# PERFORMANCE CONSIDERATIONS FOR REAL-TIME NAVIGATION WITH THE GPS

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# PREFACE

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# PERFORMANCE CONSIDERATIONS FOR REAL-TIME NAVIGATION WITH THE GPS

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# ABSTRACT

The Global Positioning System (GPS) is expected to fulfill the navigation requirements of many civilian users in the future. Whether of not it will be capable of providing sufficient navigation information to meet these requirements will depend on a variety of factors. Consideration must be given to the level of integrity, reliability and accuracy which the GPS will provide.

These concerns, and the related question of outages of the GPS, are examined with reference to the available literature. First, a review of some aspects of the GPS are given as background for further discussions. Integrity, reliability and accuracy of the GPS are then each defined and evaluated considering the planned 21-satellite constellation. Next, outages of the full constellation - both those due to bad user-satellite geometry and to satellite failures - are discussed with particular emphasis on the former. Alternatives for navigation during such outages are then briefly mentioned.

Based on the aspects considered, it is then obvious that civilian users of the GPS cannot expect the system to continuously provide all the navigation information required. Additional information will be necessary during outages of the system and to satisfy more stringent accuracy requirements.

Table of Contents

## List of Figures

List of Tables

# Chapter 1

# INTRODUCTION

When the Global Positioning System (GPS) becomes fully operational around 1988, it is expected that the system will provide a highly accurate two-dimensional and three-dimensional positioning service to suitably equipped users worldwide, in all weather conditions and on a 24-hour basis. Even if the level of service provided for civil applications is intentionally degraded, the GPS may still fulfill many air and marine navigation requirements of the civilian community. It may, therefore, have the potential to replace many of the civil navigation systems presently in use. Kalafus et al. (1983) states that this potential may be realized only if the GPS performance meets certain conditions. Four of these are:

1. Coverage matches or exceeds that of existing systems.
2. Accuracy is adequate to meet operational requirements of navigation service users.
3. System availability and reliability are adequate to meet operational requirements of navigation service users.
4. Costs of user equipment do not cause a significant economic burden on the users.

With the proposed constellation of 18 satellites and three active spares, the GPS will provide worldwide coverage essentially all of the time to users utilizing all satellites greater than 5 degrees above the local horizon. Satellites below this 5 degree minimum are considered masked out by antenna limitations, obstructions, etc . Brief outages, due to unsatisfactory user-satellite geometry will, however, occur at various locations. For many applications, the navigation service provided by the GPS alone will then be insufficient. An integrated navigation system, capable of operation with or without the full GPS service, will then be required by those users who must have continuous positioning capability.

In one recent study by Lachapelle et al. (1984), results indicate that at velocities up to 300 km/h, single point airborne navigation with the GPS may be performed to accuracies of 15-20 metres in all three coordinates using P-code dual frequency pseudoranges. This type of accuracy would satisfy requirements for a variety of applications but, since P-code access will be restricted after 1988, it cannot be considered as representative of the level of accuracy that will be available to civilian users of the GPS. The researchers also report that for C/A-code, similar accuracies are currently attainable if differential corrections to the GPS-derived latitude, longitude and height are obtained at a fixed site and used to correct the single point positions post-mission.

System availability and long term reliability are important considerations when evaluating an alternative navigation system. Availability of the GPS may be measured by the ability of the system to provide the minimum amount of information satisfying user needs at any time and at any location. System reliability deals with the question of whether or not the system can deliver this minimum amount of information at any location over an extended period of time. To do so, the long term probability of satellite failures must be kept reasonably low since any failures may adversely affect the coverage provided by the GPS.

The costs of GPS user equipment will also be an important consideration for many civilian users. It is listed here mainly because the quality of user equipment will directly affect the coverage abilities of the GPS. Low cost antennas cannot receive signals of satellites at low elevation angles and so, as recommended by the United States Federal Aviation Administration (FAA), a higher mask angle should be considered as more appropriate for civilian needs. To model this limitation, the FAA has adopted a 10-degree mask angle in all studies of the GPS performance potential (Braff et al. 1983).

From the preceding paragraphs it is then obvious that several considerations must be examined in evaluating the potential role of the GPS. Foremost among these is the question of outages of the system, both those due to poor geometry and those due to satellite failures. In the following sections a brief description of some aspects of the GPS is given as background to further discussions.

## 1.1 SYSTEM OVERVIEW

The Global Positioning System is a satellite based radionavigation system currently under development by the United States Department of Defence (DoD). Originally started in 1974, the concept validation and full-scale engineering and development phases are now complete, and production should be concluded in 1988 when full global capability will be reached. Now, as in the future, three main segments may be indentified which when taken together constitute the GPS. Shown in Figure 1-1, these are the control, space, and user segments.

### 1.1.1 Control Segment

The operational control segment will consist of monitor, master control and ground antenna stations located at the sites shown in Figure 1-2. The monitor stations will monitor the satellite signals, provide time tags with cesium clocks and, together with the local meteorological conditions, transmit this information to the master control stations. There, the satellite ephemeris, clock correction coefficients and coefficients to correct for atmospheric

Figure 1-1 The Global Positioning System (from Lachapelle (1985)

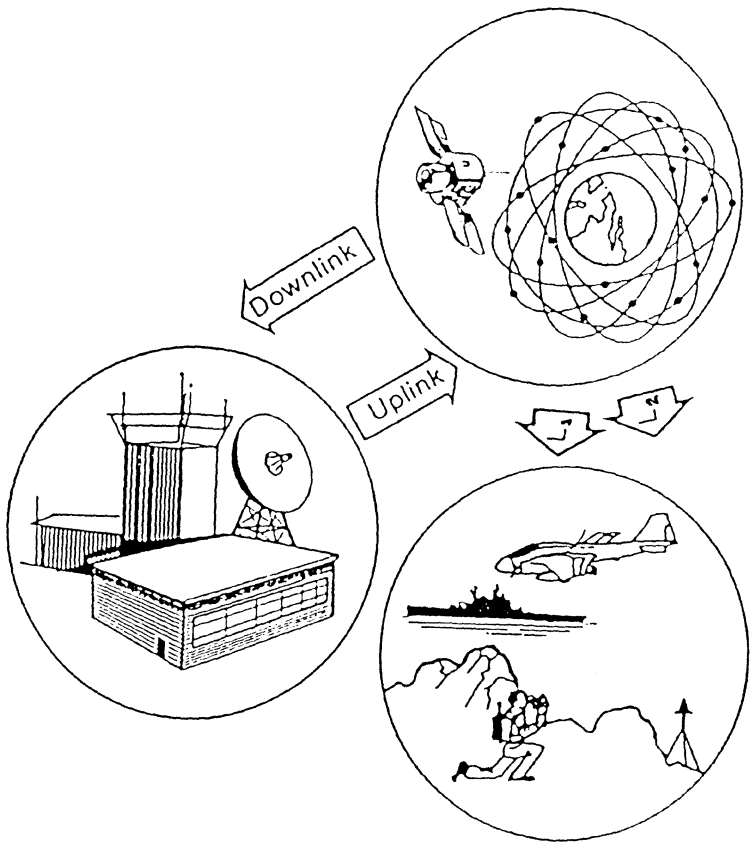


Figure 1-2 Operational Control Segment (from Lachapelle (1985))

delays are computed for each satellite. These, along with information on satellite health, are then uploaded to each space vehicle.

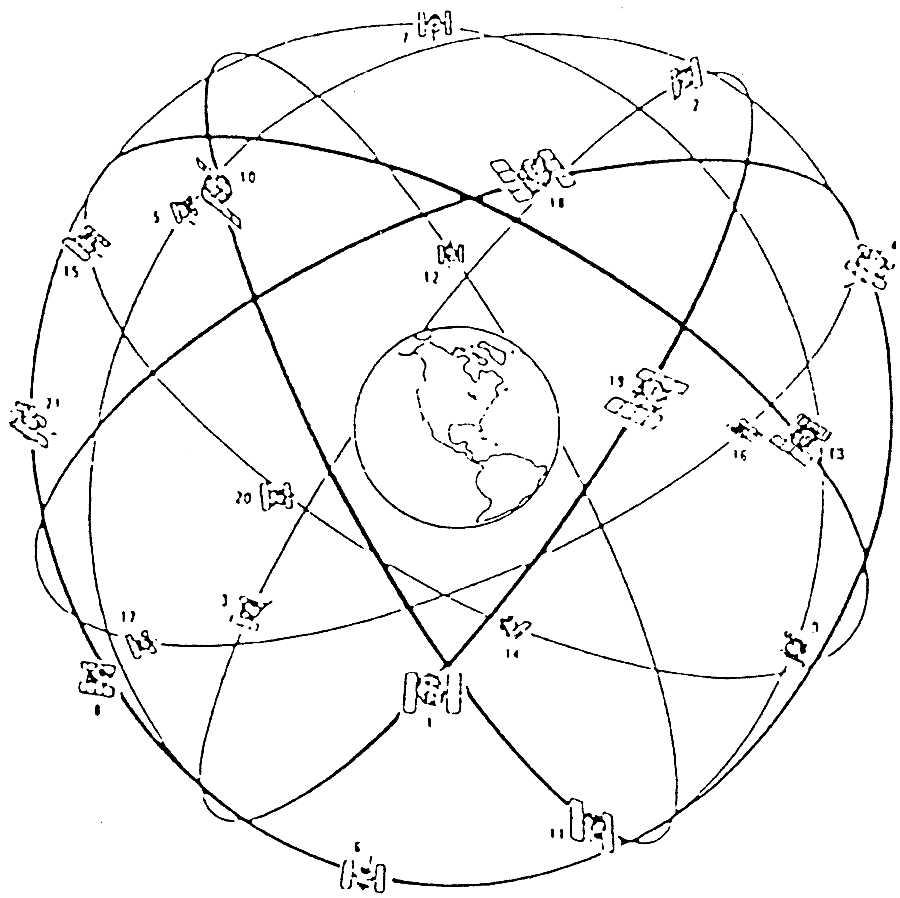
### 1.1.2 Space Segment

The originally planned operational constellation consisted of 24 satellites, distributed over three orbital planes, with periods of 12 sidereal hours and, therefore, altitudes of approximately 20,000 kilometres. Each plane was to be inclined 63 degrees to the equator, contain eight equally spaced satellites (45 degrees), and was to have an ascending node 120 degrees to the west of the adjacent plane. Relative phasing of satellites in adjacent planes was to be 30 degrees, i.e., satellites in the second plane would cross the equator 30 degrees behind those in plane one. The Walker constellation index, developed by Walker (1977), was then 24/3/2: 24 for the number of satellites in the constellation, 3 for the number of planes, and 2 indicating the number of pattern units obtained by dividing the relative phasing of the satellites by 360/24.

In 1980, as a cost cutting measure, the U.S DoD decided to plan for an operational constellation of 18 rather than 24 satellites, with provisions for possible expansion to 24 satellites at a later date. In subsequent studies, characteristics of various alternative 18 satellite constellations were then evaluated and, based on an overall better performance during simulations, a uniform constellation of six orbital planes containing three satellites each was selected (Kruh 1981). The orbital planes will have inclination angles of 55 degrees, will be spaced 60 degrees along the equator, and relative phasing of the satellites will be 40 degrees. The Walker constellation index is then 18/6/2. Three spare satellites will be included sometime after 1988 to increase the probability of 18 or more satellites being available over extended periods of time. This final constellation may then be designated by 18/6/2+3. Figures 1-3 and 1-4 show the planned distributions of the 21 satellites.

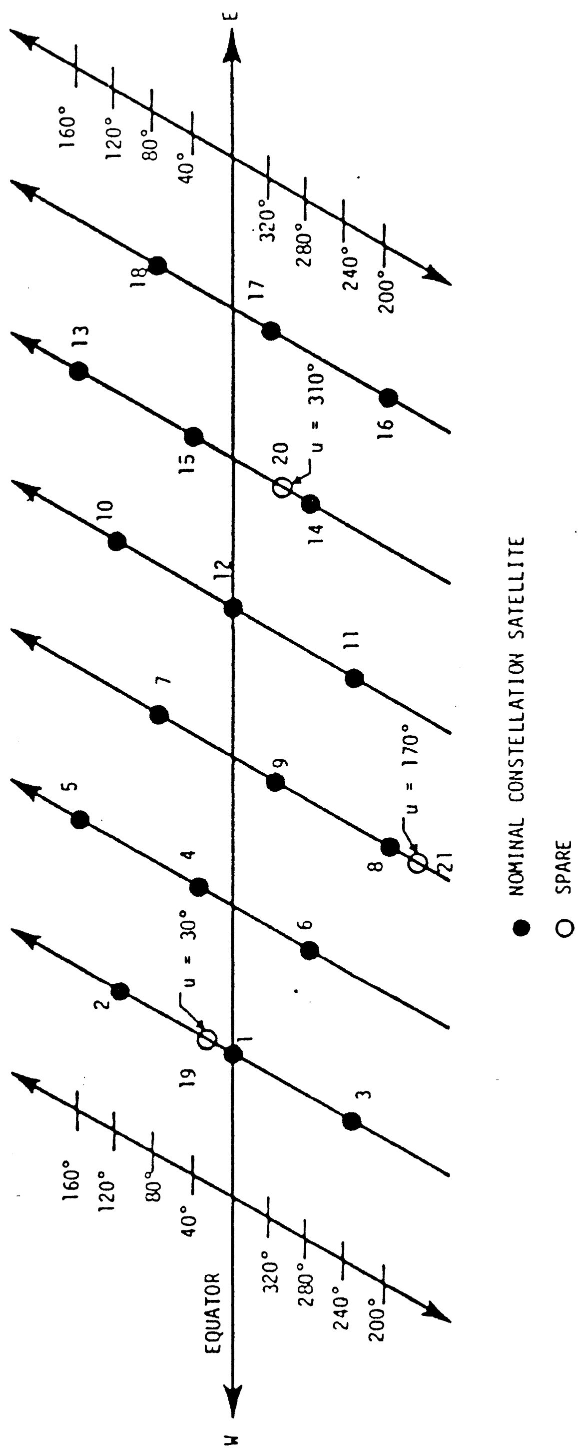
The satellites of the GPS transmit signals at two carrier frequencies, 1575.42 MHZ (L1) and 1227.60 MHZ (L2), each of which has various kinds of modulation superimposed. The cosine wave of both carriers is modulated by a pseudorandom sequence of step functions known as the P-code. The sine wave of L1 is also modulated by another sequence known as the C/A-code or S-code. Each of these codes is unique in pattern to a satellite and so may be used for satellite identification. By generating the identical codes in the receiver, the transit time of the signals may be determined by measuring the phase shift required to match the generated and received codes. In addition to these modulations, both

Figure 1-3 The GPS Operational Constellation (from Lachapelle (1985)).



- 18 satellites
  (plus 3 active spares)
- 6 orbital planes
  spaced in long. by 60$^\circ$
  inclination 55$^\circ$
  altitude 20 000 km
  orbital period 12 hours

Figure 1-4 18/6/2+3 Relative Satellite Positions (from Kalafus et al. (1983))



L1 and L2 are also continuously modulated by a data-bit stream containing the navigation message. This consists of the information uploaded by the master control stations. A more detailed discussion of the satellite signals may be found in Milliken and Zoller (1978).

### 1.1.3 User Segment

For real-time navigation with the GPS, the user receiver must be able to select the satellites to be used, acquire and decode the satellite signals, compute satellite positions and then combine all this information to obtain estimates of the user's position and receiver clock bias (the bias due to non-synchronization of the receiver clock with that of the satellites).

As mentioned previously, to obtain the transit time of the satellite signals, the GPS receiver must be able to duplicate the P-code or S-code for each satellite. This may then be used to obtain the pseudoranges to the satellites. In addition, the receiver must make these measurements of pseudorange simultaneously (or nearly so) to several satellites. This requires either a multichannel receiver (very expensive) or one capable of acquiring the satellite signals sequentially at a fairly fast rate (Wells et al. 1982).

## 1.2 NAVIGATION SOLUTION

Real-time navigation with the GPS will typically involve measurement of pseudoranges to four satellites so that latitude, longitude, ellipsoidal height and receiver clock bias may all be solved for. If, however, one or more of these parametres are known sufficiently close to their actual value, alternative solutions are then possible. For marine navigation, examples of this are ellipsoidal heights obtained from geoidal maps or synchronization of an onboard cesium clock with the GPS satellite clocks. For each alternative solution, an appropriate measure of the strength of the solution may be obtained using the Dilution of Precision (DOP) factors defined in Appendix B. Four possible solutions and the corresponding DOP factor representing all the unconstrained parametres are then as follows:

1. Solve for latitude and longitude with heights and time bias held fixed. A minimum of two visible satellites will then be required and the corresponding DOP factor is HDOP (Horizontal DOP).
2. Solve for latitude, longitude and time bias with the height held fixed. A minimum of three visible satellites will then be required and the corresponding DOP factor is HTDOP (Horizontal/Time DOP).

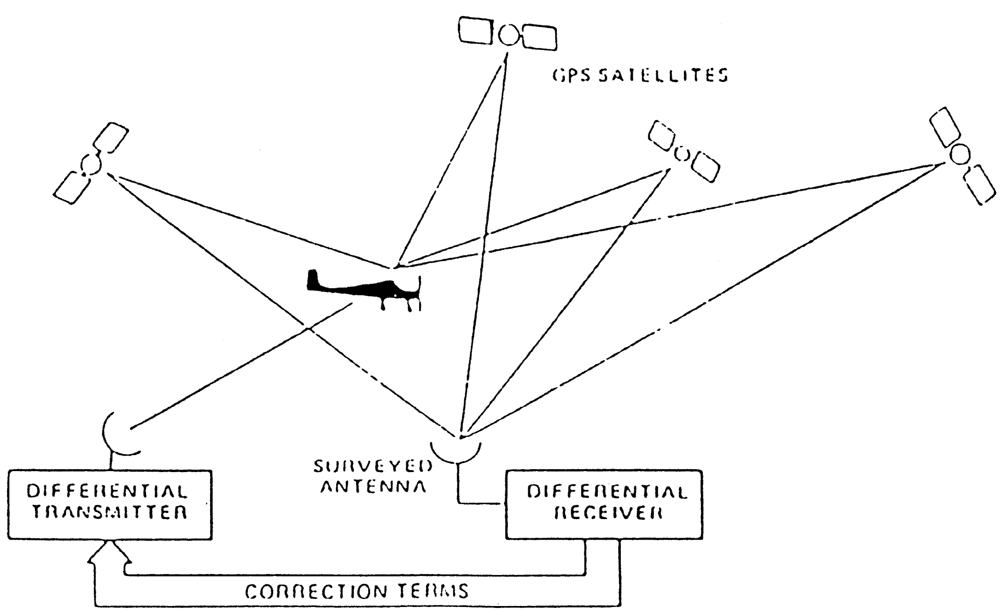
3. Solve for latitude, longitude and height with the time bias held fixed. A minimum of three visible satellites will then be required and the corresponding DOP factor is PDOP (Position DOP).
4. Solve for latitude, longitude, height and time bias. A minimum of four visible satellites will then be required and the corresponding DOP factor is GDOP (Geometric DOP).

Note that for any of these or other solutions, DOP factors other than that representing the unconstrained parametres may be computed and will then provide partial information on the geometric strength of the solution. For example, the HDOP for solution 4 represents the horizontal strength only of the solution. The results presented in Chapter 2 are for various kinds of DOP, but only for the unconstrained case (solution 4).

The equations for navigation using pseudoranges to the satellites and a method for selecting the four satellites providing the best geometry are given in Appendix A and Appendix B, respectively.

Using S-code in a diferential mode currently gives accuracies similar to those attainable using P-code dual frequency pseudoranges. This differential GPS concept will be of much interest to civilian users in the near future since it is expected that single point positioning accuracies with S-code will be intentionally degraded (Kalafus et al. 1983). With differential GPS the accuracy currently available will be maintained. As shown in Figure 1-5, the basic idea consists of a fixed monitor site from which corrections to position or measured ranges are determined and then transmitted to users in real time.

Figure 1-5 Differential GPS Operation (from Lachapelle (1985))



# Chapter 2

# INTEGRITY, RELIABILITY AND ACCURACY OF THE GPS

The GPS may have the potential to fulfill many navigation requirements of the civilian community. The quality of the service which will be provided, however, will depend on a variety of factors. In addition to coverage, reliability and accuracy another aspect which should be examined is the level of integrity that may be expected. Based on information presently available and by simulations of the future constellation of operational satellites, all of these factors may be examined to some extent.

## 2.1 SYSTEM INTEGRITY

According to Braff et al. (1983) a navigation system's integrity may be measured by a) its ability to detect malfunctions affecting the level of performance, and b) the time delay from the initial occurrence of the malfunction to notification of the pilot or navigator. Considered in the context of non-precision and precision approaches, the time delay between detection and notification or correction becomes especially critical.

The detection of malfunctions by the GPS should be sufficient for all but the most stringent requirements since, signal integrity will be monitored extensively by the ground control segment and to a lesser degree by the GPS receiver and the satellites themselves. The time delay between detection of the malfunction and notification of the user may, however, be inadequate for some applications. For civilian users, this function will be primarily the responsibility of the control segment. Depending on the circumstances, this whole process may take from 5 minutes to a half hour or more. Delays of this duration would be totally inadequate for landing guidance of civil aircraft where notification delays of 10 seconds or less are required. Braff et al. (1983) summarize the methods employed by the system and concludes that although GPS integrity is sufficient for en route navigation, an independent network for monitoring signal integrity would be required for precision and non-precision civilian air approach requirements.

## 2.2 SYSTEM RELIABILITY

System reliability refers to the ability of a navigation system to provide sufficient information for position determination of the required accuracy at any location, at any time, and over any time period. Consideration must be given to the extent of coverage provided by the system and the frequency of scheduled or unscheduled outages.

Reliability of the GPS may be evaluated on the basis of a) whether or not the minimum required satellites are visible from any location, and b) what frequency of satellite failures can be expected, this of course affecting a). In addition, the user-satellite geometry should be such that geometric effects do not degrade navigation accuracy beyond acceptable tolerances.

Since navigation with the GPS will in many cases require four visible satellites, this is the minimum number that must be continuously available to all users. Further, visibility may be defined with respect to a minimum acceptable satellite elevation above the user's horizon. Jorgensen (1981) states that satellites below a five-degree elevation angle may be masked out by terrain, antenna limitations, foliage, obstructions, etc. In many studies a five-degree mask angle has been considered appropriate. Table 2-1 gives a satellite visibility comparison for seven alternative constellations considering evenly distributed points worldwide. Braff et al (1983), in order to allow for possible limitations of low cost user equipment, compared satellite visibilty of the 18, 21, and 24 satellite constellations using a 10-degree mask angle. Table 2-2 lists the results based on averages for fifteen U.S cities, while Table 2-3 gives percentages for Chicago only. Table 2-4 gives a visibility comparison of two 18-satellite constellations using mask angles of 5, 10, and 20 degrees.

**Table 2-1**
Satellite Visibility Comparison
Based on Evenly Distributed Points Worldwide
5 Degree Elevation Mask

| Constellation | Number of Visible Satellites (% Time) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 24/3/2 | - | - | - | 11.8 | 10.7 | 34.2 | 38.9 | 4.3 | 0.1 |
| 18/3/3 | - | 6.3 | 19.1 | 38.9 | 29.9 | 5.7 | 0.1 | - | - |
| 18/18/2 | - | 2.1 | 20.4 | 46.0 | 28.4 | 3.0 | 0.045 | - | - |
| 18/6/2 | - | 0.7 | 21.6 | 48.8 | 24.9 | 3.9 | - | - | - |
| 18/6/2+3 | - | 0.03 | 1.0 | 23.5 | 44.8 | 24.8 | 5.6 | 0.3 | - |
| 18/6/2+6 | - | - | 0.1 | 3.8 | 22.6 | 39.0 | 26.2 | 7.7 | 0.6 |
| 24/6/1 | - | - | - | 1.0 | 23.6 | 43.9 | 25.1 | 5.3 | 1.1 |

From Spilker (1985)

**Table 2-2**
Satellite Visibility Comparison
Based on Averages Over 15 U.S. Cities
10 Degree Elevation Mask

| Constellation | Number of Visible Satellites (% Time) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 18/6/2 | - | 17.7 | 58.7 | 22.5 | 1.0 | 0.1 | - | - | - |
| 18/6/2+3 | - | 2.1 | 26.4 | 52.0 | 18.4 | 1.1 | - | - | - |
| 24/3/2 | - | - | 7.4 | 45.2 | 18.1 | 23.5 | 5.8 | - | - |

From Braff et al. (1983)

**Table 2-3**
Satellite Visibility Comparison
For Chicago Airport
10 Degree Elevation Mask

| Constellation | Number of Visible Satellites (% Time) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 18/6/2 | - | 22.0 | 54.0 | 24.0 | - | - | - | - | - |
| 18/6/2+3 | - | 2.0 | 32.0 | 47.0 | 19.0 | - | - | - | - |
| 24/3/2 | - | - | 8.0 | 42.0 | 30.0 | 11.0 | 9.0 | - | - |

From Braff et al. (1983)

Table 2-4
Satellite Visibility Comparison
Based on Evenly Distributed Points Worldwide
For 5, 10 and 20 Degree Mask Angles

Probability (%) of n or more satellites
being visible (n=4,7).

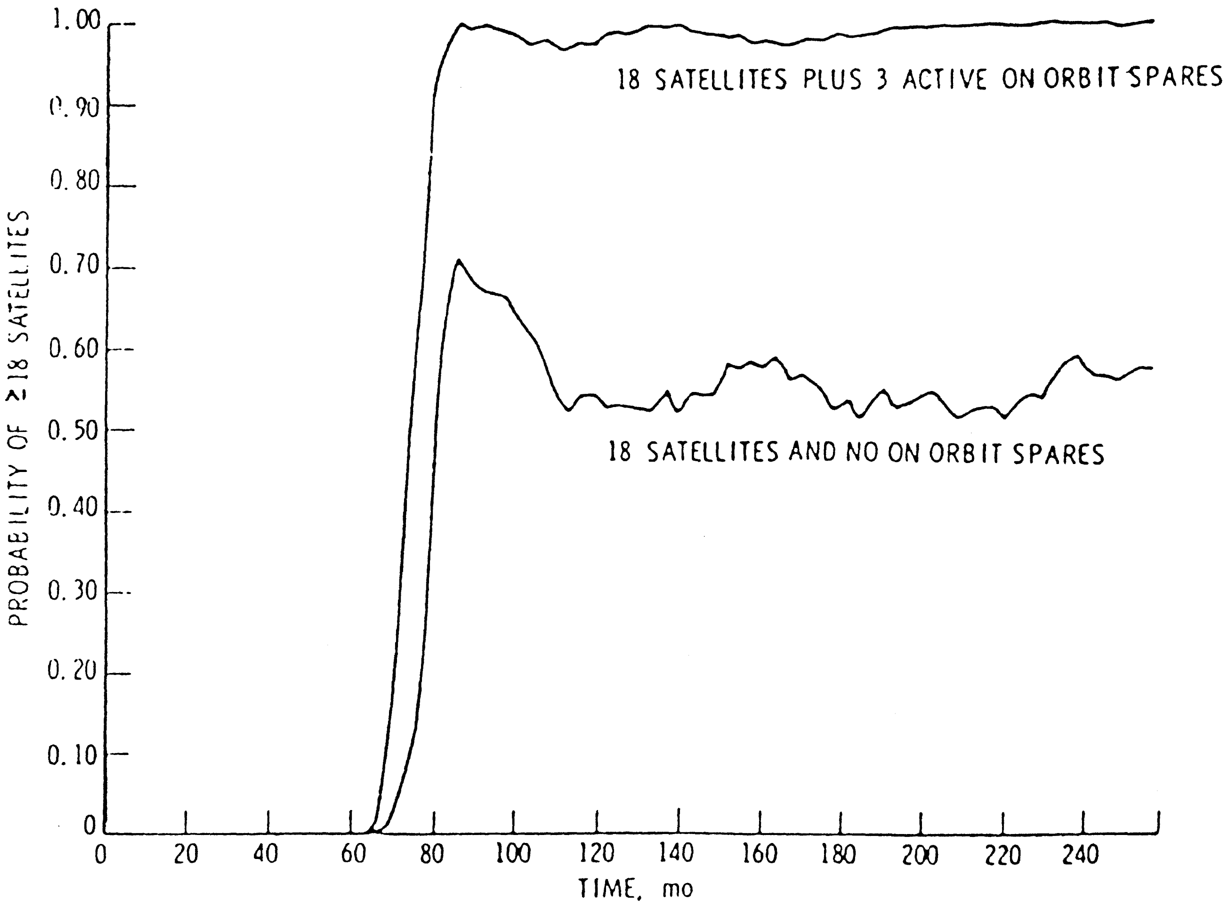
| Constellation | Mask Angle | No. of Satellites 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 18/6/2 | 5 | 100.00 | 99.32 | 78.36 | 28.72 |
| | 10 | 99.98 | 90.78 | 41.42 | 5.98 |
| | 20 | 81.11 | 27.05 | 1.61 | 0.0 |
| 18/3/2 | 5 | 99.99 | 91.39 | 71.24 | 39.94 |
| | 10 | 97.47 | 76.61 | 47.31 | 16.54 |
| | 20 | 71.65 | 33.26 | 8.33 | 0.33 |

From Kruh (1981)

Based on the information given in these four tables, it may be concluded that users equipped with receivers capable of acquiring signals from satellites above a mask angle of five degrees will always have a minimum of four satellites in view. For a ten-degree mask angle the level of availability is still quite high (99.98%) but quickly deteriorates for larger mask angles.

The above conclusions are, of course, only valid if satellite failures do not result in less than 18 properly functioning satellites. Studies by the Aerospace Corporation indicate that in order to have a 98% probability of eighteen or more operational satellites a minimum of three spares must be included in the constellation (Kruh 1983). Figure 2-1 illustrates the significant improvement over an eighteen-

Figure 2-1 Probability of $\geq$ 18 Satellites vs Time (from Kruh(1983))



satellite constellation provided by this strategy. A brief discussion of outages due to satellite failures is given in Chapter 3.

As mentioned previously, not only must a minimum of four satellites be available to all users of the GPS but, in addition, the geometric arrangement of the satellites relative to the user must not unduly degrade navigation accuracy. Studies have shown that reduction from 24 to 21 satellites will result in occasional outages of the system due solely to poor geometry. Details of these outages are given in Chapter 3.

## 2.3 SYSTEM ACCURACY

System accuracy is primarily dependant on two factors: observation uncertainties, and geometry. Uncertainties in the observations arise from random errors, systematic errors (biases), and blunders. The effect of geometry is termed Geometric Dilution of Precision (GDOP), since the contribution of observation errors to navigation accuracy is magnified or diluted according to the geometry of the situation.

Standard positioning with the GPS will consist of simultaneous ( or nearly so) pseudorange measurements to the four satellites providing the best geometric arrangement relative to the user. With this information, three coordinates of position and the user's receiver clock bias are then estimated, all to an accuracy dictated by the errors in the range measurements magnified or diluted by the user-satellite geometry.

Errors associated with the pseudorange measurements arise from three main sources: from the satellites themselves, from signal propagation effects, and from the user's GPS receiver. Satellite errors consist of errors in the predicted satellite positions and those due to non-synchronization of the satellite clocks with GPS time. Propagation errors are associated with modelling of ionospheric and tropospheric refraction and the effects of multipath. Finally, receiver errors consist of measurement noise, measurement truncation and computation errors. Table 2-5, from Wells et al. (1982), gives some estimates of the magnitude of the measurement errors for both P- and Scodes. Although some of the figures given may be out of date - current receiver and antenna designs have decreased the effects of multipath, measurement truncation and computation - the estimated range measurement errors (4m and 12m RSS) may still be considered reasonable.

Under the assumption that all measured ranges are uncorrelated with equal standard deviations, the GDOP parametres are solely a function of user-satellite geometry and, as such, are then a measure of the quality of that geometry. Small values of these parametres reflect good geometry and correspondingly small errors in position and time fixes. Figures 2-2 and 2-3 indicate the probability,

**Table 2-5**
GPS Range Measurement Errors (Standard deviations of uncorrelated equivalent range errors) in metres. Values in brackets refer to the prototype satellites.

| Error Source | P-code | S-code |
|---|---|---|
| Satellite | | |
| ephemeris | 1.5 (3.6) m | 1.5 (3.6) m |
| clock | 0.9 (2.7) | 0.9 (2.7) |
| Propogation | | |
| ionosphere - dual frequency | 3.0 | |
| ionosphere - models | | 0.5 - 15.0 |
| troposphere | 1.0 | 1.0 |
| multipath | 1.0 | 5.0 |
| Receiver | | |
| measurement noise | 1.0 | 10.0 |
| measurement truncation | 0.3 | 3.0 |
| computation | 1.0 | 1.0 |
| Combined effect (rss) | 4.0 (5.8) | 12 - 20 |

From Wells et al. (1982)

over a 24-hour period, of PDOP and HDOP being less than a given value assuming a 5-degree mask angle and a four-best strategy. Figure 2-2, from Spilker (1985), shows a comparison of the 18/6/2 and 18/18/2 constellations based on PDOP values generated for 862 user locations evenly distributed worldwide. Kalafus et al. (1983) constructed a similar plot of HDOP distributions for the 21-satellite constellation but based only on data generated for locations within the continental United States. Table 2-6, constructed of data obtained from these two studies and from Bogen (1974), gives the probabilities of GDOP, PDOP and HDOP being less than a given value for the 24/3/2, 18/6/2 and 18/6/2+3 constellations.

Figure 2-2 Probability of PDCP ≤ Given Value (from Spilker (1985))

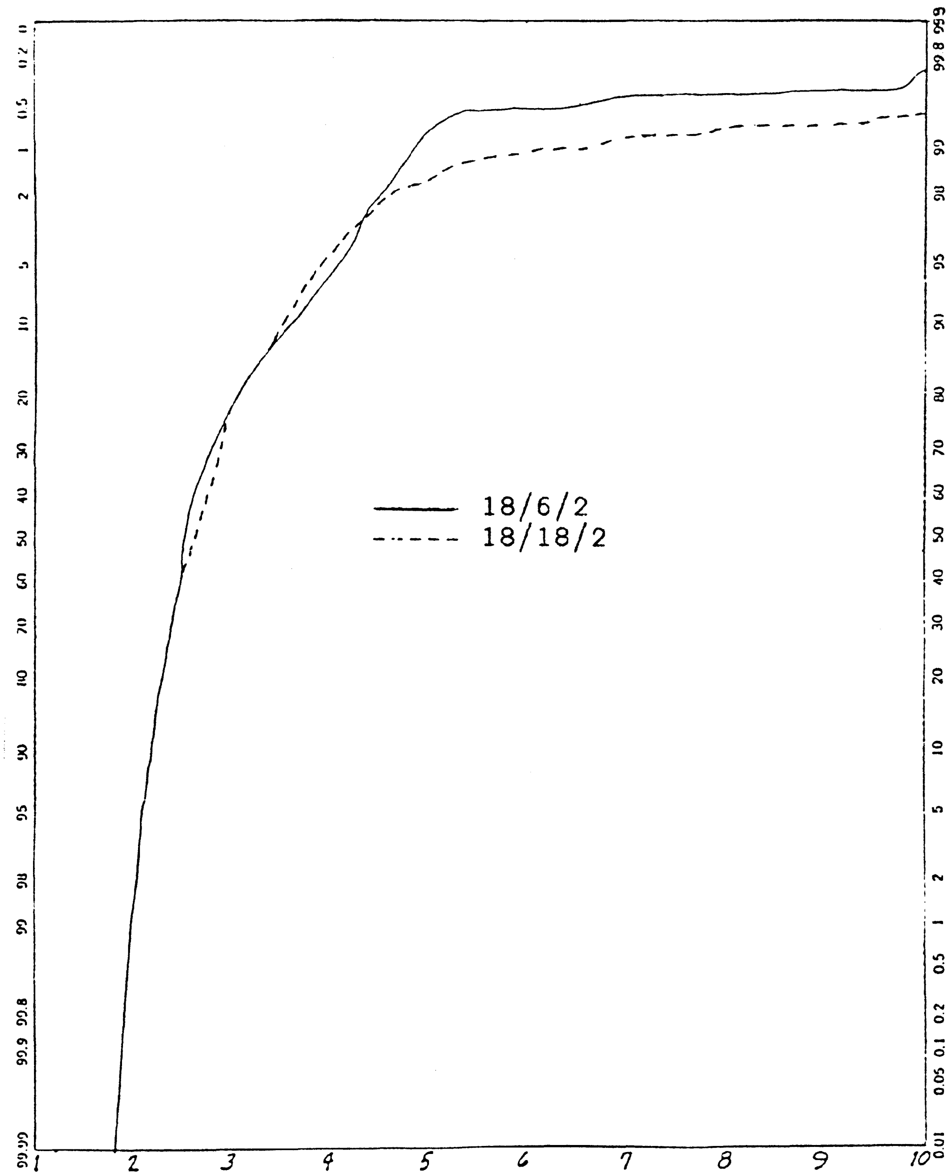


Figure 2-3 Probability of HDOP ≤ Given Value (from Kalafus et al.(1983))

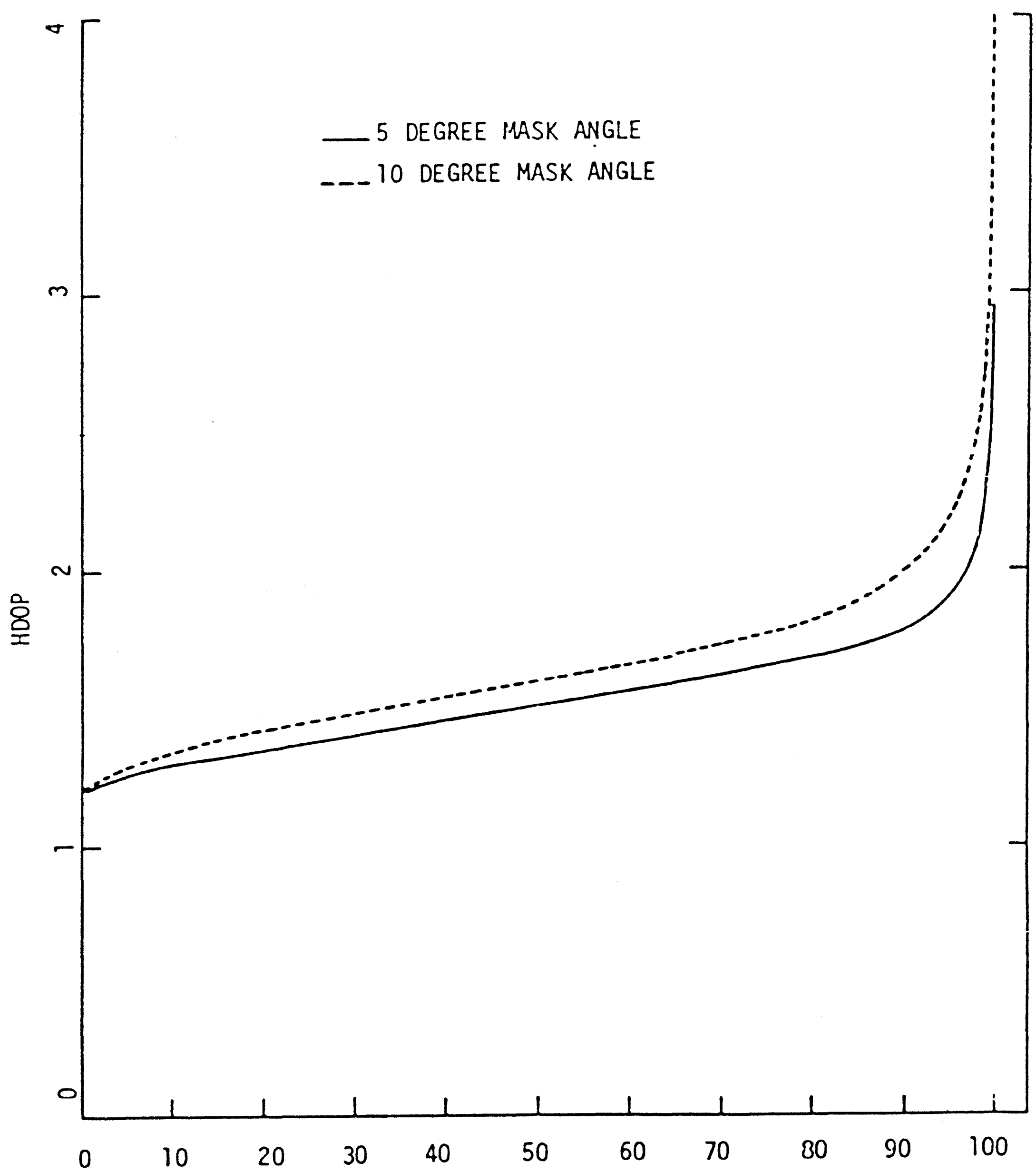


Table 2-6
Probability of DOP $\leq$ Given Value

| DOP | Constel-lation | Mask Angle | Probability 50 | 90 | 95 | 99.0 | 99.5 | 99.6 | 99.7 | 99.8 | 99.9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| GDOP | 24/3/2 | 5 | 2.6 | 3.9 | 4.2 | 4.3 | 4.7 | 4.7 | 4.7 | 4.8 | 4.9 |
| | | 10 | 3.1 | 4.3 | 4.5 | 5.1* | | | | | |
| | 18/6/2 | 5 | 2.7 | 4.1 | - | - | - | 10.0* | | | |
| PDOP | 24/3/2 | 5 | 2.4 | 3.4 | 3.7 | 4.0 | 4.1 | 4.2 | 4.2 | 4.3 | 4.3 |
| | | 10 | 2.8 | 3.8 | 3.9 | 4.3* | | | | | |
| | 18/6/2 | 5 | 2.5 | 2.5 | 4.1 | 4.8 | 6.0 | 7.5 | 10.0* | | |
| | 18/6/2+3 | 5 | 2.5 | 3.2 | 3.5 | - | - | - | - | - | 6.0* |
| HDOP | 24/3/2 | 5 | 1.4 | 1.8 | 1.9 | 2.1 | 2.2 | 2.2 | 2.3 | 2.3 | 2.3 |
| | | 10 | 1.6 | 2.0 | 2.3 | 2.6 | 2.7 | 2.9 | 3.5* | | |
| | 18/6/2 | 5 | 1.5 | 1.7 | - | - | - | - | - | | |
| | 18/6/2+3 | 5 | 1.5 | 1.8 | 1.9 | 2.3 | 2.5 | - | - | | |
| | | 10 | 1.6 | 2.0 | 2.2 | 3.1 | 3.5* | | | | |

* Above the corresponding probability level the DOP factor becomes infinitely large.

Combining the estimated magnitudes of the GDOP parametres and the expected errors in the pseudorange measurements given in Table 2-5, potential navigation accuracies of the GPS with 21 operational satellites may now be evaluated. Considering only the three-dimensional GDOP parameter, PDOP, from Table 2-4, it may be expected that 95% of the time PDOP will be less than or equal to 3.5. In terms of navigation accuracy, users of the GPS should then have a 95% probability of obtaining three-dimensional positions to 14.0 m and 42.0-70.0 m (1 sigma) using P- and S-codes, respectively. In fact, according to these statistics, PDOP values should be less than or equal to six 99.9 % of the time with corresponding navigation accuracies of 24.0 m and

78.0-120.0 m, again at one sigma levels. There is, however, 0.1% of the time when the PDOP values become very large due to poor geometry. During these outage periods, and those due to satellite failures, users of the GPS may have to rely on alternative navigation systems or supplement the information provided by the GPS with data from other sources. In addition, it should be noted once more that accuracies attainable with S-code will be intentionally degraded at some time in the future.

## Chapter 3

## OUTAGES OF THE PHASE III GPS CONSTELLATION

Kruh (1983) defines the value of a constellation as the fraction of all sample points on the earth for which PDOP is less than or equal to 6.0 over a 24 hour period. From Table 2-6, based on this definition, the constellation value for the originally planned configuration of 24 satellites in 3 orbital planes is then very nearly 1.0 considering a 5-degree mask angle. Unfortunately, reducing the number of satellites may also reduce the coverage capabilities of a constellation. Again, from Table 2-6, the 18-satellite constellation with no active spares will have a constellation value of 0.995. In other words, brief outages due to poor geometry (geometric outages) may be expected 0.5% of the times. The addition of three active spares does improve the constellation value of the GPS to 0.999 and so the problem is somewhat alleviated. The frequency, extent and duration of these geometric outages and the effect upon them of using a mask angle greater than 5 degrees requires examination. Also, since satellite failures will degrade the service provided by the GPS, the potential outages due to such failures must be considered.

## 3.1 MATHEMATICAL DEFINITION OF THE GEOMETRIC OUTAGES

When the four or more satellites visible to the user all lie in the same plane, i.e., are coplanar, the navigation equations relating pseudorange measurements to user position and receiver clock bias become indeterminant. For the majority of the outages, this coplanarity condition occurs when only four satellites are in view, but this condition may exist with four and more visible satellites (Stein 1985). The effect of coplanarity of the satellites may be seen by examining the linearized navigation equations. These may be expressed as follows:

$$\frac{(Xn - Xi)}{Rni - Tn}\Delta X + \frac{(Yn - Yi)}{Rni - Tn}\Delta Y + \frac{(Zn - Zi)}{Rni - Tn}\Delta Z + \Delta T = \Delta Ri$$

where $Xn, Yn, Zn, Tn$ are the nth estimates of the receiver coordinates and clock bias
$\Delta X, \Delta Y, \Delta Z, \Delta T$ are the corrections to the estimated coordinates and clock bias
$Xi, Yi, Zi$ are the coordinates of ith satellite
$Rni$ is the nth estimate of the range to the ith satellite
$\Delta Ri$ is the difference between the actual and the estimated range
$i = 1,4$

In these equations the coefficients of the corrections to the estimates of the receiver position are, in fact, the direction cosines to the satellites. The linearized equation may then be expressed as:

$$\alpha i1\Delta X + \alpha i2\Delta Y + \alpha i3\Delta Z + \Delta T = Ri$$

Using matrix notation, the complete system to be solved, considering four visible satellites, is then:

$$\begin{bmatrix} \alpha 11 & \alpha 12 & \alpha 13 & 1 \\ \alpha 21 & \alpha 22 & \alpha 23 & 1 \\ \alpha 31 & \alpha 32 & \alpha 33 & 1 \\ \alpha 41 & \alpha 42 & \alpha 43 & 1 \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \\ \Delta T \end{bmatrix} = \begin{bmatrix} \Delta R1 \\ \Delta R2 \\ \Delta R3 \\ \Delta R4 \end{bmatrix}$$

Or, $$A\Delta X = \Delta R$$

Jorgensen (1981) illustrates how the coplanarity condition causes divergence of the navigation equations by examining the design matrix A. When the four satellites are in a common plane, if the Z axis is chosen such that it is perpendicular to that plane, the direction cosines in the third column of A of the four unit vectors to the satellites all become equal. The determinant of A is then zero, and no solution to the above system is then possible. Further, as the determinant of A goes to zero, the GDOP parametres become infinitely large since they are obtained from $(\mathrm{trace}(A^T A)^{-1})^{1/2}$.

## 3.2 GEOMETRIC OUTAGES OF THE 18/6/2 CONSTELLATION

Figures 3-1 and 3-2, from Kruh (1981), indicate the location and duration of outages of the 18-satellite constellation with no active spares. Figure 3-1 shows the location of outages over a 24-hour period. These occur in sets of four and repeat twice daily at the indicated locations. Going eastward from set #1, each new set (2,3,...,18) will occur 40 minutes after the beginning of the previous outage and will last between 5 and 30 minutes depending on the position within an outage area. Concurrent with the 40 minute interval, outages along the same latitude are then separated by 2 hours and 40 minutes in time and 20 degrees in longitude, centre to centre. Sample plots of the time variation of PDOP are given in Figure 3-2.

Some fundamental symmetries of the GPS outages, as derived by Chen (1984), are given in Table 3-1. It should also be noted that the longitude of outages will vary depending on the orbital reference point (Kruh 1981). For the outage locations given in Figure 3-1, the right of ascension of the ascending nodes of the orbital planes are referenced as given in Table 3-2.

Table 3-1
Fundamental Symmetries of The GPS Outages

| Symmetry | | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Time | $t$ | $t+12h$ | $t+40m$ | $7h20m-t$ | $10h-t$ |
| User Latitude | $\phi$ | $\phi$ | $-\phi$ | $\phi$ | $\phi$ |
| User Longitude | $\lambda$ | $\lambda$ | $\lambda+50$ | $10-\lambda$ | $30-\lambda$ |

From Chen (1984)

Figure 3-1 Composite Outages of The 18/6/2 Constellation (from Kruh (1981))

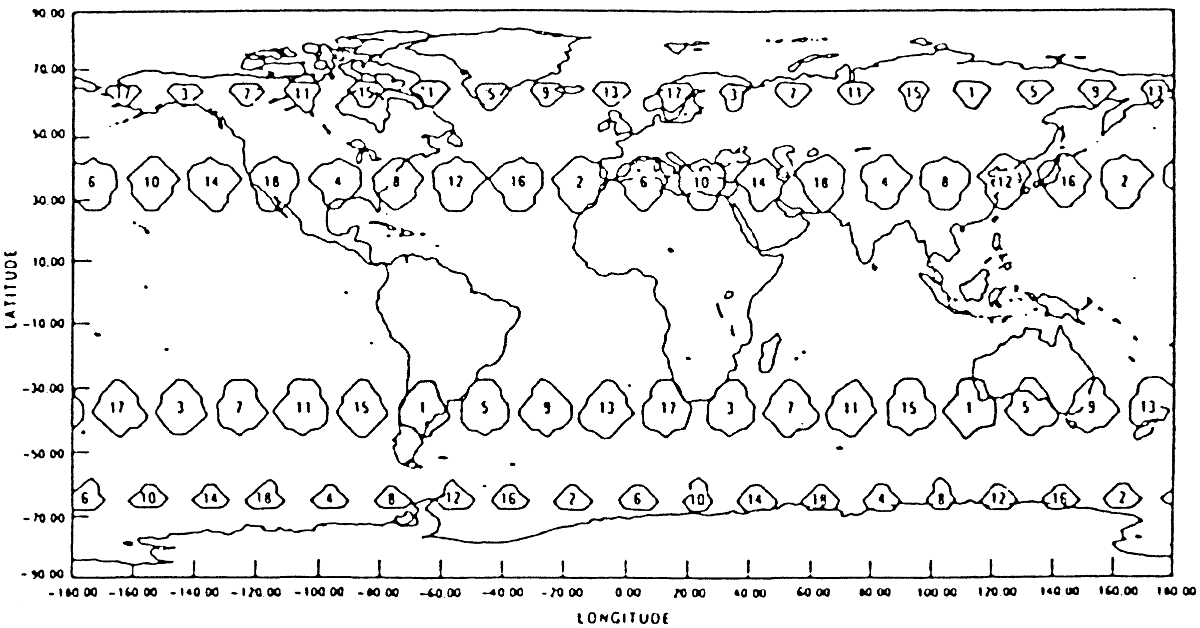
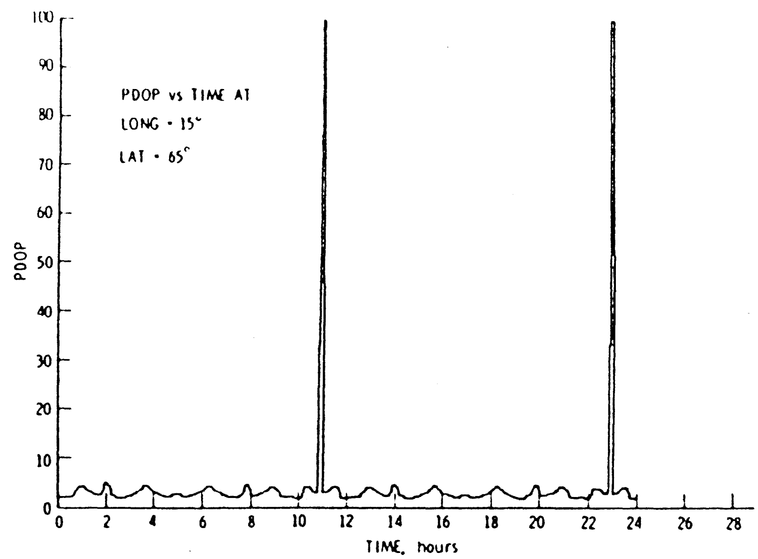


Figure 3-2  Sample PDOP vs Time Plot of The 18/6/2 Constellation (from Kruh (1981))



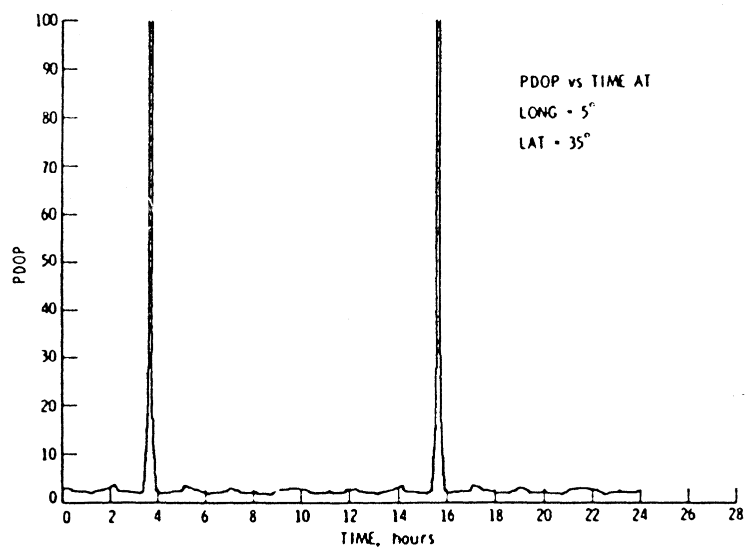



Table 3-2
18/6/2 Orbit Description Relative to
Earth and Astronomic Coordinates

| Satellite Number | Orbital Plane | Longitude of the Ascending Node (deg) | Right Ascension of the Ascending Node (deg) * |
|---|---|---|---|
| 1 | 1 | 0,180 | 30 |
| 2 | 1 | 240, 60 | 30 |
| 3 | 1 | 300,120 | 30 |
| 4 | 2 | 260, 80 | 90 |
| 5 | 2 | 320,140 | 90 |
| 6 | 2 | 20,200 | 90 |
| 7 | 3 | 340,160 | 150 |
| 8 | 3 | 40,220 | 150 |
| 9 | 3 | 100,280 | 150 |
| 10 | 4 | 60,240 | 210 |
| 11 | 4 | 120,300 | 210 |
| 12 | 4 | 180, 0 | 210 |
| 13 | 5 | 140,320 | 270 |
| 14 | 5 | 200, 20 | 270 |
| 15 | 5 | 80,260 | 270 |
| 16 | 6 | 220, 40 | 330 |
| 17 | 6 | 280,100 | 330 |
| 18 | 6 | 160,340 | 330 |
| Spares | | | |
| 19 | 1 | 195, 15 | 30 |
| 20 | 3 | 215, 35 | 270 |
| 21 | 5 | 25,205 | 150 |

* Referenced to astronomical coordinates of 1950.0 as of 1 July 1985, 0 hr 0 min GMT and regressing at -0.04009 deg/day.
From Porter et al. (1984).

To investigate the time behaviour of these outages, a software package developed at Department of Surveying Engineering of the University of New Brunswick (Mertikas et al. 1985), was modified to compute DOP values at grid points in the area of interest. Using a 5 degree mask angle, the outage in the vicinity of Hudson Bay (#15) was then plotted at ten minute intervals from the start of the outage. As may be seen in Figures 3-3 to 3-10, the outages

Figure 3-3 Hudson Bay Outage Plot
Time = 0 min , Mask Angle = 5 degrees

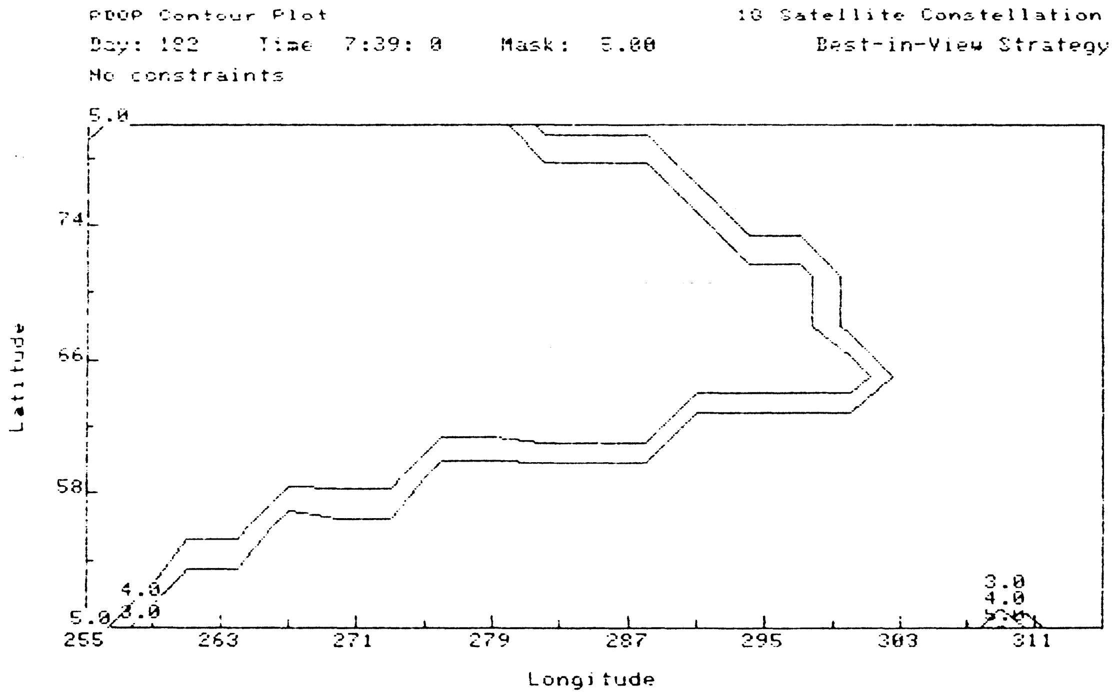



Figure 3-4 Hudson Bay Outage Plot
Time = 10 min , Mask Angle = 5 degrees

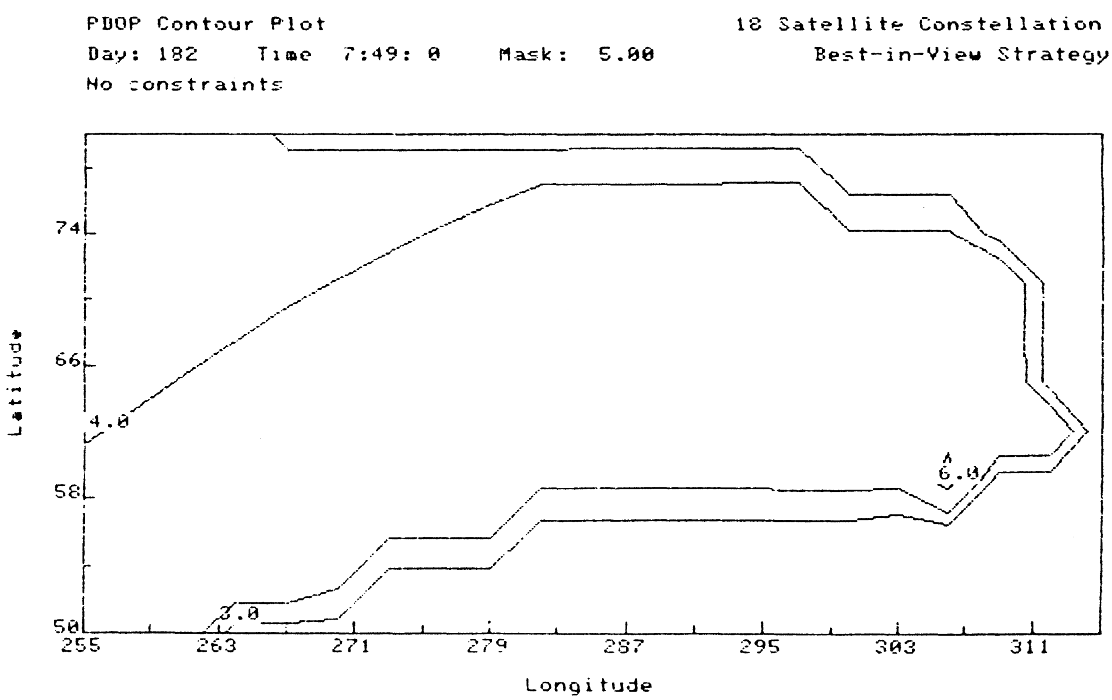


Figure 3-5 Hudson Bay Outage Plot
Time = 20 min , Mask Angle = 5 degrees

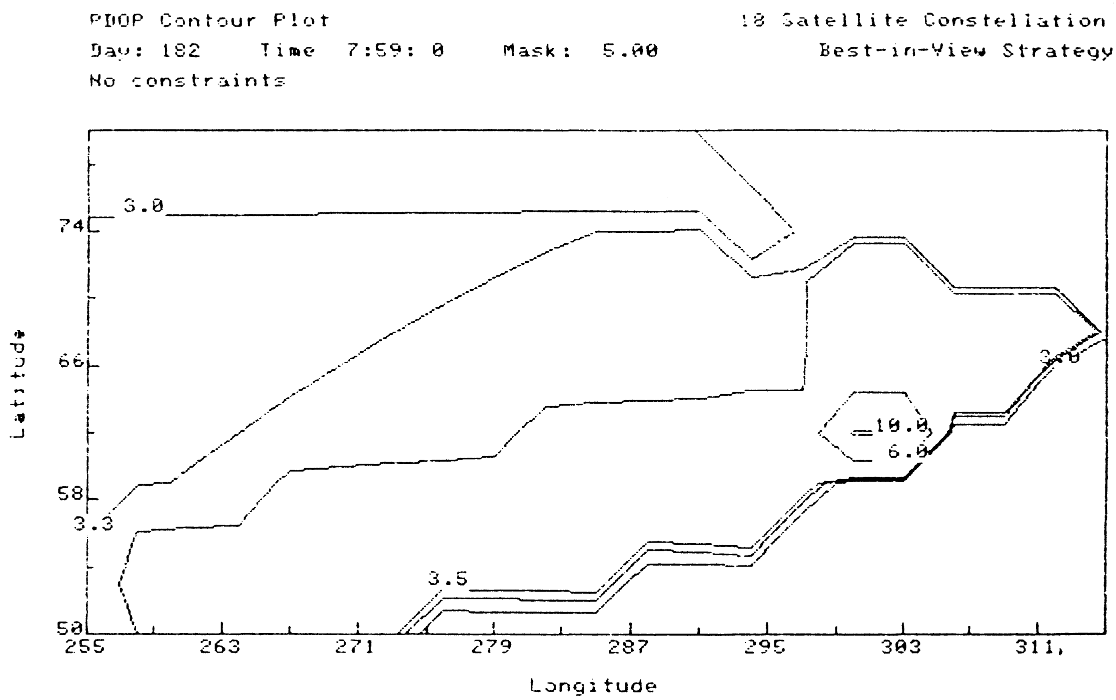



Figure 3-6 Hudson Bay Outage Plot
Time = 30 min , Mask Angle = 5 degrees

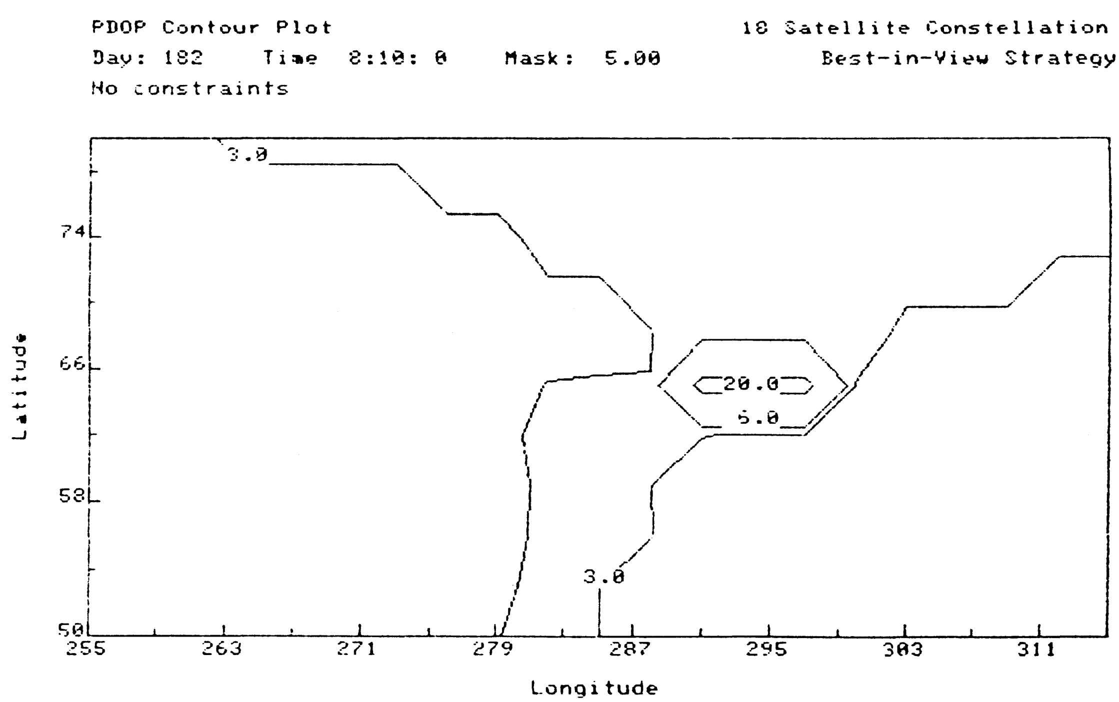


Figure 3-7 Hudson Bay Outage Plot
Time = 40 min , Mask Angle = 5 degrees

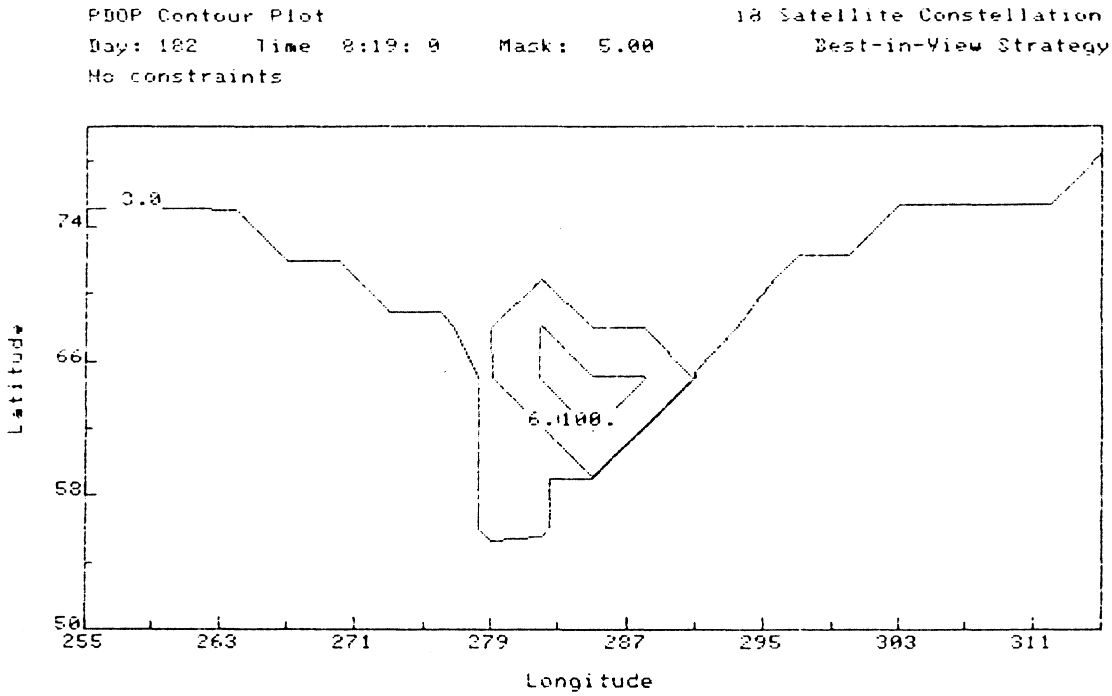



Figure 3-8 Hudson Bay Outage Plot
Time = 50 min , Mask Angle = 5 degrees

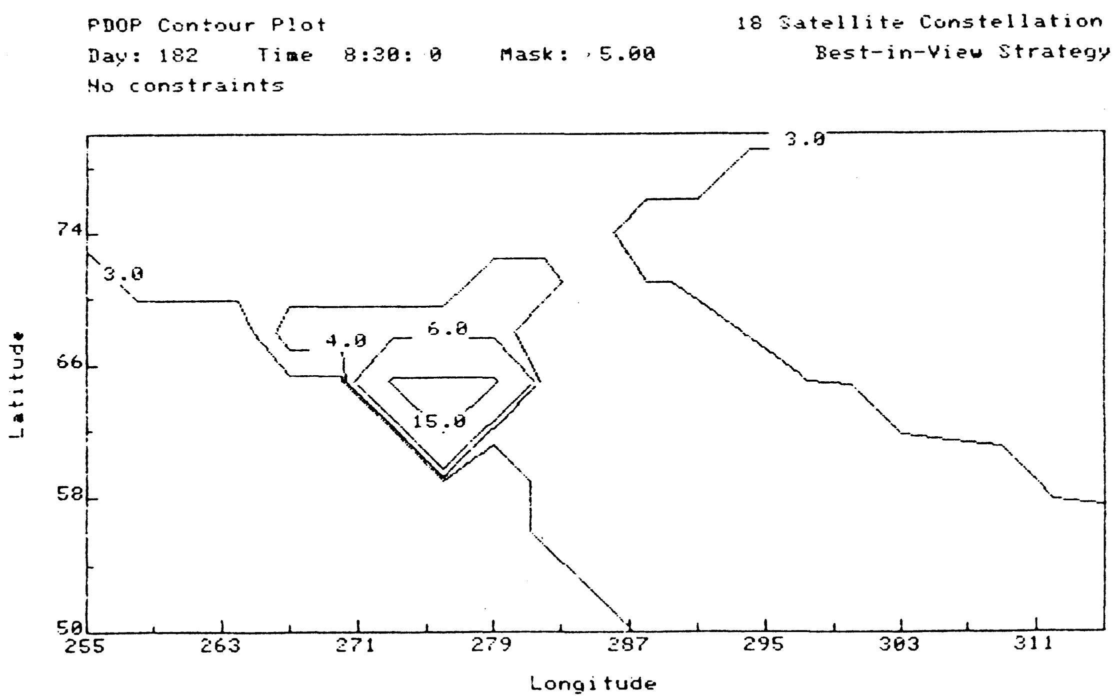


Figure 3-9 Hudson Bay Outage Plot
Time = 60 min , Mask Angle = 5 degrees

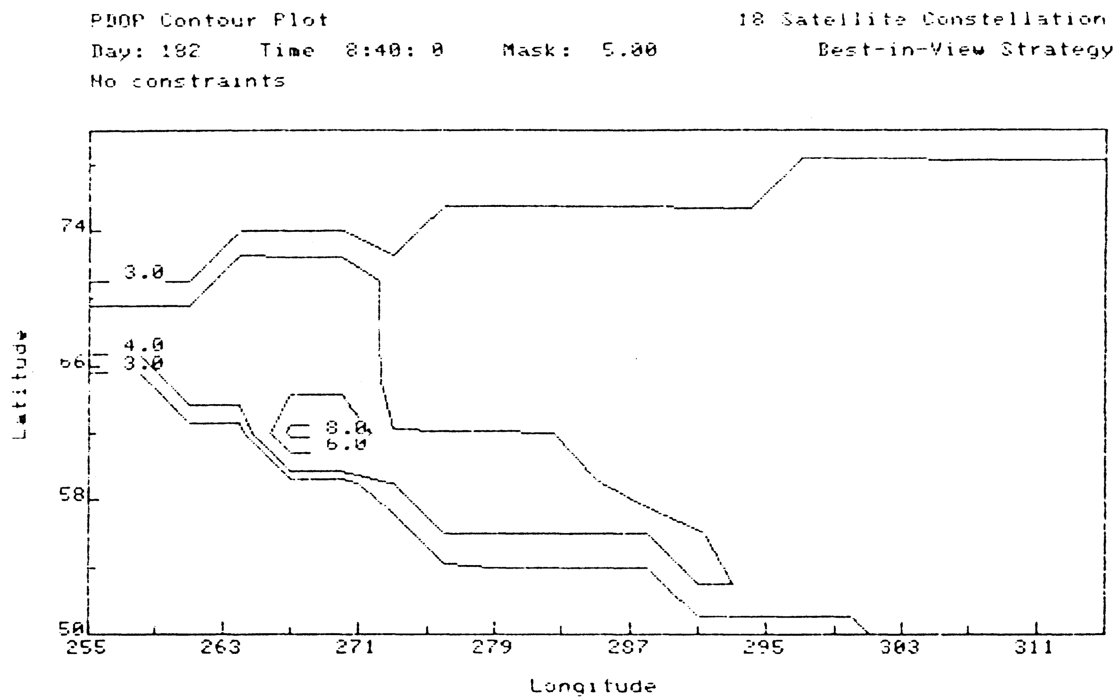
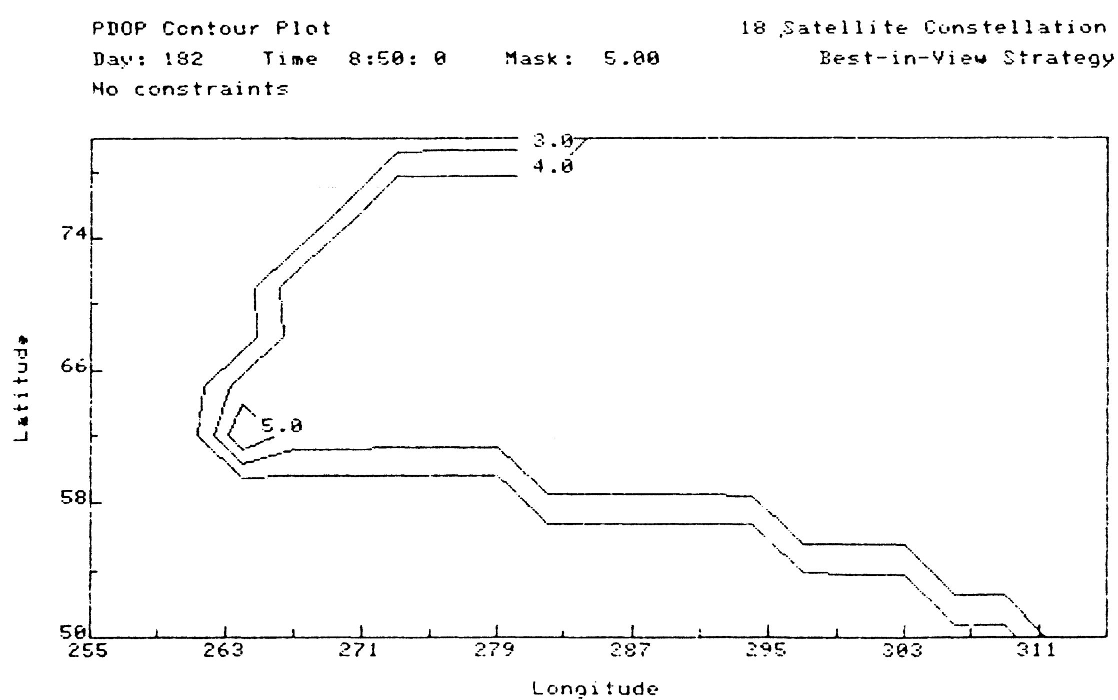



Figure 3-10 Hudson Bay Outage Plot
Time = 70 min , Mask Angle = 5 degrees



are not static entities but move from east to west. Users in the outage area will therefore experience outages of different duration and at different times depending on their position.

### 3.3 GEOMETRIC OUTAGES OF THE 18/6/2+3 CONSTELLATION

As discussed previously, three spare satellites will be included in the operational constellation of satellites to increase the probability of 18 or more properly functioning satellites always being available. Since the lifetime of a GPS satellite is not appreciably lengthened by its being kept in a dormant state, current plans are that these spares will be active. This then increases the coverage that may be expected from the GPS. Depending on the sparing strategy, certain outages will then be either eliminated entirely or reduced in frequency, duration or extent.

The currently proposed sparing stategy, given in Figure 1-5, is designed so as to eliminate outages over the continental United States for users capable of receiving signals from all satellites greater than 5 degrees over the local horizon (Kalafus et al. 1983). This is seen in Figure 3-11 where half-hour outages are eliminated at a location in the mid-western U.S. Figure 3-12 gives the location of outages over a 24 hour period after the addition of the active spares. These generally occur once daily and at any

Figure 3-11 Comparison of PDOP Plot For 18/6/2 and 18/6/2+3
(from Kalafus et al.(1983))

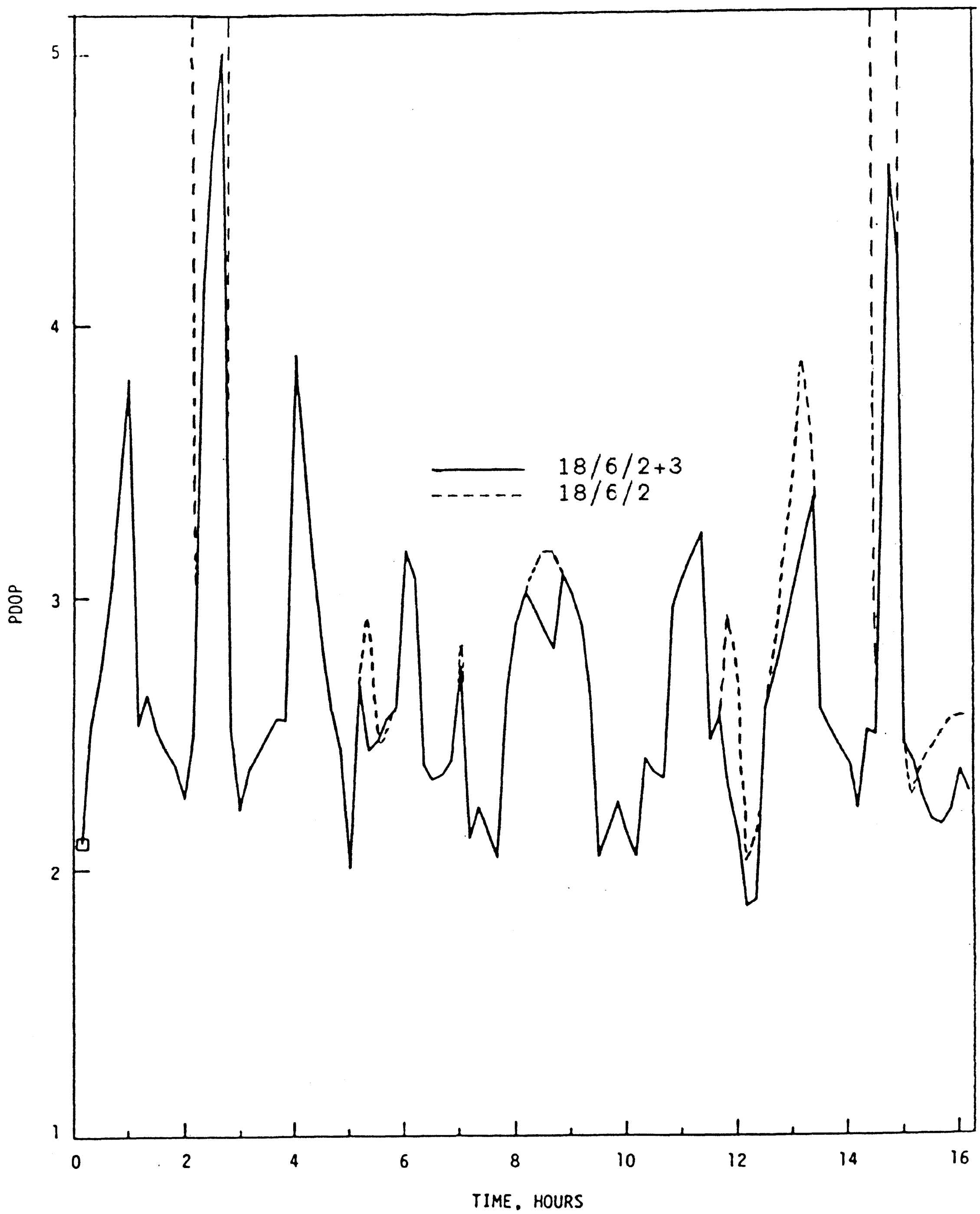
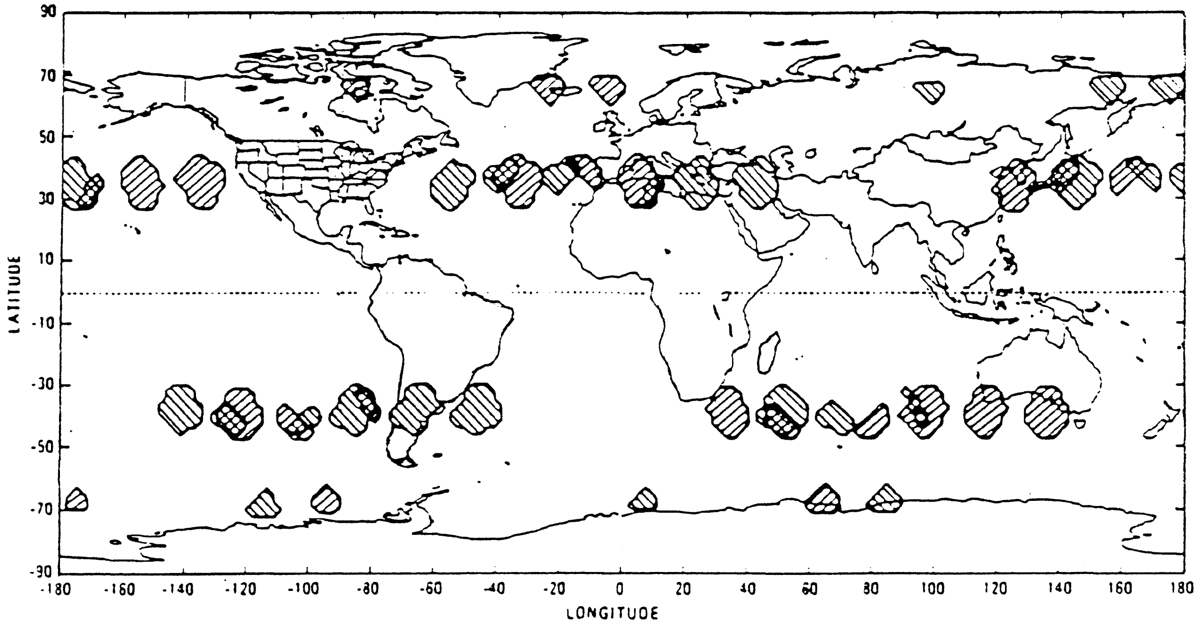


Figure 3-12 Composite Outages of The 18/6/2+3 Constellation
(from Porter et al.(1984))



point in time one to four will exist at various locations. Some points do experience outages twice daily but the second is usually less severe. Sample plots of the time variation of PDOP for the 21-satellite constellation are given in Figure 3-13.

## 3.4 GEOMETRIC OUTAGES OF THE 18/6/2+3 FOR HIGHER MASK ANGLES

The location and duration of the outages described so far have been primarily based on the assumption that all users can observe satellites 5 degrees above their local horizon. The FAA has adopted a 10 degree mask angle to allow for possible limitations of low-cost civil user antennas. This limitation must be considered for many other application as well.

Figure 3-14, from Klein and Parkinson (1984), illustrates the effect on the outages of the 21-satellite constellation of increasing the mask angle to 7.5 degrees. Outages previously eliminated by the addition of three active spares once again reappear and, in addition, all outage areas are increased in size. For example, the outage off the west coast of the U.S. is increased from approximately 18 degrees in diameter (1940 km) to 30 degrees (3240 km) when using a mask angle of 7.5 degrees. Kalafus et al. (1983) reports that if a 10 degree mask angle is used, five outages lasting between 1 and 15 minutes still exist over North America.

Figure 3-13 Sample PDOP vs Time Plot of The 18/6/2+3 Constellation
(from Porter et al.(1984))

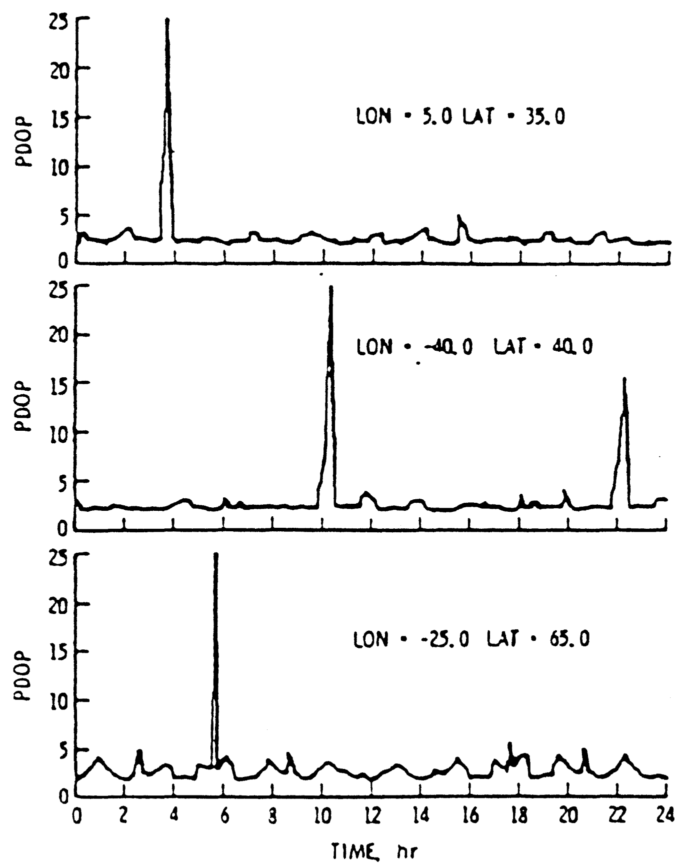


Figure 3-14 Composite Outages of the 18/6/2+3 Constellation Using a 7.5 Mask Angle (from Klein and Parkinson(1984)).

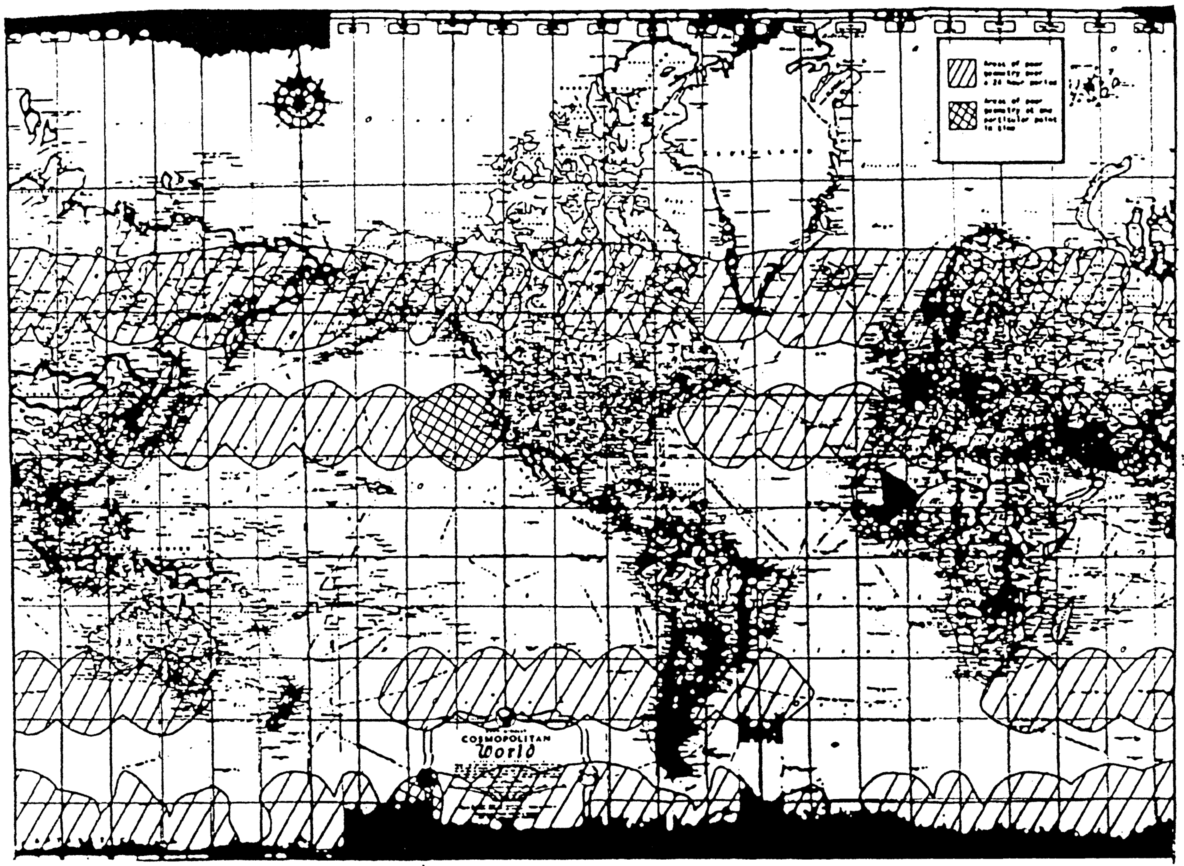


Figure 3-15 HDOP vs Time Plot for 18/6/2+3 Using a 10 Mask Angle (from Kalafus et al. (1983))

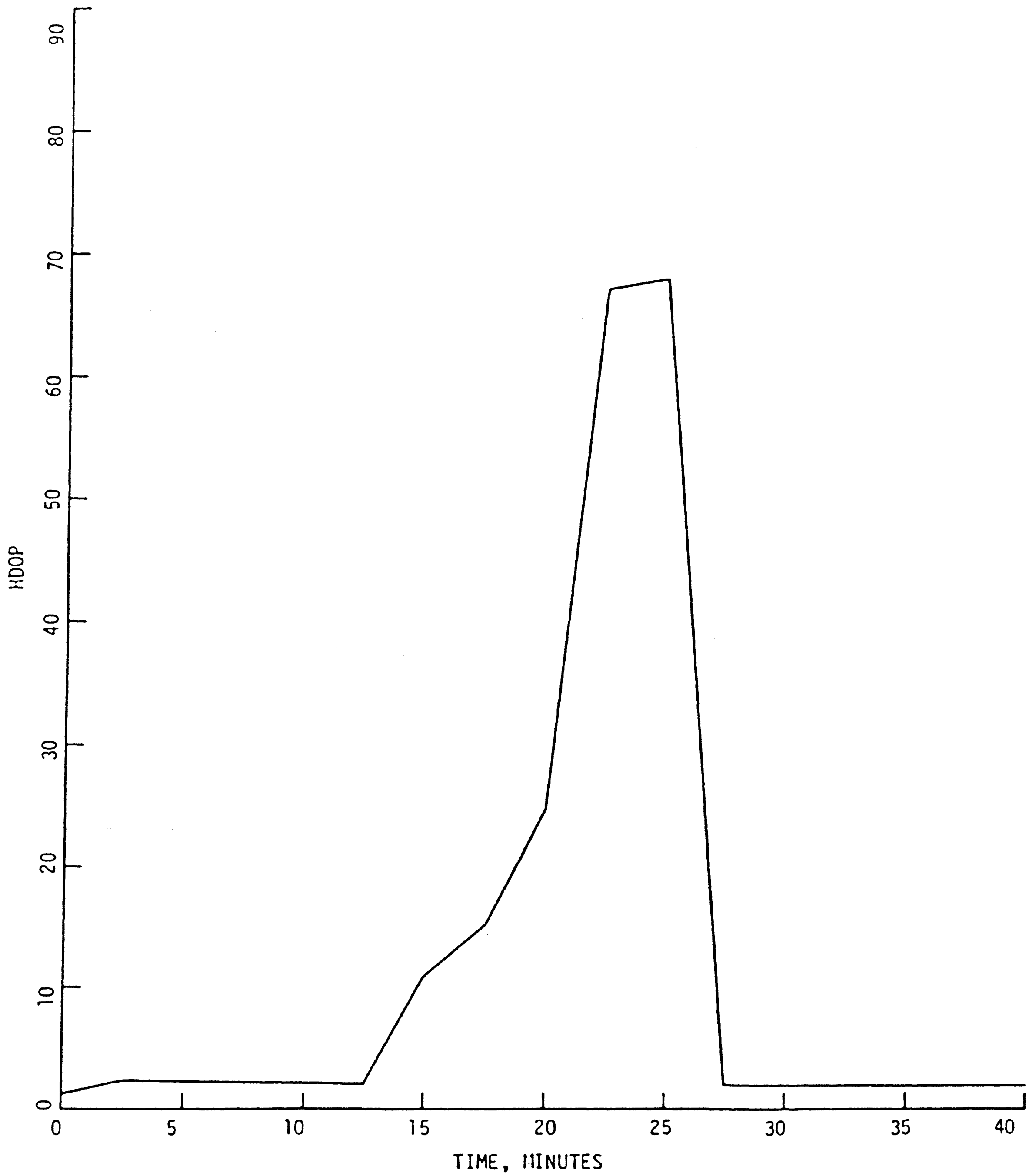


Figure 3-15 shows one such outage at 50 degrees north latitude and 85 degrees west longitude.

In addition to increasing the frequency and extent of the outages, using a higher mask angle will also result in outages of longer duration. Again, considering outage #15, this effect was examined by computing PDOP values versus time during the outage using mask angles of 10, 12.5, and 15 degrees. These values are plotted in Figures 3-16 to 3-18. As may be seen, the duration increases from 35 to 50 and 62 minutes for these three mask angles, repectively.

## 3.5 OUTAGES DUE TO SATELLITE FAILURES

Failure of one or more of the GPS satellites will result in periods of interrupted service to users requiring four satellites. Kalafus (1984), based on a simulation study in which each of the 21 satellites was removed in turn, reports that single satellite failures will result in outages lasting as long as 36 minutes over areas within the continental United States. These results were, however, based on a mask angle of 10 degrees. He further states that if a 5 degree mask angle could be used, no such degradations of the GPS service will occur.

Figure 3-16 Hudson Bay Outage Plot of PDOP vs Time, Mask=10.0

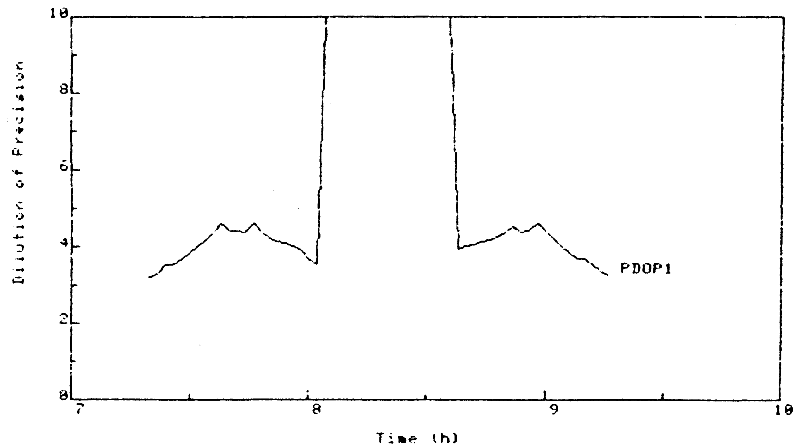



Figure 3-17 Hudson Bay Outage Plot of PDOP vs Time, Mask=12.5

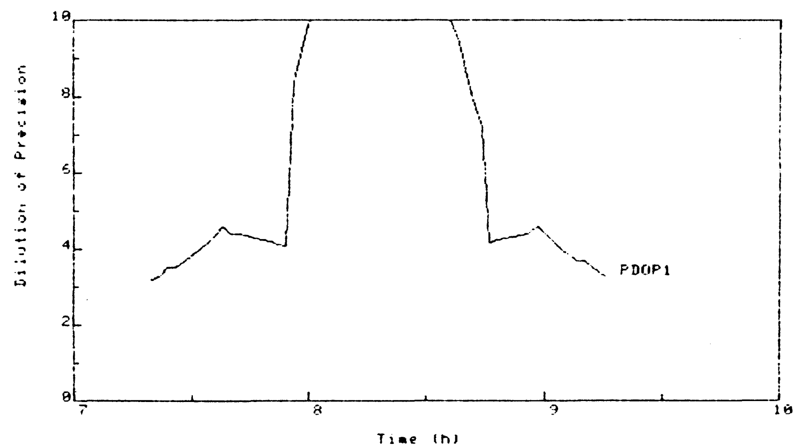
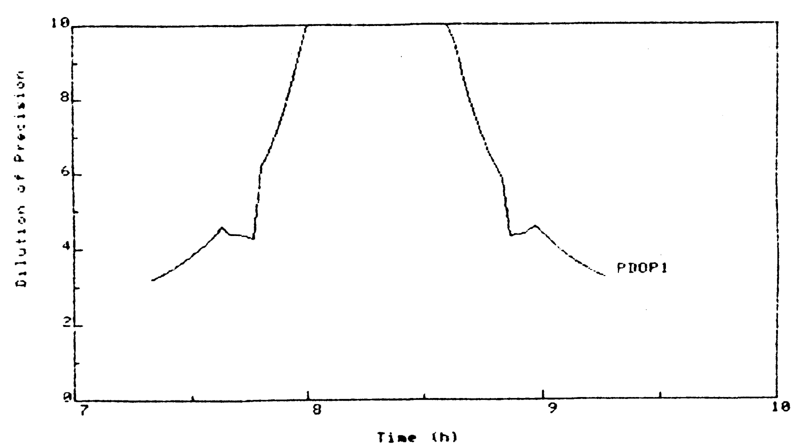



Figure 3-18 Hudson Bay Outage Plot of PDOP vs Time, Mask=15.0



## 3.6 NAVIGATION DURING OUTAGES OF THE GPS

As has been discussed, the frequency, duration and extent of the outages of the GPS will depend on a variety of factors. Since most of the outages will be relatively short in duration (10 minutes or less), many users may simply cease operations during such periods. For those requiring continuous navigation capabilities, however, additional information from some external device will be required to 'coast' through these periods.

Altimetry and/or clock - solutions in which height or time or both are constrained (solutions type 1,2 and 3 in Chapter 1) - may be used for navigation with less than four satellites or during the outages of the GPS. These techniques will not be successful during all outages, however, since, in some situations, the satellites and the user are all in the same plane. Even the aided solution will then become indeterminate. More details concerning these aided solutions may be found in Kalafus (1984), Sturza (1983), Stein (1984) and Wells et al. (1982). Differential GPS, in addition to providing higher accuracies using S-code, may also solve the problem of navigation during outages of the GPS. If the monitor station transmits position corrections using a GPS like signal, the user may then observe the pseudorange to this pseudo-satellite or pseudolite. An additional benefit of such a system is that the pseudolite stations may also monitor the integrity of the satellite signals and then provide an integrity message in the broadcast signal. A detailed discussion of the pseudolite concept and some test results may be found in Klein and Parkinson (1984).

# Chapter 4

# SUMMARY AND CONCLUSIONS

Some aspects of the GPS performance capabilities have been examined and it is evident that in various situations the GPS cannot solely provide sufficient information for safe and accurate navigation.

System integrity was not a major consideration of this report but for safety of navigation - especially for users having high dynamics - it is an important factor. Without some type of external aid (e.g. pseudolite) a user may expect time delays of 5 to 30 minutes between detection and notification or correction of satellite malfunctions. Whether or not this is sufficient will depend on the application.

The reliability of the GPS was evaluated based on whether or not a minimum of four satellites would be visible at all possible locations and what probability of satellite failures could be expected. Availability is a function of the receiver mask angle and so must be judged on that basis. It was found that for mask angles up to 10 degrees a minimum of four satellites will be visible on a continuous basis at all locations - unless satellite failures occur. The probability of satellite failure does exist, however, and so a minimum of four satellites cannot be guaranteed. The GPS reliability may not, therefore, be sufficient.

Single point positioning with the GPS currently satisfies most requirements for real-time navigation. When it is degraded, however, various alternatives, such as constrained solutions or differential GPS, may be required.

With only 21 satellites in the operational constellation, outages, due to poor geometry as well as satellite failures, will occur at various locations. The frequency, location, duration and extent of these outages are all a function of the receiver mask angle and so must be evaluated accordingly. For receivers using a 5 degree mask angle, outages due to poor geometry are widely scattered and of short duration. In addition, for single satellite failures, outages due to less than four available satellites will not occur over the continental United States. For higher receiver mask angles, however, outages occur with more frequency, over more locations, last longer and effect larger areas.

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# APPENDIX A

User Navigation Solution

Extract from:
Mertikas et al. (1985)

## 2. USER NAVIGATION SOLUTION

As mentioned previously, the GPS is a radionavigation system. Signals are emitted from the satellites and are received by a user. To obtain a position fix ($X$, $Y$, $Z$, $\Delta t$), the user must be equipped with a receiver capable of tracking at least four satellite signals simultaneously (multichannel receiver) or sequentially (single channel receiver). In the receiver, the received satellite signal is matched against an exact replica of the emitted signal in an attempt to determine the time-of-arrival, i.e., the time spent by the signal to cover the distance from the satellite to the user.

By multiplying this time by the speed of light, the user can determine his distance or range ($\tilde{\rho}_{ij}(t_k)$) from the satellite. This distance, which is in terms of raw receiver measurements and includes atmospheric delays and user clock bias, is called a pseudorange. Pseudoranges will be designated by the small Greek letter $\tilde{\rho}_{ij}$. The subscript i denotes the satellites and j the receiver.

In our computations, we shall consider an earth-fixed, earth-centred reference system, as shown in Figure 2.1, characterized by the following properties:

1. The x-axis passes through the intersection of the equator and the Greenwich meridian.
2. The z-axis passes through the North Pole.
3. The y-axis completes the right-handed coordinate system.

The basic GPS measurement is pseudoranges ($\tilde{\rho}_{ij}$) but, because of the satellite motion, the system allows us to measure another quantity; integrated Doppler range differences or delta range observations. In this

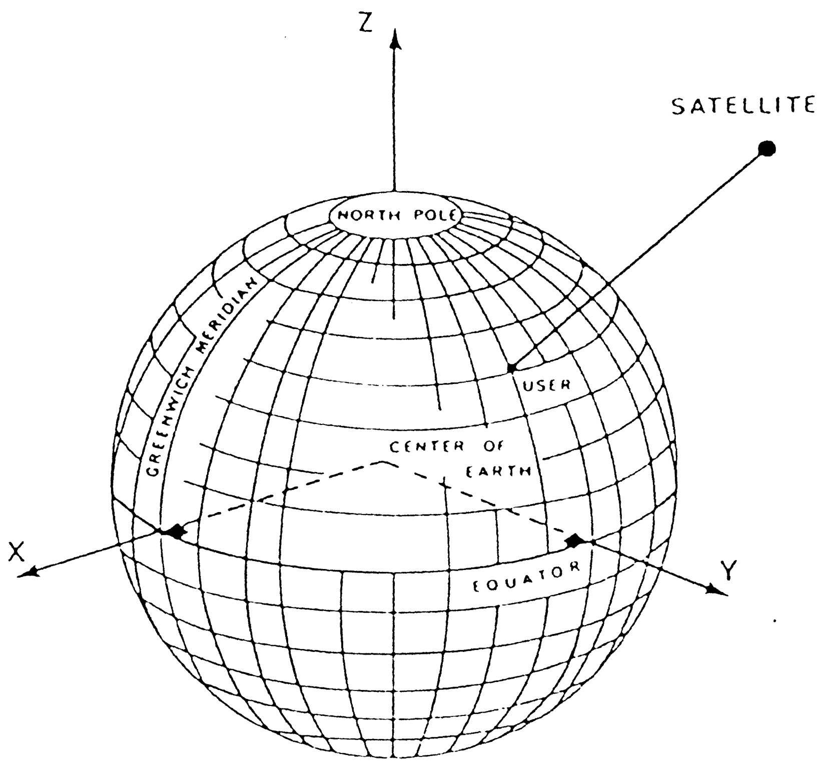



FIGURE 2.1

WGS-72 Reference System.

report, we will be dealing only with the first basic measurement of the GPS, that is, pseudoranges ($\tilde{\rho}_{ij}$). Observation equations will be developed for the determination of a position fix based on these measurements.

At the present time, the operational satellite constellation is planned to consist of 18 satellites in six orbital planes, with three satellites equally spaced in each plane, the planes to have an inclination of 55$^{o}$, and the nominal period of a satellite revolution to be 11 hr, 57 min, 57.26 sec [Payne, 1982]. This 18-satellite constellation may sometimes experience occasional outages. An outage is when a user can only obtain data from less than four satellites, and a complete three-dimensional navigation solution (X, Y, Z, $\Delta t_u$) is not possible.

For a complete three-dimensional position fix, at least four range measurements are needed. Three measurements are required for the position determination ($X_j$, $Y_j$, $Z_j$) and one for the user clock bias ($\Delta t_{u_j}$). The user is usually equipped with a fairly inexpensive crystal clock, which means that another unknown ($\Delta t_{u_j}$) is added to the computations.

To compute a position from satellite range data measurements, the following information is required for each measurement:

1. Position of the tracked satellites and time of signal transmission $[x_i(t_k),\ y_i(t_k),\ z_i(t_k)]$.
2. Time of transmission of the received signal $[t_{s_i}]$.
3. Estimates of the deterministic time delays.

The position of each tracked satellite with respect to our reference system (WGS-72) can be computed as a function of time from the six orbital elements. The most current information (taken from van Dierendonck et al. [1978]) is given as:

| | | |
|---|---|---|
| $\mu = 3.986\ 008 \cdot 10^{14}\ m^3/sec^2$ | Universal Gravitational Constant(WGS-72) | |
| $\omega_e = 7.292\ 115\ 147 \cdot 10^{-5}$ rad/sec | Earth's rotation rate (WGS-72) | |
| $a = (\sqrt{a})^2$ | Semi-major axis | |
| $n_o = \sqrt{\mu/a^3}$ | Computed mean motion | |
| $t_k = t - t_{oe}$ | Time from reference epoch | |
| $n = n_o + \Delta n$ | Corrected mean motion | |
| $M_k = M_o + n t_k$ | Mean anomaly | |
| $M_k = E_k - e \sin E_k$ | Kepler's equation for eccentric anomaly | |
| $\cos V_k = (\cos E_k - e)/(1 - e \cos E_k)$ | True anomaly | |
| $\sin V_k = \sqrt{1-e^2} \sin E_k/(1 - e \cos E_k)$ | | |
| $\phi_k = V_k + \omega$ | Argument of latitude | |
| $\delta u_k = C_{uc} \cos 2\phi_k + C_{us} \sin 2\phi_k$ | Correction to argument of latitude | 2nd harmonic perturbations |
| $\delta r_k = C_{rc} \cos 2\phi_k + C_{rs} \sin 2\phi_k$ | Correction to orbit radius | |
| $\delta i_k = C_{ic} \cos 2\phi_k + C_{is} \sin 2\phi_k$ | Correction to inclination angle | |
| $u_k = \phi_k + \delta u_k$ | Corrected argument of latitude | |
| $r_k = a(1 - e \cos E_k) + \delta r_k$ | Corrected orbit radius | |
| $i_k = i_o + \delta i_k$ | Corrected inclination | |
| $x_k = r_k \cos u_k$ | Position in orbital plane | |
| $y_k = r_k \sin u_k$ | | |
| $\Omega_k = \Omega_o + (\dot{\Omega} - \omega_e) t_k - \omega_e t_{oe}$ | Corrected longitude of ascending node | |
| $X_k = x_k \cos\Omega_k - y_k \cos i_k \sin\Omega_k$ | Earth fixed coordinates | |
| $Y_k = x_k \sin\Omega_k + y_k \cos i_k \cos\Omega_k$ | | |
| $Z_k = y_k \sin i_k$ | | |

The above satellite ephemeris, along with system time, satellite clock behaviour data, and transmitter status information, is supplied by means of the GPS navigation message [van Dierendonck, 1978].

Let us consider the jth ground station and the ith satellite. The position vector of the ground station is

$$\underline{R}_J = [X_J, Y_J, Z_J]^T \quad .$$

The position vector and Cartesian coordinates of the ith satellite, at some epoch $t_k(\tau)$ (a function of the conventional GPS time) are

$$\underline{r}_i(t_k(\tau)) = [x_i(t_k), y_i(t_k), z_i(t_k)]^T \quad .$$

The Cartesian coordinates of the geometric range vector between the ith satellite and the jth ground station are

$$\underline{\rho}_{ij}(t_k) = [\xi_{ij}(t_k), \eta_{ij}(t_k), \zeta_{ij}(t_k)]^T \quad .$$

The length of $\underline{\rho}_{ij}$ is denoted by $\rho_{ij}$. From Figure 2.2, the geometric range vector is

$$\underline{\rho}_{ij}(t_k) = \underline{r}_i(t_k) - \underline{R}_j \quad . \tag{2.1}$$

The pseudorange $\tilde{\rho}_{ij}$, which a receiver can measure, is defined as

$$\tilde{\rho}_{ij}(t_k) = \rho_{ij}(t_k) + c(\Delta t_{u_j} - \Delta t_{s_i}) + c\cdot\Delta t_{A_i} = \rho_{ij}(t_k) + \Delta\rho_{RCV} - \Delta\rho_{SAT} + \Delta\rho_{ATM} \tag{2.2}$$

where

$\tilde{\rho}_{ij}$ = pseudorange (metres)

$\rho_{ij}$ = geometric (true) range (metres)

$c$ = speed of light (metres/second)

$\Delta t_{u_j}$ = user clock time bias (seconds)

$\Delta t_{s_i}$ = satellite i clock time bias (seconds)

$c\Delta t_{A_i}$ = atmospheric delays (ionospheric, tropospheric)(metres).

The atmospheric delays $c\Delta t_{A_i}$ are introduced by propagation error due to the atmosphere, specifically the ionospheric and tropospheric delay.

Ionospheric delays are estimated by the user (j) by measuring pseudoranges $\tilde{\rho}_{ij}(t_k)$ at two different frequencies ($L_1$=1575 MHz;$L_2$=1227 MHz).

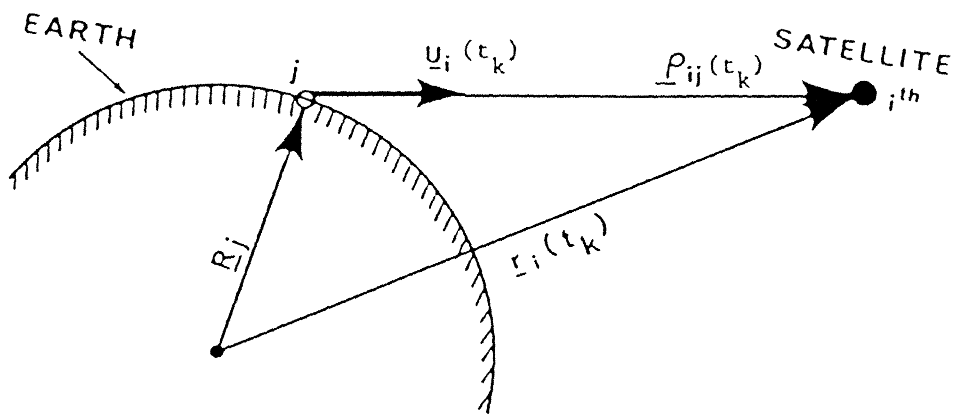
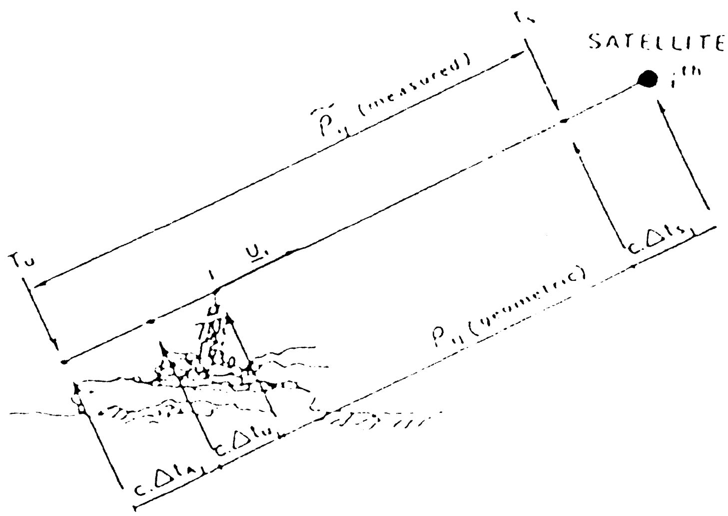



FIGURE 2.2
Navigation Solution Geometry




$c \cdot \Delta t_{A_j}$ = atmospheric delays (metres)

$c \cdot \Delta t_{u_j}$ = user clock time bias (metres)

$c \cdot \Delta t_{S_i}$ = satellite i clock time bias (metres)

$\rho_{ij}$ = geometric range (metres)

$c(T_u - T_s) = \tilde{\rho}_{ij}$ = pseudorange (metres)

FIGURE 2.3

Pseudorange Observation.

This is done because the ionosphere has a delay effect which is approximately inversely proportional to the square of the frequency ($\sim 1/f^2$) [van Dierendonck et al. 1978]. For single channel receivers, ionospheric delays must be estimated from a mathematical model. Coefficients of a polynomial model are provided by means of the navigation satellite message.

Tropospheric delays are frequency independent. Estimation models for the troposphere are based on geometry and altitude. Approximation models for the estimation of ionospheric and tropospheric delays are given in Ward [1981].

The satellite clock time bias $\Delta t_{s_i}$ is again provided by the satellite message, whereas the user clock time bias $\Delta t_{u_j}$ is considered unknown and is solved for through the navigation solution.

The mathematical model $F(\underline{X}, \underline{L}) = 0$ for an observation of pseudorange is in the form

$$F(\underline{X}, \underline{L}) = \rho_{ij} + c(\Delta t_{u_j} - \Delta t_{s_i}) + c\Delta t_{A_i} - \tilde{\rho}_{ij} = 0 \quad , \tag{2.3}$$

where i designates the satellite and j the ground station.

The geometric (true) range at some epoch $t_k$ is

$$\rho_{ij}(t_k) = \sqrt{\{X_j - x_i(t_k)\}^2 + \{Y_j - y_i(t_k)\}^2 + \{Z_j - z_i(t_k)\}^2} \quad . \tag{2.4}$$

Substituting the above in the general mathematical model of the pseudorange:

$$F(\underline{X},\underline{L}) = \sqrt{\{X_j - x_i(t_k)\}^2 + \{Y_j - y_i(t_k)\}^2 + \{Z_j - z_i(t_k)\}^2}$$
$$+ c(\Delta t_{u_j} - \Delta t_{s_i}) + c\Delta t_{A_i} - \tilde{\rho}_{ij} = 0 \quad . \tag{2.5}$$

Expanding the above equation of pseudorange observation in a Taylor series about an initial approximate user's position and clock bias

$$\underline{X}_j^{(o)} = [X_j^{(o)} \; Y_j^{(o)} \; Z_j^{(o)} \; \Delta t_{u_j}^{(o)}]^T$$

and using the measured values of the observation vector

$$\underline{L}^{(o)} = [\tilde{\rho}_{1j}^{(o)}\ \tilde{\rho}_{2j}^{(o)}\ \tilde{\rho}_{3j}^{(o)}\ \tilde{\rho}_{4j}^{(o)}\ \ldots]^T ,$$

we get

$$\{\frac{\partial F}{\partial \underline{X}_j}\Big|_{\underline{X}_j^{(o)}}\}(\underline{X}_j - \underline{X}_j^{(o)}) + (-I)(\tilde{\underline{\rho}}_{ij} - \tilde{\underline{\rho}}_{ij}^{(o)}) + F(\underline{X}_j^{(o)}, \tilde{\underline{\rho}}_{ij}^{(o)}) = 0 \qquad (2.6)$$

In our familiar notation of surveying, the above can be written as

$$A \cdot \underline{\delta X} + B \cdot \underline{V} + \underline{W} = \underline{0} , \qquad (2.7)$$

where

$$A = \{\frac{\partial F}{\partial \underline{X}_j}\Big|_{\underline{X}_j^{(o)}}\} = \text{design matrix}$$

$$\underline{\delta X} = \underline{X}_j - \underline{X}_j^{(o)} = \text{correction vector}$$

$$B = -I = \{\frac{\partial F}{\partial \underline{\rho}_{ij}}\Big|_{\underline{\rho}_{ij}^{(o)}}\} = \{\frac{\partial F}{\partial \underline{L}}\Big|_{\underline{L}^{(o)}}\} = \text{design matrix}$$

$$\underline{V} = \underline{L} - \underline{L}^{(o)} = \text{residual vector};\ \tilde{\rho}_{ij} = \tilde{\rho}_{ij}^{(o)} + v_i$$

$$\underline{W} = F(\underline{X}_j^{(o)}, \tilde{\underline{\rho}}_{ij}^{(o)}) = \text{misclosure vector.}$$

The above equation (2.7) is the linearized equation which relates pseudorange measurements to the desired user navigation information, either $[X_j, Y_j, Z_j]$ or $[\phi_j, \lambda_j, h_j]$, as well as the user clock bias $\Delta t_{u_j}$.

When four satellites are available ($i = 1, 2, 3, 4$), the linearized equations can be written as

$$A \cdot \underline{\delta X} + B \cdot \underline{V} + \underline{W} = \underline{0} , \qquad (2.8)$$

where

$$A = \text{design matrix} = \left\{\frac{\partial F}{\partial \underline{X}_j}\bigg|_{\underline{X}_j^{(o)}}\right\} = \begin{bmatrix} \dfrac{X_j^{(o)}-x_1(t_k)}{\rho_{1j}^{(o)}} & \dfrac{Y_j^{(o)}-y_1(t_k)}{\rho_{1j}^{(o)}} & \dfrac{Z_j^{(o)}-z_1(t_k)}{\rho_{1j}^{(o)}} & c \\ \dfrac{X_j^{(o)}-x_2(t_k)}{\rho_{2j}^{(o)}} & \dfrac{Y_j^{(o)}-y_2(t_k)}{\rho_{2j}^{(o)}} & \dfrac{Z_j^{(o)}-z_2(t_k)}{\rho_{2j}^{(o)}} & c \\ \dfrac{X_j^{(o)}-x_3(t_k)}{\rho_{3j}^{(o)}} & \dfrac{Y_j^{(o)}-y_3(t_k)}{\rho_{3j}^{(o)}} & \dfrac{Z_j^{(o)}-z_3(t_k)}{\rho_{3j}^{(o)}} & c \\ \dfrac{X_j^{(o)}-x_4(t_k)}{\rho_{4j}^{(o)}} & \dfrac{Y_j^{(o)}-y_4(t_k)}{\rho_{4j}^{(o)}} & \dfrac{Z_j^{(o)}-z_4(t_k)}{\rho_{4j}^{(o)}} & c \end{bmatrix} \tag{2.9}$$

$$\underline{\delta X} = \text{correction vector} = \underline{X}_j - \underline{X}_j^{(o)} = \begin{bmatrix} X_j - X_j^{(o)} \\ Y_j - Y_j^{(o)} \\ Z_j - Z_j^{(o)} \\ \Delta t_{u_j} - \Delta t_{u_j}^{(o)} \end{bmatrix} \tag{2.10}$$

$$\underline{V} = \text{residual vector} = \underline{L} - \underline{L}^{(o)} = \begin{bmatrix} \tilde{\rho}_{1j} - \tilde{\rho}_{1j}^{(o)} \\ \tilde{\rho}_{2j} - \tilde{\rho}_{2j}^{(o)} \\ \tilde{\rho}_{3j} - \tilde{\rho}_{3j}^{(o)} \\ \tilde{\rho}_{4j} - \tilde{\rho}_{4j}^{(o)} \end{bmatrix} \tag{2.11}$$

$$B = \text{design matrix} = \{\frac{\partial F}{\partial \underline{L}}\Big|_{\underline{L}}(o)\} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -I \qquad (2.12)$$

$$\underline{W} = \text{misclosure vector} = F(\underline{X}^{(o)}, \underline{L}^{(o)}) = \begin{bmatrix} \rho_{1j}^{(o)} + c\cdot(\Delta t_{u_j}^{(o)} - \Delta t_{s_1}) + c\cdot\Delta t_{A_1} - \tilde{\rho}_{1j}^{(o)} \\ \rho_{2j}^{(o)} + c\cdot(\Delta t_{u_j}^{(o)} - \Delta t_{s_2}) + c\cdot\Delta t_{A_2} - \tilde{\rho}_{2j}^{(o)} \\ \rho_{3j}^{(o)} + c\cdot(\Delta t_{u_j}^{(o)} - \Delta t_{s_3}) + c\cdot\Delta t_{A_3} - \tilde{\rho}_{3j}^{(o)} \\ \rho_{4j}^{(o)} + c\cdot(\Delta t_{u_j}^{(o)} - \Delta t_{s_4}) + c\cdot\Delta t_{A_4} - \tilde{\rho}_{4j}^{(o)} \end{bmatrix} \qquad (2.13)$$

where $\rho_{ij}^{(o)}(t_k) = \sqrt{(X_j - x_i(t_k^2)) + (Y_j - y_i(t_k^2)) + (Z_j - z_i(t_k^2))}$ . (2.14)

The quantities to be computed ($\delta X_j$, $\delta Y_j$, $\delta Z_j$, $\delta\Delta t_{u_j}$) are corrections that the user will make to his current estimate of position ($X_j^{(o)}$, $Y_j^{(o)}$, $Z_j^{(o)}$) and his clock bias $\Delta t_{u_j}^{(o)}$.

It should be noted that the coefficients in the first three columns of the design matrix A are the negative direction cosines of the line of sight from the user (j) to the satellite (i). For all four equations, the coefficient of $\Delta t_{u_j}$ is the speed of light c.

Let $\underline{U}_i(t_k) = (u_i(t_k), v_i(t_k), w_i(t_k))^T$ be the unit vector from the user position (j) to the ith satellite. $u_i$, $v_i$, and $w_i$ are the x, y, and z components of this unit vector $\underline{U}_i$, as shown in Figures 2.4 and 2.5. It is known from analytical geometry that the components of the unit vector $\underline{U}_i$

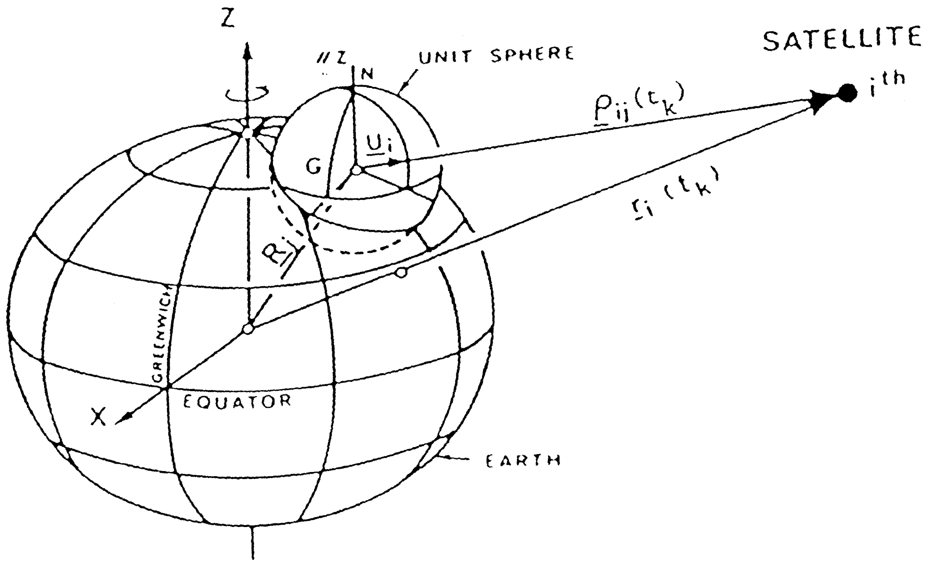



FIGURE 2, 4
Satellite Geometry

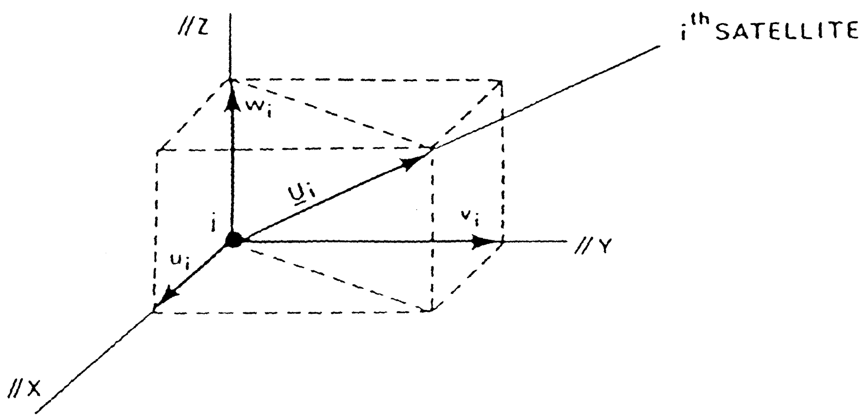



FIGURE 2.5
Unit Vectors $\underline{U}_i$.

are:

$$\left(\frac{x_i(t_k) - X_j}{\rho_{ij}}, \frac{y_i(t_k) - Y_j}{\rho_{ij}}, \frac{z_i(t_k) - Z_j}{\rho_{ij}}\right)^T = (u_i(t_k), v_i(t_k), w_i(t_k))^T .$$

Therefore the design matrix

$$A = \left\{\frac{\partial F}{\partial \underline{X}_j}\bigg|_{\underline{X}_j^{(o)}}\right\}$$

can be expressed in an equivalent form with direction cosines as

$$\begin{bmatrix} -u_1 & -v_1 & -w_1 & c \\ -u_2 & -v_2 & -w_2 & c \\ -u_3 & -v_3 & -w_3 & c \\ -u_4 & -v_4 & -w_4 & c \end{bmatrix} = A = \left\{\frac{\partial F}{\partial \underline{X}_j}\bigg|_{\underline{X}_j^{(o)}}\right\} . \qquad (2.15)$$

Assuming that the weight matrix of the observations is known, an estimate of the correction vector $\underline{\delta X} = \underline{X}_j - \underline{X}_j^{(o)}$, based on the least squares principle, is given by:

$$\underline{\delta X} = \underline{X}_j - \underline{X}_j^{(o)} = (A^T PA)^{-1} A^T P\underline{W} . \qquad (2.16)$$

The final solution vector is

$$\underline{X}_j = \underline{X}_j^{(o)} + \underline{\delta X} = \underline{X}_j^{(o)} + (A^T PA)^{-1} A^T P\underline{W} . \qquad (2.17)$$

It is obvious that the above process is iterative and this final vector $\underline{X}_j$ can be used as a new approximation for another iteration. The number of iterations depends on an error criterion. Usually three iterations are adequate.

When a solution in latitude ($\phi$), longitude ($\lambda$), and height (h) of the user position is required, either a simple conversion of the ($X_j$, $Y_j$, $Z_j$, $\Delta t_{u_j}$) is applied into a ($\phi_j$, $\lambda_j$, $h_j$, $\Delta t_{u_j}$) solution after the above procedure is performed, or the design matrix should have rows of four elements such that

$$A_\ell = [\frac{\partial F}{\partial \phi} \;\; \frac{\partial F}{\partial \lambda} \;\; \frac{\partial F}{\partial h} \;\; \frac{\partial F}{\partial \Delta t_u}] \qquad (\ell = 1, 2, 3, \ldots) \quad . \tag{2.18}$$

For marine navigation, we can consider our height as known (usually it is taken as equal to 10 metres), and determine only two coordinates of position and the user clock bias. In such a case, the sought receiver solution would be

$$X_j = \begin{bmatrix} \phi_j \\ \lambda_j \\ \Delta t_{u_j} \end{bmatrix} \quad . \tag{2.19}$$

and the design matrix $A = \{\frac{\partial F}{\partial X_j}\Big|_{X_j^{(o)}}\}$

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial \phi} & \frac{\partial F_1}{\partial \lambda} & \frac{\partial F_1}{\partial \Delta t_u} \\ \frac{\partial F_2}{\partial \phi} & \frac{\partial F_2}{\partial \lambda} & \frac{\partial F_2}{\partial \Delta t_u} \\ \frac{\partial F_3}{\partial \phi} & \frac{\partial F_3}{\partial \lambda} & \frac{\partial F_3}{\partial \Delta t_u} \\ \frac{\partial F_4}{\partial \phi} & \frac{\partial F_4}{\partial \lambda} & \frac{\partial F_4}{\partial \Delta t_u} \end{bmatrix} \quad . \tag{2.20}$$

The partial derivatives of the general mathematical model F with respect to $\phi$, $\lambda$, and $\Delta t_u$ are

$$\frac{\partial F}{\partial \phi} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial \phi} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial \phi} + \frac{\partial F}{\partial Z} \cdot \frac{\partial Z}{\partial \phi}$$

$$\frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial \phi} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial \lambda} + \frac{\partial F}{\partial Z} \cdot \frac{\partial Z}{\partial \lambda}$$

$$\frac{\partial F}{\partial \Delta t_u} = c$$

or in a matrix notation

$$\begin{bmatrix} \frac{\partial F}{\partial \phi} \\ \\ \frac{\partial F}{\partial \lambda} \\ \\ \frac{\partial F}{\partial \Delta t_u} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \phi} & \frac{\partial Y}{\partial \phi} & \frac{\partial Z}{\partial \phi} & 0 \\ \\ \frac{\partial X}{\partial \lambda} & \frac{\partial Y}{\partial \lambda} & \frac{\partial Z}{\partial \lambda} & 0 \\ \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial X} \\ \\ \frac{\partial F}{\partial Y} \\ \\ \frac{\partial F}{\partial Z} \\ \\ c \end{bmatrix} . \tag{2.21}$$

The partial derivatives involved are given in McCaskill et al. [1976] as

$$\frac{\partial X}{\partial \phi} = [\frac{a}{(1-e^2 \sin^2 \phi)^{1/2}} + h] \sin\phi \cdot \cos\lambda + \frac{a \cdot e^2 \cdot \sin\phi \cdot \cos^2\phi \, \cos\lambda}{(1-e^2 \sin^2\phi)^{3/2}} \tag{2.22}$$

$$\frac{\partial Y}{\partial \phi} = -[\frac{a}{(1-e^2 \sin^2 \phi)^{1/2}} + h] \sin\phi \cdot \sin\lambda + \frac{a \cdot e^2 \cdot \sin\phi \cdot \cos^2\phi \cdot \sin\lambda}{(1-e^2 \sin^2\phi)^{3/2}} \tag{2.23}$$

$$\frac{\partial Z}{\partial \phi} = [\frac{a}{(1-e^2 \sin^2 \phi)^{1/2}} + h - \frac{a \cdot e^2}{(1-e^2 \sin^2\phi)^{1/2}}] \cos\phi + \frac{a \cdot e^2 (1-e^2) \sin^2\phi \cdot \cos\phi}{(1-e^2 \sin^2\phi)^{3/2}} \tag{2.24}$$

$$\frac{\partial X}{\partial \lambda} = [\frac{a}{(1-e^2 \sin^2 \phi)^{1/2}} + h] \; \cos\phi \cdot \sin\lambda \tag{2.25}$$

$$\frac{\partial Y}{\partial \lambda} = [\frac{a}{(1-e^2 \sin^2 \phi)^{1/2}} + h] \; \cos\phi \cdot \cos\lambda \tag{2.26}$$

$$\frac{\partial Z}{\partial \lambda} = 0 \quad . \tag{2.27}$$

# APPENDIX B

## Accuracy Measures For GPS Performance

Extract from:
Mertikas et al. (1985)

## 3. ACCURACY MEASURES FOR GPS PERFORMANCE

The accuracy measures for the GPS performance are much simpler than the conventional ones associated with the error ellipse and the ellipsoid of constant probability. The use of those conventional measures is complicated by the orientation of the axes and the propagation of the elliptical errors. Instead, a circular form is employed which is easier to use and understand.

This section is devoted to the establishment of a meaningful accuracy statement for the GPS performance for a uniform interpretation. User accuracy is dependent upon various factors; however there are two primary ones: The range error, and geometry. The former is expressed by the User Equivalent Range Error (UERE), and the latter by the Geometric Dilution of Precision (GDOP).

UERE is based on the assumption that there is no correlation between satellite measurements. It represents the combined accuracy parameter of satellite measurements and reflects the total error contribution of the GPS system. UERE involves "system" errors, such as uncertainties of the ephemeris data, propagation errors, clock errors, etc. In other words, each pseudorange observation toward a specific satellite is associated with an observed range error, known as UERE.

The use of the GDOP value was originally developed in LORAN navigation systems [Swanson, 1978]. The GDOP value is a measure of the satellite geometry. It is a quantity which is used extensively in determining the information content due to satellite geometry and results in a measure of the overall geometrical strength to the solution. It provides a method of quantitatively determining whether a particular satellite geometry is good or bad.

It was found before that an estimate of the correction vector is given by equation (2.16) as

$$\underline{\delta \chi} \doteq \chi_j - \chi_j^{(o)} = (A^T PA)^{-1} A^T P\underline{W} \quad .$$

The cofactor matrix [Vanicek and Krakiwsky, 1982] of the estimate $\underline{\delta \chi}$ ($\delta\phi$, $\delta\lambda$, $\delta h$, $\delta\Delta t_u$) is given by

$$Q_{\underline{\delta \chi}} = (A^T \; PA)^{-1} = \begin{bmatrix} q_{\phi}^2 & q_{\phi\lambda} & q_{\phi h} & q_{\phi\Delta t_u} \\ q_{\phi\lambda} & q_{\lambda}^2 & q_{\lambda h} & q_{\lambda\Delta t_u} \\ q_{h\phi} & q_{h\lambda} & q_{h}^2 & q_{h\Delta t_u} \\ q_{\Delta t_u \phi} & q_{\Delta t_u \lambda} & q_{\Delta t_u h} & q_{\Delta t_u}^2 \end{bmatrix} \quad . \tag{3.1}$$

Geometric Dilution of Precision is defined as the square root of the trace of the above cofactor matrix after setting the weight matrix P equal to the identity matrix, that is

$$GDOP = \sqrt{\text{trace } (A^T A)^{-1}} = \sqrt{q_{\phi}^2 + q_{\lambda}^2 + q_{h}^2 + q_{\Delta t_u}^2} \quad . \tag{3.2}$$

Other quantities of interest, along with the Geometric Dilution of Precision, are the horizontal, the vertical, the positional, the time and the horizontal time dilution of precision defined as follows:

$$\begin{aligned} HDOP &= \sqrt{q_{\phi}^2 + q_{\lambda}^2} \\ VDOP &= \sqrt{q_{h}^2} = q_h \\ PDOP &= \sqrt{q_{\phi}^2 + q_{\lambda}^2 + q_{h}^2} \\ TDOP &= q_{\Delta t_u} \quad . \\ HTDOP &= \sqrt{q_{\phi}^2 + q_{\lambda}^2 + q_{\Delta t_u}^2} \end{aligned} \tag{3.3}$$

For a complete three-dimensional position fix ($\phi$, $\lambda$, $h$, $\Delta t_u$) the covariance matrix of the estimate $\underline{\delta\chi}$ is

$$C_{\underline{\delta\chi}} = \sigma_o^2 Q_{\underline{\delta\chi}} = \begin{bmatrix} \sigma_\phi^2 & \sigma_{\phi\lambda} & \sigma_{\phi h} & \sigma_{\phi\Delta t_u} \\ \sigma_{\lambda\phi} & \sigma_\lambda^2 & \sigma_{\lambda h} & \sigma_{\lambda\Delta t_u} \\ \sigma_{h\phi} & \sigma_{h\lambda} & \sigma_h^2 & \sigma_{h\Delta t_u} \\ \sigma_{\Delta t_u\phi} & \sigma_{\Delta t_u\lambda} & \sigma_{\Delta t_u h} & \sigma_{\Delta t_u}^2 \end{bmatrix} , \qquad (3.4)$$

where $\sigma_o^2$ is the variance factor [Vanicek and Krakiwsky, 1982].

By setting the weight matrix P equal to the identity matrix, equation (3.4) gives

$$C_{\underline{\delta\chi}} = \sigma_{range}^2 Q_{\underline{\delta\chi}} = (UERE)\ Q_{\underline{\delta\chi}} \quad . \qquad (3.5)$$

From the above relationship an approximate measure in the total user error would be:

$$\text{user error} = \sqrt{\sigma_\phi^2 + \sigma_\lambda^2 + \sigma_h^2 + \sigma_{\Delta t_u}^2} = (UERE)\sqrt{q_\phi^2 + q_\lambda^2 + q_h^2 + q_{\Delta t_u}^2} \ . \qquad (3.6)$$

The product of the DOP factors by an estimate in the range measurements ($\sigma_{range}$ = UERE) results in a user error such that

$$\text{user error} = (UERE)\ (GDOP) \quad . \qquad (3.7)$$

The same is true for the other DOP factors. For example, a PDOP value of 2.5 and a UERE of $\pm$ 4 m ($1\sigma$) would result in a user position error ($1\sigma$) of

$$(PDOP) \times (UERE) = 2.5(\pm\ 4\ m) = \pm\ 10 \text{ metres}.$$

It is mentioned in Ward [1981] that a PDOP value of 3 or less is expected in the full 18-satellite constellation.

Geometric dilution of precision values can be described as a measure of the navigator's position uncertainty per unit of measurement noise. It has been conceded that GDOP values are statistically distributed in a non-Gaussian fashion [Jorgensen, 1980].

So far, GDOP has been derived in an analytical way. Another way for the determination of GDOP is based on the computation of the volume of a special tetrahedron formed by the satellites and the user's location.

Let $\underline{\rho}_{ij}((t_k)$ be the geometric range vector, $\underline{R}_j$ the position vector of the jth user, and $\underline{r}_i(t_k)$ the position vector of the ith satellite, as shown in Figure 3.1.

The magnitude of the cross-product of the pair $\underline{R}_j$ and $\underline{\rho}_{ij}$ is defined as follows:

$$|\underline{R}_j \times \rho_i| = |\underline{R}_j||\underline{\rho}_{ij}| \sin(90^\circ + E) \quad , \tag{3.8}$$

whereas the dot product for the same vectors is

$$\underline{R}_j \cdot \underline{\rho}_{ij} = |\underline{R}_j||\rho_{ij}| \cos(90^\circ + E) \quad . \tag{3.9}$$

Dividing (3.9) by (3.8), we obtain:

$$\frac{\cos(90^\circ + E)}{\sin(90^\circ + E)} = \frac{-\sin E}{\cos E} = \frac{-\dfrac{\underline{R}_j \cdot \underline{\rho}_{ij}}{|\underline{R}_j|\ |\underline{\rho}_{ij}|}}{\dfrac{|\underline{R}_j \times \underline{\rho}_{ij}|}{|\underline{R}_j|\ |\underline{\rho}_{ij}|}} \quad . \tag{3.10}$$

Therefore the elevation angle E of the ith satellite can be obtained by

$$\tan E = \frac{-\dfrac{\underline{R}_j \cdot \underline{\rho}_{ij}}{|\underline{R}_j|\ \ |\underline{\rho}_{ij}|}}{\dfrac{|\underline{R}_j \times \underline{\rho}_{ij}|}{|\underline{R}_j|\ \ |\underline{\rho}_{ij}|}} = -\frac{\underline{R}_j \cdot \underline{\rho}_{ij}}{|\underline{R}_j \times \underline{\rho}_{ij}|} \tag{3.11}$$

An allowable elevation angle for the determination of whether a satellite is considered visible is

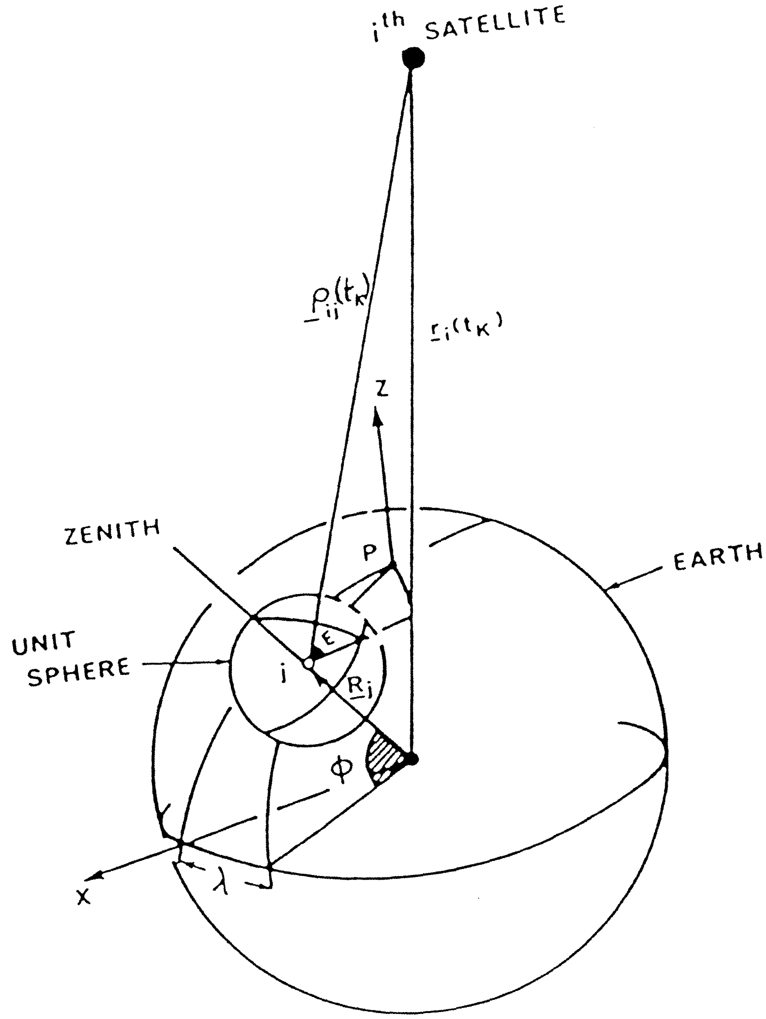



FIGURE 3.1
Elevation Angle E of the ith Satellite.

$$E \geq 5^{\circ} \quad . \tag{3.12}$$

Therefore, candidate satellites to be considered for visibility are those whose elevation angle E is greater or equal to $5^{\circ}$. Any satellite with an elevation angle of less than $5^{\circ}$ is masked out by terrain, antenna limitations, foliage, obstructions, etc. Based on the criterion of (3.12), one can determine the number of visible satellites for a particular user (j) and time ($t_k$).

Let $\underline{U}_i$ be the unit vector from the user (j) to the ith visible satellite, as shown in Figure 3.2. When the full 18 satellites are in operation, four to seven satellites will be visible, on a continuous basis, at any site on the globe [USDOD, 1982]. All unit vectors $\underline{U}_i$ are centred at the user's location (j) and enclosed within a unit sphere. If we calculate all the combinations of unit vectors $\underline{U}_i$ of four satellites, we end up with a set of four unit vectors each time. It can be seen from Figure (3.3) that a special tetrahedron (e.g., 1-2-3-4) is formed by those four unit vectors.

Variability of satellite geometry depends on the orientation of the four satellite positions available. This is, in turn, a function of the user's location (j) and time ($t_k$) because of satellite motion and earth rotation. It has been shown that the GDOP value is inversely proportional to the volume of this special tetrahedron (1-2-3-4) [Bogen, 1974]. Hence, the largest volume yields the smallest value of GDOP and vise versa.

Determination of the maximum volume of a tetrahedron among all other volumes formed by all the other combinations of four satellites also implies the determination of those satellites with the best navigation performance. The best navigation performance relies on the geometry of the four satellites and the smallest value of GDOP.

The volume (V) of the tetrahedron (1-2-3-4) can be computed using the scalar triple product

$$V = 1/6\ \underline{C}(\underline{A} \times \underline{B}) \quad . \tag{3.13}$$

The previous account takes into consideration geometrical aspects related to satellite geometry and the user's location, which has as a final goal the selection of satellites with the best navigation performance. It can seen that the geometrical interpretation is easier to understand and visualize.

The minimum number of observations constitutes the necessary and sufficient elements for a unique set of estimates for the solution. Any additional observations, which are said to be redundant with respect to the

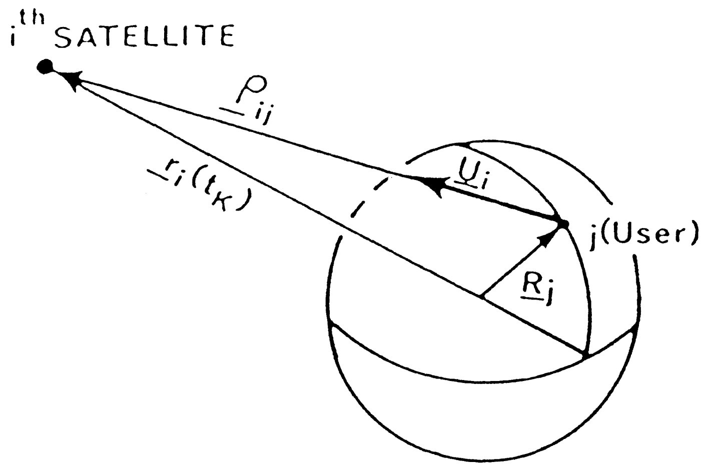



FIGURE 3.2

Unit Vector $\underline{U}_i$ to ith Satellite.

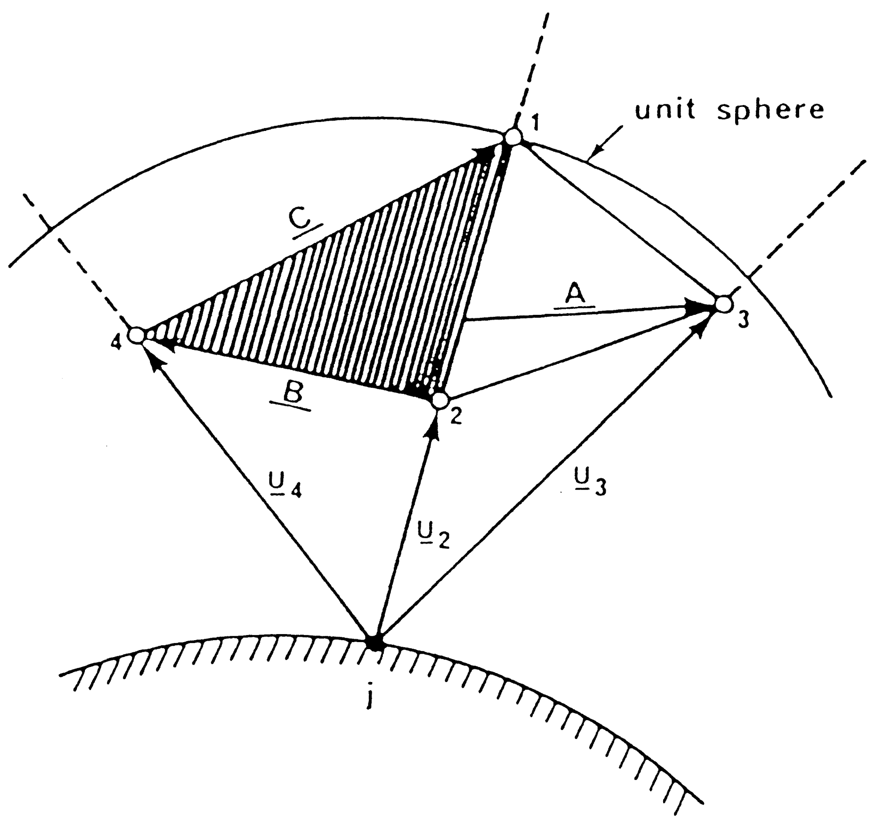



FIGURE 3.3

Tetrahedron Formed by Four Unit Vectors $\underline{U}_i$.

model (four-parameter, three-parameter, two-parameter solution), should always be taken into consideration for a more precise and reliable solution [Mikhail, 1976; Vanicek and Krakiwsky, 1982]. In this case, the corresponding GDOP value will not only incorporate four satellites but all those used for the solution (multi-dimensional GDOP).