case: level ellipsoid $u = b$):

If $(\mathrm{D}_f w / \mathrm{D}_u w)^2 / a^2 = 1$ holds where $w = w(f,u)$ is the reference potential field of *type Somigliana-Pizzetti* on the surface of the level ellipsoid $\mathrm{E}^2_{a,b}$, then its gravity field intensity $g(f,b)$ can be represented up to the order $\mathrm{O}(2)$ by

$$g(b,f) = \frac{\left|a\ \mathrm{G}_b \sin^2 f + b\ \mathrm{G}_a \cos^2 f\right|}{\sqrt{a^2\sin^2 f + b^2\cos^2 f}} \quad (19)$$

subject to

$$\Gamma_a := (g_0 + g_c)\sqrt{1+e^2/b^2} =$$
$$-\frac{GM}{ab} - \frac{1}{3}\frac{a}{b}\Omega^2 \frac{3(b^2/e^2+1)b\,\mathrm{arccot}(b/e) - e(3b^2/e^2+2)}{(3b^2/e^2+1)\,\mathrm{arccot}(b/e) - 3b/e} \quad (20)$$
$$+\,\Omega^2 a\sqrt{1+e^2/b^2}$$

$$\Gamma_b := g_0 + g_s$$
$$= -\frac{GM}{a^2} + \frac{2}{3}\Omega^2 \frac{3(b^2/e^2+1)b\,\mathrm{arccot}(b/e) - e(3b^2/e^2+2)}{(3b^2/e^2+1)\,\mathrm{arccot}(b/e) - 3b/e} \quad (21)$$

Corollary (Reference gravity intensity of type Somigliana-Pizzetti, special cases: (i) $\phi = 0$ and (ii) $\phi = \pm\frac{\pi}{2}$):

If $(\partial_\phi w / \partial_u w)^2/(u^2+e^2) = 1$ holds where $w = w(\phi, u)$ is the reference potential field of *type Somigliana-Pizzetti*, then its gravity field intensity $g(\phi, b)$ can be represented up to the order $O(2)$ by

(i) at the equator $\phi = 0$

$$g(u, \phi = 0) = |\Gamma_a| \quad (22)$$

(ii) at the poles $\phi = \pm\frac{\pi}{2}$

$$g(u, \phi = \pm\tfrac{\pi}{2}) = |\Gamma_b| \quad (23)$$

(iii) at the equator / level ellipsoid: $u = b, \phi = 0$:

$$g(b, \phi = 0) = |\Gamma_a|(u = b) \quad (24)$$

(iv) at the poles / level ellipsoid: $u = b, \phi = \pm\frac{\pi}{2}$:

$$g(b, \phi = \pm\tfrac{\pi}{2}) = \Gamma_b(u = b) \quad (25)$$

A short summary

Indeed by means of $g(u, \phi)$ in (17), (18) we have found an ideal closed form representation of the reference gravity intensity of type Somigliana-Pizzetti. The related innovative computer program requires only the information about the spheroidal coordinates (ϕ, u) as well as the semi-major axis *a* and semi-minor axis *b* of the *International Reference Ellipsoid* (E.Grafarend and A.Ardalan, 1999). In addition we have to take reference to $\{g_0, g_C, g_S\}$ of type (7)-(10) subject to the linear excentricity $e := \sqrt{a^2 - b^2}$ as well as to $\{g_a, g_b\}$, $\{G_a, G_b\}$. In summarizing, such a computer programm is easily established avoiding all types of series expansions.

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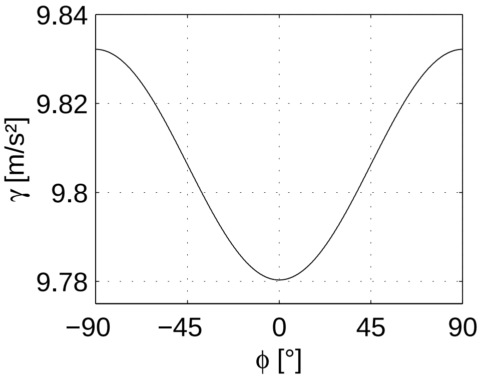

Fig. 1. Reference gravity intensity of type Somigliana-Pizzetti $g(u = b, f)$.

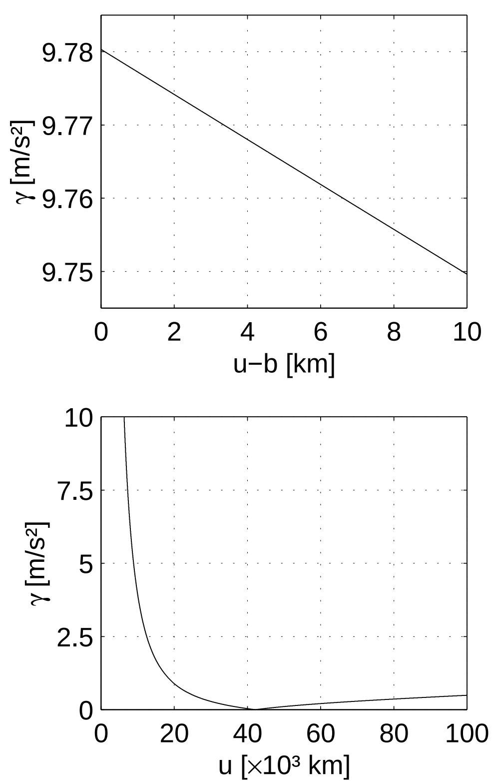

Fig. 2. Reference gravity intensity of type Somigliana-Pizzetti $g(u, f = 0)$ above the ellipsoid, near zone (horizontal axis $u - b$), far zone (horizontal axis u).

Geoid Determination From Ground and Aerial Gravity

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ABSTRACT

In this contribution two mathematical models for determination of the regional geoid from discrete observations of gravity on ground and at low-altitude aircraft are reviewed. Both cases are dealt with as completely independent, i.e. no data combination is considered herein. It is shown that the mathematical model for geoid determination from both ground and aerial gravity can rely on a single integral formula: in the case of ground data an inverse of the Fredholm integral equation of the first kind must numerically be solved while aerial data allow for a direct use of specifically developed Green's integral formula. Thus both models may be based on a single-step integration that has several favorable characteristics among which the most important are fast computer evaluation and efficient data utilization. Numerical attributes of the two models are tested using a set of high-frequency synthetic geopotential coefficients.

1. Introduction

The geoid is an equipotential surface of the Earth's gravity field corresponding to idealized calm oceans (Gauss, 1828). Its vertical deviations from the reference ellipsoid are called geoidal undulations (Listing, 1873). Their determination from discrete observations of gravity is based on potential theory and its boundary-value problems to the Laplace differential equation, see e.g. (Kellogg, 1929).

If ground gravity is used for geoid determination, an inverse one-step procedure based on transformation of ground gravity anomalies into the disturbing gravity potential at the sea level can be formulated. In this approach, the harmonic downward continuation of gravity anomalies to the sea level (roughening step) and their transformation into the disturbing gravity potential at the sea level (smoothing step) are merged in a single integral equation. As a result, a highly efficient and relatively stable mathematical model is obtained. In contrary, aerial data, due to their specific characteristics, allow for formulation of a direct one-step integration model (Novák and Heck, 2002). This approach is based on the direct integration formula while the approach for ground gravity must unfortunately rely on the inverse of the

Fredholm integral formula of the first kind.

The mathematical models for ground and aerial data are described in detail in Section 2. Since both ground and aerial gravity data are always available over limited regions only, a spherical cap integration is applied. Omitted global gravity is considered by calculating so-called truncation errors on the basis of a global geopotential model (GGM). The models are tested for accuracy and stability using noisy synthetic gravity data in Section 3. Results are then discussed in Section 4 that contains also conclusions of the presented research.

2. Theory

Assuming discrete observations of gravity g, the problem is to determine the disturbing gravity potential T at the sea level (if orthometric heights of observation points are known) or the reference ellipsoid (if geodetic heights are known). It is assumed in the following that the potential T is a harmonic function everywhere outside the geoid/reference ellipsoid, i.e. both T and its derivatives can be expanded at every point outside the geoid/reference ellipsoid into a convergent series of some harmonic base functions (such as spherical herein). Moreover, all harmonics of degree 0 and 1 are assumed to be missing. While the former requirement is achieved through the remove step of gravity reduction, the latter assumption can be satisfied by a selection of a properly oriented and positioned reference ellipsoid that generates the normal gravity field.

The so-called spherical approximation of used boundary surfaces is employed in the definition of boundary-value problems, i.e. the geoid/reference ellipsoid is approximated by a geocentric sphere of radius R. This geometric approximation makes all formulae simpler but also less accurate. The argument of lower accuracy is, however, not that important due to the local character of geoid computations (the radius R can be selected in such a way that the reference sphere fits the actually boundary in a local sense). The spherical approximation is used for providing readers with a simplified idea of the one-step models that can be applied in the local computations with a relatively high level of accuracy. If higher accuracy is required for global applications, the formulations can be changed correspondingly for any higher degree geometric approximation of the actual boundary surfaces. The geocentric spherical coordinates $(r,\varphi,\lambda)=(r,\Omega)$, that are used in formulations of mathematical models, can be defined in terms of their transformations into the Cartesian system (e.g. Heiskanen and Moritz, 1967, Eq. 1-36)

$$(x,\ y,\ z)^T = (r\ \cos\varphi\ \cos\lambda,\ r\ \cos\varphi\ \sin\lambda,\ r\ \sin\varphi)^T\ . \tag{1}$$

2.1 Ground gravity

In the spherical approximation of the geoid (the geoid is in the following approximated by the geocentric sphere of radius *R*), the following formulation can be given to the problem of geoid determination from ground gravity data

$$\nabla^2 \; T(r,\Omega) = 0 \;, \quad \text{for } r > R \;,$$

$$\Delta g(r,\Omega) = -\frac{\partial T(r,\Omega)}{\partial r}\bigg|_r \; -\frac{2}{r} \; T(r,\Omega) \;, \quad \text{for } r = R + H(\Omega) \;, \tag{2}$$

$$T(r,\Omega) = O(r^{-3}) \;, \quad \text{for } r \to \infty \;.$$

O stands for the Landau symbol, Δg for gravity anomalies and *H* for known orthometric heights (function of the geocentric direction Ω) of the Earth's surface. Equation (2) describes only approximately the actual problem: the definition of the ground gravity anomaly is slightly idealized and might not be sufficient for real data computations. Since this problem is handled elsewhere (Martinec, 1998, Eq. 11.2), the simplified spherical form of the boundary condition in Eq. (2) is kept without any additional correcting terms.

The unknown in the above problem is the function $T(R,\Omega)$, i.e. the disturbing gravity potential at the reference sphere. The transformation of observed values $\Delta g(R+H,\Omega)$ into the sought function $T(R,\Omega)$ can be done in one step by an inverse of the following integral equation

$$\Delta g(R+H,\Omega) = \frac{1}{4\pi R} \iint_{\Theta} T(R,\Omega') \; J(H,\psi) \; d\Omega' \;, \tag{3}$$

where the corresponding Green function is of the spectral form

$$J(H,\psi) = \sum_{n=2}^{\infty} \; (n-1) \; (2n+1) \; \left(\frac{R}{R+H}\right)^{n+2} \; P_n(\cos\psi) \;. \tag{4}$$

Equation (3) can easily be derived from the upward continuation integral for the gravity anomaly expanding the function Δg as well as the Poisson kernel function into a series of the Legendre polynomials. The spherical distance ψ between geocentric directions Ω = (φ,λ) and Ω' = (φ',λ') can be computed by spherical trigonometry (law of cosines)

$$\cos\psi = \cos\varphi \;\; \cos\varphi' + \sin\varphi \;\; \sin\varphi' \;\; \cos(\lambda - \lambda') \;, \tag{5}$$

and P_n stands for the Legendre polynomials of degree n (e.g. Heiskanen and Moritz, 1967, Eq. 1-57'). Introducing the unitless parameter

$$\xi = \frac{R}{R+H} \ , \tag{6}$$

the spatial form of the integration kernel J can be derived as follows

$$J(\xi,\psi) = \frac{-\xi^2 + 5\xi^3 \cos\psi - 5\xi^4 - \xi^5 \cos\psi + 2\xi^6}{L^5(\xi,\psi)} \ + \ \xi^2 \ . \tag{7}$$

The normalized distance function L is

$$L(\xi,\psi) = \sqrt{1+\xi^2 - 2\xi \cos\psi} \ . \tag{8}$$

The global integration over the full spatial angle Θ in Eq. (3) cannot be, however, carried out in the global sense due to the lack of observed data. For practical computations, Eq. (3) can only be approximated by applying for example a discrete integration over a spherical cap centered at the computation point and calculating the truncation errors by the aid of the GGM. A simple quadrature rule yields for the i-th computation point

$$\Delta g(R+H,\Omega_i) - \frac{GM}{2R^2} \sum_{k=2}^{M_k} U_k(H_i,\psi_o) \ T_k(\Omega_i) = \frac{1}{4\pi R} \sum_{j=1}^{M_j} T(R,\Omega_j) \ J(H_j,\psi_{ij}) \ \Delta\Omega_j \ . \tag{9}$$

GM denotes the geocentric gravitational constant, $\Delta\Omega_j$ is the area of the trapezoidal cell corresponding to the j-th geographical grid, M_j is the number of gravity points within the spherical cap of radius ψ_o and T_k are the Laplace harmonics of the disturbing gravity potential derived for example from the EGM96 (Lemoine et al., 1998) of maximum degree M_k=360. The truncation coefficients U_k in Eq. (9) can be derived according to Molodenskij's theory as follows (Molodenskij et al., 1960)

$$U_k(H_i,\psi_o) = \int_{\psi=\psi_o}^{\pi} J(H_i,\psi) \ P_k(\cos\psi) \ \sin\psi \ d\psi \ , \quad \text{for } 2 \le k \le M_k \ . \tag{10}$$

The matrix form of Eq. (3) reads

$$\mathbf{\Delta g}(R+H) = \mathbf{A} \ \mathbf{T}(R) \ , \tag{11}$$

where the known vector $\boldsymbol{\Delta g}(R+H)$ is reduced by the effect of data outside the spherical cap given by the spherical harmonic series on the left-hand side of Eq. (9). The system of linear equations (11) for the unknown column vector $\boldsymbol{T}(R)$ is uniquely solvable only if $\boldsymbol{A}$ is a square matrix, i.e. if the dimensions of $\boldsymbol{T}(R)$ and $\boldsymbol{\Delta g}(R+H)$ are identical. The solution of Eq. (11) can be complicated by a possibly ill-conditioned matrix operator $\boldsymbol{A}$. Moreover, small variations in $\boldsymbol{\Delta g}(R+H)$, such as observation errors, can cause large and unrealistic variations in calculated $\boldsymbol{T}(R)$. In other words, the solution can be numerically unstable and regularization of the model (smoothing) may be the only alternative to arrive at some reasonable numerical solution.

2.2 Aerial gravity

The situation is slightly different in the case of aerial gravity. Due to the noisy nature of aerial gravity observations, caused by flight dynamics, the recorded gravity signal must be low-pass filtered which results in recovery of frequency-limited gravity information - the maximum spatial resolution (full wavelength) of currently available aerial gravity is at the level of 5 to 10 km. This frequency limitation has a positive impact on the mathematical model, however, stabilizing significantly its numerical evaluation. Assuming that frequency-limited gravity disturbances δg^{ℓ} are collected at the flight level (assumed constant), the problem is to determine the corresponding disturbing gravity potential T^{ℓ} at the reference ellipsoid. In the spherical approximation, the following formulation can be given to this problem

$$\nabla^2 T^{\ell}(r,\Omega) = 0 \ , \quad \text{for } r > R \ ,$$

$$\delta g^{\ell}(r,\Omega) = -\left.\frac{\partial T^{\ell}(r,\Omega)}{\partial r}\right|_r \ , \quad \text{for } r = R + D = const. \ , \tag{12}$$

$$T^{\ell}(r,\Omega) = O(r^{-3}) \ , \quad \text{for } r \to \infty \ ,$$

where ℓ is the maximum degree in the spherical harmonic representation of the frequency-limited gravity disturbances Δg^{ℓ} and D is a known constant ellipsoidal height of the flight trajectory. The definition of the aerial gravity disturbance is also slightly idealized and the missing correcting terms to the directional derivative must be included for higher accuracy.

A direct solution of the pseudo-boundary value problem (12) for the unknown frequency-limited disturbing gravity potential T^{ℓ} at the reference sphere of radius R can be derived. This procedure results in the following solution formula (Novák and Heck, 2002)

$$T^{\ell}(R,\Omega) = \frac{R+D}{4\pi} \iint_{\Theta} \delta \mathrm{g}^{\ell}(R+D,\Omega')\; K^{\ell}(D,\psi)\; d\Omega' \;. \tag{13}$$

The frequency-limited integration kernel K^{ℓ} reads (ibid.)

$$K^{\ell}(D,\psi) = \sum_{n=2}^{\ell} \frac{2n+1}{n+1} \left(\frac{\mathrm{R+D}}{\mathrm{R}}\right)^{n+2} P_n(\cos\psi) \;. \tag{14}$$

The solution (13) can obviously be used for certain fixed values of parameters ℓ and D only. In the case of actual airborne data, approximate maximum values of 6 km for D and 4000 for ℓ can be expected (for minimum data resolution of 4 km full wavelength). Both values would guarantee stable numerical evaluation of the integral (13). In contrary to the kernel function J, a closed-form expression for the kernel function K^{ℓ} cannot be derived.

Applying the quadrature rule and considering the truncation error resulting from the spherical cap integration, the discretized integral (13) takes the form, cf. Eq. (9),

$$\begin{aligned} &T^{\ell}(R,\Omega_i) - \frac{GM}{2(R+D)} \sum_{k=2}^{M_k} (k+1) \left(\frac{R}{R+D}\right)^{k+1} V_k(D,\psi_o)\; T_k(\Omega_i) \\ &= \frac{R+D}{4\pi} \sum_{j=1}^{M_j} \delta g^{\ell}(R+D,\Omega_j)\; K^{\ell}(D,\psi_{ij})\; \Delta\Omega_j \;. \end{aligned} \tag{15}$$

The truncation coefficients V_k of the function K^{ℓ} are

$$V_k(D,\psi_o) = \sum_{n=2}^{\ell} \frac{2n+1}{n+1} \left(\frac{R+D}{R}\right)^{n+1} R_{n,k}(\psi_o) \;, \quad \text{for } 2 \le k \le M_k \;. \tag{16}$$

The coefficients $R_{n,k}$

$$R_{n,k}(\psi_o) = \int_{\psi=\psi_o}^{\pi} P_n(\cos\psi)\; P_k(\cos\psi)\; \sin\psi\; d\psi \;, \tag{17}$$

can conveniently be computed by an iterative expression (Paul, 1973, Eq. 5).

The matrix form of Eq. (13) can concisely be written as

$$\mathbf{T}^{\ell}(R) = \mathbf{B}\ \boldsymbol{\delta}\mathbf{g}^{\ell}(R+H)\ , \tag{18}$$

where the unknown vector $\boldsymbol{T}^{\ell}(R)$ is reduced by the contribution of the corresponding remote-zone data given by the spherical harmonic series on the left-hand side of Eq. (15). Since no inverse is involved in Eq. (18), no instabilities are expected in the numerical evaluation of these relationships. It should also be mentioned that - in contrast to the approach for ground data - the number of unknowns in the vector $\boldsymbol{T}^{\ell}(R)$ and data points in the vector $\boldsymbol{\delta g}^{\ell}\ (R + H)$ may generally be different.

3. Numerical tests

To test the numerical accuracy and stability of the one-step model for the ground gravity data, a synthetic GGM (SGM) is used for computation of discrete frequency-limited ground gravity anomalies

$$\Delta g(R+H,\Omega) = \frac{GM}{R^2} \sum_{n=181}^{2160} (n-1) \left(\frac{R}{R+H} \right)^{n+2} T_n(\Omega)\ . \tag{19}$$

The minimum degree 181, that corresponds to the reference field of degree 180, reflects the radius of the integration domain $\psi_o = 1°$. The maximum degree 2160 then corresponds to discretization of the gravity field at the 5 arcmin level (approximate full wavelength of 9 km). The test area is limited by the parallels of 49° and 52° northern latitude, and by the meridians of 240° and 245° eastern longitude. Topography of this area represents one of the most rugged parts of the Canadian Rocky Mountains with heights H ranging between 400 and 3000 m. This area corresponds to 2160 discrete values of the ground gravity anomalies defined at the homogeneous geographical grid of 5 x 5 arcmin. This resolution is used due to memory limitations related to the evaluation of the inverse of the Fredholm integral equation (3). Due to the integration radius $\psi_o = 1°$, the computation area is bounded by latitude of 50° and 51°, and by longitude of 242° and 243° in order to avoid the edge effects in the results. This area corresponds to 144 computation points at the grid of 5 x 5 arcmin. The random noise ε^g of 1 mGal is finally added to the synthetic ground gravity data.

The SGM is used also in the case of aerial gravity disturbances

$$\delta g(R+D,\Omega) = \frac{GM}{R^2} \sum_{n=181}^{2160} (n+1) \left(\frac{R}{R+D} \right)^{n+2} T_n(\Omega)\ . \tag{20}$$

Due to the direct solution of the integral formula (13), the aerial data can be generated at the denser grid of 2.5 x 2.5 arcmin. D = 4000 m is used in the testing computations as a typical flight height. The test area is now limited by the parallels of 49° and 52° northern latitude, and by the meridians of 238° and 243° eastern longitude. This area corresponds to 8640 values of the aerial gravity disturbances with the resolution of 2.5 arcmin. Due to the integration radius ψ_o = 1° used in the integration, the computation area is bounded by latitude of 50° and 51°, and by longitude of 240° and 241° in order to avoid the edge effects in the results. This corresponds to 576 computation points with the spatial resolution of 2.5 arcmin. The random noise ε^g of 1.5 mGal is added to the synthetic aerial gravity data reflecting a higher noise level of aerial gravity.

The unknown disturbing gravity potential T at the reference sphere is then solved for using the two one-step models based on the single integral formulas in Eqs. (9) and (15). Obtained results for the disturbing gravity potential based on the ground or aerial data are compared against reference values of the disturbing gravity potential computed directly from the SGM by the series expansion

$$T(R,\Omega) = \frac{GM}{R} \sum_{n=181}^{2160} T_n(\Omega) \ . \tag{21}$$

The differences ε^T between the reference and estimated values are converted into geoidal errors using the Bruns formula that reads in the spherical approximation (e.g. Heiskanen and Moritz, 1967, Eq. 2-144)

$$\varepsilon^N = \frac{\varepsilon^T(R,\Omega)}{\gamma} \ , \tag{22}$$

with normal gravity γ. Results are tabulated in Table 1. The propagated noise can be characterized by the magnitude of the root mean square (r.m.s.) error of differences ε^N, see the last column in Table 1.

Table 1 : *Geoid noise ε^N (m)*

data	minimum	maximum	Mean	sigma	r.m.s.
ground	-0.024	+0.036	+0.006	0.010	0.011
aerial	-0.020	+0.025	+0.006	0.009	0.011

The magnitude of the propagated noise ε^N is relatively small for both the ground and aerial data, see Table 1. It can be characterized by the value of 1 cm. This result is quite surprising especially in the case of the ground gravity data. Although the inverse of the Fredholm integral formula of the first kind is solved in the case of the ground data, the propagated noise for the ground and aerial data do not differ dramatically. The noise ε^N also remains relatively random although some correlation with the magnitude of input data can be found in the case of the ground gravity data. Both models are then numerically very efficient requiring only one step of integration for the computation of the solution.

4. Conclusions

The determination of the disturbing gravity potential at the reference ellipsoid/sea level from discrete ground and aerial gravity data is discussed in this contribution. Simulated ground and aerial gravity data, generated from the SGM and distorted by the random noise, are used for numerical testing of the two one-step models. Obtained results indicate a very good accuracy and stability of numerical solutions for both ground and aerial gravity data.

In both cases, a significantly higher numerical efficiency of geoid determination using the one-step formulas can be achieved. Instead of two integrals in the classical approach for geoid determination (harmonic downward continuation and Stokes's or Hotine's integration), only one integral equation has to be evaluated that results in faster computer algorithms. Another significant advantage of the one-step models lies in the more efficient utilization of input gravity data. Since only one integration must be numerically evaluated, the actual computation area is obtained by reducing the input data area only once (to avoid edge effects in the results). Thus the amount of gravity data required for geoid determination over a certain area can be significantly reduced. Last but not least, one should emphasize the advantage of higher numerical stability of the one-step models, especially in the case of ground gravity. Although the input gravity data are distorted by random errors at 10% level of their original values, the solution does not differ dramatically from the reference values.

The r.m.s. errors in Table 1 could indicate that the accuracy of frequency-limited geoid computations is in one-centimetre range. It would be a great mistake, however, to expect such an accuracy from actual geoid computations. Actual gravity data contain correlated noise that originates in data collection and preprocessing procedures. It is impossible to simulate such errors in all their possible complexity and variety. Moreover, the final geoid models suffer from some additional errors, such as those originating in topographical reduction and the reference field, which vary largely with geographical locations. While the errors associated with the global models are expected to decrease dramatically in the near future due to new satellite missions dedicated to global mapping of the Earth's gravity field, topography related effects will most likely remain the major obstacle in achieving higher accuracy of residual geoid determination.

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Accuracy of Analytical Downward Continuation

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ABSTRACT

Through synthetic tests and comparisons with the Poisson downward continuation, it is shown that the analytical downward continuation can introduce an error of 10% of the total downward continuation effect into the geoid in the Canadian Rocky Mountains.

1. Introduction

Downward continuation is a fundamental problem in physical geodesy due to the fact that gravity measurements are usually performed on the Earth's surface, in the air and space, while they are required on the geoid or telluroid for the solution of the geodetic boundary value problem. It is classified as an ill- or improperly-posed problem in geodetic literature limiting its practical applications, especially to the geoid determination. Under the leadership of Prof. Vaníček, the geodesy group at UNB started to solve this problem more than a decade ago. After a continuous effort [e.g. Vaníček et al. 1996; Sun and Vaníček 1998; Wong 2001; Huang 2002, etc.], the downward continuation has become a routine technique for the determination of the precise geoid. The research work presented in this paper was part of the effort.

The analytical downward continuation (ADC) is based on the Taylor series expansion of the gravity anomaly. One open question is whether it leads to a solution that is as good as the discrete Poisson downward continuation (DDC) when the disturbing potential is harmonic in the domain above the geoid. Previous studies focused on comparisons of the planar approximation that introduces a few decimeters of error into the geoid (e.g., Wang [1988]; Sideris et al. [1999]). In this study, a numerical analysis is conducted to evaluate the accuracy of the analytical downward continuation of the spherical approximation to verify if it meets the need for the precise geoid determination.

2. Theory

If $\Delta g(r_g,\Omega)$ is known on the geoid at (r_g,Ω) and has derivatives of all orders $\Delta g^{(n)}(r_g,\Omega)$ in the closed interval $[r_g, r_g + H]$, then

$$\Delta g(r_g + H, \Omega) = \sum_{n=0}^{\infty} \frac{\Delta g^{(n)}(r_g, \Omega)}{n!} H^n, \tag{1}$$

where Ω is geocentric angle denoting the pair (λ, ϕ), r_g is the radius of point on the geoid, and H is the orthometric height of a point above the geoid. This series may be symbolically written as

$$\Delta g(r_g + H, \Omega) = U \Delta g(r_g, \Omega), \tag{2}$$

where the symbol U denotes the upward continuation operator which is applied to the function $\Delta g(r_g, \Omega)$.

Given $\Delta g(r_g + H, \Omega)$ at the Earth's surface, the solution of $\Delta g(r_g, \Omega)$ on the geoid can be expressed as follows [Moritz 1980, section 45]

$$\Delta g(r_g, \Omega) = \sum_{n=0}^{\infty} g_n, \tag{3}$$

where

$$\begin{aligned} g_0 &= \Delta g(r_g + H, \Omega), \\ g_1 &= -H \;\; L_1(g_0), \\ g_2 &= -H \;\; L_1(g_1) - H^2 \;\; L_2(g_0) \\ &\cdots, \\ g_n &= -\sum_{r=1}^{n} H^r L_r(g_{n-r}). \end{aligned} \tag{4}$$

with an L operator

$$L(g) = \frac{R^2}{2\pi} \int_{\Omega'} \frac{g - g_p}{d^3} d\Omega' - \frac{1}{R} g_p, \tag{5}$$

$$L_n = \frac{1}{n!} L^n = \frac{1}{n} L L_{n-1}, \tag{6}$$

and $d = 2R\sin(\psi/2)$ is the distance between the computation point P and an integration point.

The operator L may be interpreted as a vertical gradient on a spherical surface if the gravity anomaly is known on the surface. However, for the downward continuation, it merely represents the first-order approximation to the gradient since the gravity anomaly is known on the irregular Earth surface. Equation (6) shows that the higher-order terms L_n can be recursively evaluated from the lower-order terms L_{n-1}. This relation can be used in the computation.

The analytical downward continuation is a forward problem. It is divergent with increasing resolution [Vaníček, pers. comm. 2000]. What kind of relation exists between the instability of the inverse Poisson problem and the divergence of the analytical continuation is still an open question. As far as the convergence is concerned, a deep and extensive discussion can be found in the literature [Moritz 1980; Jekeli 1981]. This issue will not be discussed here. The main question to be answered by this research is whether both methods are numerically equivalent.

3. Computational Method

The analytical downward continuation requires a number of evaluations of the convolution integral in equation (5) depending on the number of terms taken. If we take n up to 5, the integral will be evaluated 15 times in the region of interest. Thus, an efficient numerical algorithm is needed for its evaluation.

Similar to the Stokes kernel, the L_n operator kernel is a function of the spherical distance ψ that can be expressed as

$$\psi = \arccos[\sin\varphi \sin\varphi' + \cos\varphi \cos\varphi' \cos(\lambda' - \lambda)] \tag{7}$$

where φ and λ are the latitude and longitude of the computation point P, respectively, φ' and λ' are the latitude and longitude of the integration point, respectively. The 1-D FFT method has been applied by Sideris [1987] for the evaluation of the integral in equation (5).

Recently, an alternative algorithm was suggested to evaluate this type of integral [Huang et al. 2000]. Its computational complexity is $O(N)$ in contrast to $O(N \log N)$ of the FFT. Its practical speed is comparable to the FFT technique. The basic idea is to make use of the isotropic and symmetrical properties of the kernel with respect to longitude. It can be seen that the spherical distance does not depend on the longitude of the computation and integration points; it only depends on the longitude difference between the two points. This means that the kernel values need only be computed once for the evenly spaced points at the same latitude. In other words, all computation points at the same latitude use the same set of kernel values. In addition, the kernel values are symmetrical with respect to the meridian of the computation point, and thus only one-half of the kernel values need be evaluated. Furthermore, for constant grid steps $d\lambda', d\phi'$, the surface element $d\Omega' = \cos\phi' d\lambda' d\phi'$ evidently depends only on the latitude of the integration point, and needs be computed only once for each integration latitude. Compared to the 1-D FFT, this method is more straightforward and suitable for the evaluation of the L integral. It has been implemented in a computer software package for the evaluation of the analytical downward continuation in this research.

4. Numerical Comparisons with the Discrete Poisson Downward Continuation

The point-point model of the Poisson downward continuation [Martinec 1996; Huang 2002] is used for the comparisons to be consistent with the analytical downward continuation that is usually formulated as the point-point model.

4.1 Synthetic comparisons

Synthetic fields have successfully been adopted to quantify the accuracy of the geoid determination in various studies [e.g. Tziavos 1996; Novák et al. 2001; Huang 2002]. They provide an experimental means to test a theoretical method. To verify the accuracy of the analytical downward continuation, one data set was generated from GPM98a [Wenzel 1998] on a 5’ by 5’ grid as point values. The synthetic data from GPM98a were computed from degree and order 21 to degree and order 1800. It includes the synthetic gravity anomalies both on the Earth’s surface and on the geoid. The synthetic data on the Earth’s surface were used as input to the downward continuation computation, while the synthetic data on the geoid were used to verify the accuracy of the methods. The first 20 degrees of harmonic components were excluded from the computation by following the combined technique (or the remove-restore technique).

The test region covers the Rocky Mountains, delimited by latitudes 41°N and 62°N and longitudes 100°W and 138°W. The mean 5' by 5' heights range from 0 m to 3576 m with a mean of 711 m and a standard deviation of 622 m. The first row in Table 1 contains statistical information of the two synthetic data sets on the geoid.

Table 1 shows the point-point DDC results from the GPM98a synthetic data. In the Table, superscript ‘syn’ indicates synthetic data, ‘ddc’ indicates discrete Poisson downward continuation, $D\Delta g$ stands for downward continuation contribution, Dg_T stands for the far zone contribution of the DDC [Vaníček et al. 1996; Huang 2002], and δ indicates the error of the downward continuation. From this table, we can see that the DDC gives the results that agree with the synthetic data within (-0.73 mGal, 0.73 mGal). After the downward-continued synthetic gravity anomalies are transformed into geoid heights by Stokes integration, the maximum error of the geoid heights is about 0.2 cm, while the maximum downward continuation contribution is about 14 cm (see Table 3).

Table 2 shows the ADC results. Superscript 'adc' indicates analytical downward continuation. In the computation, the L integral was truncated to 6°. It can be seen that the ADC demonstrates a rapid convergence to the synthetic data from the g_1 to g_3 terms within an error range of (-1.29 mGal, 2.32 mGal). The maximum ADC error on the geoid heights is about 1.3 cm, five times greater than the DDC one (see Table 3). This error may be caused by the truncation of the series, the truncation of the L integral and the ADC method itself.

Table 1. The point-point DDC of the synthetic field of GPM98a (21-1800) in the test region. Unit: mGal.

Parameter	Min.	Max.	Mean	StdDev	R.M.S.
Δg_g^{syn}	-69.692	124.378	0.176	13.986	13.987
Δg_g^{ddc}	-69.268	124.096	0.175	13.971	13.973
$D\Delta g^{syn}$	-7.271	16.048	0.046	0.760	0.762
$D\Delta g^{ddc}$	-6.881	15.588	0.045	0.728	0.729
Dg_T	-0.196	0.192	0.003	0.026	0.026
δ	-0.726	0.726	0.000	0.046	0.046

Table 2. The ADC of the synthetic field of GPM98a (21-1800) in the test region. Unit: mGal.

Parameter	Min.	Max.	Mean	StdDev	R.M.S.
Δg_g^{syn}	-69.292	124.378	0.176	13.386	13.987
Δg_g^{adc}	-69.263	122.935	0.170	13.929	12.931
$D\Delta g^{syn}$	-7.271	16.048	0.046	0.760	0.762
g_1 g_2 g_3	-5.344 -0.719 -0.096	12.495 1.452 0.263	0.040 0.00 0.00	0.615 0.049 0.005	0.617 0.049 0.005
$D\Delta g^{adc}$	-6.150	13.864	0.040	0.659	0.660
δ	-1.288	2.324	0.006	0.110	0.110

Table 3. Accuracy of the point-point DDC and the ADC by using the synthetic field of GPM98a (21-1800) in the test region. Unit: m.

Parameter	Min.	Max.	Mean	StdDev	R.M.S.
N_{dc}^{syn}	0.004	0.141	0.040	0.021	0.045
δ_N^{ddc}	-0.001	0.002	0.000	0.000	0.000
δ_N^{adc}	-0.003	0.013	0.003	0.003	0.005

4.2 Comparisons using the Helmert gravity anomaly

Helmert's 2nd condensation has been used in the practical determination of the geoid [e.g. Vaníček and Kleusberg 1987; Véronnueau 1996, etc.]. Following this approach, the Helmert gravity anomaly needs to be downward continued from the Earth's surface to the geoid. The procedure and formulae for evaluation of the Helmert gravity anomaly can be found in Vaníček et al. (1999). In this section, the DDC and the ADC are compared by using the Helmert gravity anomaly.

The mean 5' by 5' Helmert gravity anomalies were retrieved from the UNB Helmert gravity data set in the same region as the one for the synthetic comparisons (Janák, J., pers. comm. 2000). The mean 30" by 30" DEM data were used for the evaluation of the mean 5' by 5' Helmert gravity anomalies. Table 4 shows statistical information of the residual Helmert gravity anomalies above degree and order 20 of EGM96 [Lemoine et al. 1998] and the DEM data in the test region. Compared to the synthetic data generated from GPM98a, the residual Helmert gravity anomaly is significantly larger in both magnitude and RMS.

Table 4. Statistics of the residual Helmert gravity anomaly above degree 20 of EGM96 and the height data in the test region.

Parameter	Min.	Max.	Mean	StdDev	R.M.S.
Δg_t^h (mGal)	-132.620	260.181	-2.241	23.623	23.729
H (m)	0	3567	711	623	945

Table 5 shows the DDC and the ADC gravity results. In the DDC, the far-zone contribution was neglected since it required the Helmertized global geopotential model that was not available. This omission results in a larger DDC effect as expected. The far-zone contribution from the non-Helmertized EGM96 is about 1 cm on average in the test region. Further work is needed to estimate the far-zone contribution from the Helmertized global geopotential model. The ADC was truncated up to the 5th term, displaying a rapid convergence. In the computation, the L integral was truncated to 6°. Since the ADC is a forward problem, omission of the far-zone effect may weaken the ADC effect, therefore, the ADC may appear smaller than what it should be. The last row in Table 5 shows the statistics of the differences between the DDC and ADC gravity results. It shows that the ADC is about 10% smaller than the DDC on average.

Table 6 shows the effects of the DDC and the ADC on the geoid. It is noticeable that the ADC geoid result converges significantly faster than the ADC gravity one. The terms above the 3rd are invisible in the geoid result at the millimeter level. The DDC geoid result is 5 cm larger, on average, than the ADC one (see Figures 1 and 2).

Table 5. The gravity results of the DDC and the ADC from the residual Helmert gravity anomaly in the test region. Unit: mGal.

Parameter	Min.	Max.	Mean	StdDev	R.M.S.
$D\Delta g^{ddc}$	-23.086	56.582	0.355	3.506	3.524
g_1	-16.533	38.775	0.320	2.639	2.659
g_2	-3.370	6.891	0.000	0.374	0.374
g_3	-0.566	1.349	0.000	0.052	0.052
g_4	-0.112	0.246	0.000	0.007	0.007
g_5	-0.021	0.039	0.000	0.001	0.001
$D\Delta g^{adc}$	-19.500	47.300	0.320	3.010	3.027
$D\Delta g^{ddc} - D\Delta g^{adc}$	-4.569	10.132	0.035	0.549	0.550

Table 6. The geoid height results of the DDC and the ADC from the residual Helmert gravity anomaly in the test region. Unit: m.

Parameter	Min.	Max.	Mean	StdDev	R.M.S.
N_{ddc}	0.175	0.798	0.542	0.124	0.556
$N(g_1)$	0.172	0.703	0.494	0.103	0.506
$N(g_2)$	-0.013	0.018	-0.002	0.003	0.004
$N(g_3)$	-0.002	0.003	0.000	0.000	0.000
$N(g_4)$	0.000	0.000	0.000	0.000	0.000
$N(g_5)$	0.000	0.000	0.000	0.000	0.000
N_{adc}	0.171	0.712	0.492	0.109	0.504
$N_{ddc} - N_{adc}$	0.004	0.092	0.050	0.017	0.053

5. Summary

In addition to the discrete Poisson downward continuation (DDC), there exists a well-developed method: the analytical downward continuation (ADC) that is based on the Taylor series expansion of the gravity anomaly.

The synthetic tests on the basis of GPM98a show that the ADC introduces an error of about 10% of the downward continuation effect, while the error of the DDC is smaller than 1 cm in the geoid height. The test using the residual Helmert anomaly in the Rocky Mountains shows that the maximum difference between the two methods can reach 10 mGal in gravity, and about 10 cm in the geoid height. The DDC values are 5 cm larger than the ADC ones on average in the geoid height. The difference accounts for about 10% of the downward continuation contribution.

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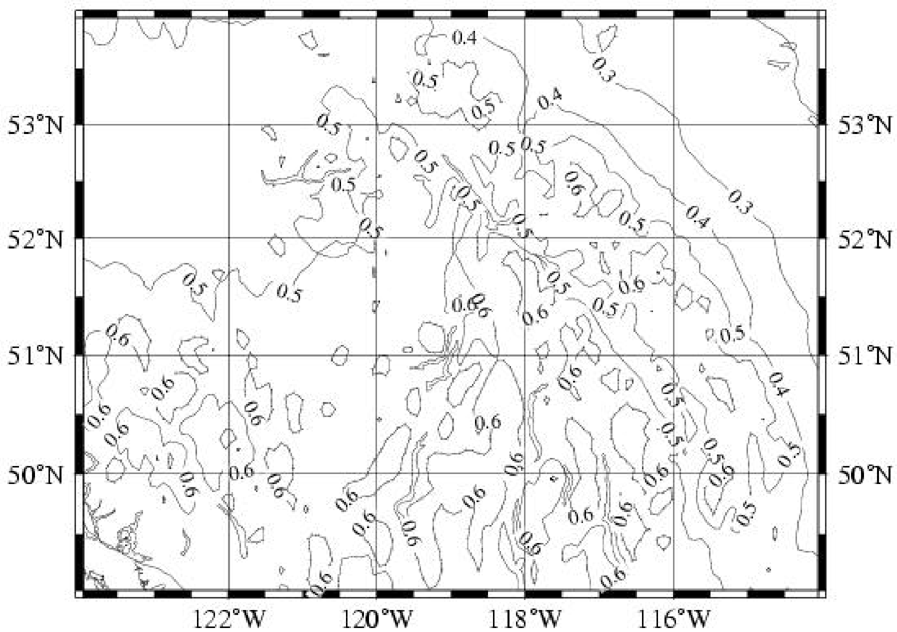

Figure 1. The ADC effect on the geoid, using the residual Helmert gravity anomalies above degree 20 of EGM96. Contour interval: 0.1 m.

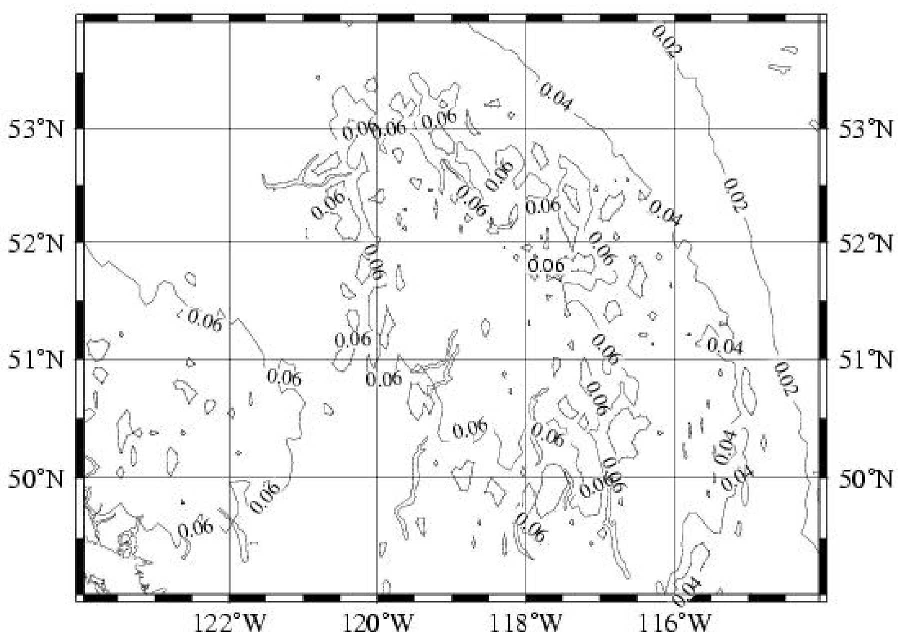

Figure 2. The difference between the DDC and ADC effects on the geoid, using the residual Helmert gravity anomalies. Contour interval: 0.02 m.

The Correction to the Modifed Stokes Formula for an Ellipsoidal Earth

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ABSTRACT

The ellipsoidal correction to Stokes's formula is derived. The correction includes a scale correction from the transformations of the gravity anomaly from the Mean Earth Ellipsoid (MEE) to the Mean Earth Sphere (MES), where the Stokes integration is carried out, and the disturbing potential from the MES to the MEE. A spherical harmonic representation is most suitable for numerical estimation of the ellipsoidal correction from an Earth Gravity Model (EGM).

Finally, the ellipsoidal correction is derived for the modified Stokes formula, and the correction is presented as a Stokes type integral. It is concluded that this correction is significant at the cm or even dm level in areas with very large gravity anomalies. The correction assumes that the contribution to the modified Stokes formula from the EGM is applied at the MEE rather than at the MES as one usually assumes. Thereby one avoids an ellipsoidal correction of the order of metres in the areas with the largest gravity anomalies and geoidal heights.

1. Introduction

For several years the modification of Stokes's formula and its refinements to precise geoid determination has been a favourable research topic for Petr Vaníček. In the struggle for "the 1-cm geoid" there are a number of necessary corrections needed to the original Stokes formula, but it is rather commonly believed that several of these corrections are of minor significance and can be neglected or at least be paid less attention and rigour in the modified Stokes formula. One such correction is caused by the fact that the Earth's shape is rather ellipsoidal than spherical, as assumed in Stokes's formula. This implies that the gravity anomaly, observed on the ellipsoid must be upward or downward continued to a sphere (e.g. by Poisson's integral as studied by Petr) before Stokes's integral can be employed. Another problem, also emphasised by Petr, is that the observed gravity anomaly is not consistent with the boundary condition of physical geodesy, which implies that it needs a correction to be rigorously applicable in Poisson's and Stokes's integrals.

The ellipsoidal correction to the original Stokes formula has been extensively studied in the geodetic literature. See e.g. the reference list of Sjöberg [2002]. However, today Stokes's formula is most frequently combined with a set of long-wavelength potential coefficients, and the Stokes integration is then limited to a spherical cap around the computation point. Considering that the ellipsoidal correction to the original Stokes formula is of the order of some decimetre, it is rather commonly believed that the correction is negligible in the modified version of the formula. However, as we are not aware of such a proof, the main task of this article will be to derive the effect.

2. The Gravity Anomaly And Stokes's Formula

The original Stokes formula can be written

$$N^0 = \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\Delta g d\sigma \,, \tag{1}$$

where R is the mean Earth radius, γ is normal gravity at the reference ellipsoid, S(ψ) is Stokes's function, Δg is the gravity anomaly and σ is the unit sphere. Disregarding the Earth's topography and atmosphere, Eq. (1) would provide the correct geoidal height, if Δg was located on the Mean Earth Sphere (MES) and was satisfying "the boundary condition" in its spherical approximation:

$$\Delta g^0 = -\frac{\partial T}{\partial r} - 2\frac{T}{r} \,, \tag{2}$$

where T is the disturbing potential and r is the geocentric radius vector. However, gravity is measured in the direction of the plumb line, which deviates from the radial direction by order e^2 (the eccentricity of the Earth ellipsoid squared). As a result of some minor discrepancies between Δg, observed on the Mean Earth Ellipsoid (MEE), and Δg^0, referring to the MES one obtains [cf. Cruz 1985]

$$\Delta g^0_{r=R} = \Delta g + \delta g^0 \,, \tag{3a}$$

where

$$\delta g^0 = \delta g^1 + (R - r_e)(\frac{\partial \Delta g^0}{\partial r})_{r=R} \tag{3b}$$

and

$$\delta g^1 = e^2(2 - 3\cos^2\theta - \sin\theta\cos\theta\frac{\partial}{\partial\theta})\frac{T}{a}. \tag{3c}$$

Here r_e is the radius, θ is the geocentric co-latitude and a is the semi-major axis of the MEE. The last term of (3b) continues the anomaly from the ellipsoid to the mean sphere (of radius R).

Correcting Δg of formula (1) by δg^0 we thus obtain the corrected geoidal height

$$N = \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)(\Delta g + \delta g^0)d\sigma + \frac{r_e - R}{\gamma}(\frac{\partial T}{\partial r})_{r=R} \ , \tag{4}$$

where the last term corrects the disturbing potential counted at the MES to the MEE. Hence, the geoidal height can be written as the sum of the spherical Stokes formula (1) and its ellipsoidal correction δN_{corr}:

$$N = N^0 + \delta N_{corr} = N^0 + \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\delta g^0 d\sigma + \frac{r_e - R}{\gamma}(\frac{\partial T}{\partial r})_{r=R}. \tag{5}$$

Before closing this chapter we will present the ellipsoidal correction δN_{corr} on a more suitable form for numerical analysis. This is obtained by performing Stokes's integration on the bounding sphere of radius a, yielding the alternative formula

$$N = \frac{a}{4\pi\gamma}\iint_\sigma S(\psi)\Delta G^0 d\sigma + \frac{r_e - a}{\gamma}(\frac{\partial T}{\partial r})_{r=a} \ , \tag{6}$$

where the last term transfers the disturbing potential from the sphere to the ellipsoid, and

$$\Delta G^0 = \Delta g + \delta G^0 = \Delta g + \delta g^1 + (a - r_e)(\frac{\partial \Delta g^0}{\partial r})_{r=a} \tag{7}$$

is the gravity anomaly on the bounding sphere consistent with the boundary condition.
In this case the ellipsoidal correction can be expressed

$$\delta N_{corr} = N - N^0 = kN^0 + \frac{a}{4\pi\gamma}\iint_\sigma S(\psi)\Delta G^0 d\sigma - ae^2\frac{\cos^2\theta}{2\gamma}(\frac{\partial T}{\partial r})_{r=a}, \tag{8}$$

where the last term is the last term of (6) rewritten by the approximation

$$r_e = a\sqrt{1 - e^2 \cos^2\theta} \approx a - ae^2 \frac{\cos^2\theta}{2} \tag{9}$$

and we have also introduced the scale factor

$$k = (a - R)/R \tag{10}$$

which is the needed link to transform the integral over the MES to the MEE. We have not found this correction in other papers on the subject, which we find rather remarkable, as it is most significant.

Finally we like to emphasize that the two corrections to the geoidal height given by Eqs. (5) and (8) yield the same result to order e^2, which is not difficult to prove.

3. A Spherical Harmonic Representation of the Correction

To get the harmonic expansion of the geoid correction (8) we start from the harmonic expansion of the disturbing potential:

$$T = \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^{n+1} T_n(\theta,\lambda) = \frac{GM}{a} \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=-n}^{n} C_{nm} Y_{nm}(\theta,\lambda) \tag{11}$$

where GM is the terrestrial gravitational constant, C_{nm} is the harmonic coefficients related with the bounding sphere of radius a and the fully normalized spherical harmonic Y_{nm}, obeying [cf. Heiskanen and Moritz 1967, p. 31]

$$\frac{1}{4\pi} \iint_{\sigma} Y_{nm} Y_{pq} d\sigma = \begin{cases} 1 & \text{if} \quad n = p \quad \text{and} \quad m = q \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

(Notice that Y_{nm} need not be defined as complex!) From (11) and the boundary condition (2) one obtains the expansion of the gravity anomaly in space:

$$\Delta g^0 = \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^{n+2} \frac{n-1}{a} T_n(\theta,\lambda) \tag{13}$$

and its radial derivative

$$\frac{\partial \Delta g^0}{\partial r} = -\sum_{n=2}^{\infty} \frac{(n-1)(n+2)}{a^2} \left(\frac{a}{r}\right)^{n+3} T_n(\theta, \lambda) \tag{14}$$

Inserting Eqs. (11) and (14) into the gravity anomaly correction δG^0 defined by (7) and taking advantage of the following relations for spherical harmonics [Martinec 1998, pp. 184-185; see also the Appendix]

$$\sin\theta \cos\theta \frac{\partial}{\partial \theta} Y_{nm} = A_{nm} Y_{n+2,m} + B_{nm} Y_{nm} + D_{nm} Y_{n-2,m} \tag{15a}$$

and

$$\cos^2\theta Y_{nm} = E_{nm} Y_{n+2,m} + F_{nm} Y_{nm} + G_{nm} Y_{n-2,m} \ , \tag{15b}$$

where the coefficients, given in the Appendix, are related by the equations

$$D_{n+2,m} = -(n+3) G_{n+2,m} = -(n+3) E_{nm} = -\frac{n+3}{n} A_{nm} \tag{16}$$

$$2B_{nm} = -3F_{nm} + 1 \tag{17}$$

and (for $H_{00} = H_{1m} = 0$)

$$\sum_{n=2}^{\infty} \sum_{m=-n}^{n} H_{nm} Y_{n+2,m} = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} H_{n-2,m} Y_{nm} \tag{18a}$$

and

$$\sum_{n=2}^{\infty} \sum_{m=-n}^{n} H_{nm} Y_{n-2,m} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} H_{n+2,m} Y_{nm} \tag{18b}$$

one obtains after a few manipulations

$$\delta G^0 = \sum_{n=0}^{\infty} \delta G_n^0 \ , \tag{19a}$$

where

$$\delta G_n^0 = e^2 \frac{GM}{2a^2} \sum_{m=-n}^{n} \begin{matrix} Y_{nm} \{[3-(n^2+n+1)F_{nm}]C_{nm} - (n^2-n+2)G_{nm}C_{n-2,m} \\ -(n^2+3n+4)E_{nm}C_{n+2,m}\} \end{matrix} \tag{19b}$$

Eq. (18), used in the derivation of (19), is valid if (as we assume) $C_{00} = C_{1m} = 0$.

Moreover, one can show by using (15b) that the effect on the geoid by continuing the disturbing potential from the bounding sphere to the Earth ellipsoid can be written:

$$\frac{\delta T^d}{\gamma} = -e^2 a \frac{\cos^2\theta}{2\gamma}\left(\frac{\partial T}{\partial r}\right)_{r=a} = e^2 \frac{\cos^2\theta}{2\gamma}\sum_{n=0}^{\infty}(n+1)T_n = \sum_{n=0}^{\infty}\frac{\delta T_n^d}{\gamma} \ , \tag{20a}$$

where

$$\delta T_n^d = e^2 \frac{GM}{2a}\sum_{m=-n}^{n} Y_{nm}[(n+1)F_{nm}C_{nm} + (n-1)G_{nm}C_{n-2,m} + (n+3)E_{nm}C_{n+2,m}] \tag{20b}$$

is the Laplace harmonic of the difference of the disturbing potential at the bounding sphere and the MEE. Hence, the spectral form of formula (8) becomes

$$\delta N_{corr} = kN^0 + \frac{\delta T_0^d + \delta T_1^d}{\gamma} + \frac{a}{\gamma}\sum_{n=2}^{\infty}\left(\frac{\delta G_n^0}{n-1} + \frac{\delta T_n^d}{a}\right), \tag{21}$$

and by inserting (19b) and (20b) one arrives at the final harmonic form of the geoidal correction:

$$\delta N_{corr} = kN^0 + \frac{\delta T_0^d + \delta T_1^d + \delta T_e}{\gamma} \ , \tag{22a}$$

where

$$\delta T_e = \sum_{n=2}^{\infty}\left(\delta T_e\right)_n \tag{22b}$$

with

$$\left(\delta T_e\right)_n = e^2\frac{GM}{2a}\sum_{m=-n}^{n}\frac{Y_{nm}}{n-1}\{[3-(n+2)F_{nm}]C_{nm} - (n+1)G_{nm}C_{n-2,m} - (n+7)E_{nm}C_{n+2,m}\} \ . \tag{22c}$$

Notice that the series of (21) with terms δG_n^0 starts at degree 2 as a consequence of that Stokes's formula is blind to the zero- and first-degree terms. By formulas (22) one can thus study the ellipsoidal effect of the original Stokes formula. However, our primary concern is the effect on the modified Stokes formula, which we will consider in the next chapter.

4. The Ellipsoidal Correction to the Modified Stokes Formula

Stokes's modified formula can be written [Sjöberg 1991; cf. also Vaníček and Sjöberg 1991, where a higher order reference field was used]:

$$N^M = \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_M(\psi)\Delta g d\sigma + c\sum_{n=2}^{M}\left(Q_{Mn}(\psi_0)+s_n\right)\Delta g_n^S, \tag{23a}$$

where σ_0 is the integration area, limited to a spherical cap around the computation point with geocentric angel ψ_0, $S_M(\psi)$ is the modified Stokes function given by

$$S_M(\psi) = S(\psi) - \sum_{k=2}^{M}\frac{2k+1}{2}s_k P_k(\cos\psi), \tag{23b}$$

s_k being the arbitrary modification parameters and $P_k(\cos\psi)$ is a Legendre's polynomial, M the maximum degree of modification, $c = R/(2\gamma)$,

$$Q_{Mn}(\psi_0) = Q_n(\psi_0) - \sum_{k=2}^{M}\frac{2k+1}{2}s_k e_{nk}(\psi_0) \tag{23c}$$

$$Q_n(\psi_0) = \int_{\psi_0}^{\pi} S(\psi)P_n(\cos\psi)\sin\psi d\psi \tag{23d}$$

$$e_{nk}(\psi_0) = \int_{\psi_0}^{\pi} P_n(\cos\psi)P_k(\cos\psi)\sin\psi d\psi \tag{23e}$$

and Δg_n^S is the Laplace harmonic of the gravity anomaly determined from the Earth Gravity Model (EGM). Here it is essential to define at which radius Δg_n^S should be determined. In the spherical application of the modified Stokes formula one frequently assumes that the radius is the same as for the MES, i.e. R. In such a case one obtains in the limits $\sigma_0 \to 0$ and $M \to \infty$ that the geoidal height estimator becomes

$$N^M = \frac{T_{r=R}}{\gamma}, \tag{24}$$

and therefore the needed ellipsoidal correction becomes

$$\delta N_{corr} = \frac{T_{r=r_e} - T_{r=R}}{\gamma} \approx \frac{r_e - R}{\gamma}\left(\frac{\partial T}{\partial r}\right)_{r=R}, \tag{25}$$

which, by taking advantage of (9) and (2), can explicitly be written (to order e^2)

$$\delta N_{corr} = (\frac{ae^2 \cos^2 \theta}{2R} - k)\left(\frac{R\Delta g}{\gamma} + 2N^0\right) \tag{26}$$

For a = 6378.140 km, R = 6371 km, e^2 = 0.0067 and γ = 981 Gal, the last formula becomes

$$\delta N_{corr} = \left(3.4\cos^2\theta - 1.1\right)\left(6.49\Delta g + 2N^0\right) \quad [\mathrm{mm}] \ , \tag{27}$$

where Δg and N^0 are given in units of mGal and m, respectively. As the magnitudes of the last two quantities may well reach $\pm$ 200 mGal and $\pm$ 100 m, δN_{corr} may reach $\pm$ 3 m and $\pm$ 1.5 m in polar and equatorial regions, respectively. Hence, to avoid this very big correction we should always apply the EGM derived Laplace gravity anomaly Δg_n^S not on the MES but at the MEE. That is, the anomaly should explicitly be written

$$\Delta g_n^S = \frac{GM}{aR}(n-1)\left(\frac{a}{r_e}\right)^{n+1} \sum_{m=-n}^{n} C_{nm} Y_{nm} \ , \tag{28}$$

which representation avoids the huge ellipsoidal correction.

The correct geoidal height (6) can also be modified to an integral over the bounding sphere of radius a:

$$N = \frac{a}{4\pi\gamma}\iint_{\sigma_0} S_M(\psi)\Delta G^0 d\sigma + \frac{a}{2\gamma}\sum_{n=2}^{\infty}\left(Q_{Mn}(\psi) + s_n^*\right)(\Delta G^0)_n + \frac{\delta T^d}{\gamma} , \tag{29a}$$

where

$$\left(\Delta G^0\right)_n = \Delta g_n + \delta G_n^0 \tag{29b}$$

$$s_n^* = \begin{cases} s_n & \text{if} \quad n \le M \\ 0 & \text{otherwise} \end{cases} \tag{29c}$$

and the last term continues the disturbing potential from the bounding sphere to the Earth ellipsoid. Alternatively, as $\Delta G^0 = \Delta g + \delta G^0$, one can rewrite (29a) on the form

$$N = \frac{\delta T_0^d + \delta T_1^d}{\gamma} + \frac{a}{4\pi\gamma}\iint_{\sigma_0} S_M(\psi)(\Delta g + \delta G^0 + \delta g^d)d\sigma + \frac{a}{2\gamma}\sum_{n=2}^{\infty}[Q_{Mn}(\psi_0) + s_n^*]\left((\Delta G^0)_n + \delta g_n^d\right) \tag{30}$$

where

$$\delta g_n^d = \frac{n-1}{a}\delta T_n^d \tag{31}$$

and δT_n^d was introduced in Eqs. (20a,b). As

$$a\left(\Delta G_n^0 + \delta g_n^d\right) = R\Delta g_n^S \ , \tag{32}$$

and by taking the difference between (30) and (23a), with Δg_n^S defined by (28), one arrives at the correction

$$\delta N_{corr}^M = N - N^M = \frac{\delta T_0^d + \delta T_1^d}{\gamma} + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_M(\psi)[k\Delta g + \frac{a}{R}(\delta G^0 + \delta g^d)d\sigma + \frac{a}{2\gamma}\sum_{n=M+1}^{\infty} Q_{Mn}\Delta g_n^S \ , \tag{33}$$

where the last term is the correction for the truncation error due to limiting the EGM to degree M and the Stokes integration to a cap. This term is therefore not part of the ellipsoidal correction but of the total geoid error.

From Eqs. (21) and (22a) it follows that

$$\delta T_e = \sum_{n=2}^{\infty}(\frac{a\delta G_n^0}{n-1} + \delta T_n^d) \tag{34}$$

which yields the corresponding gravity anomaly

$$\delta g_e = \sum_{n=2}^{\infty}\frac{n-1}{a}(\delta T_e)_n = \delta G^0 + \delta g^d, \tag{35}$$

and from Eq. (22c) it can be expressed by the spherical harmonic series

$$\delta g_e = e^2\frac{GM}{2a^2}\sum_{n=2}^{\infty}\sum_{m=-n}^{n} Y_{nm}\{[3-(n+2)F_{nm}]C_{nm} - (n+1)G_{nm}C_{n-2,m} - (n+7)E_{nm}C_{n+2,m}\} \tag{36}$$

Our intention now is to derive an approximation to the geoidal height correction (33) (disregarding the truncation bias). For this purpose we use (11) and (15b) in (36) to get the following approximate gravity anomaly correction:

$$\delta g_e \approx \frac{e^2}{2a}[3T - (a\Delta g + 3T)\cos^2\theta]. \tag{37}$$

Considering also that in a near-zone around the computation point the modified Stokes kernel and the infinitesimal integration area can be approximated by

$$S_M(\psi) \approx S(\psi) \approx \frac{2}{\psi} \quad \text{and} \quad d\sigma \approx \psi d\psi d\alpha, \tag{38}$$

where α is the azimuth, and neglecting the terms with δT_0^d and δT_1^d, we arrive at the following approximate correction to the geoidal height

$$\begin{aligned} \delta N_{corr}^M &= \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_M(\psi)\left(k\Delta g + \frac{a}{R}\delta g_e\right)d\sigma \approx \\ &\approx \psi_0^{\circ}[\left(0.12 - 0.38\cos^2\theta\right)\Delta g + 0.17N^0\sin^2\theta] \end{aligned} \tag{39}$$

where the approximate formula is given in units of mm when Δg is given in units of mGal, N^0 in m and ψ_0 in grades. As the surface gravity anomaly and geoidal height may well reach the magnitudes 200 mGal and 100 m, respectively, it follows that the ellipsoidal correction to the geoidal height may reach about $\pm$ 5 cm already for an integration cap of $1°$. From formula (39) we also notice that the correction increases with the integration cap radius.

5. Conclusions

We have derived the ellipsoidal correction to the modified Stokes formula. We have shown that this correction is not negligible, but for an integration cap size of 4 grades and bigger it is as significant as the correction to the original Stokes formula, i.e. of the order of 2 dm. For small cap sizes the error is proportional to the cap size. To avoid further corrections, of the order of metres, these results require that the contributions to the geoidal height from the EGM are applied at the MEE and not at the MES as frequently assumed in the application of the modified version of Stokes's formula.

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Appendix

From Martinec (1998, pp.184-185) we get

$$\sin\theta\cos\theta\frac{\partial}{\partial\theta}Y_{nm} = A_{nm}Y_{n+2,m} + B_{nm}Y_{nm} + D_{nm}Y_{n-2,m} \tag{A1}$$

and

$$\cos^2\theta Y_{nm} = E_{nm}Y_{n+2,m} + F_{nm}Y_{nm} + G_{nm}Y_{n-2,m} \tag{A2}$$

where

$$A_{nm} = \frac{n}{2n+3}\sqrt{\frac{[(n+1)^2 - m^2][(n+2)^2 - m^2)]}{(2n+1)(2n+5)}} \tag{A3}$$

$$B_{nm} = -\frac{n(n+1) - 3m^2}{(2n-1)(2n+3)} \tag{A4}$$

$$D_{nm} = -\frac{n+1}{2n-1}\sqrt{\frac{[(n-1)^2 - m^2](n^2 - m^2)}{(2n-3)(2n+1)}} \tag{A5}$$

$$E_{nm} = \frac{1}{2n+3}\sqrt{\frac{[(n+1)^2 - m^2][(n+2)^2 - m^2]}{(2n+1)(2n+5)}} \tag{A6}$$

$$F_{nm} = \frac{2n(n+1) - 6m^2}{3(2n-1)(2n+3)} + \frac{1}{3} \tag{A7}$$

and

$$G_{nm} = \frac{1}{2n-1}\sqrt{\frac{[(n-1)^2 - m^2](n^2 - m^2)}{(2n-3)(2n+1)}} \tag{A8}$$

The coefficients are related as follows

$$D_{n+2,m} = -(n+3)G_{n+2,m} = -(n+3)E_{nm} = -\frac{n+3}{n}A_{nm} \tag{A9}$$

and

$$F_{nm} = -\frac{2B_{nm}}{3} + \frac{1}{3} \tag{A10}$$

A Wavelet Compression Method for Computing Terrain Corrections

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Abstract

The paper presents a method of evaluating the terrain correction integral using wavelets. Due to the wavelets' localization properties in both of the time (space) and frequency (scale) domains, and because the kernel of the terrain correction integral has strong singularities and decays smoothly and quickly away from the singularities, a large number of wavelet transform coefficients of the kernel become zeros or negligible, and only a small number of wavelet transform coefficients are significant. It is thus possible to significantly compress the kernel in a wavelet basis by neglecting the zero coefficients and the small coefficients below a certain threshold. Therefore, wavelets provide a convenient way for efficiently evaluating the terrain correction integral in terms of fast computation and savings in computer memory. In this contribution, an algorithm for the wavelet evaluation of terrain corrections is presented. Numerical examples illustrate the efficiency and accuracy of the wavelet method.

Key words: Wavelets – Kernel Compression – Terrain Corrections – Singularity – Computational Efficiency

1 Introduction

During the last decade, the wavelet transform has proven to be a valuable tool in many application fields. Wavelets are functions that satisfy certain mathematical requirements and are used in representing data or other functionals. Because wavelets are of local nature in both frequency (scale) and time (space), they can easily "detect" local features of the represented functions. This localization feature makes many functionals and operators "sparse" when transformed into the wavelet domain. With multiresolution analysis (Mallat 1989), we can retain the large and important coefficients, and remove the small and unimportant coefficients whose absolute values are below a certain threshold.

The localization property of wavelets can be utilized more efficiently if the functional, which is represented in a wavelet basis, contains singular points or lines, since the functional

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can then be compressed with a larger compression ratio without loss of essential information. It is the sparse representation of geodetic singular integral operators in a wavelet basis that makes the application of wavelets in geodesy attractive. Benciolini (1994) suggested that the integral for the potential of a single layer could be computed fast by sparse representation of the integral kernel in a wavelet basis. Barthelmes et al. (1994) also mentioned that the wavelet transform may be used to reduce the computational cost of matrix-vector multiplication and the computer memory requirements by sparse representation of a weakly singular integral operator in a wavelet basis. Salamonowicz (1999, 2000) applied the two-dimensional wavelet transform to the fast computation of the singular Stokes integral point by point by using a special algorithm developed in Beylkin et al. (1991). Liu and Sideris (2002) modified the algorithm and applied it to compute the Stokes and Verning Meinesz integrals one meridian by one meridian in local or regional basis by the three-dimensional wavelet transform. In this paper, we further apply the method to compute the singular terrain correction integral, which has stronger singularities than Stokes's and Verning Meinesz integrals. The new formulas have been programmed in a computer and some preliminary numerical results are given here. We also include a comparison between the wavelet method with different compression ratios and the 2D planar FFT methods in term of computational efficiency and accuracy.

Before we proceed, some fundamental wavelet concepts are reviewed.

2. Wavelet overview

A wavelet is a small wave function $\psi \quad L^2(\quad)$ with finite energy (i.e., $\int |\psi(t)|^2 dt < +\infty$) and a zero average (Mallat 1997; Keller 2000)

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0 \tag{1}$$

where indicates the real number set. It is normalized so that $\|\psi\| = 1$ and centered in the neighborhood of $t = 0$. Formula (1) shows that a wavelet has to be oscillating and decaying with the increase of time (space) and the oscillation has to tend to zero after a certain time (space).

A family of wavelet functions $\{\psi_{m,n} | m, n \quad Z\}$, where Z indicates the integer number set, is generated from a single prototype wavelet called mother wavelet $\psi \quad L^2(\quad)$ by scaling and translating it. One prescription used to generate a wavelet family $\{\psi_{m,n} | m, n \quad Z\}$ from a mother wavelet $\psi \quad L^2(\quad)$ is

$$\psi_{m,n} = a_0^{-2/m}\psi\left(a_0^{-m}x - b_0 n\right), m,n \quad Z \tag{2}$$

where $a_0^m \quad R$ but excluding the zero element is the scaling parameter, and $b_0 n \quad R$ is the translating (shifting) parameter (Meyer 1993; Chan 1995).

For some very special choices of $\psi \quad L^2(\ )$ and a_0, b_0, the wavelet set $\{\psi_{m,n} | m,n \quad Z\}$ constitutes an orthogonal basis for $L^2(\ )$. In particular, for $a_0 = 2$, $b_0 = 1$ there exist $\psi \quad L^2(\ )$ such that

$$\psi_{m,n} = 2^{-m/2}\psi\left(2^{-m}x - n\right), m,n \quad Z \tag{3}$$

form an orthogonal basis for $L^2(\ )$ (Randy andYoung 1993).

Such an orthogonal wavelet family is usually constructed through the so-called multiresolution analysis (Mallat 1989; Chui 1992) in the language of signal processing, though there also exist some other techniques for this purpose. It consists of breaking up the space $L^2(\ )$ into a sequence of nested subspaces based on another basis function, the so-called scaling function $\phi \quad L^2(\ )$, which is then used to construct wavelets. The scaling and shifting of the scaling function also generate an orthogonal basis $\{\phi_{m,n} | m,n \quad Z\}$ for $L^2(\ )$; and $\{\phi_{m,n} | m,n \quad Z\}$ and $\{\psi_{m,n} | m,n \quad Z\}$ are also orthogonal to each other. We will not go into details here, since we are not aiming to construct wavelets in this application. The interested readers are referred to Mallat (1997) and Goswami and Chan (1999), among others.

Once we construct (chose) the orthogonal wavelets basis $\{\psi_{m,n} | m,n \quad Z\}$ for $L^2(\ )$, a function $f \quad L^2(\ )$ can be decomposed onto $\{\psi_{m,n} | m,n \quad Z\}$ and reconstructed as:

$$f(t) = \sum_{m \quad Z}\sum_{n \quad Z}\langle \psi_{m,n} \quad f \rangle \; \psi_{m,n}(t) \tag{4}$$

The inner products $\langle \psi_{m,n} \quad f \rangle$ are the wavelet transform coefficients. This reconstruction formula holds only if the decomposition is done in the whole integer space from -∞ to +∞. If a function (data) has compact support, then the series is truncated at some fixed maximum scaling values. To have a complete representation of the original function (data), we need the scaling function as a complement of information in formula (4). Let the maximum scaling value be *N*. Then the above reconstruction is modified as (Chui 1992; Mallat 1997)

$$f(t) = \sum_{m=-\infty}^{N} \sum_{n \in Z} \langle \psi_{m,n} \mid f \rangle \psi_{m,n} + \sum_{n \in Z} \langle \phi_{N,n} \mid f \rangle \phi_{Nn} \tag{5}$$

By combining equation (3) and equation (5), we can find that at smaller scale 2^m (higher resolution), detailed and higher frequencies of the function $f \in L^2(\mathbb{R})$ are detected; while at larger scale 2^m (lower resolution) coarse and lower frequencies of the function $f \in L^2(\mathbb{R})$ are detected. This is the reason wavelets are characterized as localized functions. The above decomposition is done by computing all of these inner products, and the reconstruction consists of summing up all the orthogonal projections of the function onto the wavelets. The computational effort of these coefficients is enormous if these inner products are evaluated numerically directly. This is because when *m* becomes larger the support of $\{\psi_{m,n} | m, n \in Z\}$ will grow very large and the computation would require numerical quadrature formulas with very many nodes (Strang 1993, 1994; Olivier and Vetterli 1991). To overcome this huge computational burden Mallat (1997) developed a fast algorithm, which provides a convenient tool to compute the coefficients recursively. The algorithm requires only order *n* operations to transform a vector with *n* elements or an *n*-sample vector of a function. In order to develop the fast algorithm, wavelets $\{\psi_{m,n} | m, n \in Z\}$ and scaling functions $\{\phi_{m,n} | m, n \in Z\}$ are fully characterized and represented by two so-called quadrature mirror filters $\{h(k) | k \in Z\}$ and $\{g(k) | k \in Z\}$ (Graps 1995). More interesting is that wavelets having compact support, such as the well-known compactly supported Daubechies wavelet family (Daubechies 1988), can be fully characterized by a finite number of filter coefficients. The sequence $\{h(k) | k \in Z\}$ is known as a low pass or low band filter while $\{g(k) | k \in Z\}$ is known as the high pass or high band filter in the language of digital signal processing (Graps 1995; Rioul and Vetterli 1991). The following three formulas can help us understand the relationships between the wavelet and scaling function and the two filters (Vetterli and Herley 1992; Strang and Nguyen 1996; Keller 2000):

$$\phi(x) = \sqrt{2} \sum_{k \in Z} h_k \phi(2x - k) \tag{6}$$

$$\psi(x) = \sqrt{2} \sum_{k \in Z} g_k \phi(2x - k) \tag{7}$$

$$g_k = (-1)^k h_{1-k} \tag{8}$$

From these three formulas, we can clearly see that wavelet can be constructed from the scaling function, which is the basis function in multiresolution analysis. It is also obvious that the two filters can fully characterize and represent the wavelets and scaling functions in any algorithm for wavelet decomposition and reconstruction (Phillies 1996; Hoffman 1996; Bruce et al. 1996). As we will employ the two-dimensional wavelet transform, we demonstrate the two-dimensional fast algorithm on the heights $\Delta g_0[n,m]$. Data are assumed to be known only at the discrete points of a regular grid with sampling intervals $\Delta x, \Delta y$. The maximum values for *m* and *n* may thus be computed by

$$M = \frac{X}{\Delta x} \ , \ N = \frac{Y}{\Delta y} \tag{9}$$

where *X* and *Y* are the record lengths along the *x* and *y* directions, respectively.

The two-dimensional transform of the heights at scale 2^j is composed of three wavelet coefficients plus the remaining approximation at this scale (Mallat 1997)

$$b^1_{j+1}[p,q] = \sum_n \sum_m g[n-2p]g[m-2q]\Delta g_j \tag{10.1}$$

$$b^2_{j+1}[p,q] = \sum_n \sum_m g[n-2p]h[m-2q]\Delta g_j \tag{10.2}$$

$$b^3_{j+1}[p,q] = \sum_n \sum_m h[n-2p]g[m-2q]\Delta g_j \tag{10.3}$$

$$\Delta g_{j+1}[p,q] = \sum_n \sum_m h[n-2p]h[m-2q]\Delta g_j \tag{10.4}$$

b_{j+1} and Δg_{j+1} are computed by taking every other sample of the convolution of Δg_j with gg , gh , hg and hh respectively. The filters hg , gh , and gg remove the higher frequencies of the sequence Δg_j in the vertical, horizontal and diagonal direction respectively; whereas hh is a high-pass filter which collects the remaining highest frequencies. It is computed from Δg_0 by iterating the above formulas for $0 \le j < J$, where 2^J is the largest scale.

After computing all of these 3J+1 coefficients, the original heights are recovered from this wavelet representation by iterating the following reconstruction formula for $J > j \ge 0$ (Chui 1992; Mallat 1997)

$$\Delta g_j[n,m] = \sum_p \sum_q g[n-2p]g[m-2q]b^1_{j+1} + \sum_p \sum_q g[n-2p]h[m-2q]b^2_{j+1} + \sum_p \sum_q h[n-2p]g[m-2q]b^3_{j+1} + \sum_p \sum_q h[n-2p]h[m-2q]\Delta g^4_{j+1} \tag{11}$$

The reconstruction is an interpolation that inserts zeros to expand Δg_{j+1} and b_{j+1} and filters these signals with hh, hg, gh, and gg, respectively (Mallat 1999).

We may not need to construct wavelets in an application, but we do need to choose the best wavelet for a particular application among an infinite sets of wavelets. A wavelet is optimum in terms of compression and fast computation if it produces a maximum number of wavelet coefficients $\left\langle \psi_{m,n} \quad f \right\rangle$ that are close to zero and a minimum number of high amplitude coefficients. This depends mostly on the regularity of the function $f \quad L^2(\quad)$, the number of vanishing moments of the wavelet $\psi \quad L^2(\quad)$ and the size of its support.

$\psi \quad L^2(\quad)$ is said to have vanishing moments p if

$$\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0 \quad \text{for } 0 \le k < p \tag{12}$$

From the definition of the vanishing moments, it is seen that the higher the vanishing moments of the wavelet, the more zero wavelet coefficients it produces in formula (5). To maximize the number of zero coefficients, we must increase the vanishing moments of the wavelet (Chui 1992; Goswami and Chan 1999).

The support size of the wavelet is the number of non-zero coefficients of its filters. Wavelets that overlap the singularities of a function $f \quad L^2(\quad)$ create high amplitude coefficients (Mallat 1997). If the function has singularities, the larger the size of the support of the wavelet, the more the opportunities the wavelet overlaps the singularities of the function and consequently the higher the number of high amplitude coefficients it produces. To minimize the number of high amplitude coefficients we must reduce the support size of the wavelet.

There is an unfortunate relationship between the support size and vanishing moments: if a wavelet has p vanishing moments then its support is at least 2p-1 (Mallat 1997; Victor 1994). Daubechies wavelets are optimum in the sense that they have a minimum support size for a given number of vanishing moments (Daubechies 1988, 1996; Rowe and Abbott 1995). Thus we face a tradeoff between the number of vanishing moments and the support size when we choose a particular wavelet for a function with singularities.

Different Daubechies wavelets with different vanishing moments have been tested in transforming the singular Stokes and Verning Meinesz integrals (Liu and Sideris 2002) and the terrain correction integral to be discussed in the next section. It was found that the

Daubechies wavelet with vanishing moment 4 (’db4’ in MATLAB language) is best suited for transforming these integrals in the sense that it maximizes the number of zero coefficients and minimizes the number of high amplitude coefficients.

3. The Wavelet Method for Computing Terrain Corrections

As stated in the first section, the reason we employ wavelets to evaluate geodetic integrals is that the kernels of most geodetic integrals have singular points and they decay smoothly and quickly away from the singularities in the non-singular parts of the integral. Because a large number of wavelet coefficients become zeros, there is only a small number of coefficients which are significant due to the localized nature of wavelets. And therefore the integral kernel can be sparse and compressed in a wavelet basis by neglecting these zero coefficients and the small coefficients below a certain threshold. Due to the sparse representation of the kernel in a wavelet basis and because of the fast wavelet transform algorithm the geodetic integrals can be computed efficiently by using wavelets. For the wavelet evaluations of the Stokes and Verning Meinesz integrals, see Liu and Sideris (2002). The wavelet method for evaluating the terrain correction is treated in details in the following.

If the height data $H[n,m]$ are given in a limited area *E*, we take the integration area as a plane and define a local rectangular coordinate system. The integral formula giving the terrain correction *C* at a point *P* on the plane reference surface may be written as (Sideris 1985)

$$C(x_2, y_2) = \frac{1}{2} G\rho \iint_E \frac{[H(x_1, y_1) - H(x_2, y_2)]^2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{3/2}} dx_1 dy_1 \tag{13}$$

(x_1, y_1) and (x_2, y_2) are local Cartesian coordinates of the running point and computation point, respectively, *G* is the gravitional constant, ρ is the mean density of the topographical masses (assumed constant) and *H* is the height of the points.

From formula (13) we can immediately recognize that the kernel

$$K(x_1, y_1, x_2, y_2) = \frac{1}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{3/2}} \tag{14}$$

has a singularity at the computation point, and it decreases smoothly away from the singular point as observed from Figure 1 which shows the kernel when the computation point is at the center of the data grid, Figure 2 which show its wavelet coefficients and Figure 3 which is the histogram plot showing the number of elements at different wavelet coefficient values in each column of the data grid. The data used in plotting these figures is the same as in the numerical example of next section. This is the very reason that wavelets can be applied to compress the terrain correction kernel and efficiently compute the integral. Note that, in practical computation, *K* is set to zero at the computation point since a line mass at the computation point itself contributes nothing to the terrain correction (Sideris 1985).

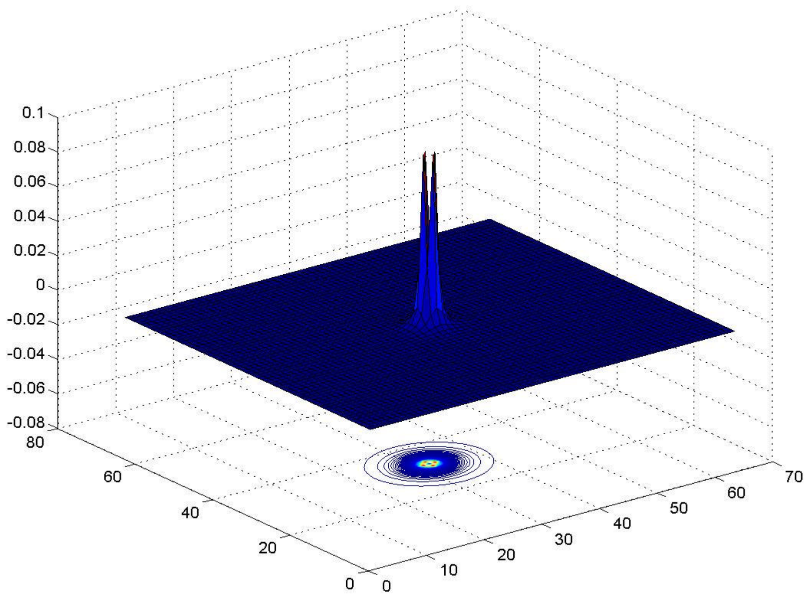

Figure 1. The kernel of terrain correction integral (set to zero at the origin).

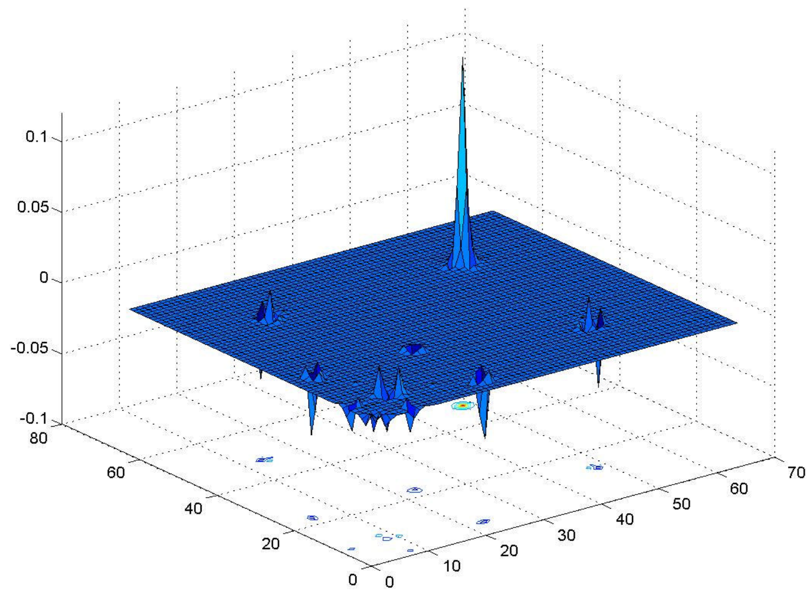

Figure 2. The wavelet transform of the terrain correction integral.

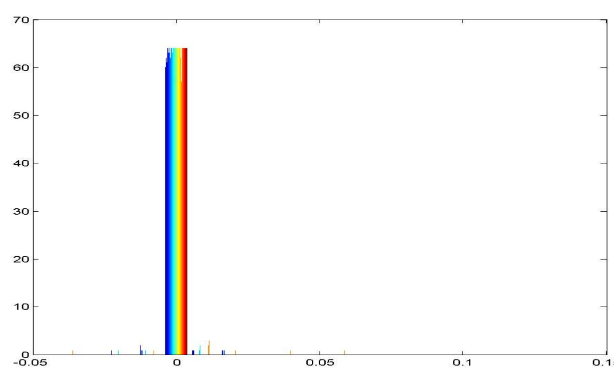

Figure 3. Histogram plot of terrain correction wavelet coefficients in each column (represented by different colors) of the data grid.

In the following, the evaluation of equation (13) using wavelet is presented. After expanding its numerator, equation (13) becomes:

$$C(x_2, y_2) = \frac{1}{2} G\rho \iint_E K(x_1, y_1, x_2, y_2) H(x_1, y_1)^2 dx_1 dy_1 -$$

$$G\rho H(x_2, y_2) \iint_E K(x_1, y_1, x_2, y_2) H(x_1, y_1) dx_1 dy_1 + \frac{1}{2} G\rho H(x_2, y_2)^2 \iint_E K(x_1, y_1, x_2, y_2) dx_1 dy_1 \tag{15}$$

The three individual integrals in the above equation can be computed by wavelet separately. The only difference is that one of the operand in the product with the kernel is the square of height, the height, or the constant 1. Since the algorithm for computing the three integrals is the same, in the following we only develop the algorithm for computing the second integral in the above equation for the sake of simplicity. The fast algorithm for computing the discrete wavelet transform and the reconstruction have already been applied to discrete height in the previous section. Analogously, they can be applied to the square of the height and constant matrix composed of elements 1. In the following, we only express the two-dimensional wavelet transform coefficients of the height at scale 2^j in their continuous forms:

$$b^1_{pq(j)} = \iint H(x_1, y_1) \psi_p(x_1) \psi_q(y_1) dx_1 dy_1 \tag{16.1}$$

$$b^2_{pq(j)} = \iint H(x_1, y_1) \psi_p(x_1) \phi_q(y_1) dx_1 dy_1 \tag{16.2}$$

$$b^3_{pq(j)} = \iint H(x_1, y_1) \phi_p(x_1) \psi_q(y_1) dx_1 dy_1 \tag{16.3}$$

$$b^4_{pq(j)} = \iint H(x_1, y_1) \phi_p(x_1) \phi_q(y_1) dx_1 dy_1 \tag{16.4}$$

ψ_p, ψ_q and ϕ_p, ϕ_q represent the orthogonal wavelet and scaling functions at the dyadic intervals at the scale 2^j when referring to a running point in the *x* and *y* directions, respectively. *H* in the discrete formula (10.4) is renamed as b^4 for the sake of easy notation. The continuous forms of two-dimensional wavelet transform coefficients $b2^1, b2^2, b2^3, b2^4$ of the square of the height and $b0^1, b0^2, b0^3, b0^4$ of constant matrix composed of element 1 can be computed in the same manner and their explicit forms are omitted.

We apply now the wavelet transform and reconstruction to the kernel. The three-dimensional wavelet transform is applied to the kernel, and the kernel is compressed in a three-dimensional basis.

The kernel is the same as formula (15), but the coordinate component x_2 of the computation point becomes constant at each meridian. Subsequently, we denote the kernel at each meridian as

$$K(x_1, y_1, y_2) = \frac{1}{[(x_c - x_1)^2 + (y_2 - y_1)^2]^{3/2}} \tag{17}$$

where the constant x_c is the x -coordinate of every computation point in the direction of the computation meridian. The kernel has a singular line at the computation meridian.

Analogously to the continuous forms of the two-dimensional wavelet transform of the heights as shown in equation (16), the continuous forms of the three-dimensional wavelet transform of the kernel at each meridian at the scale 2^j can be expressed as

$$d^1_{pqs(j)} = \iiint K(x_1, y_1, y_2)\psi_p(x_1)\psi_q(y_1)\psi_s(y_2)dx_1dy_1dy_2 \tag{18.1}$$

$$d^2_{pqs(j)} = \iiint K(x_1, y_1, y_2)\psi_p(x_1)\psi_q(y_1)\phi_s(y_2)dx_1dy_1dy_2 \tag{18.2}$$

$$d^3_{pqs(j)} = \iiint K(x_1, y_1, y_2)\psi_p(x_1)\phi_q(y_1)\psi_s(y_2)dx_1dy_1dy_2 \tag{18.3}$$

$$d^4_{pqs(j)} = \iiint K(x_1, y_1, y_2)\phi_p(x_1)\psi_q(y_1)\psi_s(y_2)dx_1dy_1dy_2 \tag{18.4}$$

$$d^5_{pqs(j)} = \iiint K(x_1, y_1, y_2)\phi_p(x_1)\phi_q(y_1)\psi_s(y_2)dx_1dy_1dy_2 \tag{18.5}$$

$$d^6_{pqs(j)} = \iiint K(x_1, y_1, y_2)\phi_p(x_1)\psi_q(y_1)\phi_s(y_2)dx_1dy_1dy_2 \tag{18.6}$$

$$d^7_{pqs(j)} = \iiint K(x_1, y_1, y_2)\psi_p(x_1)\phi_q(y_1)\phi_s(y_2)dx_1dy_1dy_2 \tag{18.7}$$

$$d^8_{pqs(j)} = \iiint K(x_1, y_1, y_2)\phi_p(x_1)\phi_q(y_1)\phi_s(y_2)dx_1dy_1dy_2 \tag{18.8}$$

where ψ_s and ϕ_s represent the orthogonal wavelet and scaling functions at the dyadic interval at the scale 2^j referring to a computation point in the meridian direction, i.e., the y direction. The above three-dimensional wavelet coefficients can also be computed recursively with a fast algorithm similar to the two-dimensional fast algorithm shown in the last section.

After computing all of these 7J+1 coefficients, where 2^J is the largest scale, the original kernel at each meridian is recovered from this wavelet representation by the following reconstruction formula:

$$
\begin{aligned}
&K(x_1, y_1, y_2) = \\
&\sum_j \sum_p \sum_q \sum_s d^1_{pqs(j)} \psi_p(x_1)\psi_q(y_1)\psi_s(y_2) + \sum_j \sum_p \sum_q \sum_s d^2_{pqs(j)} \psi_p(x_1)\psi_q(y_1)\phi_s(y_2) + \\
&\sum_j \sum_p \sum_q \sum_s d^3_{pqs(j)} \psi_p(x_1)\phi_q(y_1)\psi_s(y_2) + \sum_j \sum_p \sum_q \sum_s d^4_{pqs(j)} \phi_p(x_1)\psi_q(y_1)\psi_s(y_2) \\
&\sum_j \sum_p \sum_q \sum_s d^5_{pqs(j)} \phi_p(x_1)\phi_q(y_1)\psi_s(y_2) + \sum_j \sum_p \sum_q \sum_s d^6_{pqs(j)} \phi_p(x_1)\psi_q(y_1)\phi_s(y_2) + \\
&\sum_j \sum_p \sum_q \sum_s d^7_{pqs(j)} \psi_p(x_1)\phi_q(y_1)\phi_s(y_2) + \sum_p \sum_q \sum_s d^8_{pqs(J)} \phi_p(x_1)\phi_q(y_1)\phi_s(y_2)
\end{aligned}
\tag{19}
$$

The last term is the remaining coarse kernel approximation at scale 2^J. Substituting the reconstructed kernel into the second term of formula (15) yields the second integral of the terrain corrections in the computation meridian:

$$
\begin{aligned}
&N(x_c)_2 = \frac{G\rho H(x_2 - y_2)}{2} \iint [\sum_j \sum_p \sum_q \sum_s d^1_{pqs(j)} \psi_p(x_1)\psi_q(y_1)\psi_s(y_2) \\
&+ \sum_j \sum_p \sum_q \sum_s d^2_{pqs(j)} \psi_p(x_1)\psi_q(y_1)\phi_s(y_2) + \\
&\sum_j \sum_p \sum_q \sum_s d^3_{pqs(j)} \psi_p(x_1)\phi_q(y_1)\psi_s(y_2) + \sum_j \sum_p \sum_q \sum_s d^4_{pqs(j)} \phi_p(x_1)\psi_q(y_1)\psi_s(y_2) + \\
&\sum_j \sum_p \sum_q \sum_s d^5_{pqs(j)} \phi_p(x_1)\phi_q(y_1)\psi_s(y_2) + \sum_j \sum_p \sum_q \sum_s d^6_{pqs(j)} \phi_p(x_1)\psi_q(y_1)\phi_s(y_2) + \\
&\sum_j \sum_p \sum_q \sum_s d^7_{pqs(j)} \psi_p(x_1)\phi_q(y_1)\phi_s(y_2) + \sum_p \sum_q \sum_s d^8_{pqs(J)} \phi_p(x_1)\phi_q(y_1)\phi_s(y_2)] H(x_1, y_1) dx_1 dy_1
\end{aligned}
\tag{20}
$$

Interchanging the order of integration and summation, moving H into each integration and comparing formula (16) and formula (20), we obtain

$$
\begin{aligned}
N(x_c)_2 = {} & \frac{G\rho H(x_2, y_2)}{2} \times \sum \sum \sum \left(d^1_{pqs(1)} b^1_{pq(1)} + d^3_{pqs(1)} b^2_{pq(1)} + d^4_{pqs(1)} b^3_{pq(1)} + d^5_{pqs(1)} b^4_{pq(1)} \right) \psi_s(y_2) + \\
& \frac{G\rho}{2} \times \sum \sum \sum \left(d^2_{pqs(1)} b^1_{pq(1)} + d^6_{pqs(1)} b^3_{pq(1)} + d^7_{pqs(1)} b^2_{pq(1)} \right) \phi_s(y_2) + d^8_{pqs(J)} b^4_{pq(J)} \phi_s(y_2)
\end{aligned}
\tag{21}
$$

The first term $N(x_c)_1$ and third term $N(x_c)_3$ of formula (15) can also be computed as in the above procedure. The final terrain correction can then be obtained as the sum of the three integrals.

The summation of the products of $d^1_{pqs}b^1_{pq}$, $d^3_{pqs}b^2_{pq}$, $d^4_{pqs}b^3_{pq}$, $d^5_{pqs}b^4_{pq}$ and $d^2_{pqs}b^1_{pq}$, $d^6_{pqs}b^3_{pq}$, $d^7_{pqs}b^2_{pq}$, $d^8_{pqs}b^4_{pq}$ is the set of wavelet coefficients of the basis functions $\psi \quad L^2(\quad)$ and $\phi \quad L^2(\quad)$ in the scale 2^j for representing the second integral of terrain corrections in one meridian. The derived formula is fully consistent with the wavelet decomposition and reconstruction formula (5).

The above formula is rather symbolic because the wavelet transform and its reconstruction are done scale by scale. Some restructuring is necessary before we could practically implement the above-derived formula. The reconstructed kernel is a three-dimensional matrix, the first two indexes are translating parameters p and q in the x_1 and y_1 directions of the running point, respectively; the third is the translating parameter s in the y_2 direction of the computation point. In order to employ the reconstructed kernel in the terrain correction formula together with the wavelet coefficients of the heights the square of the heights or the constant matrix composed of elements 1 which are two-dimensional matrices, their structure needs to be rearranged. The original two-dimensional matrix corresponding to each s is rearranged as a vector, which is composed of all the rows of the original matrix in increasing order of columns from north to south. Therefore, in the rearranged two-dimensional matrix, the first index is the translating parameter s in the y_2 direction of the computation point from north to south; and the second index is the number of all the wavelet coefficients of the kernel corresponding to each point in the computation meridian. The two-dimensional matrix of the wavelet coefficients of H, H^2, or the constant matrix composed of elements 1 needs to be rearranged as a vector. It is composed of all the rows of the original matrix in increasing order of columns.

4. Numerical Examples

To test the feasibility and effectiveness of the wavelet approach in computing terrain corrections, new software code was implemented in MATLAB. Considering the reason that the wavelets are introduced in physical geodesy is their expected high efficiency of evaluation in terms of savings of memory and computation speed, it is obviously necessary to compare wavelets with the highly developed and widely accepted efficient FFT approach in evaluating terrain corrections. For this reason a MATLAB version of the 2D FFT algorithm in the plane was also written so that the two approaches can be compared in the same language and platform.

Terrain corrections were computed in the Rocky Mountains of British Columbia, Canada. The data used is a 64 by 64 grid of 5' heights in the area bounded by latitude 49˚N to 54.4˚N and longitude 236˚E to 241.4˚E, as shown in Figure 4. Table 1 gives the statistical information of the heights.

Table 1. Statistics of heights, in metres

Data	max	min	mean	rms	σ
H	2823	0	1223	1232	147

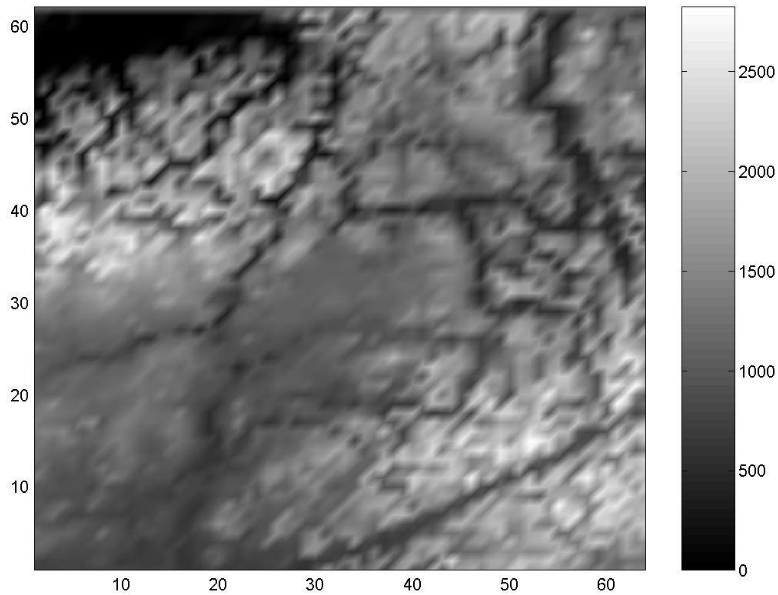

Figure 4. The topography in the test area, in metres.

The computations were done on a PC (Windows 2000) running MATLAB version 6. the terrain corrections were computed using the linemass method by pointwise integration, and by 2D FFT and wavelet method on the plane. The kernels were pre-computed first before applying pointwise integration and wavelets. The Daubechies wavelet with 4 vanishing moments was chosen for transforming the kernel because it maximizes the number of zero coefficients and minimizes the number of high amplitude coefficients as stated in section 2. Different threshold values for the wavelet coefficients of the kernel, which result in different kernel compression ratios, were employed. The coefficients whose absolute values were larger than the threshold, and their indexes were saved into two separate vectors in the first stage. They were retrieved to compute the terrain corrections in the second stage. The wavelet methods with different compression rations are compared with pointwise integration and FFT in terms of computational efficiency and accuracy.

The bulk of memory storage was needed for the gravity data and the 100% zero-padding needed for FFT (Sideris and Li 1993; Liu et al. 1996). Memory was also allocated for retrieving the kernels (or the compressed kernels for wavelets).

Table 2 shows the statistical information of the terrain corrections computed by pointwise integration on the plane, by 2D FFT on the plane, and by wavelets with different kernel compression ratios. The CPU time in seconds and the memory allocation required for these computations are also listed in the same table. Note that the number of elements needed for reading the compressed kernel when applying the wavelet method is just an approximation because the number is not constant for different points; and the listed number of elements include the number of elements for the indexes of the compressed kernel coefficients, which are read into a different vector. Figure 5 presents the terrain corrections computed with the pointwise integration (PT) method. Figure 6 gives the differences between the PT and wavelet method W7, which has a kernel compression ratio of 7%. Figure 7 shows the differences between the PT and the W3 method, which has a kernel compression ratio of 3%. Figure 8 displays the differences between the PT and the W0.7 method, which has a kernel compression ratio of 0.7%. Figure 9 displays the differences between the PT and the W0.1 method, which has a kernel compression ratio of 0.1%. The statistical information of these differences is also presented in Table 2. The values following the names of the wavelet methods in the first column of Table 2 are the wavelet coefficient thresholds for compressing the kernels.

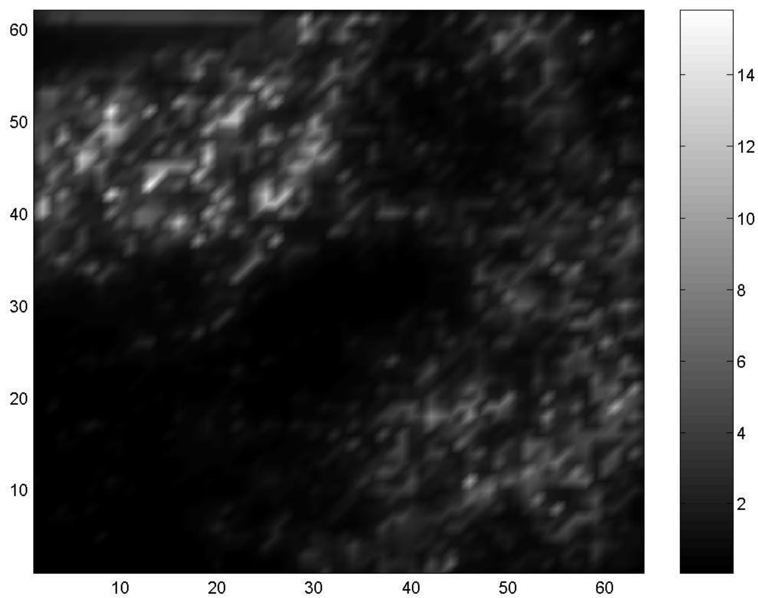

Figure 5. Terrain corrections by pointwise integration, in mGal.

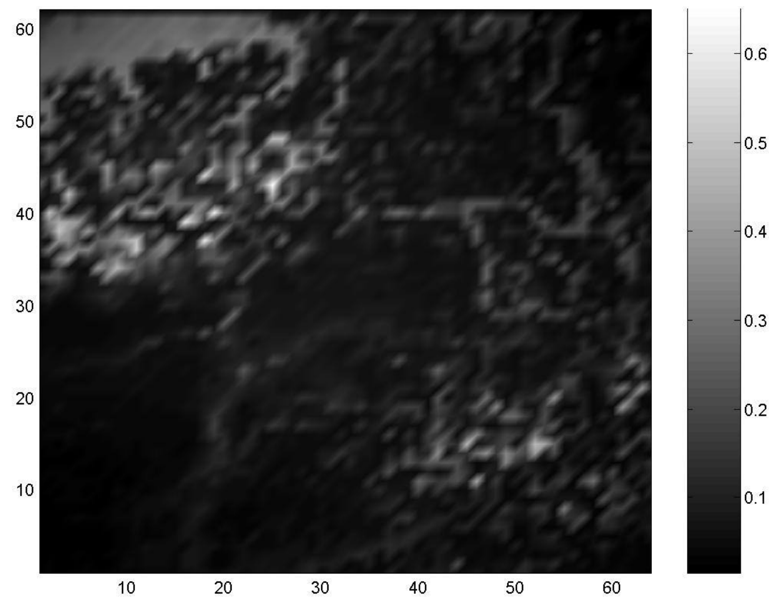

Figure 6. Difference of terrain corrections between PT and W7, in mGal.

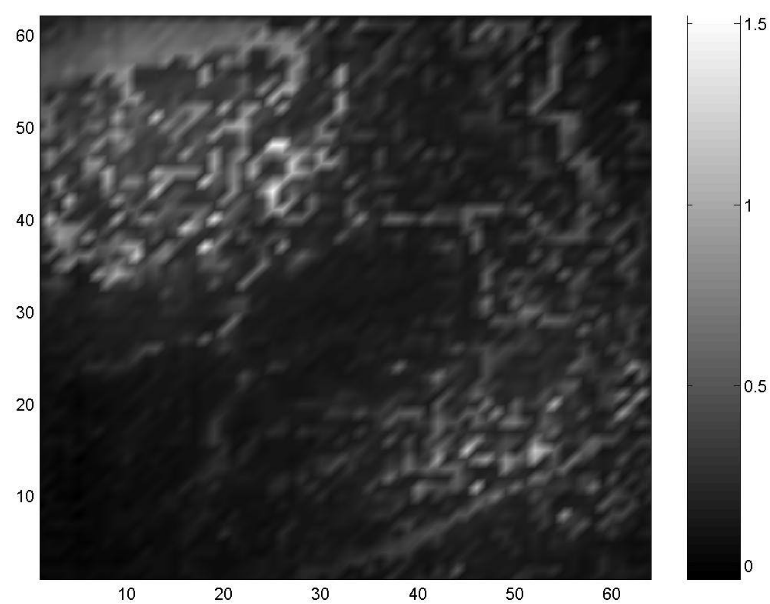

Figure 7. Difference of terrain corrections between PT and W3 in mGal.

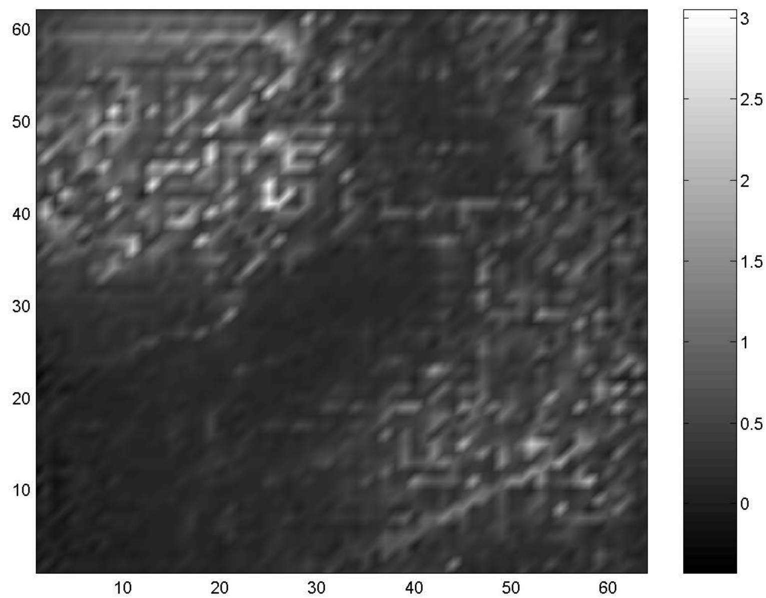

Figure 8. Difference of terrain corrections PT and W0.7, in mGal.

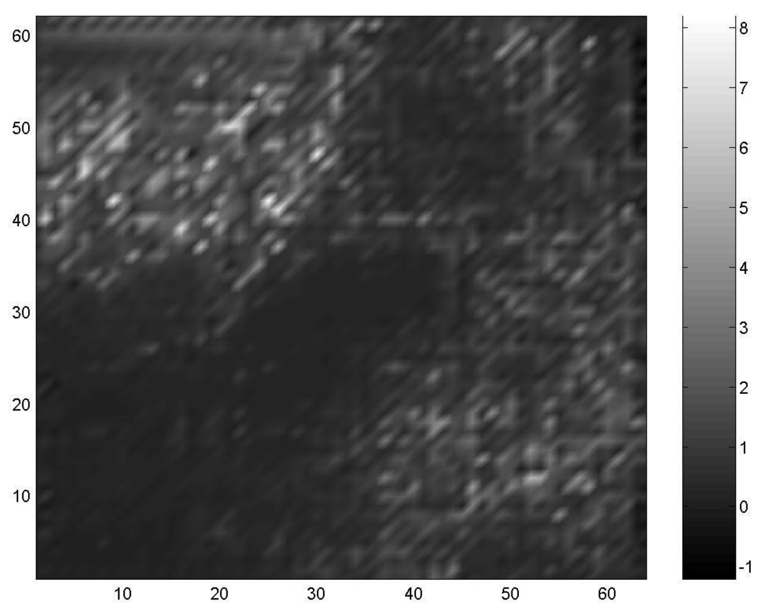

Figure 9. Difference of terrain corrections PT and W0.1, in mGal.

Table 2. Comparison of terrain corrections from different methods.

Evaluation method	max	min	mean mGal	rms	σ	CPU sec	No. of height elements	No. of kernel elements
Pointwise (PT)	15.8	0.1	1.8	2.5	1.8	14.3	64×64	64×64
2D FFT (FFT)	15.8	0.1	1.8	2.5	1.8	3.4	128×128	128×128
Wavelet (W7) 0.0001	15.6	0.0	1.7	2.4	1.8	4.1	64×64	2×286
Wavelet (W3) 0.001	14.6	-0.0	1.5	2.2	1.7	2.8	64×64	2×122
Wavelet (W0.7) 0.01	13.0	-0.3	1.3	2.0	1.5	2.2	64×64	2×28
Wavelet (W0.1) 0.05	12.6	-0.9	0.8	1.4	1.1	1.9	64×64	2×4
PT- FFT	0.0	0.0	0.0	0.0	0.0			
PT-W7	0.7	0.0	0.1	0.1	0.0			
PT-W3	1.5	-0.0	0.2	0.2	0.0			
PT-W0.7	3.0	-0.4	0.4	0.4	0.1			
PT-W0.1	8.2	-1.2	0.9	0.9	0.2			

Table 2 shows that the computer time required by the FFT is 3.4 seconds and by the wavelet methods ranges from 4.1 to 1.9 seconds. It is obvious that the wavelet method is slower than FFT method when the kernel compression ratio is over 7%, and has a comparable speed when the kernel compression ratio is down to 7%, and is faster than the FFT method when the kernel compression ratio is down to 3%. Table 2 also shows that the memory allocation for wavelets is only about 3.5%, 1.5%, 0.3% and 0.04% of the memory allocation for FFT when the kernel compression ratio is 7%, 3%, 0.7% and 0.1%, respectively.

A close inspection of Table 2, Figure 5, Figure 6, Figure 7, Figure 8 and Figure 9 shows that wavelets are a very powerful tool in evaluating the terrain corrections. The result by FFT is identical to the result by the PT method. This is because the pointwise integration is performed on the plane and the use of FFT with 100% zero-padding gives identical results to the pointwise integration (Sideris and Li 1993; Liu et al. 1996). The maximum differences between the results obtained by the W7 with a kernel compression ratio of 7%, which has a comparable speed to FFT, and the PT method can reach up to 0.6 mGal in extreme cases. As an average value, though, they generally remain below 0.1 mGal, and their difference in terms of rms and σ, are about 0.1 mGal. The differences between the PT and the W3 with a compression ratio of 3%, which is faster than FFT, are about 0.2 and 0.0 mGal in terms of rms and σ, though the maximum difference can reach 1.5 mGal in extreme cases . Even more surprising is that the differences between the results by W0.7 with a compression ratio of 0.7%, which is even faster than FFT, and the result by PT integration are still very small (0.4 and 0.1 mGal in terms both of rms and σ), though the maximum difference can reach 3 mGal in extreme cases. Even when the compression ratio is down to 0.1%, the differences between the results by W0.1 and the PT integration (0.9 mGal and 0.2 mGal difference in rms and σ) are still acceptable in many practical applications, though the maximum difference can reach 8 mGal in extreme cases. These results indicate that the kernel can be compressed down to 7% with the wavelet method if one wants to obtain the terrain corrections with better that 0.1 mGal accuracy and with a comparable speed as the FFT. The kernel can be compressed down to 3% with the wavelet method if one wants to obtain the terrain corrections with a sub mGal accuracy and with a faster speed than the FFT. If the kernel is further compressed to 0.7% then one can obtain the terrain corrections with a one mGal accuracy and an even faster speed than the FFT. Therefore, in practical applications the wavelet method appears to be more efficient than the FFT method both in terms of speed and especially in terms of memory allocation. This is due to the fact that the singularity of the terrain correction integral is very strong, so the vast majority of wavelet coefficients are very small (cf. Figure 1, Figure 2 and Figure 3) and a very small number of larger wavelet coefficients dominate the contribution of the kernel to the computation of terrain corrections. Consequently, the kernel can be significantly compressed.

5. Conclusion

From the theoretical developments and numerical experiments in this paper, it is seen that the singular integral of terrain correction can be compressed in a wavelet basis. It can be evaluated by the two-dimensional wavelet transform in a point by point basis or by the three-dimensional wavelet transform in a meridian by meridian basis. Terrain corrections with sub mGal and one mGal accuracy and faster speed than FFT can be obtained by the wavelet method with a compression ratios of 3%, 0.7%, respectively. It is shown that the wavelet evaluation of the terrain correction integral needs much less memory allocation than the FFT methods. It is also shown that wavelets can be faster than FFT if one wants to obtain the terrain correction with one mGal accuracy. Therefore, when computing the terrain corrections with an acceptable accuracy, the wavelet method is a wise alternative choice in place of FFT if one wants to save computer memory and computer time. These results refer to hights on a 5' grid. To generalize these conclusions, the computation should be performed on denser grid.

The current approach is developed on the plane. With the introduction of harmonic wavelets (Freeden 1998; Blairs and Provins 2002), it is possible to compute the geodetic integrals (Stokes and Verning Meinesz integrals and terrain corrections) in the harmonic wavelet basis on the sphere. This will be investigated in a future paper.

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A Review of the UNB Approach for Precise Geoid Determination Based on the Stokes-Helmert Method

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ABSTRACT

Many methods for geoid determination exist and are in use around the world. One of the most advantageous methods is the Stokes-Helmert approach developed at the University of New Brunswick. The main theoretical developments of this method is attributed to Vaníček, along with the contribution of other authors, such as Martinec, Sjöberg, Kleusberg, Heck and Grafarend. The theoretical aspects of the UNB approach were published in more than fifty contributions (see References) and the general principles are summarized in Vaníček and Martinec (1994), Vaníček et al. (1999), Novák (2000), and Vaníček and Janák (2001). The main idea of this contribution is to offer to readers, in a more detailed form, the basic theoretical aspects of the Stokes-Helmert approach for geoid determination. Another purpose is to summarize publications related to this topic.

Key words: Atmospheric Effect – Bruns's Formula – Density of Topographical and Atmospheric Masses – Geodetic Boundary-Value Problem – Geoid – Gravitational Attraction – Gravitational Potential – Gravity – Gravity Anomaly – Helmert Space – Helmert's Second Condensation Method – Newton's Integral – Orthometric Height – Poisson's Integral – Stokes's Integral – Terrain – Topographical Effect

1 Introduction

In the classical sense of Gauss and Listing, the geoid is defined as an equipotential surface of the Earth's gravity field with the gravity potential value W_0. Gauss (1828) was the first to define this surface in the strict mathematical sense as a surface which is intersected everywhere by directions of gravity at right angle and which best approximates the mean sea level over the whole Earth. Later, Bessel (1837) stipulated this equipotential surface as a reference for all geodetic applications. Finally, Listing (1873) called this surface "geoid".

Stokes (1849) derived a theorem, which forms a theoretical foundation for estimation of the geoid based on gravity observations that refer to the geoid (assuming as harmonic the space above the geoid). The requirement of harmonicity was difficult to fulfill in practical applications of the Stokes theory since the distribution of actual topographical density between the geoid and the Earth's surface is not known with sufficient accuracy.

The first attempt to satisfy this requirement can be attributed to Helmert (1884). Helmert suggested that the Earth's topographical masses can be replaced by an infinitesimal condensation layer of a surface density that is equal to the product of the mean topographical density and height of the Earth's surface above the geoid. This layer could be located anywhere on or beneath the geoid without violating the required assumption of harmonicity. In the second condensation method that Helmert formulated, the condensation layer is placed right on the geoid (Lambert, 1930; Heck, 1992; Martinec et al., 1993).

According to Newton's theory of gravity, Martinec and Vaníček formulated principles for the description of the effect of topographical masses on the gravitational potential and attraction in the case of laterally varying topographical density distribution and for the spherical approximation of the geoid (Martinec, 1993; Martinec and Vaníček, 1994a, b). Sjöberg (1998 and 1999) and Novák (2000) studied the effect of the atmospheric masses in the Stokes-Helmert method of geoid determination.

Based on Molodensky's theory (Molodensky et al., 1960), Vaníček and Kleusberg (1987) introduced the idea of modification of the Stokes function to separate the reference and higher-degree gravity field. Theory of the reference gravity field and the spheroid, and the reformulation of Stokes's boundary-value problem for the higher-degree reference spheroid. were described by Vaníček and Kleusberg (1987), Vaníček and Sjöberg (1991), Vaníček et al. (1995), Vaníček and Featherstone (1998).

The solution of Dirichlet's boundary-value problem by applying the Poisson integral equation for the downward continuation of Helmert's gravity anomalies was investigated by Martinec (1996), Vaníček et al. (1996), Sun and Vaníček (1998) and Huang (2002).

The principle of the Stokes-Helmert scheme of geoid determination can be summarized in the following scheme (Vaníček et al., 1999; Vaníček and Janák, 2001):

- Formulation of the boundary-value problem of the third kind on the Earth's surface.
- Transformation of the boundary-value problem into a harmonic space, i.e., transformation of gravity anomalies from the real to Helmert space (according to the second condensation technique where the topographical and atmospheric masses are condensed directly onto the geoid).

- Solution of Dirichlet's boundary-value problem by applying the Poisson integral equation, i.e., the downward continuation of Helmert's gravity anomalies from the Earth's surface to the geoid.
- Reformulation of the geodetic boundary-value problem by decomposition of Helmert's gravity field into a low and high-frequency gravity field.
- Solution of the Stokes boundary-value problem for the high-frequency Helmert gravity field (by using the modified spheroidal Stokes kernel) and evaluation of Helmert's reference spheroid (from a satellite geopotential model).
- Transformation of the equipotential surface from the Helmert space back into the real space.

2. Geodetic boundary-value problem in the real space

Let us begin with the definition of the disturbing gravity potential $T(r_t(\Omega))$ which is reckoned at the Earth's surface, $\Omega \in \Omega_{\mathrm{O}} : r_t(\Omega) = r_g(\Omega) + H^{\mathrm{O}}(\Omega)$, as the difference of the Earth's gravity potential $W(r_t(\Omega))$ and the normal gravity potential $U(r_t(\Omega))$ generated by the reference geocentric ellipsoid of revolution (Somigliana, 1929; Pizzeti, 1894 and 1911)

$$\Omega \in \Omega_{\mathrm{O}} : T(r_t(\Omega)) = W(r_t(\Omega)) - U(r_t(\Omega)), \tag{2.1}$$

where $r_t(\Omega)$ stands for the geocentric radius of the Earth's surface, $r_g(\Omega)$ is the geocentric radius of the geoid and $H^{\mathrm{O}}(\Omega)$ is the orthometric height. A pair of the geocentric coordinates ϕ and λ represent the geocentric direction $\Omega = (\phi, \lambda)$ while Ω_{O} stands for the total solid angle $\left[\phi \in \langle -\pi/2, \pi/2 \rangle, \lambda \in \langle 0, 2\pi \rangle\right]$. Eqn. (2.1) is valid only if the normal gravity potential U_o on the reference ellipsoid equals to the gravity potential W_o on the geoid.

Approximating the geoid by the geocentric sphere of radius R, i.e., $\Omega \in \Omega_{\mathrm{O}} : r_g(\Omega) \approx R$, the radial derivative of the disturbing gravity potential $T(r_t(\Omega))$ reads (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :$$

$$\left.\frac{\partial\, T(r,\Omega)}{\partial\, r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = \left.\frac{\partial\, W(r,\Omega)}{\partial\, r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} - \left.\frac{\partial\, U(r,\Omega)}{\partial\, r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} =$$

$$= \left|\mathbf{grad} W(r_t(\Omega))\right| \cos\left(\mathbf{grad} W(r_t(\Omega)), \mathbf{r^o}\right) - \left|\mathbf{grad} U(r_t(\Omega))\right| \cos\left(\mathbf{grad} U(r_t(\Omega)), \mathbf{r^o}\right), \tag{2.2}$$

where $\mathbf{r^o}$ is the unit vector in the radial direction. The vertical gradient of the gravity potential $W(r_t(\Omega))$ and the vertical gradient of the normal gravity potential define gravity $g(r_t(\Omega))$ and normal gravity $\gamma(r_t(\Omega))$:

$$\forall\, \Omega \in \Omega_O : \left| \mathbf{grad} W(r_t(\Omega)) \right| = -\left| \mathbf{g}(r_t(\Omega)) \right| = -g(r_t(\Omega)), \tag{2.3}$$

$$\forall\, \Omega \in \Omega_O : \left| \mathbf{grad} U(r_t(\Omega)) \right| = -\left| \boldsymbol{\gamma}(r_t(\Omega)) \right| = -\gamma(r_t(\Omega)). \tag{2.4}$$

The angle between the plumb line and the radial direction $\angle\left(-\mathbf{g}\,,\mathbf{r^o}\right)$, and the angle between the normal to the reference ellipsoid and the radial direction $\angle\left(-\boldsymbol{\gamma}\,,\mathbf{r^o}\right)$ can be written with sufficient accuracy as follows (Vaníček et al., 1999) :

$$\cos\left(-\mathbf{g}\,,\mathbf{r^o}\right) \cong 1 - \frac{\beta_g^2}{2}, \tag{2.5}$$

$$\cos\left(-\boldsymbol{\gamma}\,,\mathbf{r^o}\right) \cong 1 - \frac{\beta_\gamma^2}{2}. \tag{2.6}$$

Substituting Eqns. (2.3-2.6) back to Eqn. (2.2), the radial derivative of the disturbing gravity potential becomes (Vaníček et al., 1999)

$$\begin{aligned} &\forall\, \Omega \in \Omega_O : \\ &\left.\frac{\partial\, T(r,\Omega)}{\partial\, r}\right|_{r=R+H^O(\Omega)} = -g(r_t(\Omega)) + \gamma(r_t(\Omega)) + \frac{g(r_t(\Omega))}{2}\beta_g^2(r_t(\Omega)) - \frac{\gamma(r_t(\Omega))}{2}\beta_\gamma^2(r_t(\Omega)) = \\ &= -\delta g(r_t(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)), \end{aligned} \tag{2.7}$$

where the difference of gravity $g(r_t(\Omega))$ and normal gravity $\gamma(r_t(\Omega))$ defines the gravity disturbance, $\delta g(r_t(\Omega)) = g(r_t(\Omega)) - \gamma(r_t(\Omega))$, $\varepsilon_{\delta g}(r_t(\Omega))$ is the „*ellipsoidal correction to the gravity disturbance*" (ibid)

$$\begin{aligned} &\forall\, \Omega \in \Omega_O : \\ &\varepsilon_{\delta g}(r_t(\Omega)) = \delta g(r_t(\Omega))\frac{\beta_\gamma^2(r_t(\Omega))}{2} + g(r_t(\Omega))\beta_\gamma(r_t(\Omega))\xi(r_t(\Omega)) + \frac{g(r_t(\Omega))}{2}\theta^2(r_t(\Omega)), \end{aligned} \tag{2.8}$$

and ξ, η stand for the components of the deflection of a vertical θ, $\theta = \sqrt{\xi^2 + \eta^2}$.
The angle β_γ is the difference of the geodetic latitude φ and geocentric latitude ϕ that is given by Bomford (1971)

$$\beta_\gamma \cong f \sin 2\varphi \,, \tag{2.9}$$

where $f = (a-b)/a$ is the first geometric flattening of the reference ellipsoid.

Considering only the second term on the right-hand side of Eqn. (2.8), the ellipsoidal correction to the gravity disturbance $\varepsilon_{\delta g}(r_t(\Omega))$ can be evaluated with sufficient accuracy by (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \varepsilon_{\delta g}(r_t(\Omega)) \cong g(r_t(\Omega))\, f \sin 2\varphi\; \xi(r_t(\Omega)) \cong -\frac{f \sin 2\varphi}{r_t(\Omega)} \frac{\partial T(r,\Omega)}{\partial\, \varphi} \,. \tag{2.10}$$

Since the computation of normal gravity $\gamma(r_t(\Omega))$ on the Earth's surface requires the knowledge of the geodetic height $h(\Omega)$ above the reference ellipsoid, the gravity disturbance $\delta g(r_t(\Omega))$ is transformed into the gravity anomaly $\Delta g(r_t(\Omega))$. Gravity anomaly is given as a difference of gravity $g(r_t(\Omega))$ on the Earth's surface and normal gravity $\gamma(H^{\mathrm{N}}(\Omega))$ on the telluroid, i.e., $\forall\, \Omega \in \Omega_{\mathrm{O}} : r(\Omega) \cong r_o(\Omega) + H^{\mathrm{N}}(\Omega)$, see (Vaníček et al., 1999)

$$\begin{aligned} &\forall\, \Omega \in \Omega_{\mathrm{O}} : \\ &\Delta g(r_t(\Omega)) = g(r_t(\Omega)) - \gamma(H^{\mathrm{N}}(\Omega)) = \delta g(r_t(\Omega)) + \gamma(r_t(\Omega)) - \gamma(H^{\mathrm{N}}(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)), \end{aligned} \tag{2.11}$$

where $r_o(\Omega)$ is the geocentric radius of the reference ellipsoid and $H^{\mathrm{N}}(\Omega)$ stands for the normal height (Molodensky, 1945).

Considering Molodensky's approach (Molodensky et al., 1960), the difference of normal gravity $\gamma(r_t(\Omega))$ on the Earth's surface, $\Omega \in \Omega_{\mathrm{O}} : r_t(\Omega) = r_g(\Omega) + H^{\mathrm{O}}(\Omega) \cong r_o(\Omega) + h(\Omega)$, and normal gravity $\gamma(H^{\mathrm{N}}(\Omega))$ on the telluroid can be defined as

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \gamma(r_t(\Omega)) - \gamma(H^{\mathrm{N}}(\Omega)) = \left| \mathbf{grad}\, \gamma(r_t(\Omega)) \right| \varsigma(\Omega) = \left. \frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}} \right|_{r = R + H^{\mathrm{O}}(\Omega)} \varsigma(\Omega), \tag{2.12}$$

where the derivative of normal gravity is taken with respect to the normal n to the reference ellipsoid and $\varsigma(\Omega)$ is the height anomaly (Molodensky et al., 1960). Using Bruns's spherical formula (Bruns, 1878), the expression on the right-hand side of Eqn. (2.12) can be rewritten as (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \varsigma(\Omega) = \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \frac{T(r_t(\Omega))}{\gamma(H^{\mathrm{N}}(\Omega))}. \tag{2.13}$$

Substituting Eqn. (2.13) into Eqn. (2.11), the fundamental boundary condition takes the following form

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \Delta g(r_t(\Omega)) = \delta g(r_t(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)) + \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \frac{T(r_t(\Omega))}{\gamma(H^{\mathrm{N}}(\Omega))}. \tag{2.14}$$

Applying the following spherical approximation

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \frac{1}{\gamma(H^{\mathrm{N}}(\Omega))} \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} T(r_t(\Omega)) = -\frac{2}{r_t(\Omega)} T(r_t(\Omega)) - \varepsilon_{\mathrm{n}}(r_t(\Omega)), \tag{2.15}$$

the boundary condition in Eqn. (2.14) becomes (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :$$

$$\Delta g(r_t(\Omega)) = -\left.\frac{\partial\, T(r,\Omega)}{\partial\, r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \varepsilon_{\delta g}(r_t(\Omega)) - \frac{2}{r_t(\Omega)} T(r_t(\Omega)) - \varepsilon_{\mathrm{n}}(r_t(\Omega)). \tag{2.16}$$

The „*ellipsoidal correction for the spherical approximation*“ $\varepsilon_{\mathrm{n}}(r_t(\Omega))$ can be derived in the following form (Vaníček and Martinec, 1994)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \varepsilon_{\mathrm{n}}(r_t(\Omega)) \cong 2\left[m + f\left(\cos 2\varphi - \frac{1}{3}\right)\right] \frac{T(r_t(\Omega))}{R}, \tag{2.17}$$

where $m = \omega^2 a^3 / GM$ stands for the Clairaut constant (Heiskanen and Moritz, 1967), GM is the geocentric gravitational constant, ω is the mean angular velocity of the Earth's rotation, and the mean radius of the Earth R can be evaluated by the following formula (Vaníček and Krakiwsky, 1986)

$$R = \sqrt[3]{a^2 b}\,. \tag{2.18}$$

3. Geodetic boundary-value problem in the Helmert space

To investigate the geodetic boundary-value problem in the Helmert space, „*Helmert's disturbing gravity potential*“ $T^{\mathrm{H}}(r_t(\Omega))$ referred to the Earth's surface is defined by Vaníček et al. (1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}\ :\ T^{\mathrm{H}}\left(r_t(\Omega)\right) = T\left(r_t(\Omega)\right) - \delta V^{t}\left(r_t(\Omega)\right) - \delta V^{a}\left(r_t(\Omega)\right), \tag{3.1}$$

where $\delta V^{t}\left(r_t(\Omega)\right)$ and $\delta V^{a}\left(r_t(\Omega)\right)$ are the so-called residual gravitational potentials of topographical and atmospheric masses.

Assuming the mean angular velocity ω of the Earth's rotation is equal to the mean angular velocity of rotation of the reference ellipsoid, the disturbing gravity potential $T^{\mathrm{H}}(r,\Omega)$ is harmonic everywhere above the geoid, i.e., $\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > r_g(\Omega)$: $\Delta T^{\mathrm{H}}(r,\Omega) = 0$ in the Helmert space.

„*Helmert's gravity*" $g^{\mathrm{H}}\left(r_t(\Omega)\right)$ is related to actual gravity $g\left(r_t(\Omega)\right)$ as follows (Vaníček et al., 1999):

$$\forall\ \Omega \in \Omega_{\mathrm{O}}\ :\ g^{\mathrm{H}}\left(r_t(\Omega)\right) = g\left(r_t(\Omega)\right) + \left.\frac{\partial \delta V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \tag{3.2}$$

„*Helmert's gravity disturbance*" $\delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ defined as the negative vertical gradient of the Helmert disturbing gravity potential can be described as a sum of the negative radial derivative of the Helmert disturbing gravity potential $T^{\mathrm{H}}\left(r_t(\Omega)\right)$ and the ellipsoidal correction $\varepsilon_{\delta g}\left(r_t(\Omega)\right)$ to the gravity disturbance

$$\forall\ \Omega \in \Omega_{\mathrm{O}}\ :$$

$$\delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = -\left.\frac{\partial T^{\mathrm{H}}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \varepsilon_{\delta g}\left(r_t(\Omega)\right) =$$

$$= g\left(r_t(\Omega)\right) - \gamma\left(r_t(\Omega)\right) + \varepsilon_{\delta g}\left(r_t(\Omega)\right) + \left.\frac{\partial \delta V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}. \tag{3.3}$$

The relation between the gravity disturbance $\delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ and gravity anomaly $\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ in the Helmert space can be obtained from the boundary condition (Heiskanen and Moritz, 1967)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}\ :$$

$$\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = -\left.\frac{\partial T^{\mathrm{H}}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \gamma(r,\Omega)}{\partial \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \frac{T^{\mathrm{H}}\left(r_t(\Omega)\right)}{\gamma\left(H^{\mathrm{N}}(\Omega)\right)} + \varepsilon_{\delta g}\left(r_t(\Omega)\right) =$$

$$= \delta g^{\mathrm{H}}\big(r_t(\Omega)\big) + \gamma\big(r_t(\Omega)\big) - \gamma_o(\phi) - \frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\bigg|_{r=r_o(\Omega)} H^{\mathrm{N}}(\Omega) -$$

$$- \frac{1}{\gamma\big(H^{\mathrm{N}}(\Omega)\big)} \frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\bigg|_{r=R+H^{\mathrm{O}}(\Omega)} \Big[\delta V^{t}\big(r_t(\Omega)\big) + \delta V^{a}\big(r_t(\Omega)\big)\Big] + \varepsilon_{\delta g}\big(r_t(\Omega)\big), \tag{3.4}$$

where $\gamma_o(\phi)$ is normal gravity on the reference ellipsoid (Somigliana, 1929).

If Helmert's orthometric height $H^{\mathrm{O}}(\Omega)$ is used (Helmert, 1890), the „*geoid-quasigeoid correction*" has to be applied to the boundary condition formulated in the Helmert space (Vaníček et al., 1999). The geoid-quasigeoid correction, i.e., the difference of the normal and orthometric heights, can be approximately described as a function of the simple Bouguer gravity anomaly $\Delta g^{\mathrm{SB}}\big(r_t(\Omega)\big)$, see (Martinec, 1993),

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :\ H^{\mathrm{N}}(\Omega) - H^{\mathrm{O}}(\Omega) \cong H^{\mathrm{O}}(\Omega)\, \frac{\Delta g^{\mathrm{SB}}\big(r_t(\Omega)\big)}{\gamma_o(\phi)}. \tag{3.5}$$

The formula for the simple Bouguer gravity anomaly $\Delta g^{\mathrm{SB}}\big(r_t(\Omega)\big)$ reads (Heiskanen and Moritz, 1967)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :\ \Delta g^{\mathrm{SB}}\big(r_t(\Omega)\big) = g\big(r_t(\Omega)\big) - \gamma\big(H^{\mathrm{O}}(\Omega)\big) - 2\pi\, G \rho_o H^{\mathrm{O}}(\Omega), \tag{3.6}$$

where G is the Newton (universal) gravitational constant. The third term on the right-hand side of Eqn. (3.6) stands for gravitational attraction generated by the infinite Bouguer plate (with the mean topographical density ρ_o and thickness equal to the orthometric height $H^{\mathrm{O}}(\Omega)$ at the computation point). Substituting Eqn. (3.5) into the boundary condition in Eqn. (3.4), Helmert's gravity anomaly $\Delta g^{\mathrm{H}}\big(r_t(\Omega)\big)$ becomes (Vaníček et al., 1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$\Delta g^{\mathrm{H}}\big(r_t(\Omega)\big) = \delta g^{\mathrm{H}}\big(r_t(\Omega)\big) + \gamma\big(r_t(\Omega)\big) - \gamma_o(\phi) - \frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\bigg|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega)\left[1 + \frac{\Delta g^{\mathrm{SB}}\big(r_t(\Omega)\big)}{\gamma_o(\phi)}\right] -$$

$$- \frac{1}{\gamma\big(H^{\mathrm{N}}(\Omega)\big)} \frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\bigg|_{r=R+H^{\mathrm{O}}(\Omega)} \Big[\delta V^{t}\big(r_t(\Omega)\big) + \delta V^{a}\big(r_t(\Omega)\big)\Big] + \varepsilon_{\delta g}\big(r_t(\Omega)\big) =$$

$$= g\big(r_t(\Omega)\big) - \gamma\big(r_t(\Omega)\big) + \frac{\partial \delta V^{t}(r,\Omega)}{\partial\, r}\bigg|_{r=R+H^{\mathrm{O}}(\Omega)} + \frac{\partial \delta V^{a}\big(r_t(\Omega)\big)}{\partial\, r}\bigg|_{r=R+H^{\mathrm{O}}(\Omega)} +$$

$$+\gamma\left(r_t(\Omega)\right)-\gamma_o(\phi)-\left.\frac{\partial\gamma(r,\Omega)}{\partial\,\mathrm{n}}\right|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega)\left[1+\frac{\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)}{\gamma_o(\phi)}\right]-$$
$$-\frac{1}{\gamma\left(H^{\mathrm{N}}(\Omega)\right)}\left.\frac{\partial\gamma(r,\Omega)}{\partial\,\mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)}\left[\delta V^t\left(r_t(\Omega)\right)+\delta V^a\left(r_t(\Omega)\right)\right]+\varepsilon_{\delta g}\left(r_t(\Omega)\right). \tag{3.7}$$

Introducing the free-air gravity anomaly (Heiskanen and Moritz, 1967)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$
$$\Delta g^{\mathrm{FA}}\left(r_t(\Omega)\right)=g\left(r_t(\Omega)\right)-\gamma\left(H^{\mathrm{O}}(\Omega)\right)=g\left(r_t(\Omega)\right)-\gamma_o(\phi)-\left.\frac{\partial\gamma(r,\Omega)}{\partial\,\mathrm{n}}\right|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega), \tag{3.8}$$

and applying the spherical approximation from Eqn. (2.15), the boundary condition in Eqn. (3.7) can subsequently be written in the form (Vaníček et al., 1999)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$
$$\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)=\Delta g^{\mathrm{FA}}\left(r_t(\Omega)\right)+\left.\frac{\partial\delta V^t(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}+\left.\frac{\partial\delta V^a(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}+\varepsilon_{\delta g}\left(r_t(\Omega)\right)+$$
$$+\frac{2}{R}H^{\mathrm{O}}(\Omega)\,\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)+\frac{2}{r_t(\Omega)}\delta V^t\left(r_t(\Omega)\right)+\frac{2}{r_t(\Omega)}\delta V^a\left(r_t(\Omega)\right)-\varepsilon_{\mathrm{n}}\left(r_t(\Omega)\right). \tag{3.9}$$

The second and third term on the right-hand side of Eqn. (3.9) are the direct topographical and atmospheric effects on gravitational attraction. The fifth term stands for the „*geoid-quasigeoid correction to the boundary-value problem*“, and the sixth and seventh terms represent the secondary indirect topographical and atmospheric effects on gravitational attraction.

Helmert's gravity anomaly can be also formulated as a function of the Bouguer gravity anomaly. The complete Bouguer gravity anomaly $\Delta g^{\mathrm{CB}}\left(r_t(\Omega)\right)$ is defined by the following formula (Heiskanen and Moritz, 1967)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$
$$\Delta g^{\mathrm{CB}}\left(r_t(\Omega)\right)=\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)+\delta g^{tc}\left(r_t(\Omega)\right)=g\left(r_t(\Omega)\right)-\gamma\left(H^{\mathrm{O}}(\Omega)\right)-2\pi\,G\rho_o H^{\mathrm{O}}(\Omega)+$$
$$+\delta g^{tc}\left(r_t(\Omega)\right)=\Delta g^{\mathrm{FA}}\left(r_t(\Omega)\right)-2\pi\,G\rho_o H^{\mathrm{O}}(\Omega)+\delta g^{tc}\left(r_t(\Omega)\right), \tag{3.10}$$

where $\delta g^{tc}\left(r_t(\Omega)\right)$ is the „*gravimetric terrain correction*“ (Vaníček et al., 1999), i.e., the correction for gravitational attraction of topography taken relative to the height of the evaluation point (r,Ω). The curvature effect $-8\pi\, G\rho_o \left[H^{\mathrm{O}}(\Omega)\right]^2/R$ is usually not considered in the definition of the complete Bouguer gravity anomaly (Vaníček and Krakiwsky, 1986).

Substituting Eqn. (3.10) into Eqn. (3.9), the relation between the Helmert gravity anomaly $\Delta g^{\mathrm{H}}(r_t(\Omega))$ and the complete Bouguer gravity anomaly $\Delta g^{\mathrm{CB}}(r_t(\Omega))$ is given by Vaníček et al. (1999)

$$\begin{aligned}
&\forall\, \Omega \in \Omega_{\mathrm{O}}: \\
&\Delta g^{\mathrm{H}}(r_t(\Omega)) = \Delta g^{\mathrm{CB}}(r_t(\Omega)) + 2\pi\, G\rho_o H^{\mathrm{O}}(\Omega) - \delta g^{tc}(r_t(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)) - \varepsilon_{\mathrm{n}}(r_t(\Omega)) + \\
&+ \left.\frac{\partial \delta V^t(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^a(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \\
&+ \frac{2}{R} H^{\mathrm{O}}(\Omega)\,\Delta g^{\mathrm{SB}}(r_t(\Omega)) + \frac{2}{r_t(\Omega)}\,\delta V^t(r_t(\Omega)) + \frac{2}{r_t(\Omega)}\,\delta V^a(r_t(\Omega)).
\end{aligned} \tag{3.11}$$

4. Effect of topographical masses on gravitational attraction

To evaluate the Helmert gravity anomaly $\Delta g^{\mathrm{H}}(r_t(\Omega))$ on the Earth's surface according to Eqn. (3.9), the topographical effect on gravitational attraction has to be computed. The topographical effect on gravitational attraction, which is reckoned on the Earth's surface, is represented by the direct and secondary indirect topographical effects (Martinec, 1993; Martinec and Vaníček, 1994a, b; Martinec et al., 1995 and 1996; Vaníček et al., 1995a and 1999; Novák et al., 2001; Huang et al., 2001).

4.1 Residual gravitational potential of topographical masses

The „*residual gravitational potential of topographical masses*“ $\delta V^t(r_t(\Omega))$ is defined as a difference of the gravitational potential $V^t(r,\Omega)$ of topographical masses and gravitational potential $V^{ct}(r,\Omega)$ of topographical masses condensed according to the Helmert second condensation method directly onto the geoid (Martinec et al., 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^+ : \delta V^t(r,\Omega) = V^t(r,\Omega) - V^{ct}(r,\Omega). \tag{4.1}$$

The „*gravitational potential of topographical masses*“ $V^t(r,\Omega)$ is given by the Newton volume integral (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$
$$V^{t}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=r_g(\Omega')}^{r_g(\Omega')+H^{\mathrm{O}}(\Omega')} \rho(r',\Omega')\, l^{-1}[r,\psi(\Omega,\Omega'),r']\, r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega', \tag{4.2}$$

where $\rho(r,\Omega)$ is the actual density of topographical masses (i.e., masses between the geoid and the Earth's surface). The spatial distance $l[r,\psi(\Omega,\Omega'),r']$ between two points with the geocentric positions (r,Ω) and (r',Ω') reads

$$\forall\ \Omega,\Omega' \in \Omega_{\mathrm{O}},\ r,r' \in \Re^{+}:\ l[r,\psi(\Omega,\Omega'),r'] = \sqrt{r^{2} + r'^{2} - 2r\ r'\cos\psi(\Omega,\Omega')}, \tag{4.3}$$

and the spherical distance $\psi(\Omega,\Omega')$, $\psi \in \langle\ 0,\pi\ \rangle$, is given by the law of cosines

$$\forall\ \Omega,\Omega' \in \Omega_{\mathrm{O}}:\ \cos\psi(\Omega,\Omega') = \sin\phi'\sin\phi + \cos\phi'\cos\phi\cos(\lambda'-\lambda). \tag{4.4}$$

The „*gravitational potential of condensed topographical masses*“ $V^{ct}(r,\Omega)$ can be computed by the Newton surface integral (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{ct}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \sigma(\Omega')\, l^{-1}[r,\psi(\Omega,\Omega'),r_g(\Omega')]\, r_g^{2}(\Omega')\,\mathrm{d}\Omega', \tag{4.5}$$

where $\sigma(\Omega)$ is the surface density of topographical masses condensed onto the geoid.

Approximating the geoid by the geocentric sphere of radius R, i.e., $\forall\ \Omega \in \Omega_{\mathrm{O}}:\ r_g(\Omega) \approx R$, and the actual density $\rho(r,\Omega)$ of topographical masses by the laterally varying topographical density $\rho(\Omega)$, see (Martinec, 1993),

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:\ \rho(\Omega) = \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} \rho(r,\Omega)\, r^{2}\ \mathrm{d}r, \tag{4.6}$$

the gravitational potential $V^{t}(r,\Omega)$ of topographical masses in Eqn. (4.2) takes the following form (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{t}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}[r,\psi(\Omega,\Omega'),r']\, r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'. \tag{4.7}$$

The surface density $\sigma(\Omega)$ of condensed topographical masses is according to the principle of mass-conservation of topographical masses (Wichiencharoen, 1982), i.e., the mass of the condensation layer is equal to the mass of actual topographical masses, in an integral representation (Martinec, 1993)

$$\forall\ \Omega \in \Omega_O : \iint_{\Omega} \rho(\Omega) \int_{r=R}^{R+H^O(\Omega)} r^2 \ \mathrm{d}r \ \mathrm{d}\Omega = R^2 \iint_{\Omega} \sigma(\Omega)\,\mathrm{d}\Omega\,. \tag{4.8}$$

According to Eqn. (4.8) the surface density $\sigma(\Omega)$ becomes (Martinec and Vaníček, 1994a)

$$\forall\ \Omega \in \Omega_O :$$

$$\sigma(\Omega) = \frac{\rho(\Omega)}{R^2} \int_{r=R}^{R+H^O(\Omega)} r^2 \ \mathrm{d}r = \rho(\Omega) H^O(\Omega) \left[1 + \frac{H^O(\Omega)}{R} + \frac{\left[H^O(\Omega)\right]^2}{3R^2} \right] = \rho(\Omega) \frac{r_t^3(\Omega) - R^3}{3R^2}\,. \tag{4.9}$$

The gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses, see Eqn. (4.5), is then (Martinec, 1993)

$$\forall\ \Omega \in \Omega_O,\ r \in \Re^+ : V^{ct}(r,\Omega) = G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega'\,. \tag{4.10}$$

Considering the gravitational potential $V^t(r,\Omega)$ of topographical masses in Eqn. (4.7) and the gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses in Eqn. (4.10), the residual gravitational potential $\delta V^t(r,\Omega)$ of topographical masses (in the spherical approximation of the geoid $\forall\ \Omega \in \Omega_O : r_g(\Omega) \approx R$) becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_O,\ r \in \Re^+ :$$

$$\delta V^t(r,\Omega) = G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^2 \ \mathrm{d}r' \ \mathrm{d}\Omega' -$$

$$- G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega'\,. \tag{4.11}$$

The radial integral of the reciprocal spatial distance $l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right]$ multiplied by r'^2 can be described by the analytical form (Gradshteyn and Ryzhik, 1980)

$$\forall\ \Omega \in \Omega_O :$$

$$\int_{r'=R}^{R+H^O(\Omega')} l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^2 \ \mathrm{d}r' = \left| \frac{1}{2}\left[r' + 3r\cos\psi(\Omega,\Omega')\right] l\left[r, \psi(\Omega,\Omega'), r'\right] + \right.$$

$$+\frac{r^2}{2}\left(3\cos^2\psi(\Omega,\Omega')-1\right)\ln\left| r'-r\cos\psi(\Omega,\Omega')+l\left[r,\psi(\Omega,\Omega'),r'\right]\right| \Bigg|_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} . \tag{4.12}$$

4.2 Direct topographical effect

The radial derivative of the residual gravitational potential $\delta V^t(r,\Omega)$ of topographical masses in Eqn. (4.1) referred to the Earth's surface defines the „*direct topographical effect on gravitational attraction*" (Martinec, 1993; Martinec and Vaníček, 1994a)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:\ \frac{\partial\,\delta V^t(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)}=\frac{\partial V^t(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)}-\frac{\partial V^{ct}(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} . \tag{4.13}$$

The „*gravitational attraction of the topographical masses*" is given by the radial derivative of the gravitational potential $V^t(r,\Omega)$ of topographical masses, see Eqn. (4.7), referred to the Earth's surface (Martinec and Vaníček, 1994a)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$

$$\frac{\partial V^t(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)}=G\iint_{\Omega'\in\Omega_{\mathrm{O}}}\rho(\Omega')\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')}\frac{\partial\, l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega' . \tag{4.14}$$

The radial derivative of the gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses, see Eqn. (4.10), which is also reckoned on the Earth's surface, represents the „*gravitational attraction of condensed topographical masses*" (Martinec and Vaníček, 1994a)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$

$$\frac{\partial V^{ct}(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)}=GR^2\iint_{\Omega'\in\Omega_{\mathrm{O}}}\sigma(\Omega')\frac{\partial\, l^{-1}\left[r,\psi(\Omega,\Omega'),R\right]}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)}\mathrm{d}\Omega' . \tag{4.15}$$

The radial integral of the radial derivative of the reciprocal spatial distance $\partial\, l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]/\partial r$ multiplied by r'^2 can be expressed analytically as follows (Martinec, 1993):

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$

$$\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')}\frac{\partial\, l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right]}{\partial r}\Bigg|_{r} r'^2\ \mathrm{d}r'=-\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')}\frac{r-r'\cos\psi(\Omega,\Omega')}{l^3\left[r,\psi(\Omega,\Omega'),r'\right]}r'^2\ \mathrm{d}r'=$$

$$= \left| \frac{r'^2 \cos\psi(\Omega,\Omega') + 3r^2 \cos\psi(\Omega,\Omega') + r\,r' - 6\,r\,r' \cos^2\psi(\Omega,\Omega')}{l[r,\psi(\Omega,\Omega'),r']} + \right.$$

$$\left. + r\left(3\cos^2\psi(\Omega,\Omega') - 1\right)\ln\left|\, r' - r\cos\psi(\Omega,\Omega') + l[r,\psi(\Omega,\Omega'),r'] \,\right| \; \right|_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} . \tag{4.16}$$

To remove the weak singularity of the Newton integral (Kellogg, 1929) in the computation point, the gravitational attraction of the spherical Bouguer shell with the lateral topographical density $\rho(\Omega)$ and thickness $H^{\mathrm{O}}(\Omega)$ equal to the orthometric height of the computation point can be subtracted from and added to gravitational attraction of topographical masses (Martinec, 1993; Martinec et al., 1995). The gravitational potential $V^{shell}(r,\Omega)$ of the spherical Bouguer shell with the topographical density $\rho(\Omega)$ and thickness $H^{\mathrm{O}}(\Omega)$ is equal to (Wichiencharoen, 1982)

$\forall\ \Omega \in \Omega_{\mathrm{O}}$:

$$V^{shell}(r,\Omega) = G\rho(\Omega) \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega)} l^{-1}[r,\psi(\Omega,\Omega'),r']\, r'^2 \;\mathrm{d}r'\,\mathrm{d}\Omega' =$$

$$= \begin{cases} 4\pi G\rho(\Omega)\dfrac{R^2}{r} H^{\mathrm{O}}(\Omega)\left[1 + \dfrac{H^{\mathrm{O}}(\Omega)}{R} + \dfrac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{3R^2}\right], & r \geq R + H^{\mathrm{O}}(\Omega), \\ 2\pi G\rho(\Omega)\left[R^2 + 2RH^{\mathrm{O}}(\Omega) + \left[H^{\mathrm{O}}(\Omega)\right]^2 - \dfrac{2}{3}\dfrac{R^3}{r} - \dfrac{1}{3}r^2\right], & R \leq r \leq R + H^{\mathrm{O}}(\Omega), \\ 4\pi G\rho(\Omega) H^{\mathrm{O}}(\Omega)\left[R + \dfrac{1}{2}H^{\mathrm{O}}(\Omega)\right], & r \leq R. \end{cases} \tag{4.17}$$

Moreover, gravitational attraction $\partial V^{shell}(r,\Omega)/\partial\, r$ of the spherical Bouguer shell is (Vaníček et al., 2001)

$\forall\ \Omega \in \Omega_{\mathrm{O}}$:

$$\left.\frac{\partial V^{shell}(r,\Omega)}{\partial\, r}\right|_r = G\rho(\Omega) \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega)} \left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial\, r}\right|_r r'^2 \;\mathrm{d}r'\,\mathrm{d}\Omega' =$$

$$= \begin{cases} -4\pi G\rho(\Omega)\dfrac{R^2}{r^2} H^{\mathrm{O}}(\Omega)\left[1 + \dfrac{H^{\mathrm{O}}(\Omega)}{R} + \dfrac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{3R^2}\right], & r \geq R + H^{\mathrm{O}}(\Omega), \\ -\dfrac{4}{3}\pi G\rho(\Omega)\,\dfrac{r^3 - R^3}{r^2}, & R \leq r \leq R + H^{\mathrm{O}}(\Omega), \\ 0, & r \leq R. \end{cases} \tag{4.18}$$

Subtracting and adding gravitational attraction of the spherical Bouguer shell, gravitational attraction of topographical masses in Eqn. (4.14) becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\left.\frac{\partial V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = -4\pi\, G\rho(\Omega)\frac{R^2}{r_t^2(\Omega)}H^{\mathrm{O}}(\Omega)\left[1+\frac{H^{\mathrm{O}}(\Omega)}{R}+\frac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{3R^2}\right]+$$

$$+G\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left[\rho(\Omega')\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')}\left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r' -\right.$$

$$\left.-\,\rho(\Omega)\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega)}\left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r'\right]\mathrm{d}\Omega'\,. \tag{4.19}$$

A similar procedure can be applied to remove the weak singularity of the Newton surface integral from gravitational attraction of condensed topographical masses, see Eqn. (4.15). The gravitational potential $V^{layer}(r,\Omega)$ of the spherical condensation layer with the surface density $\sigma(\Omega)$ is (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$V^{layer}(r,\Omega) = GR^2\sigma(\Omega)\iint_{\Omega'\in\Omega_{\mathrm{O}}} l^{-1}[r,\psi(\Omega,\Omega'),R]\,\mathrm{d}\Omega' = \begin{cases} 4\pi G\sigma(\Omega)\dfrac{R^2}{r}, & r > R\,, \\ 4\pi G\sigma(\Omega)R, & r \le R, \end{cases} \tag{4.20}$$

and gravitational attraction $\partial V^{layer}(r,\Omega)/\partial\, r$ of the spherical condensation layer is

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\left.\frac{\partial V^{layer}(r,\Omega)}{\partial r}\right|_{r} = GR^2\sigma(\Omega)\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),R]}{\partial r}\right|_{r}\mathrm{d}\Omega' = \begin{cases} -\,4\pi G\sigma(\Omega)\dfrac{R^2}{r^2}, & r > R\,, \\ 0, & r < R\,. \end{cases} \tag{4.21}$$

Subtracting and adding gravitational attraction of the spherical condensation layer to Eqn. (4.15), gravitational attraction of condensed topographical masses takes the following form (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\left.\frac{\partial V^{ct}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = -4\pi\, G\sigma(\Omega)\frac{R^2}{r_t^2(\Omega)}+$$

$$+ GR^2 \iint_{\Omega' \in \Omega_O} \left[\sigma(\Omega') - \sigma(\Omega) \right] \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), R \right]}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} \mathrm{d}\Omega' \,. \tag{4.22}$$

Comparing gravitational attraction of the spherical Bouguer shell referred to the Earth's surface, that is given by the first term on the right-hand side of Eqn. (4.19), with gravitational attraction of the spherical condensation layer referred to the Earth's surface, that is given by the first term on the right-hand side of Eqn. (4.22), they are equal. Substituting gravitational attraction of topographical masses in Eqn. (4.19) and gravitational attraction of condensed topographical masses in Eqn. (4.22) back into Eqn. (4.13), the direct topographical effect on gravitational attraction becomes (Martinec, 1993)

$$\forall\, \Omega \in \Omega_O :$$

$$\frac{\partial \delta V^t(r,\Omega)}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} = G \iint_{\Omega' \in \Omega_O} \left[\rho(\Omega') \int_{r'=R}^{R + H^O(\Omega')} \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r' \right]}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} r'^2 \, \mathrm{d}r' - \right.$$

$$\left. - \rho(\Omega) \int_{r'=R}^{R + H^O(\Omega)} \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r' \right]}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} r'^2 \, \mathrm{d}r' \right] \mathrm{d}\Omega' -$$

$$- GR^2 \iint_{\Omega' \in \Omega_O} \left[\sigma(\Omega') - \sigma(\Omega) \right] \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), R \right]}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} \mathrm{d}\Omega' \,. \tag{4.23}$$

Separating the laterally varying topographical density $\rho(\Omega)$ into the mean value $\rho_o = 2.67 \text{ g.cm}^{-3}$ and the laterally varying anomalous topographical density $\delta\rho(\Omega)$:

$$\forall\, \Omega \in \Omega_O : \rho(\Omega) = \rho_o + \delta\rho(\Omega), \tag{4.24}$$

and substituting them into Eqn. (4.9), the surface density $\sigma(\Omega)$ becomes (Martinec, 1993)

$$\forall\, \Omega \in \Omega_O : \sigma(\Omega) = \sigma_o + \delta\sigma(\Omega) = \left[\rho_o + \delta\rho(\Omega) \right] H^O(\Omega) \left[1 + \frac{H^O(\Omega)}{R} + \frac{\left[H^O(\Omega) \right]^2}{3R^2} \right]. \tag{4.25}$$

Applying the above mentioned decomposition of densities into gravitational attraction of topographical masses and of condensed topographical masses, the direct topographical effect on gravitational attraction finally takes the following form (Martinec, 1993)

$$\forall\, \Omega \in \Omega_O :$$

$$\frac{\partial \delta V^t(r,\Omega)}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} = G\rho_o \iint_{\Omega' \in \Omega_O} \int_{r' = R + H^O(\Omega)}^{R + H^O(\Omega')} \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r' \right]}{\partial\, r} \bigg|_{r = R + H^O(\Omega)} r'^2 \, \mathrm{d}r' \mathrm{d}\Omega' -$$

$$- G\rho_o \iint\limits_{\Omega' \in \Omega_O} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} \frac{\partial\, l^{-1}[r, \psi(\Omega, \Omega'), R]}{\partial r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega' +$$

$$+ G \iint\limits_{\Omega' \in \Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} \frac{\partial\, l^{-1}[r, \psi(\Omega, \Omega'), r']}{\partial r}\bigg|_{r=R+H^O(\Omega)} r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' -$$

$$- G \iint\limits_{\Omega' \in \Omega_O} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} \frac{\partial\, l^{-1}[r, \psi(\Omega, \Omega'), R]}{\partial r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega' . \tag{4.26}$$

The first term on the right-hand side of Eqn. (4.26) is the so-called „*spherical terrain correction*", and the second term stands for the „*spherical condensed terrain correction*" (Martinec and Vaníček, 1994a). The third and fourth terms represent together the contribution of the laterally varying topographical density to the direct topographical effect.

4.3 Secondary indirect topographical effect

The „*secondary indirect topographical effect on gravitational attraction*", which refers to the Earth's surface, is given by the following equation (Martinec and Vaníček, 1994b)

$$\forall\, \Omega \in \Omega_O : \frac{2}{r_t(\Omega)} \delta V^t(r_t(\Omega)) = \frac{2}{r_t(\Omega)} V^t(r_t(\Omega)) - \frac{2}{r_t(\Omega)} V^{ct}(r_t(\Omega)). \tag{4.27}$$

Dividing the laterally varying topographical density $\rho(\Omega)$ into the mean and laterally varying anomalous topographical density, see Eqn. (4.24), the gravitational potential $V^t(r_t(\Omega))$ of topographical masses given by Eqn. (4.7) takes the following form (Martinec, 1993)

$$\forall\, \Omega \in \Omega_O :$$

$$V^t(r_t(\Omega)) = 4\pi G\rho_o \frac{R^2}{r_t(\Omega)} H^O(\Omega) \left[1 + \frac{H^O(\Omega)}{R} + \frac{[H^O(\Omega)]^2}{3R^2} \right] +$$

$$+ G\rho_o \iint\limits_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega)}^{R+H^O(\Omega')} l^{-1}[r_t(\Omega), \psi(\Omega, \Omega'), r'] \, r'^2 \, \mathrm{d}r' \mathrm{d}\Omega' +$$

$$+ G \iint\limits_{\Omega' \in \Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} l^{-1}[r_t(\Omega), \psi(\Omega, \Omega'), r'] \, r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' , \tag{4.28}$$

where the first term stands for the gravitational potential of the spherical Bouguer shell referred to the Earth's surface, see Eqn. (4.17). Similarly, the gravitational potential $V^{ct}(r_t(\Omega))$ of condensed topographical masses, see Eqn. (4.10), becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$V^{ct}\left(r_t(\Omega)\right) = 4\pi\ G\sigma_o \frac{R^2}{r_t(\Omega)} + G\rho_o \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' +$$

$$+ G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' , \tag{4.29}$$

where the first term on the right-hand side represents the gravitational potential of the spherical condensation layer at the point above the geoid, see Eqn. (4.20).

Considering the gravitational potential $V^t\left(r_t(\Omega)\right)$ of topographical masses, see Eqn. (4.28), and the gravitational potential $V^{ct}(r, \Omega)$ of condensed topographical masses, see Eqn. (4.29), the secondary indirect topographical effect in Eqn. (4.27) can be written as follows (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\frac{2}{r_t(\Omega)} \delta V^t\left(r_t(\Omega)\right) = \frac{2}{r_t(\Omega)} G\rho_o \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega)}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), r'\right] r'^2\ \mathrm{d}r' \mathrm{d}\Omega' -$$

$$- \frac{2}{r_t(\Omega)} G\rho_o \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' +$$

$$+ \frac{2}{r_t(\Omega)} G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} \frac{\partial\, l^{-1}\left[r, \psi(\Omega, \Omega'), r'\right]}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega' -$$

$$- \frac{2}{r_t(\Omega)} G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' . \tag{4.30}$$

The gravitational potential of the spherical Bouguer shell and the gravitational potential of the spherical condensation layer are subtracted from Eqn. (4.30), because, if reckoned on the Earth's surface, they are equal.

5. Effect of atmospheric masses on gravitational attraction

Transforming the boundary-value problem, as formulated in the real space by Eqn. (2.16), into the Helmert space according to Eqn. (3.9), the effect of atmospheric masses on gravitational attraction is represented by the direct and secondary indirect atmospheric effects.

5.1 Residual gravitational potential of atmospheric masses

Similarly to the residual gravitational potential of topographical masses, the „*residual gravitational potential of atmospheric masses*“ $\delta V^{a}(r,\Omega)$ is given by the difference of the gravitational potential $V^{a}(r,\Omega)$ of atmospheric masses and the gravitational potential $V^{ca}(r,\Omega)$ of atmospheric masses condensed (according to the Helmert second condensation method) onto the geoid (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^{+} :\, \delta V^{a}(r,\Omega) = V^{a}(r,\Omega) - V^{ca}(r,\Omega)\,. \tag{5.1}$$

Under the spherical approximation of the geoid ($\forall\, \Omega \in \Omega_{\mathrm{O}} : r_g(\Omega) \approx R$), the „*gravitational potential of atmospheric masses*“ $V^{a}(r,\Omega)$ reads (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^{+} :$$
$$V^{a}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r,\Omega')\, l^{-1}[r,\psi(\Omega,\Omega'),r']\, r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'\,, \tag{5.2}$$

where $\rho^{a}(r,\Omega)$ is the actual atmospheric density and $r_{\lim}$ is the upper limit of the atmosphere where the atmospheric density becomes negligible (approximately 50 km above the sea level).

The „*gravitational potential of condensed atmospheric masses*“ $V^{ca}(r,\Omega)$ is (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^{+} :\, V^{ca}(r,\Omega) = GR^{2} \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \sigma^{a}(\Omega')\, l^{-1}[r,\psi(\Omega,\Omega'),R]\, \mathrm{d}\Omega'\,, \tag{5.3}$$

where $\sigma^{a}(\Omega)$ is the surface density of condensed atmospheric masses.

Using the laterally homogenous atmospheric density distribution

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \left\langle R + H^{\mathrm{O}}(\Omega),\, r_{\lim} \right\rangle :\, \rho^{a}(r) = \frac{1}{\Omega} \iint\limits_{\Omega} \rho^{a}(r,\Omega)\, \mathrm{d}\Omega\,, \tag{5.4}$$

the gravitational potential $V^{a}(r,\Omega)$ of atmospheric masses in Eqn. (5.2) can be written in the following form (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{a}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'\,. \tag{5.5}$$

According to the principle of the mass conservation, the atmospheric surface density $\sigma^{a}(\Omega)$ is defined by (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:\ \sigma^{a}(\Omega) = \frac{1}{R^{2}} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \rho^{a}(r)\, r^{2}\ \mathrm{d}r\,. \tag{5.6}$$

Substituting Eqn. (5.6) for the atmospheric surface density $\sigma^{a}(\Omega)$ into Eqn. (5.3), the gravitational potential $V^{ca}(r,\Omega)$ of condensed atmospheric masses takes the following form (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{ca}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r')\, r'^{2}\ \mathrm{d}r'\ l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega'\,. \tag{5.7}$$

Formally, the Earth's atmospheric masses can be split into the spherical shell bounded by the maximum geocentric radius of the topography and of the upper limit of the atmosphere, $\forall\ \Omega \in \Omega_{\mathrm{O}}: r = r_{\lim}$, and the roughness term bounded by the Earth's surface, $\forall\ \Omega \in \Omega_{\mathrm{O}}: r_t(\Omega) = R + H^{\mathrm{O}}(\Omega)$ and the maximum geocentric radius of the topography (Novák, 2000).

The gravitational potential $V^{a}(r,\Omega)$ of atmospheric masses can then be described by (Novák, 2000)

$$\begin{aligned}
&\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}: \\
&V^{a}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' + \\
&+ G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'\,.
\end{aligned} \tag{5.8}$$

Dividing also the integration domain of the atmospheric surface density $\sigma^{a}(\Omega)$ in Eqn. (5.6) as follows

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:\ \sigma^{a}(\Omega) = \frac{1}{R^{2}} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{R+H_{\lim}} \rho^{a}(r)\, r^{2}\ \mathrm{d}r + \frac{1}{R^{2}} \int_{r(\Omega)=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r)\, r^{2}\ \mathrm{d}r\,, \tag{5.9}$$

the gravitational potential $V^{ca}(r,\Omega)$ of condensed atmospheric masses becomes (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$
$$V^{ca}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\mathrm{lim}}} \rho^{a}(r')\, r'^{2}\ \mathrm{d}r'\ l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega' +$$
$$+ G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H_{\mathrm{lim}}}^{r_{\mathrm{lim}}} \rho^{a}(r')\, r'^{2}\ \mathrm{d}r'\ l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega'\,. \tag{5.10}$$

5.2 Direct atmospheric effect

The „*direct atmospheric effect on gravitational attraction*“ is defined as the radial derivative of the residual gravitational potential $\delta V^{a}(r_t(\Omega))$ of atmospheric masses referred to the Earth's surface (Vaníček et al., 1999; Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}: \left.\frac{\partial\, \delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = \left.\frac{\partial V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} - \left.\frac{\partial V^{ca}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}. \tag{5.11}$$

Since the gravitational attraction of the atmospheric spherical shell (bounded by the geocentric radii of the upper limit of topography and of the upper limit of the atmosphere) at the inner point $r < R + H_{\mathrm{lim}}$ is equal to zero (Mac Millan, 1930)

$$\forall\ r < R + H_{\mathrm{lim}}: G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H_{\mathrm{lim}}}^{r_{\mathrm{lim}}} \rho^{a}(r') \left.\frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right]}{\partial r}\right|_{r} r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' = 0\,, \tag{5.12}$$

the „*gravitational attraction of atmospheric masses*“ is given as the radial derivative of the gravitational potential of the atmospheric roughness term. The roughness term, which represents gravitational attraction of the atmosphere between the topography ($\forall\ \Omega \in \Omega_{\mathrm{O}} : r_t(\Omega) = R + H^{\mathrm{O}}(\Omega)$) and the upper limit of topography ($\forall\ \Omega \in \Omega_{\mathrm{O}}$: $r = R + H_{\mathrm{lim}}$), is given by (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\left.\frac{\partial V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\mathrm{lim}}} \rho^{a}(r') \left.\frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right]}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'\,. \tag{5.13}$$

Considering that gravitational attraction of the spherical condensation layer with the surface atmospheric density $\sigma^a(\Omega)$ at the outer point above the condensation layer $r > R$ is equal to a constant (Mac Millan, 1930)

$$\forall \; r > R:$$
$$G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), R]}{\partial\, r}\bigg|_r \mathrm{d}\Omega' = -4\pi G \frac{R^2}{r^2} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \,, \qquad (5.14)$$

„*gravitational attraction of condensed atmospheric masses*“ becomes

$$\forall \; \Omega \in \Omega_O:$$
$$\frac{\partial V^{ca}(r,\Omega)}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} = -4\pi G \frac{R^2}{r_t^2(\Omega)} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' +$$
$$+ G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \, \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), R]}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega' \,. \qquad (5.15)$$

Substituting gravitational attraction of atmospheric masses in Eqn. (5.13) and the gravitational attraction condensed atmospheric masses in Eqn. (5.15) back into Eqn. (5.11), the direct atmospheric effect on gravitational attraction takes the following form

$$\forall \; \Omega \in \Omega_O:$$
$$\frac{\partial\, \delta V^a(r,\Omega)}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} = G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\lim}} \rho^a(r') \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), r']}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' +$$
$$+ 4\pi G \frac{R^2}{r_t^2(\Omega)} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' -$$
$$- G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \, \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), R]}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega' \,. \qquad (5.16)$$

5.3 Secondary indirect atmospheric effect

The „*secondary indirect atmospheric effect on gravitational attraction*“, stipulated as being on the Earth’s surface, can be described by the following expression (Novák, 2000)

$$\forall \; \Omega \in \Omega_O: \frac{2}{r_t(\Omega)} \delta V^a(r_t(\Omega)) = \frac{2}{r_t(\Omega)} V^a(r_t(\Omega)) - \frac{2}{r_t(\Omega)} V^{ca}(r_t(\Omega)). \qquad (5.17)$$

If Eqns. (5.8) and (5.10) are considered, the residual gravitational potential $\delta V^{a}\left(r_t(\Omega)\right)$ in Eqn. (5.17) takes the following form

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\delta V^{a}\left(r_t(\Omega)\right) = 4\pi\, G \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, r'\, \mathrm{d}r' \;-4\pi\, G \frac{R^2}{r_t(\Omega)} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, r'^2\, \mathrm{d}r' \;+$$

$$+\, G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, l^{-1}\left[r_t(\Omega), \psi(\Omega,\Omega'), r'\right] r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega' \,-$$

$$-\, G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, r'^2\, \mathrm{d}r'\, l^{-1}\left[r_t(\Omega), \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega' . \tag{5.18}$$

6. Downward continuation of Helmert's gravity anomalies

To obtain gravity anomalies on the geoid, that are needed for solving the Stokes boundary-value problem, the downward continuation of gravity anomalies from the Earth's surface to the geoid in the Helmert space has to be evaluated. The downward continuation is evaluated by the Poisson integral equation, which is the inverse operation to Poisson's integral.

Since topographical and atmospheric masses are condensed onto the geoid, the Helmert space above the geoid (approximated by the geocentric sphere of radius R, i.e., $\forall\ \Omega \in \Omega_{\mathrm{O}} : R \approx r_g(\Omega)$) is harmonic. Helmert's gravity anomaly $\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ multiplied by the geocentric radius of the Earth's surface $r_t(\Omega)$ then satisfies the Laplace differential equation in the space everywhere above the geoid $\forall\ \Omega \in \Omega_{\mathrm{O}},\ r_t(\Omega) > R : \Delta\left[r_t(\Omega)\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)\right] = 0$ (Vaníček et al., 1996). „*Poisson's integral*" is given by the following formula (Kellogg, 1929)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r_t(\Omega) \geq R :$$

$$\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = \frac{R}{4\pi\, r_t(\Omega)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \mathrm{K}\left[r_t(\Omega), \psi(\Omega,\Omega'), R\right] \Delta g^{\mathrm{H}}(R,\Omega')\, \mathrm{d}\Omega' , \tag{6.1}$$

where $\mathrm{K}\left[r_t(\Omega), \psi(\Omega,\Omega'), R\right]$ is the „*spherical Poisson integral kernel*" (Sun and Vaníček, 1998)

$$\forall\ \Omega, \Omega' \in \Omega_{\mathrm{O}},\ r_t(\Omega) \geq R :$$

$$\mathrm{K}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right] = \sum_{n=2}^{\infty}(2n+1)\left[\frac{R}{r_t(\Omega)}\right]^{n+1}\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) =$$
$$= R\left[\frac{r_t^2(\Omega)-R^2}{l^3\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right]} - \frac{1}{r_t(\Omega)} - \frac{3R}{r_t^2(\Omega)}\cos\psi(\Omega,\Omega')\right]. \tag{6.2}$$

The discrete form of „*Poisson's integral equation*" which generic form is the Fredholm integral equation of the first kind, can be expressed as (Martinec, 1996; Huang, 2002)

$$\mathbf{\Delta g^H}\left(r_t(\Omega)\right) = \mathbf{K}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right]\ \mathbf{\Delta g^H}\left(R,\Omega'\right), \tag{6.3}$$

where $\mathbf{\Delta g^H}\left(r_t(\Omega)\right)$ is the vector of Helmert's gravity anomalies on the Earth's surface, $\mathbf{\Delta g^H}\left(R,\Omega'\right)$ is the vector of Helmert's gravity anomalies on the co-geoid (approximated again by the reference sphere), and $\mathbf{K}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right]$ is the matrix of values of the Poisson integral kernel multiplied by the factor $R / r_t(\Omega)$ and constant $1/4\pi$.

According to Jacobi's iteration approach (Ralston, 1965) for solution of a system of linear algebraic equations, the matrix $\mathbf{K}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right]$ can be expressed in the form

$$\mathbf{K}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right] = \mathbf{E} - \mathbf{B}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right], \tag{6.4}$$

where $\mathbf{E}$ is the unit matrix. Substituting Eqn. (6.4) into Eqn. (6.3), the following system of algebraic equations is obtained (Martinec, 1996)

$$\mathbf{\Delta g^H}\left(R,\Omega'\right) = \mathbf{\Delta g^H}\left(r_t(\Omega)\right) - \mathbf{B}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right]\ \mathbf{\Delta g^H}\left(R,\Omega'\right). \tag{6.5}$$

The system of Eqns. (6.5) may be solved iteratively starting with the vector $\mathbf{\Delta g^{FA}}\left(r_t(\Omega)\right)$ of free-air gravity anomalies on the Earth's surface (because of free-air gravity anomalies on the Earth's surface are similar to Helmert's gravity anomalies on the geoid)

$$\forall\ \Omega \equiv \Omega':\ \mathbf{\Delta g^H}\left(R,\Omega'\right)\Big|_0 = \mathbf{\Delta g^{FA}}\left(r_t(\Omega)\right). \tag{6.6}$$

The k-th stage of iteration ($k > 0$) $\mathbf{\Delta g^H}\left(R,\Omega'\right)\Big|_k$ is carried out according to equation (Martinec, 1996)

$$\mathbf{\Delta g^H}\left(R,\Omega'\right)\Big|_k = \mathbf{B}\left[r_t(\Omega),\psi(\Omega,\Omega'),R\right]\ \mathbf{\Delta g^H}\left(R,\Omega'\right)\Big|_{k-1}. \tag{6.7}$$

When the difference of results from two successive steps $\left| \mathbf{\Delta g^H}(R,\Omega')\Big|_k - \mathbf{\Delta g^H}(R,\Omega')\Big|_{k-1} \right|$ is smaller than some tolerance ε, the iterative process stops. The result of this operation yields the solution of Eqn. (6.3), see (Martinec, 1996),

$$\mathbf{\Delta g^H}(R,\Omega') = \mathbf{\Delta g^H}(r_t(\Omega)) + \sum_{k=1}^{\bar{k}} \mathbf{\Delta g^H}(R,\Omega')\Big|_k , \tag{6.8}$$

where $\bar{k}$ is the final number of iteration steps.

7. Reference field and spheroid in the Helmert space

To solve the Stokes boundary-value problem, the gravity anomalies over the entire boundary surface are required. To reduce the truncation errors, i.e., the far-zone contribution in the Stokes integration, the low and high-frequency parts of Helmert's gravity field are defined (Vaníček and Sjöberg, 1991).

The reference gravity field of degree $\bar{n}$ can be expressed by the „*reference gravity potential*" $W_{\mathrm{ref}}(r,\Omega)$ as (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \geq r_t(\Omega):\ W_{\mathrm{ref}}(r,\Omega) = \frac{GM}{r} - \sum_{n=2}^{\bar{n}} \left(\frac{a_o}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{W}_{\mathrm{n,m}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega), \tag{7.1}$$

where $\mathrm{W}_{\mathrm{n,m}}$ are the geopotential coefficients of the harmonic expansion of the Earth's gravity field, $\mathrm{Y}_{\mathrm{n,m}}$ are the normalized spherical functions of degree n and order m, a_o is an arbitrary parameter of length (usually the major semi-axis of the reference ellipsoid), and $\bar{n}$ stands for the maximum degree of retained harmonics. In the Helmert space the reference gravity potential $W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)$ reads

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega) = W_{\mathrm{ref}}(r,\Omega) - \delta V^{t}_{\mathrm{ref}}(r,\Omega) - \delta V^{a}_{\mathrm{ref}}(r,\Omega)\,, \tag{7.2}$$

where $\delta V^{t}_{\mathrm{ref}}(r,\Omega)$ and $\delta V^{a}_{\mathrm{ref}}(r,\Omega)$ are the reference residual gravitational potentials of the topographical and atmospheric masses.

7.1 Reference residual gravitational potential of topographical masses

According to Eqn. (4.1) the „*reference residual gravitational potential of topographical* *masses*“ $\delta V_{\mathrm{ref}}^{t}(r,\Omega)$ can be defined as the difference of the „*reference gravitational potential of topographical masses*“ $V_{\mathrm{ref}}^{t}(r,\Omega)$ (Vaníček et al., 1995)

$$\forall\ \Omega\in\Omega_{\mathrm{O}},\, r>R+H_{\mathrm{lim}}:$$

$$V_{\mathrm{ref}}^{t}(r,\Omega)\approx G\rho_o \iint_{\Omega'\in\Omega_{\mathrm{O}}} \int_{H'=0}^{H^{\mathrm{O}}(\Omega')} \frac{1}{r}\sum_{n=0}^{\bar{n}}\left(\frac{R+H'}{r}\right)^{n} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)(R+H')^2\,\mathrm{d}H'\,\mathrm{d}\Omega', \tag{7.3}$$

and the „*reference gravitational potential of condensed topographical masses*“ $V_{\mathrm{ref}}^{ct}(r,\Omega)$ (Novák, 2000)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}, r>R: V_{\mathrm{ref}}^{ct}(r,\Omega)\approx GR \iint_{\Omega'\in\Omega_{\mathrm{O}}} \sigma(\Omega')\sum_{n=0}^{\bar{n}}\left(\frac{R}{r}\right)^{n+1} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'. \tag{7.4}$$

For points $\forall\ \Omega\in\Omega_{\mathrm{O}}: r>R+H_{\mathrm{lim}}$ outside the Brillouin sphere (minimal geocentric sphere containing all the Earth’s mass), the reference gravitational potential $V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses in Eqn. (7.3) takes the following form (Vaníček et al., 1995)

$$\forall\ \Omega\in\Omega_{\mathrm{O}},\, r>R+H_{\mathrm{lim}}:$$

$$V_{\mathrm{ref}}^{t}(r,\Omega)=G\rho_o R^2\sum_{n=0}^{\bar{n}}\left(\frac{R}{r}\right)^{n+1}\frac{1}{n+3}\sum_{k=1}^{n+3}\binom{n+3}{k}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left[\frac{H^{\mathrm{O}}(\Omega')}{R}\right]^{k}\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'. \tag{7.5}$$

Differencing the reference gravitational potential $V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses, see Eqn. (7.5), and the reference gravitational potential $V_{\mathrm{ref}}^{ct}(r,\Omega)$ of condensed topographical masses, see Eqn. (7.4), the reference residual gravitational potential $\delta V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses becomes (Novák, 2000)

$$\forall\ \Omega\in\Omega_{\mathrm{O}},\, r>R+H_{\mathrm{lim}}:$$

$$\begin{aligned}\delta V_{\mathrm{ref}}^{t}(r,\Omega)\cong G\rho_o R^2\sum_{n=0}^{\bar{n}}\left(\frac{R}{r}\right)^{n+1}&\left\{\frac{1}{n+3}\sum_{k=1}^{n+3}\binom{n+3}{k}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left[\frac{H^{\mathrm{O}}(\Omega')}{R}\right]^{k}\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'-\right.\\ &\left.-\iint_{\Omega'\in\Omega_{\mathrm{O}}}\frac{H^{\mathrm{O}}(\Omega')}{R}\left[1+\frac{H^{\mathrm{O}}(\Omega')}{R}+\frac{\left[H^{\mathrm{O}}(\Omega')\right]^2}{3R^2}\right]\mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'\right\}.\end{aligned} \tag{7.6}$$

Since for $H^{\mathrm{O}}(\Omega') << R$ the summation over k converges very quickly (Vaníček et al., 1995), Eqn. (7.6) can be rewritten into the following form (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\ r > R + H_{\mathrm{lim}}:$$

$$\delta V_{\mathrm{ref}}^{t}(r,\Omega) \cong G\rho_o R^2 \sum_{n=1}^{\bar{n}} \frac{n}{2}\left(\frac{R}{r}\right)^{n+1} \left\{ \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{\left[H^{\mathrm{O}}(\Omega')\right]^2}{R^2} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' + \right.$$

$$\left. + \frac{n+3}{3} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{\left[H^{\mathrm{O}}(\Omega')\right]^3}{R^3} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \right\}. \tag{7.7}$$

Expressing the surface harmonics of the orthometric height as (Kellogg, 1929)

$$\iint_{\Omega' \in \Omega_{\mathrm{O}}} H^{\mathrm{O}}(\Omega')\, \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} \mathrm{H_{n,m}}(\Omega)\, \mathrm{Y_{n,m}}(\Omega), \tag{7.8}$$

the reference residual gravitational potential $\delta V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses becomes (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\ r > R + H_{\mathrm{lim}}(\Omega):$$

$$\delta V_{\mathrm{ref}}^{t}(\Omega) \cong 2\pi G\rho_o \sum_{n=1}^{\bar{n}} \frac{n}{2n+1}\left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H_{n,m}^2}(\Omega)\, \mathrm{Y_{n,m}}(\Omega) +$$

$$+ \frac{2\pi}{3R} G\rho_o \sum_{n=1}^{\bar{n}} \frac{n(n+3)}{2n+1}\left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H_{n,m}^3}(\Omega)\, \mathrm{Y_{n,m}}(\Omega). \tag{7.9}$$

7.2 Reference residual gravitational potential of atmospheric masses

The „*reference gravitational potential of atmospheric masses*“ $V_{\mathrm{ref}}^{a}(r,\Omega)$ can be described in the form (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\ r > r_{\mathrm{lim}}:$$

$$V_{\mathrm{ref}}^{a}(r,\Omega) = G \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\mathrm{lim}}} \rho^a(r')\, r'^{\,n+2}\, \mathrm{d}r'\, \mathrm{d}\Omega'. \tag{7.10}$$

To define the reference residual gravitational potential $\delta V_{\mathrm{ref}}^{a}(r,\Omega)$ of the atmospheric masses, the atmospheric density $\rho^a(r)$ given by Eqn. (5.4) can be replaced by the laterally symmetrical density model (Sjöberg, 1998; Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, r \in \left\langle R + H^{\mathrm{O}}(\Omega),\, r_{\lim} \right\rangle, \nu > 2 \wedge \nu \in \Im^{+} : \rho^{a}(r) = \rho_{o}^{a} \left[\frac{R}{R + H^{\mathrm{O}}(\Omega)} \right]^{\nu}, \tag{7.11}$$

where ρ_{o}^{a} is the atmospheric density at the sea level, and the positive integer constant $\nu \in \Im^{+}$ ($\Im^{+} = 1, 2, \ldots$) determines the atmospheric density distribution model.

If the integration over the geocentric radius r from the Earth's surface $r_t(\Omega)$ to the upper limit $r_{\lim}$ of the atmosphere is evaluated by using the atmospheric model density from Eqn. (7.11)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, \nu > 2 \wedge \nu \in \Im^{+}, n = 1, 2, \ldots, \bar{n} :$$

$$\int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \rho^{a}(r)\, r^{n+2}\, \mathrm{d}r = \rho_{o}^{a} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \left(\frac{R}{r} \right)^{\nu} r^{n+2}\, \mathrm{d}r\,, \tag{7.12}$$

the reference gravitational potential $V_{\mathrm{ref}}^{a}(r, \Omega)$ of the atmospheric masses can be written as (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, r > r_{\lim}, \nu > 2 \wedge \nu \in \Im^{+} :$$

$$V_{\mathrm{ref}}^{a}(r, \Omega) \cong G R^{\nu} \rho_{o}^{a} \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \left| \frac{r'^{\,n-\nu+3}}{n-\nu+3} \right|_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega, \Omega')\right) \mathrm{d}\Omega' \cong$$

$$\cong G R^{\nu} \rho_{o}^{a} \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{R^{n-\nu+3}}{n-\nu+3} \sum_{k=1}^{n-\nu+3} \binom{n-\nu+3}{k} \frac{(r_{\lim} - R)^{k} - \left[H^{\mathrm{O}}(\Omega')\right]^{k}}{R^{k}} \mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega, \Omega')\right) \mathrm{d}\Omega'\,. \tag{7.13}$$

Applying the binomial theorem to the evaluation of the surface atmospheric density $\sigma^{a}(\Omega)$, see (Novák, 2000),

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, \nu > 2 \wedge \nu \in \Im^{+} :$$

$$\sigma^{a}(\Omega) = \frac{\rho_{o}^{a}}{R^{2}} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \left(\frac{R}{r} \right)^{\nu} r^{2}\, \mathrm{d}r =$$

$$= \rho_{o}^{a} \frac{R^{3-\nu}}{3-\nu} \sum_{k=1}^{3-\nu} \binom{3-\nu}{k} \frac{(r_{\lim} - R)^{k} - \left[H^{\mathrm{O}}(\Omega)\right]^{k}}{R^{k}}\,, \tag{7.14}$$

the „*reference gravitational potential of condensed atmospheric masses*“ $V_{\mathrm{ref}}^{ca}(r,\Omega)$ takes the following form (Novák, 2000)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>r_{\lim},\, \nu>2\wedge\nu\in\Im^{+}:$$

$$V_{\mathrm{ref}}^{ca}(r,\Omega)\cong\frac{G\rho_o^a}{R^{1-\nu}}\sum_{n=0}^{\bar{n}}\left(\frac{R}{r}\right)^{n+1}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left|\frac{r'^{3-\nu}}{3-\nu}\right|_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}}\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'=$$

$$=\frac{G\rho_o^a}{R^{1-\nu}}\sum_{n=0}^{\bar{n}}\left(\frac{R}{r}\right)^{n+1}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\frac{R^{3-\nu}}{3-\nu}\sum_{k=1}^{3-\nu}\binom{3-\nu}{k}\frac{\left(r_{\lim}-R\right)^k-\left[H^{\mathrm{O}}(\Omega')\right]^k}{R^k}\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{d}\Omega'. \qquad (7.15)$$

The „*reference residual gravitational potential of atmospheric masses*“ $\delta V_{\mathrm{ref}}^{a}(r,\Omega)$ is then obtained as the difference of the reference gravitational potential $V_{\mathrm{ref}}^{a}(r,\Omega)$ of atmospheric masses, see Eqn. (7.13), and the reference gravitational potential $V_{\mathrm{ref}}^{ca}(r,\Omega)$ of condensed atmospheric masses, see Eqn. (7.15) and (Novák, 2000),

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>r_{\lim},\, \nu>2\wedge\nu\in\Im^{+}:$$

$$\delta V_{\mathrm{ref}}^{a}(r,\Omega)\cong-2\pi\,G\rho_o^a\sum_{n=1}^{\bar{n}}\frac{n}{2n+1}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n}\mathrm{H}_{\mathrm{n,m}}^{2}(\Omega)\,\mathrm{Y}_{\mathrm{n,m}}(\Omega)-$$

$$-2\pi\,G\frac{\rho_o^a}{3R}\sum_{n=1}^{\bar{n}}\frac{n(n-2\nu+3)}{2n+1}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n}\mathrm{H}_{\mathrm{n,m}}^{3}(\Omega)\,\mathrm{Y}_{\mathrm{n,m}}(\Omega). \qquad (7.16)$$

7.3 Reference gravity potential in the Helmert space

The reference gravity potential $W_{\mathrm{ref}}^{\mathrm{H}}(r,\Omega)$ in the Helmert space in Eqn. (7.2) can be expressed by the following formula (Vaníček et al., 1995)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>R:W_{\mathrm{ref}}^{\mathrm{H}}(r,\Omega)=\frac{GM}{r}-\sum_{n=2}^{\bar{n}}\left(\frac{a_o}{r}\right)^{n+1}\sum_{m=-n}^{n}\mathrm{W}_{\mathrm{n,m}}^{\mathrm{H}}\;\mathrm{Y}_{\mathrm{n,m}}(\Omega). \qquad (7.17)$$

Since the summation in the expansion of Helmert's reference gravity potential $W_{\mathrm{ref}}^{\mathrm{H}}(r,\Omega)$ is finite, i.e., the validity of this expression is not limited to the outside of the Brillouin sphere (in the case of the topographical effect) and of the upper limit of atmosphere (in the case of the atmospheric effect), the series in Eqn. (7.17) can be used at the geoid to evaluate the reference gravity field in the Helmert space (Vaníček et al., 1995). If this surface is unknown, the appropriate approximation of the geoid by the reference ellipsoid ($\forall\,\Omega\in\Omega_{\mathrm{O}}$: $r_g(\Omega)\approx r_o(\Omega)$) can be applied (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : r_g(\Omega) \approx r_o(\Omega) \cong a\left(1 - f \sin^2 \varphi\right). \tag{7.18}$$

Substituting the term (Vaníček et al., 1995)

$$\forall\ \ \Omega \in \Omega_{\mathrm{O}},\, n = 1, 2, \ldots, \bar{n} : \left[\frac{a_o}{r_g(\Omega)}\right]^{n+1} = 1 + (n+1) f \sin^2 \varphi - \ldots, \tag{7.19}$$

into Eqn. (7.17), „*Helmert's reference gravity potential*" in the ellipsoidal approximation takes the following form (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\, r > R : W^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) \approx \frac{GM}{r_o(\Omega)} - \sum_{n=2}^{\bar{n}} \left[1 + (n+1) f \sin^2 \varphi\right] \sum_{m=-n}^{n} \mathrm{W}^{\mathrm{H}}_{\mathrm{n,m}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{7.20}$$

7.4 Reference gravity anomaly and reference spheroid in the Helmert space

According to the boundary condition (Heiskanen and Moritz, 1967), „*Helmert's reference gravity anomaly*" can be expressed as follows

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \Delta g^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) \approx -\left.\frac{\partial T^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)}{\partial\, r}\right|_{r=r_o(\Omega)} + \frac{2}{R} T^{\mathrm{H}}_{\mathrm{ref}}\left(r_o(\Omega)\right), \tag{7.21}$$

where $T^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) \cong T^{\mathrm{H}}_{\mathrm{ref}}\left(r_o(\Omega)\right) = W^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right) - U_o(\phi)$ is „*Helmert's reference disturbing potential*". The „*reference spheroid*" is given by the reference co-geoidal heights $N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$. Applying Bruns's spherical formula (Bruns, 1878) to Helmert's reference disturbing potential $T^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right)$, the reference co-geoidal height $N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$ can be expressed by the equation

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{\mathrm{ref}}(\Omega) = \frac{T^{\mathrm{H}}_{\mathrm{ref}}\left(r_g(\Omega)\right)}{\gamma_o(\phi)}. \tag{7.22}$$

8. Stokes's boundary-value problem in the Helmert space

The equipotential boundary surface in the Helmert space, which is given by co-geoidal heights $N^{\mathrm{H}}(\Omega)$, can be evaluated from Helmert's gravity anomalies $\Delta g^{\mathrm{H}}(R,\Omega)$ referred to the reference sphere of radius R by applying the Stokes integral formula (Stokes, 1849) and the Bruns spherical formula (Bruns, 1878) into the following equation (Heiskanen and Moritz, 1967)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}(\Omega) = \frac{R}{4\pi \gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(R, \Omega')\, \mathrm{S}(\psi(\Omega, \Omega'))\, \mathrm{d}\Omega' . \tag{8.1}$$

The homogenous spherical Stokes function $\mathrm{S}(\psi(\Omega, \Omega'))$, see (Stokes, 1849), can be described in the following spectral and spatial form (Heiskanen and Moritz, 1967)
$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}}$:

$$\mathrm{S}(\psi(\Omega, \Omega')) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos \psi(\Omega, \Omega')) = 1 + \operatorname{cosec} \frac{\psi(\Omega, \Omega')}{2} - 6 \sin \frac{\psi(\Omega, \Omega')}{2} -$$
$$- 5 \cos \psi(\Omega, \Omega') - 3 \cos \psi(\Omega, \Omega') \ln\left(\sin \frac{\psi(\Omega, \Omega')}{2} + \sin^2 \frac{\psi(\Omega, \Omega')}{2} \right). \tag{8.2}$$

To evaluate the co-geoidal height $N^{\mathrm{H}}(\Omega)$ by a surface integration according to the Stokes integral in Eqn. (8.1), the gravity anomalies $\Delta g^{\mathrm{H}}(R, \Omega)$ have to be known over the entire Earth.

8.1 Spheroidal Stokes's function

In practice, the gravity anomalies over the entire Earth are not available. For this reason Vaníček and Kleusberg (1987) introduced the idea to separate the summation over n in the Stokes function in Eqn. (8.2) into low and high-degree parts:

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}} : \mathrm{S}(\psi(\Omega, \Omega')) = \sum_{n=2}^{\bar{n}} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos \psi(\Omega, \Omega')) + \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos \psi(\Omega, \Omega')). \tag{8.3}$$

The second term on the right-hand side of Eqn. (8.3) represents the „*spheroidal Stokes function*“ $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega, \Omega'))$, see (Vaníček and Kleusberg, 1987; Vaníček and Featherstone, 1998),

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}} : \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega, \Omega')) = \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos \psi(\Omega, \Omega')). \tag{8.4}$$

Substituting the decomposition of the Stokes spherical function $\mathrm{S}(\psi(\Omega, \Omega'))$ into Eqn. (8.1), the co-geoid can be split into the low and high-frequency part (Martinec, 1993)
$\forall\, \Omega \in \Omega_{\mathrm{O}}$:

$$N^{\mathrm{H}}(\Omega) = N^{\mathrm{H}}_{\mathrm{ref}}(\Omega) + N^{\mathrm{H}}_{n>\bar{n}}(\Omega) = \frac{R}{4\pi\, \gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(R, \Omega') \sum_{n=2}^{\bar{n}} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos \psi(\Omega, \Omega'))\, \mathrm{d}\Omega' +$$
$$+ \frac{R}{4\pi\, \gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(R, \Omega') \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1} \mathrm{P}_{\mathrm{n}}(\cos \psi(\Omega, \Omega'))\, \mathrm{d}\Omega' . \tag{8.5}$$

The reference co-geoid (spheroid) of degree $\bar{n}$ is given by the reference co-geoidal heights

$N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$, and $N^{\mathrm{H}}_{n>\bar{n}}(\Omega)$ represents the high-frequency part of the co-geoid (Novák et al., 2001). According to this approach the reference spheroid determined from the satellite data is assumed (Vaníček and Kleusberg, 1987). The surface integration by the Stokes integral formula can be employed to compute the high-frequency part of the co-geoid only from terrestrial data.

8.2 Modified spheroidal Stokes's function

Values of the spheroidal Stokes function $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))$ diminish with the growing spherical distance $\psi(\Omega,\Omega')$. The integration domain Ω_{O} of Stokes's integral formula can be divided into the near-zone integration sub-domain $\Omega_{\psi_{\mathrm{o}}}$ (defined on the interval $\psi \in \langle 0,\psi_o \rangle$) and the far-zone integration sub-domain $\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}$ (on the interval $\psi \in \langle \psi_o, \pi \rangle$), see (Vaníček and Kleusberg, 1987):

$$\iint_{\Omega\in\Omega_{\mathrm{O}}} \mathrm{d}\Omega = \iint_{\Omega\in\Omega_{\psi_{\mathrm{o}}}} \mathrm{d}\Omega + \iint_{\Omega\in\Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \mathrm{d}\Omega \ . \tag{8.6}$$

The near-zone contribution to the high-frequency co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}(R,\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' , \tag{8.7}$$

and far-zone contribution to the high-frequency co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is given by (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}(R,\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' . \tag{8.8}$$

According to Molodensky et al. (1960), Vaníček and Kleusberg (1987) proposed to modify the spheroidal Stokes function $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))$ so that the far-zone contribution (truncation error) $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is minimal in the least-squares sense. The „*modified spheroidal Stokes's function*" $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))$ can be expressed as (Vaníček and Kleusberg, 1987)

$$\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)=\begin{cases}0, & \psi\in\langle 0,\psi_o\rangle,\\ \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi(\Omega,\Omega')\right), & \psi\in\langle\psi_o,\pi\rangle,\end{cases} \tag{8.9}$$

and then expanded into the series of Legendre polynomials

$$\forall\,\psi\in\langle 0,\,\pi\rangle:\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)=\sum_{n=\bar{n}+1}^{\infty}\frac{2n+1}{2}\mathrm{Q}_{\mathrm{n}}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right), \tag{8.10}$$

where $\mathrm{Q}_{\mathrm{n}}\left(\psi_o,\psi(\Omega,\Omega')\right)$ are „*truncation coefficients for the modified spheroidal Stokes function*“ $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)$, see (Molodensky et al., 1960). Multiplying Eqn. (8.10) by the Legendre polynomials $\mathrm{P}_{\mathrm{m}}\left(\cos\psi(\Omega,\Omega')\right)$,

$$\begin{aligned}&\forall\,\psi\in\langle 0,\,\pi\rangle:\\&\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{m}}\left(\cos\psi(\Omega,\Omega')\right)=\\&=\sum_{n=\bar{n}+1}^{\infty}\frac{2n+1}{2}\mathrm{Q}_{\mathrm{n}}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{m}}\left(\cos\psi(\Omega,\Omega')\right),\end{aligned} \tag{8.11}$$

and integrating the result over the interval $\psi\in\langle 0,\,\pi\rangle$, the following expression can be found

$$\begin{aligned}&\int_{\psi=0}^{\pi}\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{m}}\left(\cos\psi(\Omega,\Omega')\right)\sin\psi(\Omega,\Omega')\,\mathrm{d}\psi=\\&=\sum_{n=\bar{n}+1}^{\infty}\frac{2n+1}{2}\mathrm{Q}_{\mathrm{n}}\left(\psi_o,\psi(\Omega,\Omega')\right)\int_{\psi=0}^{\pi}\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{m}}\left(\cos\psi(\Omega,\Omega')\right)\sin\psi(\Omega,\Omega')\mathrm{d}\psi\,.\end{aligned} \tag{8.12}$$

Using the orthogonality property of the Legendre polynomials (Hobson, 1931)

$$\forall\,\psi\in\langle 0,\,\pi\rangle,\,n\neq m:\int_{\psi=0}^{\pi}\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{P}_{\mathrm{m}}\left(\cos\psi(\Omega,\Omega')\right)\sin\psi(\Omega,\Omega')\mathrm{d}\psi=0\,, \tag{8.13}$$

$$\forall\,\psi\in\langle 0,\,\pi\rangle,\,n=m:\int_{\psi=0}^{\pi}\left[\mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega,\Omega')\right)\right]^2\sin\psi(\Omega,\Omega')\mathrm{d}\psi=\frac{2}{2n+1}\,, \tag{8.14}$$

and substituting for $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)$ from Eqn. (8.9), the truncation coefficients $\mathrm{Q}_{\mathrm{n}}\left(\psi_o,\psi(\Omega,\Omega')\right)$ of the modified spheroidal Stokes function become (Molodensky et al., 1960)

$\forall\,\Omega\in\Omega_{\mathrm{O}}:$

$$\mathrm{Q_n}\left(\psi_o, \psi(\Omega,\Omega')\right) = \int_{\psi=0}^{\pi} \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \sin\psi(\Omega,\Omega') \, \mathrm{d}\psi \,. \tag{8.15}$$

8.3 Near-zone contribution to the high-frequency co-geoid

Helmert's gravity anomaly referred to the co-geoid can be divided into the low-frequency (reference) gravity anomaly $\Delta g^{\mathrm{H}}_{n<\bar{n}}(R,\Omega) \equiv \Delta g^{\mathrm{H}}_{\mathrm{ref}}(R,\Omega)$ and the high-frequency (residual) gravity anomaly $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$. The low-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{\mathrm{ref}}(R,\Omega)$ are evaluated according to Eqn. (7.21). The high-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$ are evaluated by subtracting the reference gravity anomalies $\Delta g^{\mathrm{H}}_{\mathrm{ref}}(R,\Omega)$ from Helmert's gravity anomalies downward continued onto the co-geoid according to Eqn. (6.5).

Taking Eqn. (8.6) into the account, the near-zone contribution of the high-frequency Helmert gravity anomalies to the co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_o}(\Omega)$ can be described by (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_\mathrm{o}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\psi_\mathrm{o}}} \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \,. \tag{8.16}$$

The Stokes integral is only weakly singular for the spherical distance $\psi = 0$ (Martinec, 1993). A classical method for treating a removable singularity consists of adding and subtracting the value of gravity anomaly at the singular point, see (Martinec, 1993),

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :$$

$$N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_\mathrm{o}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega'\in\Omega_{\psi_\mathrm{o}}} \left[\, \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega') - \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega) \,\right] \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{d}\Omega' +$$

$$+ \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega) \iint_{\Omega'\in\Omega_{\psi_\mathrm{o}}} \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \,. \tag{8.17}$$

8.4 Far-zone contribution to the high-frequency co-geoid

The far-zone contribution of high-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$ to the co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_\mathrm{o}}}(\Omega)$ is given by

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{d}\Omega' . \tag{8.18}$$

If gravity anomalies are not available over the entire Earth, the numerical computation can be done by using the following equation (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{2} \sum_{n=\bar{n}+1,\ldots} \mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega')) \sum_{m=-n}^{n} \mathrm{T}^{\mathrm{H}}_{\mathrm{n,m}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{8.19}$$

9. Primary indirect effect on the geoidal height

After evaluation of the Stokes boundary-value problem in the Helmert space, an equipotential surface in the Helmert space, i.e., the co-geoid, is obtained. To find the geoid in the real space, the primary indirect topographical and atmospheric effects on the geoidal height have to be evaluated (Vaníček and Martinec, 1994b). Helmert's disturbing gravity potential referred on the co-geoid (in the spherical approximation) reads

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : T^{\mathrm{H}}(R,\Omega) = T(R,\Omega) - \delta V^{t}(R,\Omega) - \delta V^{a}(R,\Omega). \tag{9.1}$$

Applying Bruns's spherical formula (Bruns, 1878) to the disturbing gravity potential $T(R,\Omega)$ and Helmert's disturbing gravity potential $T^{\mathrm{H}}(R,\Omega)$:

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N(\Omega) = \frac{T(R,\Omega)}{\gamma_o(\phi)}, \tag{9.2}$$

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}(\Omega) = \frac{T^{\mathrm{H}}(R,\Omega)}{\gamma_o(\phi)} = \frac{T(R,\Omega) - \delta V^{t}(R,\Omega) - \delta V^{a}(R,\Omega)}{\gamma_o(\phi)}, \tag{9.3}$$

the following relation between the geoidal height $N(\Omega)$ and the co-geoidal height $N^{\mathrm{H}}(\Omega)$ can be found (Martinec, 1993)

$\forall\, \Omega \in \Omega_{\mathrm{O}}$:

$$\delta N(\Omega) = N(\Omega) - N^{\mathrm{H}}(\Omega) = \frac{T(R,\Omega)}{\gamma_o(\phi)} - \frac{T^{\mathrm{H}}(R,\Omega)}{\gamma_o(\phi)} = \frac{\delta V^{t}(R,\Omega)}{\gamma_o(\phi)} + \frac{\delta V^{a}(R,\Omega)}{\gamma_o(\phi)} . \tag{9.4}$$

The first term on the right-hand side of Eqn. (9.4), i.e., $\delta V^{t}(R,\Omega)/\gamma_o(\phi)$, is the „*primary indirect topographical effect on the geoidal height*“, and the second term $\delta V^{a}(R,\Omega)/\gamma_o(\phi)$ stands for the „*primary indirect atmospheric effect on the geoidal height*“.

9.1 Primary indirect topographical effect

Considering the decomposition of the laterally varying topographical density $\rho(\Omega)$ into the mean and laterally varying anomalous topographical density, as described by Eqn. (4.24), and removing the weak singularity of Newton's integral, the gravitational potential $V^t(R,\Omega)$ of topographical masses (stipulated as being on the geoid) can be written as follows (Martinec, 1993)

$$\begin{aligned}
&\forall\, \Omega \in \Omega_{\mathrm{O}} : \\
&V^t(R,\Omega) = 4\pi G\rho_o H^{\mathrm{O}}(\Omega)\left[R + \frac{1}{2}H^{\mathrm{O}}(\Omega)\right] + \\
&+ G\rho_o \iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho_o \int_{r'=R+H^{\mathrm{O}}(\Omega)}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[R,\psi(\Omega,\Omega'),r'\right] r'^2 \,\mathrm{d}r'\,\mathrm{d}\Omega' + \\
&+ G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[R,\psi(\Omega,\Omega'),r'\right] r'^2 \,\mathrm{d}r'\,\mathrm{d}\Omega' ,
\end{aligned} \tag{9.5}$$

where the first term on the right-hand side is the gravitational potential of the spherical Bouguer shell referred to the geoid, see Eqn. (4.17).

Similarly, the gravitational potential $V^{ct}(R,\Omega)$ of condensed topographical masses referred on the geoid can be described as (Martinec, 1993)

$$\begin{aligned}
&\forall\, \Omega \in \Omega_{\mathrm{O}} : \\
&V^{ct}(R,\Omega) = 4\pi G\rho_o \frac{r_t^3(\Omega) - R^3}{3R} + G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho_o \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} l^{-1}\left(R,\psi(\Omega,\Omega'),R\right)\mathrm{d}\Omega' + \\
&+ G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3}\; l^{-1}\left(R,\psi(\Omega,\Omega'),R\right)\mathrm{d}\Omega' ,
\end{aligned} \tag{9.6}$$

where the first term on the right-hand side represents the gravitational potential of the spherical condensation layer.

Substituting the gravitational potential $V^t(R,\Omega)$ of topographical masses in Eqn. (9.5) and the gravitational potential $V^{ct}(R,\Omega)$ of condensed topographical masses in Eqn. (9.6) into the residual gravitational potential of topographical masses $\delta V^t(R,\Omega)$, the primary indirect topographical effect on the geoidal height takes the following form (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :$$

$$
\begin{aligned}
\frac{\delta V^{t}(R,\Omega)}{\gamma_o(\phi)} &= \frac{G}{\gamma_o(\phi)} 4\pi\rho_o \left[R\,H^{\mathrm{O}}(\Omega) + \frac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{2} - \frac{r_t^3(\Omega) - R^3}{3R} \right] + \\
&+ \frac{G}{\gamma_o(\phi)} \rho_o \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega)}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[R, \psi(\Omega,\Omega'), r'\right] r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' - \\
&- \frac{G}{\gamma_o(\phi)} \rho_o \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} \, l^{-1}\left(R, \psi(\Omega,\Omega'), R\right) \mathrm{d}\Omega' + \\
&+ \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[R, \psi(\Omega,\Omega'), r'\right] r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' - \\
&- \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} \, l^{-1}\left(R, \psi(\Omega,\Omega'), R\right) \mathrm{d}\Omega' . \qquad (9.7)
\end{aligned}
$$

9.2 Primary indirect atmospheric effect

The primary indirect atmospheric effect on the geoidal height can be described in the following basic form (Novák, 2000)

$$
\begin{aligned}
&\forall\, \Omega \in \Omega_{\mathrm{O}} : \\
&\frac{\delta V^{a}(R,\Omega)}{\gamma_o(\phi)} = \frac{V^{a}(R,\Omega)}{\gamma_o(\phi)} - \frac{V^{ca}(R,\Omega)}{\gamma_o(\phi)} = \\
&= \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r')\, l^{-1}\left[R, \psi(\Omega,\Omega'), r'\right] r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' - \\
&- \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r')\, r'^2 \, \mathrm{d}r' \, l^{-1}\left(R, \psi(\Omega,\Omega'), R\right) \mathrm{d}\Omega' . \qquad (9.8)
\end{aligned}
$$

10. Conclusions

To solve the geodetic boundary-value problem in the Helmert space, mean values of Helmert’s gravity anomalies are evaluated on the Earth’s surface. In the UNB approach, the mean values are considered for 5’x 5’ cells. It follows from Eqn. (3.9) that the mean values of Helmert’s gravity anomalies are functions of mean values of the free-air gravity anomalies, see Eqn. (3.8), ellipsoidal correction to the gravity disturbance, see Eqn. (2.10), ellipsoidal correction for the spherical approximation, see Eqn. (2.17), direct topographical and atmospheric effects, see Eqns. (4.26) and (5.16), secondary indirect topographical and atmospheric effects, see Eqns. (4.30) and (5.18), and geoid-quasigeoid correction to the boundary-value problem given by the fifth term on the right-hand side of Eqn. (3.9).

In the case of the ellipsoidal corrections to the gravity disturbance and for the spherical approximation, discrete values of Helmert's gravity anomalies computed from the geopotential model at the mid-points of corresponding cells can be considered as mean values, because they are smooth and change the geoid only by a few centimeters (as it can be seen from numerical results shown by Vaníček et al. (1999)). Similarly, the direct atmospheric effect and secondary indirect topographical effects can be evaluated as discrete values in the regular grid of 5'x 5' (as it follows from the numerical results in Novák (2000)). The mean values of the geoid-quasigeoid correction to the boundary-value problem are sufficiently (with an error < 10 μgal) computed for the mean orthometric heights of corresponding cells. The secondary indirect atmospheric effect is negligible (Novák, 2000).

Since the free-air gravity anomalies are not suitable for interpolation (Heiskanen and Moritz, 1967), their mean values are computed from the mean complete Bouguer gravity anomalies by subtracting average values of the gravimetric terrain correction and the gravitational attraction of the Bouguer plate with the mean topographical density and mean orthometric height (Janák and Vaníček, 2002). Mean values of the complete Bouguer gravity anomalies are given by averaging a certain number of discrete values, which are predicted on the regular grid from the complete Bouguer gravity anomalies at the observation points.

Mean values of the direct topographical effect have to be averaged from a sufficient number of discrete values. The number of discrete values needed for the precise evaluation of mean values of the direct topographical effect depends on the terrain roughness. The relation between the terrain roughness and the number of discrete values was investigated (at the Canadian Rocky Mountains) by Janák et al. (2001). In some areas, hundreds of discrete values (for one cell of size 5'x 5') must be computed to obtain sufficient accuracy.

To compute the effects of topographical masses, integration is carried out over the laterally varying topographical densities. When the geoid is to be determined with high accuracy (< 1 cm), the effect of lake water must also be considered. Numerical values for the lake Superior showed that the correction to the geoidal height due to the direct topographical effect on gravitational attraction lies within –1.1 and 1.3 cm, and the correction to the primary indirect topographical effect on the geoidal height is within –0.2 and 0.0 cm (Martinec et al., 1995). On the other hand, the effect of the laterally varying anomalous topographical density can cause changes of the geoid up to 10 cm (in Canada), see (Martinec, 1993; Huang et al., 2001; Huang, 2002), so that at least the laterally varying model of topographical density has to be considered.

Solving Dirichlet's boundary-value problem, the mean Helmert gravity anomalies are downward continued to the geoid by applying the discrete Poisson integral equation, see Eqn. (6.3). The Fredholm integral equation of the first kind (generic form of Poisson's integral equation) is known to be an unstable problem due to the fact that a comparatively smooth gravity anomaly on the Earth's surface is used to obtain a rougher gravity anomaly on the geoid. Solving the downward continuation for the 5'x 5' grid of Helmert's gravity anomalies, the ill effect of the instability might be partly reduced. Heck (1993) realized that space without topography is more suitable for the downward continuation than the Helmert space. For this reason, only the effect of topographical masses on the gravitational attraction can be subtracted from the gravity anomalies on the earth surface. The gravitation attraction of condensed topographical masses is then added to gravity anomalies downward continued onto the geoid.

The reference gravity anomalies and the spheroid in the Helmert space are evaluated from the satellite geopotential coefficients up to degree 20 according to Eqns. (7.21) and (7.22).

To solve the Stokes boundary-value problem in the modification for higher than the second-degree reference field (Vaníček and Sjöberg, 1991), the Stokes integration is employed for numerical integration over the 6° spherical cap, see Eqn. (8.17). The far-zone contribution is evaluated from the combined geopotential model. Usually EGM-96 up to degree 120 of the geopotential coefficients (Novák, 2000) is used according to Eqn. (8.19).

To obtain the geoid, the co-geoid (given by the discrete co-geoidal heights) is finally transformed into the real space by evaluation of discrete values of the primary indirect topographical and atmospheric effects. The primary indirect topographical effect can be computed by Eqn. (9.7) while the primary indirect atmospheric effect given by Eqn. (9.8) can be considered constant (equal to –0.6 cm), see (Sjöberg, 1998; Novák, 2000).

Evaluating the topographical and atmospheric effects on the gravitational potential and attraction, the integration domain is split into the near and far-zone integration sub-domains, where the near zone can be given by the 3° spherical cap, i.e., $\psi \in \left\langle 0, 3^{\circ} \right\rangle$. The near-zone contributions are then evaluated by numerical integration over the sufficiently dense grid of heights from the digital terrain model (especially numerical integration of the topographical effect and condensed topographical effect requires high density of elevation data (1″ or 3″) at the intermediate area surrounding the computation point. The spectral forms of Newton's integrals for evaluation of the far-zone contributions from the global elevation model were formulated by Novák (2000).

The actual accuracy of geoid determination is limited first of all by accuracy and spatial distribution of terrestrial gravity observations and orthometric heights. Other important attributes are the correctness of theoretical formulation and accuracy of numerical solutions.

Main factors limiting the theory of geoid determination by the UNB approach are the approximation of the actual topographical density by the laterally varying topographical density, resolution of gravity data for the downward continuation and primary indirect topographical effect, and spherical approximation of the geoid in the case of evaluation of topographical effects.

Computing the topographical and atmospheric effects on the gravitational potential and attraction, the geoid is approximated by the reference sphere of the geocentric radius $R \approx r_g(\Omega)$. This approximation yields a relative error 3×10^{-3} at most which then causes errors of 6 mm at most in the geoidal heights (Martinec, 1993). Since the density distribution of topographical masses between the geoid and Earth's surface is not available, the errors of geoid determination from the approximation of actual topographical density $\rho(r,\Omega)$ by the laterally varying topographical density $\rho(\Omega)$ are difficult to predict. Considering that the effect of laterally varying anomalous topographical density can cause changes of the geoid up to 10 cm (Martinec, 1993; Huang et al., 2001; Huang, 2002), the vertical variation of topographical density may cause changes of the geoid at most a few centimeters.

The surface density $\sigma(\Omega)$ of condensed topographical masses in the definition, see Eqn. (4.8), is chosen according to the principle of mass-conservation condensation (Wichiencharoen, 1982; Martinec, 1993), i.e., the mass of the condensation layer is equal to the mass of lateral topographical masses. Under this assumption, the disturbing gravity potential $T^{\mathrm{H}}(r,\Omega)$ in the Helmert space has no spherical harmonic of degree zero but it contains spherical harmonics of the first degree (because the so-called Hörmander's condition is not satisfied, $\lim_{r\to\infty} r^2 T^{\mathrm{H}}(r,\Omega) \neq 0$). It means, that the centre of the Earth's masses is shifted from the origin of the co-ordinate system. The magnitude of this shift represents 2 cm at most in each co-ordinate component and can precisely be computed (Martinec, 1993). The accuracy of numerical solution mainly depends on the interpolation of free-air gravity anomalies, evaluation of the near-zone contribution to the direct topographical effect, and accuracy of the Poisson integral equation in the case of 5'x 5' data.

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From Heights on a Deforming Earth to the United Nations Convention on the Law of the Sea

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Scientific geodetic research is about developing a greater understanding of the space-time geometry of the Earth's surface and the Earth's gravity field. Petr Vaníček’s research conducted over the last 35 years spans the full range geodesy from methodology to positioning, to the Earth's gravity field, to temporal variations. This classification of the functions of geodesy, which now seems only natural to us and it has been long adopted as a norm by the International Association of Geodesy, was not the standard nomenclature used prior to the introduction of Geodesy: The Concepts back in 1982.

Prior to The Concepts, some of us might still remember the use of archaic terms, such as "vertical" geodesy or its "horizontal" counterpart. Some of us might still remember that the main application of statistics in geodesy related only to the adjustments of heights or two-dimensional networks without any temporal or robust statistical considerations. Correlations among observable quantities and their consideration in the robust estimation of geodetic parameters were simply unheard of. Some of us might still remember when a Fourier analysis of noiseless one-dimensional equally-spaced data was not only a tool but the sole *tour de force* for the analyses of time series. Petr Vaníček's scientific geodetic research changed all that forever.

Some of my initial research conducted with Petr Vaníček addressed positioning problems relating to the availability of multiple geodetic reference systems and their different orientations. It involved the analysis of the impact of reference ellipsoid misalignment on geodetic azimuth and deflection of the vertical components (Carrera and Vaníček, 1982; Vaníček and Carrera 1985). But for the most part, our scientific research association over the years addressed height datum reference issues and temporal variations.

One of the most fascinating qualities of Petr's scientific research is his ability to develop innovative mathematical models that are capable to reveal valuable insights from what otherwise would have been considered by most other researchers as intractable or even hopeless data sets (Carrera and Vaníček, 1984a; Carrera and Vaníček, 1985a). Who among us would have ever given serious scientific consideration to a myriad of disconnected and widely scattered re-levelled segments of different classes surveyed under different standards by different parties over several decades as an effective tool to determine recent vertical crustal movements? Yet, Petr did and for the most part the results obtained and the insight gained from those analyses still form an important framework for geodynamic analyses conducted by means of either traditional geological techniques (Carrera and Vaníček, 1987) or by modern extra-terrestrial techniques in Canada today.

Our research on vertical movements spanned different scales from a local site (Carrera et al., 1984) to all ten Provinces of Canada from the Atlantic to the Pacific Oceans (Carrera and Vaníček, 1988, 1993a; Carrera et al., 1990; Vaníček and Carrera, 1989). Our map of vertical crustal movements has been widely used by geologists and geophysicists in Canada. A. Lambert and J. Shaw, for example, have used it in the Geological Survey of Canada in support of geodynamic analyses and environmental impact studies, respectively. The mathematical model used in the determination of the map accounted, for the first time to my knowledge, for correlations among differences of linear trends between pairs of tide gauges.

Petr Vaníček has looked at recent vertical crustal movements not only as "signal" from the perspective of geodynamics and environmental coastal studies but also as "noise", systematic noise to be precise, in geodetic levelling networks. From this latter perspective, we developed the mathematical model for an epoch-dependent height reference system (Carrera, 1984; Carrera and Vaníček, 1985b). We then determined the corrections that would be needed to define an epoch-dependent height reference system in Canada. We did it actually twice. The first time, we used the map compiled by Petr Vaníček and Deszo Nagy in 1980 (Vaníček et al., 1985). The second time, we used the map compiled with our colleague Michael Craymer in 1990 (Carrera et al., 1990).

Our research on recent vertical crustal movements led us into the path of tide gauge measurements and data analyses (Carrera and Vaníček, 1984b, 1985a). Again, from the modest expectations of the determination of operational tidal constituents in support of hydrographic charting applications to the definition of reliable long-term crustal movements and height reference surfaces there is a considerable leap in statistical analysis, random and systematic error detection and removal, outlier detection and elimination, and physical modelling of the instrument and the data itself.

Our analysis of the physics of tide gauges led us to the development of physical and mathematical models of the physical behaviour of float-type tide gauge response functions (Carrera and Vaníček, 1989). We also studied the statistical behaviour of digital pressure tide gauges (Carrera et al., 1996).

Our mathematical modelling of sea level series was conducted first in the time and later in the frequency domains. Some of the most interesting questions posed by Petr Vaníček in the context of this research were: Could the frequency response of sea level be decomposed linearly into separate forcing functions, such as the thermo-haline structure of sea water, atmospheric pressure, river discharge, and the wind speed vector into smooth functions that might be constructed and extrapolated to zero frequency? And if so, Could the zero-frequency response of sea level to these forcing functions be determined reliably from a *finite* data set along the coast where the realisation of the datum of levelling networks is conducted? It would have been impossible to respond these questions positively without a mathematical tool, such as the least-squares spectral analysis method developed by Petr in the late 60's, which could handle noisy and unequally spaced data (Vaníček et al., 1995).

Our analyses of multiple sea level trends and their differences were conducted, for the first time to my knowledge, in a simultaneous multivariate analysis. We computed the correlation and probability associated matrices of sea level trends in order to determine optimally the best topology for the differentiation of sea level trends. From this perspective, we could identify that the most accurate result may not necessarily be the most precise (Vaníček and Carrera, 1993).

Over the decade of the 90s, my association with Petr took a turning point of no return in my career. We agreed to create the Committee on Geodetic Aspects on the Law of the Sea (GALOS) under the official umbrella of the International Association of Geodesy in 1989. While Petr agreed to become Chairman, I had the privilege to serve as Secretary of our new Committee for the next ten years. GALOS was responsible for the organisation of two successful International Conferences held in Bali, Indonesia, in 1992 and 1996. Petr's contributions dealt with the problem of a maritime boundary involving two geodetic datums and the new technique of maximum curvature to determine a continuous tracing of the foot of the continental slope over a three-dimensional surface of the ocean floor. Many papers followed from both of us in this field over the next years up to the 1999 Conference held in Monaco.

Geodesy had been largely unknown and ignored in the field of the international law of the sea prior to the creation of GALOS. The United Nations Convention on the Law of the Sea (UNCLOS) calls upon geologists, geophysicists and hydrographers for expertise, notwithstanding the obvious need for geodesists. The International Association of Geodesy is not mentioned explicitly in UNCLOS as a legally "competent" organisation. Petr always regarded these facts with great irony and an enviable sense of humour. He never hesitated to use the opportunity to add after a successful international presentation that according to some twisted logic geodesists could be then regarded as dubious experts, which belonged to a potentially "incompetent" international scientific organisation.

The international scientific contributions of geodesy and geodesists have now more than proven their worth in the field of international maritime boundary delimitation. Geodesy now plays a fundamental role in the implementation of baselines and the outer limits of maritime spaces under national jurisdiction. The work conducted by Petr in GALOS and the later created Advisory Board on the Law of the Sea, which he chaired with the participation of experts from the Inter-governmental Oceanographic Organization (IOC) and the International Hydrographic Organization (IHO) helped position geodesy in the high place that it deserves as one of the basic sciences of maritime boundary delimitation.

The lessons taught by Petr in science and in life to me have transcended his own work. When I conducted my work as Chairman of the Editorial Committee of the Commission on the Limits of the Continental Shelf in charge of the preparation of our Scientific and Technical Guidelines (United Nations, 1999a, b), not only did I apply the knowledge that I had acquired from him over the years, I also often asked myself how would Petr organise the project and draft the document. I devoted myself to this task with clear thinking, originality and a hard work ethic, which I learned from him. I will be always indebted to Petr for these three gifts.

Petr and I still work together. We are responsible for the preparation of a geodetic module in a Training Course on Article 76 of UNCLOS being prepared by the Division of Ocean Affairs and the Law of the Sea of the United Nations over this summer of 2003.

Finally, I understand that in a document of this nature there is hardly room for anecdotes of a personal nature. We have many, too many to be told, from driving to cooking, from profound pain to great happiness throughout our lives. I consider it a fortune to have had Petr as my professor, my colleague and my friend over the last 23 years. If I had the opportunity to turn back the clock, I would be proud to make exactly the same choices all over again. He made me a successful professional beyond my most optimistic expectations. He instilled on me and all of his students scientific and ethical values to live by the rest of our lives. The challenge from Petr to all of us has always been to pass these values on to new generations of geodesists by example.

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LIST OF PUBLICATIONS

B	Book
TB	Textbook
LN	Lecture Notes
PR	Paper in a Refereed Journal
R	Research Paper
C	Critique, Reference Paper
IP	Invited Paper to a Meeting
NP	Paper Read at a Meeting
TH	Thesis
RT	Report (non-technical)
RW	Review Paper (technical)

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