OCEAN TIDE LOADING ON A SELF-GRAVITATING, COMPRESSIBLE, LAYERED, ANISOTROPIC, VISCOELASTIC AND ROTATING EARTH WITH SOLID INNER CORE AND FLUID OUTER CORE

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PREFACE

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PREFACE

This technical report is a reproduction of a thesis submitted in July 1988 to the Department of Surveying Engineering in partial fulfilment of the requirements for the degree of Doctor of Philosophy by Spiros Demitris Pagiatakis.

ABSTRACT

A novel ocean tide loading model is developed which allows the earth to be self-gravitating, compressible, layered, anisotropic, viscoelastic and rotating, with solid inner core and fluid outer core.

The deformation equations of the earth are developed, following the analytical mechanics approach. The standard-linear-solid-type rheology, as well as the grain-boundary relaxation model for the dissipation mechanism within the earth are adopted in this study. The thermodynamic state of the earth is accounted for, through its absolute temperature, Gibbs free activation energy, viscosity and Q profiles.

For the numerical integration of the equations of deformation, the following models are considered: **a)** PREM for the elasticity parameters of the earth, appropriately modified at tidal frequencies, using dispersion relations, **b)** SL8 model for the Q profile of the earth, **c)** viscosity profile with the following viscosities: 2.5×10^{22} poise for the lower mantle, 10^{22} poise for the transition zone and 10^{17} poise for the LVZ, **d)** SAMMIS ET. AL., [1977] model for the Gibbs free activation energy profile (for the transition zone and lower mantle), with an adiabatic temperature gradient of 0.3 K / km. The value of 125 kCal/Mole for the LVZ is considered, and **e)** STACEY's [1977] thermal model for the temperature profile of the earth.

Complex load numbers h'_n, k'_n and l'_n, are calculated and the results are the following:

a) The rotation of the earth has an effect on the load numbers that can be as much as 1.8%, 3.1% and 3.3% respectively, depending on the degree of expansion. There is a weak latitude dependence of the load numbers for $n \leq 4$; when latitude varies from 0° to ±45°, its effect is of the order of 0.4%.

b) The effect of anisotropy in the upper mantle can be as much as 1.9%, 2.3% and 2.5% respectively, depending on the degree of expansion.

c) At semidiurnal periods, the load numbers on a viscoelastic earth are about 0.2% larger than their corresponding values on an elastic earth. At fortnightly periods, viscoelastic h'_{100}, k'_{100} and l'_{100} are larger than their corresponding elastic values by 0.5%, 1.5% and 1.3%, respectively. For other values of n, the effect of viscosity is smaller.

Complex Green's functions are determined for displacements, gravity and tilt; they are given in the same form as those of FARRELL [1972], for easy implementation with existing software. The predictive power of the model is tested against accurately determined M_2 gravity tide residuals at 10, globally distributed, tidal stations. It is shown that the difference between observed residual gravity and predicted load gravity tide amplitudes is reduced for all tested stations by as much as 63%, when compared to predictions on an elastic, isotropic and nonrotating earth. There is also an improvement in the phases of the predicted load gravity tide.

All the novel features of this research are included in the new version of the LOADSDP software package [PAGIATAKIS, 1982]. LOADSDP software can be used to evaluate displacements, gravity perturbations and tilt at arbitrary locations on the surface of the earth with an accuracy better than 1%.

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LIST OF LATIN SYMBOLS

A, C, N, L, F	Elasticity parameters.
A, B, C, D, E	Coefficients.
C_o	Low frequency elastic wave velocity.
C_∞	High frequency elastic wave velocity.
$\mathbf{C}_{ijkl}$	Elasticity tensor.
$\mathbf{d}$	Displacement.
D	Dissipation function.
$\mathbf{e}_{ij}$	Strain tensor.
$\mathbf{E}_{ij}$	Deviatoric strain tensor.
E	Young modulus, energy.
$\boldsymbol{F}_b$	Buoyant force density.
$\boldsymbol{F}_g$	Buoyant force density due to compressibility.
$\boldsymbol{F}^T$	Tidal force.
$\mathbf{g}$	Gravity.
G	Gravitational constant.
G*	Gibbs free activation energy.
h'_n	First load number.
J	Compliance.
k	Compressibility, Boltzmann's constant.
k'_n	Second load number.
l'_n	Third load number.
L	Lagrangean density function.
m	Mass.
M	Stress relaxation function, mass of the earth.
p	Pressure.
P_{nm}	Legendre polynomials of degree n and order m.
q_i	Generalised coordinates.
Q	Quality factor.
Q_k	Generalised forces.
r	Radial distance.
r'	Normalised radial distance (by the radius of the earth).
R	Radius of the earth.

T	Kinetic energy, absolute temperature.
$\mathbf{T}_{ij}$	Deviatoric stress tensor.
T	Kinetic energy density.
u	Radial displacement.
U	Radial displacement.
v	Tangential displacement.
$\mathbf{v}$	Velocity.
V	Tangential displacement, volume.
V	Potential energy density.
V_s	Strain energy density.
W	Work.
y_i	Y-functions.
Y_{nm}	Scalar spherical harmonics.
z_i	Z-functions.

LIST OF GREEK SYMBOLS

α, β, γ, δ, ε	Unitless constants.
δ	Gravimetric factor, variation.
δ_{ij}	Kronecker symbol.
η	Viscosity
η	Bullen's dimensionless parameter.
θ	Colatitude
Θ	Cubic (volume) dilatation.
λ	Lamé parameter.
λ	Longitude.
μ	Lamé parameter (shear modulus).
ν	Poisson's ratio.
ρ	Density.
σ_{nm}	Mass density distribution of degree n and order m.
τ_{ij}	Stress tensor.
τ_{σ}	Relaxation time.
Φ^p	Primary loading potential
Φ^s	Secondary loading potential.
Φ^l_{nm}	Total loading potential of degree n and order m.
Φ_{nm}	Total gravitational potential.
Ψ	Potential.
ω	Tidal angular velocity.
Ω	Mean angular velocity of the earth.

LIST OF OTHER SYMBOLS

$\mathfrak{I}$	Real vector space.
$\wp$	Continuum.
$\mathfrak{R}$	Set of real numbers.
$\aleph$	Metric Space.

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I pay tribute to Prof. Dr. R. D. Small, of the Department of Mathematics and Statistics at UNB, who cheerfully devoted many hours to help me out with the solution of my equations.

This research would have never seen the light of day without the encouragement and moral support of my wife Roula, who not only patiently shared my problems and worries, but she helped in many different ways, more than she can actually perceive. Her devotion and innumerable sacrifices have taught me many invaluable things in life that can not be found in books. Her effort to improve the English structure of this dissertation is highly appreciated. Our daughter Christina, with her charming manners showed me wittily how important it is to escape from work, even for a little while. Our parents, although thousands of miles away, helped us in their own way and they deserve many special heartfelt thanks.

I dedicate this research to my wife Roula, whose love, patience and encouragement made me achieve a very important goal in my life, and to our daughter Christina for cheerfully inspiring our emotions.

1

INTRODUCTION

1.1 The Ocean Tide Loading Effect

As the ocean tide waters move about, they periodically load and unload the earth, causing displacement, tilt and gravity changes. These changes are the most pronounced directly underneath the load, i.e., on the ocean floor. However, the effect is of considerable magnitude even in the middle of the continents. As an example, the amplitude of the vertical displacement in the middle of North American continent can reach 1.7 cm† [PAGIATAKIS, 1982]. At coastal stations the effect becomes even more sizable.

One of the phenomena that can be observed by terrestrial means is the relative tilt‡ of the earth's surface due to direct attraction of the moon and the sun (body tide). These observations are significantly affected by the ocean tide loading. Sometimes, ocean tide loading tilt can be larger than the body tide tilt, especially at stations very close to the shore, an example being the University of New Brunswick Earth Tides Station♠ [PAGIATAKIS AND VANÍČEK, 1985].

† *This is the combined effect of the six leading constituents M_2, S_2, K_1, O_1, N_2 and P_1 using Schwiderski's ocean tide model [SCHWIDERSKI, 1978].*

‡ *Relative to the undisturbed surface of the earth.*

♠ *For instance, the amplitude of the M_2 tilt in N-S direction is 12.290 marcsec due to body tide and 20.130 marcsec due to ocean loading (calculated on an elastic earth).*

To increase the accuracy of satellite positioning and very long baseline interferometry (VLBI) techniques to subcentimetre level, a good knowledge of the loading deformations is indispensable. Furthermore, with the advent of the superconducting gravimeter, it is possible to study† the interior of the earth, using surface observations, provided that other effects such as the ocean loading can be eliminated.

Ocean tide loading effects have been studied by many researchers in the past. However, certain of the earth's attributes, such as, viscoelasticity, anisotropy, nonhomogeneity and rotation have been neglected for various reasons. In the present study, we take into account all the aforementioned effects using the approach of analytical mechanics. However, before we present the details of the context and the contributions of this research, an outline of the existing studies of the ocean loading is in order.

At the beginning of this century, a systematic difference between the values of the diminishing factor‡ estimated along N-S and E-W directions, was discovered. The analyses of the early observations of the tidal tilt showed that the diminishing factor in E-W direction was systematically larger.

Hecker in 1907 appears to be the first who tried to explain this disagreement. He attributed it to ***indirect effects*** due to the complex influence exerted by the mass of water moving in the nearby seas. D'Abbadie in France and Darwin in England in the late 1800's mentioned a possible influence of the ocean tides on the direction of the vertical [MELCHIOR, 1983]. DARWIN [1882] attempted to evaluate this effect.

Serious studies of the indirect effect, called here ***ocean tide loading***, were initiated by SLICHTER AND CAPUTO [1960], JOBERT [1960] and CAPUTO [1961]. They considered surface loading, in contrast to KAULA [1963], who considered internal mass loading. In all the above studies, simple earth density models were assumed.

† *In combination with other geophysical methods.*

‡ *The diminishing factor is defined as [MELCHIOR, 1983]:* $\gamma = 1 + k - h$, *where* h *and* k *are the first and the second Love numbers. Love numbers are defined in the the next section.*

LONGMAN [1962] initiated a study in order to evaluate the total gravitational effect of an arbitrary configuration of the ocean tides upon a gravimeter observation. He considered a symmetrical, gravitating and elastic earth, where the Lamé coefficients and density were some functions of depth. He used the equations of the free oscillations of the earth derived by PEKERIS AND JAROSCH [1958] and by ALTERMAN ET. AL., [1959]. LONGMAN [1963] calculated load deformation coefficients [see MUNK AND MACDONALD, 1960] up to degree n=40.

FARRELL [1972] considered a homogeneous self-gravitating sphere. He formulated the problem much the same way as LONGMAN [1962] did and estimated load numbers up to degree n = 10000 for different earth models. Farrell's work is very important; it has been the standard for various investigators [e.g. ZSCHAU, 1976; CHIARUTTINI AND LIVIERATOS, 1978; GOAD, 1979; MELCHIOR ET. AL., 1981; PAGIATAKIS, 1982].

BEAUMONT AND LAMBERT [1972] used the finite element method on an axisymmetric hemispherical medium to calculate the ocean tide loading effect. They also considered a lateral change in the crust structure, pointing out that the transition from oceanic to continental structure has no effect on tilts. Their results show that beyond 200 km from the point load the tilts are insensitive to crustal structure.

PERTSEV AND IVANOVA [1976] determined load numbers up to degree n = 70000 and calculated the effect of the world ocean tides on the trans United States tidal gravity profile.

ZSCHAU [1977, 1978] calculated phase shifts of the ocean tide loading effects due to low viscosity layers in the interior of the earth. He considered that the earth is a Maxwell fluid and he used the correspondence principle† to evaluate this response. He found that,

† *The Fourier (or Laplace) transform of the equations of motion (and their associated boundary conditions) of a linear viscoelastic body is of the same form as that of linear elastic body, the only difference being that the elasticity parameters are now complex quantities and they are functions of the transform variable. Therefore, any solution of the elastic equations offers a corresponding solution for a linear viscoelastic body through the inverse transform. This is known as the correspondence principle [see for instance, PELTIER, 1982].*

loading effects on a viscoelastic earth show phase shifts with respect to an elastic earth of the order of a few degrees.

MOLODENSKIJ AND KRAMER [1980] calculated derivatives of the Love numbers with respect to the elastic modulus of the real earth in order to estimate influences on earth tides by large-scale horizontal inhomogeneities in the mantle. They concluded that these influences can induce about 0.5% change in the δ factor. Thus, δ factor must be estimated with an accuracy better than 0.2% in order to carry some information on mantle inhomogeneities. Phases must be accurate to 0.1° to 0.2°.

SASAO AND WAHR [1981] modelled the response of an elastic, rotating, elliptical and oceanless earth with a fluid outer core to a given load distribution on its surface. They showed that the earth's response to diurnal surface loading must be affected by the free core nutation eigenmode†. WAHR AND SASAO [1981] gave a procedure to determine this resonant effect of the diurnal tides in the open ocean, on body tide.

1.2 Load Deformation Coefficients

Ocean loading effects can be evaluated by convolution of the ocean tidal amplitude with appropriate Green's functions‡. These Green's functions reflect the response of the earth to loading and depend, among other variables♠, on the properties of the earth. The properties of the earth enter into the Green's functions through some dimensionless

† *Consider an elliptic rotating earth with fluid core. When an external torque is applied to the earth, its rotation axis tips and the mantle pushes against the elliptical bulge of the core. Since the core rotates with the earth, it resists this deformation by an opposite torque. The result is a periodic, relative rotation between the core and the mantle. This is known as free core nutation (FCN) and has an eigenfrequency of about 1 + 1/460 cycles/day [WAHR, 1982]. Apparently, FCN affects only the diurnal tides. For theoretical discussions of the FCN see JEFFREYS AND VICENTE, [1957a, b]; MOLODENSKIJ, [1961]; TOOMRE, [1974]; SHEN AND MANSINHA [1976].*

‡ *See for instance PAGIATAKIS, [1982].*

♠ *They depend on the extent of load, as well as on time. For time dependent load numbers see PELTIER, [1982].*

quantities called ***load deformation coefficients***†, introduced by MUNK AND MACDONALD [1960]. A definition of these coefficients, using geometrical quantities of the deformation [see for example PAGIATAKIS, 1982], is as follows: Let u^{l}_{nm} denote the vertical displacement of the earth's surface due to the load, u^{a}_{nm} the displacement of the gravity equipotential surface induced by the attraction of the load masses and u^{i}_{nm} the vertical displacement of the surface of the earth due to the disturbed density field. Then

$$\left.\begin{aligned} h'_n &= u^{l}_{nm} / u^{a}_{nm}, \\ k'_n &= u^{i}_{nm} / u^{a}_{nm}, \\ l'_n &= v^{l}_{nm} / v^{a}_{nm}, \end{aligned}\right\} \tag{1.1}$$

where v^{l}_{nm} and v^{a}_{nm} are the horizontal components of the deformation. For this study, it is convenient to use physical quantities to define the load deformation coefficients. These alternative definitions‡ are as follows [e.g. WAHR, 1982]

$$\left.\begin{aligned} h'_n &= g\, u^{l}_{nm} / \Phi^{l}_{nm}, \\ k'_n &= \Phi_{nm} / \Phi^{l}_{nm} - 1, \\ l'_n &= g\, v^{l}_{nm} / \Phi^{l}_{nm}, \end{aligned}\right\} \tag{1.2}$$

where Φ^{l}_{nm} is the potential of the load, Φ_{nm} is the total gravitational potential (gravitational plus loading) and g is gravity. The load deformation coefficients do not depend on m and they are essentially independent of tidal frequency ♠.

† *Also known as load Love numbers or Love numbers.*

‡ *This can be done by using the definitions (1.1) and Bruns formula* $u^{a}_{nm} = \Phi^{l}_{nm} / g$ *from physical geodesy.*

♠ *We will see in the following chapters that this is only true for a purely elastic earth. However, for a viscoelastic earth the load deformation coefficients depend strongly on frequency, that is a viscoelastic medium is dispersive.*

1.3 Context of This Study

The primary objective of this research is to study the response of a more realistic earth, than used so far, to external forces and in particular to ocean tide loading. For the study of this response, many attributes of the earth must be taken into account, when developing the equations of deformation. One of the most important attributes in this study is the rheology of the earth. There is strong observational evidence that the rheology of the earth is not purely elastic and that tidal energy is dissipated in the earth. In order to understand the rheology of the earth, the concepts of stress and strain, as well as their relationship through a constitutive law are of primary importance. More specifically, of all the linear viscoelastic models, the standard linear viscoelastic model is examined in more detail, as this model is assumed to describe more realistically the response of the earth to tidal forces. In close relation to the standard linear viscoelastic model, the grain-boundary relaxation model is presented, which describes the dissipation mechanism within the earth. Moreover, the thermodynamical state of the earth modifies its rheology and it is taken into account. All the above concepts, along with the most recent models of the earth that are used to solve the equations of deformation of the earth are presented in Chapter 2.

In Chapter 3, the basic concepts of Lagrangean mechanics that are used to derive the equations of deformation are presented. We emphasise that our interest is the determination of the deformations of the earth; these deformations are considered as dependent variables of position and time, as opposed to the classical equations of motion that consider the position a dependent variable of time. Consequently, the Lagrangean equations of motion in this study acquire a different character; they are second order partial differential equations in the displacements (deformations) and thus, they are called "equations of deformation." Furthermore, we present the concepts of "tangent" and "cotangent bundle spaces", in which the equations of deformation are developed. Although, we do not take full advantage of this exposition in this study, we indicate its importance in future research on the deformations of the earth.

In Chapter 4, the equations of deformation are developed, following the Lagrangean mechanics approach, as described in Chapter 3. The equations are three partial differential equations of second order in the displacements.

In Chapter 5, the equations of deformation are transformed into 6 linear ordinary differential equations (ODEs) of first order. Subsequently, the equations are transformed into 12 ODEs of first order, by considering that the deformations on a viscoelastic earth are complex variables.

In Chapter 6, the equations of deformation are solved using the finite difference method of numerical integration. Load deformation coefficients are obtained, and subsequently, Green's functions are developed for the evaluation of the effect of the ocean tide loading on deformation, gravity and tilt observations.

Finally, in Chapter 7, the main conclusions of this research are presented, along with recommendations for future research.

1.4 Contributions of This Study

In this research, a number of important contributions are made, which can be summarized as follows:

1) The equations of deformation of the earth are developed from basic principles of physics, following the analytical mechanics approach (Lagrangean mechanics). The development of the equations is presented within the context of modern developments of mathematical physics.

2) We consider the earth to be layered, self-gravitating, compressible, anisotropic, rotating and viscoelastic under dynamic† surface loading. No other study has included all these features.

3) We assume that the earth is anisotropic (more specifically, laterally isotropic‡) in the uppermost layers. We indicate that a more general anisotropy can be easily incorporated into the equations of deformation, as long as an earth model is available for the equations' solution. At present, only the Parametric Reference Earth Model (PREM) by DZIEWONSKI AND ANDERSON [1981] allows for lateral (transverse) isotropy in the upper mantle.

4) It is known that the earth's rotation introduces problems in the expansion of the equations of motion into spherical harmonics [WAHR, 1981a]. We have found a partial solution to these problems, a simple method of expanding the equations into sectorial spherical harmonics (semidiurnal tides), when the properties of the earth possess rotational symmetry.

5) To allow for imperfections in elasticity, we consider that the earth has a standard-linear-solid-type rheology. We also accept that the dissipation mechanism is described by the grain-boundary relaxation model and we account for the thermodynamic state of the interior of the earth. We draw important new conclusions about the sensitivity of the load numbers to the viscosity, the quality factor Q and the thermodynamic profile of the earth. Finally, we stress the possibility of studying the interior of the earth from surface observations of the loading effects. No other study has included the above features.

6) We check the stability of the solution of the equations of motion using simple criteria and we indicate the prospects of a thorough stability investigation, using Lyapunov's stability theory.

† *By "dynamic" we mean that the frequency of the applied load is present in the equations as it is common practice in the equations of free oscillations of the earth. Some investigators have considered static deformations by simply rejecting the dynamic terms [e.g. LONGMAN, 1962; 1963]. This leads to a number of inconsistencies and "paradoxes" [see DAHLEN, 1974; CHINNERY, 1975].*

‡ *When a material possesses one axis of symmetry in the sense that all directions perpendicular to this axis are equivalent, it is said to be laterally (transversely) isotropic. The term "transverse isotropy" was introduced by Voigt in 1886 and it is being used as such, in seismology and crystal physics. We adopt the term "lateral isotropy" in this study, however.*

7) We evaluate new Green's functions for the load effects, which take into account the novel model developed in this study. More importantly, we give these Green's functions in exactly the same form as in FARRELL [1972]. This is advantageous to the many users of Farrell's Green's functions, as no serious modification in the existing software will be needed to account for the novel model developed in this study. Since the load numbers on a viscoelastic earth are complex, we calculate their imaginary part, as well. These "phase shift Green's functions" show that viscoelasticity in the earth introduces phase shifts of the order of a few degrees.

8) We give a new version of the LOADSDP software package [PAGIATAKIS 1982] for the evaluation of the loading effects that includes all the new features of this research. Moreover, we test the predictive power of the model against accurately observed gravity residuals at different tidal stations and we conclude that the present model is very promising, indeed. LOADSDP software can be used to calculate displacements, gravity perturbations and tilt at arbitrary locations on the surface of the earth and it is available from the Department of Surveying Engineering, upon request.

2

RHEOLOGY AND PROFILE OF THE EARTH

The knowledge of the relation of stress to strain through a constitutive law is essential in modelling the response of the earth to external forces. Nowadays, it is believed that the response of the earth to external forces, of periods from a few minutes to thousands of years, is not perfectly elastic. This "imperfect" behaviour "deteriorates" to a purely viscous behaviour, as the period of the deformation increases.

Studies of the imperfect elastic response of the earth at seismic frequencies have shown, that even at these high frequencies, the behaviour of the earth departs slightly from perfect elasticity [PELTIER, ET. AL., 1981]. The inelastic behaviour of the earth is extremely complicated. Among others, it depends on its chemical constitution, phase transformations and thermodynamical state.

In order to study the response of the earth to ocean tide loading, the earth's interior structure must be known to a certain extent. Of particular importance in the development of an ocean tide loading model, is the structure of the crust and the upper mantle of the earth. In addition, the capabilities of each layer to support positive and negative loads must be taken into account. The rheological properties of the crust and the upper mantle can be estimated from modelling observations of loads, such as, glaciation and deglaciation [CATHLES, 1975], volcanic seamounts [MCNUTT AND MENARD, 1978, RUNDLE, 1982] and topographic rises at ocean trenches [MELOSH, 1978].

The viscoelastic behaviour of the earth can be described by the amount of energy dissipated within the earth when it is subjected to stress. This amount of dissipated energy can be expressed as a function of the "quality factor" Q and depends on the dissipation mechanism within the earth.

In this chapter, we introduce the concepts of stress and strain in the solid earth, as well as, their changes in time. We elaborate on the profile of the earth from three points of view: anelasticity, rheology and thermodynamical state.

2.1 Stress and Strain

Stress is defined as force per unit area. It is transmitted through a material by interatomic force fields [TURCOTTE AND SCHUBERT, 1982]. Stresses, that are transmitted perpendicular to a surface of interest, are known as ***normal*** stresses; those, that are transmitted parallel to a surface of interest, are called ***shear*** stresses.

When dealing with deformations of a solid, stress must be defined in three dimensions. Since stress changes with position, even when the accompanied displacements are infinitesimal, it is necessary to recognise three triplets (on the three faces of an infinitesimal cube), leading to nine components of stress; these nine components are the independent elements of the symmetric ***stress tensor.*** In Figure 2.1a the nine components of stress on the faces of a finite element are shown. The first subscript of a component of stress denotes the direction of the normal to the surface, on which the force acts and the second subscript denotes the direction of the force.

Tensile stress is a normal force per unit area tending to extend the finite element. ***Compressive*** stress is normal force per unit area tending to contract the finite element. Conventionally, tensile stress is positive and compressive stress is negative.

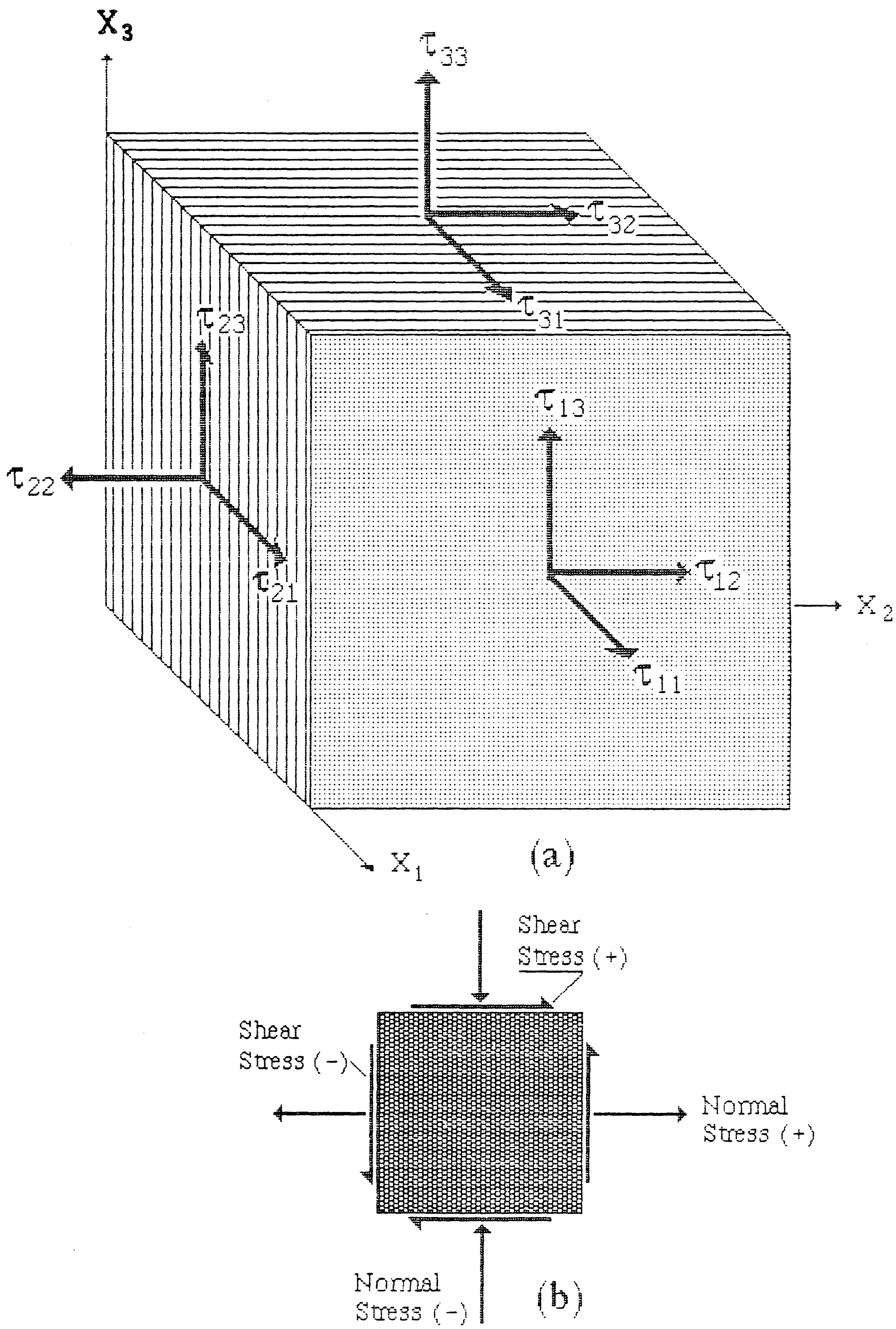


Fig. 2.1. The finite element and the stress conventions.

Shear stress is considered to be positive when it tends to rotate the finite element clockwise. The above conventions are illustrated in Figure 2.1b.

Strain is defined as the measure of differential deformation. The nine possible strains (corresponding to the nine components of stress) form a second rank tensor called ***strain tensor*** [EIRICH, 1956]. Normal strain is defined as the ratio of the change in length of a solid to the original length. Shear strain is defined as one-half of the decrease in a right angle in a solid when it is deformed. In the sequel, unless otherwise indicated, the summation convention applies, when an index is repeated twice.

The state of stress and strain in a solid can be described completely by the stress and strain tensors, respectively. If τ_{ij} is the stress tensor at a point, then it can be shown [BULLEN, 1975] that the trace of τ_{ij} is independent of the orientation of the coordinate axes†. Hence, $\tau_{ij}\delta_{ij}/3$ (whether referred to the principal axes or not) is equal to the mean of the three principal stresses (normal stresses). It is conventional to denote this mean by -p. Therefore,

$$p = -\tau_{ij}\delta_{ij}/3, \qquad i, j = 1, 2, 3. \tag{2.1}$$

Quantity p is called ***pressure***, or ***hydrostatic stress***.

The ***deviatoric*** stress tensor $\mathbf{T}_{ij}$ is defined as [BULLEN, 1975]

$$\mathbf{T}_{ij} = \tau_{ij} - \sum_k \tau_{kk}\delta_{ij}/3 = \tau_{ij} + p\delta_{ij} \qquad i, j, k = 1, 2, 3. \tag{2.2}$$

Deviatoric stress does not include the hydrostatic pressure induced by the neighbouring mass elements. Its trace is equal to zero.

† *This is called "invariance in coordinate transformation."*

Dilatation is defined as

$$\Theta = \mathbf{e}_{ij}\boldsymbol{\delta}_{ij} = \mathrm{trace}(\mathbf{e}_{ij}), \qquad i, j = 1, 2, 3, \tag{2.3}$$

where $\mathbf{e}_{ij}$ is the strain tensor. ***Compression*** is defined as negative dilatation. Similar to deviatoric stress tensor, the deviatoric strain tensor can be defined as

$$\mathbf{E}_{ij} = \mathbf{e}_{ij} - \sum_k \mathbf{e}_{kk}\boldsymbol{\delta}_{ij}/3 = \mathbf{e}_{ij} - \Theta\boldsymbol{\delta}_{ij}/3, \qquad i, j, k = 1, 2, 3. \tag{2.4}$$

Changes in stress in any material are accompanied, in general, by changes in deformation. A first step in deformation theory is to arrive empirically at a suitable set of model relations connecting $\boldsymbol{\tau}_{ij}$ and $\mathbf{e}_{ij}$. The relation between stress and strain tensors is referred to as the ***constitutive law***. The constitutive law depends on the rheology, on the thermodynamical conditions of the material at hand and on the time scale over which the stress is applied [PELTIER, 1974].

For a perfect (ideal) elastic material, Hooke's law defines the constitutive relation. For pure uniaxial deformation (deformation of a linear element), we can write [NOWICK AND BERRY, 1972]

$$\tau = Ee, \tag{2.5}$$

where E is the ***Young modulus*** and τ and e are the uniaxial stress and strain respectively. The reciprocal J of E is called ***modulus of compliance***.

Incompressibility or ***bulk modulus*** of a solid is denoted by k; it is defined as the ratio of pressure to compression. When the mode of deformation is pure hydrostatic deformation, Young modulus E in (2.5) changes to bulk modulus k. Compressibility is the reciprocal of incompressibility. The order of magnitude of k discriminates between gasses and liquids† [BULLEN, 1975].

Rigidity or ***shear modulus*** of a solid is the measure of the strain produced by an assigned deviatoric stress and it is denoted by μ. When the mode of deformation is pure shear deformation, Young's modulus in (2.5) changes to shear modulus μ. The order of magnitude of μ discriminates between fluids and solids. For fluids, μ is negligibly small (zero for ideal fluids). For most metals and rocks under normal conditions, μ is of the order of 10^9 to 10^{11} Nm^{-2}. A perfectly elastic material is called solid when μ is not negligible (when $\mu > 10^9\ Nm^{-2}$). A material is called fluid, when the evidence shows that μ does not exceed $10^9\ Nm^{-2}$.

Dynamic viscosity, or simply ***viscosity*** of a fluid is a measure of its resistance to deformation. Viscosity arises from cohesion of molecules and from the transfer of momentum, as molecules diffuse from one position to another [OBERT, 1960]. Viscosity is denoted by η and has units of $ML^{-1}T^{-1}$ (in CGS units, 1 poise=$1 gcm^{-1}sec^{-1}$)‡. A perfect, or ideal fluid, has zero viscosity. It is called ***Pascal fluid***, or ***inviscid fluid.*** Therefore, neither shear stress, nor internal friction can be transmitted in a Pascal fluid.

A simple, true, or ***Newtonian fluid*** has a coefficient of viscosity independent of the shear stress or rate of deformation. Hence, for a Newtonian fluid, the rate of strain is directly proportional to the applied stress. This constant of proportionality is the reciprocal of viscosity of the fluid.

† *Experimental values for incompressibility of gases can be obtained by several ways, most of which are indirect. The incompressibility of gases depends strongly on the way the compression takes place (e.g. isothermally or isendropically) and its value is usually close to unity. For liquids, incompressibility approaches infinity.*

‡ *From "kinetic theory" that explains various phenomena due to kinetic motion and elastic collisions of atoms and molecules, viscosity is given by:* $\eta = A\rho c\lambda$. *A is a coefficient depending on the forces between molecules,* ρ *is density, c is the mean speed of the molecules and* λ *is the mean free path of the molecules. Mean free path is the average distance traversed by a molecule between collisions.*

Very often in dynamics, the terms ***effective viscosity*** and ***kinematic viscosity*** are used. Effective viscosity η_{eff} is the ratio of stress to the rate of strain

$$\eta_{eff} = \tau_{ij}/\dot{e}_{ij} \tag{2.6}$$

and has units of $ML^{-1}T^{-1}$ (same as the dynamic viscosity). Kinematic viscosity η_{kin} is the ratio of viscosity to density [TURCOTTE AND SCHUBERT, 1982]

$$\eta_{kin} = \eta/\rho. \tag{2.7}$$

Kinematic viscosity has units of L^2T^{-1} (in CGS units, 1 stoke=$1 cm^2 sec^{-1}$) and expresses how momentum diffuses.

2.2 Linear Elastic Constitutive Law

The constitutive relation for a perfectly elastic and isotropic† material (Hookean solid) can be written as [BULLEN, 1975]

$$\tau_{ij} = (k - 2\mu/3)\Theta\delta_{ij} + 2\mu e_{ij}, \quad i, j = 1, 2, 3, \tag{2.8}$$

where τ_{ij} and e_{ij} are the stress and strain tensors respectively, k and μ are the incompressibility and rigidity respectively. The term in parentheses of (2.8) is called the first Lamé parameter λ, thus,

$$\lambda = k - 2\mu/3. \tag{2.9}$$

Parameters λ and μ are known as ***Lamé parameters***.

† *Isotropic material is the material whose rheological properties are the same in any direction.*

Young's modulus E and Poisson's ratio ν constitute an alternative pair of (position dependent) coefficients to λ and μ in expressing the stress-strain relations of a perfectly elastic and isotropic material. Coefficients E and ν are commonly interpreted by means of the deformation of a homogeneous cylindrical wire, subjected to uniform normal stresses at the ends. If e_1, e_2 and e_3 are the principal strains produced (e_1 is longitudinal and e_2 and e_3 are lateral with $e_2 = e_3$), then (according to Hooke's law):

$$E = \tau_1/e_1, \quad \nu = -e_2/e_1. \tag{2.10}$$

For perfectly elastic and isotropic material we can write [BULLEN, 1975]

$$E = 9k\mu/(3k+\mu) = \mu(3\lambda+2\mu)/(\lambda+\mu), \quad \nu = \lambda / \{2(\lambda+\mu)\} = (3k-2\mu)/(6k+2\mu). \tag{2.11}$$

Relations (2.9) and (2.11) establish the equivalence of the pairs $\{E, \nu\}$, $\{\lambda, \mu\}$ and $\{k, \mu\}$. For metals, ν varies between 0.3 and 0.4. For polycrystalline metals ν is about 0.25. The value of ν increases as μ/k decreases and is 0.5 for perfect fluid.

There are three implicit conditions in equations (2.8) and (2.11) that define perfect (ideal) elasticity. These are [NOWICK AND BERRY, 1972]:

1) The strain response at each level of applied stress (or vice versa) has a **unique equilibrium value**.

2) The equilibrium response is achieved **instantaneously**.

3) The response is **linear**.

The above three conditions may be lifted in various combinations to give different behaviour. Of these combinations, two are of importance in this study. When condition (2) above is lifted, the behaviour is called ***anelastic*** When conditions (1) and (2) are lifted, the behaviour is called ***linear viscoelastic***. Thus, linear viscoelasticity includes anelasticity as a special case. Since the absence of condition (1) implies a hysteresis loop, viscoelastic behaviour is characterised by dissipation (attenuation) of a fraction of the deformation energy. This subject is presented in more detail in section 2.4.

2.3 Linear Viscoelastic Constitutive Relations

Let us consider a linear viscoelastic material to which a stress is applied. In response to this stress, the material deforms and due to its viscous component, some time is required before it reaches an equilibrium. Similarly, if a strain change is imposed on the material, stress will not be transmitted through the material instantaneously. In both cases, the viscoelastic material does not "adjust" itself instantaneously to the changes of its state. We say then, that the material exhibits ***relaxation*** In the sequel, we examine separately the above two cases considering, for simplicity that, either stress, or strain are applied abruptly.

When stress τ_o is applied abruptly and held constant, strain changes as a function of time. This yielding of the viscoelastic material is called ***creep*** or ***transient anelasticity*** Therefore, creep is a special case of relaxation; it is called ***strain relaxation*** For one dimensional case, the ratio of strain (as a function of time), to the applied constant stress is called ***creep function***, or ***creep compliance*** [NOWICK AND BERRY, 1972]. Thus, the creep compliance is equal to:

$$J(t) = e(t) / \tau_o. \tag{2.12}$$

For elastic material, there is no creep; strain is observed instantaneously and J(t) is constant. As we watch the creep process progress, before the material reaches an equilibrium state and there is still time left for the creep process to continue, we say that the material has not yet relaxed. The measure of its deformation is described through the ***unrelaxed creep function*** J_U. As time progresses and the material approaches the relaxed state, J tends to become constant, i.e. it approaches the ***relaxed compliance*** J_R.

In summary,

$$\left.\begin{aligned} J_U &= \lim_{t \to 0} J(t), \\ J_R &= \lim_{t \to \infty} J(t). \end{aligned}\right\} \tag{2.13}$$

Let us assume that after some time has elapsed, not necessarily sufficient for the material to reach equilibrium, the stress is abruptly removed. Then, there will be a time-dependent decay of the strain in addition to the immediate elastic response. This phenomenon is called ***elastic aftereffect***, or ***creep recovery***. Creep and elastic aftereffect are illustrated in Fig. 2.2 [after NOWICK AND BERRY, 1972].

Let us suppose now, that a linear viscoelastic material is at rest (no external stresses applied), when an abrupt change in its strain is imposed. Apart from its immediate response due to the elastic component, there will be a time dependent change of stress in the material. This change of stress (as a function of time) is called ***stress relaxation*** and it is described by the ***stress relaxation function*** M(t). M(t) is the reciprocal of J(t)†. Similar to the creep function,

$$\left.\begin{aligned} M_U &= \lim_{t \to 0} M(t), \\ M_R &= \lim_{t \to \infty} M(t). \end{aligned}\right\} \tag{2.14}$$

The time required for a viscoelastic material to reach the relaxed state when it is either at constant stress, or at constant strain, is called ***relaxation time***. Relaxation time depends strongly on the chemical composition of the material as well as on its thermodynamic state.

In Table 2.1, some of the most common rheological models are shown along with their corresponding constitutive relations [FLÜGGE, 1975; PELTIER, ET. AL., 1981]. In these relations, τ_{ij} and $\mathbf{e}_{ij}$ are the stress and strain tensors respectively, λ and μ are the

† *Do not confuse M with Young's modulus E. M, as it is indicated, is a function of time, whereas E is not.*

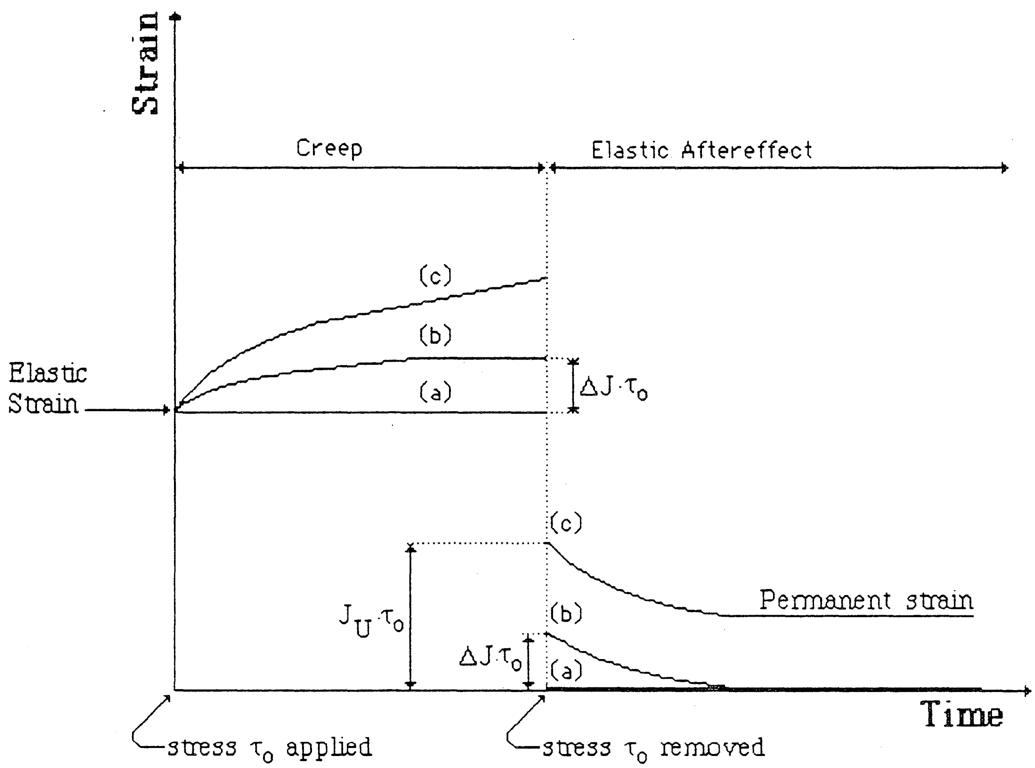


Fig. 2.2. Creep and elastic aftereffect for (a) ideal elastic solid, (b) anelastic solid and (c) linear viscoelastic solid [after NOWICK AND BERRY, 1972].

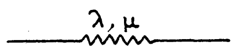
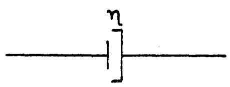
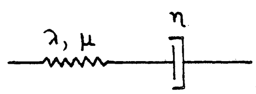
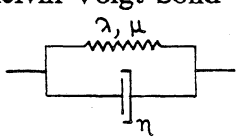
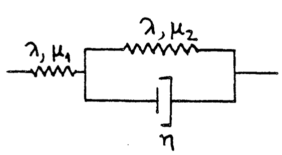
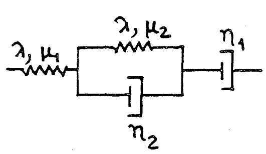
ANALOGUE	CONSTITUTIVE LAW
Hookean Solid 	$\tau_{ij} = 2\mu e_{ij} + \lambda e_{ii}\delta_{ij}$
Viscous Fluid 	$\tau_{ij} = 2\eta\dot{e}_{ij}$
Maxwell Solid 	$\dot{\tau}_{ij} = \mu/\eta\ (\tau_{ij} - \tau_{ii}\delta_{ij}/3) + 2\mu\dot{e}_{ij} + \lambda\dot{e}_{ii}\delta_{ij}$
Kelvin-Voigt Solid 	$\tau_{ij} = 2\mu_1 e_{ij} + \lambda e_{ii}\delta_{ij} + 2\eta\dot{e}_{ij}$
Standard Linear Solid 	$\dot{\tau}_{ij} + (\mu_1 + \mu_2)/\eta\ (\tau_{ij} - \tau_{ii}\delta_{ij}/3) =$ $2\mu_1\dot{e}_{ij} + \lambda\dot{e}_{ii}\delta_{ij} + 2\mu_1\mu_2/\eta\ (e_{ij} - e_{ii}\delta_{ij}/3)$
Burgers Body 	$\ddot{\tau}_{ij} + (\mu_1+\mu_2)/\eta + \mu_1/\eta_1(\dot{\tau}_{ij} - \tau_{ii}\delta_{ij}/3)$ $+ \mu_1\mu_2/(\eta_1\eta_2)(\tau_{ij} - \tau_{ii}\delta_{ij}/3) =$ $2\mu_1\ddot{e}_{ij} + \lambda\ddot{e}_{ii}\delta_{ij} + 2\mu_1\mu_2/\eta_2\ (\dot{e}_{ij} - e_{ii}\delta_{ij}/3)$

Table 2.1. Linear viscoelastic models and their constitutive law [After FLÜGGE, 1975, PELTIER, ET. AL., 1981].

Lamé parameters and η is viscosity. Dots indicate time differentiation. The dashpot analogue of a Newtonian fluid is characterised by a steady state creep behaviour. Spring and dashpot in series (Maxwell solid) is characterised by an instantaneous elastic response and a long term steady state creep. Spring and dashpot in parallel (Kelvin-Voigt solid) have transient anelastic response (creep). Other combinations of springs and dashpots exhibit more complicated responses.

2.4 Hysteresis

Theory of elasticity can account for solids, which have the capacity to store all the mechanical energy supplied by external forces [CHRISTENSEN, 1971]. On the other hand, Newtonian viscous fluids are characterised by their property to dissipate all the energy supplied by external forces and thus, have no capacity to store energy. Viscoelastic solids fall between the elastic solids and viscous fluids. They are described by their property to dissipate a fraction of the mechanical energy supplied by any external forces, while most of the energy is stored in the form of elastic or strain energy.

Let us assume that a certain viscoelastic material is deformed under a slowly varying periodic load and exhibits linear viscoelasticity. An application of a tensile load will cause the material to extend. On a stress-strain graph, the behaviour of the material is described by curve #1 (Fig. 2.3). When unloading the material, its stress will be higher than the stress during loading, for the same magnitude of strain (curve #2). Finally, when a full cycle of loading is completed, the behaviour of the viscoelastic material under periodic stress is depicted by the closed line (loop). This behaviour of the material is included in the class of ***hysteresis phenomena*** The closed line on the stress-strain diagram is called a ***hysteresis loop*** [LOVE, 1927].

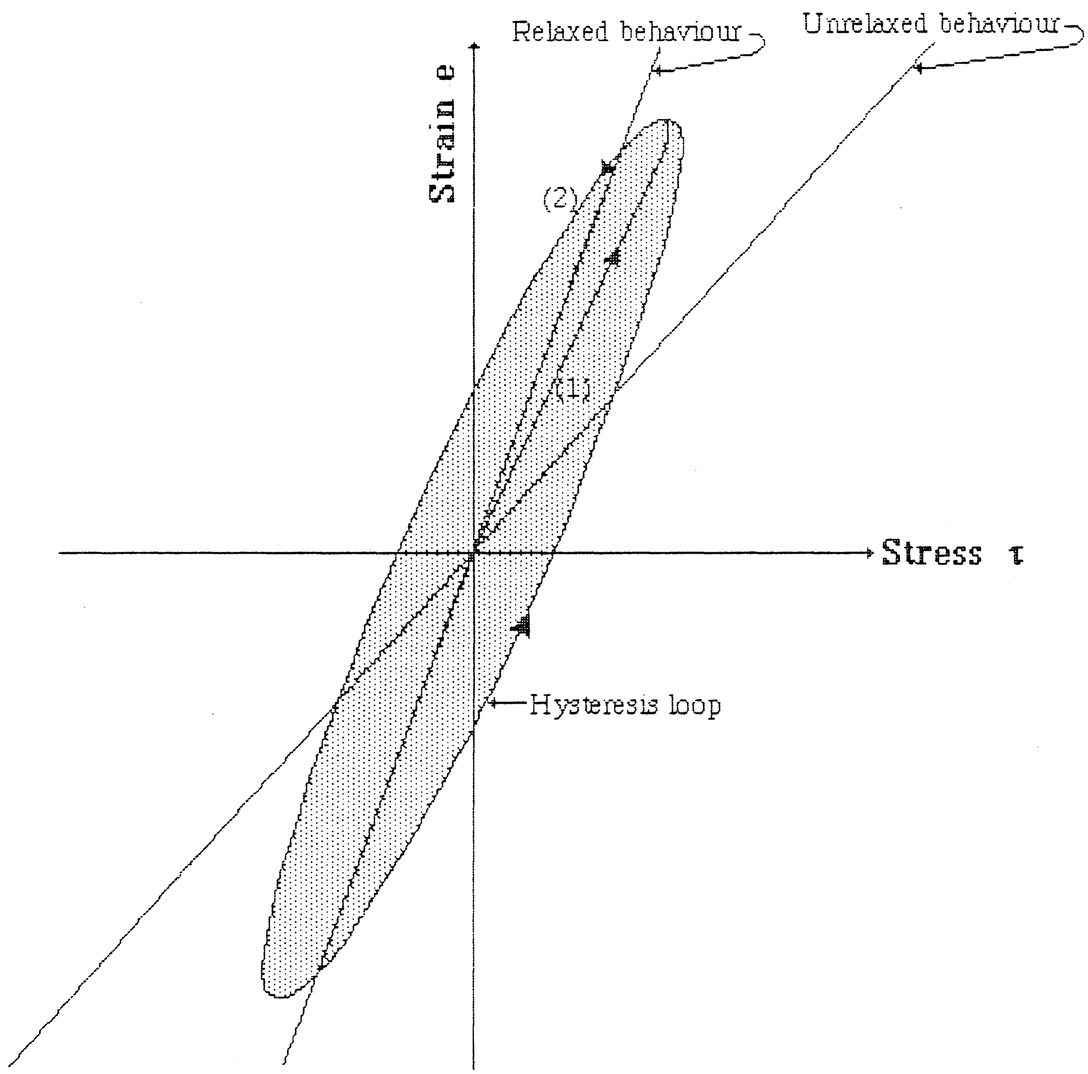


Fig.2.3. For long period sinusoidal stress (similar to earth tides), the slope of the major axis of the hysteresis loop (ellipse for linear viscoelastic bodies) is very close to the slope defined by the relaxed modulus of compliance [after MINSTER, 1980].

Elliptical hysteresis loops indicate linear viscoelastic behaviour† [BRENNAN, 1981]. The area inside the loop is proportional to the amount of energy dissipated during one cycle of loading and it is related to the phase difference between stress and strain [MINSTER, 1980].

2.5 Q Profile of the Earth

It has been established experimentally, that the earth's crust and mantle exhibit viscoelastic behaviour. Evidence for this behaviour is abundant, both from phenomena with short characteristic periods, such as dispersion of body wave velocities, spatial attenuation of surface waves as well as from longer period processes, such as postglacial rebound, crustal bending and polar wander [PELTIER ET. AL., 1981]. This viscoelastic behaviour of the earth is possibly due to relaxation phenomena with mechanisms accounting for slippage of grain boundaries (dislocation), partial melting of the material, phase transformations and thermoelasticity [LELIWA-KOPYSTYNSKI AND TEISSEYRE, 1984].

Quality factor Q expresses the amount of energy that is irreversibly lost by dissipation during a complete cycle of stress. Its reciprocal is defined as [LAMBECK, 1980]

$$Q^{-1} = 1/(2\pi)\Delta E/E, \tag{2.15}$$

where E is the maximum value of the elastic energy (peak elastic energy) stored during a complete cycle of straining and ΔE is the amount of energy dissipated during a complete cycle of straining.

Other definitions for Q are more convenient to use. For example, an alternative definition to (2.15) is [LAMBECK, 1980]

† *There are cases in which the hysteresis loop is cusped. This indicates that the material exhibits non-linear behaviour; its constitutive law is not a linear differential equation anymore. Since we are only interested in linear viscoelastic constitutive relations, cusped hysteresis loops are not of concern in this study.*

$$Q^{-1} = 1/(2\pi)\Delta E/(2\langle E\rangle), \tag{2.16}$$

where $\langle E\rangle$ is the average elastic energy stored. Another useful definition is

$$Q^{-1} = 1/(2\pi)\Delta T/T, \tag{2.17}$$

where T is the peak kinetic energy and ΔT is the change of T over a complete cycle of straining. For low attenuation (Q>100), however, the above definitions are essentially equivalent [DZIEWONSKI, 1979; JORDAN, 1980].

Energy dissipation within the earth can occur during pure compression, or pure shear, or both. Therefore, the necessity for the definition of two quality factors arises. Dissipation in pure compression, or ***bulk dissipation***, is described by a specific quality factor Q_k. Dissipation in pure shear, or ***shear dissipation***, is described by a specific quality factor Q_μ. Quality factor Q and specific quality factors Q_k and Q_μ are related through the following formula [JORDAN, 1980]:

$$Q^{-1} = \langle E_k\rangle/\langle E\rangle\, Q_k^{-1} + \langle E_\mu\rangle/\langle E\rangle\, Q_\mu^{-1}, \tag{2.18}$$

where $\langle E_k\rangle$ and $\langle E_\mu\rangle$ are the average elastic energies in compression and shear respectively stored in the earth during a complete cycle of straining and $\langle E\rangle$ is the average total energy stored in the earth during a complete cycle of straining. Losses in pure compression within the earth are usually small compared to losses in shear. Bulk dissipation can be actually neglected in the solid regions of the earth [JORDAN, 1980].

The frequency dependence of Q in the earth is an exceedingly controversial subject. There is some observational evidence, that Q depends on frequency. However, there is not enough evidence of how Q might depend on frequency [MINSTER, 1980]. LAMBECK's [1977] estimated values of Q at tidal periods, as well as, SAILOR AND DZIEWONSKI's [1978] Q values for free oscillation periods are consistent with a weakly frequency-dependent Q. This weak dependence of Q on frequency is valid for periods between 14 months (for the Chandler wobble), to 54 minutes (for the spheroidal free oscillation mode). The above weak dependence of Q on frequency is also consistent with the $\omega^{0.2}$ (where ω is the angular velocity of the applied stress) dependence, proposed by JEFFREYS [1970]. However, within a limited frequency band, it is safe to consider constancy in Q.

ANDERSON AND HART [1978b] constructed a model Q, as a function of depth, compatible with the normal mode data set (including overtones) and with teleseismic body and surface wave observations. The above model is consistent with a frequency independent Q†. The construction of the depth profile of Q was based on standard linear solid type rheology. It is commonly known as "***Q model SL8***" [Fig. 2.4].

A more recent depth profile of Q is found in the earth model, known as PREM and given by DZIEWONSKI AND ANDERSON [1981]. This model was obtained by inversion of a large set of observational data, allowing for anisotropy and dissipation. This model is discussed in section 2.6.

In this study, we make the hypothesis that the earth exhibits standard-linear-type rheology at tidal frequencies and that the quality factor Q at all depths is independent of frequency within the tidal band. We must stress here that the above is only an assumption, which has been shown to be satisfactory at least within the seismic band [DZIEWONSKI AND ANDERSON, 1981].

† *However, this frequency independence of Q is not required.*

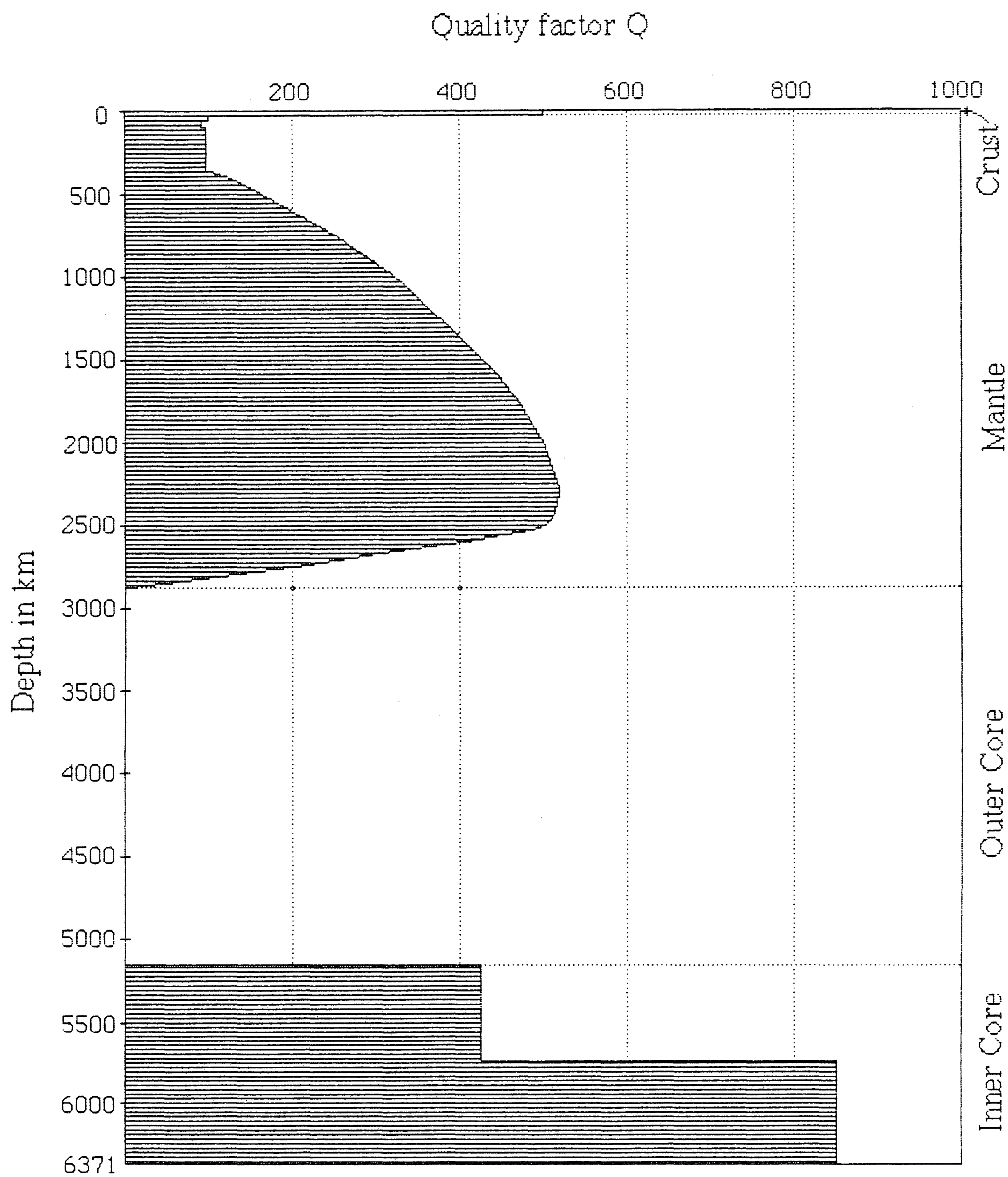


Fig. 2.4. Q Model SL8

2.6 Rheological Profile of the Earth

The division of the earth into crust, mantle and core by two major seismic discontinuities at the depths of about 35 km (Mohorovicic discontinuity) and about 2900 km, has been very well known for many decades [LELIWA-KOPYSTYNSKI AND TEISSEYRE, 1984]. Nevertheless, their exact character is still uncertain [MAXWELL, 1984]. Other minor seismic discontinuities have been recognised at various depths. These discontinuities further refine the layering of the earth.

In the middle of this century, the study of seismic wave propagation reached a stage, that permitted various investigators to develop more realistic earth models. These series of earth models, later referred to as the A-type models, included distributions of earth's density, incompressibility, rigidity, pressure, gravity, as well as other derived variables (such as Young's modulus and Poisson's ratio), as functions of depth. Improvements of these models were obtained in the early 1960's and these models are known as A' and A″ earth models [BULLEN, 1975]. The basic assumptions made in constructing the A-type models are:

1) The density of the earth just below the Mohorovicic discontinuity is 3.32 gcm^{-3}.

2) The velocities of compressional (p) and shear (s) seismic waves are known .

3) The Adams-Williamson† condition holds true in some deep regions of the earth.

BULLEN [1946] noticed that for the A-type models, there exists a remarkable feature in the behaviour of incompressibility k, in the vicinity of the core-mantle boundary. The changes of k, as well as the changes of the ratio dk/dp (where p is pressure) were small and smooth, despite the drastic changes in density and rigidity. The above behaviour was verified by laboratory experiments for a wide class of materials under pressure up to 10^{10} Nm^{-2}‡.

† *The Adams-Williamson condition is one of the equations of state in the earth [BULLEN, 1975]. It is given by* $d\rho/dz = \rho g/\phi$, *where* ρ *is density,* z *is depth,* g *is gravity and* ϕ *is the seismic parameter depending on the seismic wave velocities.*

‡ *This is one order of magnitude less than the pressure at the core-mantle boundary.*

BULLEN [1946] formulated the ***incompressibility-pressure (k-p) hypothesis*** as follows:

«*Throughout the earth's lower mantle (below 1000 km depth) and core, irrespective of variations of composition as may occur inside this entire region, k varies continuously and smoothly with P.*» Observational evidences for the reliability of the k-p hypothesis were supplied later [BULLEN, 1975].

The k-p hypothesis was the main feature for the construction of the second generation of earth models, known as B-type models. The first difference between A- and B-type models is that the inner core of the B-type models is modelled to be solid. The second difference is that B-type models have a larger density gradient in the lower mantle than A-type models. The most serious difference is that B-type models exhibit considerably larger density in the upper mantle than A-type [BULLEN, 1975]. Revised estimates of the moments of inertia of the earth, revised seismic wave velocities as well as continuity of incompressibility and the ratio dk/dP in the lower mantle and core contributed to the improvement of the B-type models by BULLEN AND HADDON [1967a; 1968].

HB_1 earth model was the first model constructed by taking into account free oscillation data [BULLEN AND HADDON, 1967b] and thus it marks a specific stage in the evolution of the earth models. There have been subsequent earth models too, based on free oscillation data and overtone periods of the free oscillations of the earth. A good description of these is given in LELIWA-KOPYSTYNSKI AND TEISSEYRE [1984].

DZIEWONSKI ET. AL., [1975] constructed three parametric earth models (PEM) in which radial variations of the density and seismic velocities are represented by piecewise continuous analytical functions (algebraic polynomials of order not higher than three) of the normalised radial distance from the centre of the earth. These three models are:

1) Oceanic parametric earth model (PEM-O),

2) Continental parametric earth model (PEM-C),

3) Average parametric earth model (PEM-A).

The data used for the construction of these models consisted of observations of eigenperiods of 1064 normal modes, 246 travel times of body waves for five different phases and regional surface-wave dispersion data, extending to periods as short as 20 seconds. The agreement of the model with seismic wave velocities, free oscillation data and with the Adams-Williamson condition below 670 km depth, is better than 0.2% [DZIEWONSKI ET. AL., 1975]. Models PEM-O and PEM-C reflect the properties of the oceanic and continental upper mantles respectively. PEM-A represents the average earth. PEM-A was obtained by using weighted means of PEM-O and PEM-C with weights 2/3 and 1/3, respectively. All three models are identical below the depth of 420 km.

More recent models have been developed, allowing for attenuation and anisotropy in the earth. DZIEWONSKI AND ANDERSON [1981], following the guidelines established by the Standard Earth Model (SEM) Committee†, composed of members from the International Association of Geodesy (IAG) and the International Association of Seismology and Physics of the Earth's Interior (IASPEI), presented a new parametric earth model, called the ***Preliminary Reference Earth Model*** (PREM). For the construction of this model, a large data set of about 1000 normal mode periods, 500 travel time observations, 100 normal mode Q values, mass and moments of inertia of the earth was inverted. The introduction of lateral isotropy for the outer 220 km of the mantle improved the agreement among the different data sets. In addition, the assumption of the frequency independence of Q was incorporated, giving satisfactory results [DZIEWONSKI AND ANDERSON, 1981]. PREM can be described as follows (see also Fig. 2.5):

† *Working group of International Union of Geodesy and Geophysics (IUGG).*

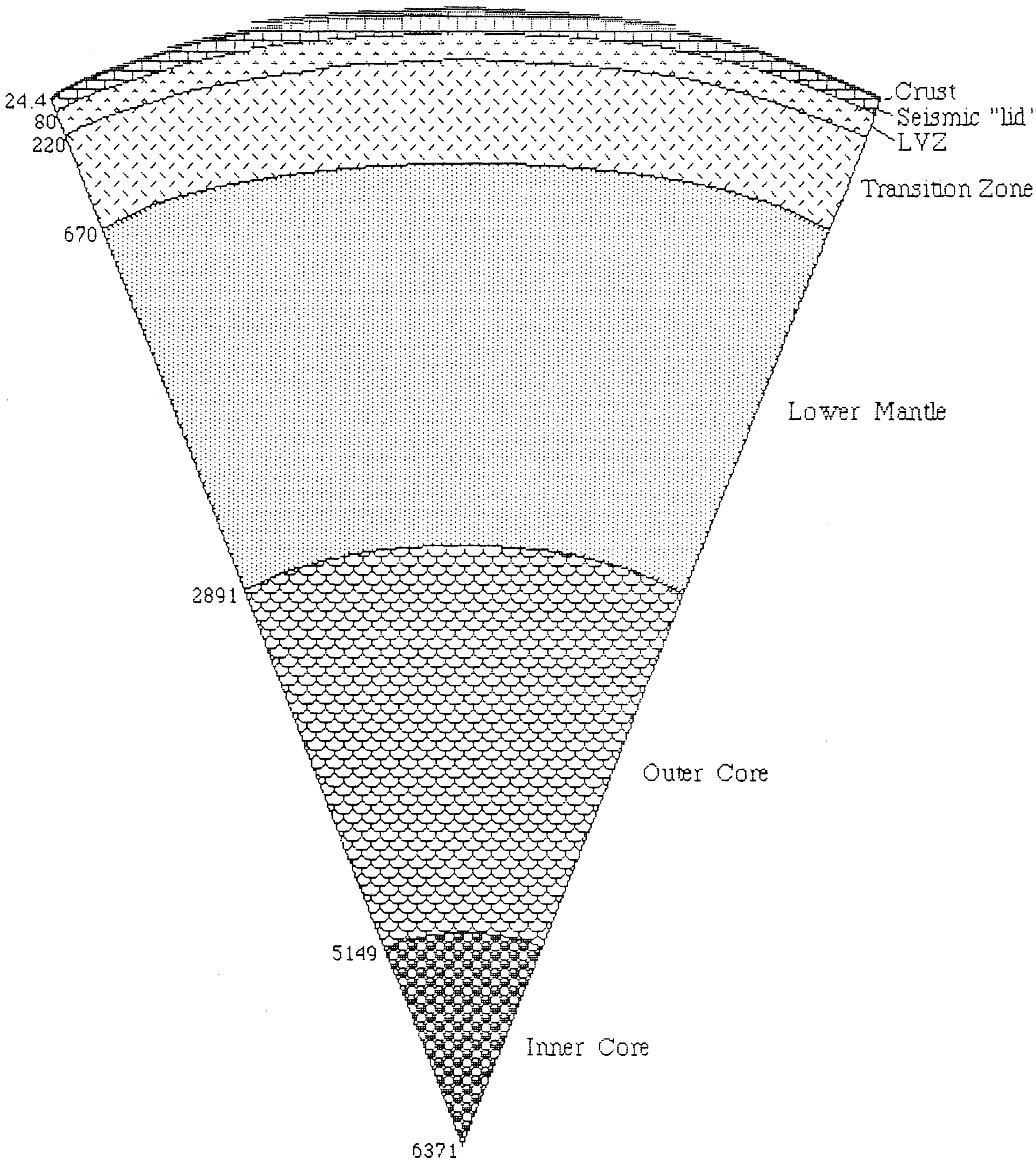


Fig. 2.5. Preliminary Reference Earth Model. Depths in km.

1) ***Crust*** is the outer shell of the earth that extends to a depth of 24.4 km.

2) The region above the low velocity zone, also known as seismic "lid", extends from 24.4 km to 80 km depth. The crust and the seismic "lid" constitute what is known as seismic lithosphere†.

3) ***Low Velocity Zone*** (LVZ) is the weakest zone of the earth and sometimes it is equated with the asthenosphere. It extends from 80 km to 220 km depth and is characterised by lateral isotropy.

4) ***Transition zone*** is the deepest part of the upper mantle and is characterised by phase transformations‡. It can be considered as a fairly homogeneous and isotropic layer [LELIWA-KOPYSTYNSKI AND TEISSEYRE, 1984]. It extends from 220 km to 670 km depth.

5) ***Lower mantle*** is considered to be in a solid state with some discontinuities. Little is known about the nature of these discontinuities and their extension on a global or regional scale. Lower mantle extends from 670 km to 2891 km depth.

6) ***Outer core*** is characterised by its liquid state and extends from 2891 km to 5149.5km depth.

7) ***Inner core*** is in a solid state. Although this is still a subject of research, it appears to be consistent with observational data [BOLT AND UHRHAMMER, 1981; BOLT, 1987].

† *The lithosphere, as any other layer of the earth, can be defined from different points of view. For instance, lithosphere can be defined by its elastic or flexural characteristics, or by its thermal state, or by its chemical and mineralogical constitution. Thus, we have "elastic lithosphere", "thermal lithosphere" and "chemical lithosphere", respectively. Seismic lithosphere is the lithosphere determined from seismic observations.*

‡ *In general, phase transformations refer to the changes of the state of the matter, such as melting, solidification, condensation, evaporation, etc. Phase transformations are related to the thermodynamical state of the matter.*

2.7 Thermodynamics and Profile of the Earth

In order to understand the dissipation mechanism within the earth, some basic concepts from the theory of thermodynamics must be presented. More specifically, the notions "adiabatic process" and "equations of state" are of primary importance in this work.

Adiabatic process is a process during which the state of matter changes without exchange of heat with the surroundings. It is found from experience, that the work required to change the state of a thermally insulated (adiabatic) system depends only on the initial and final states of the system and not on the path of the change of the state.

The ***equation of state*** of a system interrelates different thermodynamic properties. It can be written as

$$f(P, V, T, m, \text{Universal Constants}) = 0, \tag{2.19}$$

where P is the pressure, V is the volume, T is the absolute temperature and m is the mass of the system. In some instances, it is necessary to include properties other than those included in (2.19) to describe completely the state of the system [LEE AND SEARS, 1963].

The equation, which expresses the internal energy of a system as a function of any pair of its thermodynamic properties, is called the ***energy equation*** of the system. The equation of state and the energy equation together determine completely all the thermodynamic properties of a system.

Isothermal process is a constant temperature process and it follows Boyle's law PV = constant. ***Thermodynamic cycle*** is a sequence of processes, that eventually returns any system to its original state. The thermodynamic cycle is the concept applicable to a closed system. ***Reversible process*** is an ideal process, that can be stopped at any stage and reversed, so that the system and surroundings are exactly restored to their initial states. During a reversible process, the system must pass through the same states on the reversed path, as were initially visited on the forward path. An ***irreversible*** process is a real process; it cannot be reversed so as to follow exactly the forward path because a fraction of the energy of the system is transformed into heat.

2.8 Dissipation Mechanism within the Earth

The dominance of dissipation in shear over bulk dissipation can be explained by a grain-boundary relaxation model. This model was developed by ZENER [1941] and it was based on the evidence of viscous sliding of adjacent crystals. Although this theory was developed already in 1941, still there is no satisfactory quantitative theory of the phenomenon. In fact, even qualitative concepts are still in doubt [NOWICK AND BERRY, 1972].

Before we describe the model of the grain-boundary relaxation process, the introduction of the definition of Gibbs free activation energy is in order. ***Gibbs free activation energy*** is the excess energy over the ground state, which must be acquired by an atomic or molecular system, in order that a particular process may occur [VAN NOSTRAND'S SCIENTIFIC ENCYCLOPEDIA, 1976].

The mechanism of the grain-boundary relaxation model considers spherical elastic grains that are bound together with viscous material. The application of a shear stress causes the grains to slide over other grains. During sliding, there is a shear stress build-up, opposing the applied stress. When the stress is removed, the deformation generates a reverse shear stress and produces an elastic aftereffect. The grain-boundary relaxation model resembles the behaviour of the standard linear solid. The elastic aftereffect is governed by the dynamic viscosity of the material that holds the grains together, which in turn depends strongly on the temperature and pressure [O'CONNELL, 1977].

In the interior of the earth (below about 150 km), temperatures are higher than one-half of melting temperature. In this regime, known as ***high temperature background*** the relaxation process is believed to be thermally activated and the relaxation time can be estimated from an Arrhenius equation

$$\tau_\sigma = \tau_0 \exp\{G^* / (kT)\}, \tag{2.20}$$

where τ_0 is a characteristic time related to atomic jump frequency†, G^* is Gibbs free activation energy‡, k is Boltzmann's constant and T is absolute temperature.

We consider SAMMIS ET. AL. [1977] model for G^* for an adiabatic temperature gradient of 0.3 K/km that agrees with Stacey's thermal model [STACEY, 1977] used in the estimation of T in (2.20). However, we apply a correction to G^* model of - 45 kcal/mole throughout the mantle♠, so as to be consistent with ANDERSON's [1967] calculations of the activation volume and the values of viscosity for the lower mantle, namely 10^{22} poise, a value derived also from postglacial rebound data.

† *$\tau_0 = h / (kT)$, where h is Planck's constant (6.63×10^{-34} joule sec), k is Boltzmann's constant (1.38×10^{-23} joule/K) and T is the absolute temperature.*

‡ *G^* depends on the internal energy of the matter, its entropy, pressure and absolute temperature.*

♠ *Actually, G^* model of SAMMIS ET. AL. [1977] was used by the same authors to evaluate the viscosity and activation volume in the mantle. Their calculations gave unusually high values for both. They concluded, that if a correction of about - 45 kcal/mole was applied to G^*, viscosity profile as well as activation volume agree with Anderson's calculations. However, as we shall see in chapter 6 of this work, our calculation of load numbers is more or less insensitive to this correction.*

3

LAGRANGEAN MECHANICS

The response of the earth to ocean tide loading is described by some equations of motion, hereafter called the ***equations of deformation***. The study of this response is complicated, even when the earth is considered elastic, homogeneous and isotropic. Furthermore, if we wish to consider a more realistic earth, the development as well as the solution and interpretation of the equations of deformation will become extremely difficult, if not impossible. In such complicated cases, the Lagrangean approach appears to be the most suitable for the development of mathematical models.

Lagrangean mechanics is a powerful tool for the study of the behaviour of complicated mechanical systems. No matter how complex the system is, it may be represented by a single scalar function: the Lagrangean. In addition, the application of Hamilton's principle of least action to the Lagrangean function leads to an invariant set of differential equations, known as Lagrangean equations of motion. Lagrangean equations of motion are second order partial differential equations; the position of the mechanical system is the dependent variable and time is the independent variable.

In this research we are interested in the determination of the deformations of the earth; these deformations are considered as dependent variables of position and time, as opposed to the classical equations of motion that consider the position as dependent variable of time. Consequently, the Lagrangean equations of motion in this study acquire a different character; they become second order partial differential equations (PDEs) in the displacements (deformations) and thus, they are called "equations of deformation." In this Chapter, we present Lagrangean equations of motion in their classical form, for a mechanical system of n particles; the equations of motion in a 3-D space are 3n second order PDEs in the positions, the time being the independent variable. Subsequently, we show that the equations of deformation of the earth, have the same form of the equations of motion, the substantial difference being that the dependent variables are the displacements (deformations) and the independent variables are the position and the time. Therefore, we arrive at three PDEs of second order, although the earth is considered as a continuous body, consisting of an infinite number of particles.

We present the concepts of "tangent" and "cotangent bundle spaces", in which the equations of deformation are developed. Although, we do not take full advantage of this exposition in this study, we indicate its importance in future research on the deformations of the earth.

3.1 Definitions

The subject matter of the present research is the study of the response of the earth to external loads. The study of such a response becomes possible by assuming that the earth is totally continuous. The earth's molecular structure is to be disregarded and the earth pictured as a body without gaps or empty spaces; the earth is viewed macroscopically in the sense that its smallest characteristic unit (part) is much larger than the size of an atom or a molecule. This is an excellent approximation when the study of a body, such as the earth, under the influence of external forces, is of interest [ERINGEN, 1967]. The study of a continuous medium, also known as ***continuum***, can be accomplished by applying the classical laws of mechanics and a realistic constitutive law.

Bodies are described by their configurations, also known as manifolds. A ***manifold*** is a higher-dimensional analogue of a smooth curve or surface [POSTON AND STEWART, 1978]; it can be regarded as a real vector space $\mathfrak{I}$ that has the following properties (as any real vector space) [KREYSZIG, 1978; ODEN, 1979]:

a) It consists of elements, called vectors.

b) The operation of "addition" between vectors is defined, following the usual rules of arithmetic.

c) The operation of "multiplication" between a vector and a real number is defined, following the usual rules of arithmetic.

The above properties of $\mathfrak{I}$ mean that if $\boldsymbol{\varepsilon}_1, \ \boldsymbol{\varepsilon}_2 \in \mathfrak{I}$ and $\lambda_1, \ \lambda_2 \in \mathfrak{R}$, ($\mathfrak{R}$ is the set of real numbers), then $\boldsymbol{\varepsilon}$ defined by: $\boldsymbol{\varepsilon} = \lambda_1 \ \boldsymbol{\varepsilon}_1 + \lambda_1 \ \boldsymbol{\varepsilon}_2$ is also a member of the real vector space, i.e., $\boldsymbol{\varepsilon} \in \mathfrak{I}$.

In order to realise geometrically the configuration (configuration manifold) of a body, i.e., to be able to determine, either its size, or the size of its deformation, the concept of the real vector space is not adequate; the size of any vector in $\mathfrak{I}$ cannot be determined, simply because the way of measuring it is not known. Thus, the necessity of the definition of a more appropriate space arises. This new space, called ***metric space***, has all the "ingredients" of $\mathfrak{I}$ with the addition of a metric. In general, a ***metric*** in a space is a generalisation of the familiar concept of distance between two points. The introduction of a metric in $\mathfrak{I}$ is equivalent to the introduction of a coordinate system in which the configuration of the body is to be determined. Many different ways of measuring distances (metrics) can be considered; yet, every one of them must satisfy certain conditions. If we denote the metric space by $\aleph$ and a distance function $d: \aleph \times \aleph \Rightarrow \mathfrak{R}$, that associates (maps) pairs of elements of $\aleph$ with real numbers in $\mathfrak{R}$, then the distance function (metric) must satisfy the following four conditions [ODEN, 1979]:

a) $d(\mathbf{x}, \mathbf{y}) \geq 0, \ \forall \, \mathbf{x}, \mathbf{y} \in \aleph.$

b) $d(\mathbf{x}, \mathbf{y}) = 0,$ if and only if $\mathbf{x} = \mathbf{y}$.

c) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}), \ \forall \, \mathbf{x}, \mathbf{y} \in \aleph.$

d) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}), \ \forall \, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \aleph.$

The application of the theory of continuum mechanics† to the study of the response of a physical body to external forces, requires some fundamental postulates to be made. Denoting with $\mathfrak{p}$ the body under investigation, these fundamental postulates are as follows [FREDERICK AND CHANG, 1965; TRUESDELL, 1966]:

a) The configuration manifold of $\mathfrak{p}$ is assumed to be smooth, i.e., it has a unique tangent plane at each point,

b) $\mathfrak{p}$ is divisible into a finite number of elements,

c) $\mathfrak{p}$ is embedded in a metric space $\aleph$ of finite dimensions.

According to the above postulates, the quantitative description of the configuration of a continuum $\mathfrak{p}$ can be accomplished in a metric space, hereafter called the ***configuration space*** $\aleph$ of $\mathfrak{p}$. The configuration space is spanned by n quantities $q_1, q_2, q_3, \ldots, q_n$ known as ***generalised coordinates*** [NEIMARK AND FUFAEV, 1972]. Generalised coordinates, as the name implies, are not necessarily Euclidean coordinates; they can be of different entity. For example, potential, strain, gravity, temperature, pressure, or any other quantity that is needed to define the configuration of a continuum, can be considered as a generalised coordinate. Yet, according to postulate (c) above, the number of generalised coordinates is always finite.

† *As presented in Chapter 2 of the present study.*

The external forces that influence a continuum can be of different character. Forces originating from contacts with other bodies, gravitation, thermal changes, chemical reactions and environmental changes are the most common. In the present study we consider forces of mechanical origin, i.e., the first two categories. Continua that are influenced by the above forces will be called ***mechanical systems***.

3.2 Holonomic and Nonholonomic Mechanical Systems

The position, the motion, as well as the equilibrium state of a mechanical system may be required to satisfy a number of conditions and restrictions. It is conventional to say that ***constraints*** are imposed on the system. These constraints may be either ***geometrical constraints***, or ***kinematical constraints***, when they represent restrictions on the geometrical position or on the motion of the system, respectively. However, geometrical and kinematical constraints are not independent. Geometrical constraints are essentially kinematical constraints. This happens because geometrical constraints can be differentiated with respect to time to give kinematical constraints. The opposite is not always valid; kinematical constraints may not be integrable with respect to time to impose geometrical constraints.

A mechanical system with non-integrable kinematical constraints is called a ***nonholonomic system*** [NEIMARK AND FUFAEV, 1972]. When set in motion, a non-holonomic system follows a geometrical path (trajectory) that is not restricted by any non-integrable kinematical constraint; the motion of the system may violate the kinematical constraints. In contrast, if the kinematical constraints are integrable with respect to time, the motion of the system will not violate the kinematical constraints. Such a system is called ***holonomic.***

When the dimension of the configuration space is n and the number of non-integrable (nonholonomic) constraints is m, then the ***degrees of freedom*** of the system are n-m.

For a holonomic system the degrees of freedom are n. When a mechanical system is holonomic and, in addition, time t appears explicitly in the equations of motion, then the system is called ***rheonomic*** or ***nonautonomous***. When time t does not appear explicitly in the equations of motion, then the system is called ***scleronomic***, or ***autonomous*** [SANTILLI, 1978]. We must stress here that the generalised coordinates are dependent variables, whereas time t is an independent variable.

3.3 Tangent Space and Tangent Bundle Space

The set of all tangent vectors at a point x of an n-dimensional configuration manifold M, forms an n-dimensional ***tangent space*** Tx [ARNOLD, 1978]. Tangent space is a metric space.

The union of the tangent spaces to M at all points, is a smooth (differentiable) manifold, the dimension of which is twice the dimension of M. This manifold is called the ***tangent bundle space*** of M and is denoted by TM. Tx and TM are both metric spaces.

3.4 Cotangent Space and Cotangent Bundle Space

The set of all linear transformations f, such that f: $Tx \Rightarrow \Re$, forms a new metric space that is called the ***cotangent space***; it is denoted by T^*x. In order to visualise such a linear transformation, we consider a velocity vector that is tangent to the configuration manifold M. Similarly, the momentum vector is also tangent to M at the same point; the velocity vector and the momentum vector are simultaneously tangent, i.e., they are cotangent to M, at the same point. However, velocity and momentum are different entities; momentum can be obtained from velocity by scaling it with mass. Therefore, velocity belongs to Tx and momentum belongs to T^*x.

The union of cotangent spaces to M at all of its points is called the ***cotangent bundle space*** of M, it is denoted by T^*M and has twice as many dimensions as the cotangent space [ARNOLD, 1978]. Tangent bundle and cotangent bundle spaces are dual†.

3.5 Lagrangean and Hamiltonian Mechanics

Lagrangean mechanics describes the motion of a mechanical system in the ***configuration space*** [ARNOLD, 1978]. For a holonomic system of n degrees of freedom, the Lagrangean equations of motion are formulated on the configuration manifold, also called ***Lagrangean configuration manifold***, and consist of n second-order partial differential equations (PDEs), which are generally non-linear in the generalised coordinates q_k. In general, to solve such a system of equations is extremely difficult, if not impossible. Yet, even if an analytical solution existed, we would be faced with two problems:

a) Having analytical expressions only for the generalised coordinates would not help us visualise geometrically the motion of the system. We need to have analytical expressions for the velocity or the momentum.

b) In complicated mechanical systems it is imperative that we study the stability of the motion of the system. However, certain reliable and advanced stability theories, such as Lyapunov's stability theory, are only applicable to systems of linear, first order ordinary differential equations.

To overcome the above problems we must transform the equations of motion into first-order coupled ordinary differential equations (ODEs). This can be achieved by augmenting the state vector of the second-order PDEs by the generalised velocities. Then, the n non-linear second-order PDEs, which are valid in the configuration space, are

† *In the case of a metric space, the set of all bounded linear functionals on it constitutes a second metric space, which is called the dual space [ODEN, 1979].*

transformed into 2n first-order ODEs in the tangent bundle space of 2n dimensions. However, this transformation is not unique. We can augment the state vector by the generalised momenta p_k instead of generalised velocities. Since p_k is cotangent to generalised velocity, the Lagrangean equations of motion are transformed into 2n first order simultaneous ODEs in the cotangent bundle space. These equations are known as ***Hamilton's canonical equations of motion*** [D'SOUZA AND GARG, 1984]. State variables q_k, p_k are called ***canonically conjugate variables*** [SANTILLI, 1978].

Since the tangent bundle space and the cotangent bundle space are the dual of each other, we can begin with the Lagrangean equations of motion in the tangent bundle space and arrive at Hamilton's canonical equations of motion in the cotangent bundle space by the ***Legendre transformation*** [ARNOLD, 1978]. Thus, Lagrangean and Hamiltonian formulations are equivalent.

The cotangent bundle space spanned by the generalised coordinates q_k and the generalised momenta p_k is called ***phase space*** [NEIMARK AND FUFAEV, 1972]. When the state of a system is required as a function of time, the equations of motion can be represented in a (2n+1-m) dimensional space called the ***state space*** [D'SOUZA AND GARG, 1984], or ***extended phase space*** [ARNOLD, 1978]. The above transformations are shown schematically with the use of a commutative diagram in Figure 3.1

Hamilton's equations are equivalent to Lagrangean equations, when the former are the Legendre transform of the latter. The converse is also true: Hamilton's equations in phase space, unlike Lagrangean equations in configuration space, **are not invariant** to all possible transformations. Only canonical transformations preserve the form of Hamilton's equations in phase space. Canonical transformations of phase space are desirable because they can simplify Hamilton's equations further. However, carrying out

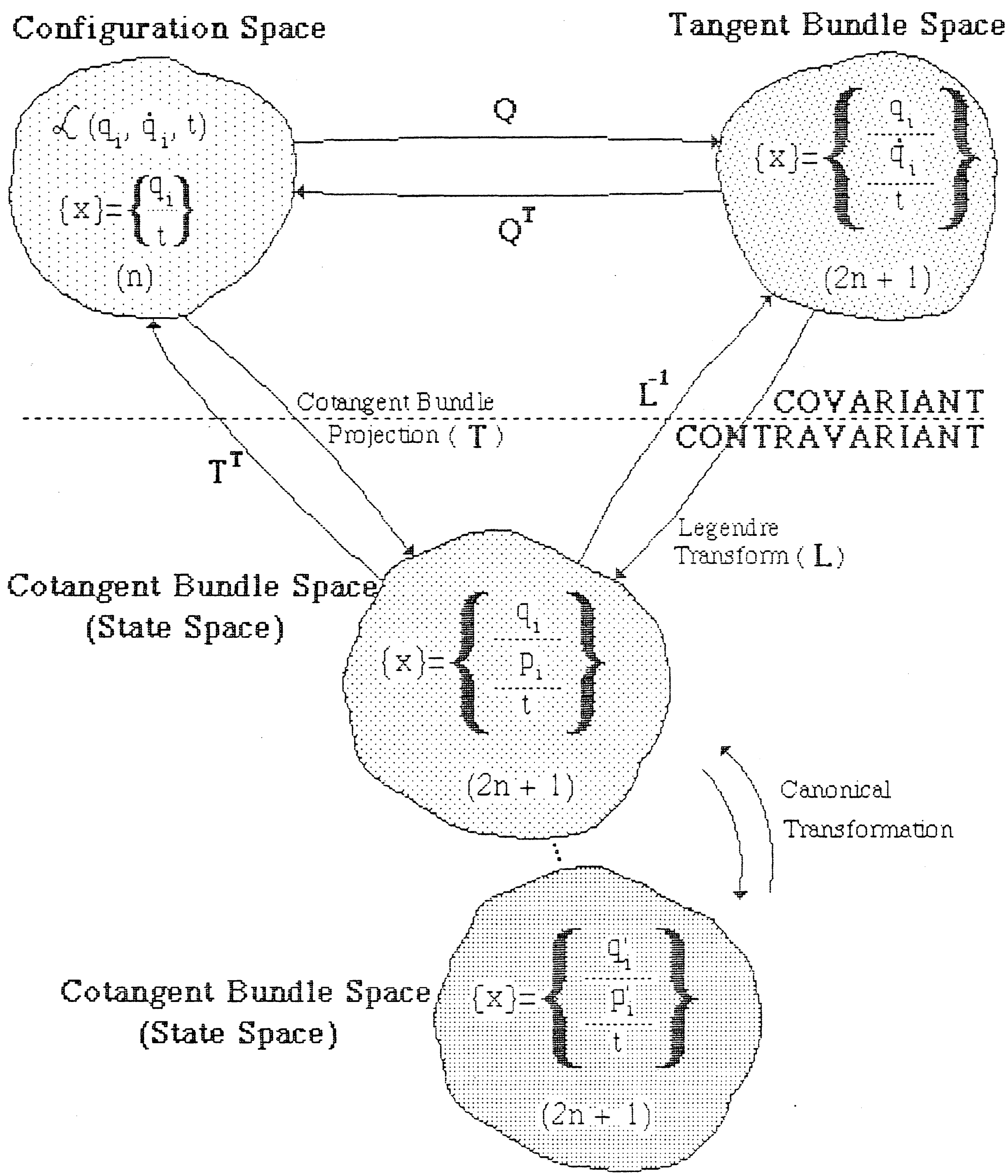


Fig. 3.1. Commutative diagram showing the relation of configuration, tangent bundle and cotangent bundle spaces. "Q" is a projection operation. Using canonical transformations the equations of motion in the state space (Hamilton's equations) can be simplified further. This can be achieved by tranformation of the state variables. Note that the state vector includes time t as independent variable.

such transformations† is not an easy task.

Any arbitrary infinitesimal changes δq_k in the generalised coordinates of a mechanical system that introduce small variations in the tangent space, compatible with the constraints of the system, are called ***virtual displacements*** [NEIMARK AND FUFAEV, 1972]. Virtual displacements are not true displacements of the system under consideration because they arise from the displacement of the coordinate system used. Therefore, there is no time intrinsically associated with them.

We are now in a position to present Hamilton's principle from which the equations of motion of a mechanical system can be derived. Let us suppose that external forces act on a mechanical system. As a result of this action, the system will be set into motion. Yet, of all possible paths, only one is followed for which no virtual displacements will be present. In other words, the available energy to the system is being spent in the most efficient way with no unnecessary displacements (virtual displacements). Thus we can say, that the integral of the virtual work in time over a path (trajectory) is equal to zero.

If T is the kinetic energy of a mechanical system and W is the work done by external forces acting on the system, then the virtual work in the interval $[t_1,t_2]$ will be zero.

$$\int_{t_1}^{t_2} \delta(T+W)dt = 0. \tag{3.1}$$

The above equation expresses ***Hamilton's principle*** in its most general form. In the special case of conservative external forces, i.e., forces that produce work independent of the path followed and dependent only on the end points, Hamilton's principle becomes

† *These transformations can be achieved with the use of a generating function, which is a solution to the Hamilton-Jacobi equation. Although Hamilton-Jacobi equation is a first order partial differential equation, most of the time it is unsolvable.*

$$\delta\int_{t_1}^{t_2} L dt = 0, \tag{3.2}$$

where L = T - V is the Lagrangean of the system and V = - W is the potential energy [MEIROVITCH, 1967].

3.6 Lagrangean Equations of Motion for Nonconservative Holonomic Systems

The Lagrangean equations of motion, for a rheonomic mechanical system with n degrees of freedom are [for detailed derivation see Appendix I]:

$$\partial T/\partial q_k - d[\partial T/\partial \dot{q}_k]/dt = Q_k, \qquad k = 1, 2,..., n \tag{3.3}$$

where Q_k are generalised forces acting on the system. Equations (3.3) were derived without assuming the character of the generalised forces Q_k. Generalised forces Q_k can be, either conservative, or nonconservative, or both. Furthermore, we can say that among the various kinds of forces acting on a particle of the system, it is possible to recognise a special type of friction force F arising from the motion of the particle in a viscous medium. This nonconservative force is assumed to be proportional to some power of velocity [MEIROVITCH, 1967]. Therefore, we can write that

$$Q_k = Q_k^c + Q_k^{nc} + F, \qquad k = 1, 2,..., n, \tag{3.4}$$

where Q_k^c and Q_k^{nc} are conservative and nonconservative (other than F) generalised forces, respectively. For the conservative forces Q_k^c we can write

$$Q_k^c = - \partial\Phi/\partial q_k, \qquad k = 1, 2,..., n, \tag{3.5}$$

where Φ is a potential. If D is a function that gives the amount of energy per unit time (units: ML^2T^{-3}) dissipated in the mechanical system, then

$$F = -\partial D/\partial \dot{q}_k, \qquad k = 1, 2, \ldots, n. \tag{3.6}$$

Function D is called the dissipation function and it is treated in detail in the next Chapter. Introducing equations (3.4), (3.5) and (3.6) into (3.3), yields

$$d[\partial T/\partial \dot{q}_k]/dt - \partial T/\partial q_k + \partial D/\partial \dot{q}_k = -\partial \Phi/\partial q_k + Q_k^{nc}, \quad k = 1, 2, \ldots, n. \tag{3.7}$$

Q_k^{nc} are forces stemming neither from a potential field, nor from friction. In addition, forces introduced by T and Φ are conservative. Since V is not a function of the generalised velocities (by definition V depends only on the position of the system, i.e., it depends only on the generalised coordinates), then

$$\partial T/\partial \dot{q}_k = \partial L/\partial \dot{q}_k, \qquad k = 1, 2, \ldots, n. \tag{3.8}$$

Thus, equation (3.7) becomes

$$d[\partial L/\partial \dot{q}_k]/dt - \partial L/\partial q_k + \partial D/\partial \dot{q}_k = -\partial \Phi/\partial q_k + Q_k^{nc}, \quad k = 1, 2, \ldots, n. \tag{3.9}$$

Equation (3.9) is the most general form of the Lagrangean equations of motion of any holonomic mechanical system, which exhibits dissipation properties, defined in an n-dimensional configuration space, when excited by external nonconservative forces. For continuous systems that consist of an infinite number of particles, the number of degrees of freedom becomes infinite and the number of equations (3.9) becomes infinite.

For the study of the response of the earth to ocean tide loading, the earth is considered to be continuous. However, we are interested in the deformation of the earth, rather than its motion in time. Therefore, the generalised coordinates are deformations and not positions. There are only three generalised coordinates that correspond to the three deformations, along the radial, N-S and E-W directions; they are dependent variables on position and time. Thus, the equations of deformation are substantially different in character from the equations of motion, although they are of the same form. Considering three generalised coordinates to represent the three displacements in a Cartesian coordinate system Oxyz, the equations of deformation can be written analogously to (3.9), as [see also BÅTH, 1968; p. 302, 303]

$$\partial[\partial L/\partial \dot{q}_k]/\partial t - \partial L/\partial q_k + \partial[\partial L/\partial q_{k(x)}]/\partial x + \partial[\partial L/\partial q_{k(y)}]/\partial y + \partial[\partial L/\partial q_{k(z)}]/\partial z + \partial D/\partial \dot{q}_k =$$
$$- \partial \Phi/\partial q_k + Q_k^{nc}, \qquad q_k = q_k(x, y, z, t), \qquad k = 1, 2, 3, \qquad (3.10)$$

where, for instance, $\partial q_{k(x)} = \partial q_k/\partial x$. The aggregate of the third, fourth and fifth terms of the above equation denote the divergence of a generalised force field Σ†, and we can write

$$\partial[\partial L/\partial \dot{q}_k]/\partial t - \partial L/\partial q_k + \mathrm{div}(\Sigma) + \partial D/\partial \dot{q}_k = - \partial \Phi/\partial q_k + Q_k^{nc},$$
$$k = 1, 2, 3. \qquad (3.11)$$

Equations (3.11) are used in Chapter 4 to derive the equations of deformation of the earth.

† *The nature of the generalised force field Σ is not immediately obvious. In the case of the study of the response of a deformable earth to external forces, the generalised coordinates are displacements and Σ becomes a strain field arising from the resistance of the earth to the deformation, i.e., Σ expresses some internal (to the earth) constraints, which depend upon the rheology of the earth. When we assume no deformations of the earth (or of any physical body), i.e., when we are interested in its motion in space, the divergence of Σ will vanish identically and the equations of deformation will become equations of motion.*

4

THE EQUATIONS OF DEFORMATION IN THE LAGRANGEAN CONFIGURATION MANIFOLD

The equations of deformation are developed in the lagrangean configuration manifold, following the analytical approach (lagrangean mechanics). An elementary volume element is followed in its motion, therefore, its position, velocity and acceleration are written as functions of time. In order to make the derivations of the equations of deformation easier, we consider energy densities per unit volume, rather than energies. The rheology of the earth is considered to be that of the standard linear viscoelastic solid. Moreover, the earth is regarded to be rotating, stratified, inhomogeneous, anisotropic, compressible, self-gravitating and rotating. Finally, ocean loading deformations are considered to be adiabatic, i.e., during a cycle of loading there is no heat transfer within the earth. This a valid assumption since ocean tide loading is a fast phenomenon when compared to convection, which may be the primary source for the thermal changes in the earth [O'CONNELL AND HAGER, 1980; PELTIER, 1980]. It is certain however, that in the seismic band an adiabatic thermal state is consistent with the observations [DAVIS, 1974].

In this chapter, it is shown that the response of the earth to a surface load can be described by three second order partial differential equations on a three dimensional configuration manifold. The state vector consists of three generalised coordinates, namely the vertical and horizontal displacements and the total loading potential. Furthermore, it is shown that the earth-load system is a holonomic and autonomous†.

† *Recall that a holonomic system is described by integrable equations of motion and by integrable constraints (boundary conditions). A system is autonomous when time does not appear explicitly in the equations of motion.*

For a nonrotating earth, expansions of the solution of the equations of motion into a series of spherical harmonics have been applied succesfully for the study of the free oscillations of the earth [ALTERMAN ET. AL., 1959] and the response of the earth to surface loading [LONGMAN, 1962; 1963]. Moreover, normal mode expansions have been applied to the seismic excitation of a nonrotating earth, to investigate the properties of the earth [SAITO, 1967; GILBERT, 1970]. In all the above cases, the solution of the equations of motion can be expressed as a sum of linear decoupled normal modes†.

Rotation in the equations of motion introduces Coriolis forces that couple the coefficients of the normal modes severely, i.e., any normal mode is a function of every other mode [DAHLEN AND SMITH, 1975]. WAHR [1979; 1981a; 1981b] developed an expansion formalism for a rotating earth which decouples completely the normal modes.

For the study of the effects of anisotropy, rotation and viscoelasticity on the response of the earth to ocean tide loading, we choose semidiurnal periods for simplicity reasons and we show that under the assumption of rotational symmetry in the properties of the earth, the equations of deformation on a rotating earth can be expanded completely into series of spherical harmonics, without applying the expansion of WAHR [1981a, 1981b]. Furthermore, for the study of the diurnal loading, the results from the semidiurnal loading can be extended into the diurnal band, allowing for corrections to be made, due to free-core nutation eigenfrequency, as presented by WAHR AND SASAO [1981]. For the study of the long period‡ ocean loading, we neglect the rotation of the earth altogether, under the assumption that long period oceanic loading is far from the rotational eigenmodes of the earth and thus it is practically unaffected by rotation.

† *"Decoupled normal modes" means that the coefficients of the normal modes are functions of the same (n,m) values of the spherical harmonic expansion of degree n and order m.*

‡ *Fortnightly, semiannual and annual periods.*

4.1 Response of a "Real Earth" to Ocean Tide Loading

The presence of inhomogeneities and anisotropy in the earth, as well as the earth's departure from perfect elasticity introduce complications in the evaluation of its response to surface loading. This is particularly evident when ocean tide loading deformations are evaluated at points very close to the shore [PAGIATAKIS AND VANÍČEK, 1985].

As it has been demonstrated in section 2.5, the global response of the earth departs from being elastic; this fact will be considered in the development of the equations of deformation. The first step will be to establish the constitutive law that governs the response. We could possibly consider the constitutive law as non-linear, which would complicate further the study. Fortunately, there is evidence which suggests that, for low strain amplitudes, the viscoelastic behaviour of the earth can be described by a linear constitutive law [BRENNAN, 1981; BRENNAN AND SMYLIE, 1981]. This evidence comes from laboratory experiments, as well as from observations of the real earth. Laboratory experiments are performed, in general, at a considerably lower temperature than that of the earth's mantle. However, low temperature and pressure experiments performed by BRENNAN [1981], indicate that stress-strain hysteresis loops produced for a number of different materials, which have been stressed with periods between 5 sec and 8.5 min and up to a strain amplitude of 10^{-6} (1 μstrain), have an elliptical shape. This suggests that the response can be described by linear viscoelastic constitutive relations.

BERCKHEMER ET. AL., [1979] reported linear transient creep experiments on mantle peridotite at a temperature of 1250°C for strain levels up to 5×10^{-5}. AGNEW [1981] analysed 5.7 years of strain tide records from Piñon Flat, California, in order to detect non-linearities of rock behaviour at tidal frequencies and strain levels. This analysis shows that the observations do not give any definite indication of the presence of non-linearity.

In the absence of such evidence there is no reason to reject the simpler hypothesis: the deformation of rock at small strains is considered to be linear [AGNEW, 1981].

The spatial wavelength of the harmonic load plays an important role in the evaluation of the loading deformation. For a very short wavelength†, the lithosphere appears effectively infinite horizontally, and can be considered as a half-space, i.e., the deformation takes place in the lithosphere alone. However, a very long wavelength load cannot be supported by the lithosphere alone. Consequently, the lower substrata exert reactive forces to support the load. It appears, therefore, that the stratification of the earth plays a significant role for a long wavelength load.

4.2 The Lagrangean Density Function

The ***lagrangean density function*** (lagrangean function per unit volume) for a point of the earth that is loaded by ocean tide waters can be written as [see Chapter 3 of this study]

$$L = T - V \tag{4.1}$$

where T is the ***kinetic energy density*** and V is the ***potential energy density*** at the point of interest. Equation (4.1) holds true for a perfectly elastic earth.

4.2.1 Kinetic Energy Density

Let us consider a compressible volume element in the earth, of mass m, defined by its position vector **r** with respect to the conventional terrestrial system‡. If the volume element undergoes deformations induced by a periodic load, its density will also change periodically, when mass conservation is assumed. If ρ_o is the density of the element at

† *Compared to the thickness of the lithosphere.*

‡ *Conventional terrestrial system is the system whose origin is at the centre of mass of the earth, the z-axis points to the Conventional International Origin (CIO), the xz-plane contains the mean Greenwich Observatory and the y-axis is selected to make the system right-handed [VANÍČEK AND KRAKIWSKY, 1986].*

equilibrium, its density ρ, at time t, will be given by

$$\rho(t) = \rho_o + \delta\rho(t). \tag{4.2}$$

Since the cubic dilatation Θ expresses the relative change of the density of the volume element, under the assumption of mass conservation, we can write that [EWING ET. AL., 1957]

$$\Theta = - \delta\rho / \rho_o. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\rho = \rho_o(1 - \Theta). \tag{4.4}$$

It is known from classical mechanics that the kinetic energy T of a mass m moving with velocity $\mathbf{v}$ is given by

$$T = m\mathbf{v}\mathbf{v} / 2. \tag{4.5}$$

The volume element under consideration has a translational velocity as well as rotational velocity, since it rotates with the earth. Thus,

$$\mathbf{v} = \dot{\mathbf{d}} + \boldsymbol{\Omega} \times \mathbf{d} \tag{4.6}$$

where $\mathbf{d}$ is the displacement vector and $\boldsymbol{\Omega}$ is the angular velocity of the earth. Combining (4.4), (4.5) and (4.6) the kinetic energy density can be written as

$$T = 1/2\rho_o(1 - \Theta)(\dot{\mathbf{d}} + \boldsymbol{\Omega} \times \mathbf{d})^2 \tag{4.7}$$

4.2.2 Elastic Potential Energy Density

Tidal work is mainly stored as elastic potential energy. The stored elastic energy density V_s can be written as [JEFFREYS, 1961]

$$V_s = \mathbf{C}_{ijkl}\mathbf{e}_{ij}\mathbf{e}_{kl} / 2, \tag{4.8}$$

where $\mathbf{C}_{ijkl}$ is a tensor of rank four which depends on the elasticity parameters of the earth and $\mathbf{e}_{ij}$ is the strain tensor. Equation (4.8) is known as ***strain energy function*** [LOVE, 1927]. Since $\mathbf{C}_{ijkl}$ is a tensor of rank four in 3-D space, it has $3^4 = 81$ independent elements. However, we can reduce the number of its independent elements, by imposing certain restrictions. For example, we can impose symmetry† in $\mathbf{C}_{ijkl}$ in the sense that $\mathbf{C}_{ijkl}$ remains unalterable when we interchange i and j or k and l ; the number of independent elements is then reduced to 36. Furthermore, we can accept that the tidal deformations are adiabatic, and $\mathbf{C}_{ijkl}$ becomes symmetric in the pair (i, j), interchanged with the pair (k, l). This assumption reduces the number of independent elements to 21 [BULLEN, 1975].

The form of $\mathbf{C}_{ijkl}$ tensor depends on the various symmetries in the properties of the earth. When examining the response of the earth to applied stresses, there exist various degrees of geometrical symmetries in the internal structure that can be introduced, which allow elastic properties in particular directions to become identical. When such symmetries are introduced, the number of independent elements of $\mathbf{C}_{ijkl}$ can be reduced to less than 21. The so called ***triclinic*** structure, which involves 21 independent elements, is the most general, whereas the ***isotropic*** structure involves only two independent elements [WASLEY, 1973; JURETSCHKE, 1974]. Of particular interest in this study is the ***lateral (transverse) isotropy*** for which the symmetry axis is the vertical (radial). Using Love's notation [LOVE, 1927; p. 160], the strain energy function (potential energy density) can be written as

† *These restrictions are related to the symmetries in the properties of the material.*

$$V_s = (1/2)\{A(e_{\theta\theta} + e_{\lambda\lambda})^2 + Ce_{rr}^2 + 2F(e_{\theta\theta} + e_{\lambda\lambda})e_{rr} + L(e_{\lambda r}^2 + e_{r\theta}^2) + N(e_{\theta\lambda}^2 - 4e_{\theta\theta}e_{\lambda\lambda})\}, \tag{4.9}$$

where A, C, F, L and N are elastic moduli†. A and C can be determined from measurements of the velocity of compressional waves propagating horizontally, as well as vertically. We have [DZIEWONSKI AND ANDERSON, 1981]

$$A = \rho v_{ph}^2, \quad C = \rho v_{pv}^2. \tag{4.10}$$

Similarly, N and L are functions of the shear wave velocity and F is a function of both, compressional and shear velocities:

$$N = \rho v_{sh}^2, \quad L = \rho v_{sv}^2, \quad F = \eta(A - 2L), \tag{4.11}$$

where ρ is density, v_{sh} , v_{sv} are the shear wave velocities in the horizontal and vertical directions respectively and η is a dimensionless parameter‡ [ANDERSON, 1961]. Numerical values for the above parameters can be obtained from the PREM. For the isotropic regions of the earth, we have [BULLEN AND BOLT, 1985]

$$A = C = \lambda + 2\mu, \quad L = N = \mu, \quad F = \lambda. \tag{4.12}$$

4.2.3 Buoyant Potential Energy Density

The earth can be regarded as being hydrostatically prestressed, where the hydrostatic pressure is induced by self-gravitation. The effect of self-gravitation on the deformation of the earth, that is induced by external forces (ocean tides) can be taken into account very easily if the earth is considered as fluid. Nonetheless, this assumption will hold equally well for a solid earth if the deformations are very small and they do not affect the

† *Do not confuse elasticity parameter* L *with the Lagrangean function.*

‡ *Note that for the dimensionless parameter* η *we use italic style to distinguish it from viscosity.*

continuity. Consequently, elements of the earth in the disturbed state will be surrounded by different density material from that of the undisturbed state. This will result in a ***buoyant*** or ***Archimedes force***, which will tend to restore the deformation to the equilibrium state. If the earth is allowed to be stratified and its density ρ increases with increasing depth, we can write that

$$d\rho/dr < 0, \tag{4.13}$$

where r is the radial distance from the earth's centre of mass. During the application of a point surface load, the earth material will experience displacement. This buoyant force density per unit volume can be written as [TOLSTOY, 1973]

$$\boldsymbol{F}_b = -\, \mathbf{g}\Delta\rho, \tag{4.14}$$

where $\mathbf{g}$ is gravity and $\Delta\rho$ is the difference in density between the surrounding and displaced elements. If the element moves in the radial direction by u, then

$$\Delta\rho = \rho(r) - \rho(r+\mathrm{u}). \tag{4.15}$$

Here, u is positive upwards. If the vertical displacement u is small, we can assume that $\Delta\rho$ is a linear function of r,

$$\Delta\rho = -\,\mathrm{u}\; d\rho/dr. \tag{4.16}$$

Combining (4.14) and (4.16), we obtain

$$\boldsymbol{F}_b = \mathrm{gu}\; d\rho/dr \tag{4.17}$$

and the corresponding potential energy density will be

$$V_1 = -\, F_b \mathrm{u}\,/\,2 \quad \text{or,} \quad V_1 = -\, 1/2\mathrm{gu}^2 d\rho/dr. \tag{4.18}$$

In the above derivations, **g** was considered positive downwards and $\boldsymbol{F_b}$ positive upwards. The negative sign in (4.18) shows that V_1 is positive when it results in an uplift (after unloading), as it intuitively should.

In the above, we considered an incompressible stratified earth. In the case of a compressible stratified earth, though, there is an additional term arising from the compression of the material itself, induced by the displacement field. In other words, the element of density ρ_o in the undisturbed state is compressed by the surrounding material, when displaced downwards. This results in a change in its density, when conservation of mass is assumed. If $\delta\rho$ is the density variation, the force density will be

$$F_g = - g(\rho - \rho_o), \tag{4.19}$$

where ρ is the density of the element in the compressed state. Combining (4.3) and (4.19), yields

$$F_g = g\rho_o\Theta \tag{4.20}$$

and the potential energy density will be

$$V_2 = - F_g u = - gu\rho_o\Theta. \tag{4.21}$$

Finally, the total potential energy density of gravitational origin can be written as

$$V_g = V_1 + V_2 = - gu(u/2\, d\rho/dr + \rho_o\Theta). \tag{4.22}$$

4.2.4 <u>The Lagrangean Density Function</u>

The elastic potential energy density given by (4.9) must be corrected for the buoyant potential energy density. Therefore, combining (4.1), (4.9) and (4.22) yields

$$L = T - V = 1/2\rho_o(1 - \Theta)(\dot{\mathbf{d}} + \boldsymbol{\Omega} \times \mathbf{d})^2 - \mathbf{C}_{ijkl}\, \mathbf{e}_{ij}\mathbf{e}_{kl} / 2 + gu(u/2\, d\rho/dr + \rho_o\Theta), \tag{4.23}$$

where u is the component of the displacement along the vertical. The above Lagrangean density function holds true for a conservative, stratified, compressible and rotating earth under the influence of self-gravitation. The first term of L, i.e., the kinetic energy density, gives rise to inertial and Coriolis forces, relative to the conventional terrestrial system. In general, the inertial forces are very small compared to the elastic forces at tidal frequencies. This is because the tidal frequencies are much lower than the free oscillation eigenfrequencies. However, in the derivation of the equations of deformation, both inertial and Coriolis forces are considered under the simplifying assumption that their component arising from the incremental changes in the density field (indirect effect) are negligibly small. As a consequence, dilatation becomes negligibly small and the Lagrangean density function can be written as

$$L = T - V \cong 1/2\rho_o(\dot{\mathbf{d}} + \boldsymbol{\Omega} \times \mathbf{d})^2 - \mathbf{C}_{ijkl}\, \mathbf{e}_{ij}\mathbf{e}_{kl} / 2 + gu(u/2\, d\rho/dr + \rho_o\Theta). \qquad (4.24)$$

4.2.5 <u>The Forcing Terms</u>

At this point, it is necessary to make a distinction between free and forced motion. In the case of free motion (i.e., free oscillations), after the disturbing force (e.g. earthquake) is removed, the earth regains its original shape by the action of elastic restoring forces, as well as from forces arising from the disturbed density field (indirect effect). Problems of this type are described by homogeneous second order partial differential equations in the displacements; non-zero solutions exist only for certain values of the forcing frequency (eigenfrequencies), i.e., we have an eigenvalue problem. For a forced motion, such as tidal deformation, the motion of the earth is described by nonhomogeneous partial differential equations of second order, the right-hand-side being the forcing term. For these types of problems we must consider a forcing term of pertinent frequency to solve for the displacements.

The deformation of the earth's surface due to ocean tide loading arises from the direct pressure of the tidal waters on the ocean floor, from the direct attraction of the ocean waters (Newtonian), as well as from the attraction of the indirect deformation (indirect effect). Since the direct pressure of the tidal waters will be taken into account in the boundary conditions (see section 4.5.3), the forcing terms account only for the direct attraction of the tidal waters and the indirect effect. Thus,

$$\boldsymbol{F}^{T} = \rho_o \nabla \Phi, \tag{4.25}$$

$$\Phi = \Phi^{p} + \Phi^{s}, \tag{4.26}$$

where, Φ is the total ocean tide deformation potential, Φ^{p} is the primary tidal potential due to the direct attraction and Φ^{s} is the secondary tidal potential due to the deformation.

4.3 The Dissipation Function for a Standard Linear Solid (SLS)

Following the notation found in [NOWICK AND BERRY, 1972; LAPWOOD AND USAMI, 1981], we can write the constitutive relation for the SLS as follows:

$$\mathbf{e}_{ij} + \tau_\sigma \dot{\mathbf{e}}_{ij} + \lambda J_U \tau_\sigma \dot{\Theta} \boldsymbol{\delta}_{ij} = J_R \boldsymbol{\tau}_{ij} + \tau_\sigma J_U \dot{\boldsymbol{\tau}}_{ij}, \tag{4.27}$$

where J_U and J_R are defined by (2.13) and τ_σ is the ***relaxation time*** given by

$$\tau_\sigma = \eta(J_R - J_U). \tag{4.28}$$

Dividing (4.27) by τ_σ and rearranging, we obtain

$$\dot{\mathbf{e}}_{ij} = (J_R/\tau_\sigma)\boldsymbol{\tau}_{ij} + J_U \dot{\boldsymbol{\tau}}_{ij} - \lambda J_U \dot{\Theta} \boldsymbol{\delta}_{ij} - (1/\tau_\sigma)\mathbf{e}_{ij}. \tag{4.29}$$

For the derivation of the dissipation function, we assume incompressibility†, i.e., Θ=0. Then, (4.29) becomes

$$\dot{\mathbf{e}}_{ij} = (J_R/\tau_\sigma)\boldsymbol{\tau}_{ij} + J_U \dot{\boldsymbol{\tau}}_{ij} - (1/\tau_\sigma)\mathbf{e}_{ij}, \tag{4.30}$$

Equation (4.30) is a linear ordinary differential equation of first order in $\mathbf{e}_{ij}$ and has a solution given by [REKTORYS, 1969; p.744]

$$\mathbf{e}_{ij} = \left\{ \int [(J_R/\tau_\sigma)\boldsymbol{\tau}_{ij} + J_U \dot{\boldsymbol{\tau}}_{ij}]\, exp(\textstyle\int dt/\tau_\sigma)\, dt \right\} exp(-\textstyle\int dt/\tau_\sigma). \tag{4.31}$$

Considering that the applied stress is periodic, with angular velocity ω then

$$\begin{aligned} \boldsymbol{\tau}_{ij} &= \boldsymbol{\tau}_{ij,o} \cos \omega t, \\ \dot{\boldsymbol{\tau}}_{ij} &= -\, \omega\, \boldsymbol{\tau}_{ij,o} \sin \omega t, \end{aligned} \tag{4.32}$$

† *WU AND PELTIER [1982] and WOLF [1985] have shown that the effect of compressibility on the relaxation of Maxwell continua is insignificant and can be neglected. We assume here that the same holds true for SLS.*

where $\tau_{ij,o}$ is the strain amplitude (at t = 0). Substituting (4.32) into (4.31) yields

$$\mathbf{e}_{ij} = \left\{ \int [(J_R/\tau_\sigma)\tau_{ij,o}\cos\omega t - J_U\omega\tau_{ij,o}\sin\omega t]\ exp(\int dt/\tau_\sigma)\ dt \right\} exp(-\int dt/\tau_\sigma). \tag{4.33}$$

Carrying out the integration [BOIS, 1961], (4.33) reduces to

$$\mathbf{e}_{ij} = \tau_{ij,o}(J_1\cos\omega t + J_2\sin\omega t). \tag{4.34}$$

J_1 and J_2 are compliances given by

$$J_1 = (J_R + \tau_\sigma^2\omega^2 J_U)(1 + \tau_\sigma^2\omega^2)^{-1}, \tag{4.35}$$

$$J_2 = \tau_\sigma\omega(J_R - J_U)(1 + \tau_\sigma^2\omega^2)^{-1}. \tag{4.36}$$

Quantities J_1 and J_2 are known as the real and imaginary parts of the ***complex compliance*** of the general linear solid [FLÜGGE, 1975]. Equations (4.35) and (4.36) are often called ***Debye equations*** [NOWICK AND BERRY, 1972]. In equation (4.34), the first term, being a function of J_1, describes purely elastic behaviour and the second, being a function of J_2, describes viscoelastic behaviour.

The work done by the stresses can be written as [FLÜGGE, 1975]

$$W = \int \dot{\mathbf{e}}_{ij}\,\tau_{ij}\,dt, \tag{4.37}$$

where the summation convention applies for the repeated indices. Differentiating (4.34) with respect to time and considering (4.32), (4.37) becomes

$$W = \sum_{i,j} \left\{ -J_1\omega\tau_{ij,o}^{\ 2}\int \sin\omega t\cos\omega t\,dt + J_2\omega\tau_{ij,o}^{\ 2}\int \cos^2\omega t\,dt \right\}, \tag{4.38}$$

where the implied summation in (4.38) was interchanged with the integration. Integration of (4.38) in the interval $[0, 2\pi/\omega]$† reveals that the first integral becomes zero.

† *What we are interested in here is the average energy (stored or dissipated) during a cycle of straining and not the energy as a function of time. The reason is that average stored and dissipated energies are used in the definition of quality factor Q (see Chapter 2 of this work). Most importantly, using average energy over a cycle of straining, we avoid the problem of having time dependent coefficients in the equations of deformation.*

The integration of the second term of (4.38) in the interval $[0, 2\pi/\omega]$ gives

$$W' = \pi J_2 \sum_{i,j} \tau_{ij,o}{}^2, \tag{4.39}$$

which is the amount of dissipated energy in a full cycle of straining and has units of $ML^{-2}T^{-2}$. Further observation of (4.39) reveals that W' is a function of frequency, proportional to $\tau_\sigma\omega(1+\tau_\sigma{}^2\omega^2)^{-1}$. Any function having this property is called a ***Debye peak*** [NOWICK AND BERRY, 1972].

For the evalution of the dissipation terms of the equations of deformation, (4.39) is not useful. We need to express W' as a function of the strain rate. The average dissipated energy D per unit time can be obtained from (4.39) by dividing W' by $2\pi/\omega$,

$$D = 1/2J_2\omega \sum_{i,j} \tau_{ij,o}{}^2, \tag{4.40}$$

where D has units of $ML^{-2}T^{-3}$ and expresses the amount of the average dissipated energy density per unit time. Substituting (4.32) into (4.30) and solving for $\tau_{ij,o}$, we obtain

$$\tau_{ij,o} = (\dot{\mathbf{e}}_{ij} + \mathbf{e}_{ij}/\tau_\sigma)/(J_R/\tau_\sigma \cos\omega t - J_U\omega\sin\omega t). \tag{4.41}$$

Setting $t = 0^\dagger$, we obtain

$$\tau_{ij,o} = (\dot{\mathbf{e}}_{ij}\tau_\sigma + \mathbf{e}_{ij})/J_R \tag{4.42}$$

and (4.40) becomes

$$D = 1/2(J_2\omega/J_R{}^2)\sum_{i,j} (\dot{\mathbf{e}}_{ij}\tau_\sigma + \mathbf{e}_{ij})^2, \tag{4.43}$$

† *Equation (4.41) is valid for any t. For convenience we choose t=0.*

In (4.43) we need to evaluate the complex compliance J_2. Since J_2 reflects the relaxation process, it is a function of the thermodynamic state of the earth, as well as, function of the quality factor Q. Let us start with the equation [ANDERSON AND HART, 1978a]

$$Q^{-1}(\omega) = \{C_\infty / C_o - 1\}\, \omega\tau_\sigma (1 + \tau_\sigma{}^2\omega^2)^{-1}, \tag{4.44}$$

which relates a frequency dependent Q and relaxation time τ_σ. Here, C_∞ and C_o are the high and low frequency elastic wave velocities. We can see that (4.44) is valid for a standard linear solid [cf: (4.35) and (4.36)], i.e., $Q^{-1}(\omega)$ is a Debye peak. We accept the simple hypothesis of a grain-boundary relaxation model [NOWICK AND BERRY, 1972] for which the ratio C_∞ / C_o can be estimated from [ZENER, 1941]†

$$C_\infty / C_o = \sqrt{\{(35 - 20\nu) / (14 + 10\nu)\}}, \tag{4.45}$$

where ν is the Poisson's ratio. Combining (4.36) and (4.44) we obtain

$$J_2 = \Delta J / Q \{C_\infty / C_o - 1\}^{-1}, \tag{4.46}$$

where $\Delta J = J_R - J_U$‡. From (4.28) we have

$$\Delta J = \tau_\sigma / \eta \tag{4.47}$$

and the calculation of J_2 reduces to the calculation of relaxation time, provided that depth profiles of Q, η and ν are available and C_∞ / C_o known. Relaxation time can be calculated from the Arrhenius equation (2.23)

† *Zener's theory assumes spherical elastic grains; under this assumption, eqn. (4.45) holds true. However, O'CONNELL AND BUDIANSKY [1974] assume more realistic grain geometries, such as dodehahedrons for the evaluation of the ratio C_∞ / C_o.*

‡ *ΔJ is known as relaxation strength.*

4.4 The Equations of Deformation

It is convenient to derive the equations of deformation in a spherical coordinate system (r, θ, λ), where $\theta = \pi/2-\phi$ is the colatitude of the point of interest and λ the east longitude. The strain tensor $\mathbf{e}_{ij}$ can be written as a function of spherical coordinates and displacements u, v, w along the vertical, meridian and parallel respectively. These relations are [LOVE, 1927, p. 56; BEN-MENAHEM AND SINGH, 1981]

$$\left.\begin{aligned}
&e_{rr} = \partial_r u, \qquad e_{\theta\theta} = r^{-1}(\partial_\theta v + u), \qquad e_{\lambda\lambda} = r^{-1}\,[(1/\sin\theta)\partial_\lambda w + u + v\cot\theta],\\
&e_{\theta\lambda} = e_{\lambda\theta} = r^{-1}\,[\partial_\theta w - w\cot\theta + (1/\sin\theta)\partial_\lambda v],\\
&e_{\lambda r} = e_{r\lambda} = (1/r\sin\theta)\partial_\lambda u + \partial_r w - r^{-1}w,\\
&e_{r\theta} = e_{\theta r} = \partial_r v - r^{-1}v + r^{-1}\partial_\theta u,\\
&\Theta = e_{rr} + e_{\theta\theta} + e_{\lambda\lambda} = \partial_r u + r^{-1}[\partial_\theta v + 2u + (1/\sin\theta)\partial_\lambda w + v\cot\theta].
\end{aligned}\right\} \quad (4.48)$$

In the above equations, ∂_{*}¥ denotes "partial derivative of ¥ with respect to *." Furthermore, the divergence of the generalised force density filed $\boldsymbol{\Sigma}$ in spherical coordinates is given by [COURANT AND HILBERT, 1970; VOL 1, p. 224].

$$\mathrm{div}\boldsymbol{\Sigma} = \partial_r\Sigma^1 + \partial_\theta\Sigma^2 + \partial_\lambda\Sigma^3 + 2\Sigma^1/r + \Sigma^2\cot\theta, \qquad (4.49)$$

where $\Sigma^1, \Sigma^2, \Sigma^3$ are the contravariant components of $\boldsymbol{\Sigma}$. Considering that u, v and w are the dependent variables (generalised coordinates) and t, r, θ and λ are the independent variables, we obtain from (3.11) and (4.49) the equations of deformation

$$\partial_t[\partial L/\partial \dot{u}] - \partial_u L + \partial_r[\partial L/\partial u_r] + \partial_\theta[\partial L/\partial u_\theta] + \partial_\lambda[\partial L/\partial u_\lambda] + 2/r\,[\partial L/\partial u_r] + [\partial L/\partial u_\theta]\cot\theta + \partial D/\partial \dot{u}$$
$$= \rho_o \partial_r \Phi$$

$$\partial_t[\partial L/\partial \dot{v}] - \partial_v L + \partial_r[\partial L/\partial v_r] + \partial_\theta[\partial L/\partial v_\theta] + \partial_\lambda[\partial L/\partial v_\lambda] + 2/r\,[\partial L/\partial v_r] + [\partial L/\partial v_\theta]\cot\theta + \partial D/\partial \dot{v}$$
$$= r^{-1}\rho_o \partial_\theta \Phi$$

$$\partial_t[\partial L/\partial \dot{w}] - \partial_w L + \partial_r[\partial L/\partial w_r] + \partial_\theta[\partial L/\partial w_\theta] + \partial_\lambda[\partial L/\partial w_\lambda] + 2/r\,[\partial L/\partial w_r] + [\partial L/\partial w_\theta]\cot\theta$$
$$+ \partial D/\partial \dot{w} = (r\sin\theta)^{-1}\rho_o \partial_\lambda \Phi, \tag{4.50}$$

The next step is to evaluate the individual terms of (4.50), taking into account the Lagrangean L, given by (4.23) and the dissipation function D given by (4.43). In the above equations (4.50) the first two terms give the inertial and Coriolis force densities. We start with the evaluation of these terms using vector notation:

$$\begin{aligned}
\boldsymbol{F}_{i+c} &= \partial_t[\partial L/\partial \dot{\mathbf{d}}] - \partial L/\partial \mathbf{d} \\
&= \partial_t[\rho_o(\dot{\mathbf{d}} + \boldsymbol{\Omega} \times \mathbf{d})] - \rho_o(\dot{\mathbf{d}} + \boldsymbol{\Omega} \times \mathbf{d}) \times \boldsymbol{\Omega} \\
&= \rho_o(\ddot{\mathbf{d}} + \boldsymbol{\Omega} \times \dot{\mathbf{d}}) - \rho_o(-\boldsymbol{\Omega} \times \dot{\mathbf{d}} - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{d}) \\
&= \rho_o\ddot{\mathbf{d}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{d}}
\end{aligned} \tag{4.51}$$

The first term in (4.51) is the inertial force density and the second term is the Coriolis force density. The components of $\boldsymbol{F}_{i+c}$ along the vertical, north-south and east-west directions, respectively, are [LAPWOOD AND USAMI, 1981]

$$\boldsymbol{F}_{i+c} = \begin{cases} \rho_o\ddot{u} - 2\Omega\rho_o\sin\theta\,\dot{w}, \\ \rho_o\ddot{v} - 2\Omega\rho_o\cos\theta\,\dot{w}, \\ \rho_o\ddot{w} + 2\Omega\rho_o[\cos\theta\,\dot{v} + \sin\theta\,\dot{u}]. \end{cases} \tag{4.52}$$

Evaluating the rest of the terms of the equations of deformation (4.50) we obtain

$$
\left.
\begin{aligned}
\partial_u L &= -\,\partial_u V_s + gud_r\rho + g\rho_o\Theta + 2g\rho_o u r^{-1} = -\,\partial V_s/\partial e_{\theta\theta}\,\partial_u e_{\theta\theta} \\
&\quad -\,\partial V_s/\partial e_{\lambda\lambda}\,\partial_u e_{\lambda\lambda} + gud_r\rho + g\rho_o\Theta + 2g\rho_o u r^{-1} \\
&= -r^{-1}\,[2(A-N)(e_{\theta\theta}+e_{\lambda\lambda}) + 2Fe_{rr}] + gud_r\rho_o + g\rho_o\Theta + 2g\rho_o u r^{-1} \\[2ex]
\partial_v L &= -\,\partial_v V_s = -\,[\,\partial V_s/\partial e_{\lambda\lambda}\,\partial_v e_{\lambda\lambda} + \partial V_s/\partial e_{r\theta}\,\partial_v e_{r\theta}\,] \\
&= -r^{-1}\,\left\{\cot\theta\,[A(e_{\theta\theta}+e_{\lambda\lambda}) + Fe_{rr} - 2Ne_{\theta\theta}\,] + Le_{r\theta}\right\} \\[2ex]
\partial_w L &= -\,\partial_w V_s = -\,[\,\partial V_s/\partial e_{\theta\lambda}\,\partial_w e_{\theta\lambda} + \partial V_s/\partial e_{r\lambda}\,\partial_w e_{r\lambda}] \\
&= -r^{-1}\,[N\,e_{\theta\lambda}\cot\theta + L\,e_{r\lambda}] \\[2ex]
\partial_r[\partial L/\partial u_r] &= -\partial_r[\partial V_s/\partial e_{rr} - g\rho_o u] = -\,\partial_r[Ce_{rr} + F\,(e_{\theta\theta}+e_{\lambda\lambda})] + \rho_o\partial_r[gu] + gud_r\rho_o \\[2ex]
\partial_\theta\,[\partial L/\partial u_\theta\,] &= -\,\partial_\theta\,[\partial V_s/\partial e_{r\theta}\,\partial e_{r\theta}/\partial u_\theta] = -\,(L/r)\,\partial_\theta e_{r\theta} \\[2ex]
\partial_\lambda\,[\partial L/\partial u_\lambda] &= -\partial_\lambda\,[\partial V_s/\partial e_{r\lambda}\,\partial e_{r\lambda}/\partial u_\lambda] \\
&= -\,\partial_\lambda\,[L\,e_{r\lambda}(2r\sin\theta\,)^{-1}] = -\,L/(r\sin\theta\,)\,\partial_\lambda e_{r\lambda} \\[2ex]
\partial_r[\partial L/\partial v_r] &= -\,\partial_r[\partial V_s/\partial e_{r\theta}\,\partial e_{r\theta}/\partial v_r] = -\,\partial_r\,[L\,e_{r\theta}] \\[2ex]
\partial_\theta\,[\partial L/\partial v_\theta\,] &= -\,\partial_\theta\,[\partial V_s/\partial\,e_{\theta\theta}\;\partial e_{\theta\theta}\,/\partial v_\theta - g\rho_o u] \\
&= -\,(1/r)\,\partial_\theta\,[A\,(e_{\theta\theta}+e_{\lambda\lambda}) + Fe_{rr} - 2N\,e_{\lambda\lambda} - g\rho_o u] \\[2ex]
\partial_\lambda[\partial L/\partial v_\lambda] &= -\,\partial_\lambda[\partial V_s/\partial\,e_{\theta\lambda}\,\partial e_{\theta\lambda}/\partial v_\lambda] = -\,(N/r\sin\theta\,)\,\partial_\lambda e_{\theta\lambda}
\end{aligned}
\right\}
\qquad (4.53)
$$

$$\partial_r[\partial L/\partial w_r] = -\,\partial_r[\partial V_s/\partial e_{r\lambda}\ \partial e_{r\lambda}/\partial w_r] = -\,\partial_r[L\, e_{r\lambda}]$$

$$\partial_\theta[\partial L/\partial w_\theta] = -\,\partial_\theta[\partial V_s/\partial e_{\theta\lambda}\ \partial e_{\theta\lambda}/\partial w_\theta] = -\,(N/r)\,\partial_\theta e_{\theta\lambda}$$

$$\begin{aligned}\partial_\lambda[\partial L/\partial w_\lambda] &= -\,\partial_\lambda[\partial V_s/\partial e_{\lambda\lambda}\ \partial e_{\lambda\lambda}/\partial w_\lambda - g\rho_o u] \\ &= -\,(r\sin\theta)^{-1}\partial_\lambda[A\,(e_{\theta\theta}+e_{\lambda\lambda}) + F e_{rr} - 2N\, e_{\theta\theta} - g\rho_o u]\end{aligned}$$

For the derivation of the dissipation terms we have that (because of no bulk dissipation)

$$\partial D/\partial\dot{u} = (\partial D/\partial\dot{e}_{\theta\theta})(\partial\dot{e}_{\theta\theta}/\partial\dot{u}) + (\partial D/\partial\dot{e}_{\lambda\lambda})(\partial\dot{e}_{\lambda\lambda}/\partial\dot{u}) = 0 \tag{4.54}$$

Furthermore,

$$\partial D/\partial\dot{v} = (\partial D/\partial\dot{e}_{\lambda\lambda})(\partial\dot{e}_{\lambda\lambda}/\partial\dot{v}) + (\partial D/\partial\dot{e}_{r\theta})(\partial\dot{e}_{r\theta}/\partial\dot{v}) \tag{4.55}$$

Once again, if we consider that only the shear stresses contribute to the dissipation, the first term in (4.55) will be zero. Therefore,

$$\partial D/\partial\dot{v} = -\,J_2\omega\tau_\sigma/(J_R{}^2 r)(\tau_\sigma\,\dot{e}_{r\theta} + e_{r\theta}). \tag{4.56}$$

Similarly,

$$\partial D/\partial\dot{w} = (\partial D/\partial\dot{e}_{\theta\lambda})(\partial\dot{e}_{\theta\lambda}/\partial\dot{w}) + (\partial D/\partial\dot{e}_{r\lambda})(\partial\dot{e}_{r\lambda}/\partial\dot{w}) \tag{4.57}$$

If we consider only spheroidal deformation (deformation that has a component in the radial direction), the radial component of the curl of the displacement ($\nabla\times\mathbf{d}$) vanishes, [ALTERMAN ET. AL., 1959]. This is equivalent to $e_{\theta\lambda} = \dot{e}_{\theta\lambda} = 0$ [BEN-MENAHEM AND SINGH, 1981; eqn. A.124] and (4.57), after differentiation, becomes

$$\partial D/\partial\dot{w} = -\,J_2\omega\tau_\sigma/(J_R{}^2 r)(\tau_\sigma\,\dot{e}_{r\lambda} + e_{r\lambda}). \tag{4.58}$$

Neglecting the small term $2g\rho_o u r^{-1}$ the equations of deformation become

$$\left.\begin{aligned}
&\rho_o\ddot{u} - 2\Omega\rho_o\sin\theta\,\dot{w} - \rho_o\partial_r\Phi + \rho_o\partial_r[gu] - g\rho_o\Theta \\
&\quad + (2/r)\{(A\text{-}N\text{-}F)(e_{\theta\theta} + e_{\lambda\lambda}) + (F\text{-}C)\Theta\} - \partial_r[Ce_{rr} + F(e_{\theta\theta} + e_{\lambda\lambda})] \\
&\quad - (L/r)[\partial_\theta e_{r\theta} + \sin^{-1}\theta\,\partial_\lambda e_{r\lambda} + e_{r\theta}\cot\theta] = 0, \\
&\rho_o\ddot{v} - 2\Omega\rho_o\cos\theta\,\dot{w} - r^{-1}\rho_o\partial_\theta\Phi \\
&\quad - \partial_r[Le_{r\theta}] - (1/r)\partial_\theta[A(e_{\theta\theta} + e_{\lambda\lambda}) + Fe_{rr} - 2Ne_{\lambda\lambda} - g\rho_o u] \\
&\quad - N(r\sin\theta)^{-1}\partial_\lambda e_{\theta\lambda} \\
&\quad - (N/r)[2\cot\theta\,(e_{\theta\theta} - e_{\lambda\lambda})] - (3L/r)\,e_{r\theta} - J_2\omega\tau_\sigma/(J_R{}^2 r)(\tau_\sigma\dot{e}_{r\theta} + e_{r\theta}) = 0, \\
&\rho_o\ddot{w} + 2\Omega\rho_o[\cos\theta\,\dot{v} + \sin\theta\,\dot{u}] - (r\sin\theta)^{-1}\rho_o\partial_\lambda\Phi - \partial_r(Le_{r\lambda}) - (N/r)\partial_\theta e_{\theta\lambda} \\
&\quad - (1/r\sin\theta)\partial_\lambda[A(e_{\theta\theta} + e_{\lambda\lambda}) + Fe_{rr} - 2Ne_{\theta\theta} - g\rho_o u] - (3L/r)\,e_{r\lambda} \\
&\quad - (2N/r)e_{\theta\lambda}\cot\theta - J_2\omega\tau_\sigma/(J_R{}^2 r)\,(\tau_\sigma\dot{e}_{r\lambda} + e_{r\lambda}) = 0.
\end{aligned}\right\} \tag{4.59}$$

Introducing the total loading potential Φ, we can see that by virtue of secondary potential Φ^s being unknown, Φ is unknown and must be transferred to the left-hand-side of the equations of deformation. Thus, the equations of deformation become homogeneous. However, as we will see in detail in the next chapter, this does not imply that we are faced with an eigenvalue problem as we should not be.

Equations (4.59) are three simultaneous, partial differential equations of second order in the dependent variables u, v, w and Φ. In order to be able to solve the above system, we need one additional equation. This fourth equation is Poisson's equation given by [LONGMAN, 1962]

$$\Delta\Phi = 4\pi G[\rho_o\Theta + u\partial_r\rho_o]. \tag{4.60}$$

For an elastic earth, the last terms in the second and third equations of deformation (4.59) become zero. Taking also into account (4.12), the equations of deformation of an elastic, isotropic and nonrotating earth reduce to

$$\left.\begin{aligned}
&\rho_o\ddot{u} - \rho_o\partial_r\Phi^s - g\rho_o\Theta + \rho_o\partial_r[gu] - \partial_r[\lambda\Theta + 2\mu e_{rr}] - (\mu/r)\partial_\theta e_{r\theta} \\
&\qquad - (\mu/r\sin\theta)\partial_\lambda e_{r\lambda} - (\mu/r)[(4e_{rr} - 2e_{\theta\theta} - 2e_{\lambda\lambda}) + e_{r\theta}\cot\theta] = \rho_o\partial_r\Phi^p, \\[2ex]
&\rho_o\ddot{v} - (\rho_o/r)\partial_\theta\Phi^s - \partial_r\mu e_{r\theta} - (1/r)\partial_\theta[- g\rho_o u + \lambda\Theta + 2\mu e_{\theta\theta}] \\
&\quad - \mu(r\sin\theta)^{-1}\partial_\lambda e_{\theta\lambda} \\
&\quad - (\mu/r)[2\cot\theta(e_{\theta\theta} - e_{\lambda\lambda}) + 3e_{r\theta}] = r^{-1}\rho_o\partial_\lambda\Phi^p, \\[2ex]
&\rho_o\ddot{w} - (\rho_o/r\sin\theta)\partial_\lambda\Phi^s - \partial_r\mu e_{r\lambda} - (\mu/r)\partial_\theta e_{\theta\lambda} \\
&\quad - (r\sin\theta)^{-1}\partial_\lambda[- g\rho_o u + \lambda\Theta + 2\mu e_{\lambda\lambda}] \\
&\quad - (3\mu/r)e_{r\lambda} - (2\mu/r)e_{\theta\lambda}\cot\theta = (r\sin\theta)^{-1}\rho_o\partial_\lambda\Phi^p,
\end{aligned}\right\} \quad (4.61)$$

which in the absence of the forcing terms reduce to the equations of free oscillations of the earth as given by ALTERMAN ET. AL., [1959]. This gives us a check on the basic formulation of the loading deformations.

4.5 Boundary Conditions

4.5.1 Continuity of the Total Loading Potential

For the formulation of the boundary conditions for the total loading potential, we follow LONGMAN [1962] and PEKERIS [1978]. The primary loading potential $\Phi_{nm}{}^{p}$ of degree n and order m, generated by a surface harmonic distribution σ_{nm}, of the tidal waters, can be written as [LAMB, 1945, p. 305; PEKERIS, 1978]

$$ {}^{e}\Phi_{nm}{}^{p} = 4\pi GR\,(2n+1)^{-1}\,(R/r)^{n+1}\,\sigma_{nm}, \tag{4.62} $$

$$ {}^{i}\Phi_{nm}{}^{p} = 4\pi GR\,(2n+1)^{-1}\,(r/R)^{n}\,\sigma_{nm}, \tag{4.63} $$

where R is the radius of the earth in the undisturbed state. Pre-superscripts "e" and "i" denote the external and internal potential respectively, with respect to the surface. The primary potential is harmonic everywhere, outside and inside the surface of the earth, thus,

$$ \nabla^2\Phi_{nm}{}^{p} = 0, \qquad r \neq R, \tag{4.64} $$

where ∇ is the Hamilton nabla operator. The secondary loading potential $\Phi_{nm}{}^{s}$ satisfies Poisson's equation inside the earth, thus

$$ \nabla^2\,{}^{i}\Phi_{nm}{}^{s} = 4\pi G(\rho_o\Theta_{nm} + u_{nm}\partial_r\rho_o), \tag{4.65} $$

where u_{nm} is the radial displacement of degree n and order m, due to load. Taking into account that $\Phi_{nm}=\Phi_{nm}{}^{p}+\Phi_{nm}{}^{s}$ and equation (4.64), (4.65) becomes

$$ \nabla^2\,{}^{i}\Phi_{nm} = 4\pi G(\rho_o\Theta_{nm} + u_{nm}\partial_r\rho_o). \tag{4.66} $$

At any internal boundary in the deformed state (r = c + u), the total loading potential, as well as its derivatives with respect to r must be continuous:

$$ {}^{-}\Phi_{nm} = {}^{+}\Phi_{nm}, \qquad r = c + u_{nm} \tag{4.67} $$

$$ \partial_r {}^{-}\Phi_{nm} = \partial_r {}^{+}\Phi_{nm}, \qquad r = c + u_{nm} \tag{4.68} $$

where the pre-superscripts "-" and "+" denote inside and outside the boundary surface respectively.

When the earth is at the deformed state, its surface will be at $r = R + u_{nm}$. However, the value of gravity inside the surface $r = R$ is equal to the value of gravity outside $r = R + u_{nm}$, i.e., it is equal to the value of gravity in the air, plus an increment arising: **a)** from the presence of the material heaped up over the surface $r = R$, and **b)** from the presence of the surface harmonic distribution of the tidal waters. Thus,

$$ \partial_r {}^{i}\Phi_{nm} = \partial_r {}^{e}\Phi_{nm} + 4\pi G \rho_o u_{nm} + 4\pi G \sigma_{nm}, \quad r = R, \tag{4.69} $$

or by rearranging:

$$ \partial_r {}^{i}\Phi_{nm} - \partial_r {}^{e}\Phi_{nm} = 4\pi G(\rho_o u_{nm} + \sigma_{nm}), \qquad r = R, \tag{4.70} $$

Furthermore, the total loading potential outside $r = R + u_{nm}$ is a harmonic function. Therefore,

$$ \nabla^2 \Phi_{nm} = 0, \qquad r > R + u_{nm}. \tag{4.71} $$

Equation (4.71) is satisfied when [PEKERIS, 1978]

$$ {}^{e}\Phi_{nm} = 4\pi G R\, (2n + 1)^{-1}\, (R/r)^{n+1}\, \sigma_{nm}. \tag{4.72} $$

Since differentiation of (4.72) with respect to r gives

$$ \partial_r {}^{e}\Phi_{nm} = -\, R^{-1}(n+1)\, {}^{e}\Phi_{nm}, \qquad r = R, \tag{4.73} $$

equation (4.70) becomes

$$\partial_r{}^i\Phi_{nm} + R^{-1}(n+1)^e\Phi_{nm} = 4\pi G(\rho_o u_{nm} + \sigma_{nm}), \quad r = R, \tag{4.74}$$

Since we must have continuity of the total loading potential at the surface $r = R$, then ${}^i\Phi_{nm} = {}^e\Phi_{nm}$, and equation (4.74) becomes

$$\partial_r\Phi_{nm} + R^{-1}(n+1)\Phi_{nm} = 4\pi G(\rho_o u_{nm} + \sigma_{nm}), \qquad r = R, \tag{4.75}$$

Equation (4.75) is one of the boundary conditions of the equations of deformation.

4.5.2 <u>Continuity of the Total Loading Potential at Internal Boundaries</u>

We can use (4.70) to express the continuity of the total loading potential at any internal boundary, the only difference being in the absence of σ_{nm}. Thus,

$$\partial_r{}^-\Phi_{nm} - \partial_r{}^+\Phi_{nm} = 4\pi G({}^-\rho_o - {}^+\rho_o)u_{nm}, \tag{4.76}$$

where ${}^-\rho_o$ is the density inside the boundary and ${}^+\rho_o$ the density outside the boundary.

4.5.3 <u>State of Stress at the Deformed Surface</u>

At the **deformed** surface of the earth the pressure of the tidal waters $-g_o\sigma_{nm}$, introduces only a normal stress, which is equal to the pressure term of the first of the equations of deformation. Therefore,

$$Ce_{rr} + F(e_{\theta\theta} + e_{\lambda\lambda}) = -g_o\sigma_{nm}. \tag{4.77}$$

Equation (4.77) is a first order approximation since $g_o\sigma_{nm}$ is applied at the deformed surface $r = R+u$ and not at $r = R$.

All the tangential stresses vanish identically at $r = R$†. This is a valid approximation when the loading mass is water. Therefore, the pressure terms of the second and third equations of deformation become zero. Thus,

$$e_{r\theta} = e_{r\lambda} = 0, \qquad r = R. \tag{4.78}$$

We must note here that boundary conditions (4.76) and (4.77) are non-homogeneous equations in the sense that they contain the known function σ_{nm} of the tidal water distribution, i.e. they depend on the forcing term.

4.6 Expansion of the Equations of Deformation into Spherical Harmonics

The independent variables in the equations of deformation (4.59) are: time t, angular velocity of deformation ω and the spherical coordinates (r, θ, λ). Variable ω can be fixed to 2 cycles/day, 1 cycle/day, or lower angular velocities, when the response of the earth to these frequencies is desired. The solution of the equations of deformation will be attempted by the method of separation of variables.

Assuming that the displacements u_{nm}, v_{nm} and w_{nm} of degree n and order m, as well as the total loading potential Φ_{nm}, can be expanded into the series of spherical harmonics $Y_{nm}(\theta, \lambda)$, we can write for spheroidal deformations:

† *This is an excellent approximation when considering ocean tides in the open ocean. However, MERRIAM [1986] showed that tangential stresses caused by ocean tide loading of the continental slope, can generate strain tides of the order of a few percent of the total tidal strain. This is comparable in magnitude to perturbations from local topography, cavity and structural effects.*

$$
\begin{aligned}
u_{nm} &= U_{nm}(r,t)\, Y_{nm}(\theta,\lambda),\\
v_{nm} &= V_{nm}(r,t)\ \partial_\theta Y_{nm}(\theta,\lambda),\\
w_{nm} &= V_{nm}(r,t)\ \sin^{-1}\theta\ \partial_\lambda Y_{nm}(\theta,\lambda),\\
\Phi_{nm} &= \Psi_{nm}(r,t)\ Y_{nm}(\theta,\lambda),
\end{aligned}
\tag{4.79}
$$

where,

$$
Y_{nm}(\theta,\lambda) = (A_{nm}\cos m\lambda + B_{nm}\sin m\lambda)P_{nm}(\cos\theta), \tag{4.80}
$$

and P_{nm} are the associated Legendre polynomials of degree n and order m, given by VANICEK AND KRAKIWSKY, [1986]

$$
P_{nm}(\cos\theta) = \frac{(1-\cos^2\theta)^{m/2}}{n!\ 2^n}\ \frac{d^{n+m}}{d(\cos\theta)^{n+m}}(\cos^2\theta-1)^n, \tag{4.81}
$$

In the sequel, we drop indices (n, m) from U_{nm}, V_{nm}, Ψ_{nm} and Θ_{nm} for simplicity. Substituting (4.79) into the last of the equations (4.48), we obtain for dilatation Θ

$$
\Theta = \partial_r U Y_{nm} + (1/r)\partial_\theta[V\partial_\theta Y_{nm}] + (2/r)U\, Y_{nm} + V(r\sin^2\theta)^{-1}\partial_\lambda{}^2 Y_{nm} + (r^{-1}\cot\theta)V\partial_\theta Y_{nm}, \tag{4.82}
$$

or, by rearranging

$$
\Theta = \partial_r U Y_{nm} + (2/r)U\, Y_{nm} + (V/r)[\partial_\theta{}^2 Y_{nm} + \sin^{-2}\theta\ \partial_\lambda{}^2 Y_{nm} + \cot\theta\ \partial_\theta Y_{nm}]. \tag{4.83}
$$

Considering the equation of Laplace for spherical harmonics [HEISKANEN AND MORITZ, 1979, p. 20]

$$
\partial_\theta{}^2 Y_{nm} + \cot\theta\,\partial_\theta Y_{nm} + (\sin^{-2}\theta)\,\partial_\lambda{}^2 Y_{nm} + n(n+1)Y_{nm} = 0, \tag{4.84}
$$

equation (4.83) can be written as

$$\Theta = [\partial_r U + (2/r)U - (V/r)n(n+1)]Y_{nm}, \tag{4.85}$$

or,

$$\Theta = X(r, t)Y_{nm},$$

$$X(r, t) = \partial_r U + (2/r)U - (V/r)n(n+1). \tag{4.86}$$

For the surface distribution of mass we have [LONGMAN, 1962]

$$\sigma_{nm} = (2n+1)(4\pi R^2)^{-1}\, Y_{nm}, \tag{4.87}$$

which is a point load with unit mass, expanded into a series of spherical harmonics. For any mass m, we have simply that

$$\sigma_{nm} = m(2n+1)(4\pi R^2)^{-1}\, Y_{nm}. \tag{4.88}$$

For our calculations it is convenient to take m as the mass of the earth, i.e.,

$$m = R^2 g_o|_R \,/\, G \tag{4.89}$$

and (4.88) becomes

$$\sigma_{nm} = \{g_o|_R /(4\pi G)\}(2n+1)Y_{nm}. \tag{4.90}$$

For the rotational terms, if we consider that we have only sectorial tides (semidiurnal), i.e., n = m, we have [BEN-MENAHEM AND SINGH, 1981, p.980, eqn. D.126]

$$P_{nn}(\cos\theta) = (2n - 1)!\, \sin^n\theta. \tag{4.91}$$

Differentiating (4.91) with respect to θ we obtain

$$d_\theta P_{nn}(\cos\theta) = (2n-1)!\, n \sin^{n-1}\theta \, \cos\theta$$
$$= n(2n-1)! \sin^n\theta \cot\theta$$
$$= n \cot\theta\, P_{nn}(\cos\theta) \qquad (4.92)$$

and from (4.80) we obtain

$$\partial_\theta Y_{nn} = n\cot\theta\, Y_{nn}. \qquad (4.93)$$

Furthermore, by differentiating (4.80) with respect to λ we get

$$\partial_\lambda Y_{nm}(\theta,\lambda) = -\,m\,(A_{nm}\sin m\lambda \;-\; B_{nm}\cos m\lambda)P_{nm}(\cos\theta), \qquad (4.94)$$

or considering rotational symmetry of 90° for the properties of the earth we can write

$$m\lambda = 90^\circ + m\lambda \qquad (4.95)$$

and (4.94) becomes

$$\partial_\lambda Y_{nm}(\theta,\lambda) = -\,m\; Y_{nm}(\theta,\lambda). \qquad (4.96)$$

Relations (4.93) and (4.94) can be used in the expansion of the equations into spherical harmonics.

The remaining terms of the equations of deformation can be transformed similarly [Appendix II], and the equations of deformation reduce to

$$\left.\begin{array}{l}
\rho_o\ddot{U} + 2n\Omega\rho_o\dot{V} + \rho_o\partial_r[g_oU] - \rho_o g_o X - \rho_o\partial_r\Psi \\
\qquad + 2r^{-1}\left\{[A - N - F][2r^{-1}U - n(n+1)r^{-1}V] + [F - C]\partial_r U\right\} \\
\qquad + Lr^{-1}[r^{-1}U - r^{-1}V + \partial_r V]n(n+1) \\
\qquad - \partial_r\left\{C\partial_r U + F[2r^{-1}U - n(n+1)r^{-1}V]\right\} = 0
\end{array}\right\rfloor$$

$$\left.\begin{aligned}
&\rho_o\ddot{V} + 2\Omega\rho_o\dot{V} - r^{-1}\rho_o\Psi + r^{-1}\rho_o g_o U \\
&\qquad - r^{-1}\left\{A[2r^{-1}U - n(n+1)r^{-1}V] + F\partial_r U\right\} \\
&\qquad - 2Nr^{-1}[r^{-1}V - r^{-1}U] - \partial_r\left\{L[r^{-1}U - r^{-1}V + \partial_r V]\right\} \\
&\qquad - 3Lr^{-1}[r^{-1}U - r^{-1}V + \partial_r V] \\
&\qquad - J_2\omega\tau_\sigma/(J_R{}^2 r)\left\{\tau_\sigma\,[r^{-1}\dot{U} - r^{-1}\dot{V} + \partial_r\dot{V}] + [r^{-1}U - r^{-1}V + \partial_r V]\right\} = 0 \\
&\sin^{-1}\theta\,\rho_o\ddot{V} - 2\Omega\rho_o\sin^{-1}\theta\;[\cos^2\theta\,\dot{V} + n^{-1}\sin^2\theta\;\dot{U}] \\
&\qquad - (r\sin\theta)^{-1}\rho_o\Psi + (r\sin\theta)^{-1}\,\rho_o g_o U \\
&\qquad - (r\sin\theta)^{-1}\left\{A[2r^{-1}U - n(n+1)r^{-1}V] + F\partial_r U\right\} \\
&\qquad - 2N\,(r\sin\theta)^{-1}[r^{-1}V - r^{-1}U] - \partial_r\left\{L[r^{-1}U - r^{-1}V + \partial_r V]\right\} \\
&\qquad - 3L\,(r\sin\theta)^{-1}\,[r^{-1}U - r^{-1}V + \partial_r V] \\
&\qquad - J_2\omega\tau_\sigma/(J_R{}^2 r\sin\theta)\left\{\tau_\sigma\,[r^{-1}\dot{U} - r^{-1}\dot{V} + \partial_r\dot{V}]\right. \\
&\qquad \left. + \,[r^{-1}U - r^{-1}V + \partial_r V]\right\} = 0 \\
&\partial_r{}^2\Psi + 2r^{-1}\partial_r\Psi - r^{-2}n(n+1)\Psi = 4\pi G(\rho_o X + U\partial_r\,\rho_o)
\end{aligned}\right\}\quad (4.97)$$

Multiplying the third equation of (4.97) by $-\sin\theta$ and adding it to the second equation, we obtain

$$(\cos^2\theta + 1)\dot{V} + n^{-1}\sin^2\theta\;\dot{U} = 0,$$

or

$$\dot{V} = -\,n^{-1}\sin^2\theta\;(\cos^2\theta + 1)^{-1}\,\dot{U}. \qquad (4.98)$$

Substituting (4.98) into the first of (4.97), we obtain the new set of deformation equations

$$
\left.
\begin{aligned}
&\rho_o\ddot{U} - 2\Omega\rho_o \sin^2\theta\,(\cos^2\theta + 1)^{-1}\dot{U} + \rho_o\partial_r[g_o U] - \rho_o g_o X - \rho_o\partial_r\Psi \\
&\qquad + 2r^{-1}\left\{[A - N - F][2r^{-1}U - n(n+1)r^{-1}V] + [F - C]\partial_r U\right\} \\
&\qquad + Lr^{-1}[r^{-1}U - r^{-1}V + \partial_r V]n(n+1) \\
&\qquad - \partial_r\left\{C\partial_r U + F[2r^{-1}U - n(n+1)r^{-1}V]\right\} = 0, \\[2ex]
&\rho_o\ddot{V} + 2\Omega\rho_o\dot{V} - r^{-1}\rho_o\Psi + r^{-1}\rho_o g_o U \\
&\qquad - r^{-1}\left\{A[2r^{-1}U - n(n+1)r^{-1}V] + F\partial_r U\right\} \\
&\qquad - 2Nr^{-1}[r^{-1}V - r^{-1}U] - \partial_r\left\{L[r^{-1}U - r^{-1}V + \partial_r V]\right\} \\
&\qquad - 3Lr^{-1}[r^{-1}U - r^{-1}V + \partial_r V] \\
&\qquad - J_2\omega\tau_\sigma/(J_R^2 r)\left\{\tau_\sigma[r^{-1}\dot{U} - r^{-1}\dot{V} + \partial_r\dot{V}] + [r^{-1}U - r^{-1}V + \partial_r V]\right\} = 0, \\[2ex]
&\partial_r^2\Psi + 2r^{-1}\partial_r\Psi - r^{-2}n(n+1)\Psi = 4\pi G(\rho_o X + U\partial_r\rho_o).
\end{aligned}
\right\} \quad (4.99)
$$

Similarly, boundary conditions (4.75), (4.77) and (4.78) reduce respectively to

$$
\left.
\begin{aligned}
&\partial_r\Psi + R^{-1}(n+1)\Psi = 4\pi G[\rho_o U + (2n+1)g_o|_R/(4\pi G)], && r = R, \\
&C\partial_r U + F[2r^{-1}U - r^{-1}n(n+1)V] = -(2n+1)\,g_o^2|_R/(4\pi G), && r = R, \\
&r^{-1}U - r^{-1}V + \partial_r V = 0, && r = R.
\end{aligned}
\right\} \quad (4.100)
$$

We note that the equations of deformation (4.99) are three partial differential equations of second order. They describe the deformation of earth under harmonic surface loads, on a three dimensional configuration manifold, with a state vector consisting of the generalised coordinates U, V and Ψ. The constraints applied to the system, namely the boundary conditions (4.100), are integrable partial differential equations of first order in the generalised coordinates and thus, they are purely geometrical constraints. This implies that the system is holonomic. In addition, since time does not appear explicitly in either the equations, or in the constraints, the system is autonomous.

5

THE EQUATIONS OF DEFORMATION IN THE TANGENT BUNDLE SPACE

The presence of dissipation introduces delay in the displacements and to the secondary loading potential. Therefore it is expedient to consider the state variables (generalised coordinates) spanning the 3-D Lagrangean configuration manifold to be complex. This expands the real dimension of the configuration manifold to six. We transform the equations of deformation and the boundary conditions from the 6-D Lagrangean configuration manifold into a 12-D tangent bundle space, using appropriate substitutions for the complex generalised coordinates. The equations of deformation in this 12-D tangent bundle space are 12 simultaneous linear ordinary differential equations of first order (ODEs). Furthermore, we normalise all the variables in order to make the solution more stable numerically.

5.1 Reduction of the PDEs of Second Order to ODEs of First Order

Since the ocean load is periodic and the response of the earth is assumed linear, the resulting load displacements will be periodic of the same angular velocity and we can write:

$$\mathrm{U}(r, \mathrm{t}) = [\mathrm{U}_1(r) - i\,\mathrm{U}_2(r)]\exp(i\,\omega\mathrm{t}), \qquad (5.1)$$

$$\mathrm{V}(r, \mathrm{t}) = [\mathrm{V}_1(r) - i\,\mathrm{V}_2(r)]\exp(i\,\omega\mathrm{t}) \qquad (5.2)$$

and the complex total periodic loading potential

$$\Psi(r, t) = [\Psi_1(r) - i\Psi_2(r)]\exp(i\omega t). \tag{5.3}$$

Taking the first and second time derivatives of (5.1) and (5.2) we obtain

$$\dot{U} = \omega\,[iU_1 + U_2]\,\exp(i\omega t), \tag{5.4}$$

$$\ddot{U} = -\,\omega^2\,[U_1 - iU_2]\,\exp(i\omega t), \tag{5.5}$$

$$\dot{V} = \omega\,[iV_1 + V_2]\,\exp(i\omega t), \tag{5.6}$$

$$\ddot{V} = -\,\omega^2\,[V_1 - iV_2]\,\exp(i\omega t). \tag{5.7}$$

It should be remembered, that the components of the state vector, namely U_1, U_2, V_1, V_2, Ψ_1, Ψ_2 are functions of the degree of expansion n, i.e., for each value of n, there corresponds a different solution. However, for simplicity, we have dropped the subscript n. Substituting (5.1) - (5.7) into the equations of deformation (4.99) and omitting the common factor $\exp(i\omega t)$, we obtain

$$\left.\begin{aligned}
&-\omega^2\rho_o(U_1 - iU_2) - 2\Omega\omega\rho_o\sin^2\theta\,(\cos^2\theta + 1)^{-1}(iU_1 + U_2) + \rho_o\partial_r[g_o(U_1 - iU_2)] \\
&\quad - \rho_o g_o\{\partial_r(U_1 - iU_2) + 2r^{-1}(U_1 - iU_2) - n(n+1)r^{-1}(V_1 - iV_2)\} \\
&\quad - \rho_o\partial_r(\Psi_1 - i\Psi_2) + 2r^{-1}\{[A - N - F][2r^{-1}(U_1 - iU_2) - n(n+1)r^{-1}(V_1 - iV_2)] \\
&\quad + [F - C]\partial_r(U_1 - iU_2)\} + Lr^{-1}[r^{-1}(U_1 - iU_2) - r^{-1}(V_1 - iV_2) + \partial_r(V_1 - iV_2)]n(n+1) \\
&\quad - \partial_r\{C\partial_r(U_1 - iU_2) + F[2r^{-1}(U_1 - iU_2) - n(n+1)r^{-1}(V_1 - iV_2)]\} = 0 \\[1ex]
&-\omega^2\rho_o(V_1 - iV_2) + 2\Omega\omega\rho_o(iV_1 + V_2) - r^{-1}\rho_o(\Psi_1 - i\Psi_2) + r^{-1}\rho_o g_o(U_1 - iU_2) \\
&\quad - r^{-1}\{A[2r^{-1}(U_1 - iU_2) - n(n+1)r^{-1}(V_1 - iV_2)] + F\partial_r(U_1 - iU_2)\} \\
&\quad - 2Nr^{-1}[r^{-1}(V_1 - iV_2) - r^{-1}(U_1 - iU_2)]
\end{aligned}\right\} \tag{5.8}$$

$$
\left.
\begin{aligned}
&-\partial_r\left\{L[r^{-1}(U_1 - iU_2) - r^{-1}(V_1 - iV_2) + \partial_r(V_1 - iV_2)]\right\} \\
&-3Lr^{-1}[r^{-1}(U_1 - iU_2) - r^{-1}(V_1 - iV_2) + \partial_r(V_1 - iV_2)] \\
&-J_2\omega\tau_\sigma/(J_R^2 r)\left\{\omega\tau_\sigma[r^{-1}(iU_1 + U_2) - r^{-1}(iV_1 + V_2) + \partial_r(iV_1 + V_2)]\right. \\
&\left.+ [r^{-1}(U_1 - iU_2) - r^{-1}(V_1 - iV_2) + \partial_r(V_1 - iV_2)]\right\} = 0 \\
&\partial_r^2(\Psi_1 - i\Psi_2) + 2r^{-1}\partial_r(\Psi_1 - i\Psi_2) - n(n+1)r^{-2}(\Psi_1 - i\Psi_2) \\
&\quad = 4\pi G\left\{\rho_o[\partial_r(U_1 - iU_2) + 2r^{-1}(U_1 - iU_2) - n(n+1)r^{-1}(V_1 - iV_2)]\right. \\
&\qquad\left. + (U_1 - iU_2)\partial_r\rho_o\right\}
\end{aligned}
\right|
$$

Equating separately the real and imaginary parts of (5.8) to zero, we obtain the following six equations for the six unknowns U_1, U_2, V_1, V_2, Ψ_1 and Ψ_2:

$$
\left.
\begin{aligned}
&-\omega^2\rho_o U_1 - 2\Omega\omega\rho_o\sin^2\theta(\cos^2\theta + 1)^{-1}U_2 + \rho_o\partial_r[g_o U_1] \\
&\quad -\rho_o g_o[\partial_r U_1 + 2r^{-1}U_1 - n(n+1)r^{-1}V_1] \\
&\quad -\rho_o\partial_r\Psi_1 + 2r^{-1}\left\{[A - N - F][2r^{-1}U_1 - n(n+1)r^{-1}V_1] + [F - C]\partial_r U_1\right\} \\
&\quad + Lr^{-1}[r^{-1}U_1 - r^{-1}V_1 + \partial_r V_1]n(n+1) - \partial_r\left\{C\partial_r U_1 + F[2r^{-1}U_1 - n(n+1)r^{-1}V_1]\right\} = 0, \\
&-\omega^2\rho_o V_1 + 2\Omega\omega\rho_o V_2 - r^{-1}\rho_o\Psi_1 + r^{-1}\rho_o g_o U_1 \\
&\quad - r^{-1}\left\{A[2r^{-1}U_1 - n(n+1)r^{-1}V_1] + F\partial_r U_1\right\} - 2Nr^{-1}[r^{-1}V_1 - r^{-1}U_1] \\
&\quad -\partial_r\left\{L[r^{-1}U_1 - r^{-1}V_1 + \partial_r V_1]\right\} - 3Lr^{-1}[r^{-1}U_1 - r^{-1}V_1 + \partial_r V_1] \\
&\quad - J_2\omega\tau_\sigma/(J_R^2 r)\left\{\omega\tau_\sigma[r^{-1}U_2 - r^{-1}V_2 + \partial_r V_2] + [r^{-1}U_1 - r^{-1}V_1 + \partial_r V_1]\right\} = 0, \\
&\partial_r^2\Psi_1 + 2r^{-1}\partial_r\Psi_1 - n(n+1)r^{-2}\Psi_1 = 4\pi G\left\{\rho_o[\partial_r U_1 + 2r^{-1}U_1 - n(n+1)r^{-1}V_1] + U_1\partial_r\rho_o\right\}
\end{aligned}
\right\} (5.9)
$$

$$
\left.
\begin{aligned}
&\omega^2\rho_o U_2 - 2\Omega\omega\rho_o \sin^2\theta\,(\cos^2\theta + 1)^{-1}U_1 - \rho_o\partial_r[g_o U_2] \\
&\quad + \rho_o g_o[\partial_r U_2 + 2r^{-1}U_2 - n(n+1)r^{-1}V_2] \\
&\quad + \rho_o\partial_r\Psi_2 - 2r^{-1}\left\{[A - N - F][2r^{-1}U_2 - n(n+1)r^{-1}V_2] + [F - C]\partial_r U_2\right\} \\
&\quad - Lr^{-1}[r^{-1}U_2 - r^{-1}V_2 + \partial_r V_2]n(n+1) + \partial_r\left\{C\partial_r U_2 + F[2r^{-1}U_2 - n(n+1)r^{-1}V_2]\right\} = 0, \\[2ex]
&\omega^2\rho_o V_2 + 2\Omega\omega\rho_o V_1 + r^{-1}\rho_o\Psi_2 - r^{-1}\rho_o g_o U_2 \\
&\quad + r^{-1}\left\{A[2r^{-1}U_2 - n(n+1)r^{-1}V_2] + F\partial_r U_2\right\} + 2Nr^{-1}[r^{-1}V_2 - r^{-1}U_2] \\
&\quad + \partial_r\left\{L[r^{-1}U_2 - r^{-1}V_2 + \partial_r V_2]\right\} + 3Lr^{-1}[r^{-1}U_2 - r^{-1}V_2 + \partial_r V_2] \\
&\quad - J_2\omega\tau_\sigma/(J_R^2 r)\left\{\omega\tau_\sigma[r^{-1}U_1 - r^{-1}V_1 + \partial_r V_1] - [r^{-1}U_2 - r^{-1}V_2 + \partial_r V_2]\right\} = 0, \\[2ex]
&\partial_r^2\Psi_2 + 2r^{-1}\partial_r\Psi_2 - n(n+1)r^{-2}\Psi_2 = 4\pi G\left\{\rho_o[\partial_r U_2 + 2r^{-1}U_2 - n(n+1)r^{-1}V_2] + U_2\partial_r\rho_o\right\}.
\end{aligned}
\right\}
$$

At this point we transform the equations of deformation in the tangent bundle space, following a procedure similar to LONGMAN's [1963]. We introduce the following substitutions:

$$
\begin{aligned}
&\left.\begin{aligned} y_1 &= U_1, \\ y_1^* &= U_2, \end{aligned}\right\} \text{Radial displacement} \\
&\left.\begin{aligned} y_2 &= C\partial_r U_1 + F[2r^{-1}U_1 - n(n+1)r^{-1}V_1], \\ y_2^* &= C\partial_r U_2 + F[2r^{-1}U_2 - n(n+1)r^{-1}V_2] \end{aligned}\right\} \text{Normal stress} \\
&\left.\begin{aligned} y_3 &= V_1, \\ y_3^* &= V_2, \end{aligned}\right\} \text{Tangential displacement} \\
&\left.\begin{aligned} y_4 &= L[r^{-1}U_1 - r^{-1}V_1 + \partial_r V_1], \\ y_4^* &= L[r^{-1}U_2 - r^{-1}V_2 + \partial_r V_2], \end{aligned}\right\} \text{Shear stress} \\
&\left.\begin{aligned} y_5 &= \Psi_1, \\ y_5^* &= \Psi_2, \end{aligned}\right\} \text{Total loading potential}
\end{aligned}
\qquad (5.10)
$$

$$\left.\begin{aligned} y_6 &= \partial_r \Psi_1 - 4\pi G\rho_o U_1. \\ y_6^* &= \partial_r \Psi_2 - 4\pi G\rho_o U_2, \end{aligned}\right\} \text{Gravity perturbation}$$

where the asterisk denotes imaginary part. Elimination of the original unknowns results in:

$$\left.\begin{aligned} \partial_r y_1 &= C^{-1}y_2 - F\,C^{-1}[2r^{-1}y_1 - n(n+1)r^{-1}y_3], \\ \partial_r y_1^* &= C^{-1}y_2^* - F\,C^{-1}[2r^{-1}y_1^* - n(n+1)r^{-1}y_3^*], \\ \partial_r y_3 &= -r^{-1}y_1 + r^{-1}y_3 + L^{-1}y_4\,, \\ \partial_r y_3^* &= -r^{-1}y_1^* + r^{-1}y_3^* + L^{-1}y_4^*\,, \\ \partial_r y_5 &= 4\pi G\rho_o y_1 + y_6. \\ \partial_r y_5^* &= 4\pi G\rho_o y_1^* + y_6^*. \end{aligned}\right\} \quad (5.11)$$

Substituting (5.10) and (5.11) into the equations of deformation (5.9), we obtain

$$\left.\begin{aligned} \partial_r y_2 =& \left\{-4\rho_o g_o r - \rho_o r^2\omega^2 + 4C^{-1}[(A - N)C - F^2]\right\} r^{-2}y_1 \\ &+ 2C^{-1}(F - C)\, r^{-1}y_2 + n(n+1)\left\{\rho_o g_o r - 2C^{-1}[(A - N)C - F^2]\right\} r^{-2}y_3 \\ &+ n(n+1)\, r^{-1}y_4 - \rho_o\, y_6 - 2\Omega\omega\rho_o \sin^2\theta\, (\cos^2\theta + 1)^{-1} y_1^*, \\ \\ \partial_r y_2^* =& \left\{-4\rho_o g_o r - \rho_o r^2\omega^2 + 4C^{-1}[(A - N)C - F^2]\right\} r^{-2}y_1^* \\ &+ 2C^{-1}(F - C)\, r^{-1}y_2^* + n(n+1)\left\{\rho_o g_o r - 2C^{-1}[(A - N)C - F^2]\right\} r^{-2}y_3^* \\ &+ n(n+1)\, r^{-1}y_4^* - \rho_o\, y_6^* + 2\Omega\omega\rho_o \sin^2\theta\, (\cos^2\theta + 1)^{-1} y_1, \\ \\ \partial_r y_4 =& \left\{\rho_o g_o r + 2C^{-1}[F^2 - (A - N)C]\right\} r^{-2}y_1 - FC^{-1}r^{-1}y_2 \\ &+ \left\{-\rho_o r^2\omega^2 + n(n+1)C^{-1}[AC - F^2] - 2N\right\} r^{-2}y_3 - [3 + J_2\omega\tau_\sigma/(LJ_R^2)]r^{-1}y_4 \\ &- \rho_o r^{-1}y_5 + 2\Omega\omega\rho_o y_3^* - J_2\omega^2\tau_\sigma/(LJ_R^2)\, r^{-1}y_4^*, \end{aligned}\right\} \quad (5.12)$$

$$\left.\begin{aligned}
\partial_r y_4^{*} = &\left\{ \rho_o g_o r + 2C^{-1}[F^2 - (A - N)C] \right\} r^{-2} y_1^{*} - FC^{-1} r^{-1} y_2^{*} + \\
&+ \left\{ -\rho_o r^2 \omega^2 + n(n+1)C^{-1}[AC - F^2] - 2N \right\} r^{-2} y_3^{*} - [3 + J_2 \omega \tau_\sigma/(LJ_R^2)] r^{-1} y_4^{*} \\
&- \rho_o r^{-1} y_5^{*} - 2\Omega\omega\rho_o y_3 + J_2 \omega^2 \tau_\sigma/(LJ_R^2)\, r^{-1} y_4, \\
\\
\partial_r y_6 = &- 4\pi G \rho_o n(n+1) r^{-1} y_3 + n(n+1) r^{-2} y_5 - 2 r^{-1} y_6. \\
\\
\partial_r y_6^{*} = &- 4\pi G \rho_o n(n+1) r^{-1} y_3^{*} + n(n+1) r^{-2} y_5^{*} - 2 r^{-1} y_6^{*}.
\end{aligned}\right\}$$

Equations (5.11) and (5.12) form a system of 12 ODEs of first order in y_i and y_i^{*}. Substituting (5.10) and (5.11) into (4.100), the boundary conditions at the surface of the earth ($r = R$) become

$$\left.\begin{aligned}
y_2 &= -(2n+1) g_o^2|_R/(4\pi G), \\
y_2^{*} &= 0, \\
y_4 &= 0, \\
y_4^{*} &= 0, \\
y_6 + (n+1) y_5/R &= (2n+1) g_o|_R, \\
y_6^{*} + (n+1) y_5^{*}/R &= 0.
\end{aligned}\right\} \quad (5.13)$$

5.2 Normalisation of the Equations of Deformation

It is necessary to normalise the variables of the equations of deformation in order to make the numerical solution more stable. We normalise the elastic moduli A, C, L, N, F, viscosity η, density ρ_o, gravity g_o and complex compliance J_2 with their corresponding values at the centre, or at the surface of the earth or at the core-mantle boundary. The procedure followed here is similar to LONGMAN's [1963]:

$$A' = A/F^c,\; C' = C/F^c,\; L' = L/F^c,\; N' = N/F^c,\; F' = F/F^c,\; \eta' = \eta/\eta^m,$$
$$\rho_o' = \rho_o/\rho_o^c,\;\; J_2' = J_2/J_2^m,\; r' = r/R,\;\; g_o' = g_o/\,g_o|_R,\;\; \omega' = \omega/\Omega, \tag{5.14}$$

where superscripts "c" and "m" denote values at the centre and at the core-mantle boundary, respectively. Likewise, we normalise y_i and y_i^* as follows:

$$\left.\begin{array}{lll} z_1 = y_1/R, & z_2 = y_2/F^c, & z_3 = y_3/R, \\ z_4 = y_4/F^c, & z_5 = y_5/[R\, g_o|_R\,], & z_6 = y_6/\,g_o|_R. \\ z_1^* = y_1^*/R, & z_2^* = y_2^*/F^c, & z_3^* = y_3^*/R, \\ z_4^* = y_4^*/F^c, & z_5^* = y_5^*/[R\, g_o|_R\,], & z_6^* = y_6^*/\,g_o|_R. \end{array}\right\} \tag{5.15}$$

We also define the following dimensionless and constant quantities:

$$\left.\begin{array}{l} \alpha = g_o^2(R)/(4\pi F^c G), \\ \beta = 4\pi G \rho_o^c R/g_o|_R, \\ \gamma = \rho_o^c g_o|_R\, R/F^c, \\ \delta = \rho_o^c R^2 \Omega^2/F^c, \\ \varepsilon = J_2^m \Omega^2 \tau_\sigma^{m2}/(F^c J_R^{m2}). \end{array}\right\} \tag{5.16}$$

Upon substitution of the above normalised variables into the equations of deformation (5.11) and (5.12), we obtain:

$$\partial_{r'} z_1 = -\,2F'C'^{-1}r'^{-1} z_1 + C'^{-1}z_2 + F'C'^{-1}n(n+1)r'^{-1} z_3,$$

$$\begin{aligned} \partial_{r'} z_2 = &\left\{-4\gamma\rho_o' g_o' r' - \delta\rho_o' r'^2 \omega'^2 + 4C'^{-1}[(A' - N')C' - F'^2]\right\} r'^{-2} z_1 \\ &+ 2C'^{-1}(F' - C')r'^{-1} z_2 \\ &+ n(n+1)\left\{\gamma\rho_o' g_o' r' - 2C'^{-1}[(A' - N')C' - F'^2]\right\} r'^{-2} z_3 \\ &+ n(n+1)\, r'^{-1} z_4 - \gamma\rho_o' z_6 - 2\delta\omega' \rho_o' \sin^2\theta\, (\cos^2\theta + 1)^{-1} z_1^*, \end{aligned}$$

$$
\left.
\begin{aligned}
&\partial_{r'} z_3 = -r'^{-1} z_1 + r'^{-1} z_3 + L'^{-1} z_4 \, ,\\[1ex]
&\partial_{r'} z_4 = \left\{ \gamma\rho_o' g_o' r' + 2C'^{-1}[\, F'^2 - (A' - N')C'] \right\} r'^{-2} z_1 - F'C'^{-1} r'^{-1} z_2\\
&\qquad + \left\{ -\delta\rho_o' r'^2 \omega'^2 + n(n+1)C'^{-1}[A'C' - F'^2] - 2N' \right\} r'^{-2} z_3\\
&\qquad - [3 + J_2'\, \omega' \tau_\sigma'/(L'J_R'^2)\,] r'^{-1} z_4 - \gamma\rho_o' r'^{-1} z_5 + 2\delta\omega' \rho_o' z_3^*\\
&\qquad - \varepsilon J_2'\, \omega'^2 \tau_\sigma'^2/(L'J_R'^2)\, r'^{-1} z_4^*,\\[1ex]
&\partial_{r'} z_5 = \beta\rho_o' z_1 + z_6.\\[1ex]
&\partial_{r'} z_6 = -\, n(n+1)\beta\rho_o' r'^{-1} z_3 + n(n+1) r'^{-2} z_5 - 2r'^{-1} z_6 \, .\\[1ex]
&\partial_{r'} z_1^* = -\, 2F'C'^{-1} r'^{-1} z_1^* + C'^{-1} z_2^* + F'C'^{-1} n(n+1) r'^{-1} z_3^*,\\[1ex]
&\partial_{r'} z_2^* = -\left\{ 4\gamma\rho_o' g_o' r' + \delta\rho_o' r'^2\, \omega'^2 - 4C'^{-1}[(A' - N')C' - F'^2] \right\} r'^{-2} z_1^*\\
&\qquad + 2C'^{-1}(F' - C') r'^{-1} z_2^*\\
&\qquad + n(n+1) \left\{ \gamma\rho_o' g_o' r' - 2C'^{-1}[(A' - N')C' - F'^2] \right\} r'^{-2} z_3^*\\
&\qquad + n(n+1)\, r'^{-1} z_4^* - \gamma\rho_o'\, z_6^* + 2\delta\omega' \rho_o' \sin^2\theta\, (\cos^2\theta + 1)^{-1} z_1,\\[1ex]
&\partial_{r'} z_3^* = -r'^{-1} z_1^* + r'^{-1} z_3^* + L'^{-1} z_4^* \, ,\\[1ex]
&\partial_{r'} z_4^* = \left\{ \gamma\rho_o' g_o' r' + 2C'^{-1}[\, F'^2 - (A' - N')C'] \right\} r'^{-2} z_1^* - F'C'^{-1} r'^{-1} z_2^*\\
&\qquad + \left\{ -\, \delta\rho_o' r'^2 \omega'^2 + n(n+1)C'^{-1}[A'C' - F'^2] - 2N' \right\} r'^{-2}\, z_3^*\\
&\qquad - [3 + J_2'\, \omega' \tau_\sigma'/(L'J_R'^2)] r'^{-1} z_4^* - \gamma\rho_o' r'^{-1} z_5^* - 2\delta\omega' \rho_o' z_3\\
&\qquad + \varepsilon J_2'\, \omega'^2 \tau_\sigma'^2/(L'J_R'^2)\, r'^{-1} z_4,
\end{aligned}
\right\} \quad (5.17)
$$

$$\partial_{r'} z_5^* = \beta\rho_o' z_1^* + z_6^*.$$

$$\partial_{r'} z_6^* = -\,n(n+1)\beta\rho_o'\, r'^{-1} z_3^* + n(n+1)\, r'^{-2} z_5^* - 2r'^{-1} z_6^*\,.$$

Equations (5.17) are 12 linear simultaneous ODEs of first order with variable coefficients in 12 state variables, namely z_i, z_i^* ($i = 1,\ldots, 6$), that span the 12-D tangent bundle space.

The boundary conditions at the surface of the earth become

$$\left.\begin{aligned}
&z_2 = -\,(2n + 1)\alpha,\\
&z_4 = 0,\\
&z_6 + (n+1)z_5 = (2n + 1),\\
&z_2^* = 0,\\
&z_4^* = 0,\\
&z_6^* + (n+1)z_5^* = 0.
\end{aligned}\right\} \quad (5.18)$$

5.3 Equations of Deformation for the Liquid Outer Core

In this section, we derive the equations of motion of the liquid outer core, since the equations (5.17) are valid only for the solid regions of the earth. The procedure we follow here is standard and can be found, for instance, in WAHR [1982].

In the liquid outer core, the shear wave vanishes and its velocity can be set to zero. From (4.10) and (4.11) we have

$$N = L = 0 \tag{5.19}$$

Furthermore, shear stresses, as well as their derivatives with respect to r' are everywhere zero. Thus, we have [cf: eqn. 5.10]

$$z_4 = z_4^* = 0, \tag{5.20}$$

$$\partial_{r'} z_4 = \partial_{r'} z_4^* = 0. \tag{5.21}$$

Because of (5.19), (5.20) and (5.21), the third and ninth of (5.17) become meaningless due to the undetermined terms z_4/L' and z_4^*/L'. In addition, (5.21) reduces the fourth and 10th of (5.17) to algebraic equations. In symbolic form we have

$$a_{0401}z_1 + a_{0402}z_2 + a_{0403}z_3 + a_{0405}z_5 + a_{0407}z_1^* = 0, \tag{5.22}$$

$$a_{1007}z_1^* + a_{1008}z_2^* + a_{1009}z_3^* + a_{1011}z_5^* + a_{1001}z_1 = 0, \tag{5.23}$$

where a_{ijkl} are coefficients†. Equations (5.22) and (5.23) are then used to eliminate z_3 and z_3^* from the equations of deformation. Thus, for the liquid outer core, the equations of deformation reduce to eight simultaneous ODEs of first order and two algebraic equations:

† *The first two subscripts denote the equation and the second two the dependent variable. For instance, a_{1007} is the coefficient of the 7^{th} dependent variable (z_1^*) in the 10^{th} equation.*

$$
\left.
\begin{aligned}
&\partial_{r'} z_1 = -\,2F'C'^{-1}r'^{-1} z_1 + C'^{-1}z_2 + F'C'^{-1}n(n+1)r'^{-1} z_3,\\[2ex]
&\partial_{r'} z_2 = \left\{-4\gamma\rho_o' g_o' r' - \delta\rho_o' r'^{2}\,\omega'^{2}\right\} r'^{-2} z_1 + n(n+1)\left\{\gamma\rho_o' g_o' r'\right\} r'^{-2} z_3\\
&\qquad - \gamma\rho_o'\, z_6 - 2\delta\omega'\rho_o'\sin^{2}\theta\,(\cos^{2}\theta + 1)^{-1} z_1^{*},\\[2ex]
&\partial_{r'} z_5 = \beta\rho_o' z_1 + z_6,\\[2ex]
&\partial_{r'} z_6 = -\,n(n+1)\beta\rho_o' r'^{-1} z_3 + n(n+1)r'^{-2} z_5 - 2r'^{-1} z_6\,,\\[2ex]
&\partial_{r'} z_1^{*} = -\,2F'C'^{-1}r'^{-1} z_1^{*} + C'^{-1}z_2^{*} + F'C'^{-1}n(n+1)r'^{-1} z_3^{*},\\[2ex]
&\partial_{r'} z_2^{*} = -\left\{4\gamma\rho_o' g_o' r' + \delta\rho_o' r'^{2}\,\omega'^{2}\right\} r'^{-2} z_1^{*} + n(n+1)\left\{\gamma\rho_o' g_o' r'\right\} r'^{-2} z_3^{*}\\
&\qquad - \gamma\rho_o'\, z_6^{*} + 2\delta\omega'\rho_o'\sin^{2}\theta\,(\cos^{2}\theta + 1)^{-1} z_1,\\[2ex]
&\partial_{r'} z_5^{*} = \beta\rho_o' z_1^{*} + z_6^{*},\\[2ex]
&\partial_{r'} z_6^{*} = -\,n(n+1)\beta\rho_o'\, r'^{-1} z_3^{*} + n(n+1)\, r'^{-2} z_5^{*} - 2r'^{-1} z_6^{*}\,,\\[2ex]
&z_3 = [\delta\rho_o' r'^{2}\omega'^{2'}]^{-1} r'^{2}\left\{[\gamma\rho_o' g_o' r']r'^{-2} z_1 - F'C'^{-1}r'^{-1}z_2 - \gamma\rho_o' r'^{-1} z_5\right.\\
&\qquad \left. - 2\delta\omega'\rho_o' z_3^{*}\right\},\\[2ex]
&z_3^{*} = [\delta\rho_o' r'^{2}\omega'^{2'}]^{-1} r'^{2}\left\{[\gamma\rho_o' g_o' r']r'^{-2} z_1^{*} - F'C'^{-1}r'^{-1}z_2^{*}\right.\\
&\qquad \left. + \gamma\rho_o' r'^{-1} z_5^{*} + 2\delta\omega'\rho_o' z_3\right\}.
\end{aligned}
\right\}\quad (5.24)
$$

6

SOLUTION OF THE EQUATIONS OF DEFORMATION

In this chapter, we develop a theoretical procedure to solve the equations in the tangent bundle space. We then apply this procedure to solve the equations numerically, using the finite difference method of numerical integration.

6.1 General Considerations

The existence of different regions in the earth, namely the solid inner core, the liquid outer core, the mantle and the crust complicate the integration of the equations of deformation. Fortunately, only the low degree load deformation penetrates the earth down to the inner core and thus the effort to obtain a solution reduces drastically.

At the centre of the earth ($r' = 0$), the equations of deformation are undefined. So long as we impose regularity of the solution at the origin, we can start the integration from a sphere with arbitrarily small radius, considering that all the material inside this sphere has been removed. This is only a simplifying assumption, as at this arbitrarily small sphere we need specify boundary conditions and we are free to impose any physically meaningfull conditions, whether the material has been removed or not. At this internal free surface, we have neither normal, nor tangential stress applied, since we deal only with loads at the earth's surface. Therefore, at the surface of the internal cavity we can impose boundary conditions similar to those at the earth's surface, the only difference being the absence of the forcing terms. The nonhomogeneous boundary conditions (5.13), which hold true at the surface of the earth, become homogeneous at the free internal surface. If the radius of the internal cavity is "a", then after normalization of (5.13) we obtain:

$$\left.\begin{aligned} &z_2 = 0, \\ &z_4 = 0, \\ &z_6 + (n+1)(R/a)z_5 = 0, \\ &z_2^* = 0, \\ &z_4^* = 0, \\ &z_6^* + (n+1)(R/a)z_5^* = 0. \end{aligned}\right\} \quad (6.1)$$

In the third and sixth boundary conditions above, the term R/a may become arbitrarily large when the radius of the internal cavity becomes arbitrarily small. Care must be exercised when selecting "a" for the numerical integration of the equations of deformation. We will discuss this issue in section 6.3.

The procedure of solving the complete system of the 12 ODE's for a viscoelastic nonhomogeneous and rotating earth is similar to the procedure of solving the six simultaneous ODEs for a purely elastic and nonrotating earth. We start with the solution of the equations of deformation on a purely elastic, anisotropic and nonrotating earth, and then we generalise it for a viscoelastic and rotating earth.

6.2 Solution for a Purely Elastic, Anisotropic and Nonrotating Earth

For a purely elastic, anisotropic and nonrotating earth we have six simultaneous ODEs of first order with three boundary conditions at the internal cavity, namely

$$z_2 = z_4 = z_6 + (n+1)(R/a)z_5 = 0, \qquad r' = R/a \qquad (6.2)$$

and three surface boundary conditions:

$$z_2 = -(2n+1)\alpha, \quad z_4 = 0, \quad z_6 + (n+1)z_5 = (2n+1), \quad r' = 1. \qquad (6.3)$$

As we recall, the equations of deformation are drastically different for the liquid outer core, therefore, a one-step integration can not be performed. Instead, we integrate the equations in steps as follows†:

At the surface of the internal cavity, the three boundary conditions require that there are only three (6 ODEs - 3 boundary conditions) independent (free) solutions. If $f_i(r')$ $g_i(r')$ and $h_i(r')$ (i = 1,...,6) are the three independent sets of (partial) solutions for z_i, then, any linear combination of those will be the general solution of the system. Therefore, for the solid inner core the solution of the equations has the form

$$z_i(r') = A\, f_i(r') + B\, g_i(r') + C\, h_i(r'), \quad i = 1, 2, ..., 6, \quad \text{for the inner core} \qquad (6.4)$$

where *A*, *B*, *C* are arbitrary constants.

At the inner core - outer core boundary (b_1), there is an additional condition, namely, $z_4 = 0$, which indicates the absence of any tangential stresses‡. This boundary condition is used to eliminate one of the arbitrary constants, for instance *C* at this boundary, i.e., constants *A* and *B* will be functions of *C*.

For the numerical solution of the deformation equations in the fluid outer core, we proceed as follows: At boundary b_1, we have two independent solutions from the integration of the equations in the inner core, and continuity of z_1, z_2, z_5 and z_6 (equivalent

† *Although the method of solving similar equations is very well known to the geophysical community, in our opinion it is very poorly explained in the literature. The procedure presented here was developed by the author, with assistance generously provided by Prof. Dr. R.D. Small, from the Department of Mathematics and Statistics at the University of New Brunswick.*

‡ *For the purpose of tidal studies in general, this is an excellent approximation. However, for the study of the deep interior of the earth using various geophysical methods, it is believed that the solid-fluid boundaries between inner core-outer core and outer core-lower mantle are not smooth and tangential stresses may exist.*

to two boundary conditions). We do not impose any continuity of z_3 to allow for slippage between inner and outer cores.

If we call $x_i(r')$ and $y_i(r')$ (i = 1, 2, 5, 6), the two partial solutions of the four ODEs in the outer core, then the general solution will be a linear combination of those. In actuality, the constants will be functions of A and B. Thus, we can write

$$z_i(r') = D(A, B)x_i(r') + E(A, B)\, y_i(r'), \quad i = 1,\ 2,\ 5,\ 6, \text{ for the outer core} \qquad (6.5)$$

At the outer core-lower mantle boundary (b_2) we have the two independent solutions from the integration in the outer core as well as the continuity of z_1, z_2, z_4, z_5, and z_6. For the solid mantle and crust, z_3 is again defined. Therefore, the two solutions at b_2 must be combined with a third, that of z_3. Therefore, if $p_i\,(r')$, $q_i\,(r')$ and $s_i\,(r')$ are the three independent solutions for the mantle and core, a general solution will be

$$z_i(r') = D(A, B)p_i(r') + E(A, B)\, q_i(r') + F s_i(r'), \qquad i = 1, 2, \ldots, 6, \quad r' > b_2, \quad (6.6)$$

where F is a new arbitrary constant, introduced to account for z_3. This solution also holds true at the surface of the earth, where the three boundary conditions will be used to determine the three constants $D(A, B)$, $E(A, B)$ and F. From D and E, arbitrary constants A and B, and subsequently C, can be determined, to provide us with the general solution of the deformation equations, valid throughout the earth.

6.3 Numerical Integration

For the numerical integration of the equations, the following steps are followed, based on the theoretical treatment above:

a) At the inner cavity we set [cf: eqn. (6.4)]:

$$z_1 = A, \quad z_3 = B, \quad z_5 = C. \qquad (6.7)$$

The three independent solutions can then be obtained by setting, for example:

$$\left.\begin{aligned} &A = 1,\ B = 0,\ \ C = 0, \\ &A = 0,\ B = 1,\ \ C = 0, \\ &A = 0,\ B = 0,\ \ C = 1. \end{aligned}\right\} \quad (6.8)$$

These three solutions†, along with the three boundary conditions at the internal cavity provide six initial values for the six unknown functions z_i; they must be propagated upwards, till we reach boundary b_1.

b) At boundary b_1, the additional condition $z_4 = 0$ reduces the independent solutions to two by setting, for example:

$$\left.\begin{aligned} &A = 1,\ B = 0,\ \ C = 1, \\ &A = 0,\ B = 1,\ \ C = 1. \end{aligned}\right\} \quad (6.9)$$

The above "conditions" (6.9) show that there are only two linear combinations of the three independent solutions at b_1, that give $z_4 = 0$.

c) The two independent solutions found at b_1 are propagated in the fluid outer core till we reach boundary b_2. At this boundary we ensure continuity for z_1, z_2, z_4, z_5 and z_6. At this point, a third solution is introduced, to account for z_3. We set $z_3 = F$ and thus, we have three independent solutions at b_2, namely

$$\left.\begin{aligned} &A = 1,\ B = 0,\ \ F = 0, \\ &A = 0,\ B = 1,\ \ F = 0, \\ &A = 0,\ B = 0,\ \ F = 1. \end{aligned}\right\} \quad (6.10)$$

† *In reality, instead of setting A, B, C equal to unity, we found that for numerical stability, we had to set these arbitrary constants equal to* 10^{-4}.

The first two cases are the solutions coming from the outer core with $z_3 = 0$. For the third solution all z_i are zero except for z_3, which takes an arbitrary value.

d) All three solutions determined in step (c) above are propagated upwards to the surface. At the surface, the three boundary conditions are used to determine constants *A*, *B*, *F* and subsequently *C*. Thus, a unique general solution to the deformation equations is obtained.

The solution of the 12 simultaneous ODEs of first order is similar to the solution of the 6 ODEs described above, the only difference being the number of boundary conditions and the number of independent solutions. More specifically, the number of boundary conditions, as well as the number of independent solutions for the determination of the initial values are doubled, and thus, the effort for the determination of a numerical solution increases dramatically. Moreover, for the ocean tide loading case, there exists an infinite number of solutions, each corresponding to different values of wave number n. Fortunately, for n>300, the solution is a slowly varying function of n and we need only a few solutions, e.g., for n = 500, 800, 1000, 2000, 5000 and 10000; for intermediate values of n, a linear interpolation can be used. More importantly, for n>10000, the solution converges to a constant value and thus, no solutions need to be calculated for n>10000. Most importantly, from our experience from the numerical integration of the equations of deformation, only the low degree surface loads (up to n = 10) introduce deformations of significant magnitude below the core-mantle boundary. For n>10, the integration of the deformation equations can start in the solid mantle; the radius of the internal cavity can be larger than that of the core-mantle boundary and thus, the boundary value problem can be solved on the computer in one step.

6.4 The Earth Models

The coefficients of the equations of deformation are functions of the elasticity parameters of the earth, namely A, C, N, L and F, as well as functions of viscosity η, density ρ_o, gravity g, Gibbs free activation energy G^* and absolute temperature T. All the above parameters are functions of the normalised radius and different models had to be combined to obtain them.

6.4.1 The PREM

PREM gives density and seismic wave velocities as piecewise continuous polynomials in the normalised radius. Gravity is given at discrete points and a least squares fit of piecewise continuous polynomials† in the normalised radius is performed.

All the parameters of the PREM are valid for a reference period of 1 sec. To use these parameters at tidal periods, a transformation was performed according to the formulæ [KANAMORI AND ANDERSON, 1977]

$$v_p|_T = v_p|_1 \{1 - E\, ln T/(\pi Q)\}, \tag{6.11}$$

$$v_s|_T = v_s|_1 \{1 - ln T/(\pi Q)\}, \tag{6.12}$$

where $v_p|_T$ and $v_s|_T$ are the compressional and shear wave velocities, respectively, for period T (in seconds), $v_p|_1$ and $v_s|_1$ are the velocities at reference period of 1 sec, Q is the quality factor and

$$E = 4/3\, (v_s|_1 / v_p|_1)^2. \tag{6.9}$$

† *Of degree no greater than four.*

6.4.2 Q Model SL8

For values of Q, model SL8 was used. For the different regions in the earth†, an average Q was taken as representative value. The inner and outer cores as well as the lithosphere‡ were considered perfectly elastic and thus $Q \rightarrow \infty$.

6.4.3 Viscosity Model

To our knowledge, there is no complete model available for viscosity within the earth. Therefore, we considered various different studies to evaluate viscosity in the mantle. According to these studies we have the following values (piecewise constant profile):

a) For the lower mantle the value of 2.5×10^{22} poise was taken [YUEN ET. AL., 1982].

b) For the transition zone the value of 10^{22} poise was taken [YUEN AND SABADINI, 1984].

c) For the LVZ the value of 10^{17} poise was taken [VETTER AND MEISSNER, 1977; VETTER, 1978].

One may argue whether the above values of viscosity, which have been estimated from long periodic phenomena, are valid at tidal periods. It is true that in the past decade or so, inferences about the viscosity of the mantle have been made almost exclusively from studies of the post-glacial rebound, assuming Maxwell rheology. However, the resolution of these techniques is inadequate to detect rapid changes in viscosity with depth [SAMMIS, ET. AL., 1977]; these rapid changes are required by the convection hypothesis [PELTIER, 1982]. It appears therefore, that the post-glacial rebound data constrain the value of the average viscosity in the mantle to be O(10^{22} poise). VETTER [1978], calculated viscosity profiles for the asthenosphere using the so called "temperature method." He assumed creep rates from plate

† *Consistent with PREM*

‡ *Crust and seismic "lid."*

tectonic movements, a typical value being $e=2\times10^{-15}$ sec^{-1}, which corresponds to a uniform motion of the plates of 2 cmy^{-1}. For the different models of the asthenosphere he used (from the viewpoint of composition and thickness), he found that viscosity differences of about 1.5 orders of magnitude between the continental and the oceanic asthenospheres are typical. In addition, viscosity in the asthenosphere, at regions away from subduction zones can be O(10^{17} poise), a value also obtained by PELTIER ET. AL. [1981], by fitting a single relaxation time SLS to the Q's of the low order fundamental normal modes of the free oscillations. In support to this short term viscosity is YAMASHITA's [1979] work. Yamashita, using an SLS rheology with viscosity O(10^{17-18} poise) was able to explain post-seismic deformations in terms of aftershock occurences.

Other studies have been carried out to infer about the viscosity of the mantle. In general, we can say that the post-glacial rebound data yield a viscosity for the mantle, which is O(10^{22} poise); this value is the average viscosity of the mantle that is required by the convection hypothesis [PELTIER, ET. AL., 1981]. Viscosity estimates obtained by SABADINI ET. AL. [1982] and YUEN ET. AL., [1982] show that values O(10^{22-23} poise) fit the polar wandering and the rotational data satisfactorily.

YUEN AND FLEITOUT [1984] examined the causes of the convective instabilities below the oceanic lithosphere. Assuming temperature and pressure dependent viscosity, they arrived at viscosities O(10^{22} poise) for the upper mantle and at viscosities O(10^{20} poise) for the LVZ.

The analyses of a data set (for instance from post-glacial rebound) using different methods and assumptions give values of viscosity that they may be different even by two orders of magnitude. Furthermore, the analyses of different data sets covering a wide range of characteristic time scales (from post-glacial rebound to convection) show that the differences in the values of viscosity are below the accuracy estimates. This may suggest that at the moment we are unable to infer about the dependence of viscosity upon frequency at long characteristic time scales. The work of YAMASHITA[1979] and PELTIER ET. AL. [1981], show that in the seismic band, an SLS rheology suggests extremly low values for viscosity in the mantle O(10^{17-18} poise), which agree well with VETTER's [1978] values for the viscosity of the asthenosphere, calculated at long characteristic time scales. It may well be that the low values of viscosity obtained by YAMASHITA[1979] and PELTIER ET. AL. [1981] reflect the presence of a low viscosity asthenosphere rather than a low viscosity mantle, i.e., attenuation of seismic waves occurs primarily in the asthenosphere. If this is true, then the viscosity in the earth may be weakly dependent upon frequency and it may be safe to assume the same values of viscosity for the entire spectrum of time scales from seismic deformations to convection. We will see later that this argument is also supported by the present study.

6.4.4 G* Model

Many independent studies have been carried out in the past to determine Gibbs free activation energy G^*. The most representative value of G^* in the LVZ appears to be 125 kCal/Mole [KOHLSTEDT AND GOETZE, 1974; WEERTMAN AND WEERTMAN, 1975]. For the transition zone and lower mantle, G^* increases almost linearly with depth. For these regions we consider the model given by SAMMIS ET. AL., [1977], for an adiabatic temperature of 0.3 K / km, consistent with the thermal model of STACEY [1977].

6.4.5 Thermal Model

We consider here Stacey's thermal model [STACEY, 1977]. We applied a least squares fit to the discrete values of temperature to obtain piecewise continuous polynomials in the normalised radius.

6.5 Computational Results

For the numerical integration of the equations of deformation we apply the finite difference method. We use subprogram DVCPR of the International Mathematical and Statistical Library (IMSL) on the I.B.M. 3090-180 VF main frame computer. The finite difference method algorithm, used by DVCPR, is described by LENTINI AND PEREYRA, [1975]. We used DVCPR to solve differential systems with known analytical solutions prior to using it for the solution of the deformation equations and we found that the accuracy estimates of DVCPR are indeed pessimistic. For the solution of the equations of deformation we used variable step-size in order to achieve uniform accuracies throughout the integration interval (from the centre of the earth to the surface). The accuracies of the final results were of the order of 10^{-6} or better.

6.5.1 Load Deformation Coefficients

The load deformation coefficients h'_n, k'_n and l'_n can be obtained directly from the solution of the equations of deformation (5.17). This can be shown easily by simple considerations. For instance, having considered that the loading mass is equal to the mass of the earth, and combining (4.62) and (4.88) the loading potential is

$$\Phi^{l}_{nm}=\{GM/R\}Y_{nm}. \tag{6.10}$$

For gravity g we have

$$g = GM / R^2. \tag{6.11}$$

Substituting (6.10) and (6.11) into the first of (1.2) and taking into account the first of (4.79) we obtain

$$h'_n = U_{nm} / R, \tag{6.12}$$

which shows that the first load number is equal to the normalised vertical displacement, i.e., it is equal to z_1 (cf: eqn. 5.16). Similarly, l'_n is the normalised tangential displacement and it is equal to z_3. In addition, the radial displacement of an equipotential surface due to terrain displacement and indirect effect is $u_{nm}^e = (1 + k'_n)\ \Phi^l_{nm}/g$ [PAGIATAKIS 1982] and $u_{nm}^e = \Phi_{nm}/g$, (Φ_{nm} is gravitational + loading potential). Combining the above formulæ with the second of (1.2) we obtain that $z_5 = 1 + k'_n$. Summarising all the above, we have

$$\left.\begin{aligned} h'_n &= z_1, \\ k'_n &= z_5 - 1, \\ l'_n &= z_3. \end{aligned}\right\} \qquad (6.13)$$

We solved the equations of deformation starting with the solution of the equations on a purely elastic, isotropic and nonrotating earth†, for different degrees of harmonic expansion. Then, we added, one at a time, anisotropy, rotation and dissipation and we re-solved the equations to determine the effects of the above components on the load deformation coefficients. The results we obtained are as follows:

a) The load numbers on a purely elastic, isotropic and nonrotating earth were compared with those of FARRELL [1972]. We found minor differences (1-3%), for $n<800$. For $n>1000$, the differences were of the order of several percent. These differences are attributed to the different earth models used for the solution of the equations. The more detailed PREM tends to increase the magnitude of the load numbers of higher degree.

b) The real part of the viscoelastic load numbers evaluated in this study was compared with ZSCHAU's [1978] load numbers. Even for the low degree h' load number ($n<10$), Zschau's values are significantly smaller than ours of the order of 4%; Zschau's values are also smaller than Farrell's load numbers by about 2.5%. This disagreement between Farrell's and Zschau's load numbers is not what one might have expected. Both researchers used the same earth model and it would be reasonable to expect that Zschau's

† *Compressibility and self-gravitation were considered in all cases.*

load numbers would be larger than Farrell's, since the former are viscoelastic.

c) The effect of anisotropy on h'_n, k'_n and l'_n, in the upper 220 km is shown in Fig. 6.1 and can be as high as (in absolute value) 1.9%, 2.3% and 2.5%, respectively.

d) The effect of rotation on h'_n, k'_n and l'_n is shown in Fig. 6.2. It can be as large as (in absolute value) 1.8%, 3.1% and 3.3%, respectively. We found a weak latitude dependence of the load numbers for $n \leq 4$. This effect amounts to a maximum of 0.4%, when latitude varies from $0°$ to $\pm 45°$.

e) The dissipation of tidal energy in the earth results in an increase in the absolute value of the load numbers. Since the effect of dissipation is frequency dependent, we calculated load numbers at different frequencies. For semidiurnal tides and all degrees of expansion, the load numbers were systematically larger than on an elastic earth. Yet, this effect never exceeded 0.2%. However, load numbers calculated at longer periods can be significantly higher than on an elastic earth. For instance, for $n=100$ at fortnightly period we found that h', k' and l' were larger than their corresponding values on an elastic earth, by 0.5%, 1.5% and 1.3%, respectively.

f) On a dissipating earth, it is the imaginary part, rather than the real part of the load numbers, that is more sensitive to Gibbs free activation energy G^*, viscosity η and quality factor Q. More specifically, the imaginary part of the load numbers of $80<n<120$ are sensitive to G^*, η and Q values in the LVZ. As an example, for $n=100$, a change of G^* by 5kcal/mole (one sigma), affects the imaginary load numbers by almost two orders of magnitude; this is equivalent to a phase shift of the order of tens of degrees. A change of η by one order of magnitude affects the load numbers by one order of magnitude. A change of Q by 10, affects the load numbers by only 10%. In all cases, the real part was practically unaffected. It is interesting to note that changes of viscosities by almost two orders of magnitude in the lower mantle did not affect the imaginary part of the load numbers, indicating that tidal load dissipation occurs primarily in the LVZ. Complex load numbers on

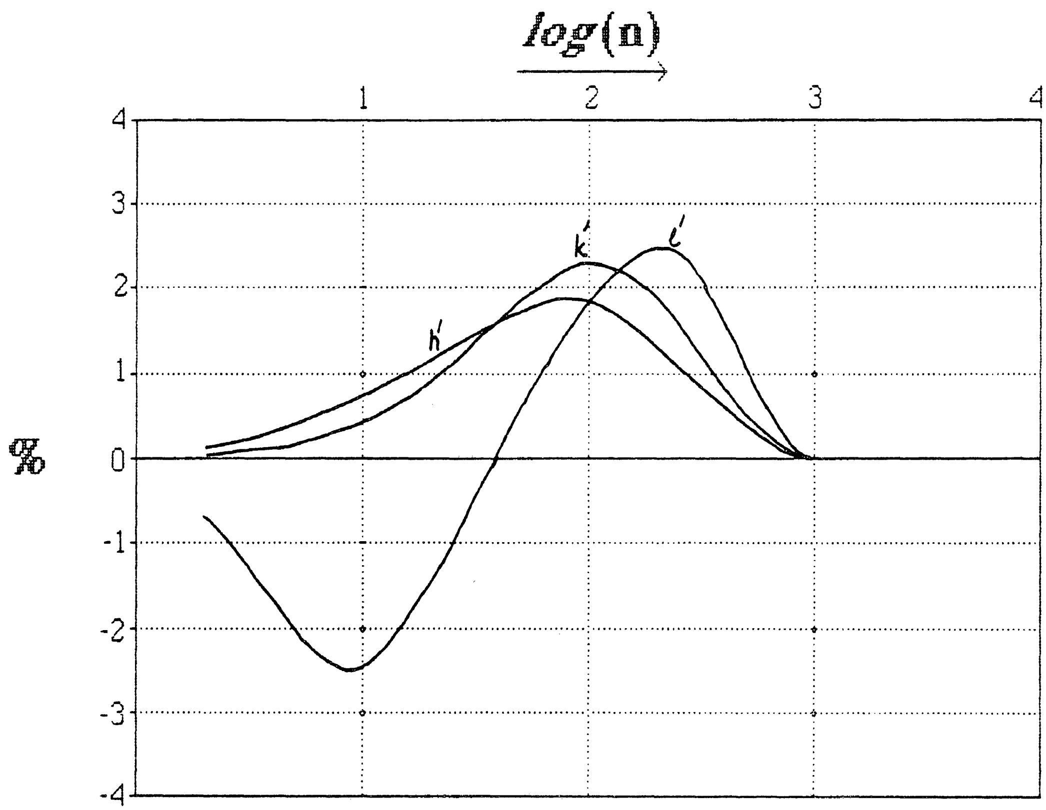


Fig. 6.1. Effect of anisotropy on the load deformation coefficients (in %).

Curves are plots of $(h'_{aniso} - h'_{iso})/h'_{iso}$, $(l'_{aniso} - l'_{iso})/l'_{iso}$ and $(k'_{aniso} - k'_{iso})/k'_{iso}$.

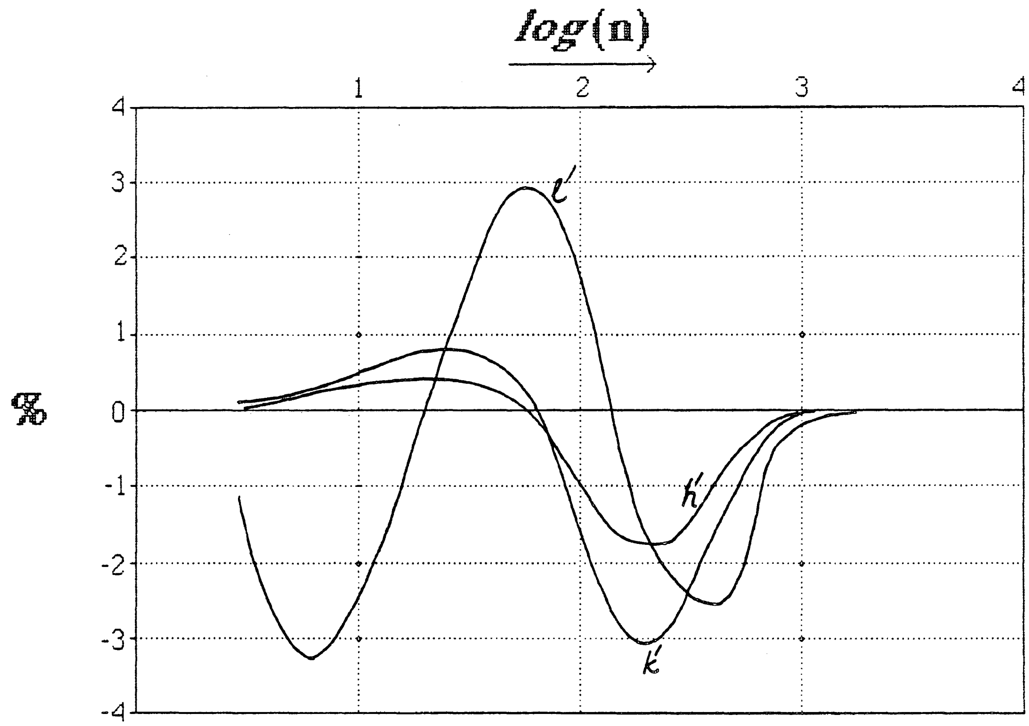


Fig. 6.2. Effect of earth's rotation on the load deformation coefficients.

Curves are plots of $(h'_{rot} - h'_{non\text{-}rot}) / h'_{non\text{-}rot}$,

$(l'_{rot} - l'_{non\text{-}rot}) / l'_{non\text{-}rot}$ and $(k'_{rot} - k'_{non\text{-}rot}) / k'_{non\text{-}rot}$.

a viscoelastic, anisotropic and rotating earth are given in Appendix III.

f) As a by-product of the integration of the equations of deformation, we obtained load numbers as functions of depth. We selected to plot h'_n and nl'_n versus depth for n = 20, 100, 200 and 500 (Fig. 6.3). We found that after a depth of about 1.2 times the wavelength of the deformation, the load numbers approach zero, asymptotically. As a rough rule-of-thumb we can say that load deformations penetrate the earth to a depth, which is twice the wavelength of the load. For n >500, we can see that the deformation takes place only in the lithosphere and the discontinuities of density and elastic parameters of the earth at the depths of 15 km and 24.4 km (Mohorovicic discontinuity) are reflected strongly in the load numbers, which is intuitively pleasing.

6.5.2 Green's Functions

For the evaluation of the effect of ocean tide loading on geodetic quantities of interest, such as, deformations, gravity and tilt, the usual procedure of convolution of appropriate Green's functions with an ocean tide model can be followed. For this reason, we evaluated Green's functions for radial and horizontal displacements, gravity and tilt (Fig. 6.4 - Fig. 6.7), following FARRELL's [1972] procedure†. Since Farrell's Green's functions are the most widely used nowadays, we decided to tabulate our Green's functions (real part) in Farrell's form so as to be easily adaptable into existing software. The imaginary part of the Green's functions is given in the form of phase shift (lag or advance) with respect to the total load effect (Fig. 6.8). Their numerical values are tabulated in Appendix III.

On a purely elastic earth, the effect of load on any geodetic quantity of interest decreases as the point of load gets further away from the point of interest; Green's functions become smaller (in magnitude) as ψ increases. This is not necessarily true on a

† *We did not use the disc factor artifice however, for two reasons. Firstly because it was introduced by Farrell to speed up the convergence, a factor very important for the computers of the early 1970's. Secondly, the disk factor has been proven to be not exactly correct [see for instance FRANCIS AND DEHANT, 1987] because it requires the use of Euler's transformation; this transformation introduces errors for small, as well for large ψ.*

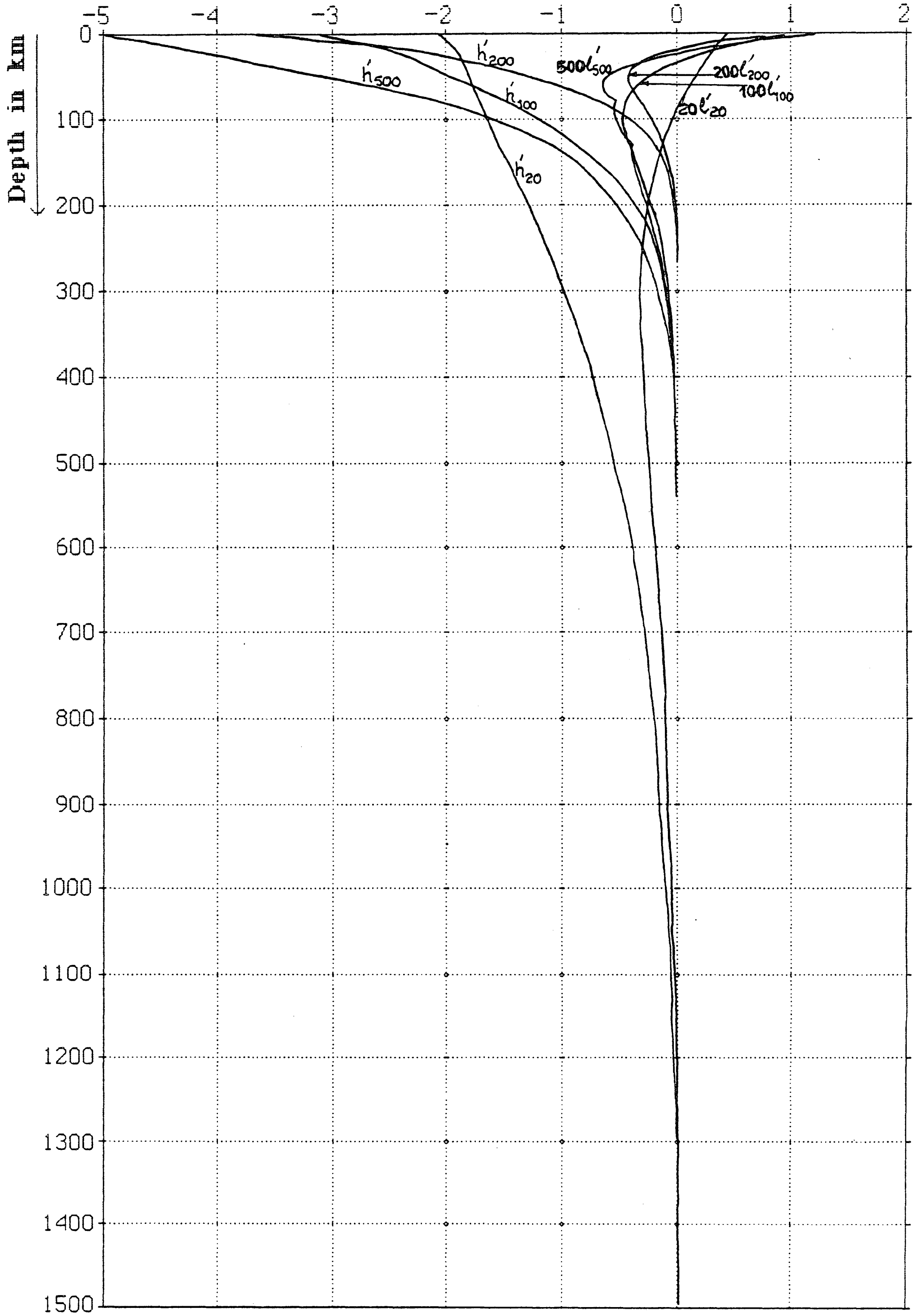


Fig. 6.3. Load deformation coefficients versus depth.

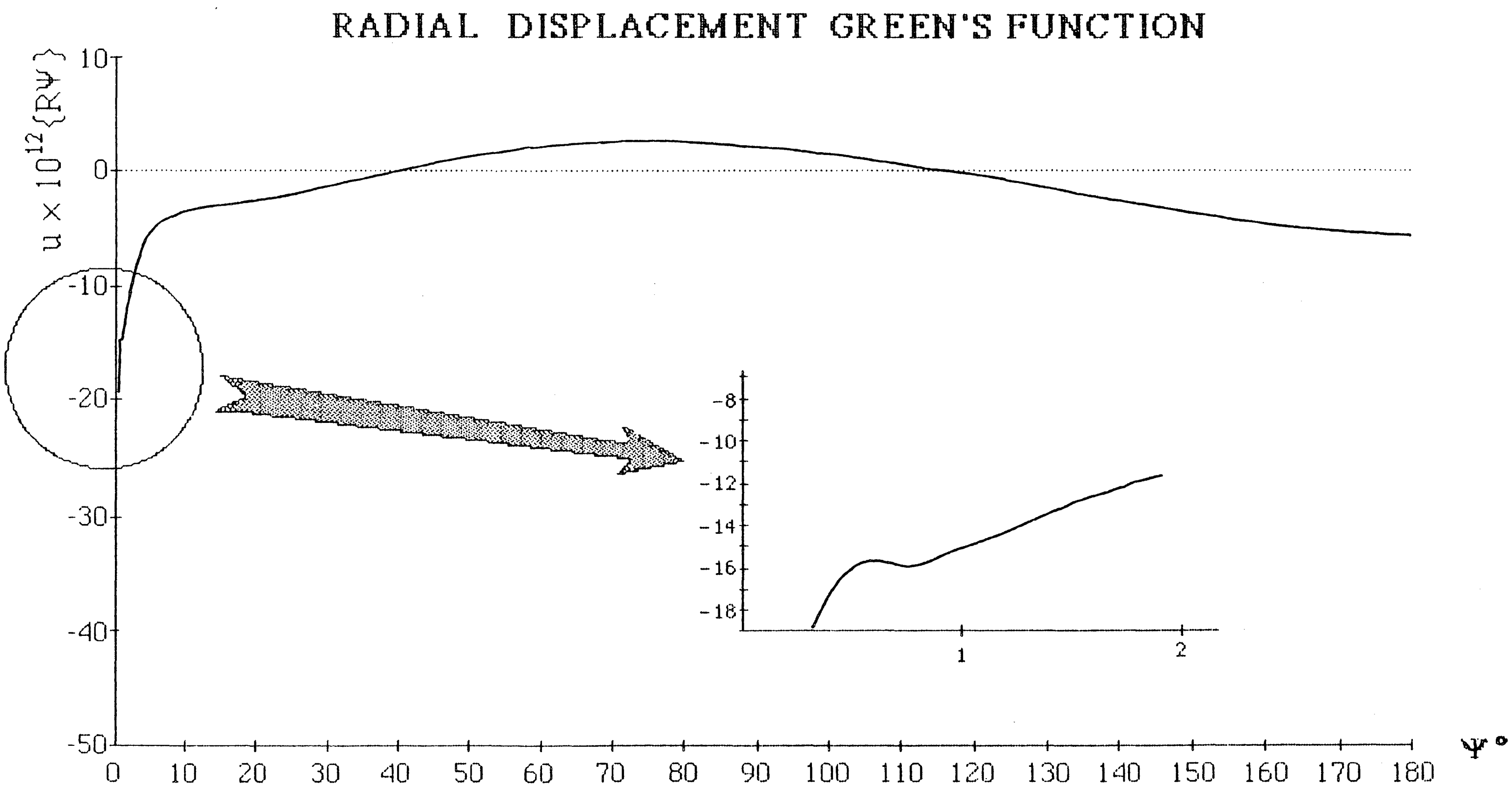


Fig. 6.4. Radial displacement Green's function (real part). u is radial displacement, R is the mean radius of the earth and ψ is the geocentric angle between point of interest and point of load.

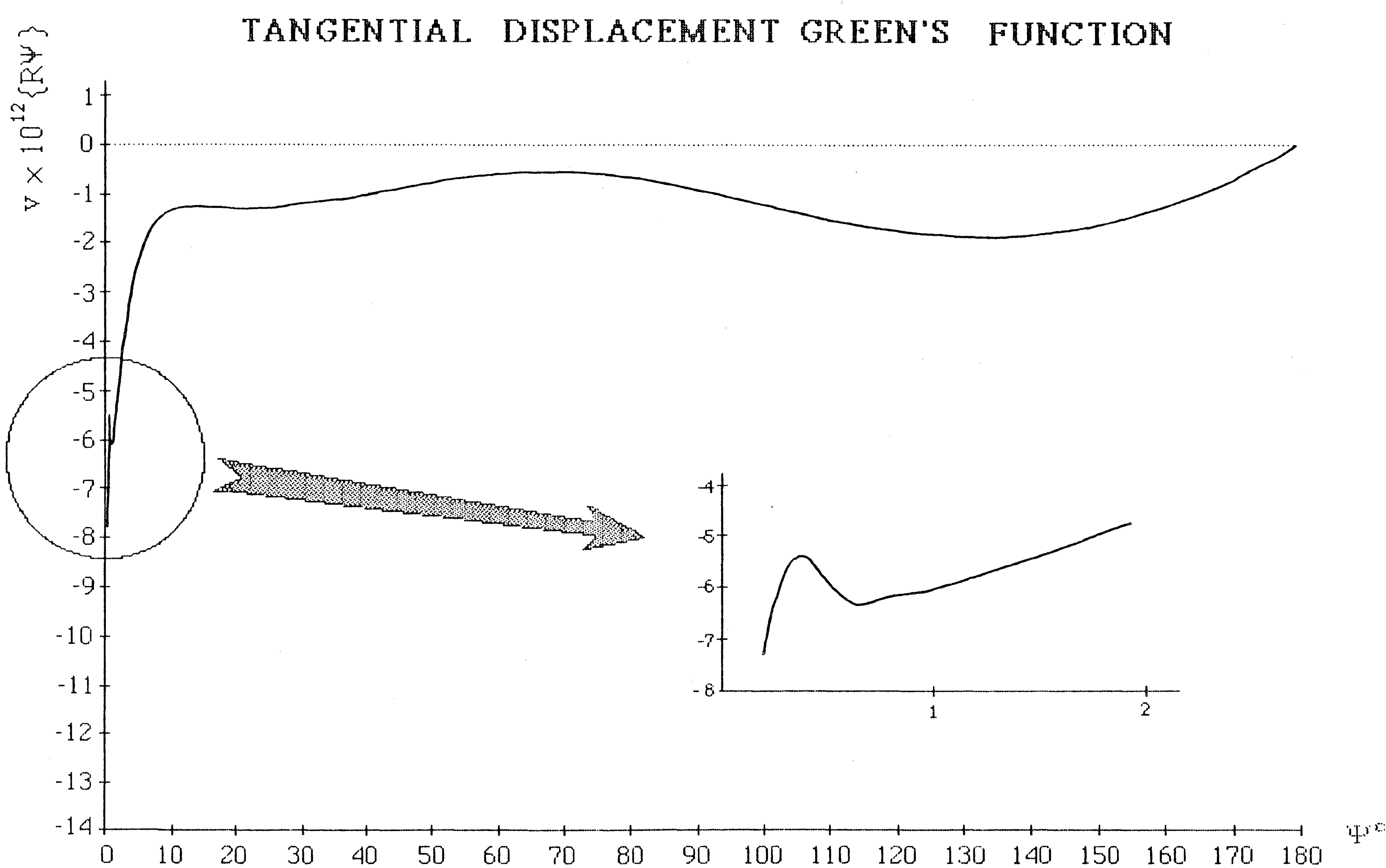


Fig. 6.5. Tangential displacement Green's function (real part). R is the mean radius of the earth and ψ is the geocentric angle between point of interest and point of load.

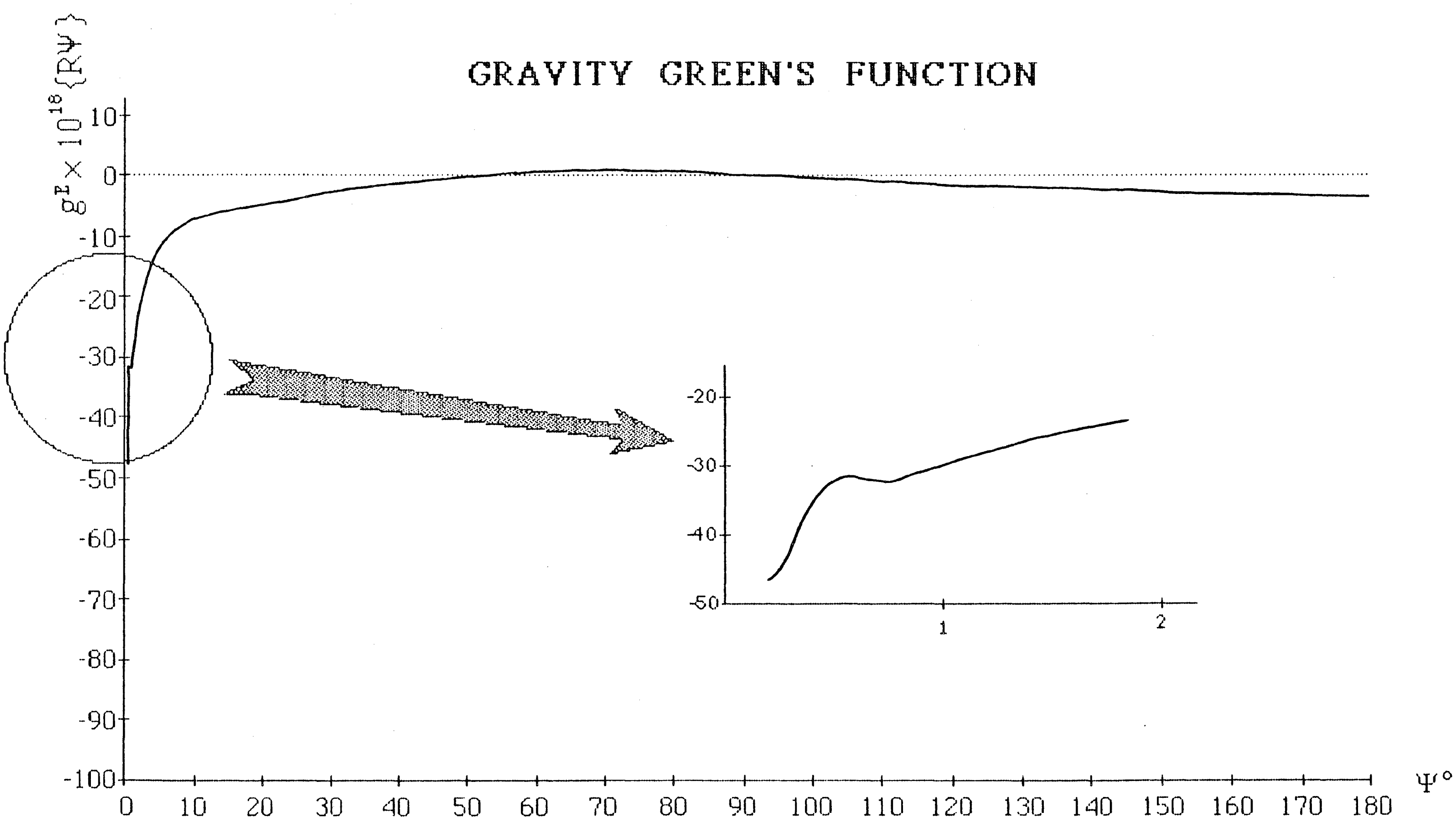


Fig. 6.6. Gravity Green's function (real part). g^E is the elastic gravity, R is the mean radius of the earth and ψ is the geocentric angle between point of interest and point of load.

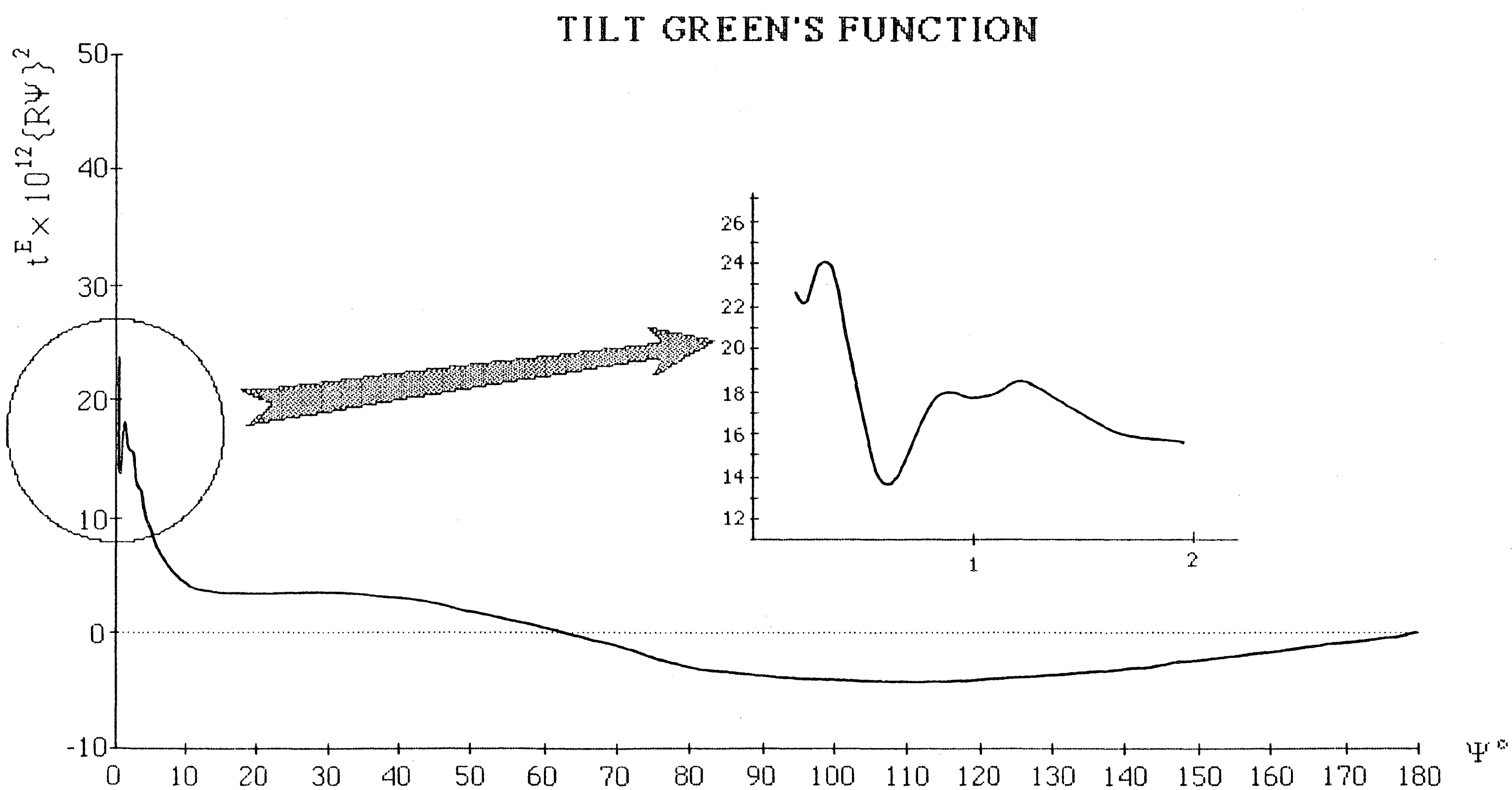


Fig. 6.7. Tilt Green's function (real part). t^E is the elastic tilt R is the mean radius of the earth, and ψ is the geocentric angle between point of interest and point of load.

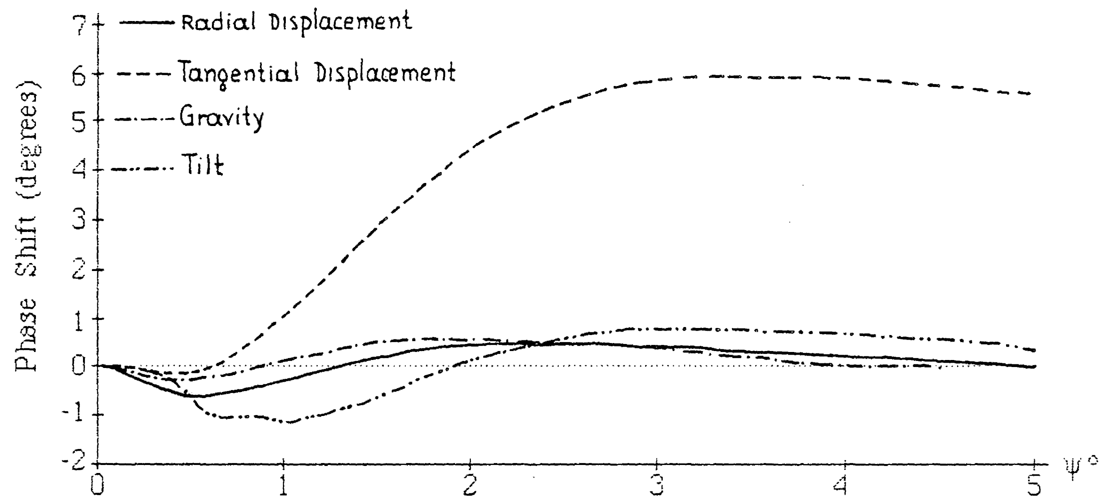


Fig. 6.8. Phase shift Green's Functions. ψ is the geocentric angle between point of interest and point of load in degrees. Phase shifts are with respect to the total elastic deformation.

viscoelastic earth. More specifically, our Green's functions grow larger (in magnitude) as ψ increases from approximately 0.5° to 1°. On the contrary, when $\psi > 1^\circ$, they decrease as ψ increases. This is explained as follows: As ψ becomes larger than 0.5°, the deformation enters deeper in the earth and when $\psi \cong 1^\circ$, it is the LVZ that supports, almost entirely, the load. Since LVZ is significantly weaker than the lithosphere, the load effect becomes more pronounced for $\psi \cong 1^\circ$, than for $\psi \cong 0.5^\circ$, where the load is practically supported only by the stronger lithosphere. The peripheral bulge that is present in the viscoelastic Green's functions vanishes almost entirely when viscoelasticity is omitted. To support this argument we also calculated tilt Green's functions (where the peripheral bulge is more pronounced) using Zschau's load numbers. It was pleasing to realise that the peripheral bulge was present as opposed to Farrell's Green's functions where the bulge was flattened out. It it worth noting that both researchers used the same earth model. In our case, the peripheral bulge is further enhanced by the more detailed earth model we used (PREM). The above arguments agree also with PELTIER [1974].

Comparisons of the real part of the Green's functions obtained in this study, with FARRELL's [1972], show that there is a difference of a few percent for $0.3^\circ < \psi < 1.5^\circ$; our values appear to be larger due to the presence of dissipation. When $\psi > 1.5^\circ$, they become almost identical.

6.6 Applications

The software for loading calculations, developed in PAGIATAKIS [1982], has been modified to account for the developments of the present study. The validity and predictive power of our model is tested against accurately determined M_2 gravity tide residuals at 10, globally distributed observational sites (Fig. 6.9, Table, 6.1). We calculate load gravity tide using: **a)** FARRELL's [1972] Green's functions on an elastic earth, and **b)** the complex Green's functions developed in this study. The following important conclusions are drawn:

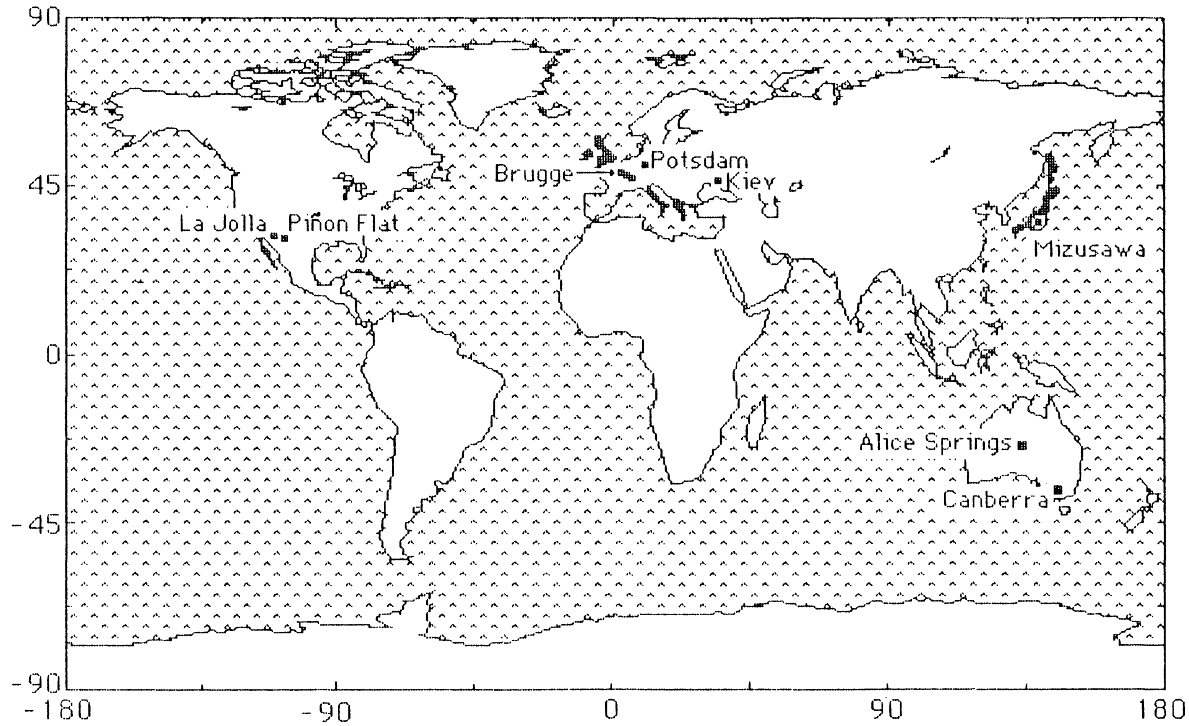


Fig. 6.9. Distribution of the test stations

M_2 LOAD GRAVITY TIDE

STATION	ELASTIC EARTH	PRESENT STUDY	OBSERVED RESIDUALS	IMPROVEMENT Amplitude	Phase
La Jolla†	2.047 (-84.5)	2.070 (-84.4)	3.640 (-81.0)	+1.0	+0.1
Piñon Flat†	0.983 (-74.2)	0.994 (-73.9)	1.490 (-74.0)	+2.0	+0.1
Alice Springs‡	0.511 (-48.8)	0.523 (-48.4)	0.530 (-48.0)	+63.0	+0.4
Canberra‡	3.463 (-39.3)	3.475 (-39.4)	3.570 (-41.0)	+11.0	+0.1
Bruxelles♥	1.548 (59.9)	1.576 (60.3)	1.760 (61.4)	+13.0	+0.3
Brugge♥	2.029 (69.4)	2.045 (70.4)	2.690 (72.0)	+2.0	+1.0
Walferdange♥	1.237 (57.0)	1.259 (56.9)	1.630 (55.0)	+6.0	+0.1
Potsdam♥	0.957 (43.2)	0.974 (43.7)	0.990 (44.6)	+52.0	+0.5
Mizusawa♦	1.873 (44.3)	1.867 (44.6)	1.820 (46.0)	+11.0	+0.3
Kiev♥	0.385 (10.8)	0.391 (13.2)	1.060 (12.5)	+1.0	+1.0

Table 6.1. M_2 Load gravity tide. All the amplitudes are in μGals and the phases are in degrees. Predictions on an elastic earth have been obtained by using FARRELL's [1972] Green's functions and SCHWIDERSKI's [1978] M_2 ocean tide model. The last two columns show the improvements in amplitudes (in %) and phases (in degrees) the present model introduced. Amplitude improvements are ratios of the difference between predictions using Farrell's Green's functions and the present model over the remaining discrepancy between Elastic predictions and observed gravity residuals.

† *Warburton et. al., (1975),*
‡ *Melchior, (1983),*
♥ *Francis and Dehant (1977),*
♦ *Hosoyama (1977).*

a) The difference between observed residual gravity and predicted load gravity tide amplitudes is reduced for all tested stations by as much as 63%, when compared to predictions on an elastic, isotropic and nonrotating earth.

b) The phases of the predicted load gravity tide are closer to the observed phases by 0.4° (in average) when compared to predictions on an elastic, isotropic and nonrotating earth. Improvements in the phases using the present model can be as high as 1°.

c) The average of the phase differences between observed gravity residuals and predicted load gravity tide is 1.2° as opposed to 1.6° on an elastic, isotropic and nonrotating earth.

d) The above results improve further if we exclude stations, such as La Jolla, Piñon Flat and Kiev, which are influenced by local tides not included in the Schwiderski's model used in the present study. It is the ocean tide model, rather than the loading model that imposes limitations in the accuracy of the predicted load tide at stations close to the shore.

7

CONCLUSIONS AND RECOMMENDATIONS

The main objective of the present research was to develop a mathematical model that would describe the earth's response to harmonic surface loading with particular emphasis to ocean tide loading. The three main issues in the development of the model were: **a)** consideration of viscoelastic rheology, **b)** anisotropy and **c)** earth's rotation. Other factors, such as, layered earth with solid inner core and fluid outer core, self-gravitation and compressibility were also considered in the model.

7.1 The Equations of Deformation

The equations of deformation, which describe the response of the earth to surface loading, were derived initially in a 6-D Lagrangean configuration manifold; the state vector consisted of the complex vertical and horizontal displacements and the complex loading potential. The equations were second order partial differential equations in the state variables. Furthermore, in order to facilitate their solution, we transformed the equations into 12 first order linear simultaneous ordinary differential equations with variable coefficients, valid in a 12-D tangent bundle space. The new 12-D state vector consisted of six complex variables namely, horizontal and vertical displacements, loading potential, normal and shear stresses and gravity disturbance (due to loading).

Perhaps, the most challenging problem that we had to solve was the determination of the dissipation function. We postulated that the rheology of the earth can be described by a standard linear solid (SLS). Standard-linear-solid-type rheology is characterised by transient anelasticity, appropriate at tidal frequencies and strain levels. In addition, we found that the dissipation function depends on the imaginary part of the complex compliance of the SLS which, in turn, depends on the dissipation mechanism within the earth. Furthermore, we postulated that the dissipation mechanism in the mantle can be described by a grain-boundary relaxation model; due to the high temperature background, the dissipation is primarily a thermally activated process. Finally, the dissipation function was derived, depending on viscosity, quality factor Q, Gibbs free activation energy and absolute temperature, quantities available from existing literature.

The inclusion of rotation of the earth in the equations of deformation complicated their expansion into spherical harmonics. However, by considering rotational symmetry in the properties of the earth and sectorial tides (semidiurnal), we succeeded in expanding the equations into spherical harmonics.

To our knowledge, the only available information about the anisotropy in the earth, (in global scale), is contained in the PREM. More specifically, PREM allows only for transverse isotropy in the upper 220 km of the mantle; that is what we considered in our model. However, when more complicated types of anisotropy become available, they can be accounted for easily by extending our present model.

While developing the equations of deformation, we also demonstrated that it is easy to account for different dissipation mechanisms. Once a specific rheological model and dissipation mechanism are accepted to represent the earth, the dissipation function, and subsequently the dissipation terms in the equations, can be derived and they can replace the corresponding terms in the equations in the tangent bundle space.

7.2 Solution of the Equations of Deformation

The solution of the equations was obtained numerically using the finite difference method of numerical integration. We considered the most recent earth models available in the literature for the distribution of density, elasticity parameters as well as thermodynamical state of the earth. Some of the above parameters, being valid only at seismic frequencies, were transformed to tidal frequencies, using dispersion relations.

In order to study the effects of anisotropy, earth rotation and dissipation of ocean loading tidal energy, on the load deformation coefficients, we solved the system of equations many times and for different degrees of harmonic expansion of load, adding to the equations the above features, one at a time. We found that, if the load effects need be evaluated with an accuracy of 1% or better, anisotropy, earth's rotation and dissipation must be considered. More specifically,

a) Anisotropy affects the load numbers by as much as 2.5%,

b) Earth rotation affects the semidiurnal load numbers by as much as 3.3%,

c) At semidiurnal frequencies, load numbers on a dissipative earth are slightly larger (about 0.2% maximum) in magnitude than their corresponding values on an elastic earth. However, our calculations showed that at longer periods, e.g., fortnightly tides, this effect can be as high as 1.5%. When solving for fortnightly tides, we disregarded the rotation of the earth.

7.3 <u>Green's Functions</u>

We evaluated complex Green's functions for vertical and horizontal displacements, for gravity and for tilt. We found that loads applied at about 100 km from the point of interest, introduce phase shifts in the calculation of the above quantities of the order of several degrees. For instance, at the U.N.B. earth tides station, which is located about 80 km from the Bay of Fundy, we found a phase shift of 1.08° for the N-S tilt and 3.65° for the E-W tilt, when compared to the purely elastic case. The combined effects of anisotropy, earth rotation and dissipation affected the N-S amplitude by 4.7% and the E-W amplitude by

5.2%. The above predictions explain the residual tilt in both directions better than the corresponding predictions on an elastic, isotropic and nonrotating earth. However, there is still some unexplained difference between observed residual tilt and predicted load tilt.

The validity and predictive power of our model was tested against accurately determined M_2 gravity tide residuals at 10, globally distributed tidal stations. We showed that for all tested stations the present model is in better agreement with observed residual gravity both, in amplitude and phase, when compared to predictions on an elastic, isotropic and nonrotating earth.

For points more than approximately 300 km from the shore, phase shifts become very small and perhaps smaller than the observational accuracies. However, the amplitudes may be affected by a few percent due to anisotropy and earth rotation. The accuracy of the calculated load effects using the present model is belived to be better than 1%.

7.4 Recommendations

The present research opens new directions in the study of the response of the earth to external forces in conjuction with studies of the interior of the earth. Some of the prospects of this research can be summarised as follows:

a) Loading calculations are very sensitive to the elasticity and thermodynamic properties of the earth and especially those of the LVZ. Due to the imperfect elastic behaviour of the earth, phase shifts of the order of a few degrees are introduced in the calculations of the loading effects. We believe that the accuracy of the observed phases of tilt tide is at least one order of magnitude better than the phase shifts introduced by the imperfect elasticity in the earth; this makes loading observations a very effective tool for the determination of the properties of the interior of the earth. The present model can be used in conjuction with observations of the ocean loading effect to provide us with further constraints on the properties of the earth.

b) We recommend that the stability of the solution of the equations of deformation must be studied in a rigorous fashion, using Lyapunov's stability theory. This will provide us with a better understanding of the loading deformations.

c) With a versatile formulation such as the one developed in this research, other plausible dissipation mechanisms within the earth can be evaluated.

d) The boundary value problem, we have dealt with here, is well defined. It may be important to obtain an analytical solution, so as more insight into the mechanism of the loading deformations is gained. It may be even necessary to transform the equations into the phase space and study the stability of the solution using methods of catastrophe theory.

e) The present model has been tested against observed gravity residuals at different tidal stations, in global scale. However, it is known that ocean tide loading deformations of the crust affect VLBI and satellite laser ranging observations. We strongly believe that the present model is a powerful tool in the analysis of such observations.

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APPENDIX I

Lagrangean Equations of Motion for Nonconservative Holonomic Systems

For the derivation of Lagrangean equations of motion, we consider that the mechanical system under investigation is rheonomic with n degrees of freedom. The position vectors $\mathbf{r}_i$ of its particles with respect to a coordinate system are dependent variables and can be written in an n-dimensional configuration space as follows [MEIROVITCH, 1967]

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2,..., q_n, t), \tag{I.1}$$

and the velocities as

$$\dot{\mathbf{r}}_i = \partial\mathbf{r}_i/\partial t + (\partial\mathbf{r}_i/\partial q_1)\dot{q}_1 + (\partial\mathbf{r}_i/\partial q_2)\dot{q}_2 + ... + (\partial\mathbf{r}_i/\partial q_n)\dot{q}_n. \tag{I.2}$$

The kinetic energy of the system can be written as

$$T = (1/2)\sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i, \tag{I.3}$$

where m_i are the masses of the particles (N in total) having position vectors $\mathbf{r}_i$. Substituting (I.2) into (I.3) yields

$$T = (1/2)\sum_{i=1}^{N} m_i \{ (\partial \mathbf{r}_i/\partial q_r)(\partial \mathbf{r}_i/\partial q_s)\dot{q}_r\dot{q}_s + 2(\partial \mathbf{r}_i/\partial t)(\partial \mathbf{r}_i/\partial q_r)q_r + (\partial \mathbf{r}_i/\partial t)(\partial \mathbf{r}_i/\partial t)\},$$

$$r,\ s = 1,\ 2, \ldots,\ n \qquad (I.4)$$

where the summation convention applies to repeated indices r, s. Introducing virtual displacements δq_k, the variation of kinetic energy can be written as

$$\delta T = (\partial T/\partial q_k)\delta q_k + (\partial T/\partial \dot{q}_k)\delta \dot{q}_k, \quad k = 1,\ 2, \ldots,\ n \qquad (I.5)$$

Integrating (I.5) with respect to time in the time interval $[t_1,t_2]$ and interchanging the implied summation with the integration yields

$$\int_{t_1}^{t_2} \delta T dt = \sum_{k=1}^{n} \int_{t_1}^{t_2} (\partial T/\partial q_k)\delta q_k dt + \sum_{k=1}^{n} \int_{t_1}^{t_2} (\partial T/\partial \dot{q}_k)\delta \dot{q}_k dt. \qquad (I.6)$$

In the second integral of the right-hand-side, variation δ can be interchanged with differentiation d/dt. Therefore,

$$\int_{t_1}^{t_2} \delta T dt = \sum_{k=1}^{n} \int_{t_1}^{t_2} (\partial T/\partial q_k)\delta q_k dt + \sum_{k=1}^{n} \int_{t_1}^{t_2} (\partial T/\partial \dot{q}_k)(d\delta q_k/dt)dt. \qquad (I.7)$$

Integrating by parts, (I.7) becomes

$$\int_{t_1}^{t_2} \delta T dt = \sum_{k=1}^{n} \int_{t_1}^{t_2} (\partial T/\partial q_k)\delta q_k dt + \sum_{k=1}^{n} \left\{ [(\partial T/\partial \dot{q}_k)\delta q_k]_{t_1}^{t_2} - \int_{t_1}^{t_2} [d(\partial T/\partial \dot{q}_k)/dt]\delta q_k dt \right\}. \quad (I.8)$$

Since the virtual displacement δq_k is supposed to vanish at t_1 and t_2, the term in the first square brackets of (I.8) vanishes. Therefore, (I.8) becomes

$$\int_{t_1}^{t_2} \delta T dt = \sum_{k=1}^{n} \left\{ \int_{t_1}^{t_2} (\partial T/\partial q_k)\delta q_k dt - \int_{t_1}^{t_2} \left[d(\partial T/\partial \dot{q}_k)/dt \right] \delta q_k dt \right\} . \qquad \text{(I.9)}$$

By interchanging variation and integration in the left-hand-side of (I.9) and rearranging the right-hand-side we obtain

$$\delta \int_{t_1}^{t_2} T dt = \sum_{k=1}^{n} \int_{t_1}^{t_2} \left\{ \partial T/\partial q_k - d(\partial T/\partial \dot{q}_k)/dt \right\} \delta q_k dt. \qquad \text{(I.10)}$$

If we consider n generalised forces Q_k acting on the mechanical system, their virtual work can be written as

$$\delta W = Q_k \delta q_k, \qquad k = 1, 2, \ldots, n. \qquad \text{(I.11)}$$

By substituting (I.10) and (I.11) into (I.9) yields

$$\sum_{k=1}^{n} \int_{t_1}^{t_2} \left\{ \partial T/\partial q_k - d[\partial T/\partial \dot{q}_k] + Q_k \right\} \delta q_k dt = 0. \qquad \text{(I.12)}$$

Since the virtual displacements δq_k are arbitrary, equation (I.12) holds true only when the term in braces is zero. Therefore,

$$\partial T/\partial q_k - d[\partial T/\partial \dot{q}_k]dt + Q_k = 0. \qquad \text{(I.13)}$$

Equations (I.13) were derived without assuming the character of the generalised forces Q_k. Generalised forces Q_k can be conservative, nonconservative or both. Furthermore, one can say that among the various kinds of forces acting on a particle of the system, it is possible to recognise a special type of friction force F arising from the motion of the particle in a viscous medium. This nonconservative force is assumed to be proportional to some power of velocity [MEIROVITCH, 1967]. Therefore, we can write that

$$Q_k = Q_k^{c} + Q_k^{nc} + F, \qquad k = 1, 2, \ldots, n, \qquad (I.14)$$

where Q_k^{c} and Q_k^{nc} are conservative, non-conservative (other than F) generalised forces. For the conservative force Q_k^{c} we can write

$$Q_k^{c} = - \partial\Phi/\partial q_k, \qquad k = 1, 2, \ldots, n, \qquad (I.15)$$

where Φ is a potential. If D is a function that gives the amount of energy per unit time (units: ML^2T^{-3}) dissipated in the mechanical system, then

$$F = - \partial D/\partial \dot{q}_k, \qquad k = 1, 2, \ldots, n. \qquad (I.16)$$

Function D is called the dissipation function. Introducing equations (I.14), (I.15) and (I.16) into (I.13), yields

$$d[\partial T/\partial \dot{q}_k]/dt - \partial T/\partial q_k + \partial D/\partial \dot{q}_k + \partial\Phi/\partial q_k = Q_k^{nc}, \qquad k = 1, 2, \ldots, n. \qquad (I.17)$$

Q_k^{nc} are forces steming neither from a potential field, nor from friction. In addition, forces introduced by T and Φ are conservative. Since V is not a function of the generalised velocities, then

$$\partial T/\partial \dot{q}_k = \partial L/\partial \dot{q}_k, \qquad k = 1, 2, \ldots, n. \qquad (I.18)$$

Thus, equation (I.17) becomes

$$d[\partial L/\partial \dot{q}_k]/dt - \partial L/\partial q_k + \partial\Phi/\partial q_k + \partial D/\partial \dot{q}_k = Q_k^{nc}, \qquad k = 1, 2, \ldots, n. \qquad (I.19)$$

APPENDIX II

Expansion of the Equations of Deformation into Spherical Harmonics

The individual terms of the equations of deformation can be written as functions of spherical harmonics $Y_{nm}(\theta, \lambda)$ as follows

$$e_{\theta\theta}+e_{\lambda\lambda} = (1/r)[\partial_\theta v + 2u + \sin^{-1}\theta\,\partial_\lambda w + v\cot\theta] =$$

$$(1/r)\{\partial_\theta [V\partial_\theta Y_{nm}] + \sin^{-1}\theta\,\partial_\lambda [V \sin^{-1}\theta\,\partial_\lambda Y_{nm}] + 2UY_{nm} + \cot\theta\, V\,\partial_\theta Y_{nm}\} =$$

$$(1/r)\{V\partial_\theta{}^2 Y_{nm} + V\sin^{-2}\theta\,\partial_\lambda{}^2 Y_{nm} + 2UY_{nm} + V\cot\theta\,\partial_\theta Y_{nm}\} =$$

$$(2U/r)Y_{nm} + (V/r)\,\{\partial_\theta{}^2 Y_{nm} + \sin^{-2}\theta\,\partial_\lambda{}^2 Y_{nm} + \cot\theta\,\partial_\theta Y_{nm}\} =$$

$$(2U/r)Y_{nm} - (V/r)\,n(n+1)Y_{nm} =$$

$$[(2U/r) - (V/r)\,n(n+1)]Y_{nm}\,,$$

or,
$$e_{\theta\theta} + e_{\lambda\lambda} = Z(r, t)Y_{nm},$$

$$Z(r, t) = (2U/r) - (V/r)\,n(n+1). \tag{II.1}$$

In addition, we have

$$e_{rr} = \partial_r u = \partial_r U Y_{nm}. \tag{II.2}$$

Similarly for the rest of the terms of the first equation of deformation we have

$$- L/r\ [\partial_\theta e_{r\theta} + \partial_\lambda e_{r\lambda} + e_{r\theta}\cot\theta] =$$

$$- L/r\,\{\partial_\theta [\partial_r v - v/r + 1/r\,\partial_\theta u] + \sin^{-1}\theta\,\partial_\lambda [(r\sin\theta)^{-1}\partial_\lambda u + \partial_r w - w/r]$$

$$+ [\partial_r v - v/r + 1/r\,\partial_\theta u]\cot\theta\,\} =$$

$$- L/r^2\{\partial_\theta [r\partial_r v - v + \partial_\theta u] + \sin^{-1}\theta\,\partial_\lambda [\sin^{-1}\theta\,\partial_\lambda u + r\partial_r w - w]$$
$$+ [r\partial_r v - v + \partial_\theta u]\cot\theta \} =$$
$$- L/r^2\{ r\partial_r V\partial_\theta{}^2 Y_{nm} - V\partial_\theta{}^2 Y_{nm} + U\partial_\theta{}^2 Y_{nm} + \sin^{-2}\theta\, U\partial_\lambda{}^2 Y_{nm}$$
$$+ r\sin^{-2}\theta\,\partial_r V\partial_\lambda{}^2 Y_{nm} - \sin^{-2}\theta\, V\partial_\lambda{}^2 Y_{nm} - r\cot\theta\partial_r V\partial_\theta Y_{nm} - \cot\theta\, V\partial_\theta Y_{nm}$$
$$+ \cot\theta\, U\partial_\theta Y_{nm}\} =$$
$$- L/r^2\{ [U - V + r\partial_r V]\partial_\theta{}^2 Y_{nm} + \sin^{-2}\theta\,[U - V + r\partial_r V]\partial_\lambda{}^2 Y_{nm}$$
$$+ \cot\theta\,[U - V + r\partial_r V]\partial_\theta Y_{nm}\} =$$
$$- L/r^2[U - V + r\partial_r V]n(n+1)Y_{nm}. \qquad \text{(II.4)}$$

Furthermore, considering (II.1) and (II.2) we obtain

$$(2/r)[(A - N - F)(e_{\theta\theta} + e_{\lambda\lambda}) + (F - C)e_{rr}] =$$
$$= 2/r\ [(A - N - F)Z + (F - C)\partial_r U]Y_{nm}. \qquad \text{(II.5)}$$

For the second equation of deformation we have

$$(\rho_o/r)\partial_\theta \Phi = (\rho_o/r)\partial_\theta [\Psi Y_{nm}] = (\rho_o/r)\Psi\partial_\theta Y_{nm} \qquad \text{(II.6)}$$

$$\partial_r[Le_{r\theta}] = \partial_r[L(\partial_r v - v/r + r^1\,\partial_\theta u)] =$$
$$\partial_r[L(\partial_r V\partial_\theta Y_{nm} - Vr^1\partial_\theta Y_{nm} + Ur^1\partial_\theta Y_{nm}] =$$
$$\partial_r[(L/r)(U - V + r\partial_r V)\partial_\theta Y_{nm}] =$$
$$\partial_r[(L/r)(U - V + r\partial_r V)]\,\partial_\theta Y_{nm} \qquad \text{(II.7)}$$

$$(1/r)\partial_\theta\ [A(e_{\theta\theta} + e_{\lambda\lambda}) + Fe_{rr}] =$$

$$(1/r)\partial_\theta\ [AZY_{nm} + F\partial_r U\, Y_{nm}] =$$

$$(1/r)[AZ + F\partial_r U]\partial_\theta Y_{nm} \qquad \text{(II.8)}$$

$$e_{\lambda\lambda} = (1/r)[\sin^{-1}\theta\,\partial_\lambda w + u + \cot\theta\, v] =$$

$$(1/r)[\sin^{-1}\theta\,\partial_\lambda(V\sin^{-1}\theta\,\partial_\lambda Y_{nm}) + UY_{nm} + \cot\theta(V\partial_\theta Y_{nm})] =$$

$$(1/r)[\sin^{-2}\theta V\partial_\lambda^2 Y_{nm} + UY_{nm} + V\cot\theta\,\partial_\theta Y_{nm}] =$$

$$(1/r)[V(\sin^{-2}\theta\,\partial_\lambda^2 Y_{nm} + \cot\theta\,\partial_\theta Y_{nm}) + UY_{nm}] =$$

$$(1/r)[V(-\partial_\theta^2 Y_{nm} - n(n+1)Y_{nm}) + UY_{nm}] =$$

$$-\ (1/r)[V\partial_\theta^2 Y_{nm} + Vn(n+1)Y_{nm} - UY_{nm}] \qquad \text{(II.9)}$$

Differentiating (II.9) with respect to θ we obtain

$$\partial_\theta e_{\lambda\lambda} = -\ (1/r)[V\partial_\theta^3 Y_{nm} + n(n+1)V\partial_\theta Y_{nm} - U\partial_\theta Y_{nm}] =$$

$$-\ (1/r)[V\partial_\theta^3 Y_{nm} + [n(n+1)V - U]\partial_\theta Y_{nm}] \qquad \text{(II.10)}$$

$$e_{\theta\theta} - e_{\lambda\lambda} = (1/r)[\partial_\theta v - \sin^{-1}\theta\,\partial_\lambda w - \cot\theta\, v] =$$

$$(1/r)[\partial_\theta(V\partial_\theta Y_{nm}) - \sin^{-1}\theta\,\partial_\lambda(V\sin^{-1}\theta\,\partial_\lambda Y_{nm}) - \cot\theta(V\partial_\theta Y_{nm})] =$$

$$(1/r)[V\partial_\theta^2 Y_{nm} - V\sin^{-2}\partial_\lambda^2 Y_{nm} - V\cot\theta\,\partial_\theta Y_{nm}] =$$

$$(V/r)[\partial_\theta^2 Y_{nm} - (\sin^{-2}\partial_\lambda^2 Y_{nm} - \cot\theta\,\partial_\theta Y_{nm})] =$$

$$(V/r)[\partial_\theta^2 Y_{nm} - (-\partial_\theta^2 Y_{nm} - n(n+1)Y_{nm})] =$$

$$(V/r)[2\partial_\theta^2 Y_{nm} + n(n+1)Y_{nm}] \qquad \text{(II.11)}$$

$$e_{\theta\lambda} = (1/r)[\partial_\theta w - \cot\theta\, w + \sin^{-1}\theta\,\partial_\lambda v] =$$

$$(1/r)[\partial_\theta(V\sin^{-1}\theta\,\partial_\lambda Y_{nm}) - \cot\theta(V\sin^{-1}\theta\,\partial_\lambda Y_{nm}) + \sin^{-1}\theta\,\partial_\lambda(V\partial_\theta Y_{nm})] =$$

$$(1/r)[V\partial_\theta(\sin^{-1}\theta\,\partial_\lambda Y_{nm}) - V\cot\theta\sin^{-1}\theta\,\partial_\lambda Y_{nm} + V\sin^{-1}\theta\,\partial_{\lambda\theta}{}^2 Y_{nm}] =$$

$$(V/r)[(-\cos\theta\,\partial_\lambda Y_{nm})\sin^{-2}\theta - \cot\theta\sin^{-1}\theta\,\partial_\lambda Y_{nm} =$$

$$(V/r)[-\cot\theta\sin^{-1}\theta\;\partial_\lambda Y_{nm} - \cot\theta\sin^{-1}\theta\,\partial_\lambda Y_{nm} =$$

$$-(2V/r\sin\theta)\cot\theta\,\partial_\lambda Y_{nm} \qquad \text{(II.12)}$$

Differentiating (II.12) with respect to λ we obtain

$$\partial_\lambda e_{\theta\lambda} = -(2V/r\sin\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm} \qquad \text{(II.13)}$$

Considering (II.10), (II.11) and (II.12) we have

$$(N/r)[2\partial_\theta e_{\lambda\lambda} - \sin^{-1}\theta\;\partial_\lambda e_{\theta\lambda} - 2\cot\theta(e_{\theta\theta} - e_{\lambda\lambda})] =$$

$$(N/r)\Big[-(2V/r)\partial_\theta{}^3 Y_{nm} - (1/r)[2n(n+1)V - 2U]\partial_\theta Y_{nm}$$

$$- \sin^{-1}\theta(2V/r\sin\theta)(\partial_\theta\partial_\lambda{}^2 Y_{nm} - \cot\theta\,\partial_\lambda{}^2 Y_{nm})$$

$$- 2\cot\theta\;(V/r)[2\partial_\theta{}^2 Y_{nm} + n(n+1)Y_{nm}]\Big] =$$

$$(N/r)\Big[-(2V/r)\partial_\theta{}^3 Y_{nm} - (1/r)(2n(n+1)V - 2U)\partial_\theta Y_{nm}$$

$$- (2V/r\sin^2\theta)\partial_\theta\partial_\lambda{}^2 Y_{nm} + (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm}$$

$$- (2V/r)\cot\theta\;[\partial_\theta{}^2 Y_{nm} - \cot\theta\,\partial_\theta Y_{nm} - \sin^{-2}\theta\,\partial_\lambda Y_{nm}]\Big] =$$

$$(N/r)\Big[-(2V/r)\partial_\theta{}^3 Y_{nm} - (1/r)(2n(n+1)V - 2U)\partial_\theta Y_{nm}$$

$$- (2V/r\sin^2\theta)\partial_\theta\partial_\lambda{}^2 Y_{nm} + (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm}$$

$$- (2V/r)\cot\theta\;\partial_\theta{}^2 Y_{nm} + (2V/r)\cot^2\theta\,\partial_\theta Y_{nm}$$

$$+ (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm}\Big]$$

$$(N/r)\ \Big[-(2V/r)\,\partial_\theta[-\cot\theta\,\partial_\theta Y_{nm} - \sin^{-2}\partial_\lambda Y_{nm} - n(n+1)Y_{nm}]$$

$$-(1/r)[2n(n+1)V - 2U]\partial_\theta Y_{nm} - (2V/r\sin^2\theta)\partial_\theta\partial_\lambda{}^2 Y_{nm}$$

$$+ (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm} - (2V/r)\cot\theta\ \partial_\theta{}^2 Y_{nm} + (2V/r)\cot^2\theta\,\partial_\theta Y_{nm}$$

$$+ (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm}\Big] =$$

$$(N/r)\ \Big[-(2V/r)\sin^{-2}\partial_\theta Y_{nm} + (2V/r)\ \cot\theta\,\partial_\theta{}^2 Y_{nm}$$

$$+ (2V/r)[\sin^2\theta\,\partial_\theta\partial_\lambda{}^2 Y_{nm} - 2\sin\theta\cos\theta\,\partial_\lambda{}^2 Y_{nm}]\sin^{-4}\theta$$

$$+ (2V/r)n(n+1)\partial_\theta Y_{nm} - (1/r)[2n(n+1)V - 2U]\partial_\theta Y_{nm}$$

$$- (2V/r\sin^2\theta)\partial_\theta\partial_\lambda{}^2 Y_{nm} + (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm} -$$

$$-(2V/r)\cot\theta\,\partial_\theta{}^2 Y_{nm} + (2V/r)\cot^2\theta\,\partial_\theta Y_{nm} + (2V/r\sin^2\theta)\cot\theta\,\partial_\lambda{}^2 Y_{nm}\Big] =$$

$$- (N/r)\ \Big[r^1 V - r^1 U]\Big]\partial_\theta Y_{nm} \qquad \text{(II.14)}$$

Furthermore,

$$e_{r\theta} = \partial_t[\partial_r v - v/r + (1/r)\partial_\theta u] =$$

$$= \partial_t[\partial_r(V\partial_\theta Y_{nm}) - (1/r)(V\partial_\theta Y_{nm}) + (1/r)(U\partial_\theta Y_{nm})] =$$

$$= [\partial_r\dot{V} - (1/r)\dot{V} + (1/r)\dot{U}]\partial_\theta Y_{nm} \qquad \text{(II.15)}$$

For the third equation we have

$$\rho_o/(r\sin\theta)\partial_\lambda\Phi = \rho_o/(r\sin\theta)\partial_\lambda\Psi Y_{nm} = \rho_o/(r\sin\theta)\Psi\partial_\lambda Y_{nm} \qquad \text{(II.17)}$$

$$(3L/r)\,e_{r\lambda} = (3L/r)[(r\sin\theta)^{-1}\,\partial_\lambda u + \partial_r w - w/r] =$$

$$(3L/r)[(r\sin\theta)^{-1}\,U\partial_\lambda Y_{nm} + \partial_r V\partial_\lambda Y_{nm} - V/(r\sin\theta)\partial_\lambda Y_{nm}] =$$

$$3L/(r^2\sin\theta)[U - V + r\partial_r V]\partial_\lambda Y_{nm} \qquad \text{(II.18)}$$

$$\partial_r[\mathrm{L}e_{r\lambda}] = \partial_r\Big[\mathrm{L}[(r\sin\theta)^{-1}\,\mathrm{U}\partial_\lambda \mathrm{Y}_{nm} + \partial_r\mathrm{V}\partial_\lambda \mathrm{Y}_{nm} - \mathrm{V}/(r\sin\theta)\partial_\lambda \mathrm{Y}_{nm}\,] =$$

$$\partial_r\Big[\mathrm{L}/(r^2\sin\theta)[\mathrm{U} - \mathrm{V} + r\partial_r\mathrm{V}]\Big]\partial_\lambda \mathrm{Y}_{nm} \qquad \text{(II.19)}$$

$$(r\sin\theta)^{-1}\partial_\lambda\,[\mathrm{A}(e_{\theta\theta} + e_{\lambda\lambda}) + \mathrm{F}e_{rr} - g\rho_o u] =$$

$$(r\sin\theta)^{-1}\,[\mathrm{AZ} + \mathrm{F}\partial r\mathrm{U} - g\rho_o\mathrm{U}]\partial_\lambda \mathrm{Y}_{nm} \qquad \text{(II.20)}$$

APPENDIX III

A. Load Deformation Coefficients

n	h'	100 H'	n *l*'	100 n L'	n k'	100 n K'
1	-0.295	0.003	0.143	0.022	0.000	0.000
2	-1.007	0.003	0.063	0.044	-0.618	0.011
3	-1.065	0.004	0.203	0.064	-0.596	0.016
4	-1.069	0.005	0.221	0.095	-0.544	0.018
5	-1.103	0.006	0.217	0.140	-0.533	0.025
6	-1.164	0.006	0.214	0.176	-0.555	0.032
7	-1.238	0.007	0.221	0.210	-0.590	0.039
8	-1.313	0.007	0.230	0.244	-0.631	0.050
9	-1.388	0.008	0.243	0.285	-0.684	0.062
10	-1.460	0.008	0.255	0.327	-0.737	0.071
12	-1.600	0.009	0.291	0.397	-0.802	0.095
14	-1.726	0.010	0.327	0.492	-0.878	0.126
16	-1.845	0.012	0.365	0.559	-0.951	0.151
18	-1.952	0.013	0.404	0.606	-1.018	0.172
20	-2.048	0.014	0.443	0.629	-1.080	0.191
25	-2.252	0.016	0.534	0.764	-1.209	0.261
30	-2.411	0.041	0.612	0.760	-1.307	0.297
40	-2.633	0.127	0.730	0.841	-1.434	0.437
50	-2.777	0.223	0.809	0.751	-1.502	0.528
60	-2.880	0.319	0.864	0.622	-1.539	0.601
70	-2.958	0.395	0.901	0.469	-1.557	0.640

80	-3.021	0.448	0.925	0.321	-1.565	0.652
90	-3.077	0.480	0.941	0.191	-1.567	0.640
100	-3.127	0.488	0.951	0.082	-1.567	0.616
120	-3.221	0.465	0.959	-0.067	-1.564	0.533
140	-3.312	0.406	0.959	-0.134	-1.564	0.434
160	-3.403	0.335	0.955	-0.162	-1.570	0.339
180	-3.496	0.265	0.951	-0.155	-1.582	0.257
200	-3.590	0.204	0.948	-0.135	-1.599	0.191
250	-3.829	0.096	0.950	-0.076	-1.662	0.084
300	-4.069	0.042	0.970	-0.035	-1.745	0.034
350	-4.320	0.017	1.005	-0.015	-1.840	0.013
400	-4.526	0.005	1.054	-0.003	-1.939	0.004
800	-5.263	0.001	1.616	-0.001	-2.675	0.001
1000	-5.600	0.000	1.714	0.000	-2.812	0.000
2000	-6.186	0.000	1.873	0.000	-3.059	0.000
3000	-6.262	0.000	1.892	0.000	-3.092	0.000
5000	-6.274	0.000	1.894	0.000	-3.097	0.000
10000	-6.274	0.000	1.894	0.000	-3.097	0.000

B. Green's Functions (Applied Load: 1 kg)

ψ	$u\times10^{12}(R\psi)$	ϑ°	$v\times10^{12}(R\psi)$	ϑ°	$g^E\times10^{18}(R\psi)$	ϑ°	$t^E\times10^{12}(R\psi)^2$	ϑ°
0.0001	-42.603	0.00	-12.875	0.00	-98.875	0.00	42.628	0.00
0.001	-42.377	0.00	-12.875	0.00	-98.361	0.00	42.628	0.00
0.01	-40.132	-0.01	-12.836	0.00	-93.260	0.00	42.601	0.00
0.02	-37.704	-0.01	-12.719	0.00	-87.757	-0.01	42.445	0.00
0.03	-35.400	-0.01	-12.535	0.00	-82.552	-0.01	42.074	0.00
0.04	-33.258	-0.02	-12.298	0.00	-77.732	-0.01	41.459	0.00
0.06	-29.522	-0.05	-11.693	0.00	-69.374	-0.03	39.585	0.00
0.08	-26.523	-0.08	-10.976	0.00	-62.674	-0.04	37.091	0.00
0.10	-24.206	-0.12	-10.225	0.00	-57.490	-0.05	34.158	0.00
0.15	-20.745	-0.21	-8.554	0.00	-49.808	-0.10	27.775	-0.01
0.20	-19.273	-0.30	-7.291	0.00	-46.668	-0.12	22.234	-0.03
0.25	-18.557	-0.35	-6.305	-0.03	-45.043	-0.13	22.645	-0.06
0.30	-17.583	-0.45	-5.669	-0.06	-41.839	-0.18	23.936	0.00
0.35	-16.517	-0.52	-5.404	-0.07	-38.048	-0.22	23.435	-0.18
0.40	-15.649	-0.60	-5.437	-0.07	-34.798	-0.24	21.298	-0.25
0.45	-15.083	-0.62	-5.654	-0.06	-32.615	-0.24	18.737	-0.41
0.50	-14.799	-0.66	-5.926	-0.06	-31.531	-0.24	16.305	-0.57
0.55	-14.736	-0.64	-6.154	-0.03	-31.335	-0.21	14.364	-0.83
0.60	-14.806	-0.63	-6.288	0.00	-31.690	-0.18	13.580	-0.98
0.80	-14.642	-0.49	-6.177	0.47	-31.763	-0.02	17.499	-1.00

1.0	-13.811	-0.31	-6.061	1.07	-29.756	0.17	17.670	-1.14
1.2	-12.922	-0.11	-5.808	1.80	-27.805	0.31	18.548	-0.97
1.4	-12.022	0.09	-5.545	2.60	-25.799	0.49	17.329	-0.75
1.6	-11.310	0.25	-5.257	3.36	-24.374	0.59	16.269	-0.44
1.8	-10.695	0.37	-4.967	3.99	-23.169	0.62	15.681	-0.11
2.0	-10.112	0.43	-4.690	4.61	-22.005	0.62	15.604	0.16
2.5	-8.708	0.45	-4.071	5.53	-19.033	0.53	14.408	0.61
3.0	-7.699	0.37	-3.542	6.00	-16.953	0.36	12.609	0.83
4.0	-6.190	0.17	-2.747	6.03	-13.671	0.03	10.181	0.71
5.0	-5.243	0.00	-2.207	5.71	-11.534	-0.19	8.422	0.38
6.0	-4.634	-0.08	-1.849	5.26	-10.081	-0.25	7.015	0.10
7.0	-4.225	-0.17	-1.610	4.92	-9.045	-0.28	5.945	-0.12
8.0	-3.949	-0.09	-1.457	4.45	-8.309	-0.26	5.149	-0.21
9.0	-3.751	-0.10	-1.360	4.24	-7.747	-0.23	4.601	-0.31
10.0	-3.602	-0.10	-1.304	3.86	-7.308	-0.20	4.170	-0.43
12.0	-3.383	0.00	-1.273	2.55	-6.667	-0.16	3.796	-0.28
14.0	-3.189	0.00	-1.276	1.41	-6.136	-0.06	3.591	-0.10
16.0	-3.013	0.00	-1.287	0.84	-5.714	0.00	3.424	0.31
18.0	-2.831	0.00	-1.300	1.11	-5.314	0.00	3.403	0.32
20.0	-2.642	0.00	-1.306	1.38	-4.946	0.00	3.369	0.11
25.0	-2.114	0.00	-1.286	0.28	-4.018	0.00	3.440	-0.21
30.0	-1.518	0.00	-1.229	1.17	-3.059	0.00	3.415	0.11
35.0	-0.892	0.00	-1.120	1.08	-2.266	0.00	3.226	0.11

40.0	-0.267	0.00	-1.012	0.00	-1.468	0.00	2.926	-0.13
50.0	0.894	0.00	-0.758	0.95	-0.456	0.00	1.936	0.00
60.0	1.720	0.00	-0.594	0.61	0.159	0.00	0.422	0.85
70.0	2.135	0.00	-0.552	0.65	0.512	0.00	-1.156	0.00
80.0	2.098	0.00	-0.666	1.08	0.181	0.00	-2.985	0.00
90.0	1.667	0.00	-0.936	0.38	-0.427	0.00	-3.684	0.10
100.0	0.955	0.00	-1.255	0.29	-0.890	0.00	-4.087	0.18
110.0	0.016	0.00	-1.557	0.00	-1.528	0.00	-4.377	0.08
120.0	-1.056	0.00	-1.809	0.00	-2.215	0.00	-3.955	0.09
130.0	-2.184	0.00	-1.932	0.00	-2.768	0.00	-3.531	0.10
140.0	-3.320	0.00	-1.887	0.00	-3.272	0.00	-3.090	0.12
150.0	-4.390	0.00	-1.684	0.00	-3.749	0.00	-2.407	0.15
160.0	-5.319	0.00	-1.312	0.00	-4.133	0.00	-1.399	0.26
170.0	-6.045	0.00	-0.745	0.00	-4.412	0.00	-0.556	0.00
180.0	-6.535	0.00	0.000	0.00	-4.660	0.00	0.000	0.00

u = radial displacement, v = tangential displacement, g^E = "Elastic" gravity, t^E = "Elastic" tilt, R = radius of the earth, taken as 6.371×10^6m and ψ = distance from load in radians. ϑ° is phase shift (degrees) with respect to the total elastic displacement.