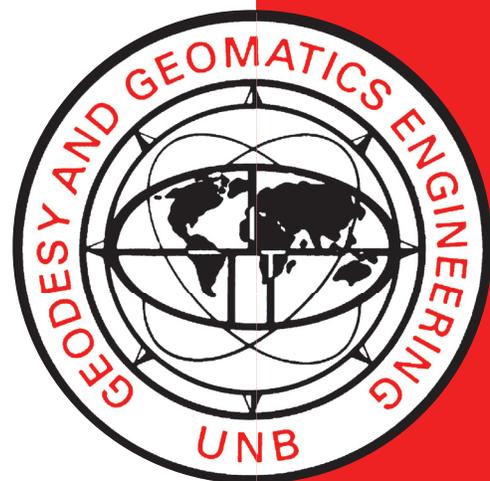


RIGOROUS DENSIFICATION OF HORIZONTAL GEODETIC NETWORKS

F. N. LUGOE

November 1985



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PREFACE

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RIGOROUS DENSIFICATION OF HORIZONTAL GEODETIC NETWORKS

Faraha Ngeregere Lugoe

Department of Surveying Engineering
University of New Brunswick
P.O. Box 4400
Fredericton, N.B.
Canada
E3B 5A3

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TABLE OF CONTENTS

	PAGE
ABSTRACT	i
TABLE OF CONTENTS	iii
LIST OF FIGURES	vi
LIST OF TABLES	viii
ACKNOWLEDGEMENTS	ix
1.0 INTRODUCTION	1
1.1 Statement of the Problem	1
1.2 Rigorous Densification in Perspective	2
1.3 Selecting the Approach to Rigorous Densification	5
1.3.1 Direct and indirect approaches	5
1.3.2 The economics of 2D densification	8
1.4 Effect of Inconsistent Observations in 2D Networks	9
1.4.1 Random errors	9
1.4.2 Systematic errors	10
1.4.3 Error analysis	13
1.4.4 Strain analysis	14
1.5 Transit, GPS, Inertial and Photogrammetric Densification of 2D Networks	15
1.6 Special Requirements and Rigorous Densification	19
1.7 Goals and Contributions	21
2.0 RIGOROUS NETWORK DENSIFICATION BY A SIMULTANEOUS ADJUSTMENT OF TWO NETWORKS	25
2.1 The Scope of the Problem	25
2.2 Mathematical Models	26
2.3 Derivation of Normal Equations	29
2.4 The Least-Squares Solution	32
2.4.1 The combined solution	32
2.4.2 The densification network solution	32
2.4.3 The partitioned solution for, S_2	33
2.4.4 The existing non-junction points \bar{s}	34
2.5 The Covariance Matrices of Correction Vectors	35
2.5.1 The covariance matrix, $C_{\delta_2}^{\wedge}$	35
2.5.2 The covariance matrices, $C_{\delta_j}^{\wedge}$ and $C_{\delta_n}^{\wedge}$	39
2.5.3 The covariance matrix, $C_{\delta_e}^{\wedge}$	40
2.6 The Covariance Matrices of Adjusted Positions	41
3.0 RIGOROUS DENSIFICATION BY NETWORK ADJUSTMENT WITH WEIGHTED POSITION CONSTRAINTS	44
3.1 The Scope of the Problem	44
3.2 Mathematical Models	45
3.3 Derivation of Normal Equations	46
3.4 The Least-Squares Solution	49

TABLE OF CONTENTS (Cont's)	PAGE
6.6 Simulation Studies and Results	114
6.6.1 Sensitivity analysis 1	115
6.6.2 Sensitivity analysis 2	121
6.6.3 Sensitivity analysis 3	126
7.0 POST-ADJUSTMENT CHANGES IN THE RIGOROUS SOLUTION	130
7.1 The Px-Matrix	130
7.2 The Initial Coordinates $x_j^{(1)}$	135
7.5 The Observation Weights	135
7.4 The Observations	137
7.5 The Covariance Matrix, C_p^{\wedge}	138
8.0 CONCLUSIONS AND RECOMMENDATIONS	140
8.1 Summary and Contributions	141
8.2 Concluding Remarks with Recommendations	146
9.0 REFERENCES	151
APPENDIX I: Minimal Constraint Adjustment of the Existing Network	160
APPENDIX II: Simplified Inversion of $(N+\Delta N)^{-1}$ for a Given N^{-1}	165
APPENDIX III: Residuals in a Rigorous Densification	167
APPENDIX IV: Solution of a Triple Partitioned System of Normal Equations	173
APPENDIX V: Computer Software	179
V.1: Program CTEST	179
V.2: Program STRAIN1	190
V.3: Program PASTEL	199
V.4: Program NETPLOT1	203
V.5: Program RANDOM	209
V.6: Program SMTRA	211
V.7: Program SIMFRA	218

LIST OF FIGURES

NUMBER		PAGE
1.1	A densified horizontal geodetic network.	5
1.2	Precision of distance determinations achieved or achievable using different techniques.	16
4.1	The existing network.	65
4.2	The densification network.	64
4.3	The densified network.	65
4.4	Absolute error ellipses in the existing network at 95% confidence level.	67
4.5	Absolute error ellipses in the rigorous densification at 95% confidence level.	68
4.6	Absolute error ellipses in overconstrained densification at 95% confidence level.	70
4.7	Distortions in the overconstrained densification.	72
4.8	Absolute error ellipses in the fixed-point adjustment at 95% confidence level.	74
4.9	Distortions in fixed-point densification.	76
4.10	Distortions in fixed-point densification with transformation.	79
5.1	Intersection of point C.	95
6.1	Change in the axes of the strain ellipse at stations 66 when the error in an observation at station 32 is changed.	117
6.2	Strain patterns of a $-4.0\sigma_{\rho}$ error in the angle (25-32-35).	118
6.3	Strain patterns of a $1.0\sigma_{\rho}$ error in the angle (25-32-35).	119
6.4	Strain patterns of a $+5.0\sigma_{\rho}$ error in the angle (25-32-35).	120
6.5	Strain induced in a rigorous densification by the existing network (junction points only).	123

LIST OF FIGURES (Cont'd)

6.6	Strain induced in rigorous densification by the existing network (all points, random errors only).	124
6.7	Strain induced in rigorous densification by the existing network (all points, random + biases).	125
6.8	Strain induced in rigorous densification by $2.5\sigma_2$ biases in 7 observations (junction points undistorted).	127
6.9	Strain induced in rigorous densification by $2.5\sigma_2$ biases in 7 observations (junction points distorted).	128
III-1	The Geometry of the least-squares residuals.	169

LIST OF TABLES

NUMBER		PAGE
1.1	Vectors and matrices involved in various rigorous densification schemes.	7
4.1	Position differences between the over-constrained and the rigorous weighted position constraint solution positions (in cms).	71
4.2	Position differences between the fixed-point and the rigorous weighted position constraint adjustment solutions (in cms).	75
4.5	Position differences between the fixed-point solution with transformation and the rigorous weighted position constraint adjustment solution (in cms).	78
6.1	Variation in the major semi-axis of strain ellipses of the existing network when the error in one observation at station 32 is changed. . . .	116

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1. INTRODUCTION

1.1 Statement of the Problem

Network densification has, until recently, been the only way of making the positions of surveyed points both technically and economically accessible to the users. It is a prerequisite for establishing an integrated survey system [Brown, 1976; Blachut et al., 1979]. The requirements imposed on points of the integrated survey system by the wide range of users include various position information and point locations, monumentation and spacing. Accuracy requirements include the stringent 5 cm and 1 cm (with 400 m spacing or less) for 1:500 large scale mapping and relocation surveys in urban areas, respectively, and the much lower 5-10 m requirement for medium scale mapping at 1:50000 scale [Lebedev, 1974; Blachut et al., 1979]. Network densification provides these requirements in several stages. The major concern is that at each stage in the hierarchy, the densification network can be improperly defined within the existing network. The prime purpose of this research is therefore to study the various techniques of adjusting 2D densification networks rigorously in the coordinate system of the existing network. However, since several techniques exist which lead to rigorous solutions this research will examine the techniques and the rigour of the solution in-context of practicality and economy. The work reported here shall embrace three

main areas: (a) Analysis of rigorous densification schemes including post-adjustment correction considerations, (b) statistical testing of densification networks as solitary networks and in conjunction with the existing networks and (c) possibility of strain analysis applications to quality control of densification networks.

1.2 Rigorous Densification in Perspective

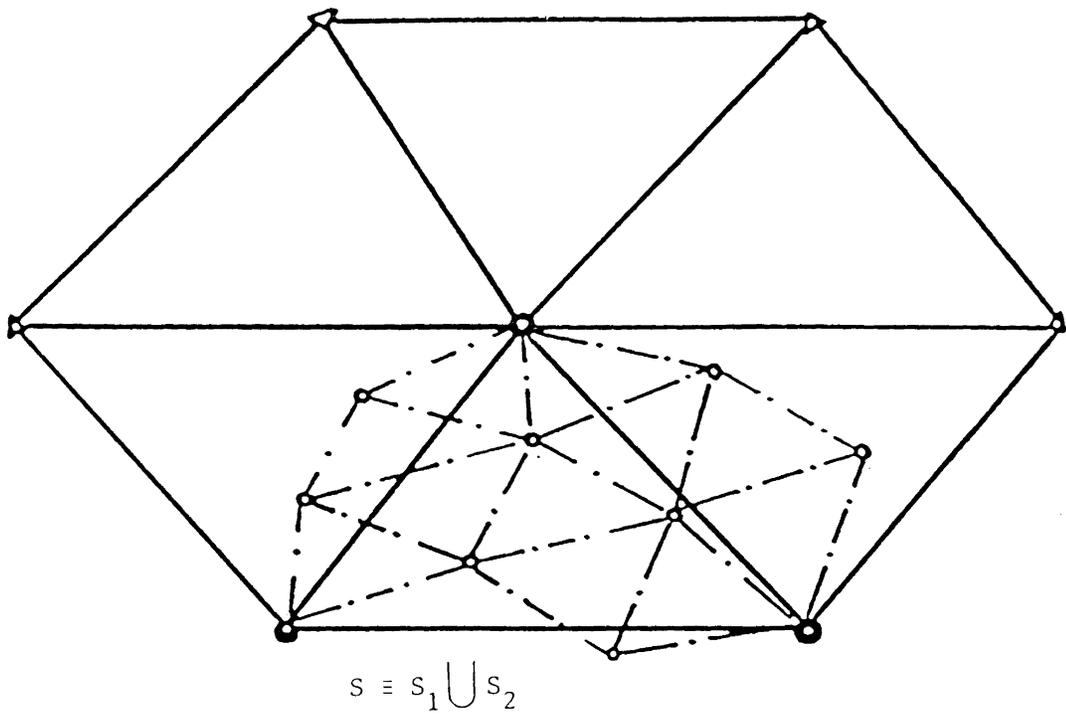
Densification in surveying and geodesy is the addition to the quantity of network points and hence to their density per unit area by designing, observing and adjusting a densification network, $S_2 \equiv \{x_j, x_n\}$ in the coordinate system of the existing network $S_1 \equiv \{x_e, x_j\}$. The result of densification is a densified network $S \equiv \{x_e, x_j, x_n\}$ consisting of the existing non-junction points x_e , the junction points x_j and the new points x_n . In the sequel the subscripts e, j and n shall be used to refer to the existing non-junction, the junction and the new points, respectively.

The positions of the points are estimated from the observation vectors ℓ_1 and ℓ_2 procured in the networks S_1 and S_2 using the method of least-squares. A subnetwork of junction points $S_3 \equiv \{x_j\}$ can be estimated from both the ℓ_1 and ℓ_2 observations. It establishes a link between S_1 and S_2 which would otherwise be disjointed, i.e.,

$$S_3 \equiv S_1 \cap S_2 \quad 1.1$$

An example of the geometrical connectivity within the densified network is shown in Figure 1.1.

A number of ways exists through which the positions of the



- \triangle Existing points (non-junction)
- \odot Junction points
- \circ New points

Figure 1.1: A Densified Horizontal Geodetic Network.

densified network can be modelled and estimated to provide a minimum norm least-squares solution [Bomford, 1971; Mikhail, 1976; Vanicek and Krakiwsky, 1982]. Rigorous densification examines one of the ways. The adjective rigorous is defined in Funk and Wagnalls New Standard Dictionary of the English Language [Funk and Wagnalls, 1963] as "logically accurate; exact; strict". A densification shall be regarded rigorous if the positions of the points in S_2 are as accurately determined from any conceivable mathematical models as they are when S_1 and S_2 are adjusted together using minimal constraints while incorporating all a priori position information available. The densification shall be known as non-rigorous if the position and error estimates of the points of S_2 otherwise adjusted are different from their estimates obtained from a combined adjustment of S_1 and S_2 .

Depending on the mathematical models, rigorous solutions $(\hat{x}_j, \hat{c}_{x_i})$ and $(\hat{x}_n, \hat{c}_{x_n})$ can be obtained from S_2 using the ℓ_2 observations alone [Papo, 1973; Blaha, 1974; Mikhail, 1976]. The logic and exactness in this case, should be sought in the mathematical, statistical and geometrical formulations that lead to:

- a) the propagation of the effect of the existing network into the densification network,
- b) minimization of the effect of the random errors of the observables on the estimated positions, and
- c) an assessment of the densification results, and testing of their statistical significance.

Although the approach focusses on S_2 , it is a reversible one.

Mathematically speaking S_1 can be treated as a densification of S_2 . The positions \hat{x}_j and \hat{x}_e can be estimated within the coordinate system of S_2 . Practically, such a process is necessary if the existing points x_e are to be updated after the rigorous densification.

The general functional relationship between the observables, with weight matrices P_1 and P_2 , and the point positions when the combined adjustment of S_1 and S_2 is contemplated is:

$$F(x_e, x_j, x_n, \ell_1, \ell_2) = 0 \quad : P_1, P_2 \quad 1.2$$

The same relationship is established when merging $S_1 \equiv \{x_e, x_j\}$ with $S_2 \equiv \{x_j, x_n\}$ or extending $S_1 \equiv \{x_e, x_j\}$ to $S \equiv \{x_e, x_j, x_n\}$. Similar to the mathematical formulation (1.2), network extension and merger do not include the point density requirement. Densification, extension and merger of networks are mathematically equivalent operations [Vanicek and Krakiwsky, 1982]. Therefore, the rigour in merging or extending two networks must also be defined within the context of a simultaneous adjustment of S_1 and S_2 .

1.5 Selecting the Approach to Rigorous Densification

1.5.1 Direct and indirect approaches

The selection of the combined (simultaneous) adjustment as the logical technique against which the rigour of the densification solution is to be judged is supported by basic principles of least-squares adjustment of overdetermined systems. We seek to estimate the positions \hat{x}_j and \hat{x}_n by minimizing the quadratic sum of the residuals in a selected, for the adjustment, coordinate system using all available observations. Even in the absence of systematic

errors, the solution $(\hat{x}_j, C_{x_j}^-)$ from the independent networks S_1 and S_2 will be different in each case and different from that obtained from the combined adjustment [Mikhail, 1976; Haymov, 1980]. This is due to difference in redundancy [Hamilton, 1964] and geometry of the junction subnetwork in S_1 and S_2 .

The independent adjustment of the densification network, S_2 , can be improved to incorporate the effect of the l_1 observations on the junction points and subsequent propagation into the new points by introducing an auxiliary mathematical model to be used together with the main mathematical model. The analytical expressions for the least-squares solution of densification networks using the weighted position constraint adjustment or the P_x -adjustment were first derived by Papo [1975]. The expressions lead to the same solution for the junction and new points as would the combined adjustment. Empirical results from comparing the two approaches given in Nickerson and Knight [1983] show that the equivalence of the two methods depends on the P_x -matrix. The weight matrix, P_x , must be the inverse of the covariance matrix of the junction points obtained by adjusting S_1 independently.

Rigorous densification can also be achieved indirectly, by correcting a non-rigorous densification solution. The method requires the computation of a correction $\nabla \hat{x}'$ that can be added to the non-rigorous estimate \hat{x}' to give the rigorous solution \hat{x} . Choosing between this approach and the direct methods, i.e., combined and weighted position constraints adjustments, will depend on two factors. First, the a priori vectors and matrices necessary to make the method feasible must have been preserved (see Table 1.1). These include

Table 1.1: Vectors and Matrices Involved in Various Rigorous
Densification Schemes.

A - Minimal Constraint Adjustment

B - Over-Constrained Adjustment

Required Vectors and Matrices	Method of Densification			
	Combined Adjustment	Px Adjustment	Correction to	
			A	B
q_1	X	-	-	-
q_2	X	X	X	X
P_1	X	-	-	-
P_2	X	X	X	X
\hat{x}_j^1	-	X	-	-
$P_x = C_{x_j}^{-1}$	-	X	X	-
\hat{x}'	-	-	X	X
$\hat{C}_{x'}$	-	-	X	X
ΔP_x	-	-	-	X
$x_j^{(0)}$	X	-	-	-
$x_n^{(0)}$	X	X	-	-

the solution \hat{x}_j^1 of the junction points obtained from the adjustment of S_1 , its covariance matrix $C_{\hat{x}_j^1} = P_x^{-1}$, the difference ΔP_x in weight matrices used in rigorous P_x and non-rigorous P_x' adjustments, the non-rigorous solution \hat{x}' and the covariance matrix, $C_{\hat{x}'}$. Second, the method must prove economical over the alternatives.

1.3.2 The economics of 2D densification

The cost of adjusting a network is the sum of resource investment in digitizing the observations whenever necessary and the cost of running the software on the computer when it is available. Additional costs of software development and even development of mathematical models must be considered in some instances. The cost of digitization and running the software is determined by the dimensions of both observation and parameter vectors. Selection of an appropriate densification scheme on the basis of cost criterion is heavily biased against the combined adjustment of S_1 and S_2 .

The choice between the P_x -adjustment and the correction of non-rigorous densification solutions depends largely on whether or not a non-rigorous densification has been completed, and whether or not the non-rigorous solution $(\hat{x}', C_{\hat{x}'})$ has been preserved. Under these circumstances, a comparison of the least-squares expressions is made to determine the total number of operations and storage requirements necessary to achieve a solution. In the absence of a non-rigorous solution and with the availability of the solution $(\hat{x}_j^1, C_{\hat{x}_j^1})$, the weighted position constraint adjustment method is the most economical one. For example, in an adjustment of 1.3 million

observations and 40,887 stations of the Land Registration and Information Service (LRIS) Maritime network with 267 weighted constraint stations [Nickerson, 1981], only 5% of the 5800 Canadian primary network points [McLellan, 1978] were used. This reduced the dimensions of the normal equations matrix by 12.64% with significant computer-time savings.

The drawbacks of the Px-adjustment compared with the combined (simultaneous) adjustment lie in the necessity to have the solution $(\hat{x}_1, \hat{C}_{x_1}^{-1})$ and S_1 stored in a retrievable form, in order that the complete Px-matrix can be extracted. This factor cannot be over-emphasized. There cannot be a substitute for the rigorous Px-matrix. A network adjustment with a diagonal Px-matrix carried out by Thomson [1976] showed that the adjustment results when the covariances are neglected are statistically compatible with adjustment results without weighted position constraints. It is therefore imperative that the Px-matrix be the fully populated inverse $C_{x_1}^{-1}$. The cost of storage must be considered to be part and parcel of the cost of the Px-adjustment.

1.4 Effect of Inconsistent Observations in 2D Networks

1.4.1 Random errors

Geodetic observations are always inconsistent with the mathematical model. The inconsistency Δ^2 has traditionally been decomposed into random and systematic components, $\Delta \ell_r$ and $\Delta \ell_s$, respectively, and studied independently of each other [Moritz, 1980; Vanicek and Krakiwsky, 1982]. The effect of inconsistencies in

observations on the estimated positions can be minimized by minimizing both components.

The effect of random error propagation can be considerably reduced by improving the statistical strength of the network at the design stage. The classical approach of using controlling baselines and Laplace azimuths have always been measures of improving the statistical strength of the network. Other post-adjustment techniques are given in Dare [1982] and Welsch [1982]. The use of Doppler points in terrestrial networks is credited with improving reliability [Thomson, 1976] and improving accuracy by up to 3 times the original accuracy [Pinch, 1974; Moose and Henriksen, 1976; Salih, 1984]. Doppler points strengthen the network both statistically and geometrically [Burford, 1980].

The effect of random errors on the estimated positions \hat{x} is fully described by the covariance matrix, $C_{\hat{x}}$. The covariance matrix of estimated positions, however, is meaningful only in the absence of systematic errors. It is for this reason that the assessment of adjustment results can only be made objective when the effect of systematic errors on the results is negligible.

1.4.2 Systematic errors

Systematic errors in terrestrial networks can be classified as observation errors and projection errors. Observation systematic errors include: errors in horizontal angles due to lateral refraction. In first order networks this is in the order of 2 arc seconds [Bomford, 1971]. Systematic error in electro-magnetic distance

measurements (EDM), due to inadequate modelling of meteorological data (pressure, humidity, temperature), account for 4 ppm of the derived distance [Deumlich, 1967; Jones, 1974; Lebedev, 1974; Laurila, 1976]. Timing errors in astronomical observations are in the order of 1.5 arc seconds [Kuznetsov, 1966; Merry, 1975; Mueller, 1977]. Errors in star positions in the star catalogue are estimated at 0.4 arc seconds [Ibid.]. All of the above are in addition to the random errors after the observations have been screened by various techniques such as trend analysis [Blais, 1976; Vanicek and Krakiwsky, 1982]. Furthermore, the analysis of systematic errors, which were not modelled before network adjustment, can be done within the adjustment procedure [Zimovnov, 1960; Sunter, 1967; Markuze, 1974]. Claim on total modelling and removal of systematic errors from the observations has not been made in geodetic literature. It no doubt remains the single most important problem in improving the accuracy of positions in geodetic networks. Systematic errors in observations often affect all similar observables in the same way. Such errors are difficult to unveil and are a major cause of distortions in estimated positions in networks.

Projection errors affect all networks computed on a reference ellipsoid. Projecting observations onto the ellipsoid requires a knowledge of the orthometric height H , the geoidal height N and the astrogeodetic deflections of the vertical ξ and η_0 at each point of the network [Clark, 1961; Zakatov, 1974; Thomson, 1976; Vanicek and Krakiwsky, 1982]. These quantities are normally not available for every point and can only be estimated. The surface

fitting technique [Merry and Vanicek, 1975] used to compute N at points not observed gives an error of up to 2 m [Merry, 1975]. A Doppler derived geoid or a combination of gravity data with either GEM10 or GEM10B [Lachapelle, 1978] gives relative accuracy in Canada for N at better than 1.0 m. This generally depends on how reliable the gravity data are. Network distortions due to uncertainties in geoidal heights in, for example, the Labrador chain of the Canadian primary network are estimated by Thomson et al. [1974]. There, rigorous reduction of the single available distance in the network is reported to change the scale of the network by -1.7 ppm. Rigorous reduction of the directions to the ellipsoid changed both the scale and rotation by -1.2 ppm and -0.065 arc seconds, respectively.

The adjustment of networks by adjusting the observations in a height-controlled spatial system of coordinates, usually in a local astronomical system, without reducing them to the ellipsoid [Vincenty and Bowring, 1978], bypasses the procedure that is accountable for the projection errors. The direction of gravity must be known, however, at every point of the network in the form of astronomical coordinates. The accuracy of this technique therefore rests with the accuracy with which astronomical positions (observed or interpolated) are determined. Canadian primary networks in the 1985 adjustment of the North American Geodetic Networks (NAD83) will be partly adjusted in a height, controlled spatial system [Steeves, 1984].

Systematic errors in networks established by satellite techniques (Doppler and NAVSTAR/GPS) have a different character from those already described. The errors can be classified into three

groups: satellite errors, propagation errors and receiver errors [Hittel and Kouba, 1971; Wells et al., 1981]. Satellite errors are errors due to the ephemerides and satellite clock. Propagation errors are errors due to unmodelled ionospheric and tropospheric refraction. Receiver errors include measurement noise, truncation and computation errors.

1.4.5 Error analysis

Rigorous densification does not imply, in any way, that the rigorous solution will be free of the effects of random and systematic errors. The two sets of position estimates of the points of the junction subnetwork S_3 offer a tool for further investigation of these errors. Thomson [1976], Beattie et al. [1978], Cooper and Leahy [1978] and McLellan [1978] have investigated terrestrial networks using more accurate Doppler Networks and were able to separate the misfit in the positions into two main components: errors in the terrestrial observables and errors due to difference in adjustment schemes. Thomson found, for example, that some of the observations in the terrestrial network were statistically incompatible with the rigorous weighted position constraint adjustment results of the same network. Six observations were flagged for rejection when the terrestrial and Doppler networks were combined. In the light of these investigations the junction subnetwork in rigorous densification offers the rare opportunity to compare the 'old' and 'new' observations, ℓ_1 and ℓ_2 , every time a densification is made. In so doing, the correlation between the existing and the

densification network must be established. The covariance matrix of the position differences must be derived unambiguously and a testing procedure must be established to check whether or not the existing and densification networks are statistically compatible.

1.4.4 Strain analysis

A novel approach to study distortions in geodetic networks is the strain analysis technique described in Vanicek et al. [1981] and Dare and Vanicek [1982]. The technique can be used to investigate causes of a non-zero displacement vector. In the absence of physical motion of the monuments, the displacement vector of the junction subnetwork point positions is caused by inconsistency between the observations ℓ_1 and ℓ_2 . The inconsistency is related linearly to the strain vector. Transforming this inconsistency into strain gives a unique view of the kind of distortion experienced by each point - in rotation, extension or compression and shear. On the other hand, given the strain parameters, it is possible to formulate an inverse strain analysis problem to study the possibility of recovering the inconsistency responsible for a particular strain. An ambiguity to be resolved is whether the inconsistency is to be connected with ℓ_1 or ℓ_2 observations. The strain analysis technique can be applied in conjunction with statistical methods. The hierarchy of the techniques should be as follows: statistical testing based on residuals in both adjustments, compatibility testing to show whether or not the networks are statistically compatible, and strain analysis to give an insight into the causes of the incompatibility

if it exists. Thus, strain analysis is only required when the two sets of points of the junction point vectors or subvectors thereof are statistically incompatible at a desired confidence level.

1.5 Transit, GPS, Inertial and Photogrammetric Densification of 2D Networks

Horizontal networks for mapping, engineering, land and resource management surveys have traditionally been established using the methods of triangulation, trilateration and traversing. Underlying these methods is the principle of working from the whole to the part [Clark, 1961], i.e., a sparse network is densified by a less accurate network until the required density of points is attained. This concept of positioning through densification using classical methods is tantamount to a step-wise increase in the density of points accompanied by loss in positioning accuracy at each step. All classical methods require visibility between adjacent network points during the observation campaign. Intervisibility limits the station separation usually to less than 50 km, determines to some extent the network geometry and increases the time and cost of the campaign [Langley et al., 1982; Vanicek et al., 1983].

The modern positioning techniques of Doppler, GPS, inertial and photogrammetric surveys are not limited by intervisibility. They offer the possibility of establishing a dense network of points in one or two steps. Their accuracy capabilities (see Figure 1.2) are much higher than is possible with the classical techniques. These techniques are therefore more cost effective than the classical

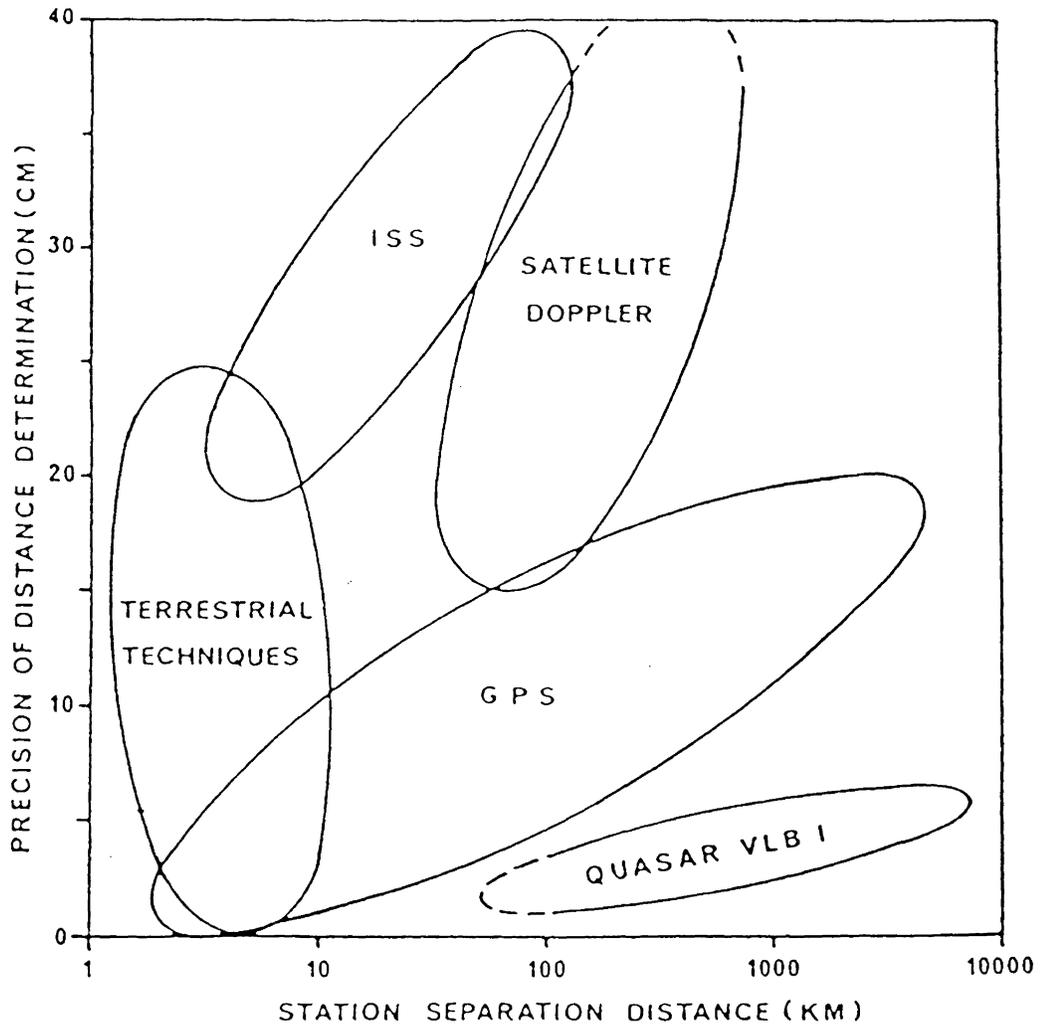


Figure 1.2: Precision of distance determinations achievable using different techniques [Langley et al. 1982].

methods [Brown, 1976; O'Brien, 1979; Langley et al., 1982]. Densification with techniques yielding higher positioning accuracy than the existing network is now a reality.

The transit system is capable of providing a network of uniform accuracy points (with approximately 100 km spacing) in 3D [EMR, 1978; Wells, 1980]. Such a network can be established using multistation multipass data and the precise ephemerides with 30 cm position repeatability [Wells, 1980]. The GPS system has the capability of providing a network of points at 10 kilometre spacing with accuracy of 1-3 ppm of baseline length [Counselman and Steinbrecher, 1982; Beutler et al., 1984; Bock et al., 1984]. The system has a potential of better than 1 ppm in accuracy of baseline length [Goad and Remondi, 1984]. Both the Doppler and GPS 2D networks defined by projecting the 3D coordinates onto a selected geocentric reference ellipsoid provide a horizontal framework for other survey systems, classical or modern [Vanicek and Krakiwsky, 1982].

Inertial Survey Systems (ISS) have a wider range of application compared to classical techniques but are generally less accurate (see Figure 1.2). Inertial positioning is versatile for rapid densification of surveys over large areas as shown by Doxey, Jr. [1977] and Mueller [1981]. Densification using the inertial survey system requires that an existing network be of 80-100 km spacing or less [Schwarz and Gauthier, 1981]. This technique has proved to be twice as productive as EDM traversing in areas of normal terrain [O'Brien, 1979]. The dependency of the inertial positioning technique on the existing network must also be viewed in light of the rigorous densification described earlier. As was stated in section 1.2, the

covariance matrix of existing points must be used rigorously. One way to improve the ISS system may be to incorporate the P_x -matrix in the onboard computer software.

Positioning by aerial triangulation gives the same advantages of the ISS in areal coverage and productivity but with higher density at comparable accuracies. The state-of-the-art method of adjustment is the bundle adjustment with self-calibration described by Brown [1976] as follows:

"The adjustment involves the simultaneous, least-squares triangulation of all bundles of rays from all exposure stations to all measured ground points in a process which also (a) recovers the elements of orientation of all participating exposures, (b) adjusts the control survey (in accordance with its postulated accuracy), and (c) estimates coefficients of error models that describe the residual systematic errors affecting the plate coordinates."

The bundle adjustment with self-calibration to obtain photogrammetrically determined ground coordinates is a rigorous one according to the definition in section 1.2. However, the method used to transform photo-coordinates into geodetic coordinates is not. Consequently, transformation using a least-squares fit as used in aerial triangulation [Ackermann, 1981; Forstner, 1981a, 1981b] does not yield the same results as, for example, the P_x -adjustment. This comparison is also made in section 4.3.5. The accuracy achievable in photogrammetric densification depends largely on the accuracy of the existing control. Densification of a GPS network by aerial triangulation can therefore provide advantages in accuracy and productivity which is unprecedented by any other two-step densification procedure.

1.6 Special Requirements and Rigorous Densification

Rigorous densification as defined earlier will yield rigorous results which may not agree with what some groups of users would like to obtain. The possibility of changes in junction point positions, their covariance matrices (or both) is indeed very likely in rigorous densification. Engineering concerns, for example, focus on obtaining as high relative accuracy between points as a special purpose network can give. Cadastral surveyors are concerned when significant changes in the land data files are contemplated.

Rigorous and non-rigorous densification have one common drawback for engineering and cadastral surveys - dual position information on the junction points. Dual positions defeat the basics of precise and reliable location and identification of property boundaries and in some cases the discrepancy may not meet cadastral standards for parcel identification. Dual positions can necessitate constant updating of land information data which is not only an expensive undertaking but also prone to confuse the user. It is a desirable condition that the geodetic framework and hence coordinates on which a cadastre is based remain unchanged and of adequate accuracy and precision to permit system operation at the parcel level. The same is required for base maps compiled over an epoch of time [Chatterton and McLaughlin, 1975; National Research Council, US, 1983].

In Engineering networks, when the main concern is on the internal consistency, special purpose networks are established. This is particularly true in deformation surveys [Chen, 1982], in

construction surveys [Lugoe, 1978; Teskey, 1979] and in municipal and utility surveys [Blachut et al., 1980]. When it becomes necessary for the networks to be tied to a higher order network, necessary design precautions are taken to minimize the effect of the higher order network on the engineering survey network. Proposals have been made to reformulate design standards of hierarchial municipal networks in such a way that higher order networks will have only marginal effects on subsequent city surveys [Lebedev, 1973, 1974; Adler et al., 1979; Blachut et al., 1980].

These cadastral and engineering concerns, coupled with data base management requirements described earlier for the rigorous weighted position constraint adjustment, have traditionally been a source of skepticism about the usefulness of a rigorous densification as performed in this research. A band-aid alternative has been to use the overconstrained schemes of adjustment. One such scheme particularly appealing, for example, to cadastral concerns is the Blaha algorithm [Blaha, 1974; Chamberlain, 1977]. The adjustment suppresses a second set of positions of junction points from being estimated, while improving the uncertainty of the existing positions to equal the would be hypothetical rigorous positions. Overconstrained solutions often result in seemingly more accurate results [Cooper and Leahy, 1978]. However, such results are not realistic because they are based on false pretenses. If absolute constraints were imposed on all points for example, the covariance matrix of estimated positions would be a null matrix which implies a perfect solution. Such a solution is wrong because parameter estimates from

non-deterministic observations must have a certain degree of uncertainty.

Special interest user concerns are regarded here to be secondary to rigour and will not be discussed further.

1.7 Goals and Contributions

The goals of this research can be stated as follows:

- 1) To derive the least-squares expressions which make the Px-adjustment solution equivalent with the combined adjustment solution for the densification network and to state clearly the limitations within which the equivalent solutions are guaranteed.
- 2) To formulate non-rigorous densification schemes similar to the Px-adjustment by giving a meaningful interpretation of a fixed-point in network adjustment. To derive simple, economical and practical expressions required to correct non-rigorous densification solutions.
- 3) To investigate the use of compatibility testing in densification networks using the two sets of solutions of the junction points. To derive the weight matrix of the position differences of the two junction solutions required in the test for compatibility and to examine the merits and demerits of such a test.
- 4) To attempt, using the novel strain analysis technique in densification networks, to study gross-errors in the observations. To investigate the sensitivity of the strain technique to unveil the presence of gross-errors in conjunction with the statistical compatibility test. To give, if possible, the mathematical formulation of an inverse strain analysis problem which, if

solved, uncovers the inconsistencies in the observations responsible for the strain in a given network.

- 5) To derive expressions which can be used to correct a densification solution for minor changes in the matrices and vectors involved in the adjustment. The matrices P_x , P_ℓ and vectors ℓ_2 and $\hat{x}^{(0)}$ will all be considered.

The contributions made in this work are as follows:

- 1) Derivation of the least-squares expressions of both the combined adjustment and the P_x -adjustment by considering a priori information on the existing and new points.
- 2) Proof of the equivalence of the combined adjustment with the P_x -adjustment solutions and derivation of the P_x -adjustment from the combined adjustment algorithms.
- 3) Application of the concept of stochastic Taylor points to densification networks and derivation of covariance matrices using a finite covariance matrix of existing positions.
- 4) Formulation of non-rigorous densification models using weighted position constraints. The limiting cases of the diagonal elements of the weight matrix, P_x has been applied.
- 5) Derivation of expressions required to economically transform non-rigorous into rigorous solutions (for the improper use of the P_x -matrix). The P_x -adjustment solution has been compared with the fixed-point, overconstrained and fixed-point with transformation solutions.
- 6) Practical applications of the statistical compatibility test to check the compatibility of the existing and densification solutions.

A computer program CTEST has been developed to perform any test for compatibility of two network solutions.

- 7) The cross-covariance matrix between the existing and densification solutions has been derived and the weight matrix of position differences of the junction points has been confirmed.
- 8) The novel strain analysis technique has been introduced to densification networks.
- 9) The strain analysis technique has been used to study strain effects of the existing network and the strain effect of gross-errors on the densification network.
- 10) A mathematical formulation of the inverse strain analysis problem has been made.
- 11) Expressions have been derived to effect changes in the rigorous densification solution for minor changes in input data (P_x and P_ℓ matrices, ℓ and $\hat{x}^{(0)}$ vectors) cost effectively.

This study presents a systematic study of the problem of rigour in the densification of horizontal networks. The most comprehensive way to incorporate new points in an existing network rigorously is by simultaneous adjustment of the existing (old) and new observations. The formulation of mathematical models and estimation of positions and corresponding covariance matrices (using a priori information) is the essence of chapter 2. The most elegant way of adjusting densification networks rigorously is by weighted position constraints - the P_x -adjustment. The mathematical models for this type of adjustment and solution for the position and error estimates of the densification network is the subject matter of chapter 3. Chapter 4 discusses various algorithms with which

non-rigorous densifications can be corrected to rigorous ones. A comparison of non-rigorous solutions with the rigorous solutions of a simulated network is also made. Chapter 5 illuminates the question of statistical testing of densification networks. The possibility of formulating and testing hypotheses on the residuals and the positions is the essence of this chapter. The merits of compatibility testing in densification networks are discussed with aid of simulation studies. The strain analysis of inconsistent observations as applied to densification networks is discussed in chapter 6 in which, the inverse strain analysis problem is also formulated. Simulation studies are also carried out to investigate the strain effect of gross-errors in densification networks. Chapter 7 examines algorithms required to correct for the effect of various blunders in the adjustment process such as blunders in data entry for the initial positions, the P_x -matrix, observations and observation weights. Algorithms to correct for the reobserved elements of ℓ_2 are also given in the same chapter. Chapter 8 concludes this study.

2.0 RIGOROUS NETWORK DENSIFICATION BY A SIMULTANEOUS ADJUSTMENT OF TWO NETWORKS

2.1 The Scope of the Problem

Any two horizontal geodetic networks can be adjusted simultaneously (i.e., together) if the networks are designed in such a way that:

- a) observations linking the two networks are procured, or
- b) a set of common junction points exists, or
- c) both a) and b) are considered.

The system of equations in each of the networks will be dependent upon the other in all three cases.

The design of densification networks $S_2 \equiv \{x_j, x_n\}$ always includes a subnetwork of junction points $S_3 \equiv \{x_j\}$ which also belongs to the existing network $S_1 \equiv \{x_e, x_j\}$. The networks S_1 and S_2 with observation vectors l_1 and l_2 respectively can therefore be adjusted simultaneously. The simultaneous adjustment of the S_1 and S_2 involves the formulation and solution of normal equations jointly. The mathematical models may be separate for each of the networks.

Linearization of the mathematical models requires some knowledge of the positions of the points in the model - the parameters. Such initial positions $x^{(0)}$, obtained by approximate methods, can

have a finite covariance matrix $C_x^{(0)}$ associated with them. A third mathematical model will therefore be formulated. The points in the network shall be constrained through this model using the weight matrix, $C_x^{-1(0)}$.

The observations in each set can be assumed to be uncorrelated leading to diagonal weight matrices P_1 of ℓ_1 and P_2 of ℓ_2 . Correlation between ℓ_1 and ℓ_2 observations procured at different time epochs with probably no overlapping observations may exist through the observation media, instrumentation and observation methodology. This correlation can, with careful design precautions, be reduced to a minimum and is of necessity, neglected in practice. It shall be assumed henceforth that the weight matrix P of the observations for the simultaneous adjustment has the following structure;

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad 2.1$$

Besides the position parameters, nuisance parameters are normally introduced into the mathematical model. For simplicity these parameters will not be considered here. Reference is made to Krakiwsky [1968], Krakiwsky and Thomson [1978] and Nickerson [1980] for detailed treatment of the subject.

2.2 Mathematical Models

The functional relationship between the observations and the positions are established as;

$$F_1(\bar{x}_e, \bar{x}_j) = \bar{\ell}_1 \quad :P_1 \quad 2.2$$

$$F_2(\bar{x}_j, \bar{x}_n) = \bar{\ell}_2 \quad :P_2 \quad 2.3$$

The constraint model that introduces a priori information $C_x^{(0)}$ into the adjustment is;

$$F_3(\bar{x}, \bar{\ell}_x) = 0 \quad :C_x^{-1(0)} \quad 2.4$$

where,

$$\bar{x} \equiv \{\bar{x}_e, \bar{x}_j, \bar{x}_n\}$$

$\bar{\ell}_x$ are pseudo-observations with covariance matrix $C_x^{(0)}$.

The mathematical models (2.2) - (2.4) can be linearized by Taylor series expansion into;

$$A_e \delta_e + A_j \delta_j - r_1 + \omega_1 = 0 \quad :P_1 \quad 2.5$$

$$A_j \delta_j + A_n \delta_n - r_2 + \omega_2 = 0 \quad :P_2 \quad 2.6$$

$$\delta - r_x + \omega_x = 0 \quad :C_x^{-1(0)} \quad 2.7$$

where,

$$A_e = \left. \frac{\partial F_1}{\partial x_e} \right|_{x_e = x_e^{(0)}} \quad 2.8$$

$$A_j = \left. \frac{\partial F_1}{\partial x_j} \right|_{x_j = x_j^{(0)}} \quad 2.9$$

$$A_n = \left. \frac{\partial F_2}{\partial x_n} \right|_{x_n = x_n^{(0)}} \quad 2.10$$

$$\delta \equiv \{\delta_e, \delta_j, \delta_n\}$$

$$r_1 = \bar{\ell}_1 - \ell_1 \quad 2.11$$

$$r_2 = \bar{\ell}_2 - \ell_2 \quad 2.12$$

$$\delta_e = \bar{x}_e - x_e^{(0)} \quad 2.13$$

$$\delta_j = \bar{x}_j - x_j^{(0)} \quad 2.14$$

$$\delta_n = \bar{x}_n - x_n^{(0)} \quad 2.15$$

$$\omega_1 = \ell_1^{(0)} - \ell_1 \quad 2.16$$

$$\omega_2 = \ell_2^{(0)} - \ell_2 \quad 2.17$$

$$\omega_x = \ell_x^{(0)} - \ell_x \quad 2.18$$

ℓ_x is a vector of pseudo-observations;

\bar{x}_i for $i \in \{e, j, n\}$ are expected position vectors;

$x_i^{(0)}$ are initial position vectors;

$\ell_1^{(0)}, \ell_2^{(0)}$ are observation vectors computed using $x_i^{(0)}$ positions.

The design matrices A_i which transform the correction vectors δ_i into a linear model space are assumed to be of full rank, i.e.,

$$\text{rank}(A_i) = \dim(x_i)$$

$$\text{and } \dim(x) < \dim(\ell)$$

The auxilliary model (2.4) is introduced, as stated, to take care of a priori information in the adjustment. In this case the linear models (2.5) - (2.7) are formulated in a differential neighbourhood of the expected estimates. A one iteration solution will be contemplated which in turn means that $\omega_x = 0$. In the event that more than one solution exist for the points to be adjusted (e.g., Doppler, GPS, ISS, Photogrammetric coordinates) then only one of the solutions shall be selected for the linearization of the mathematical models. This, of course, implies that the coordinate system of the selected solution will be adopted as the coordinate system of the combined adjustment. The other solutions can rigorously be merged with the combined adjustment solution in a separate step.

2.3 Derivation of Normal Equations

The observation vectors $\hat{\ell}_x$, $\hat{\ell}_1$ and $\hat{\ell}_2$ are to be estimated from ℓ_x , ℓ_1 and ℓ_2 respectively such that the estimated vectors are consistent with the models (2.5) and 2.6) besides satisfying the least-squares criterion;

$$\min_{\delta_e, \delta_j, \delta_n, \delta} (r_1^T P_1 r_1 + r_2^T P_2 r_2 + r_x^T C_x^{-1}(0) r_x)$$

This criterion expresses an extremal problem which shall be formulated as;

$$\begin{aligned} \min \{ & (A_e \delta_e + A_j \delta_j + \omega_1)^T P_1 (A_e \delta_e + A_j \delta_j + \omega_1) + (A_j \delta_j + A_n \delta_n + \omega_2)^T P_2 (A_j \delta_j + A_n \delta_n + \omega_2) \\ & + (\delta + \omega_x)^T C_x^{-1}(0) (\delta + \omega_x) \} \end{aligned} \quad 2.19$$

Differentiating (2.19) with respect to the unknown parameters δ , δ_e , δ_j and δ_n and setting the result to zero leads to the following system of normal equations;

$$\begin{aligned} A_e^T P_1 (A_e \hat{\delta}_e + A_j \hat{\delta}_j + \omega_1) &= 0 \\ A_j^T P_1 (A_e \hat{\delta}_e + A_j \hat{\delta}_j + \omega_1) &= 0 \\ A_j^T P_2 (A_j \hat{\delta}_j + A_n \hat{\delta}_n + \omega_2) &= 0 \\ A_n^T P_2 (A_j \hat{\delta}_j + A_n \hat{\delta}_n + \omega_2) &= 0 \\ C_x^{-1}(0) (\hat{\delta} + \omega_x) &= 0 \end{aligned}$$

which can be combined into the following two hypermatrix equations, assuming the inverses $C_x^{-1}(0)$, $C_x^{-1}(0)$ and $C_x^{-1}(0)$ exist.

$$\begin{bmatrix} \Lambda_{e1}^T P_1 \Lambda_e & \Lambda_{e1}^T P_1 \Lambda_j & 0 \\ \Lambda_{j1}^T P_1 \Lambda_e & \Lambda_{j2}^T P_2 \Lambda_j + \Lambda_{j1}^T P_1 \Lambda_j & \Lambda_{j2}^T P_2 \Lambda_n \\ 0 & \Lambda_{n2}^T P_2 \Lambda_j & \Lambda_{n2}^T P_2 \Lambda_n \end{bmatrix} \begin{bmatrix} \hat{\delta}_e \\ \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} \Lambda_{e1}^T P_1 \omega_1 \\ \Lambda_{j2}^T P_2 \omega_2 + \Lambda_{j1}^T P_1 \omega_1 \\ \Lambda_{n2}^T P_2 \omega_2 \end{bmatrix} = 0 \quad 2.20$$

and

$$\begin{pmatrix} C_{x_e}^{-1}(0) & 0 & 0 \\ 0 & C_{x_j}^{-1}(0) & 0 \\ 0 & 0 & C_{x_n}^{-1}(0) \end{pmatrix} \begin{pmatrix} \hat{\delta}_e \\ \hat{\delta}_j \\ \hat{\delta}_n \end{pmatrix} + \begin{pmatrix} C_{x_e}^{-1}(0) \omega_x \\ C_{x_j}^{-1}(0) \omega_x \\ C_{x_n}^{-1}(0) \omega_x \end{pmatrix} = 0 \quad 2.21$$

Equation (2.20) is obtained from the models (2.2) and (2.5).

Equation (2.21) is obtained entirely from the auxilliary model (2.4).

An obvious question is how and when should (2.20) and (2.21) be used.

The existing solution and therefore the pseudo-observables ℓ_x are uncorrelated with either ℓ_1 or ℓ_2 . The observation vector $\ell \equiv \{\ell_1, \ell_2, \ell_x\}$ contains more information compared to ℓ_1 and ℓ_2 alone. Intuitively, we expect better results from ℓ than from ℓ_1 and ℓ_2 . A combined hypermatrix equation of normal equations for the three models (2.5) - (2.7) is the sum of (2.20) and (2.21). Recalling that for a one step solution $\omega_x = 0$, the system of normal equation can be simplified. Let us introduce the following notations:

$$N_{ee} = \Lambda_{e1}^T P_1 \Lambda_e + C_{x_e}^{-1}(0) \quad 2.22$$

$$N_{1j} = A_{j1}^T P_1 A_j + C_{x_j}^{-1}(0) \quad 2.23$$

$$N_{jj} = A_{j2}^T P_2 A_j \quad 2.24$$

$$N_{nn} = A_{n2}^T P_2 A_n + C_{x_n}^{-1}(0) \quad 2.25$$

$$N_{ej} = N_{je}^T = A_{e1}^T P_1 A_j \quad 2.26$$

$$N_{jn} = A_{j2}^T P_2 A_n \quad 2.27$$

$$u_e = A_{e1}^T P_1 \omega_1 \quad 2.28$$

$$u_j^1 = A_{j1}^T P_1 \omega_1 \quad 2.29$$

$$u_j = A_{j2}^T P_2 \omega_2 \quad 2.30$$

$$u_n = A_{n2}^T P_2 \omega_2 \quad 2.31$$

the result is;

$$\begin{bmatrix} N_{ee} & N_{ej} & 0 \\ N_{je} & (N_{1j} + N_{jj}) & N_{jn} \\ 0 & N_{nj} & N_{nn} \end{bmatrix} \begin{bmatrix} \hat{\delta}_e \\ \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} u_e \\ u_j^1 + u_j \\ u_n \end{bmatrix} = 0 \quad 2.32$$

Equation (2.32) may be decomposed into a summation equation of the normal equations of S_1 and S_2 , i.e.,

$$\begin{bmatrix} N_{ee} & N_{ej} & 0 \\ N_{je} & N_{1j} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\delta}_e \\ \hat{\delta}_j^1 \\ 0 \end{bmatrix} + \begin{bmatrix} u_e \\ u_j^1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & N_{jj} & N_{jn} \\ 0 & N_{1j} & N_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\delta}_j^2 \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} 0 \\ u_j \\ u_n \end{bmatrix} = 0 \quad 2.33$$

As this equation shows, the solution of normal equations for one network is dependent on the other. The same dependency exists between the existing and densification networks.

2.4 The Least-Squares Solution

2.4.1 The combined solution

Let the normal equations (2.32) be written as;

$$\begin{pmatrix} N_{ee} & N_{ej} & 0 \\ N_{je} & N_{jj} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & N_{jj} & N_{jn} \\ 0 & N_{nj} & N_{nn} \end{pmatrix} \begin{bmatrix} \hat{\delta}_e \\ \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} u_e \\ u_j \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_j \\ u_n \end{bmatrix} = 0$$

or in short;

$$(N_1 + N_2)\hat{\delta} + (u_1 + u_2) = 0 \quad 2.34$$

The subscripts 1, 2 refer to the networks S_1 and S_2 respectively.

The solution of (2.34) is derived directly as;

$$\hat{\delta} = -N^{-1}(u_1 + u_2) \quad 2.35$$

$$\text{where, } N = N_1 + N_2 \quad 2.36$$

2.4.2 The densification network solution

The normal equations of the densification network can be derived by eliminating $\hat{\delta}_e$ from (2.32) using the block-elimination method [Ashkenazi, 1967] as;

$$(N_2 + Q_e)\hat{\delta}_2 + (u_2 + u_x) = 0 \quad 2.37$$

where,

$$N_2 = \begin{bmatrix} N_{jj} & N_{jn} \\ N_{nj} & N_{nn} \end{bmatrix} \quad 2.38$$

$$\hat{\delta}_2 = \begin{pmatrix} \hat{\delta}_j \\ \vdots \\ \hat{\delta}_n \end{pmatrix} \quad 2.39$$

$$u_2 = \begin{pmatrix} u_j \\ \vdots \\ u_n \end{pmatrix} \quad 2.40$$

$$u_x = u_j^1 - N_{je} N_{ee}^{-1} u_e \quad 2.41$$

$$Q_e = N_{1j} - N_{je} N_{ee}^{-1} N_{1j} \quad 2.42$$

The matrix Q_e is the normal equations matrix of the junction sub-network $S_{\bar{5}}$ in the existing network and u_x is the corresponding constant vector (Appendix I). Q_e and u_x constitute the effect of the existing network on the normal equations matrix and constant vector respectively of the densification network. The least-squares solution for $\hat{\delta}_2$ in equation (2.37) is;

$$\hat{\delta}_2 = -(N_2 + Q_e)^{-1} (u_2 + u_x) \quad 2.43$$

2.4.5 The partitioned solution

Let us introduce the identity;

$$\begin{bmatrix} N_{jj} + Q_e & N_{jn} \\ N_{nj} & N_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} H_{jj} & H_{jn} \\ H_{nj} & H_{nn} \end{bmatrix} \quad 2.44$$

which is substituted, together with (2.39) and (2.40), into (2.43) to give the partitioned solution as;

$$\hat{\delta}_j = -H_{jj} (u_j + u_x - N_{jn} N_{nn}^{-1} u_n) \quad 2.45$$

$$\hat{\delta}_n = -H_{nn} [u_n - N_{nj} (N_{jj} + Q_e)^{-1} (u_j + u_x)] \quad 2.46$$

The relationship between the submatrices in (2.44) has been proved [Fadeev and Fadeeva, 1963] to be ; ;

$$H_{jj} = (N_{jj} + Q_e - N_{jn} N_{nn}^{-1} N_{nj})^{-1} \quad 2.47$$

$$H_{nn} = [N_{nn} - N_{nj} (N_{jj} + Q_e)^{-1} N_{jn}]^{-1} \quad 2.48$$

$$H_{nj} = H_{jn}^T = -N_{nn}^{-1} N_{nj} H_{jj} \quad 2.49$$

$$N_{nn}^{-1} N_{nj} H_{jj} = H_{nn} N_{nj} (N_{jj} + Q_e)^{-1} \quad 2.50$$

Equations (2.45) and (2.46) can also be phased with respect to the observations. Factorization of the RHS gives;

$$\hat{\delta}_j = \hat{\delta}_j^{(1)} - H_{jj} (u_j - N_{jn} N_{nn}^{-1} u_n) \quad 2.51$$

$$\hat{\delta}_n = \hat{\delta}_n^{(1)} - H_{nn} [u_n - N_{nj} (N_{jj} + Q_e)^{-1} u_j] \quad 2.52$$

where,

$$\hat{\delta}_j^{(1)} = -H_{jj} u_x$$

$$\hat{\delta}_n^{(1)} = -N_{nn}^{-1} N_{nj} \hat{\delta}_j^{(1)}$$

Equations (2.51) and (2.52) show that it is possible to adjust the densification network without incorporating the ℓ_1 observations.

The mathematical model must, in such a case, be linearized about $(x_j^{(0)} + \hat{\delta}_j^{(1)})$ and $(x_n^{(0)} + \hat{\delta}_n^{(1)})$ instead of $x_j^{(0)}$ and $x_n^{(0)}$ respectively.

A problem emerges however, of how to obtain, a priori, the vectors $\hat{\delta}_j^{(1)}$ and $\hat{\delta}_n^{(1)}$ which depend upon the matrices A_n and P_2 of S_2 . Such a problem can be overcome by using $\hat{\delta}_j^*$ of the independent existing solution (Appendix I) of S_1 and weight matrix $Px_j = Q_e$ for the junction points linking the two networks together. This approach to rigorous densification is the essence of the weighted position constraint adjustment to be addressed in Chapter 3.

2.4.4 The existing non-junction points solution

Let us recall the first equation in the hypermatrix equation (2.32).

$$N_{ee} \hat{\delta}_e + N_{ej} \hat{\delta}_j + u_e = 0$$

The least-squares estimate $\hat{\delta}_e$ can then be expressed through $\hat{\delta}_j$ as;

$$\hat{\delta}_e = -N_{ee}^{-1} u_e - N_{ee}^{-1} N_{ej} \hat{\delta}_j \quad 2.53$$

which depends on $\hat{\delta}_n$ also. The effect of the vector $\hat{\delta}_n$ is embedded in $\hat{\delta}_j$, the correction vector for the junction points. The adjustment problem presented by the normal equations (2.32) is symmetrical about the junction points. The equations valid for $\hat{\delta}_n$ will be equally valid for $\hat{\delta}_e$ when the indices are interchanged and the networks S_1 and S_2 becoming the densification and existing networks respectively. The normal equation can be partitioned to include $\hat{\delta}_j$ in S_1 or S_2 . Thus the vector $\hat{\delta}_j$ can be solved for together with $\hat{\delta}_n$ as performed in section 2.4.3 or together with $\hat{\delta}_e$. In the latter case $\hat{\delta}_n$ will be expressed as;

$$\hat{\delta}_n = -N_{nn}^{-1} u_n - N_{nn}^{-1} N_{nj} \hat{\delta}_j$$

The effect of the existing correction vector $\hat{\delta}_e$ is now embedded in the vector $\hat{\delta}_j$.

2.5 The Covariance Matrices of Correction Vectors

2.5.1 The covariance matrix, $C_{\hat{\delta}_2}$

The covariance matrix $C_{\hat{\delta}_2}$ of the correction vector to initial positions in the densification network is derived by applying the covariance law [Hamilton, 1964] to equation (2.43). The result is;

$$C_{\hat{\delta}_2} = (N_2 + Q_e)^{-1} C_{u_2 + u_x} (N_2 + Q_e)^{-1} \quad 2.54$$

To evaluate the cross-covariance matrix $C_{u_2 + u_x}$ we recall equations (2.40) and (2.41) and present them in the following forms:

$$u_2 = A_2^T P_2 \omega_2 \quad 2.55$$

$$u_x = M \bar{A}_1^T P_1 \omega_1 \quad 2.56$$

where,

$$A_2 = \begin{pmatrix} A_j \\ A_n \end{pmatrix} \quad 2.57$$

$$\bar{A}_1 = (A_e \quad A_j)^T \quad 2.58$$

$$M = \begin{bmatrix} 0 & 0 \\ -N_j e^{N_e^{-1}} & I \end{bmatrix} \quad 2.59$$

We have assumed, at the beginning of the chapter, that the observations z_1 and z_2 are uncorrelated. Therefore, the matrix $C_{u_2+u_x}$ must be equal to the sum of the covariance matrices C_{u_2} and C_{u_x} which are evaluated from equations (2.55) and (2.56) respectively.

$$C_{u_2+u_x} = A_2^T P_2 C_{\omega_2} P_2 A_2 + M \bar{A}_1^T P_1 C_{\omega_1} P_1 \bar{A}_1 M^T \quad 2.60$$

Equation (2.60) can further be evaluated by evaluating the matrices C_{ω_2} and C_{ω_1} from equations (2.16) and (2.17). We consider two factors here: First, the stochasticity of the initial positions used to linearize the mathematical models is considered by using the covariance matrix $C_{x(0)}$ in the derivations of the covariance matrix of the misclosures. Second, we stick to our definition of the problem that whenever more than one set of a priori information (solution) exists for a set of points only one set of information is considered. With regard to the second factor, the junction points have two sets of a priori information, i.e., $C_{x_1(0)}$ and Q_e . In this case the a priori information $C_{x_1(0)}$ will not be used in the derivation of C_{ω_1} . Consequently,

$$C_{\omega_1} = P_1^{-1} \quad 2.61$$

$$C_{\omega_2} = P_2^{-1} + A_2 C_{x_2(0)} A_2^T \quad 2.62$$

where ,

$$C_{x_2(0)} = \begin{bmatrix} Q_e^{-1} & 0 \\ 0 & C_{x_n(0)} \end{bmatrix} \quad 2.63$$

When equations (2.61) and (2.62) are substituted into (2.60) we obtain;

$$C_{u_2+u_x} = A_2^T P_2 (P_2^{-1} + A_2 C_{x_2(0)} A_2^T) P_2 A_2 + \bar{M} \bar{A}_1^T P_1 A_1 M^T \quad 2.64$$

Equation (2.64) is substituted into (2.54) while considering (2.63) to give;

$$\begin{aligned} \hat{C}_{\delta_2} &= (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} A_2^T P_2 (P_2^{-1} + A_2 C_{x_2(0)} A_2^T) P_2 A_2 (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} \\ &\quad + (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} M N_1^* M^T (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} \end{aligned} \quad 2.65$$

where,

$$\bar{N}_2 = A_2^T P_2 A_2 \quad 2.66$$

$$N_1^* = \bar{A}_1^T P_1 \bar{A}_1 \quad 2.67$$

$$\bar{N}_2 + C_{x_2(0)}^{-1} = N_2 + Q_e \quad 2.68$$

The matrix identity [cf. Liebelt, 1967]

$$(B^T C B + A)^{-1} B^T C \equiv A^{-1} B^T (C^{-1} + B A^{-1} B^T)^{-1} \quad 2.69$$

is introduced and applied to $(\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} A_2^T P_2$. The result is;

$$(\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} A_2^T P_2 = C_{x_2(0)} A_2^T (P_2^{-1} + A_2 C_{x_2(0)} A_2^T)^{-1} \quad 2.70$$

Substituting (2.70) into (2.65) we obtain;

$$\hat{C}_{\delta_2} = C_{x_2(0)} \bar{N}_2 (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} + (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} M N_1^* M^T (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} \quad 2.71$$

The term $MN_1^*M^T$ in (2.71) is evaluated from equations (2.58) and (2.59) as;

$$MN_1^*M^T = Q_e \quad 2.72$$

which is substituted into (2.71) to give;

$$\hat{C}_{\delta_2} = C_{x_2(0)} \bar{N}_2 (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} + (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} Q_e (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} \quad 2.73$$

It can be shown that;

$$C_{x_2(0)} \bar{N}_2 (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} = C_{x_2(0)} - (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} \quad 2.74$$

Proof:

Multiply both sides of equation (2.74) by the matrix $(\bar{N}_2 + C_{x_2(0)}^{-1})$. The result is;

$$C_{x_2(0)} \bar{N}_2 = C_{x_2(0)} \bar{N} \quad \text{q.e.d.}$$

The covariance matrix (2.73) is then written using equation (2.74) as;

$$\hat{C}_{\delta_2} = C_{x_2(0)} - (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} + (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} Q_e (\bar{N}_2 + C_{x_2(0)}^{-1})^{-1} \quad 2.75$$

The matrices \bar{N}_2 and N_2 are related as in equation (2.68). Equation (2.75) can also be given as;

$$\hat{C}_{\delta_2} = C_{x_2(0)} - (N_2 + Q_e)^{-1} + (N_2 + Q_e)^{-1} Q_e (N_2 + Q_e)^{-1} \quad 2.76$$

But (c.f. Appendix II, equation II.8);

$$(N_2 + 2Q_e)^{-1} = (N_2 + Q_e)^{-1} - (N_2 + Q_e)^{-1} Q_e (N_2 + Q_e)^{-1}$$

which when substituted into (2.76) gives;

$$\hat{C}_{\delta_2} = C_{x_2(0)} - (N_2 + 2Q_e)^{-1} \quad 2.77$$

Equation (2.77) is the expression of the covariance matrix of the correction vector of the densification network estimated in a simultaneous adjustment of the networks S_1 and S_2 . It includes the

known uncertainty in existing positions. In the event that the a priori information $C_{x(0)}$ is missing then equation (2.75) will be;

$$C_{\delta_2}^{\wedge} = (\bar{N}_2 + Q_e)^{-1} A_2^T P A_2 (\bar{N}_2 + Q_e)^{-1} + (\bar{N}_2 + Q_e)^{-1} Q_e (\bar{N}_2 + Q_e)^{-1}$$

or

$$C_{\delta_2}^{\wedge} = (\bar{N}_2 + Q_e)^{-1} \quad 2.78$$

Equation (2.78) is the well known expression for $C_{\delta_2}^{\wedge}$ when the role of the Taylor Points does not go beyond that of linearizing the Mathematical Models. Stochastic Taylor points are also used in Grafarend et al. [1983]. Blaha [1976] points out that in an adjustment the a priori estimated $\hat{x}^{(0)}$ loses the nature of known constants and assume the role of quasi-observations with a weight matrix $C_{x(0)}^{-1}$. Equation (2.77) therefore considers the a priori positions as observations.

2.5.2 The covariance matrices $C_{\delta_j}^{\wedge}$ and $C_{\delta_n}^{\wedge}$

The covariance matrices of the correction vectors $\hat{\delta}_j$ and $\hat{\delta}_n$ can be obtained either by partitioning equation (2.77) or by applying the covariance law to equations (2.45) and (2.46) respectively. Both approaches have been tried and the results are the same. From equation (2.68) the partitioning technique is applied in accordance with (2.44), and (2.63) to give;

$$\begin{bmatrix} C_{\delta_j \hat{\delta}_j}^{\wedge} & C_{\delta_j \hat{\delta}_n}^{\wedge} \\ C_{\delta_n \hat{\delta}_j}^{\wedge} & C_{\delta_n \hat{\delta}_n}^{\wedge} \end{bmatrix} = \begin{bmatrix} Q_e^{-1} & 0 \\ 0 & C_{x_n(0)} \end{bmatrix} - \begin{bmatrix} H_{ij} & H_{jn} \\ H_{nj} & H_{nn} \end{bmatrix} + \begin{bmatrix} H_{jj} Q_e H_{jj} & H_{jj} Q_e H_{jn} \\ H_{nj} Q_e H_{jj} & H_{nj} Q_e H_{jn} \end{bmatrix}$$

which leads to;

$$\hat{C}_{\delta_j} = Q_e^{-1} - H_{jj} + H_{jj} Q_e H_{jj} \quad 2.79$$

and

$$\hat{C}_{\delta_n} = C_{x_n(0)} - H_{nn} + H_{nj} Q_e H_{jn} \quad 2.80$$

As in the unpartitioned case absence of the $C_{x(0)}$ information would require that equation (2.78) and not (2.76) be used. The results will be therefore;

$$\hat{C}_{\delta_j} = H_{jj} \quad 2.81$$

and

$$\hat{C}_{\delta_n} = H_{nn} \quad 2.82$$

It can be seen by comparing equations (2.79) with (2.81) or (2.80) with (2.82) that the covariance matrix of the correction vector has a smaller trace when the a priori information is used than when it is not.

2.5.3 The covariance matrix, \hat{C}_{δ_e}

The derivation of \hat{C}_{δ_e} is sought by applying the covariance law to equation (2.53) which can be written using (2.28) as;

$$\hat{\delta}_e = -N_{ee}^{-1} A_{e1}^T P_{\omega_1} - N_{ee}^{-1} N_{ej} \hat{\delta}_j \quad 2.83$$

Assuming that \hat{C}_{δ_j} exists, we obtain;

$$\begin{aligned} \hat{C}_{\delta_e} &= N_{ee}^{-1} A_{e1}^T P_{\omega_1} P_{\omega_1} A_{e1} N_{ee}^{-1} + N_{ee}^{-1} N_{ej} \hat{C}_{\delta_j} N_{je} N_{ee}^{-1} \\ &\quad + N_{ee}^{-1} A_{e1}^T P_{\omega_1} P_{\omega_1} A_{ej} N_{jj} N_{ee}^{-1} + N_{ee}^{-1} N_{ej} H_{jj} A_{ej}^T P_{\omega_1} P_{\omega_1} A_{e1} N_{ee}^{-1} \\ &\quad - N_{ee}^{-1} A_{e1}^T P_{\omega_1} P_{\omega_1} A_{e1} N_{ee}^{-1} N_{ej} H_{jj} N_{je} N_{ee}^{-1} - N_{ee}^{-1} N_{ej} H_{jj} N_{je} N_{ee}^{-1} A_{e1}^T P_{\omega_1} P_{\omega_1} A_{e1} N_{ee}^{-1} \end{aligned}$$

Equation (2.45) has been used for $\hat{\delta}_j$. In addition, the cross-covariances between ω_1 and ω_2 are, as in (2.1) taken to be equal to zero. Further simplification using equation (2.60) gives;

$$\begin{aligned} \hat{C}_{\delta_e} = & N_{ee}^{-1} + N_{ee}^{-1} N_{ej} \hat{C}_{\delta_j} N_{je}^{-1} + N_{ee}^{-1} N_{ej} H_{jj} N_{je}^{-1} \\ & + N_{ee}^{-1} N_{ej} H_{jj} N_{je}^{-1} N_{ee}^{-1} - N_{ee}^{-1} N_{ej} H_{jj} N_{je}^{-1} N_{ee}^{-1} N_{ej} H_{jj} N_{je}^{-1} + C_{x_1}^{(0)} \\ & + C_{x_1}^{(0)} N_{ej} H_{jj} N_{je}^{-1} + C_{x_1}^{(0)} N_{ej} H_{jj} N_{je}^{-1} - C_{x_1}^{(0)} N_{ej} H_{jj} N_{je}^{-1} \\ & - C_{x_1}^{(0)} N_{ej} H_{jj} N_{je}^{-1} \end{aligned}$$

or

$$\hat{C}_{\delta_e} = N_{ee}^{-1} + N_{ee}^{-1} N_{ej} \hat{C}_{\delta_j} N_{je}^{-1} + C_{x_e}^{(0)} \quad 2.84$$

which is the covariance matrix of existing non-junction points.

An alternative expression can be derived by exploiting the symmetry of the densification problem with respect to $S_3 \equiv \{x_j\}$ subnetwork. If e is made to replace n in equation (2.80) we obtain;

$$\hat{C}_{\delta_e} = C_{x_e}^{(0)} - H_{ee} + H_{ej} Q_n H_{je} \quad 2.85$$

The subscripts are interchanged in the expressions for H_{nn} , H_{nj} and Q_e . It is much more convenient to use equation (2.84) when \hat{C}_{δ_j} has been computed. The compilation of the matrices H_{ee} , H_{ej} and Q_n is avoided in this case.

2.6 The Covariance Matrices of Adjusted Positions

The least-squares process converges to the same solution whether or not the initial positions are estimated quantities. Let

us assume, for simplicity that deterministic initial values are used.

The solution for the network S_2 will then be (using equation (2.37));

$$\hat{x}_2 = x_2^{(0)} - (N + Q_e)^{-1} u_x - (N_2 + Q_e)^{-1} u_2 \quad 2.86$$

which can also be written as;

$$\hat{x}_2 = \hat{x}_2^{(0)} - (N_2 + Q_e)^{-1} u_2 . \quad 2.87$$

The covariance matrix of the estimated positions \hat{x}_2 can be obtained by applying the covariance law to either (2.86) or (2.87). It is a straight forward derivation when $C_{\hat{x}_2}^{\wedge}$ is derived from equation (2.86) than (2.87). This is because the cross-correlation between $x_2^{(0)}$ and the other two terms in (2.86) is known to be equal to zero. The computation of the cross-correlation between $\hat{x}_2^{(0)}$ and $(N_2 + Q_e)^{-1} u_2$ in (2.87) is much more involved and will not be attempted here. Applying the covariance law to (2.86) we obtain;

$$\begin{aligned} C_{\hat{x}_2}^{\wedge} &= (N_2 + Q_e)^{-1} [N_{1j} + N_2 + N_{je} N_{ee}^{-1} N_{ej} - 2N_{je} N_{ee}^{-1} N_{ej}] (N_2 + Q_e)^{-1} \\ &= (N_2 + Q_e)^{-1} [N_2 + Q_e] (N_2 + Q_e)^{-1} \\ C_{\hat{x}_2}^{\wedge} &= (N_2 + Q_e)^{-1} \end{aligned} \quad 2.88$$

Equation (2.88) is the covariance matrix of the adjusted positions of S_2 in a simultaneous adjustment of S_1 and S_2 . It is equal to the covariance matrix of the correction vector (2.78) when the a priori covariance matrix $C_{x_2}^{(0)}$ is disregarded.

The covariance matrices of the partitioned solution are obtained directly by partitioning equation (2.88). The partitioned inverse $(N_2 + Q_e)^{-1}$ is given in equation (2.44). Therefore;

$$\hat{C}_{x_j} = H_{jj} \quad 2.89$$

and

$$\hat{C}_{x_n} = H_{nn} \quad 2.90$$

We again recall the symmetry of the normal equations matrix of $S \equiv \{S_1, S_2\}$ in order that we may evaluate the covariance matrix \hat{C}_{x_e} . Interchanging subscripts n for e in the expression (2.47) of H_{nn} we obtain

$$\hat{C}_{x_e} = [N_{ee} - N_{ej}(N_{1j} + Q_n)^{-1} N_{je}]^{-1} \quad 2.91$$

where,

$$Q_n = N_{jj} - N_{jn}N_{nn}^{-1}N_{nj}$$

An alternative expression to (2.91) can be obtained using the covariance law on $\hat{\delta}_e$ since $\hat{C}_{x_e} = \hat{C}_{\delta_e}$ as shown earlier in this chapter ($\hat{\delta}_e$ obtained without $C_{x_e(0)}$). Disregarding $C_{x_e(0)}$ in (2.84) we obtain;

$$\hat{C}_{x_e} = N_{ee}^{-1} + N_{ee}^{-1} N_{ej} \hat{C}_{\delta_j} N_{je} N_{ee}^{-1} \quad 2.92$$

Equation (2.92) is more convenient to use than (2.91) since \hat{C}_{δ_j} already exists from the adjustment of S_2 .

3.0 WEIGHTED POSITION CONSTRAINT ADJUSTMENT

3.1 The Scope of the Problem

The system of normal equations in the simultaneous adjustment of the existing and densification networks is shown in equation (2.34) as a summation of two terms, each of which, is in itself, a separate system of normal equations. The second term is the would be system of normal equations if the densification network were adjusted separately and independently of the existing network.

We now seek to adjust the densification network S_2 separate from the existing network S_1 (without using the ℓ_1 observations) while rigorously propagating the effect of S_1 into S_2 at the same time. The propagation is made through the junction subnetwork $S_3 \equiv S_1 \cap S_2$; the points of which have been previously adjusted and the solution $(\hat{x}_j^{(1)}, C_{x_j^{(1)}})$ exists. The coordinates, $\hat{x}_j^{(1)}$, of S_3 and any other points determined as unbiased estimates are stochastic quantities and hence have a finite covariance matrix. The linearization of the mathematical model about stochastic coordinate values is therefore contemplated [c.f., Grafarend et al., 1983], using the position $\hat{x}_j^{(1)}$ of S_3 , and corresponding covariance matrix, $P_{x_j}^{-1}$. Initial positions of the new points can be treated in the same way when determined independently of the observation vector ℓ_2 as discussed in section 2.2. This approach must be rigorous equivalent to the simultaneous adjustment. We shall assume the

equivalence of the two approaches and derive least-squares expressions of the weighted position constraint solution required to give a solution equivalent to that obtained from a combined adjustment.

3.2 Mathematical Models

The functional relationship between the expected observation vector $\bar{\ell}_2$, positions (\bar{x}_j, \bar{x}_n) in the densification network S_2 , and corresponding constraint model is given, assuming for generality sake that \bar{x}_n are independently determined too, as;

$$F(\bar{x}_j, \bar{x}_n) = \bar{\ell}_2 \quad :P_2 \quad 3.1$$

$$F(\bar{x}_j, \bar{x}_n, \bar{\ell}_x) = 0 \quad :P_{x_j} \quad 3.2$$

Linearization of (3.1) and (3.2) using Taylor series expansion is made about initial position $\hat{x}_j^{(1)}$ and $x_n^{(0)}$ for the junction and new points respectively. The result is a system of linear equations;

$$\tilde{A}_j \tilde{\delta}_j + \tilde{A}_n \tilde{\delta}_n - r_2 + \omega_2 = 0 \quad :P_2 \quad 3.3$$

$$\tilde{A}_n \tilde{\delta}_n + \tilde{A}_j \tilde{\delta}_j - r_x + \omega_x = 0 \quad :P_{x_j} \quad 3.4$$

where,

$$P_x = \begin{bmatrix} P_{x_j} & 0 \\ 0 & P_{x_n} \end{bmatrix} \quad 3.5$$

$$P_{x_j} = C_{\hat{x}_j^{(1)}}^{-1} \quad 3.6$$

$$P_{x_n} = C_{x_n^{(0)}}^{-1} \quad 3.7$$

$$P_2 = C_{\ell_2}^{-1}$$

r_x is the residual vector to the pseudo-observables δ_j and δ_n . $\hat{x}_j^{(1)}$ estimated positions of junction points from the existing solution of S_1 .

The tilde (\sim) has been used in (3.3) and (3.4) to distinguish the design matrices and correction vectors from those of the simultaneous adjustment described in equations (2.9) - (2.15).

3.3 Derivation of Normal Equations

Two approaches of rigorous densification are equivalent if the final solutions are equal. To derive the expressions that give the same results when the Px-adjustment is used as the combined adjustment we shall assume the equality of the solutions. The initial positions in the Px-adjustment are different from those of the combined adjustment which means that the estimated correction vectors in both approaches will be different. We recall the system of normal equations for the combined adjustment given in equation (2.37). This system may be regarded as a sum of two systems of normal equations. Substituting (2.38) - (2.42) into (2.37) we obtain the partitioned form of the normal equations. This system presented as a summation equation is;

$$\begin{bmatrix} N_{jj} & N_{jn} \\ N_{nj} & N_{nn} \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} u_j \\ u_n \end{bmatrix} + \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} u_x \\ 0 \end{bmatrix} = 0 \quad 3.8$$

where, Q_e is given in equation (2.42),

$$\hat{\delta}_j = \hat{\delta}_j^* + \hat{\delta}_j \quad 3.9$$

$\hat{\delta}_j$ is the correction vector of the Px-solution,

$\hat{\delta}_j^*$ is the correction vector from the independent adjustment of S_1 (Appendix I).

Equation (3.9) assumes the equality between the correction vector of the simultaneous adjustment $\hat{\delta}_j$ and the sum of the correction

vectors of the existing and Px-adjustment solutions which is possible when the initial positions $x_j^{(0)}$ in the existing and combined adjustments are the same. On substituting for $\hat{\delta}_j$ from equation (3.9) into the second normal equations system in (3.8) we obtain;

$$\begin{bmatrix} N_{jj} & N_{jn} \\ N_{nj} & N_{nn} \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} u_j \\ u_n \end{bmatrix} + \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\delta}_j^* \\ 0 \end{bmatrix} + \begin{bmatrix} u_x \\ 0 \end{bmatrix} = 0 \quad 3.10$$

$Q_e \hat{\delta}_j^* + u_x = 0$ is the normal equations for the existing network (Appendix I). Therefore, equation (3.10) transforms into;

$$\begin{bmatrix} N_{jj} & N_{jn} \\ N_{nj} & N_{nn} \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} u_j \\ u_n \end{bmatrix} + \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} = 0 \quad 3.11$$

The first system of normal equations in (3.11) corresponds to the mathematical model (3.3) which is linearized about $\begin{pmatrix} x_j^{(0)} \\ x_n^{(0)} \end{pmatrix}$.

The second system corresponds to the mathematical model (3.4) linearized about x_j^1 and when $\omega_x = 0$. We require that both mathematical models be consistent by linearizing them about the existing solution whenever possible. The correction vector $\hat{\delta}_j$ will be changed by a value $\hat{\delta}_j^*$ and transformed to $\hat{\delta}_j$. The design matrices formed using the existing solution will have a tilde (\sim) to distinguish them from those defined at other values. The matrices $N_{nn}, N_{jj}, N_{jn}, N_{nj}, u_j$ and u_n will be transformed to $\tilde{N}_{nn}, \tilde{N}_{jj}, \tilde{N}_{jn}, \tilde{N}_{nj}, \tilde{u}_j$ and \tilde{u}_n respectively. Adopting uniform notation for $\hat{\delta}_n$, too, equation (3.11) transforms into;

$$\begin{bmatrix} \tilde{N}_{jj} + Q_e & \tilde{N}_{jn} \\ \tilde{N}_{nj} & \tilde{N}_{nn} \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} \tilde{u}_j \\ \tilde{u}_n \end{bmatrix} = 0$$

The matrix Q_e is, by definition, the weight matrix of the pseudo-observations, i.e.,

$$P_{X_j} = N_{1j} - N_{je} N_{ee}^{-1} N_{ej} \quad 3.12$$

The normal equations for the P_X -adjustment is therefore given as;

$$\begin{bmatrix} \tilde{N}_{jj} + P_{X_j} & \tilde{N}_{jn} \\ \tilde{N}_{nj} & \tilde{N}_{nn} \end{bmatrix} \begin{bmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} \tilde{u}_j \\ \tilde{u}_n \end{bmatrix} = 0 \quad 3.13$$

The same expression can be derived by minimizing the quadratic norm $(r_2^T P_2 r_2 + r_x^T P_x r_x)$ and differentiating with respect to r_2 and r_x . In so doing, the mathematical models (3.1) and (3.2) must be linearized about the existing solution in which case, since $\omega_x = 0$, it must be assumed that the linearization is made in a differential neighbourhood of the final solution. Only one iteration is contemplated in the adjustment process. Whenever more than one existing solutions are available, only one of them shall be chosen and used to linearize the model. This means that a one-step solution is considered. The other solutions can later be merged rigorously with the densification solution. It is necessary that all solutions to be merged with the rigorous densification solution must first be transformed to the coordinate system of the densification solution. In the unpartitioned form equation (3.13) is;

$$\tilde{N}_2 \tilde{\delta}_2 + \tilde{U}_2 = 0 \quad 3.14$$

where,

$$\tilde{N}_2 = \begin{bmatrix} \tilde{N}_{jj} + P_X & \tilde{N}_{jn} \\ \tilde{N}_{nj} & \tilde{N}_{nn} \end{bmatrix} \quad 3.15$$

$$\tilde{\delta}_2 = \begin{pmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{pmatrix} \quad 3.16$$

$$\tilde{u}_2 = \begin{pmatrix} \tilde{u}_j \\ \tilde{u}_n \end{pmatrix} \quad 3.17$$

The matrix Px_j in equation (3.14) is embedded in the matrix \tilde{N}_2 . The non-null submatrix of Px_j , whenever the constrained points are less than the total number of stations, is of the size of the constrained points. If the new points have a priori information as discussed in section 2.2 then the structure of the Px -matrix is;

$$Px = \begin{bmatrix} Px_j & 0 \\ 0 & Px_n \end{bmatrix} \quad 3.18$$

3.4 The Least-Squares Solution

3.4.1 The correction vectors

The expression of the least-squares solution for the correction vector in both partitioned and unpartitioned forms can be obtained from section 2.4.2 by applying the tilde ($\tilde{}$) where appropriate and setting $Q_e = Px_j$ and $u_x = 0$. The expressions are;

$$\hat{\delta}_2 = -\tilde{N}_2^{-1} \tilde{u}_2 \quad 3.19$$

$$\hat{\delta}_j = -\tilde{H}_{jj} (\tilde{u}_j - \tilde{N}_{jn} \tilde{N}_{nn}^{-1} \tilde{u}_n) \quad 3.20$$

$$\hat{\delta}_n = -\tilde{H}_{nn} [\tilde{u}_n - \tilde{N}_{nj} (\tilde{N}_{jj} + Px_j)^{-1} \tilde{u}_j] \quad 3.21$$

3.4.2 The covariance matrix, $C_{\hat{\delta}_2}$

When the covariance law is applied to equation (3.19) we obtain the covariance matrix $C_{\hat{\delta}_2}$ of the correction vector $\hat{\delta}_2$ as;

$$\hat{C}_{\delta_2} = \tilde{N}_2^{-1} \tilde{A}_2^T P_2 C_{\omega_2} P_2 \tilde{A}_2 \tilde{N}_2^{-1} \quad 3.22$$

where,

$$\tilde{A}_2 = \begin{pmatrix} \tilde{A}_j \\ \tilde{A}_n \end{pmatrix}$$

$$\tilde{N}_2 = \tilde{A}_2^T P_2 \tilde{A}_2 + P_X \quad 3.23$$

and assuming the stochasticity of initial positions (see section 2.5.1);

$$C_{\omega_2} = P_2^{-1} + A_2 P_X^{-1} A_2^T \quad 3.24$$

The covariance matrix C_{ω_2} is obtained in the same way as equation (2.62).

Substituting (3.23) and (3.24) into (3.22) gives;

$$\hat{C}_{\delta_2} = (\tilde{A}_2^T P_2 \tilde{A}_2 + P_X)^{-1} \tilde{A}_2^T P_2 (P_2^{-1} + A_2 P_X^{-1} A_2^T) P_2 \tilde{A}_2 (\tilde{A}_2^T P_2 \tilde{A}_2 + P_X)^{-1} \quad 3.25$$

The identity (2.69) is applied to the inverse of (3.23) to give;

$$(\tilde{A}_2^T P_2 \tilde{A}_2 + P_X)^{-1} \tilde{A}_2^T P_2 = P_X^{-1} \tilde{A}_2^T (P_2^{-1} + A_2 P_X^{-1} A_2^T)^{-1} \quad 3.26$$

similarly;

$$P_2 \tilde{A}_2 (\tilde{A}_2^T P_2 \tilde{A}_2 + P_X)^{-1} = (P_2^{-1} + A_2 P_X^{-1} A_2^T)^{-1} \tilde{A}_2 P_X^{-1} \quad 3.27$$

Substituting (3.26) into (3.25) leads to;

$$\hat{C}_{\delta_2} = P_X^{-1} \tilde{N}_2 (\tilde{N}_2 + P_X)^{-1} \quad 3.28$$

where,

$$\tilde{N}_2 = \tilde{A}_2^T P_2 \tilde{A}_2 \quad 3.29$$

Substituting (3.29) into (3.25) leads to;

$$\hat{C}_{\delta_2} = (\tilde{N}_2 + P_X)^{-1} \tilde{N}_2 P_X^{-1} \quad 3.30$$

The identity of equations (3.28) and (3.30) is easily established by multiplying one by the inverse of the other. The result is of course an identity matrix. Equation (3.28) and (3.30) give the expression for the covariance matrix of the correction vector \hat{C}_{δ_2} when all the points in the network are constrained and weighted by

the inverse of the covariance matrix of initial positions according to the model (3.4). A word of caution is in order. If only some of the points are taken into the auxiliary model (3.4) the normal equation (3.12) will be valid for the constrained points only. If only the junction points, for example, are included in the model (3.4), the P_X^{-1} in equation (3.28) and (3.30) for $C_{\delta_2}^{\sim}$ will be singular. A unique inverse, the Moore-Penrose inverse P_X^+ can be taken instead. For a P_X of the structure;

$$P_X = \begin{bmatrix} P_{X_j} & 0 \\ 0 & 0 \end{bmatrix} \quad 3.31$$

the Moore-Penrose inverse is [Rao and Mitra, 1971];

$$\begin{bmatrix} P_{X_j} & 0 \\ 0 & 0 \end{bmatrix}^+ = \begin{bmatrix} P_{X_j}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad 3.32$$

which is then used in equations (3.28) and (3.30). The matrix P_{X_j} is a submatrix of P_X . Using equation (3.32) instead of P_X^{-1} and (3.31) instead of P_X in these expressions is compatible with the current practice of constraining only those points which have been estimated in the existing adjustment - the junction points. It has been shown in section 2.5.1 that equation (3.28) and hence (3.30) is a difference of two inverses, i.e.,

$$P_X^{-1} \tilde{N}_2 (\tilde{N}_2 + P_X)^{-1} = P_X^{-1} - (\tilde{N}_2 + P_X)^{-1} \quad 3.33$$

Therefore,

$$C_{\delta_2}^{\sim} = P_X^{-1} - (\tilde{N}_2 + P_X)^{-1} \quad 3.34$$

The covariance matrix of the correction vector in a P_X -adjustment of S_2 equals to the difference between the covariance matrices P_X^{-1} and $(\tilde{N}_2 + P_X)^{-1}$. Equation (3.34) can be derived directly from

equation (2.71). The second term in (2.71) equals to zero when $\omega_x = 0$. Equation (2.77) will then be equal to (2.74) and transformed to (3.34) when the tilde (\sim) is introduced.

3.4.3 The covariance matrices $C_{\delta_j}^{\sim}$ and $C_{\delta_n}^{\sim}$

The covariance matrices of the partitioned solution are easily obtainable from equation (3.34). The matrix P_x^{-1} in partitioned form is the inverse of equation (3.18) while the inverse $(\tilde{N}_2 + P_x)^{-1}$ is obtained analogous to (2.44). The partitioned $C_{\delta_2}^{\sim}$ is;

$$\begin{bmatrix} C_{\delta_j \delta_j}^{\sim} & C_{\delta_j \delta_n}^{\sim} \\ C_{\delta_n \delta_j}^{\sim} & C_{\delta_n \delta_n}^{\sim} \end{bmatrix} = \begin{bmatrix} P_{x_j}^{-1} & 0 \\ 0 & P_{x_n}^{-1} \end{bmatrix} - \begin{bmatrix} \tilde{H}_{jj} & \tilde{H}_{jn} \\ \tilde{H}_{nj} & \tilde{H}_{nn} \end{bmatrix} \quad 3.35$$

The expressions for \tilde{H}_{jj} , \tilde{H}_{jn} , \tilde{H}_{nj} and \tilde{H}_{nn} are given in equations (2.17) - (2.19) when $N_{nn} = \tilde{N}_{nn} + P_{x_n}$, $Q_e = P_{x_j}$ and a tilde (\sim) is put on the other matrices involved. In equation (3.35) $C_{\delta_j \delta_j}^{\sim}$ and $C_{\delta_n \delta_n}^{\sim}$ are respectively $C_{\delta_j}^{\sim}$ and $C_{\delta_n}^{\sim}$ which lead to;

$$C_{\delta_j}^{\sim} = P_{x_j}^{-1} - \tilde{H}_{jj} \quad 3.36$$

and

$$C_{\delta_n}^{\sim} = P_{x_n}^{-1} - \tilde{H}_{nn} \quad 3.37$$

The covariance matrices of the partitioned densification solutions are equal to the differences between the respective covariance matrices of the existing positions and the respective submatrices of the inverse of the normal equations matrix of the rigorous densification solution. All matrices in (3.36) and (3.37) are positive definite

matrices. The diagonal elements of the covariance matrices and of the matrices \tilde{H}_{jj} and \tilde{H}_{nn} cannot be less than that of Px_j^{-1} and Px_n^{-1} respectively. The equations therefore make sense and provide an improvement in uncertainty to that of the existing solutions.

Similar to the unpartitioned case the expressions (3.36) and (3.37) can be derived directly from similar expressions of the combined adjustment.

3.4.4 The covariance matrix of estimated positions

The positions of S_2 adjusted using the Px -adjustment are derived in a similar way to those of the simultaneous adjustment, i.e.,

$$\hat{x}_2 = \hat{x}^{(1)} + \hat{\delta}_2 \quad 3.38$$

The initial positions $\hat{x}^{(1)}$ in equation (3.38) are least-squares derived positions with a covariance matrix Px^{-1} . These positions can be expressed in terms of the adjustment of the (existing) independent adjustment as;

$$\hat{x}^{(1)} = x^{(0)} + \hat{\delta}^*$$

which when substituted into (3.38) gives;

$$\hat{x}_2 = x^{(0)} + \hat{\delta}^* + \hat{\delta}_2$$

or using equation (3.9) while omitting the j for the sake of generality we obtain;

$$\hat{x}_2 = x^{(0)} + \hat{\delta}_2 \quad 3.39$$

The initial positions $x^{(0)}$ in (3.39), unlike $\hat{x}^{(1)}$ are constants used to linearize the mathematical models in the existing adjustment. The covariance matrix of adjusted positions \hat{x}_j can, with ease, be derived from (3.39) than from (3.38) since the cross-correlation

$C_{x(0)\delta_2}^{\wedge}$ is known to be zero. Applying the covariance law to (3.39) we obtain;

$$C_{x_2}^{\wedge} = C_{\delta_2}^{\wedge} \quad 3.40$$

The covariance matrix $C_{\delta_2}^{\wedge}$ is the one given by equation (2.78).

For $Q_e = Px$ and $A_2 = \tilde{A}_2$ (since final solutions are equal)

$$C_{x_2}^{\wedge} = (\tilde{N}_2 + Px)^{-1} \quad 3.41$$

The covariance matrices of \hat{x}_j and \hat{x}_n can now be derived by partitioning (3.41). The results are;

$$C_{x_j}^{\wedge} = \tilde{H}_{jj} \quad 3.42$$

$$C_{x_n}^{\wedge} = \tilde{H}_{nn} \quad 3.43$$

Equations (3.42) and (3.43) are respectively the covariance matrices of the junction and new point positions in a Px-adjustment. For all practical purposes, the equality of the final solutions means that the covariance matrices are also equal, which make $\tilde{H}_{jj} \approx H_{jj}$ and $\tilde{H}_{nn} \approx H_{nn}$ in the final iteration of the adjustment process.

3.4.5 The cross-covariance matrix, $C_{x(1)\delta_2}^{\wedge}$

The cross-covariance matrix $C_{x(1)\delta_2}^{\wedge}$ can now be evaluated in a very simple way. If $C_{x_2}^{\wedge}$ was obtained by applying the covariance law to equation (3.38) we would obtain the following expression;

$$C_{x_2}^{\wedge} = C_{x(1)}^{\wedge} + C_{\delta_2}^{\wedge} + 2C_{x(1)\delta_2}^{\wedge} \quad 3.44$$

Substituting equations (3.42) and (3.43) into (3.44) gives;

$$C_{x(1)\delta_2}^{\wedge} = (\tilde{N}_2 + Px)^{-1} - Px^{-1} \quad 3.45$$

or

$$C_{\hat{x}^{(1)}\hat{\delta}_2} = -C_{\hat{\delta}_2} \quad 3.46$$

Equations (3.45) and (3.46) state that the cross-covariance matrix between the existing and the densification solutions equal to the covariance matrix of the correction vector in a Px-adjustment with opposite sign. We have proved that this is infact true by deriving the matrix directly as the expectation $E(\hat{x}^{(1)}\hat{\delta}_2^T)$.

4.0 RIGOROUS DENSIFICATION BY CORRECTING NON-RIGOROUS SOLUTIONS

4.1 Application of Non-Rigorous Densification Schemes

The direct method of rigorous densification have been dealt with in Chapters 2 and 3. This chapter discusses an indirect approach based on correcting non-rigorous densification solutions. It will be seen, from examining applicable non-rigorous schemes, that the indirect approach is computationally more economical when non-rigorous solutions had been already obtained.

Non-rigorous schemes are here understood to be such schemes in which the effect of S_1 is not rigorously propagated into S_2 , whether or not the covariance matrix of initial positions is considered. Such schemes, often used in practice include the over-constrained (i.e., fixed junction points) adjustment and two of the commonly used minimum constraint adjustments - the fixed point adjustment and the free adjustment.

The use of non-rigorous adjustment schemes is especially popular when there is reason to suspect the existence of distortions in the existing network. Such distortions would naturally be propagated into the densification network by the rigorous densification [Chrzanowski and Canellopoulos, 1974; Blaha, 1982a,b]. The conventional wisdom of selecting a suitable point that is

unaffected by distortions to provide the anchor for the densification network is contrary to the rigorous densification adjustment already discussed. Although not wholly attributable to the adjustment scheme [c.f., Thomson et al., 1974], distortions revealed in the North American primary networks [Baker, 1974; McLellan, 1974; and Villasana, 1974] and in subsequent densifications [Fila and Chamberlain, 1978; Lachapelle and Mainville, 1981] indicate that non-rigorous adjustment not only perpetuates but magnifies distortions.

Free network adjustment has useful applications in analyzing the residuals in a preliminary coordinate system [Blaha, 1982a,b]. After the analysis the positions are computed by supplying known coordinates for at least one point and orientation unknowns. The coordinate system chosen often coincides with that of the existing network. The results are therefore the same as results from a fixed-point adjustment up to a translation and rotation of the points of the densification network [Meissl, 1982]. In other words, the difference between solutions of various minimal constraint adjustment schemes can be removed by a translation and rotation of the networks.

Position accuracy estimates of a minimal constraint solution can be improved by a least-squares fit to the existing network (such as a Doppler network). Accuracy improvement of 1-3 ppm in distances have been reported by Moose and Henriksen [1976], Thomson [1976], Burford [1984] and Salih [1984] to this effect. Such fitting really models the transformation parameters needed to transform one coordinate system to the other. The transformation of a network weak

in scale to a stronger network results in scale improvement. The positions of the two networks are then defined in a common coordinate system.

4.2 Mathematical Models and General Assumptions in Non-Rigorous Schemes

The mathematical models for the free and overconstrained non-rigorous adjustment schemes can be written in a form similar to that of the rigorous weighted position constraint adjustment, i.e.,

$$A_j \delta_j + A_n \delta_n - r_2 + \omega_2 = 0 \quad : P_2 \quad 4.1$$

$$\delta_j - r_x + \omega_x = 0 \quad : P_x' \quad 4.2$$

where,

P_x' is the weight matrix of the initial positions used in a non-rigorous adjustment.

Now, however, we must assume that the junction points have either null or an infinite weight matrix P_x' in order to model the non-rigorous schemes. If this was true then its inverse $P_x'^{-1}$, which is the covariance matrix of $\hat{x}^{(1)}$ would be undefined or null respectively. The junction points, having been estimated previously are (c.f., Chapter 5) known to possess a finite covariance matrix $C_{\hat{x}^{(1)}} = P_{x_j}^{-1}$ and therefore a finite weight matrix, P_{x_j} .

In the discussions that follow in the remaining sections of this chapter the weight matrix of the junction points in non-rigorous densification schemes shall be assumed to be non-zero or finite and the weight matrix shall be defined as follows:

Definition

- 1) The weight matrix P_{X_j} in a fixed-point adjustment is one in which the diagonal elements of the fixed point will be considered very large while the other elements equal to zero, i.e.,

$$P_{X_j}' = \begin{bmatrix} \infty & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & \infty \end{bmatrix} \quad 4.3$$

- 2) The weight matrix P_{X_j}' in an overconstrained adjustment (overconstrained points are fixed points) is such that the diagonal elements are all considered to be very large, i.e.,

$$P_{X_j}' = \begin{bmatrix} \infty & & & 0 \\ & \infty & & \\ & & \ddots & \\ 0 & & & \infty \end{bmatrix} \quad 4.4$$

- 3) The weight matrix P_{X_j}' in a free adjustment (i.e., free of constraints) is a null matrix, i.e.,

$$P_{X_j}' = 0 \quad 4.5$$

In all three cases the weight matrix P_{X_n} is finite leading to the normal equations

$$N_{nn} = A_n^T P_n A_n + P_{X_n}$$

The elements of the covariance matrices will therefore be close to zero (very small) but not equal to zero (i.e., $1/\infty \rightarrow 0$). The limiting cases of the elements of the weight matrices (and covariance matrices) are considered. This gives us the possibility to treat all cases of P_X and C_X uniformly and hence formulate the expressions of the non-rigorous schemes in a similar way to rigorous schemes by replacing P_X with P_X' .

4.3 The Least-Squares Solutions

4.3.1 The general expressions

Besides the improper weight matrix Px' the non-rigorous schemes use, like the weighted position constraint adjustment, only the l_2 observations. The two therefore make a much closer comparison the different Px matrices being the only difference. We shall henceforth regard the vector $x^{(0)}$, in non-rigorous schemes to be sufficiently close to $\hat{x}_j^{(1)}$ and the initial positions of the new points $x_n^{(0)}$ to be the same in both cases. In addition, we shall assume a strong network such that a slight change in the junction positions do not significantly affect the design matrices. For all intents and purposes therefore, any difference in design matrices which the tilde (\sim) is meant to exemplify can be ignored. The expressions of the rigorous weighted position constraint adjustment can be thus used without the tilde. As a result the expressions will become non-rigorous when the weight matrices (4.3) - (4.5) are used to replace the weight matrix of the weighted position constraint adjustment.

The difference in weight matrices between rigorous and non-rigorous adjustment schemes obviously does not change the structure of the normal equations (3.15). The expressions of the correction vectors (3.20) and (3.21), their covariance matrices (3.37) and (3.38) and the covariance matrices of the adjusted positions (3.43) and (3.44) remain unchanged. However, their values change. These expressions can be written directly by substituting the non-rigorous Px' -matrix for Px in the respective expressions and we

repeat them here for convenience:

$$\hat{\delta}'_j = -(N_{jj} + P_{X'} - N_{jn} N_{nn}^{-1} N_{nj})^{-1} (u_j - N_{jn} N_{nn}^{-1} u_n) \quad 4.6$$

$$C_{\hat{\delta}'_j} = P_{X'_j}^{-1} - (N_{jj} + P_{X'_j} - N_{jn} N_{nn}^{-1} N_{nj})^{-1} \quad 4.7$$

$$C_{X'_j} = (N_{jj} + P_{X'_j} - N_{jn} N_{nn}^{-1} N_{nj})^{-1} \quad 4.8$$

and,

$$\hat{\delta}'_n = -[N_{nn} - N_{nj} (N_{jj} + P_{X'})^{-1} N_{jn}]^{-1} [u_n - N_{nj} (N_{jj} + P_{X'})^{-1} u_j] \quad 4.9$$

$$C_{\hat{\delta}'_n} = P_{X'_n}^{-1} - [N_{nn} - N_{nj} (N_{jj} + P_{X'})^{-1} N_{jn}]^{-1} \quad 4.10$$

$$C_{X'_n} = [N_{nn} - N_{nj} (N_{jj} + P_{X'})^{-1} N_{jn}]^{-1} \quad 4.11$$

The symbol (') used on the correction vector and covariance matrices is in conformity with the same symbol on the weight matrix to distinguish non-rigorous solutions from the rigorous solution.

4.3.2 The over-constrained solution

The weight matrix $P_{X'}$ for the overconstrained adjustment scheme is defined by the expressions (4.4). The inverse of (4.4) states that each of the diagonal elements of $P_{X'}^{-1}$ is close to zero, which when applied to equations (4.6) - (4.11) gives;

$$\hat{\delta}'_j = 0 \quad 4.12$$

$$C_{\hat{\delta}'_j} = 0 \quad 4.13$$

$$C_{X'_j} = 0 \quad 4.14$$

$$\hat{\delta}'_n = -N_{nn}^{-1} u_n \quad 4.15$$

$$C_{\hat{\delta}'_n} = P_{X'_n}^{-1} - N_{nn}^{-1} \quad 4.16$$

$$\hat{C}_{X_n}^{-1} = N_{nn}^{-1} \quad 4.17$$

These expressions show the obvious, that positions of fixed points in this least-squares adjustment are not estimated. The obtained expression for the correction vector, $\hat{\delta}_n$ is standard [c.f., Mikhail, 1976].

4.3.3 Comparison of overconstrained and rigorous solutions

The Simulations

This comparison between results of a rigorous, P_x-adjustment, and the non-rigorous, overconstrained, adjustment is based on a simulation of triangulation networks shown in Figures 4-1 and 4-2. The networks and observations were simulated following the 1978 Specifications and Recommendations of the Surveys and Mapping Branch of the Department of Energy, Mines and Resources (EMR), Ottawa.

Figure 4-1 is a simulated first-order network. It consists of 14 points, 29 distances, 55 angles, 2 azimuths and 26 unknowns. Ten of the 14 points were designed as junction points in a subsequent densification (Figure 4.3).

The simulation of observations was done in two stages: First, the deterministic observations (error-less) were computed manually one triangle at a time. By assuming some of the observations (distances and angles) the other observations were computed using trigonometric formula. The deterministic observations were run through an adjustment program to check on the computation errors. Non-zero residuals were revealed. The adjusted observations were then taken to be the deterministic observations. Second, the deterministic

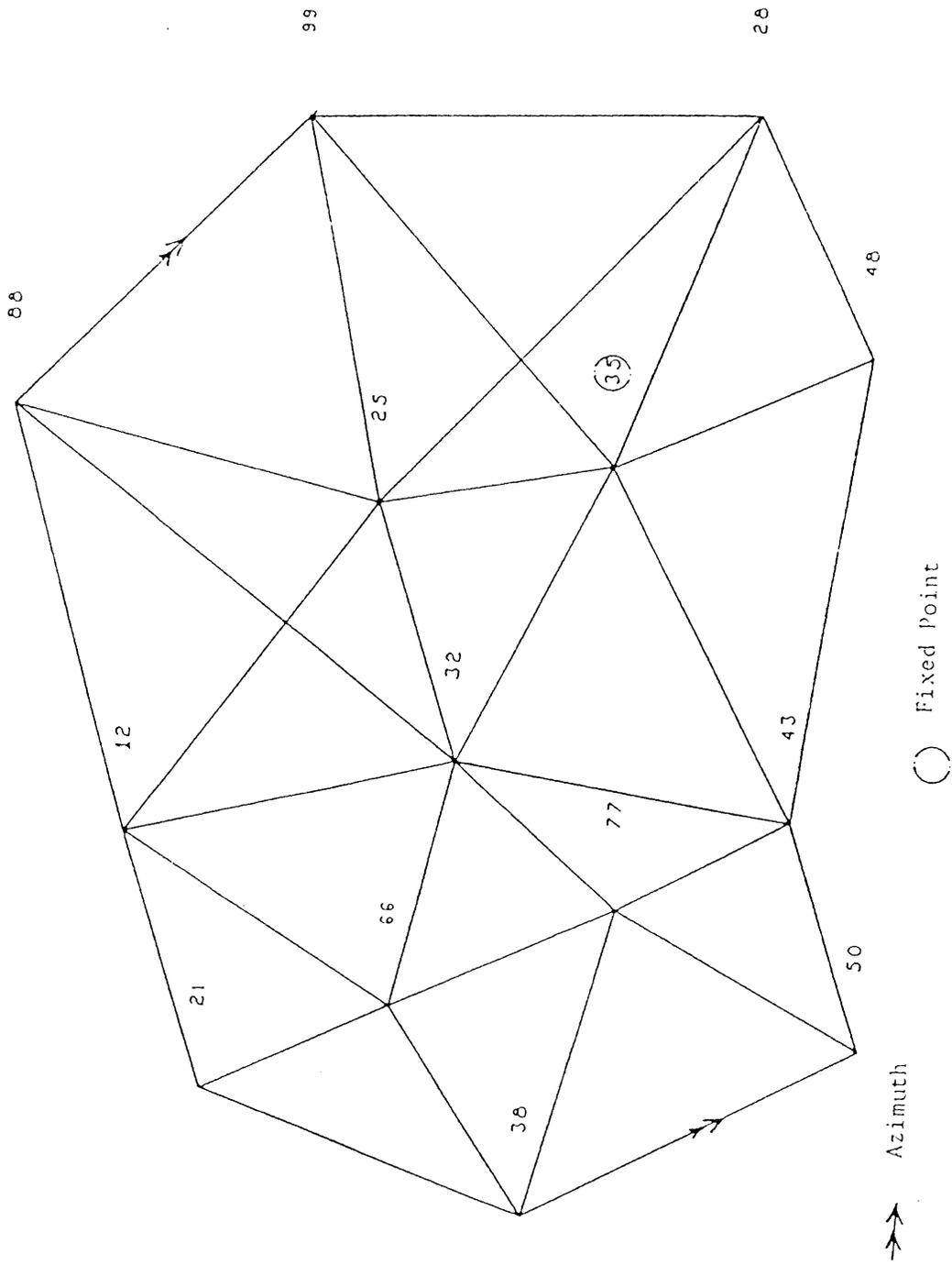


Figure 4-1: The existing Network.

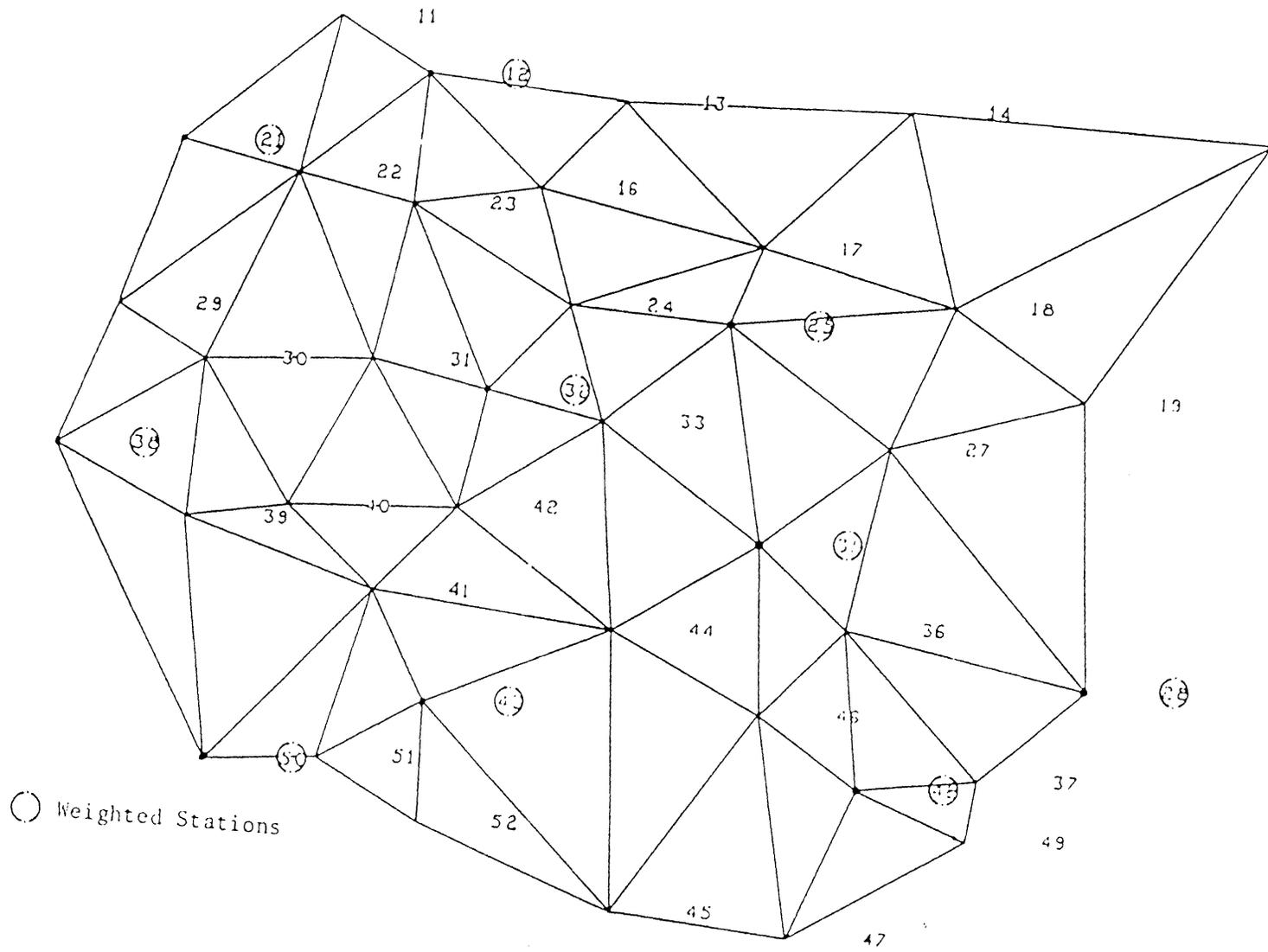


Figure 4-2: The densification Network.

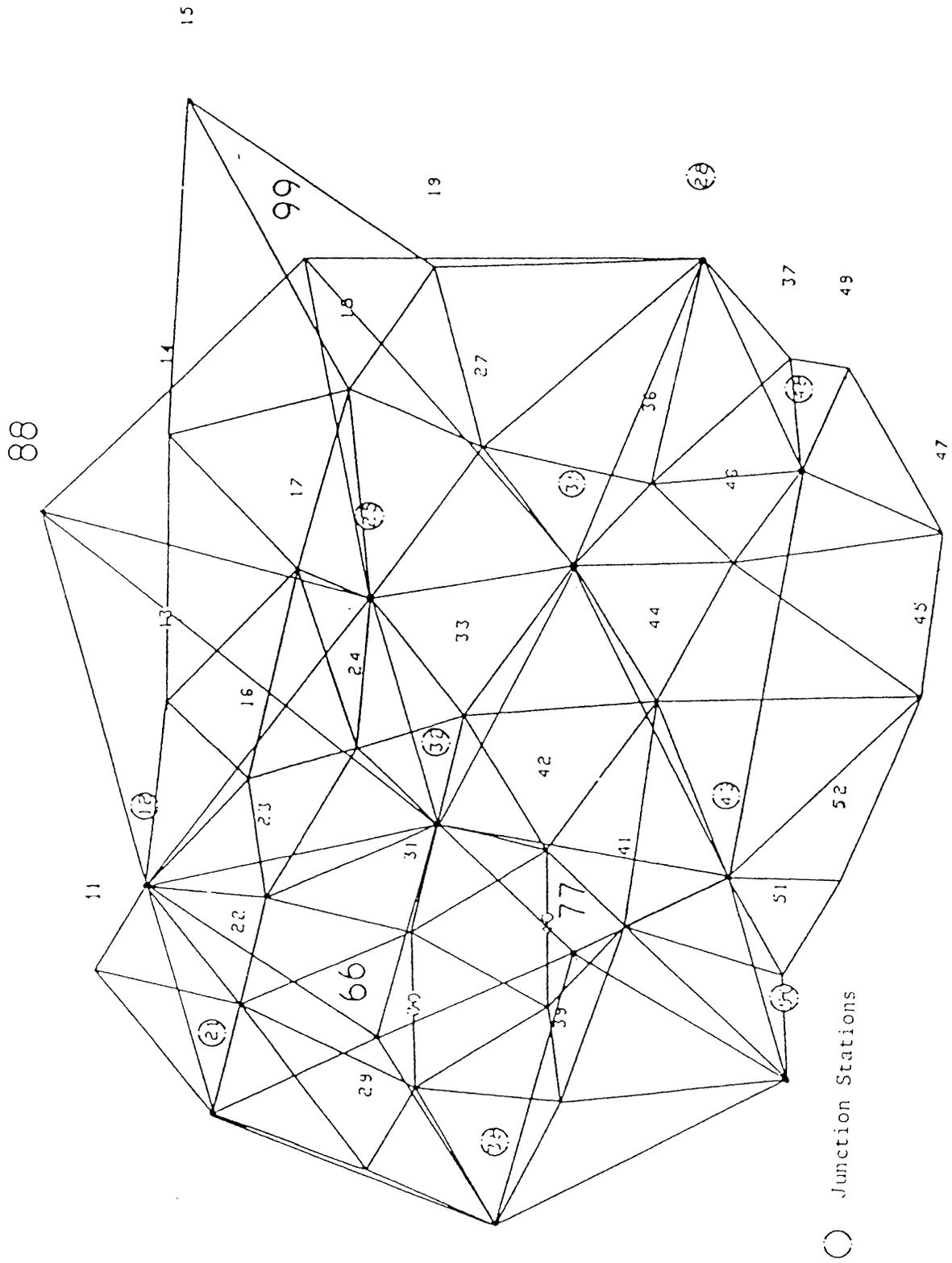


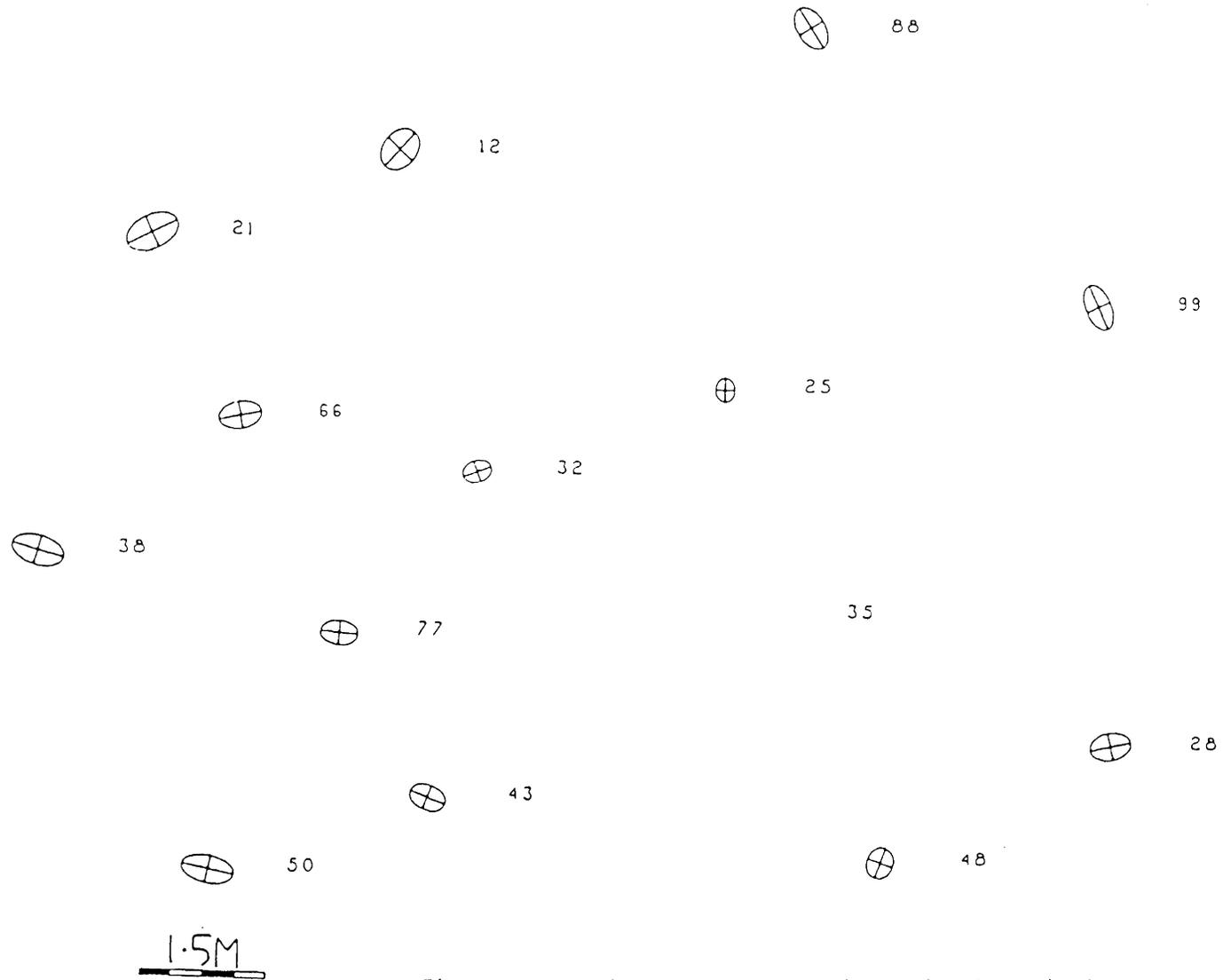
Figure 4-5: The Densified Network.

observations were randomized. A random number generating program RANDOM (Appendix V-5) was used to generate random noise using different variances at each network point. The variances were in conformity with the specifications [EMR, 1978].

Adjustment of the first order network was performed using program GEOPAN [Steeves, 1978]. Station 35 was held fixed in this adjustment. Statistical testing of the adjustment results was performed by the adjustment program. The results passed the tau-max test on the residuals, the χ^2 -test on the variance factor and the χ^2 -goodness of fit test. None of the simulated observations was therefore flagged for rejection by the program. The confidence in the adjusted positions is expressed by the error ellipses (at 95% confidence level) shown in Figure 4-4.

Figure 4-2 shows the densification network. It consists of 39 stations, 86 distances, 177 angles, 1 azimuth, 9 known positions and 78 unknowns. The number of degrees of freedom is 204.

The simulation of observations for Figure 4-2 was performed in a similar way as that of Figure 4-1. The adjustment was also performed using the same program. First, the Px-adjustment was done for the densification network by using the existing solution from the higher order network rigorously as described in Chapter 3. The results were also tested as above and none of the observations were flagged for rejection by the statistical tests. The error ellipses of the adjusted positions of the densification network (at 95% confidence level) are shown in Figure 4-5. Second, the over-constrained adjustment was performed by holding fixed all points which were weighted in the Px-adjustment. The results of the adjust-



67

Figure 4-4: Absolute error ellipses in the existing network at 95% confidence level.



Figure 4-5: Absolute error ellipses in the rigorous densification at 95% confidence level.

ment were also tested statistically. None of the observations was flagged for rejection. The error ellipses of the adjusted positions from this adjustment are shown in Figure 4-6 also at 95% confidence level.

The Comparison

A comparison of the rigorous and the overconstrained adjustments was made by computing position differences, their mean and standard deviations (Table 4.1). Position differences were also plotted as vectorial sums of the coordinate differences (Figure 4-7). The differences between the two solutions range between 1.1 cms and 9.1 cms with a mean of 5.4 cms and standard deviation of 1.87 cms. The differences in adjusted distances do not exceed 5 ppm a requirement that satisfies first order networks [EMR, 1978].

4.3.4 The one-point-fixed solution

We recall the expression (4.3) defining the weight matrix of junction points in a fixed point adjustment, and use it in equations (4.6) - (4.11). We obtain for $\hat{\delta}_j' = \begin{pmatrix} \hat{\delta}_j^f \\ \hat{\delta}_j^{nf} \end{pmatrix}$ and $P_{X_j}' = P_{X_j}^{nf}$;

$$\hat{\delta}_j^f = 0 \quad 4.18$$

$$C_{\hat{\delta}_j^f} = 0 \quad 4.19$$

$$C_{\hat{\delta}_j^{nf}} = 0 \quad 4.20$$

$$\hat{\delta}_j^{nf} = -(N_{jj}' - N_{jn} N_{nn}^{-1} N_{nj}')^{-1} (u_j - N_{jn} N_{nn}^{-1} u_n) \quad 4.21$$

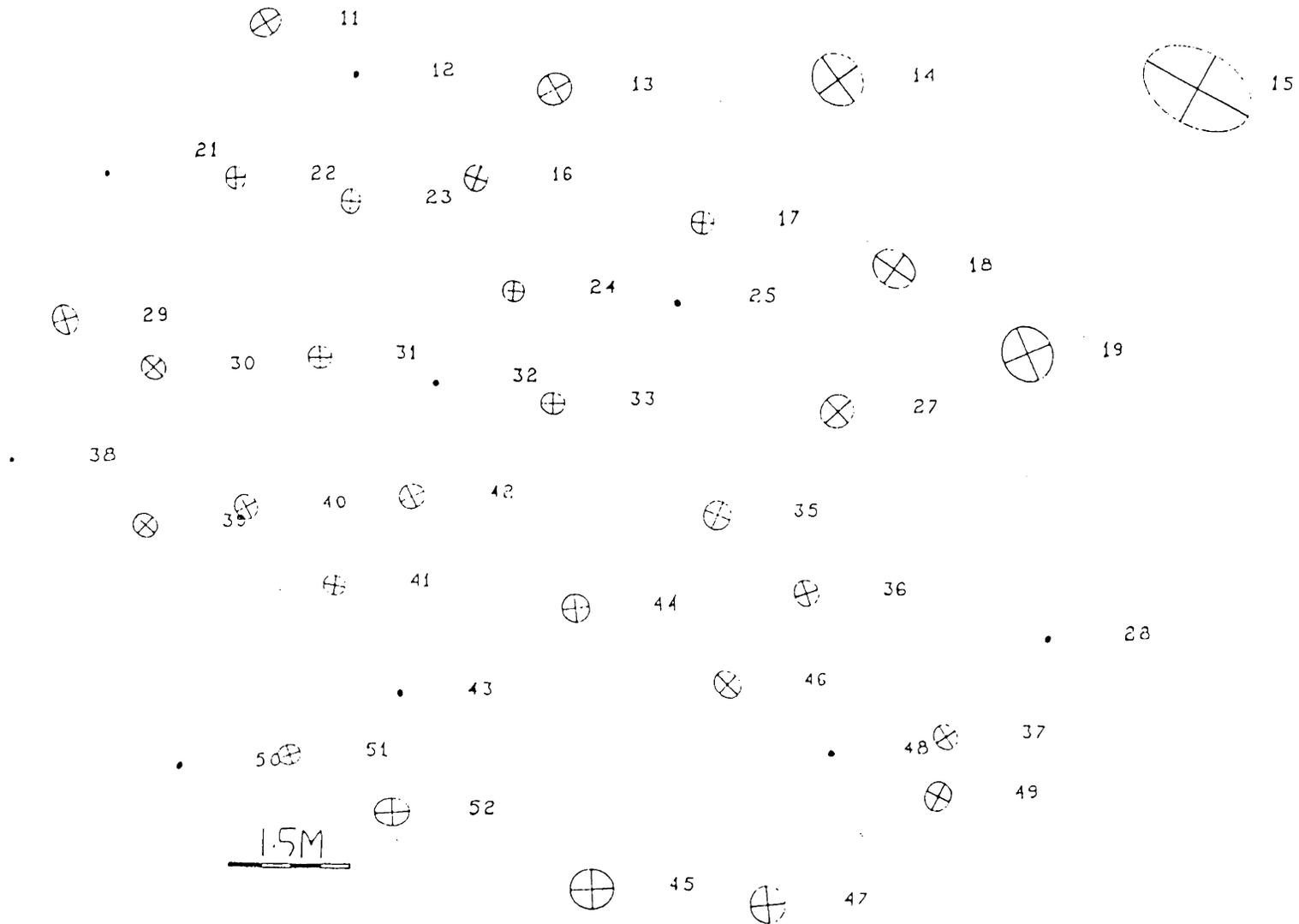


Figure 4-6: Absolute error ellipses in the overconstrained densification at 95% confidence level.

Table 4.1: Position Differences Between the Overconstrained
and the Px-Adjustment Solutions (in cms).

Station	Differences			MEAN d RMS (d)	Station	Differences		
	Δx	Δy	d			Δx	Δy	d
1	2	3	4	5	6	7	8	9
12	0.01	6.62	6.62		24	1.25	3.36	3.59
25	1.70	0.41	1.75		27	2.07	-2.93	3.59
28	0.94	-5.88	5.95		29	-0.91	-7.39	7.45
35	1.41	-2.16	2.58		30	-0.91	6.61	6.67
32	0.80	4.14	4.22		31	0.06	5.46	5.46
38	-2.41	7.07	7.47		33	0.85	2.33	2.48
21	0.53	8.08	8.10		36	1.63	-4.82	5.09
50	-1.28	5.41	5.56		37	2.41	-7.46	7.84
43	-1.14	5.75	5.86		39	1.39	6.26	6.41
48	3.90	-8.24	9.12		40	-0.93	6.00	6.07
11	0.57	6.87	6.89		41	-0.86	5.55	5.62
13	2.72	3.88	4.74		42	-0.21	4.42	4.42
14	4.85	-1.46	5.06		44	-0.14	1.12	1.13
15	5.43	-5.67	7.65		45	-3.04	-1.02	3.21
16	1.56	4.79	5.02		46	0.61	-3.56	3.61
17	2.66	0.43	2.69		47	-0.18	-5.82	5.82
18	3.08	-2.77	4.14		49	2.38	-7.97	8.32
19	2.92	-4.38	5.26		51	-1.57	5.67	5.88
22	0.09	6.59	6.59		52	-2.50	4.50	5.15
23	0.54	5.68	5.71					

MEAN of d = 5.35 cms

RMS of d = 1.87 cms

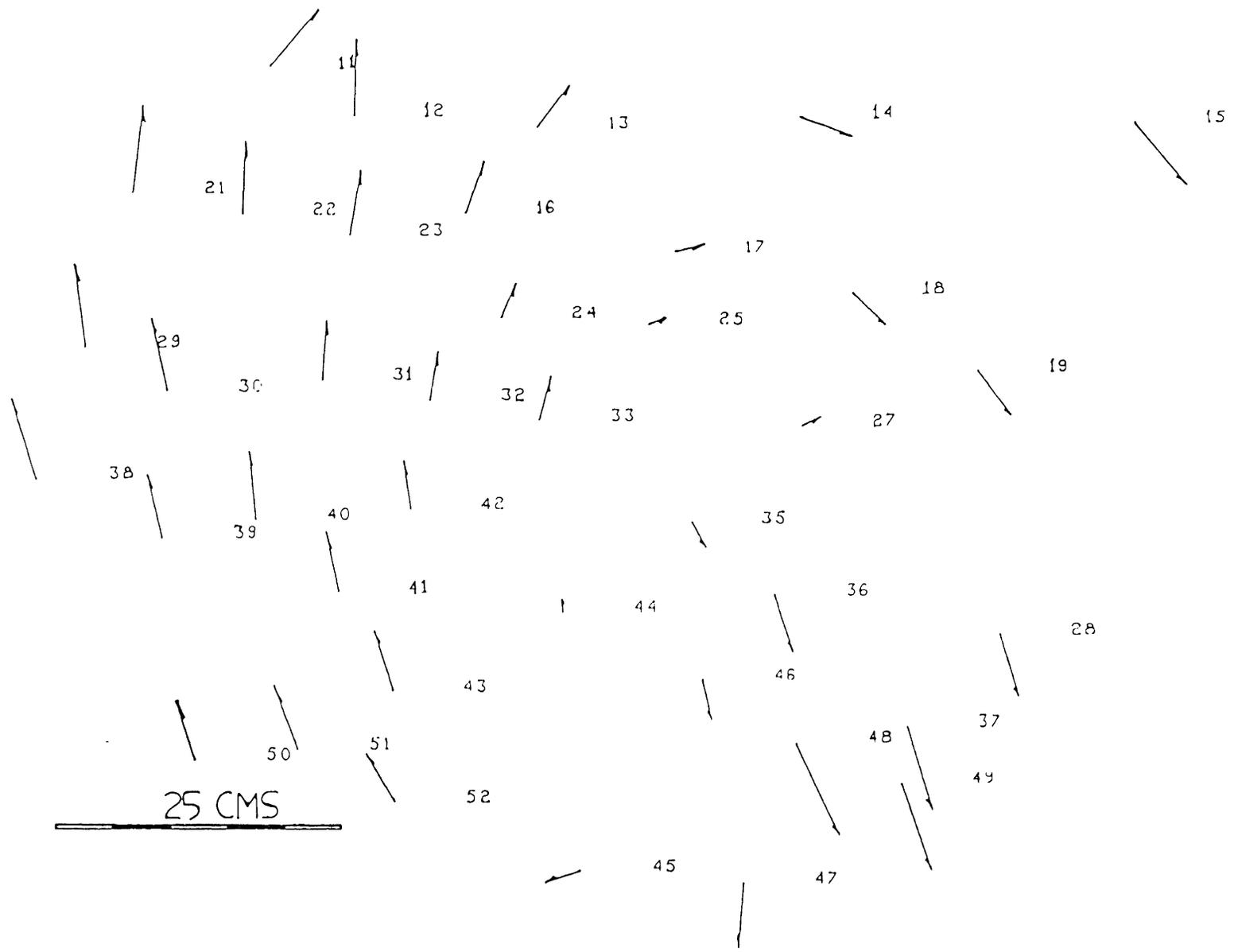


Figure 4-7: Distortions in the overconstrained densification.

$$C_{\hat{\delta}_j}^{nf} = P_{x_j}^{-1} - (N'_{jj} - N_{jn} N_{nn}^{-1} N_{nj})^{-1} \quad 4.22$$

$$\hat{\delta}'_n = - (N_{nn} - N_{nj} N'_{jj}^{-1} N_{jn})^{-1} (u_n - N_{nj} N_{jj}^{-1} u_j) \quad 4.23$$

$$C_{\hat{\delta}'_n} = P_{x_n}^{-1} - (N_{nn} - N_{nj} N'_{jj}^{-1} N_{jn})^{-1} \quad 4.24$$

$$C_{\hat{x}_j}^{nf} = (N'_{jj} - N_{jn} N_{nn}^{-1} N_{nj})^{-1} \quad 4.25$$

$$C_{\hat{x}'_n} = (N_{nn} - N_{nj} N'_{jj}^{-1} N_{jn})^{-1} \quad 4.26$$

The superscripts f and nf respectively stand for fixed and non-fixed points.

As in the overconstrained case the one fixed point is not estimated. Equations (4.23) and (4.24) are again the standard expressions of a fixed point adjustment in a partitioned form [cf., Wells and Krakiwsky, 1971; Meissl, 1982].

4.3.5 Comparison of fixed-point and rigorous results

(a) The fixed point vs rigorous solutions

The same simulations were used as in section 4.3.5. The densification network was first adjusted in a rigorous way, then by holding station 48 fixed at the position given by the first order adjustment.

Figure 4.8 presents the 'absolute' 95% confidence ellipses. The major axis of the furthest points are about 10 times larger than those in the immediate vicinity of station 48. Table 4.2 gives the position and coordinate differences between the fixed-point and rigorous adjustment results. The position differences range between

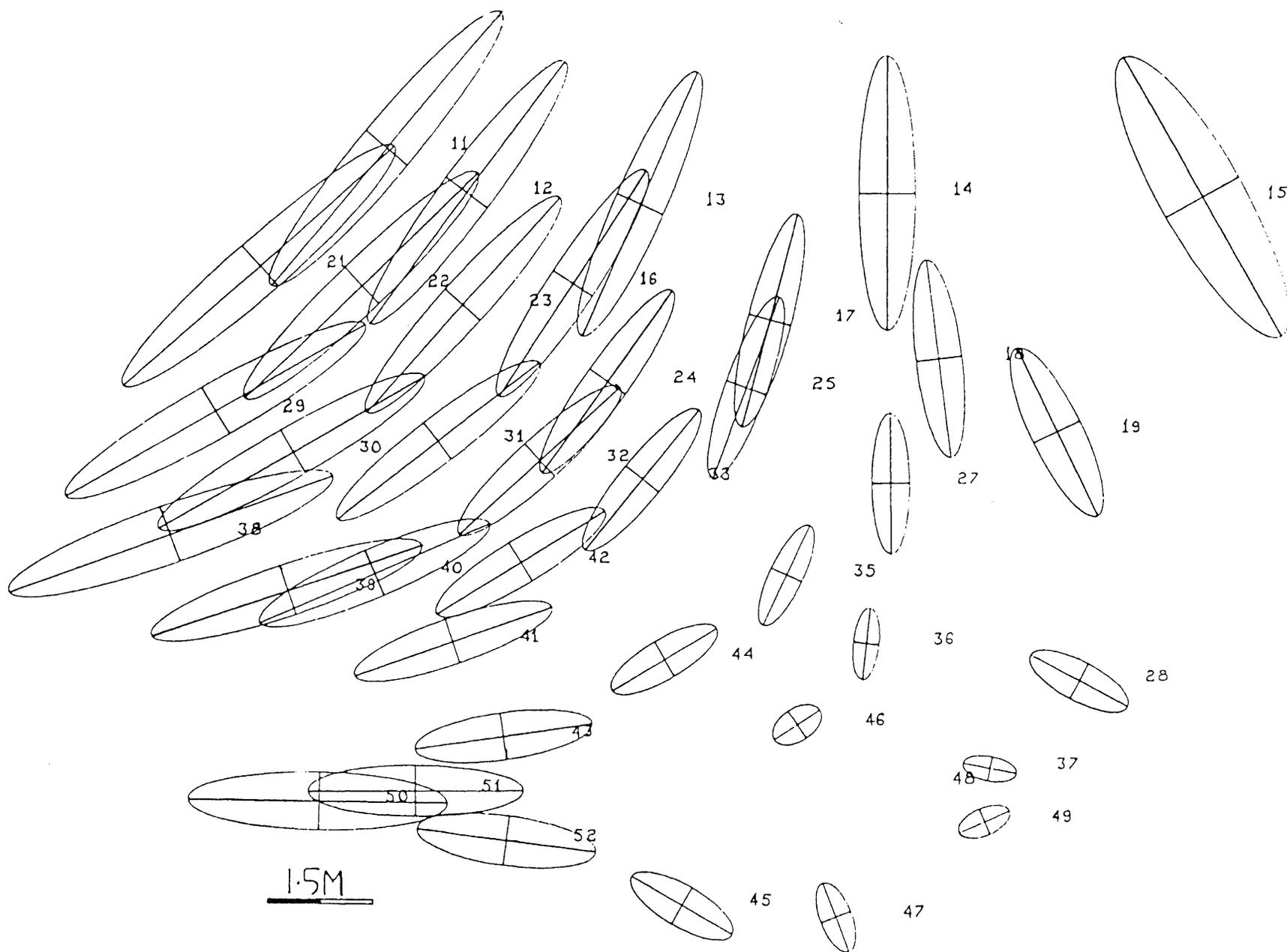


Figure 4-8: Absolute error ellipses in the fixed-point adjustment at 95% confidence level.

Table 4.2: Position Differences between Fixed-Point and Px-
Adjustment Estimates (in cms).

Station	Differences			MEAN d RMS (d)	Station	Differences		
	Δx	Δy	d			Δx	Δy	d
1	2	3	4	5	6	7	8	9
12	-63.2	-76.8	99.5		24	-46.8	-53.6	71.2
25	-45.9	-31.2	55.5		27	-37.9	-10.0	39.2
28	-15.9	26.1	30.6		29	-39.5	-102.5	109.9
35	-25.3	-25.3	35.7		30	-34.6	-92.5	98.8
32	-38.8	-62.5	73.6		31	-38.7	-75.4	84.8
38	-20.1	-107.2	109.1		33	-35.5	-47.8	59.5
21	-58.9	-96.9	113.4		36	-17.1	-12.9	21.4
50	- 1.6	-88.8	88.8		37	1.23	9.25	9.33
43	- 6.1	-66.2	66.5		39	-19.2	-92.5	94.4
48	3.9	- 8.2	9.1		40	-22.3	-82.7	85.7
11	-66.8	-85.8	108.7		41	-16.1	-73.1	74.9
13	-68.8	-51.2	85.7		42	-25.2	-64.4	69.2
14	-74.8	-12.5	75.87		44	-13.6	-43.9	46.0
15	-76.0	31.2	82.2		45	20.8	-40.9	45.9
16	-57.6	-60.6	83.7		46	- 4.61	-23.2	23.6
17	-55.7	-29.1	62.9		47	24.7	-16.9	29.9
18	-53.2	- 4.32	53.4		49	9.89	8.15	12.8
19	-46.5	13.3	48.3		51	- 0.01	-77.8	77.8
22	-54.8	-85.3	101.4		52	8.1	-66.0	66.5
23	-53.2	-73.9	91.1					

MEAN of d = 66.56

RMS of d = 29.94

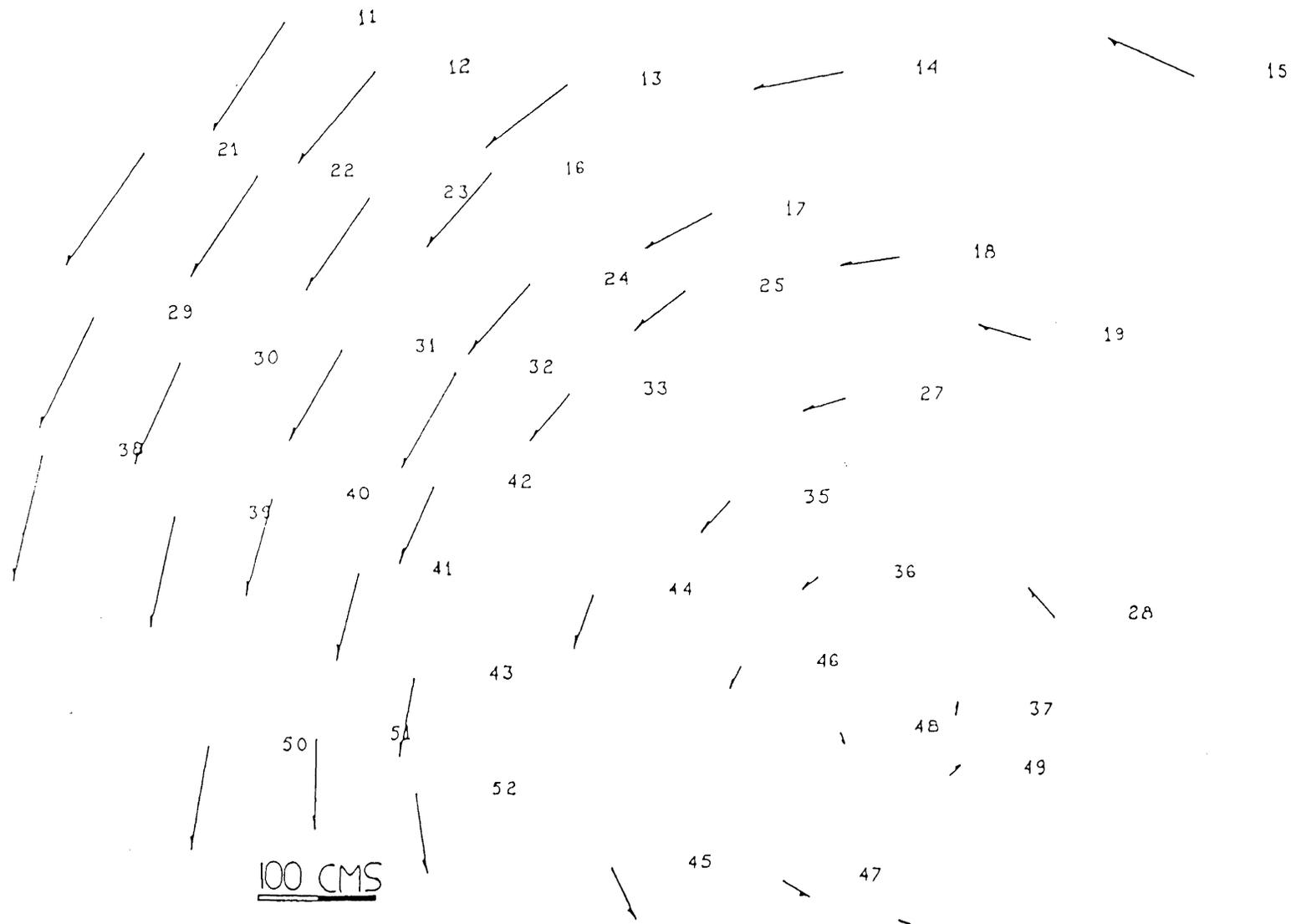


Figure 4-9: Distortions in the fixed-point densification.

9.1 cms and 113.4 cms with a mean of 66.6 cms and a standard deviation of 29.94 cms. The differences plotted in Figure 4-9 are the equivalent of 90 ppm of the mean distances in the network. The difference in adjusted distances is of the order of 1-2 ppm. These results show that the fixed-point adjustment solution is internally consistent. However it is probably translated and rotated with respect to the rigorous solution.

(b) The transformed fixed-point compared to the rigorous solution

A comparison of the transformed fixed-point solution and the rigorous solution was performed by transforming the fixed point results to the coordinate system of the rigorous results and computing the position differences. The parameters to transform the "fixed-point" adjusted coordinates were computed from the two sets of coordinates of the junction points using the programs SMTRA (Appendix V-6). SMTRA applies the least-squares fit to compute the translation parameters, rotation and scale factor. The transformation was performed using program SIMTRA (Appendix V-7).

The coordinate and position differences between the transformed solution and the rigorous solution is given in Table 4.3. The position differences lie between 1.9 cms and 20.7 cms with a mean of 9.4 cms and a standard deviation of 10.53 cms. These position shifts are of the same order as the differences in the adjusted distances of the fixed-point in Figure 4-10. The existence of these differences show therefore that even a minimal constraint solution with transformation cannot replace the rigorous adjustment in network densification.

Table 4.3: Position Differences Between Transformed Fixed-Point Adjustment Solutions (in cms).

Station	Differences			MEAN d RMS (d)	Station	Differences		
	Δx	Δy	d			Δx	Δy	d
1	2	3	4	5	6	7	8	9
12	7.66	-9.75	12.40		24	-0.63	-8.98	9.00
25	1.36	-6.35	6.49		27	-2.23	-7.07	7.41
28	-5.77	-0.41	5.78		29	-3.03	-2.75	4.09
35	-4.22	-8.78	9.74		30	-2.95	-4.15	5.09
32	-4.78	-9.38	10.53		31	-3.34	-7.60	8.30
38	-1.78	-2.77	3.29		33	-2.71	-9.45	9.83
21	1.38	-1.28	1.88		36	-4.56	-8.60	9.73
50	-18.62	-8.99	20.67		37	-2.20	-5.88	6.22
43	-11.48	-11.90	16.53		39	-7.16	-5.30	8.91
48	-3.09	-8.69	9.22		40	-6.64	-7.72	10.18
11	9.10	-6.55	11.19		41	-8.92	-10.25	13.59
13	3.00	-8.76	9.26		42	-5.96	-10.05	11.68
14	2.71	-5.01	5.44		44	-6.02	-11.29	12.79
15	4.90	-6.26	7.95		45	-6.06	-14.07	15.32
16	1.97	-9.74	9.94		46	-4.43	-10.30	11.21
17	1.48	-6.69	6.85		47	-1.74	-12.05	12.18
18	0.94	-6.45	6.52		49	-1.05	-6.82	6.90
19	-1.14	-6.59	6.69		51	-14.30	-11.66	18.45
22	1.46	-4.44	4.67		52	-11.90	-13.45	17.96
23	1.78	-7.78	7.98					

MEAN of d = 9.44 cms

RMS of d = 10.53 cms

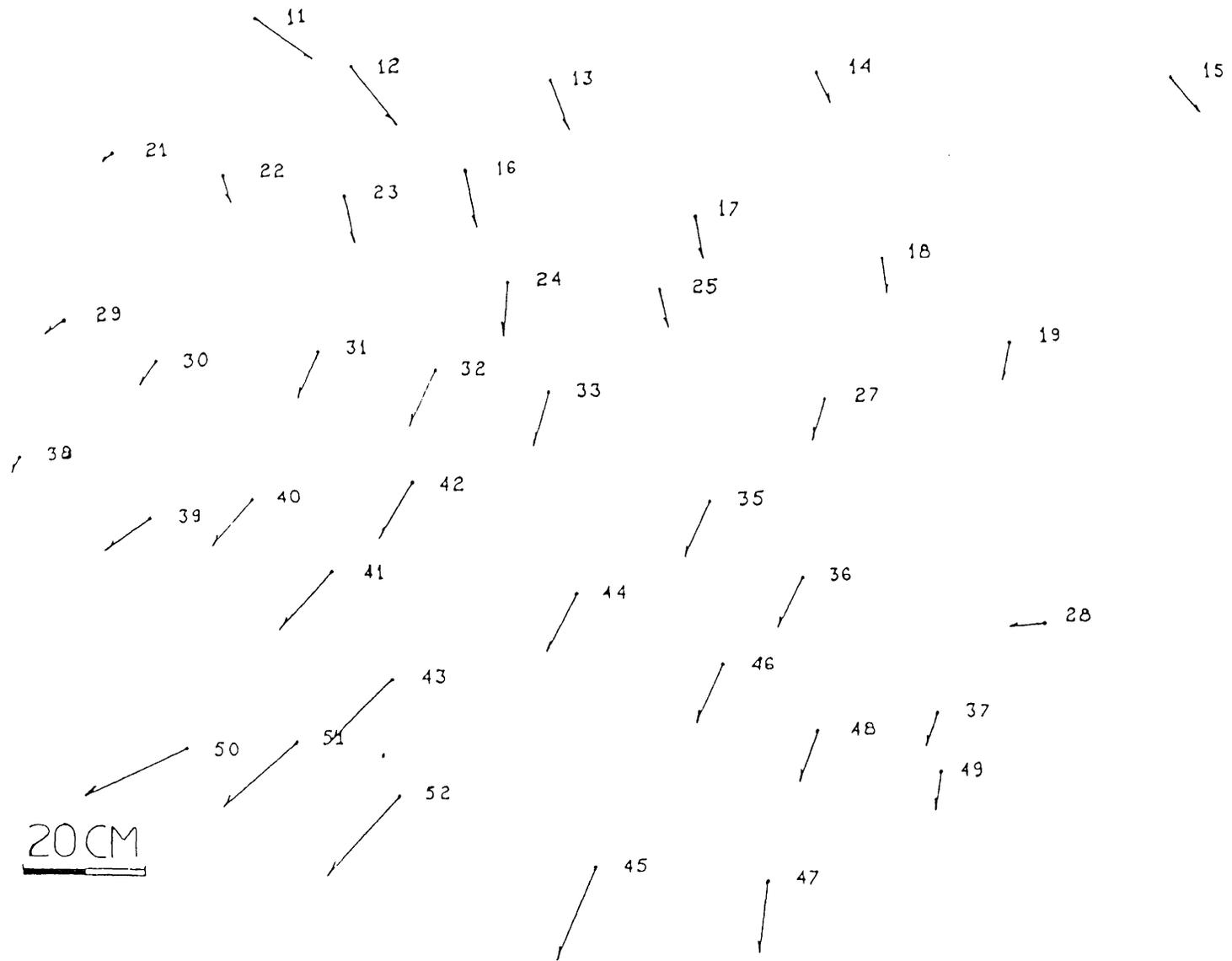


Figure 4-10: Distortions in the fixed-point densification with transformation.

4.4 Correcting the Overconstrained Solution

In the remaining part of this chapter we seek to correct the overconstrained and fixed-point position and error estimates. These estimates are made rigorous by adding a correction vector $\nabla\delta'$ which, in a partitioned form, is equal to;

$$\begin{pmatrix} \nabla\delta'_j \\ \nabla\delta'_n \end{pmatrix} = \begin{pmatrix} \hat{\delta}_j \\ \hat{\delta}_n \end{pmatrix} - \begin{pmatrix} \hat{\delta}'_j \\ \hat{\delta}'_n \end{pmatrix} \quad 4.27$$

All final expressions of corrected vectors and covariance matrices shall, for practicality, be expressed using vectors and matrices obtained from the non-rigorous adjustments.

4.1.1 The junction points

The positions of the junction points are, as seen in section 4.3.1, not estimated in an overconstrained adjustment. Substituting equation (4.12) into (4.27) gives;

$$\nabla\delta'_j = \hat{\delta}_j = -H_{jj}(u_j - N_{jn} N_{nn}^{-1} u_n) \quad 4.28$$

Consequently, the covariance matrix (4.28) is;

$$C_{\nabla\delta'_j} = C_{\hat{\delta}_j} \quad 4.29$$

Equations (4.28) and (4.29) show that a rigorous adjustment of the S_3 subnetwork of S_2 is necessary when an improvement of the overconstrained solution is contemplated.

4.4.2 The new points

The appropriate expressions for $\hat{\delta}_n$ and $\hat{\delta}'_n$ are obtained respectively from equations (3.21) and (4.15). Substituted into

the second equation of (4.27) they give;

$$\nabla \delta'_n = -H_{nn} u_n - H_{nn} N_{nj} N_{jj}^{-1} u_j + N_{nn}^{-1} u_n$$

or

$$\nabla \delta'_n = (N_{nn}^{-1} - H_{nn}) u_n + N_{nn}^{-1} N_{nj} H_{jj} u_j \quad 4.30$$

Lets us introduce the following identity [Liebelt, 1967];

$$(C - B^T A^{-1} B)^{-1} \equiv C^{-1} + C^{-1} B^T (A - B C^{-1} B^T)^{-1} B C^{-1} \quad 4.31$$

When the matrix H_{nn} (equation (2.48)) is expressed using (4.31)

we obtain;

$$H_{nn} = N_{nn}^{-1} + N_{nn}^{-1} N_{nj} H_{jj} N_{jn} N_{nn}^{-1} \quad 4.32$$

Then,

$$N_{nn}^{-1} - H_{nn} = -N_{nn}^{-1} N_{nj} H_{jj} N_{jn} N_{nn}^{-1} \quad 4.32$$

where, H_{jj} is defined in equation (2.47).

Equation (4.32) can be substituted into (4.30) and rearranged to give;

$$\nabla \delta'_n = N_{nn}^{-1} N_{nj} H_{jj} [u_j - N_{jn} N_{nn}^{-1} u_n] \quad 4.33$$

Equation (3.20) is then substituted into (4.33). The result

(neglecting "˜") is;

$$\nabla \delta'_n = -N_{nn}^{-1} N_{nj} \hat{\delta}_j \quad 4.54$$

The rigorous solution for the new points is finally obtained by substituting (4.34) into (4.27) and evaluating $\hat{\delta}_n$ as;

$$\hat{\delta}_n = \hat{\delta}'_n - N_{nn}^{-1} N_{nj} \hat{\delta}_j \quad 4.35$$

then,

$$\hat{x}_n = \hat{x}'_n - N_{nn}^{-1} N_{nj} \hat{\delta}_j \quad 4.36$$

Equation (4.28) and (4.54) can be expressed jointly as;

$$\begin{pmatrix} \nabla \delta_j' \\ \nabla \delta_n' \end{pmatrix} = \begin{pmatrix} I \\ -N_{nn}^{-1} N_{nj} \end{pmatrix} \hat{\delta}_j \quad 4.37$$

The improvement becomes computationally advantageous, using the expression (4.36), when the normal equations inverse N_{nn}^{-1} is preserved. The matrix N_{nj} is assembled while computing $\hat{\delta}_j$.

Using the existing N_{nn}^{-1} instead of formulating and inverting (N_2+Px) as required by the rigorous adjustment saves computer storage and time. The savings can be roughly estimated by comparing the number of multiplications required to obtain $(N_2+Px)^{-1}$ and H_{jj} . If say, the row dimension of H_{jj} is half that of $(N_2+Px)^{-1}$, i.e., 50% of the points in S_2 were overconstrained (in the non-rigorous adjustment) then it is estimated (in a similar way to Appendix II) that only 37.5% of the total number of multiplications required to obtain $(N_2+Px)^{-1}$ will be required to obtain H_{jj} . For a relatively smaller H_{jj} , as often encountered in practice, the savings are higher.

4.4.3 Correcting the covariance matrix, C_{x_n}'

The covariance matrix obtained by correcting the non-rigorous positions must be equal to that obtained in the rigorous adjustment. Consequently, the covariance matrix of the corrected positions can be obtained from equation (3.44) if the final positions are computed as;

$$\hat{x}_n = x_n^{(0)} + (\hat{\delta}_n' + \nabla \delta_n')$$

A practical expression for the corrected covariance matrix must be expressed in terms of the already formed non-rigorous covariance matrix \hat{C}_{x_n} . Equation (3.44) is expressed using (4.32) as;

$$\hat{C}_{x_n} = H_{nn} = N_{nn}^{-1} + N_{nn}^{-1} N_{nj} H_{jj} N_{jn} N_{nn}^{-1} \quad 4.38$$

We recall equation (4.17) which then transforms (4.38) into the form;

$$\hat{C}_{x_n} = \hat{C}_{x_n} + \hat{C}_{x_n} N_{nj} H_{jj} N_{jn} \hat{C}_{x_n} \quad 4.39$$

Equations (4.36) and (4.39) require the rigorous solution of the subnetwork of junction points to be known. Rigorous adjustment of the S_3 subnetwork of S_2 is therefore the first important step towards improving the overconstrained solution. Computationally, the adjustment still constitutes an advantage over a readjustment of the whole network as discussed in section 4.2.

4.5 Correcting the Fixed-Point Solution

In this section the vector required to correct the (non-rigorous) fixed-point solution to a rigorous one is computed as in (4.27). The rigorous expressions are obtained from the weighted position constraint adjustment. The non-rigorous expressions are derived in section 4.5.4.

4.5.1 The junction points

(a) The correction vector

The junction point held fixed in the fixed-point adjustment is not estimated. It follows therefore from section 4.4.1

that this point must be estimated prior to performing the correction of the non-rigorous solution. A network must be triple partitioned as in Appendix IV. Estimation of the correction vector then uses equation IV.40. It is assumed here that the matrix of normal equations is fully populated and is the same as in the rigorous case when the Px-matrix is omitted in both cases, i.e., the difference between the two adjustments lies in the use of the Px-matrix only.

The first equation in (4.27) is recalled and appropriate expressions for $\hat{\delta}_j$ and $\hat{\delta}'_j$ obtained from equations (3.20) and (4.21) to give;

$$\nabla \delta_j = -H_{jj}' u_j + H_{jj}' N_{jn} N_{nn}^{-1} u_n + H_{jj}' u_j - H_{jj}' N_{jn} N_{nn}^{-1} u_n \quad 4.40$$

where,

$$H_{jj}' = (N_{jj}' - N_{jn} N_{nn}^{-1} N_{nj})^{-1}$$

$$N_{jj}' = N_{jj} + P_{Xj}$$

Equation (4.40) can be written as;

$$\nabla \delta_j' = (H_{jj}' - H_{jj}) u_j - (H_{jj}' - H_{jj}) N_{jn} N_{nn}^{-1} u_n \quad 4.41$$

Let us introduce a matrix identity [cf., Mickhail, 1976];

$$(A+B)^{-1} \equiv A^{-1} (A^{-1} + B^{-1})^{-1} B^{-1} \quad 4.42$$

Let us then apply this identity to the terms within the brackets in (4.41). We obtain;

$$(H_{jj}' - H_{jj})^{-1} = -H_{jj}'^{-1} \Delta P_{Xj}^{-1} H_{jj}'^{-1}$$

where,

$$H_{jj}'^{-1} - H_{jj}^{-1} = P_{Xj} - P_{Xj}' = \Delta P_{Xj}$$

the inverse becomes;

$$H'_{jj} - H_{jj} = -H_{jj} \Delta P_{X_j} H'_{jj} \quad 4.43$$

Substituting (4.43) into (4.41) while considering the expression (4.21) for $\hat{\delta}_j^{nf}$ we obtain;

$$\nabla \delta'_j = H_{jj} \Delta P_{X_j} \hat{\delta}_j^{nf} \quad 4.44$$

when $\nabla \delta'_j$ is added to $\hat{\delta}_j^{nf}$ we obtain the rigorous $\hat{\delta}_j$, i.e.,

$$\hat{\delta}_j = (I - H_{jj} \Delta P_{X_j}) \hat{\delta}_j^{nf} \quad 4.45$$

Let us recall again the H_{jj} given in equation (2.47) and express H_{jj}^{-1} in the following form;

$$H_{jj}^{-1} = (N'_{jj} - N_{jn} N_{nn}^{-1} N_{nj}) + P_{X_j}' + \Delta P_{X_j} \quad 4.46$$

The inverse H_{jj} to be used in equation (4.45) is obtained as the RHS of (4.46) which is;

$$H_{jj} = [H_{jj}^{-1} + \Delta P_{X_j}]^{-1} \quad 4.47$$

or (cf., Appendix II),

$$H_{jj} = H'_{jj} - H'_{jj} \Delta P_{X_j} H'_{jj}$$

Equation (4.47) is much more advantageous to be used in expression (4.45) than (2.47) because both matrices under the square bracket already exist from the non-rigorous adjustment. The use of (4.47) makes it possible to avoid compilation of the matrix N_{jn} . This matrix and N_{nn}^{-1} are taken directly from the non-rigorous adjustment.

(b) The estimated positions

The correct positions of the junction points can be obtained by adding $\nabla \delta'_j$ to the non-rigorous position estimates, i.e.,

$$\hat{x}_j = \hat{x}'_j - H_{jj} \Delta P_{X_j} \hat{\delta}_j^{nf} \quad 4.48$$

The matrix H_{jj} being compiled only in a rigorous adjustment can be substituted by an expression in terms of already compiled H'_{jj} matrix of the non-rigorous adjustment, i.e., for

$$H_{jj}^{-1} = H'_{jj}{}^{-1} + \Delta P_{X_j} \quad 4.49$$

where,

$$\Delta P_{X_j} = P_{X_j} - P'_{X_j}$$

we use,

$$H_{jj} = (H'_{jj}{}^{-1} + \Delta P_{X_j})^{-1} \quad 4.50$$

applying the identity (4.42) to (4.50) gives;

$$H_{jj} = H'_{jj} (H'_{jj} + \Delta P_{X_j}{}^{-1})^{-1} \Delta P_{X_j}{}^{-1} \quad 4.51$$

Substituting (4.51) into (4.48) gives;

$$\hat{x}_j = \hat{x}'_j - H'_{jj} (H'_{jj} + \Delta P_{X_j}{}^{-1})^{-1} \hat{\delta}_j^{nf} \quad 4.52$$

$$\text{But, } (H'_{jj} + \Delta P_{X_j}{}^{-1})^{-1} = H'_{jj}{}^{-1} - H'_{jj}{}^{-1} \Delta P_{X_j}{}^{-1} H'_{jj}{}^{-1} \quad 4.53$$

which when substituted in equation (4.52) gives

$$\hat{x}_j = \hat{x}'_j - (I - \Delta P_{X_j}{}^{-1} H'_{jj}{}^{-1}) \hat{\delta}_j^{nf} \quad 4.54$$

Equation (4.54) is convenient to correct the non-rigorous \hat{x}'_j .

Inversion of matrices is not required since $H'_{jj}{}^{-1}$ already exists.

4.5.2 The new points

(a) The correction vector

The vector required to transform the fixed-point solution of the new points to rigorous position estimates is the second equation in (4.27) using proper expressions from the fixed-point adjustment, i.e.,

$$\nabla \delta'_n = \hat{\delta}'_n - \hat{\delta}'_n$$

or

$$\nabla \delta'_n = -(H'_{nn} - H'_{nn})u_n - (H'_{nn} - H'_{nn})N'_{nj}N'^{-1}_{jj}u_j \quad 4.55$$

where, Px'_j is used in the definition of N'_{jj} and H'_{nn}

$$H'_{nn} = N'^{-1}_{nn} + N'^{-1}_{nn}N'_{nj}H'_{jj}N'_{jn}N'^{-1}_{nn} \quad 4.56$$

$$H_{nn} = N'^{-1}_{nn} + N'^{-1}_{nn}N'_{nj}H_{jj}N'_{jn}N'^{-1}_{nn} \quad 4.57$$

the difference between (4.56) and (4.57) becomes;

$$H'_{nn} - H_{nn} = N'^{-1}_{nn}N'_{nj}(H'_{jj} - H_{jj})N'_{jn}N'^{-1}_{nn} \quad 4.58$$

The expression for the difference of the submatrices of the inverse of normal equations for the junction points $H'_{jj} - H_{jj}$ is derived in equation (4.43), which on substitution into (4.58) gives;

$$H'_{nn} - H_{nn} = -N'^{-1}_{nn}N'_{nj}H_{jj}\Delta Px'_jH'_{jj}N'_{jn}N'^{-1}_{nn} \quad 4.59$$

We recall equation (2.50) and write (4.55) as;

$$\nabla \delta'_n = (H'_{nn} - H_{nn})u_n - N'^{-1}_{nn}N'_{nj}(H'_{jj} - H_{jj})u_j \quad 4.60$$

Substituting (4.58) into (4.60) gives;

$$\nabla \delta'_n = N'^{-1}_{nn}N'_{nj}(H'_{jj} - H_{jj})[u_j - N'_{jn}N'^{-1}_{nn}u_n] \quad 4.61$$

The expression for the matrix difference $(H'_{jj} - H_{jj})$ is derived in equation (4.43) which on substituting into (4.61) gives;

$$\nabla \delta'_n = -N'^{-1}_{nn}N'_{nj}H_{jj}\Delta Px'_jH'_{jj}[u_j - N'_{jn}N'^{-1}_{nn}u_n] \quad 4.62$$

but from equation (4.42),

$$(H'_{jj} - H_{jj}) = H'_{jj}\Delta Px'_jH_{jj} \quad 4.63$$

which on substituting into (4.61) gives;

$$\nabla \delta'_n = -N'^{-1}_{nn}N'_{nj}H'_{jj}\Delta Px'_jH_{jj}[u_j - N'_{jn}N'^{-1}_{nn}u_n] \quad 4.64$$

On the other hand, using (4.50) and (4.53) in (4.62) gives;

$$\nabla \delta'_n = N'^{-1}_{nn}N'_{nj}(1 - \Delta Px'_jH'_{jj})\hat{\delta}'_j \quad 4.65$$

Then;

$$\hat{\delta}_n = \hat{\delta}_n^{nf} + N_{nn}^{-1} N_{nj} (I - \Delta P X_j^{-1} H_{jj}'^{-1}) \hat{\delta}_j^{nf} \quad 4.66$$

Equation (4.65) is more suited to correcting the non-rigorous solution due to the availability of all matrices, and the correction vector $\hat{\delta}_j^{nf}$ from the non-rigorous adjustment.

(b) The estimated positions

The corrected positions are obtained by adding (4.65) to the non-rigorous positions \hat{x}_n' , i.e.,

$$\hat{x}_n = \hat{x}_n' + N_{nn}^{-1} N_{nj} (I - \Delta P X_j^{-1} H_{jj}'^{-1}) \hat{\delta}_j^{nf} \quad 4.67$$

Equations (4.54) and (4.67) can be combined into one equation as,

$$\begin{pmatrix} \hat{x}_j \\ \hat{x}_n \end{pmatrix} = \begin{pmatrix} \hat{x}_j' \\ \hat{x}_n' \end{pmatrix} + \begin{pmatrix} -I \\ +N_{nn}^{-1} N_{nj} \end{pmatrix} (I - \Delta P X_j^{-1} H_{jj}'^{-1}) \hat{\delta}_j^{nf} \quad 4.68$$

Equation (4.68) is the correction expression for both junction, and new points positions of the fixed-point adjustment.

4.5.3 Correcting the covariance matrices $C_{x_j}^{\hat{\prime}}$ and $C_{x_n}^{\hat{\prime}}$

Covariance matrices of a corrected solution are evaluated below using a modified form of the expressions (3.43) and (3.44). The modification is necessary so that matrices derived in the non-rigorous adjustment can be used directly. Thus, substituting (4.50) into (3.43) gives;

$$C_{x_j}^{\hat{\prime}} = (H_{jj}'^{-1} + \Delta P X_j)^{-1} \quad 4.69$$

but (cf., equation II.8 of Appendix II),

$$(H_{jj}'^{-1} + \Delta P X_j)^{-1} = H_{jj}' - H_{jj}' \Delta P X_j H_{jj}' \quad 4.70$$

then

$$\hat{C}_{x_j} = H'_{jj} - H'_{jj} \Delta P_{x_j} H_{jj} \quad 4.71$$

On the other hand, substituting (4.50) into (4.57) gives;

$$\hat{C}_{x_n} = N_{nn}^{-1} + N_{nn}^{-1} N_{nj} (H'_{jj} + \Delta P_{x_j})^{-1} N_{jn} N_{nn}^{-1} \quad 4.72$$

Substituting (4.70) into (4.72) we write;

$$\hat{C}_{x_n} = \hat{C}_{x_n} + N_{nn}^{-1} N_{nj} H'_{jj} N_{jn} N_{nn}^{-1} - N_{nn}^{-1} N_{nj} H'_{jj} \Delta P_{x_j} H'_{jj} N_{jn} N_{nn}^{-1} \quad 4.73$$

All the matrices used in (4.73) are, as expected, obtained from the non-rigorous adjustment. Equations (4.71) and (4.73) are respectively the expressions for correcting the covariance matrices of the non-fixed junction points and the new points of the fixed-point adjustment.

5.0 STATISTICAL TESTING OF DENSIFICATION NETWORKS

5.1 Testing Considerations

Statistical tests of geodetic networks are designed as quality control on the observations and estimated parameters. The role of statistical tests in densification networks is broadened by the existence of a second set of positions for the junction points which leads to compatibility testing - a subject that has yet to be appropriately addressed. On the other hand, quality control of observations and estimated parameters is quite well covered in standard literature such as Hamilton [1967], Hogg and Craig [1970], Wells and Krakiwsky [1971], Mikhail [1976], Vanicek and Krakiwsky [1982] and Chen [1983].

Testing can be done on the observations alone or in conjunction with their fit to the formulated mathematical model. This chapter will review the latter which is particularly affected by the strain imposed on the new observations by the auxiliary model (3.4). The fit shall be discussed first in light of a postulated probability distribution function (P.D.F.), ϕ , and second in search for outliers.

Quality control of a network can also be examined in light of its compatibility with an independent determination. Rigorous densification gives the possibility of estimating a second set of

positions for the junction points. The statistical compatibility of the junction point solutions from the existing and densification networks, using an appropriately derived weight matrix of the position differences, shall be addressed.

5.2 Testing the Postulated P.D.F.

As a result of the least-squares adjustment process, a vector of estimated observations $\hat{\ell}$ is derived which is consistent with the mathematical model. The misfit of ℓ to the model is expressed by the vector of estimated residuals, \hat{r} . In rigorous densification the vectors $\hat{\ell}$ and \hat{r} are (see section 3.1);

$$\hat{\ell} = \begin{pmatrix} \hat{\ell}_2 \\ \hat{\ell}_x \end{pmatrix}$$

$$\hat{r} = \begin{pmatrix} \hat{r}_2 \\ \hat{r}_x \end{pmatrix}$$

A multivariate normal P.D.F. for the residuals \hat{r} can be written [cf., Hogg and Craig, 1970] as;

$$f_{\hat{r}} = \frac{1}{T} \exp\left[-\frac{1}{2}(\hat{r}^T \hat{C}_r^{-1} \hat{r})\right] \quad 5.1$$

$$\text{where, } T = (2\pi)^{n/2} (\det \hat{C}_r)^{1/2} \quad 5.2$$

$$\hat{C}_r = \begin{bmatrix} \hat{C}_{r_2} & \hat{C}_{r_2 r_x} \\ \hat{C}_{r_x r_2} & \hat{C}_{r_x} \end{bmatrix} \quad 5.3$$

$$n = \dim(\hat{r})$$

The two matrices $C_{r_2}^{\wedge}$ and C_r^{\wedge} in equation (5.3) are singular which make the P.D.F. (5.1) meaningless. Often the weight matrices P_2 and P are used respectively. The P.D.F. for the adjusted observations is then;

$$\phi_{\hat{r}} = \frac{1}{T} \exp\left[-\frac{1}{2}(\hat{r}^T P \hat{r})\right] \quad 5.4$$

where, $\hat{r}^T P \hat{r} = \hat{r}_2^T P_2 \hat{r}_2 + \hat{r}_x^T P_x \hat{r}_x$, assuming $C_{r_2}^{\wedge} \hat{r}_x = 0$, are the sum of the squares of the weighted residuals \hat{r}_2 and \hat{r}_x . Let

$$\hat{r}_2^T P_2 \hat{r}_2 + \hat{r}_x^T P_x \hat{r}_x = k^2 \quad 5.5$$

Equation (5.5) represents a family of ellipses. Each ellipse represents a confidence region at a prescribed probability level. One ellipse can be specified by specifying a value k^2 for k^2 [Mikhail, 1976]. Since the quadratic sum (5.5) is Chi-square distributed, the value of k^2 can be taken to be the percentile of the χ^2 -distribution function (when σ_0^2 is known) or F-distribution (when σ_0^2 is unknown) at a specified confidence level, $1-\alpha$. Equation (5.5) gives the test on the quadratic form of \hat{r} as;

$$\hat{r}_2^T P_2 \hat{r}_2 + \hat{r}_x^T P_x \hat{r}_x \leq \chi_{n,1-\alpha}^2 \quad 5.6$$

Equation (5.6) when divided by σ_0^2 and the number of degrees of freedom, df, gives;

$$\frac{\hat{\sigma}_0^2}{\sigma_0^2} \leq \frac{1}{\sigma_0^2 \cdot df} \cdot \chi_{n,1-\alpha}^2 \quad 5.7$$

The expression (5.7) is the χ^2 -test on the variance factor [Wells and Krakiwsky, 1971; Neimeier, 1979; Kok, 1977, 1980]. The test is designed to test the correctness of σ_0^2 .

Individual elements of \hat{r}_2 and \hat{r}_x can also be tested (in their standardized forms) as to whether or not they satisfy the

postulated P.D.F. (5.4). Such a test - the Chi-square goodness of fit test is described in Hogg and Craig [1970], Wells and Krakiwsky [1982]. The test statistic is;

$$\frac{\hat{r}_i}{\hat{\sigma}_i} \leq \chi_{n,1-\alpha}^2 \quad 5.8$$

The standard deviation $\hat{\sigma}_i$ of the i-th observation with a residual \hat{r}_i is extracted from the covariance matrix \hat{C}_ℓ .

5.3 Searching for Outliers

Surveyors and geodesists pay great attention to the problem of identifying outliers in the observations when their presence is suspected. If the observations ℓ of the mathematical model $A\delta + \omega = r$ are partitioned into observations with gross-errors ℓ'' and those without gross errors ℓ' , the model itself can also be partitioned as;

$$\begin{bmatrix} A' & 0 \\ A'' & I \end{bmatrix} \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} + \begin{bmatrix} \omega' \\ \omega'' \end{bmatrix} = \begin{bmatrix} r' \\ r'' \end{bmatrix} \quad 5.9$$

where,

$$\begin{aligned} A &= \begin{pmatrix} A' \\ A'' \end{pmatrix} \\ \delta &= \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix} \\ \omega &= \begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} \\ r &= \begin{pmatrix} r' \\ r'' \end{pmatrix} \end{aligned} \quad 5.10$$

Statistical tests in search for outliers seek to reveal the residual vector \hat{r}'' and hence indirectly, the parameter vector $\hat{\delta}''$ affected by the outliers ℓ'' . The null hypothesis is formulated as [Förstner, 1979; Chen, 1983] $H_0: \hat{\delta}'' = 0$. In practice it is assumed that $\ell'' = 0$ during the adjustment. Then to test the presence or absence of outliers it is first hypothesized that \hat{r}'' equals to some "boundary value" $\nabla_0 \ell_i$.

A number of techniques have been devised to test the above hypothesis by assessing each element of the vector \hat{r} against a statistic formulated for each technique. The widely acclaimed testing techniques are data snooping [Baarda, 1968], τ -test [Pope, 1976] and t-test [Heck, 1981]. A comprehensive review of these techniques is given in Van Mierlo [1981], Kavouras [1982] and Chen [1983]. The last author's "generalized method" derives a general statistic of which all the above are special cases.

The performance of the above tests for outliers depends on the geometrical strength of the network as characterized by the redundancy numbers, q_{ii} [Baarda, 1968; Förstner, 1979; Ackerman, 1981]. The average redundancy (average value of the redundancy numbers) in a particular network type is fairly constant at 0.5 for triangulation and 0.33 for levelling [Pope, 1976]. This means that the marginally detectable gross-errors are also likely to be constant for a given network type. Using, for example, data snooping marginally detectable gross-errors of $6.2\sigma_{\ell}$ at $\beta_0 = 92\%$ [Ackerman, 1981] and $4.2\sigma_{\ell}$ at $\beta_0 = 80\%$ [Kavouras, 1982] are reported for triangulation. The observations containing gross-

errors smaller than the quoted values are not regarded, by these techniques, to be outliers. In fact much larger errors than the marginal values quoted above are not detected by the above techniques, as is explained below.

5.4 Redundancy Numbers and Network Distortions

The residuals \hat{r} upon which the statistical tests are based constitute only one component of the observation errors $\Delta\ell$ [Pope, 1976]. The other component \hat{r}' is left unexamined. It is possible to write for one observation error $\Delta\ell_i$ that [Förstner, 1979; Kavouras, 1982];

$$\Delta\ell_i = \hat{r}_i + \hat{r}'_i \quad 5.11$$

where,

$$\hat{r}_i = q_{ii} \Delta\ell_i \quad 5.12$$

$$\hat{r}'_i = m_{ii} \Delta\ell_i = (1 - q_{ii}) \Delta\ell_i \quad 5.13$$

It is clear from equations (5.12) that when $q_{ii}=0$ any error $\Delta\ell_i$ will not be transformed into residual and cannot be detected. In general, outliers are more difficult to unveil when q_{ii} is small.

Example; A point C in Figure 5.1 is fixed from two known points A and B by observing the angles β_1 and β_2 .

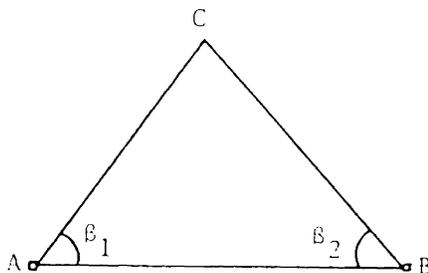


Figure 5.1: Intersection of Point C.

In this problem $C_r^{\wedge} = 0$. Therefore q_{ii} and hence $q_{ii} \Delta \ell_i = 0$. Any errors in β_1 and β_2 cannot be evaluated. Such errors will affect the position of C.

Definitions;

The redundancy numbers q_{ii} are by definition the diagonal elements of the matrix product $C_r^{\wedge} P$ [Baarda, 1968; Förstner, 1979], i.e.,

$$q_{ii} := (C_r^{\wedge} P)_{ii} \quad 5.14$$

The elements m_{ii} in equation (5.13) are the diagonal elements of a matrix M. The expression of the matrix M can now be developed. We recall the expression of the covariance matrix of the residuals C_r^{\wedge} to be [Vanicek and Krakiwsky, 1982];

$$C_r^{\wedge} = C_{\ell} - AC_x^{\wedge} A^T$$

where,

C_x^{\wedge} is the covariance matrix of adjusted positions

$$C_{\ell} = P^{-1}$$

then,

$$C_r^{\wedge} P = I - AC_x^{\wedge} A^T P \quad 5.15$$

The expression for C_x^{\wedge} given in equation (5.41) is substituted into (5.15) to give (for $N=N$);

$$C_r^{\wedge} P = I - M \quad 5.16$$

where (see also Appendix III),

$$M = A(N+Px)^{-1} A^T P \quad 5.17$$

Considering one element in (5.16) we obtain;

$$q_{ii} = 1 - m_{ii} \quad 5.18$$

which when multiplied by $\Delta \ell_i$ on both sides gives equation (5.11).

The elements m_{ii} are therefore the diagonal elements of the matrix expressed in equation (5.17).

Gross-Errors in an Adjusted Network

We now consider a gross-error $\nabla\ell_i$ in the i -th observation. In the adjustment process $\nabla\ell_i$ is decomposed into $\bar{r}_i = q_{ii}\nabla\ell_i$, the effect of the gross-error on the i -th residual and $\bar{r}'_i = m_{ii}\nabla\ell_i$, the effect of $\nabla\ell_i$ on the adjusted observation $\hat{\ell}_i$. Then similar to (5.11);

$$\nabla\ell_i = q_{ii}\nabla\ell_i + m_{ii}\nabla\ell_i \quad 5.19$$

If a given test for outliers passes the whole $\nabla\ell_i$, and not only $q_{ii}\nabla\ell_i$, is disregarded by such a test. Judging the sensitivity of a statistical test on the basis of \hat{r}_i is therefore misleading. Equation (5.19) says, in fact, that $\hat{r}_i < \nabla\ell_i$ [cf., De Heus, 1982].

A word on the search for outliers in the rigorous vis-a-vis non-rigorous adjustment is in order. The auxilliary model (3.4) introduces additional information in the form of pseudo-observations, ℓ_x , which is absent in the models for the non-rigorous densification (Chapter 4). Addition of pseudo-observations increases the number of degrees of freedom and improves the reliability of the network. Statistically speaking it becomes easier for a test to reject outlying observations. The use of Doppler points as weighted position constraints in an adjustment of a part of the Maritime Primary network by Thomson [1976] increased the number of rejected outliers from two to six. Similar results are reported by Dracup [1975].

The effect of $\nabla\ell_i$ on the adjusted observations ($m_{ii}\nabla\ell_i$) is transformed into position errors [Förstner, 1979, 1981; Ackerman, 1980, 1981; Van Mierlo, 1981; De Heus, 1982]. Therefore, if we

are to assess the effect of the gross-error ∇l_i on the adjustment results, it is necessary to assess the effect of $m_{ii} \nabla l_i$ as well. If the affected positions are also known from an independent determination, a test for statistical compatibility becomes the next logical step. Junction points of the densification network provide a possibility for such a test to be made.

5.5 Compatibility Testing

5.5.1 The test statistic

To assess whether or not the densification solution $(\hat{x}_2, C_{x_2}^{\wedge})$ is statistically compatible with the existing solution $(\hat{x}_1, C_{x_1}^{\wedge})$ we hypothesize on the position differences, Δx . The null hypothesis for this test sets the position differences to zero, i.e., $H_0: \Delta x = 0$.

In testing the hypothesis, we shall characterize the uncertainty in positions through a probability, α . If the P.D.F. in each determination is a multivariate normal function, the function $\phi_{\Delta x}$ will also be multivariate normal [Hamilton, 1964]. The function (5.4) is recalled in which (\hat{r}, P) is replaced by $(\Delta x, C_{\Delta x}^{-1})$. The result is;

$$\phi_{\Delta x} = \frac{1}{T} \exp \left(- \frac{1}{2} \Delta x^T C_{\Delta x}^{-1} \Delta x \right) \quad 5.20$$

where,

$$T = (2\pi)^{u/2} \det (C_{\Delta x})^{1/2}$$

$$u = \dim (\Delta x)$$

$C_{\Delta x}$ - the covariance matrix of position differences.

The probability statement for the quadratic sum, $\Delta x^T C_{\Delta x}^{-1} \Delta x$ at

1- α confidence level is;

$$\Pr(\Delta x^T C_{\Delta x}^{-1} \Delta x \leq K^2) = 1-\alpha \quad 5.21$$

The term in brackets defines a confidence region at 1- α confidence level for the quadratic sum. The test of the quadratic sum is made analogous to (5.6) as;

$$\Delta x^T C_{\Delta x}^{-1} \Delta x \leq K^2 \quad 5.22$$

K^2 is the percentile of the χ^2 -distribution at 1- α confidence level and ν , the degrees of freedom.

5.5.2 The covariance matrix, $C_{\Delta x}$

The mathematical model for rigorous densification by weighted position constraints have been formulated in a differential neighbourhood of the existing solution \hat{x}_1 . As such, the vector of position differences for the junction points equals to the correction vector of the same points, $\hat{\delta}_j$, i.e.,

$$\Delta x = \hat{\delta}_j \quad 5.23$$

Consequently,

$$C_{\Delta x} = C_{\hat{\delta}_j} \quad 5.24$$

The expression for $C_{\hat{\delta}_j}$ is given in equation (3.45). We note that $P_{x_j}^{-1} = C_{x_1}^{\wedge}$ and $H_{jj} = C_{x_2}^{\wedge}$. Equation (5.24) can then be expressed as;

$$C_{\Delta x} = C_{x_1}^{\wedge} - C_{x_2}^{\wedge} \quad 5.25$$

Equation (5.25) was derived by Steeves [1983] by considering Δx as the residual vector of the pseudo-observations $\hat{\delta}_j$ in a weighted position constraint adjustment. The equation is in agreement with that of Blaha [1976] and Grafarend et al., [1983]. This equation

underscores the fact that the rigorous densification solution is an improvement over the existing solution, i.e., the least-squares norm of $C_{\Delta x}$ satisfies the inequality;

$$|| C_{\Delta x} || \geq 0 . \quad 5.26$$

The matrix $C_{\Delta x}$ is non-negative definite.

5.5.3 The outcome of the compatibility test

The densification network is statistically compatible with the existing network at $1-\alpha$ confidence level when the inequality (5.22) is satisfied and incompatible otherwise. The result reflects the effect of the gross-errors on the densification solution. This statement is somewhat misleading however, because the effect of gross-errors in the existing network on the junction points is also assessed by the same test.

A statement of the type given in (5.19) for more than one observation must consider the correlation imposed on the residual vector by the mathematical model. Forstner [1979] orthogonalized one of the two terms with respect to the other. In either case, the correlations make it impossible to pinpoint the offensive observations. One of the ways one can get to individual outliers is through compatibility testing of subvectors of Δx or even individual elements of Δx whenever possible.

5.5.4 Compatibility testing for subvectors of Δx

Generally, the probability of any member Δx_j of Δx to be in a given confidence region is higher than that of all the

m-members simultaneously. The probability statement for a sub-vector Δx_j of Δx is;

$$\Pr[\Delta x_j^T C_{\Delta x_j}^{-1} \Delta x_j \leq K^2] > 1-\alpha \quad 5.29$$

The inequality (5.27) expands the individual confidence region of the subvector Δx_j to keep the test of each subvector in-context of that of Δx . The probability statement (5.21) can be used for the subvectors Δx_j if the significance level of the test α is changed, i.e.,

$$\Pr[\Delta x_j^T C_{\Delta x_j}^{-1} \Delta x_j \leq K^2] = 1-\alpha' \quad 5.28$$

The significance level α' is suggested to be equal to [Thompson, 1938; Pope, 1976];

$$\alpha' = \sum_{j=1}^m \alpha \quad 5.29$$

Equation (5.29) restricts the significance level α' . In turn it restricts the number of subvectors Δx_j to be tested in context of Δx . Such a restriction does not exist when Δx_j is tested out-of-context of Δx .

5.6 Simulation Study

Simulation studies were conducted to test the statistical compatibility of a densification network with the existing network when both networks had been found statistically acceptable as solitary networks. The same observations (excluding distances) were simulated for the networks given in Figures 4.1 and 4.2 as described in section 4.3.3. The adjustment and statistical testing of individual networks was performed using program GEOPAN [Steeves, 1978]. The tests included the χ^2 goodness of fit test, the χ^2 -test

on the variance and the tau-max test (the routines of which are built in the program) all of which passed.

The adjustment and testing of the densification network was repeated after one angle at each of the stations 22, 24, 27, 35, 38, 42, 45, 50 and 52 were burdened with additional errors of $2.5\sigma_i$. As in the first case, the network passed the prescribed tests. Clearly, the additional $2.5\sigma_i$ errors in the nine selected observations were statistically acceptable to the testing techniques used, i.e., gross-errors were regarded as non-existent.

The study proceeded to test the statistical compatibility of the densification and existing solutions as described in section 5.5. First, the original densification solution (without the $2.5\sigma_i$ errors) was tested using program CTEST (Appendix V) at 95% confidence level. Second, the solution with the additional observation errors was tested, also using the same program and level of confidence as in the first case.

The first test passed for all the junction points together and all the individual points respectively. The second test passed when all junction points were tested simultaneously. However, an out-of-context test of individual points of the whole junction vector when additional errors were simulated, failed the test on 40% of all junction points. The points which failed the test were directly connected to the points at which the $2.5\sigma_i$ error was injected. Unlike the tests in the solitary networks therefore, the out-of-context compatibility test was sensitive to the additional errors as expected from the discussion in section 5.3. The $2.5\sigma_i$ errors are

regarded by this test to be gross-errors and the observations in which the errors were injected are regarded as outliers.

An additional test was performed to check the validity of the results given in the previous paragraph. This test was to assess the statistical compatibility of the two densification solutions and show whether or not the same conclusions as above could be reached. The networks were found to be compatible when all the 59 points were tested simultaneously. Testing subvectors of 13 and 5 points out-of-context of the 39 point vector showed that some of the subvectors were not compatible. If more than 20% of the subnetwork consisted of, or were connected to, the points burdened with the $2.5\sigma_i$ errors then such a subvector failed the test.

It is interesting to note that the position differences in the densification network caused by the $2.5\sigma_i$ errors in the nine observations were equivalent to 100 ppm of adjusted distances. The maximum anticipated error in adjusted distances accepted in the second order network by the Surveys and Mapping Branch, Ottawa, is 50 ppm [EMR, 1978]. In the above simulations the testing of the solitary networks proved to have no power towards achieving the acceptable quality for second order network. Compatibility testing has not only questioned the quality of the network, it has localized the source of gross-errors to the subnetwork level.

It must also be pointed out that network distortions are best characterized by the amount of deformation experienced by the network rather than position errors. Such deformation can be presented in the form of strain [Thapa, 1980; Vanicek et al., 1981;

Vanicek and Krakiwsky, 1982]. The next logical analysis is to perform a strain analysis of the subnetworks which fail the compatibility test. This subject is addressed in the next chapter.

6.0 APPLICATION OF STRAIN FOR THE DETECTION OF GROSS-ERRORS IN DENSIFICATION NETWORKS

In this chapter we shall describe analytically, the displacement and strain fields in densification networks. The strain field which shall be computed in a series of simulation studies and presented by various strain patterns (strain ellipses and rotation arcs) is that of inconsistencies in observations. We shall proceed to formulate the inverse strain analysis problem. Solution of the inverse problem which includes the computation and interpretation of inconsistencies from given strain shall not be attempted.

6.1 The Feasibility of the Novel Strain Analysis Technique

From the time it was introduced into geodesy, about 55 years ago, the strain analysis technique has been mostly applied in connection with deformation and geodynamic problems. Strain accumulation in a physically deformed part of the earth can be evaluated from geodetic observations procured at different epochs. It can also be evaluated from position differences of displaced geodetic monuments. In both approaches a physical motion is quantified, resolved into components (parallel to given coordinate axes) and finally transformed into strain. All methods of transforming discrete position displacement field into continuous strain field

and ultimate computation of meaningful strain parameters (such as total strain, shear, rotation, dilatation) are described in Pope [1966], Schneider [1982] and Chen [1983].

Recently, strain analysis of geodetic networks (in the absence of physical motion) has been successfully attempted by Thapa [1981] and Dare [1982]. Both authors generated displacement fields from assumed inconsistencies in the observations. An inconsistency in an observation linking two points directly affect the two points. In these attempts, the strain field is regarded to be continuous within local bounds of the affected stations.

The type of strain to be expected in a network can be predicted from the types of observations in the network. Empirical investigations by Dare and Vanicek [1982] have shown, for example, that rotations are to be expected when azimuths are observed in a network. Total strain and shear are sensitive respectively to distance and angle observations. These analysis are substantiated by derivations in Grafarend et al. [1979, p. 342] in which an attempt to link strain parameters and observations is made. Geodetic observation equations expressed in terms of the elements of the strain matrix, e , have two distinctive characteristics. First, observation equations for distances and angles are free of any rotations. Secondly, rotations are inherent in observation equations for both directions and azimuths. Strain analysis of any network in which scale and orientation is resolved shall inevitably produce differential rotations, total strain, shear and dilatation [cf., Frank, 1966].

Strain analysis of the junction subnetwork, S_3 of the densification network (in the absence of physical motion) is possible whenever a non-zero displacement vector is obtained. The analysis can be extended to other points of the network when another determination of their positions is made by using a different adjustment scheme or changing observations (e.g., reobserving) significantly to give non-zero displacements.

6.2 The Displacement Field of the Junction Subnetwork

The displacement Δx_j at a point p_j has two components (u_j, v_j) parallel to the coordinate axes (x, y) . These components can be expressed as a continuous function of the coordinates. It has been shown [Vanicek et al., 1982] that for the strain analysis of inconsistent observations (in the absence of physical motion) the displacement components can satisfactorily be expressed as a linear function of local coordinates as first proposed by Terada and Miyabe [1929], i.e.,

$$u(x, y) = e_{xx}x + e_{xy}y + u(x_0, y_0) \quad 6.1$$

$$v(x, y) = e_{yx}x + e_{yy}y + v(x_0, y_0) \quad 6.2$$

where, (x, y) local coordinates with origin at p_j ;

$e_{xx}, e_{yy}, e_{xy}, e_{yx}$ are partial derivatives of displacement components along the local coordinate axes;

$u(x_0, y_0), v(x_0, y_0)$ are displacement components at a point p_0 with coordinates (x_0, y_0) .

The coordinates (x, y) in (6.1) and (6.2) are known. Therefore the unknown parameters in equations (6.1) and (6.2) are the partial

derivatives e_{xx}, e_{yy}, e_{xy} and e_{yx} . The displacement components at p_0 are not important to the analysis. They constitute a set of nuisance parameters that must be eliminated in the computation procedure. If p_j is connected by observations to $(k-1)$ points, then k pairs of equations must be compiled. The equations can uniquely be solved if $k = 5$. The least-squares method may be used when $k > 5$.

Equations (6.1) and (6.2) can be combined into the following observation equation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{bmatrix} \quad 6.3$$

or,

$$\Delta x = [B \quad F] \begin{bmatrix} C \\ e \end{bmatrix} \quad 6.4$$

where the structure and dimensions of the matrices and vectors (for k points) are as follows;

$$\Delta x_{2k \times 1} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$B_{2k \times 2} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$F_{2k \times 4} = \begin{bmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{bmatrix}$$

$$C_{2 \times 1} = \begin{bmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{bmatrix}$$

$$e_{4 \times 1} = [e_{xx} \quad e_{xy} \quad e_{yx} \quad e_{yy}]^T$$

The vector C is eliminated by block partitioning during the adjustment process. In a network of n junction points, n separate systems of equations (6.4) will be required for strain analysis of inconsistent observations to be performed in the whole junction subnetwork, S_3 .

The displacement vector on the junction points equals to the correction vector $\hat{\delta}_j$. The components of the displacement vector are hence the components of the correction vector which is analytically derived in equation (3.20) as;

$$\Delta x_j = -(N_{jj} + P_X - N_{jn} N_{nn}^{-1} N_{nj})^{-1} (A_j^T - N_{jn} N_{nn}^{-1} A_n) P_2 \omega_2$$

It is this equation that is to be transformed into a hypervector of strain vectors e_i ($i=1, \dots, n$) for further analysis.

6.3 The Strain Field of the Junction Subnetwork

We shall assume that each point for which strain is to be computed is connected by observations to more than two points (i.e., $k > 3$). A least-squares solution for the strain vector can be obtained from (6.4). First, we obtain the solution as;

$$\begin{bmatrix} C \\ e \end{bmatrix} = - \begin{bmatrix} B_{\Delta X}^T C_{\Delta X}^{-1} B & B_{\Delta X}^T C_{\Delta X}^{-1} F \\ F_{\Delta X}^T C_{\Delta X}^{-1} B & F_{\Delta X}^T C_{\Delta X}^{-1} F \end{bmatrix}^{-1} \begin{bmatrix} B_{\Delta X}^T C_{\Delta X}^{-1} \Delta X \\ F_{\Delta X}^T C_{\Delta X}^{-1} \Delta X \end{bmatrix} \quad 6.6$$

where, $C_{\Delta X}$ is a submatrix of the covariance matrix of displacements of junction points derived in equation (5.25); the cap (^) on estimated vectors has deliberately been left out.

Second, the technique of block partitioning is used to eliminate the nuisance parameter vector, C . The strain vector is then estimated

as;

$$e = G[F^T C_{\Delta x}^{-1} B (B^T C_{\Delta x}^{-1} B)^{-1} B^T - F^T] \cdot C_{\Delta x}^{-1} \Delta x \quad 6.7$$

$$\dim(G) = (4,4)$$

where,

$$G = [F^T C_{\Delta x}^{-1} F - F^T C_{\Delta x}^{-1} B (B^T C_{\Delta x}^{-1} B)^{-1} B^T C_{\Delta x}^{-1} F]^{-1}$$

The covariance matrix C_e of the strain vector is derived by applying the covariance law to equation (6.7). Derivations in Chapters 2 and 3 show that C_e is the appropriate submatrix of the normal equations inverse in (6.6), i.e.,

$$C_e = G \quad 6.8$$

The size of the C_e for each point is of the size of the matrix G (i.e., 4x4) and provides the uncertainty in the determination of the four elements of the strain vector.

The strain vector, e can be presented as a strain matrix, E , used in equation (6.3) as;

$$E = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} \quad 6.9$$

which is a square matrix. It is well known from matrix algebra [Thompson, 1969] that a square matrix E can be written as a sum of a symmetric matrix $\epsilon = \frac{1}{2}(E+E^T)$ and a skew-symmetric matrix $\underline{\omega} = \frac{1}{2}(E-E^T)$, i.e.,

$$E = \epsilon + \underline{\omega} \quad 6.10$$

where,

$$\epsilon = \begin{bmatrix} e_{xx} & \frac{1}{2}(e_{xy} + e_{yx}) \\ \frac{1}{2}(e_{xy} + e_{yx}) & e_{yy} \end{bmatrix} \quad 6.11$$

$$\underline{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \quad 6.12$$

$$\omega = \frac{1}{2}(e_{xy} - e_{yx}) \quad 6.13$$

ω is the average differential rotation.

The types of strain relevant to the analysis of inconsistencies in observation can, as in other cases, be deduced from (6.11) and (6.12) [Nye, 1960; Timoshenko and Goodier, 1970; Dare, 1982].

- a) Pure shear $\tau = \frac{1}{2}(e_{xx} - e_{yy})$
- b) Simple shear $v = \frac{1}{2}(e_{xx} + e_{yy})$
- c) Total shear $\gamma = (\tau^2 + v^2)^{1/2}$
- d) Total strain $\lambda = (a^2 + b^2)^{1/2}$

The parameters a and b are the major and minor semi-axes of the strain ellipse computed as eigenvalues of the matrix ϵ (equation (6.11)).

The strain field described by the elements of the strain vector e can therefore be transformed into physically meaningful parameters (τ, v, γ, λ or ω) describing the local state of strain at various points of the network. The transformation of inconsistencies into strain can be made directly without first computing the displacements [Dare and Vanicek, 1982]. The transformation for densification networks is described below in detail.

6.4 The Strain Response to Inconsistencies in the Observations

The strain vector e given in equation (6.7) can be written as;

$$e = Q \Delta x_j \quad 6.14$$

where, $\dim(Q) = (4, 2k)$

$$Q = G[F^T C_{\Delta x}^{-1} B (B^T C_{\Delta x}^{-1} B)^{-1} B^T - F^T] C_{\Delta x}^{-1}$$

The matrix Q transforms the displacements (or position errors) into a vector of strain components. The expression for displacement vectors given in equation (6.5) can be substituted into (6.14) to give;

$$C = -Q(N_{jj} + P_x - N_{jn} N_{nn}^{-1} N_{nj})^{-1} (A_j^T - N_{jn} N_{nn}^{-1} A_n^T) P_2 (\ell_2^{(0)} - \ell_2)$$

where, we have substituted $(\ell_2^{(0)} - \ell_2)$ for ω_2

We can also write;

$$e = Q T_j \ell_2 - C_j \quad 6.15$$

where,

$$T_j = (N_{jj} + P_x - N_{jn} N_{nn}^{-1} N_{nj})^{-1} (A_j^T - N_{jn} N_{nn}^{-1} A_n^T) P_2 \quad 6.16$$

$$C_j = Q T_j \ell_2^{(0)} \quad 6.17$$

Equation (6.15) is a transformation of the observation vector ℓ_2 procured in the densification network to a strain vector e. T_j is the least-squares operator for the junction points. C_j is a constant.

The change in the elements of the strain vector due to finite changes $\delta \ell_2$ in ℓ_2 can be evaluated by differentiating equation (6.15). The result is;

$$\delta e = Q T_j \delta \ell_2 \quad 6.18$$

or

$$\delta e = R_j \delta \ell_2 \quad 6.19$$

where,

$$R_j = Q T_j \quad 6.20$$

Except for the subscripts, the matrix R_j , as is T_j , is used to make the notation consistent with that in Dare and Vanicek [1982]. R_j is the strain response matrix to the finite changes $\delta \ell_2$ (inconsistencies) in the observation vector ℓ_2 . A change $\delta \ell_2$ can be taken to mean a change or an error in any single element, group or entire vector of observations ℓ_2 .

Strain analysis of inconsistent observations has so far been performed for a known vector $\delta \ell_2$ computed from repetitively simulating measurements in the same network. The scenario differs from that of densification networks in two aspects: First, the design of the junction subnetwork, S_j , in the existing and densification networks is different. Second, the ultimate object of strain analysis in densification networks is to investigate the vector $\delta \ell_2$ which leads directly to an inverse problem to (6.18).

6.5 The Inverse Strain Analysis Problem

The inverse strain analysis problem states: Given the strain vector δe derive a vector of associated observation inconsistencies $\delta \ell$. If D is the inverse of R_j , then the inverse strain problem is formulated as;

$$\delta \ell = D \delta e \quad 6.21$$

The strain response matrix R_j in equation (6.20) for $(k-1)$ observations linking n given points to other points of the network is a $(4n \times \sum_{l=1}^n 2k-2)$ matrix. R_j is then a singular (for $k > 3k$) matrix of rank $4n$, i.e.,

$$\text{rank}(R_j) = 4n$$

Although R_j has full rank it has a number of possible inverses, D ,

one of which is appropriate to the problem at hand. One such inverse is the Moore-Penrose inverse R_j^+ . It is here selected to be equal to D . This selection of the Moore-Penrose inverse is made to be consistent with the inverse of the normal equations matrix embedded in the matrix R_j . Therefore,

$$D = R_j^+ \quad 6.22$$

Substitution of (6.22) into (6.21) gives the desired form of the inverse strain problem as;

$$\delta \ell = R_j^+ \delta e \quad 6.23$$

A (4nx1) strain vector changes is transformed into observation differences (inconsistencies) by equation (6.23). As in the direct problem, only those columns of R_j corresponding to the observations linking the points under investigation to other network points are considered [Dare, 1982].

6.6 Simulation Studies and Results

Solution of the inverse strain problem where possible will add to the advantages of the compatibility test in two respects: First, it will be possible to compute the component of the gross-error ($m_{ii} \Delta \ell_i$) given in equation (5.13) which is otherwise not possible to obtain. The compatibility test can only tell us whether or not this component is significant at a given level of confidence. Second, strain analysis provides a 2D view of the effect of errors in observations, i.e., whether the errors deform the network by rotating, expanding and/or contracting it. Judging from the strain patterns it is possible to compute the errors responsible or contributing to a deformation or distortion of a certain type. The effect of

observation errors on the state of strain of densification networks can also be investigated graphically. The purpose of these investigations is to establish the sensitivity of the strain technique to inconsistencies in observations.

6.6.1 Sensitivity analysis 1

The first-order network described in section 4.3.3 was used in this analysis. The angle 35-32-25 was selected for the analysis as an observation at a point situated close to the centroid of the network. The observation was perturbed by an error equal to a multiple of its standard deviation ($\sigma = 0.60$) thirteen times. The change in strain was computed and examined for the thirteen perturbations in the observation in the range -4.0σ to 5.0σ . Strain for the perturbations out of the given range were not computed as the observation was flagged for rejection by the tau-max test at 95% probability. Computation of strain and subsequent plotting was made possible by programs STRAIN1, PASTEL and NETPLOT1 (Appendix V-2 to V-4). The programs are modifications of STRAIN, EVALUE and NETPLOT [Thapa, 1980] respectively. Solid lines in the plots show local extension and positive rotation while dashed lines and arcs refer to contraction and negative rotation at each point.

Figures 6.2, 6.3 and 6.4 show the strain patterns (ellipses and arcs) for the -4.0σ , σ and 5σ perturbations respectively. The ellipses and arcs seem to differ in scale and sign only. Negative perturbations induce the same, for the same perturbations, strains with opposite effect. This relationship best depicted by the changes in the semi-major axes of the strain ellipses (Table 6.1) and as

Table 6.1: Variation in the major semi-axis of strain ellipses when the error in one observation (25-32-35) at station 32 is changed.

Station	σ_{ℓ}	Change in Error (σ_{ℓ} 's)										Remarks
		-4	-3	-2	-1	0	+1	+2	+3	+4	+5	
12	0.80	-0.55	-0.45	-0.33	-0.23	-0.12	-0.01	+0.10	+0.21	+0.31	+0.43	i
88	0.80	+1.76	+1.67	+1.59	+1.51	+1.42	+1.33	+1.25	+1.16	+1.07	+0.98	d
25	0.65	+0.42	+0.41	+0.41	+0.41	+0.40	+0.40	+0.41	+0.41	+0.41	+0.41	c
99	0.80	-0.15	-0.20	-0.25	-0.29	-0.34	-0.39	-0.43	-0.49	-0.54	-0.58	i
21	0.75	0.0	+0.01	+0.03	+0.04	+0.06	+0.07	+0.08	+0.10	+0.12	+0.13	i
66	0.75	+0.24	+0.25	+0.24	+0.24	+0.24	+0.24	+0.24	+0.24	+0.24	+0.25	c
32	0.60	+3.31	+2.71	+2.10	+1.50	+1.50	+0.30	-0.31	-0.89	-1.50	-2.10	d
35	0.65	+1.90	+1.89	+1.89	+1.88	+1.88	+1.89	+1.89	+1.88	+1.89	+1.89	c
28	0.80	-0.14	-0.16	-0.19	-0.21	-0.21	-0.26	+0.28	-0.31	-0.33	-0.36	d
38	0.80	-0.08	-0.05	-0.04	-0.01	-0.01	+0.05	+0.08	+0.09	+0.12	+0.16	i
77	0.70	+0.66	+0.67	+0.66	+0.66	+0.66	+0.66	+0.67	+0.66	+0.66	+0.67	c
50	0.85	+0.37	+0.40	+0.42	+0.43	+0.43	+0.46	+0.48	+0.49	+0.51	+0.52	i
43	0.80	+1.06	+1.12	+1.19	+1.25	+1.25	+1.38	+1.43	+1.50	+1.55	+1.64	i
48	0.85	+0.26	+0.20	+0.12	+0.03	+0.03	-0.08	-0.14	-0.21	-0.29	-0.35	d
Sum		+9.06	+8.47	7.84	+7.21	+7.21	+6.04	+5.47	+4.84	+4.22	+3.69	d

c - constant
i - monotonically increasing
d - monotonically decreasing

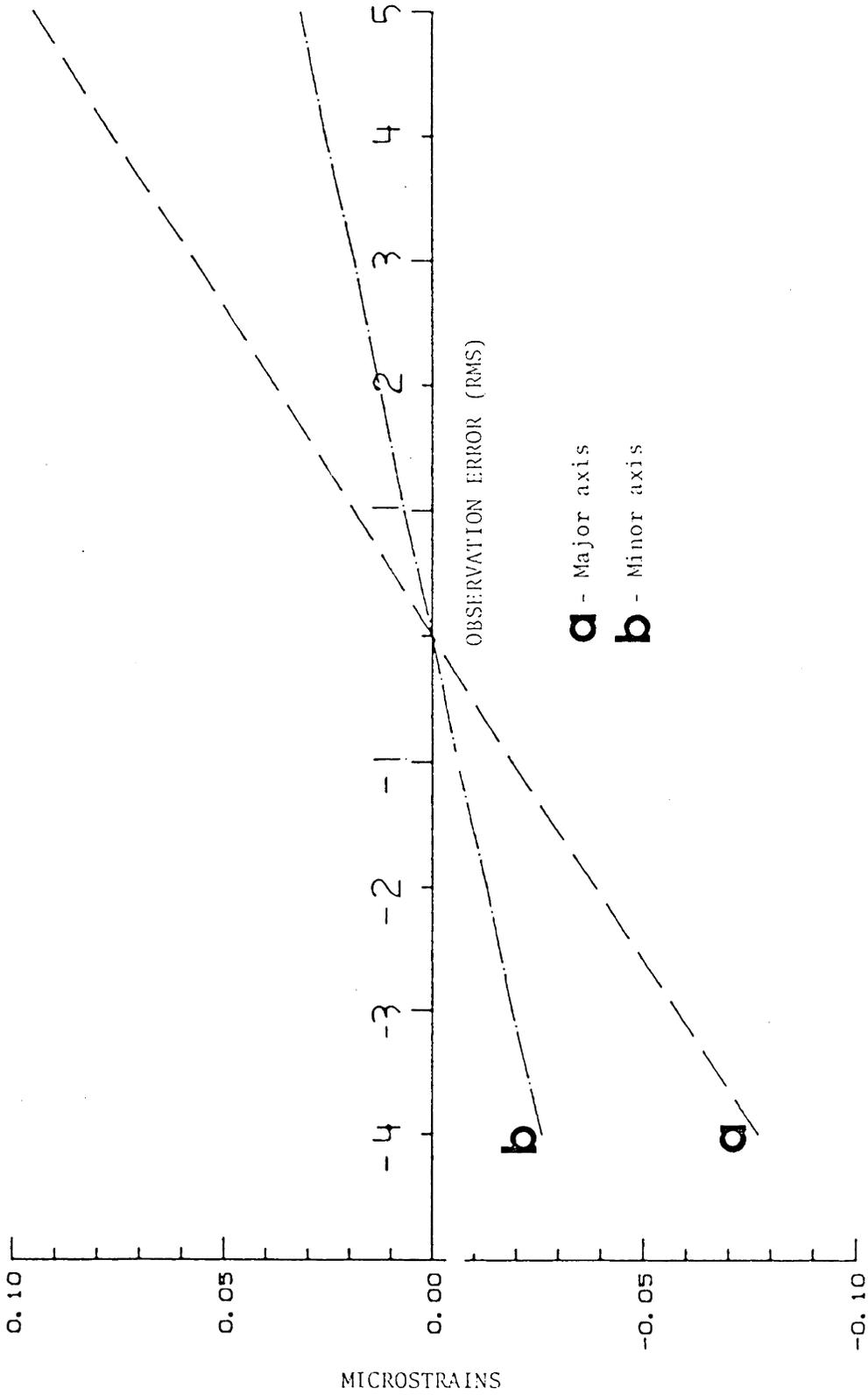


Figure 6.1: Change in axes of the strain ellipse at station 66 when the error of angle 25-32-35 changes.

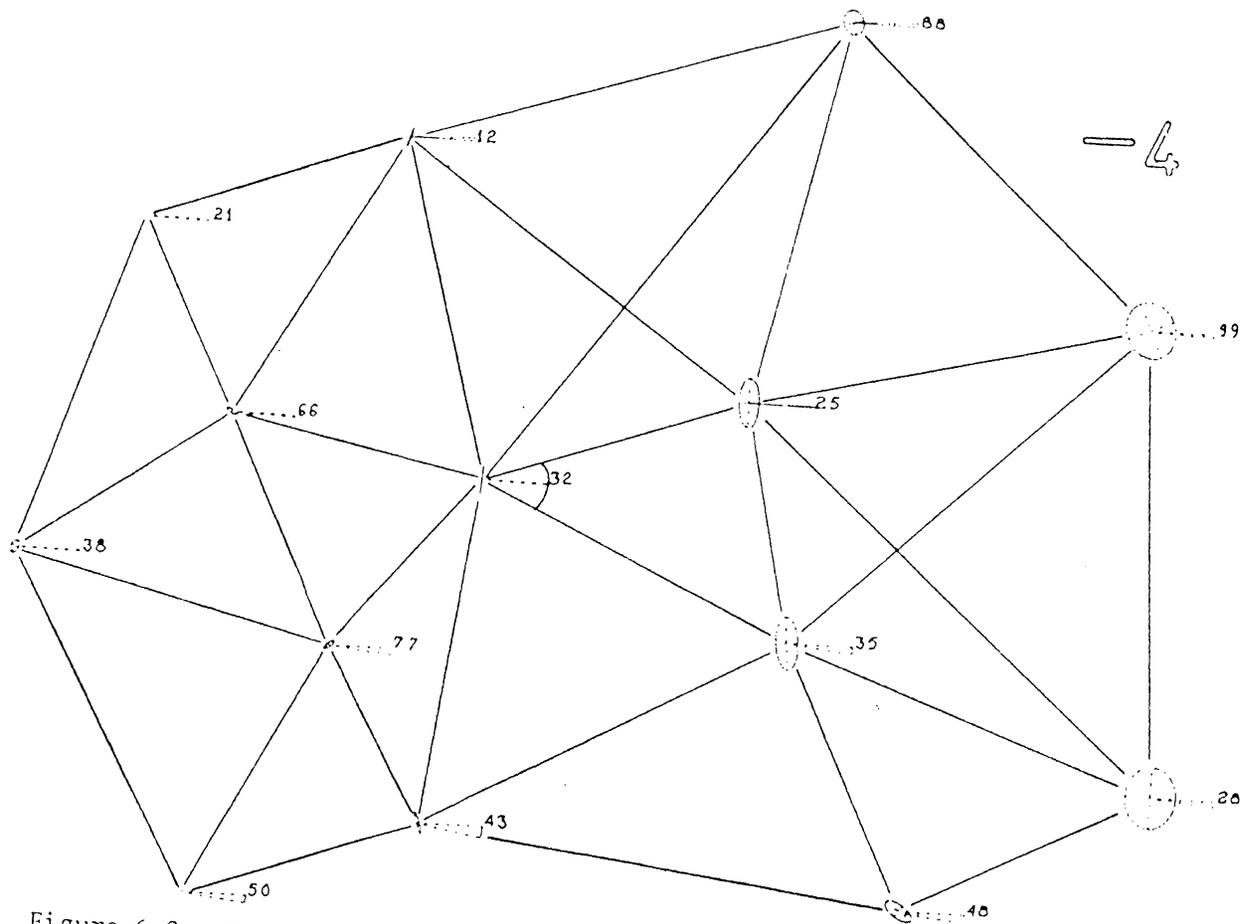


Figure 6.2: Strain patterns of a $-4.0\sigma l$ error in angle (25-32-35).
 Scale of strain ellipse: 1 cm = 0.27 μ strain. Scale of ω , 1:1

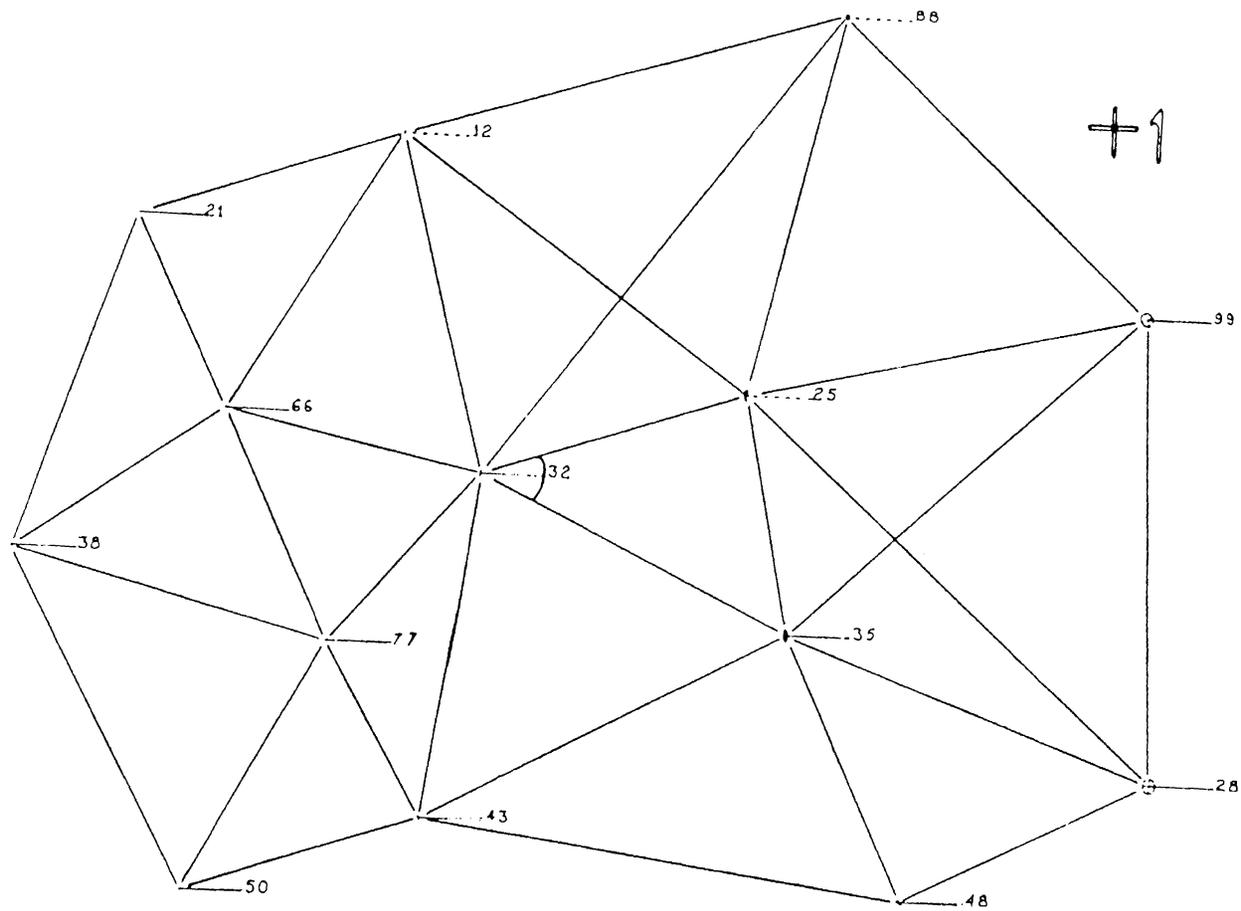


Figure 6.3: Strain patterns of a 1.00% error in angle (25-32-35). Scale of strain ellipse axes: 1 cm = 0.27 μ strains. Scale of ω , 1:1

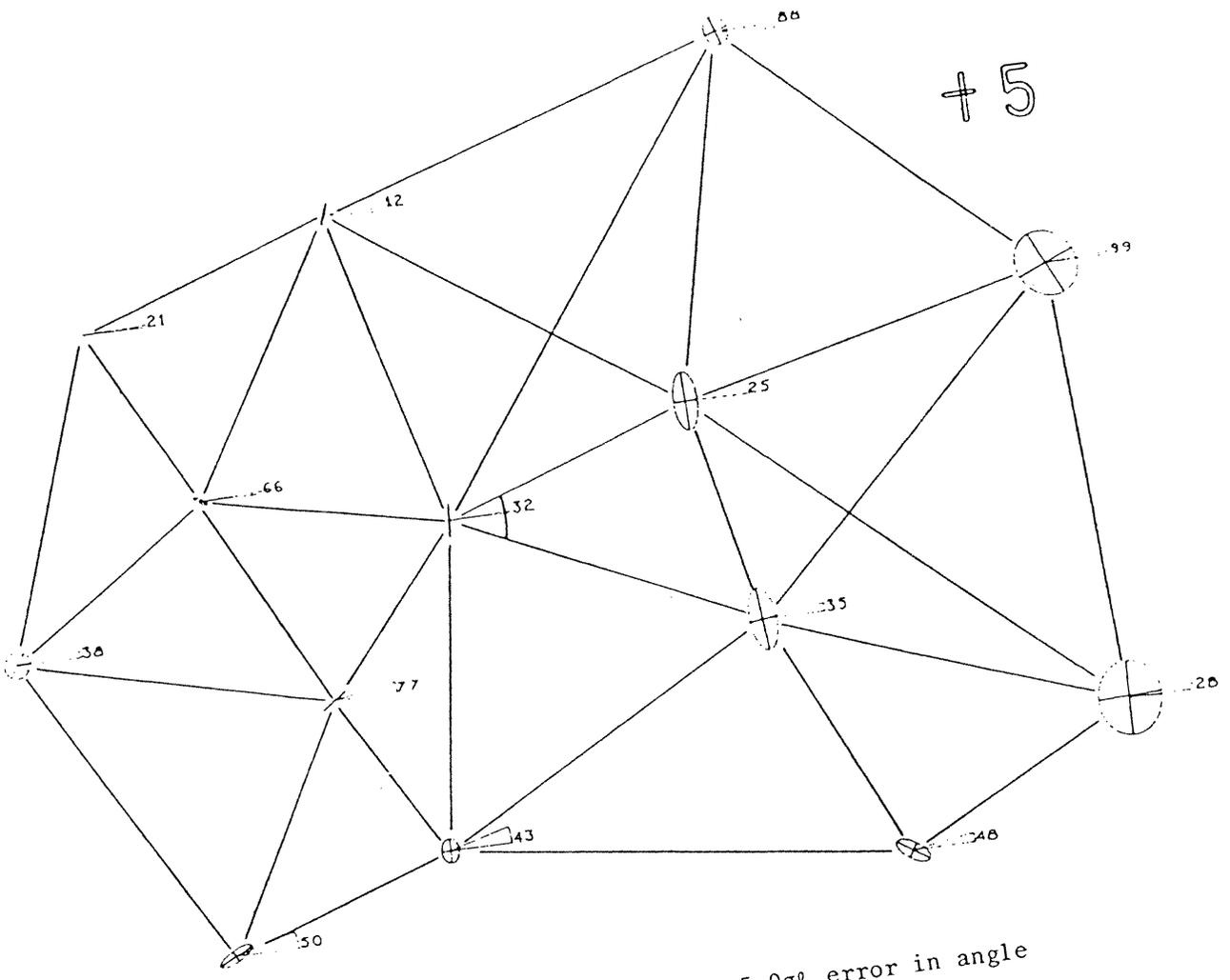


Figure 6.4: Strain patterns of a $+5.0\omega$ error in angle (25-32-35). Scale of strain ellipse axes: $1 \text{ cm} = 0.27 \text{ } \mu\text{strain}$. Scale of ω , 1:1

portrayed graphically in Figure 6.1 for station 66 is a reflection of equation (6.19). As the graph shows, any small change in the observation can be sensed by the network. There is therefore no limit as to the magnitude of observation errors for which the strain analysis technique can be useful, as long as such errors can produce a non-zero displacement of the network points. The importance of this outcome cannot be over-emphasized. The technique enables us to investigate errors in observations smaller than those investigated by statistical tests. The inverse of this statement is also true. We shall use this versatile tool therefore to investigate strain patterns in rigorously densified networks.

6.6.2 Sensitivity analysis 2

The effect that the distorted points of the existing network can have on the densification network can be investigated. The first order network used in section 6.6.1 was simulated with additional errors of 2.5σ in angles (38-66-77), (43-35-48), (25-12-32) and (25-28-99). As the previous simulation (section 6.6.1) shows, each of the errors will displace the network points appreciably as to induce strain in all points of the network. With the exception of stations 66, 77, 88 and 99, all stations have also been included in the design of the densification network as described in section 4.3.3. We shall compute the strain induced in the densification network. We shall use the solutions of the first-order network (i.e., before and after the perturbations) as a priori position in two separate Px-adjustments of the densification network. Computation of strain and subsequent plotting shall be done as described in the previous

section.

The strain patterns plotted as a result of this simulation are displayed in Figures 6.5 and 6.6. Figure 6.5 gives the strain patterns of the junction subnetwork only. Stations 28 and 48 seem to have been strained the most. All points of the densification network are strained by the distortions in the junction points as the strain patterns in Figure 6.6 reveal. An obvious question is whether or not the densification network is strained differently by a distorted junction subnetwork when the new observations are contaminated with gross-error.

To answer this question, eight angles in the densification network were corrupted with additional errors of 2.5σ . The corrupted angles were; (29-38-50), (53-42-44), (25-24-50), (41-50-51), (43-45-52), (19-27-28) and (43-35-45). The least-squares adjustment process, strain computation and plotting procedures were repeated as described in the last two paragraphs. The strain patterns plotted in Figure 6.7 are identical to those in Figure 6.6. The reason for this behaviour is yet to be established. It is suspected to be due to the local nature of the strain. The identity of the strain patterns in these two figures shows the effect of the gross errors in the existing network on the densification network to be invariant with respect to gross-errors in the new observations. This does not by any means imply that gross-errors in the new observations do not strain the densification networks. However, whether or not this is the case can be seen in the simulation study that follows hereafter.

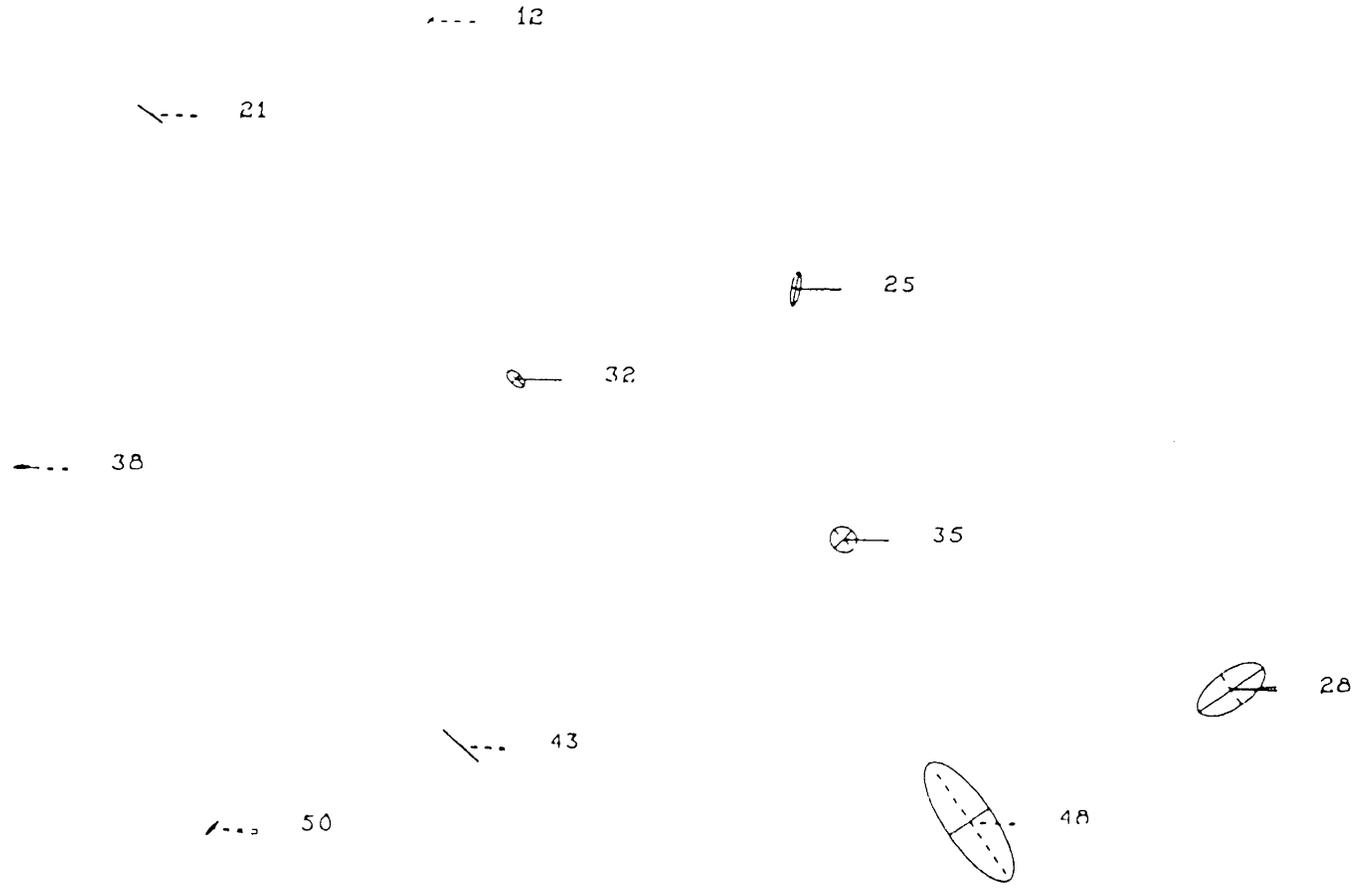


Figure 6.5: Strain induced in a rigorous densification (triangulation by the existing network (junction points only). Scale of $\omega, 1:1$. Scale of strain ellipse axes: 1 cm = 9 μ strains.



Figure 6.6: Strain induced in rigorous densification by the existing network (all points random errors only). Scale of strain ellipse axes: 1 cm = 7 μ strains. Scale of ω , 1:1.



Figure 6.7: Strain induced in rigorous densification by the existing network (all points, random and biases). Scale of strain ellipse axes: 1 cm = 7 μ strain. Scale of ω , 1:1.

6.6.3 Sensitivity analysis 3

Results of two determinations of the densification network for which the compatibility test was performed in section 5.6 were also subjected to the strain analysis. The existing solution and weight matrix P_x being the same in both adjustment means that the differences on positions were a result of the additional errors in the observations. The strain patterns in Figure 6.8 present the strain in the densification network using the undistorted solution of the existing network and corresponding covariance matrix for the two densification adjustments. The strain patterns in Figure 6.9 describe the strain patterns using the distorted solution of the existing network and its covariance matrix. As shown in the previous section, the two figures are also identical. Similar to the results in the last section, we find the strain in densification network due to gross-errors in the new observations in these figures are invariant with respect to the distortions in existing network.

Figures 6.8 and 6.9 show also that the strain can be more conspicuous at some points than others even though the same gross-errors were simulated at all selected stations. It was discussed in section 5.3 that the component of the observation error which is transformed into position errors by the adjustment process, is a function of the geometrical strength of the network. The variability of the strain response portrayed here is therefore not surprising. Also not surprising is the fact that of the stations simulated with the 2.50 error the largest strain ellipse and the smallest of the largest residuals at a station are at the same station (station 50).

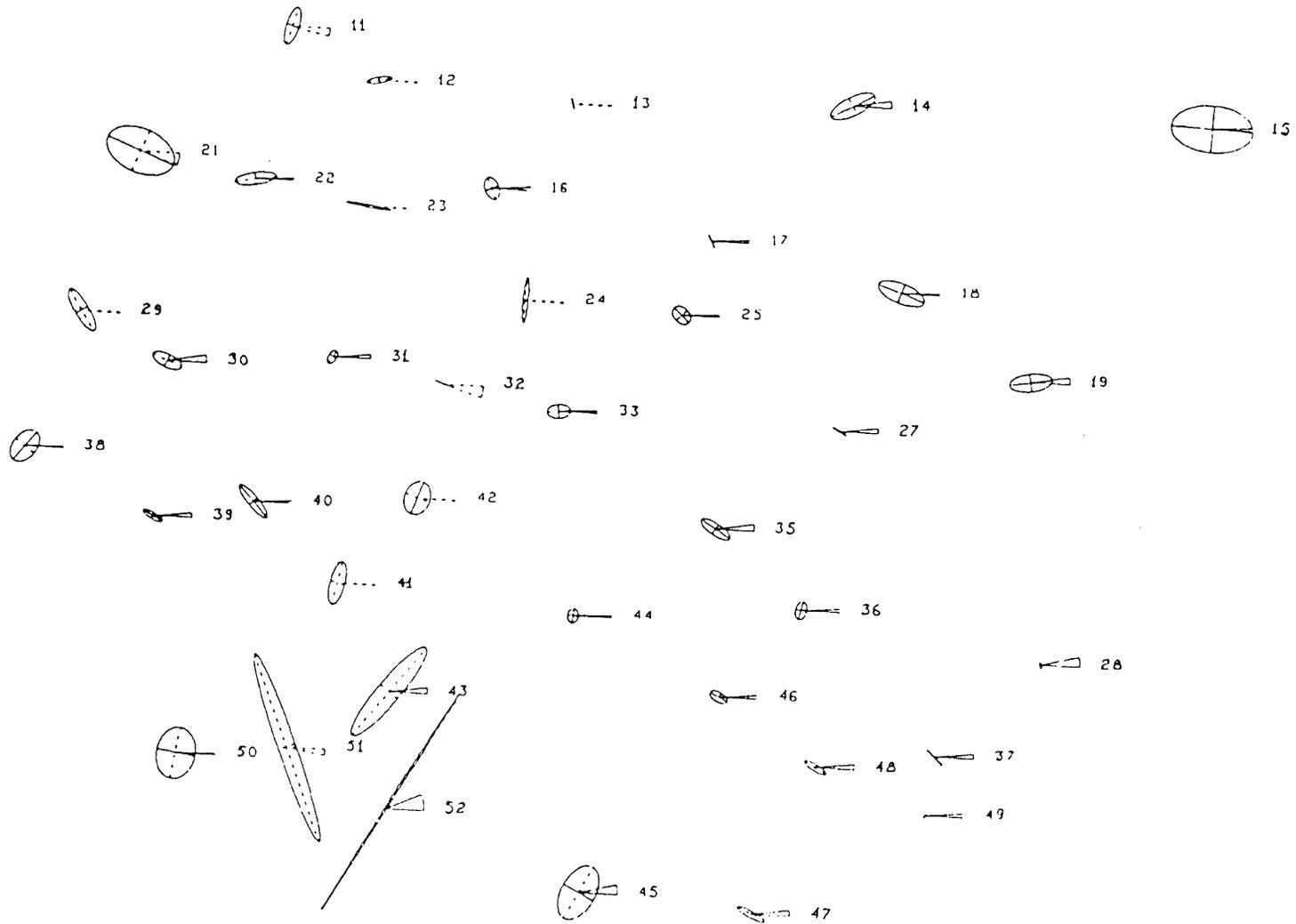


Figure 6.8: Strain induced in rigorous densification by 2.5ω bias in 7 observations (junction points undistorted). Scale of strain ellipse axes: 1 cm = 7 μ strains. Scale of ω , 1:1.

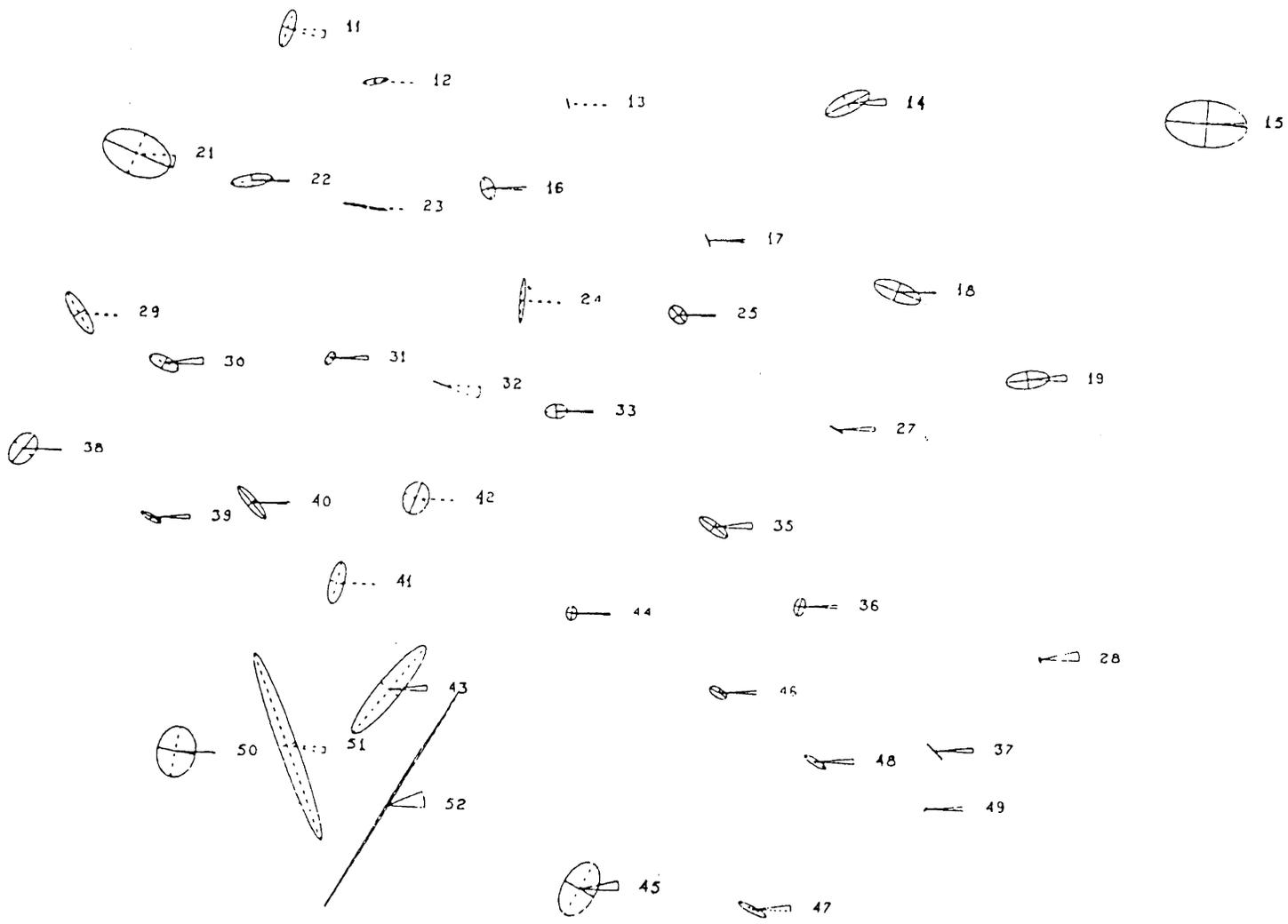


Figure 6.9: Strain induced in rigorous densification by 2.5% bias in 7 observation (junction points distorted). Scale of strain ellipse axes: 1 cm = 7 μ strains. Scale of ω , 1:1.

The largest of the largest residuals at a station is at station 35 which as the strain ellipses show is not much different from those at stations with uncorrupted observations.

The strain technique here reveals that it is capable of unveiling gross-errors which are less than half the marginally detectable error by data snooping as computed by Ackermann [1981] and Kavouras [1982]. It also confirms the idea that hypothesis testing on both residuals and positions whenever possible improves the threshold of gross-error detection in observations. Stations with conspicuous strain ellipses (i.e., 21, 45, 50, 51, 52) are included in the subnetworks that failed the out-of-context compatibility test at 95% confidence level.

7.0 POST-ADJUSTMENT CHANGES IN THE RIGOROUS SOLUTION

The expressions developed in Chapters 2-4 and the checks described in Chapter 5 and 6 are necessary to ensure rigorous densification results. Yet to be addressed is the question of blunders which can be discovered by, for example, compatibility testing or strain analysis after the adjustment has been completed. This Chapter will give analytical expressions required to apply corrections to the least-squares solution of a rigorously adjusted densification network for minor changes in observations, observation weights, P_2 , the P_x -matrix and the initial coordinates $\hat{x}_j^{(1)}$. Assumption is made that the affected submatrices of P_x and P_2 are limited to a few stations and observations respectively. This Chapter does not discuss changes in more than one matrix and vector.

7.1 The P_x -Matrix

We assume that because of punching or other mechanical errors a matrix P_{x_1} was entered into the adjustment instead of the correct matrix P_x . The difference between them is ΔP_x such that;

$$\Delta P_x = P_x - P_{x_1} \quad 7.1$$

and

$$\Delta P_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta P_x \\ 0 & 0 & 0 \end{bmatrix} \quad 7.2$$

The matrix ΔP_X is embeded in a null matrix, and
 $\dim(\Delta P_X) \ll \dim(P_X)$.

The Approximate Correction

We recall the correction vector $\hat{\delta}$ from equation (3.19)
 and present it in the form;

$$\hat{\delta} = - (N+P_X)^{-1} A^T P \omega \quad 7.5$$

The expression of the correction vector when P_{X_1} has been used is;

$$\hat{\delta}' = - (N+P_{X_1})^{-1} A^T P \omega \quad 7.4$$

The difference between the correction vectors $\hat{\delta}$ and $\hat{\delta}'$ is;

$$\begin{aligned} \Delta \hat{\delta}' &= \hat{\delta} - \hat{\delta}' \\ &= - [(N+P_X)^{-1} - (N+P_{X_1})^{-1}] A^T P \omega \end{aligned} \quad 7.5$$

We introduce an identity similar to (4.42) as;

$$(A-B)^{-1} \equiv -A^{-1} (A^{-1} - B^{-1})^{-1} B^{-1}$$

for $A = (N+P_X)^{-1}$ and $B = (N+P_{X_1})^{-1}$ we obtain;

$$[(N+P_X)^{-1} - (N+P_{X_1})^{-1}]^{-1} = - (N+P_X) (P_X - P_{X_1})^{-1} (N+P_{X_1})$$

the inverse of which is;

$$(N+P_X)^{-1} - (N+P_{X_1})^{-1} = - (N+P_X)^{-1} \Delta P_X (N+P_{X_1})^{-1} \quad 7.6$$

where,

$$\Delta P_X = P_X - P_{X_1}$$

which on substitution into (7.5) gives;

$$\Delta \hat{\delta}' = (N+P_X)^{-1} \Delta P_X (N+P_{X_1})^{-1} A^T P \omega \quad 7.7$$

The matrices A^{-1} and B^{-1} in the identity given above can be inter-
 changed without changing the left hand side of the expression, which
 in turn leads to;

$$\Delta \hat{\delta}' = (N+P_{X_1})^{-1} \Delta P_X (N+P_X)^{-1} A^T P \omega \quad 7.8$$

Substitution of (7.3) and (7.4) into (7.8) and (7.7) respectively we obtain;

$$\Delta\delta' = - (N+P_{X_1})^{-1} \Delta P_X \hat{\delta} \quad 7.9$$

and

$$\Delta\delta' = - (N+P_X)^{-1} \Delta P_X \hat{\delta}' \quad 7.10$$

Equations (7.9) and (7.10) suggest that in order to correct δ' for ΔP_X it is necessary, as pointed out in Chapter 4, to compile the rigorous normal equations matrix and obtain its inverse. This task is equivalent to adjusting the network all over again. Let us assume that the norm of P_X and that of P_{X_1} are much smaller than that of N , i.e.,

$$\|P_X\| \ll \|N\| \quad 7.11$$

and

$$\|P_{X_1}\| \ll \|N\| \quad 7.12$$

Then [c.f., Fox, 1964 and Appendix II]

$$(N+P_X)^{-1} \approx N^{-1} - N^{-1} P_X N^{-1} \quad 7.13$$

and

$$(N+P_{X_1})^{-1} \approx N^{-1} - N^{-1} P_{X_1} N^{-1} \quad 7.14$$

The rigorous form of (7.13) and (7.14) is obtained in Appendix II by replacing P_X , P_{X_1} by P_X^* and $P_{X_1}^*$ respectively. The star indicates a matrix computed similar to equation II.4 (Appendix II). The difference between (7.13) and (7.14) is;

$$(N+P_X)^{-1} - (N+P_{X_1})^{-1} \approx - N^{-1} \Delta P_X N^{-1} \quad 7.15$$

Comparing equations (7.6) and (7.15) we can say that under the conditions stipulated by the inequalities (7.11) and (7.12) the inverses $(N+P_X)^{-1}$ and $(N+P_{X_1})^{-1}$ can be assumed to be equal, i.e.,

$$(N+Px)^{-1} \approx (N+Px_1)^{-1} \quad 7.16$$

The correction for the blunder in Px_1 can be computed from equation (7.10) using the normal equations inverse with the weight matrix Px_1 , i.e.,

$$\Delta\delta' = -(N+Px_1)^{-1} \Delta Px \hat{\delta}' \quad 7.17$$

However, because of (7.16) the covariance matrix of the corrected positions $\hat{x} = \hat{x}' + \Delta\delta'$ will remain unchanged, within the approximation in equation (7.15). The covariance matrix pertaining to $V\delta'_j$ and to the points to which the matrix Px refers can be improved by adding the correction $-N^{-1} \Delta Px N^{-1}$ to $C_X^{\wedge'}$, i.e.,

$$C_X^{\wedge} = C_X' - N^{-1} \Delta Px N^{-1} \quad 7.18$$

Equation (7.18) is similar to (4.71). The latter is however confined to the junction points only. Both equations (7.17) and (7.18) make use of matrices and the vector δ' formed in the non-rigorous adjustment. The savings in computer storage and time obtained as a result of the approximation are tremendous. These savings equal to the cost of storage and time required for the computer to perform n^3 multiplication when n is the row dimension of the matrix inverse. These savings are however worth considering only in the event that the approximate correction is not significantly different from the rigorous solution. This difference has been investigated.

The approximate correction equations (7.17) and (7.18) have been tested by using the data provided in section 4.2.1 of Nickerson and Knight [1983]. The approximate correction vector and the rigorous correction vector are identical up to the third place of decimal. All vectors and matrices in (7.17) are known.

The Rigorous Correction

Let P_x be assumed to refer to the junction points only, i.e., $P_x = P_{x_j}$. Let the matrix P_{x_1} used in this chapter be the P_x' in Chapter 4. By partitioning the matrix inverse, $(N+P_x)^{-1}$, as performed in equation (2.44) we obtain for equation (7.10);

$$\Delta\delta'_j = -H_{jj} \Delta P_{x_j} \hat{\delta}'_j \quad 7.19$$

where, j refers to the junction points.

Equation (7.19) is the same as equation (4.44) for improving non-rigorous densification schemes, at least in mathematical form if not in philosophy. P_{x_1} differs however from P_x' in Chapter 4 by the fact that the latter is connected with all non-fixed junction points. The larger submatrix thereof tends to zero as equation (4.3) shows.

The expression for the correction to the correction vector of the new points is obtained analogous to (7.19) as;

$$\Delta\delta'_{\mathbf{n}} = -H_{\mathbf{n}j} \Delta P_{x_j} \hat{\delta}'_j \quad 7.20$$

which is equivalent to the expression (4.62) for correcting the non-rigorous solution of new points in a fixed-point adjustment. The matrices H_{jj} and $H_{\mathbf{n}j}$ contain the correct P_{x_j} matrix which implies that for a rigorous $\Delta\delta_j$ and $\Delta\delta_{\mathbf{n}}$ the matrix H_{jj} must be computed.

Equation (7.17) is an approximation while (7.19) and (7.20) are accurate expressions. These expressions were tested also using

the same data used for the approximate expression (7.17). The results of the corrected positions showed to be identical with those obtained from the Px-adjustment. When using the corrected expression it is suggested that the set of points for which (7.19) applies must be points for which $\Delta Px \neq 0$. The remaining points are grouped together with the new points. Rigorous correction of the covariance matrices is discussed in section 4.5.3.

7.2 The Initial Coordinates, $\hat{x}_j^{(1)}$

The weighted positions in a Px-adjustment are known to assume the role of observations (pseudo-observations) as well as that of initial positions. Any changes in these observations are blunders which may be unveiled by statistical testing on the residuals. These changes distort the set-up of the Px-adjustment and may require more than one iteration for the adjustment process to converge. The $C_{\Delta x}$ matrix will not be given by equation (5.25) and is non-meaningful for compatibility testing. Using network simulations, we have shown that these claims are indeed true. Changes in $\hat{x}_j^{(1)}$ must, therefore, be treated as the changes in l_2 .

7.3 The Observation Weights

Changing observation weights may be necessary when parts of a network have been, for some reason, reobserved and a new solution is sought. The observation weights can be perceived as having changed the weight matrix P_2 before reobservation to $(P_2 + \Delta P)$ after reobservation. We shall assume for a moment that the change in observation

weights is a result of blunder committed in compiling the input data to an adjustment program and that the observations were not in any way affected (i.e., no change in the misclosure vector, ω_2). The normal equations matrix N will also be affected and will change from N to $(N+\Delta N_p)$. The correction vector changes from $\hat{\delta}$ to $(\hat{\delta}+\Delta\hat{\delta})$ which, with the necessary changes in equation (7.3), reads;

$$\hat{\delta} + \Delta\hat{\delta} = - (N+\Delta N_p)^{-1} A^T (P+\Delta P)\omega \quad 7.21$$

Substituting (7.13) into (7.21) while bearing in mind that N in (7.21) is the same as $(N+Px)$ in (7.3) we obtain;

$$\hat{\delta} + \Delta\hat{\delta} = - (N^{-1} - N^{-1}\Delta N_p N^{-1}) A^T (P+\Delta P)\omega \quad 7.22$$

$$\begin{aligned} &= - N^{-1} A^T P\omega + N^{-1}\Delta N_p N^{-1} A^T P\omega - N^{-1} A^T \Delta P\omega \\ &\quad + N^{-1}\Delta N_p N^{-1} A^T \Delta P\omega \end{aligned} \quad 7.23$$

or by ignoring second order terms;

$$\Delta\hat{\delta} = N^{-1}\Delta N_p \hat{\delta}' - N^{-1} A^T \Delta P\omega \quad 7.24$$

where,

$$\hat{\delta}' = - N^{-1} A^T P\omega$$

Equation (7.24) is the expression for the change in the positions when a change in the weight matrix, P , occurs. This equation has been obtained and used by Vanicek [1984]. The expression uses the already computed inverse N^{-1} and offers time and storage savings in computation as discussed in section 4.4.2 and Appendix II.

The new covariance matrix is obtained as the inverse $(N+\Delta N_p)^{-1}$. This inverse can also be computed cost effectively using (7.13) as;

$$C_{X'}^{\wedge} = C_{X'}^{\wedge} - C_{X'}^{\wedge} \Delta N_p C_{X'}^{\wedge} \quad 7.25$$

where,

$$C_{X'}^{\wedge} = N^{-1}$$

The matrices ΔP and ΔN_p in equations (7.24) and (7.25) are structured such that the dimensions of the non-null submatrices are very small compared to the full matrices.

7.4 The Observations

The observation vector used in the adjustment often contains gross-errors $\nabla \ell$ which can only be revealed after the adjustment has been completed through statistical testing. Is it always necessary to repeat the adjustment when gross-errors in the observations are unveiled? We shall attempt to answer this question. We shall assume that the new observation used to replace the one with gross-errors has the same observation weight. Let us write an observation with a gross-error as $\ell_2 + \nabla \ell$. The misclosure vector defined, for example, in equation (2.17) becomes;

$$\omega_2' = \ell_2^{(0)} - \ell_2 - \nabla \ell \quad 7.26$$

or

$$\omega_2' = \omega_2 - \nabla \ell \quad 7.27$$

where,

ω_2' includes the gross-errors, $\nabla \ell$.

The correction vector (7.3) in the presence of gross-errors is therefore;

$$\hat{\delta}' = - (N+PX)^{-1} A^T P \omega_2' \quad 7.28$$

Assume that the correction vector with the gross-error equals to;

$$\hat{\delta}' = \hat{\delta} + \Delta \hat{\delta}$$

where, $\Delta \hat{\delta}$ is due to the gross errors $\nabla \ell$, then by substituting (7.27) into (7.28) we obtain;

$$\Delta \hat{\delta} = - (N+Px)^{-1} A^T P \nabla \ell \quad 7.29$$

Equation (7.29) suggests that a network need not be re-adjusted when the inverse $(N+Px)^{-1}$ is preserved. Only an appropriate submatrix of $A^T P$ that multiplies with $\nabla \ell$ need be compiled. The remaining procedure boils down to a multiplication of matrices.

7.5 The Covariance Matrix, C_{ℓ}^{\wedge}

The covariance matrix can be derived by applying the covariance law to equation (7.28). In this case the covariance matrix of the misclosure vector (7.26) need to be known, i.e.,

$$C_{\omega_2} = C_{\ell_2(0)} + C_{\ell_2} + C_{\nabla \ell} + 2C_{\ell_2 \nabla \ell} \quad 7.30$$

The vector $\nabla \ell$ is vector of changes of some of the observations (in the vector ℓ_2) without changing the observation weights of those observations. Their error characteristics and hence the vectors ℓ_2 and $(\ell_2 + \nabla \ell)$ are indistinguishable from the second statistical moment's point of view. If $\nabla \ell$ were blunders, for example, it would correspond to a change in the mean (the first moment) without change in dispersion. In both cases it means that;

$$C_{\ell} = C_{\ell + \nabla \ell} \quad 7.31$$

and

$$C_{\ell, \ell} = C_{\ell, \ell + \nabla \ell} \quad 7.32$$

which is possible if the covariance matrix $C_{\nabla \ell}$ is a null matrix, i.e.,

$$C_{\nabla \ell} = C_{\ell_2 \nabla \ell} = 0 \quad 7.33$$

Equation (7.30) can therefore be presented in the form identical

with C_{ω_2} , i.e.,

$$C_{\omega_2'} = C_{\omega_2} \tag{7.34}$$

which in fact, leads to the identity between the covariance matrices of the adjusted positions before and after the observations are corrected for the errors, $\forall \lambda$. This conclusion says also that if the adjustment is to be corrected for blunders in observations, the covariance matrix of adjusted positions should not be changed. It is important to check or establish equality (7.31) and (7.32) before (7.34) is accepted. The a posteriori variance factor may however be different leading to different covariance matrices when scaled by $\hat{\sigma}_0^2$.

8.0 CONCLUSIONS AND RECOMMENDATIONS

We set out to lay down a mathematical foundation and derive least-squares expressions necessary to rigorously adjust a 2D densification network. The ideas expounded here can, as well, be applied to 3D networks. The term "rigorous densification" was clearly defined, thus setting the boundaries within which the theory of rigorous densification would be applied. It was found necessary to triple partition a densified network and examine each subnetwork in context of the rest of the network. The junction points have been found to propagate the information from the existing network into the densification network and vice versa. The dual position information that can be obtained on the junction points and its covariance matrix can be used to extend statistical testing of the residuals into compatibility testing of two network solutions. The dual positions offer a possibility to perform strain analysis in densification networks hence expanding the error analysis problem into the strain space.

The objectives of this study have been achieved. In the course of the research it was found necessary to address a number of other problems related to rigorous densification. A broader view of statistical testing in search of outliers, correcting of non-rigorous solutions to rigorous and post-adjustment changes in the solution, for blunders or observations rejected by statistical

testing, have been addressed and the merits of the solutions discussed. Below is a summary, followed by conclusions and recommendations, made as a result of this work.

8.1 Summary and Contributions

Network densification is a procedure of adding information into the existing network and integrating the new information with the existing information. The integration can be done correctly or incorrectly. The correct way of densification described in this dissertation starts with the formulation of the mathematical models. When formulating the models all points which have been estimated prior to the densification (hence have a finite covariance matrix) must be taken to be pseudo-observations. An auxiliary mathematical model is formulated for all such points. This model is then used in conjunction with the main mathematical model (or models).

One main mathematical model is formulated when the Px-adjustment is contemplated. It establishes the functional relationship between all the positions of the densification network S_2 and the observation vector ℓ_2 . The linearization of the model is made using the existing solution. In so doing, it is assumed that the model is linearized in a differential neighbourhood of the final solution, \hat{x}_2 . The misclosure vector w_x in the auxiliary model will then be equal to zero. Defining the problem in a differential neighbourhood of \hat{x}_2 also means that the least-squares process will converge in one iteration. The expressions developed for the Px-adjustment cannot be guaranteed to work beyond the prescribed conditions.

Two main mathematical models are formulated when a combined adjustment of the existing, S_1 and the densification networks is to be performed. The mathematical models establish the functional relationships between the positions and the observations in the two networks separately. The junction points are therefore related to both the ℓ_1 and ℓ_2 observations. These points establish the correlation between the two networks S_1 and S_2 . Linearization of the mathematical models is made at initial values $x^{(0)}$ which may or may not be deterministic quantities. If $x^{(0)}$ are estimated quantities then they are taken to be pseudo-observations and treated like the junction point positions of the existing network in the Px-adjustment. The possibility of multiple determinations of the network points (possibly using different techniques) exists. However, only one determination may be used to linearize the models. Other determinations are considered separate from the densification problem - more appropriately as a merger problem.

This dissertation has assumed the equality of the Px-adjustment with the combined adjustment results. The least-squares expressions for the Px-adjustment were derived, as a result of this assumption, from the expressions of the combined adjustment. Thus, the expressions in the Px-adjustment are those that would give the same position and error estimates as the combined adjustment.

It is also possible to obtain rigorous solutions indirectly from the non-rigorous solutions. Correction vectors to the non-rigorous solutions (for improper use of the weight matrix Px) are expressed in terms of the already computed vectors and matrices

of the non-rigorous adjustment. It is estimated, for example, that for a $P \times x$ matrix which is 10% the size of the normal equations matrix N , the inverse $(N+Px)^{-1}$ can be computed in at least 87.8% less multiplications for a given N^{-1} . The alternative, of course, would be to re-adjust the network.

All the tests performed in search of outliers in a solitary network (data snooping, τ -test, etc.) are quality control measures on the model made through the residuals. Residuals are however one of two components of observation errors. The other component can be investigated only through the estimated positions whenever possible. Quality control in densification networks can be taken a step further by testing the statistical compatibility of the existing and densification solutions on the junction points. Compatibility testing is a test on the significance of observation errors on the estimated positions. It must always be performed whenever two solutions are given. The weight matrix of the position differences of the junction points is the difference in covariance matrix inverses of the solutions. The weight matrix of position differences must always be established prior to the test. Strain analysis can be performed for all points which fail the compatibility test. Strain patterns portray the local deformations and rotations experienced by the network as a result of the inconsistency in observations. Unfortunately all results shown by simulations cannot easily be realized in a practical network analysis. The future of strain analysis is bright and may solve our quality control problems at a new level.

We have earlier encountered corrections to adjustment results to correct for the improper use of the matrix, P_x . Other corrections of relatively minor nature include making post-adjustment changes in the solution due to minor changes in the $P_{\hat{x}}$ -matrix, P_x -matrix, ℓ_2 -vector or $\hat{x}^{(0)}$ -vector. Such changes may be necessitated only after the adjustment process is completed. Formulations which lead to more cost-effective computations than readjusting the network have proved to be realistic when used in practice.

The contributions made in this work are;

- 1) The concept of stochastic Taylor points has been applied to densification networks. A priori information has been assumed for all points old and new. The covariance matrices of the misclosures and correction vectors have been derived. These matrices are not equal to those of the observations and estimated positions respectively as the case is when a priori information is not considered.
- 2) The equivalence of the combined adjustment and the P_x -adjustment has been proved by transforming the expressions of the combined adjustment into the P_x -adjustment. It has also been proved that for the two solutions to be equal the mathematical models of the P_x -adjustment must be linearized in a differential neighbourhood of the final solution using the existing solution.
- 3) A comparison of the rigorous P_x -adjustment with the fixed-point, overconstrained and fixed-point with transformation has been made. Expressions have been derived for correcting non-rigorous densification solutions at significant savings in computer time and storage.

- 4) Statistical compatibility of the solutions obtained on junction points after rigorous densification has been performed. A special program CTEST has been written, tested and used for this purpose.
- 5) The cross-covariance matrix between the existing and the rigorous densification solutions has been derived. The weight matrix of position differences has been confirmed to be the difference of the covariance matrix of the solutions.
- 6) The mathematical models and subsequent least-squares expressions of the non-rigorous adjustment schemes have been formulated analogous to those of the P_x -adjustment. A uniform treatment of the limiting cases of the matrices P_x and C_x as well as arbitrary finite representations thereof has been made with regard to weighted position constraints.
- 7) Strain analysis has been introduced to densification networks and used to study the strain effect of the existing network on the densification network and the strain effect of gross-errors on the densification network. The inverse strain analysis problem has been formulated.
- 8) Expressions have been developed to correct the rigorous solution when a few observations are changed or changes in the solution for minor changes in the P_ℓ and P_x matrices and ℓ and $\hat{x}^{(0)}$ vectors are contemplated without attempting to readjust the network.

8.2 Concluding Remarks with Recommendations

1) Point positions computed in a weighted position constraint adjustment of horizontal geodetic networks using the expressions given in Chapter 3 are identical to those obtained by adjusting the existing and densification networks simultaneously and are therefore rigorous. Two conditions must however be satisfied.

- a) The initial coordinates for the linearization of the main mathematical model must be equal to the least-squares estimated positions of the junction points derived in the existing network, and suitable initial position coordinates for the new points.
- b) The weight matrix of the initial junction point positions is equal to the appropriate submatrix of the normal equations inverse from the adjustment of the existing network.

In general therefore, the initial point positions in the weighted constraint adjustment are stochastic variables with a finite covariance matrix.

2) The mathematical models and normal equations of the simultaneous adjustment are symmetrically formulated with respect to the new and existing non-junction points. The existing network affects the densification network and vice versa. The extent of the effect of the new network on the existing network requires more study. Such a study could, for example establish some "rule-of-thumb" that stipulates the region of significant influence of the densification network on the existing network. The same expressions developed for the new points can be used for the

existing points by interchanging the subscripts n and e .

- 3) It is possible and cost effective to correct the non-rigorous fixed-point and overconstrained adjustment results to rigorous ones, when the covariance matrix (normal equations inverse) of the non-rigorous solution has been preserved. The correction algorithms require however that a rigorous solution be obtained for the fixed-points of the junction subnetwork by rigorous adjustment of these points.
- 4) Correcting the rigorously adjusted results for blunders committed while assembling the input data for an adjustment program is also possible and cost effective without a complete readjustment of the network. Algorithms to correct for such blunders require the normal equations inverse and the corrupted matrices and/or vectors to be preserved.
- 5) It is mathematically wrong to assign zero weights to initial positions in a weighted position constraint adjustment. Zero weights assume the covariance matrix to be undefined and our trust in their values to be zero. The effect of the existing network cannot be propagated into the densification network under such an assumption. Elements of weight matrix corresponding to fixed positions can be considered to be large with very small reciprocal values which are close to zero.
- 6) The marginally detectable errors in observations depend on the geometrical strength of the network as the redundancy numbers show. It has been established in this study that a densification network can be found to be incompatible with the existing network

when errors smaller than those marginally detectable by statistical tests on the residuals exist in the observations. It is recommended that out-of-context compatibility tests be part of the statistical analysis of each densification solution.

- 7) The covariance matrix of the position differences of the junction points before and after densification has been confirmed to be equal to the covariance matrix of existing positions minus the covariance matrix of the densification solution. These results confirm the fact that a rigorous densification strengthens the existing network.
- 8) a) Strain due to inconsistent observations is a linear function of the inconsistency. Depending on the network geometry, strain on points that are statistically incompatible can be quite conspicuous.
- b) Mathematically speaking, it is possible to recover the inconsistency in the observations associated with a particular type of strain by solving the inverse strain analysis problem as derived but not tested in this study.
- c) The potentials of the strain analysis technique in horizontal geodetic networks is far from having been fully exploited. It is recommended that geodesists take advantage of the versatility of the technique and its capacity to treat a network as a deformable body in scale, rotation and shape. Very little effort is required to obtain these indicators. However, the assumption of no physical motion to the network made with regard to analysis of observations is not always valid,

especially when the junction points lie in a seismically active area. More work has to be done in this area. The numerical solution of the inverse strain analysis problem and its significance to geodetic networks is a new area open for research.

- 9) The least-squares expressions developed in this study for the rigorous densification and error analysis of horizontal geodetic networks make it possible to:
- a) adjust densification networks rigorously and cost effectively using the raw observations or by correcting non-rigorous solutions,
 - b) test the compatibility of the existing and densification after searching for outliers using the residuals.
 - c) perform strain analysis of the network when the existing and densification networks are statistically incompatible, or when investigating the strain effects of the existing network on the densification network and vice versa.

Implementation of these routines requires;

- a) The development of ways and means of storing all adjustment results (positions and covariance matrices) in a retrievable form by improving existing data banks and establishing some where they are non-existent.
- b) the development of computer software with the capability to implement the stipulated routines. State-of-the-art software (e.g., TRAV10, MANOR, GHOST) are not equipped with this possibility.

The software for compatibility testing and strain analysis is

given in Appendix V-1. It is recommended that similar routines be incorporated into all software for network adjustment.

- 10) Developments made in this dissertation do not claim to have solved all the problems of network densification. The contributions made with regard to quality control, are confined to only a subnetwork of the densified network. Even here, the ambiguity of identifying the offensive network (S_1 or S_2) makes it impossible, at least for now, to pinpoint the responsible observations. One approach which can help to define the cause for incompatibility, if and when it exists, includes the use of the following procedure: First, a network is adjusted as a free network. The literature on free network adjustment is quite rich, i.e., Blaha [1971, 1982a,b,c], Markuze [1971], Mittermayer [1972], Grafarend and Schaffrin [1974], Perelmuter [1979] and Welsch [1979] all of which address different problems associated with free network adjustment. The purpose of the free adjustment is to perform an analysis on the residuals. Here, the residuals are free from the strain that would otherwise be imposed on the adjustment by constraints. The search for outliers is then done using the statistical tests described in Chapter 5. Secondly, the computation of rigorous positions is done by transforming the free adjustment results into rigorous results using the expressions developed in Chapter 4.

The above procedures ensures quality control of the new observations. If the network S_2 is incompatible with S_1 it is then suggested that we look out in S_1 for outliers.

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APPENDIX I
MINIMAL CONSTRAINT ADJUSTMENT OF
THE EXISTING NETWORK

APPENDIX I

MINIMAL CONSTRAINT ADJUSTMENT OF THE EXISTING NETWORKI.1 The Mathematical Model

$$F(\bar{x}_e, \bar{x}_j) = \bar{l} \quad :P_1 \quad I.1$$

In linear form (cf., section 2.2 for notation)

$$A_e \delta_e + A_j \delta_j^* - r + \omega_1 = 0 \quad :P_1 \quad I.2$$

I.2 The System of Normal Equations

$$N_1 \delta_1 + u_1 = 0 \quad I.3$$

where,

$$N_1 = A_1^T P_1 A_1$$

$$u_1 = A_1^T P_1 \omega_1$$

$$A_1 = (A_e \ A_j)^T$$

In partitioned form (cf., section 2.3 for notation)

$$\begin{bmatrix} N_{ee} & N_{ej} \\ N_{je} & N_{jj} \end{bmatrix} \begin{bmatrix} \hat{\delta}_e \\ \hat{\delta}_j^* \end{bmatrix} + \begin{bmatrix} u_e \\ u_j^1 \end{bmatrix} = 0 \quad I.4$$

I.3 Solution of Normal Equations for $\hat{\delta}_j^*$

(a) The correction vector (obtained by block-partitioning I.4)

$$\hat{\delta}_j^* = -Q_e^{-1} u_x \quad I.5$$

where,

$$Q_e = N_{jj} - N_{je} N_{ee}^{-1} N_{ej}$$

$$u_x = u_j^1 - N_{je} N_{ee}^{-1} u_e$$

(b) The covariance matrix $C_{\delta_1}^{\hat{*}}$

$$C_{\delta_j}^{\hat{*}} = Q_e^{-1} N_{1j} Q_e + Q_e^{-1} N_{je} N_{ee}^{-1} N_{ej} Q_e^{-1} - 2 Q_e^{-1} N_{je} N_{ee}^{-1} N_{ej} Q_e^{-1}$$

$$C_{\delta_j}^{\hat{*}} = Q_e^{-1} [N_{1j} - N_{je} N_{ee}^{-1} N_{ej}] Q_e^{-1}$$

or

$$C_{\delta_j}^{\hat{*}} = Q_e^{-1} \tag{I.6}$$

(c) The covariance matrix, $C_{x_j}^{\hat{*}}$

$$\hat{x}_j^1 = x^{(0)} + \hat{\delta}_j^*$$

The model (I.2) assumes $C_{x^{(0)}} \approx 0$. Therefore

$$C_{x_j}^{\hat{*}} = C_{\delta_j}^{\hat{*}} \tag{I.7}$$

or

$$C_{x_j}^{\hat{*}} = Q_e^{-1} \tag{I.8}$$

APPENDIX II
SIMPLIFIED INVERSION OF $(N+\Delta N)^{-1}$
FOR A GIVEN N^{-1}

APPENDIX II

SIMPLIFIED INVERSION OF $(N+\Delta N)^{-1}$ FOR A GIVEN N^{-1}

Lets recall the matrix identity [Liebelt, 1967];

$$(C^{-1} + A^T B^{-1} A)^{-1} = C - C A^T (B + A C A^T)^{-1} A C \quad \text{II.1}$$

Let ,

$$C = N^{-1}$$

$$B = I$$

$$A = a^T$$

Equation (II.1) can now be expressed as;

$$(N + a a^T)^{-1} = N^{-1} - N^{-1} a (I + a^T N^{-1} a)^{-1} a^T N^{-1} \quad \text{II.2}$$

If $\Delta N = a a^T$ then equation (II.2) can be expressed as;

$$(N + \Delta N)^{-1} = N^{-1} - N^{-1} \Delta N^* N^{-1} \quad \text{II.3}$$

where,

$$\Delta N^* = a (I + a^T N^{-1} a)^{-1} a^T \quad \text{II.4}$$

The above formulation defines "a" to be the Choleski root of ΔN .

Equation (II.3) is the rigorous expression of the inverse $(N + \Delta N)^{-1}$.

Equation (II.2) can be expanded further. Let equation (II.1) be applied

to the matrix inverse $(I + a^T N^{-1} a)^{-1}$. The result is;

$$(I + a^T N^{-1} a)^{-1} = I - a^T (N + a a^T)^{-1} a \quad \text{II.5}$$

Equation (II.2) can be substituted repetitively into (5) to give;

$$(I + a^T N^{-1} a)^{-1} = I - a^T N^{-1} a + a^T N^{-1} a a^T N^{-1} a - \dots \quad \text{II.6}$$

Substituting equation (II.6) back into (II.4) and considering

$N = a a^T$ we obtain;

$$(N + \Delta N)^{-1} = N^{-1} - N^{-1} \Delta N N^{-1} + N^{-1} \Delta N N^{-1} \Delta N N^{-1} - \dots \quad \text{II.7}$$

The Series expansion (II.7) can be continued. The next member of the

series is obtained by pre-multiplying the last term by $-N^{-1}\Delta N$ or, which is the same thing post-multiplying by $-\Delta NN^{-1}$. Equation (II.7) can be truncated when $\|\Delta N\| \ll \|N\|$ to;

$$(N+\Delta N)^{-1} = N^{-1} - N^{-1}\Delta NN^{-1} \quad \text{II.8}$$

Savings in Computation

Both the rigorous expression (equation (II.3)) and the approximate expression (equation (II.8)) give tremendous savings in computation when compared to the direct inversion of $(N+\Delta N)^{-1}$. This is especially true when the matrix N^{-1} is available. On the other hand the approximate equation (II.8) offers computation advantages over the rigorous expression, especially when the size of the non-null matrix of ΔN is significant say, 1% - 10% of the size of N . The matrix product $N^{-1}\Delta NN^{-1}$ is of the size of the non-zero submatrix of ΔN the number of multiplications required to obtain the inverses (II.3) and for (II.8) for $\dim(\Delta N) = (p \times p)$ include [Ashkenazi, 1967; Wells, 1985];

Computation	Dimension	Multiplications
$a^T N^{-1}$	(p, n)	$p^2 n$
$a^T N^{-1} a$	(p, p)	p^3
$(I + a^T N^{-1} a)^{-1}$	(p, p)	p^3
$N^{-1} a (I + a^T N^{-1} a)^{-1} a^T N^{-1}$	(n, n)	$n^2 p + p^2 n$
Total for $N^{-1} - N^{-1} \Delta N^* N^{-1}$	(n, n)	$n^2 p + 2p^2(p+n)$
Total for $N^{-1} - N^{-1} \Delta NN^{-1}$	(n, n)	$n^2 p + p^2 n$
Total for $(N+\Delta N)^{-1}$	(n, n)	n^3

The total number of multiplications for $p = 0.1n$ and $p = 0.01n$ are given in the table below:

Matrices	Multiplications		Multiplications	
	$p=0.01n$	Difference	$p=0.1n$	Difference
$(N+\Delta N)^{-1}$	n^3		n^3	
$N^{-1} - N^{-1} \Delta N^* N^{-1}$	$1.0202n^3 \times 10^{-2}$	$0.989798n^3$	$1.22n^3 \times 10^{-1}$	$0.878n^5$
$N^{-1} - N^{-1} \Delta NN^{-1}$	$1.01n^3 \times 10^{-2}$	$1.02n^3 \times 10^{-4}$	$1.001n^3 \times 10^{-1}$	$2.19n^3 \times 10^{-2}$

The total number of multiplications for $p=0.1n$ and $p=0.01n$ when $n = 10^4$ are given in the table below.

Matrices	Multiplications		Multiplications	
	$p=0.01n$ $n=10^4$	Difference	$p=0.1n$ $n=10^4$	Difference
$(N+\Delta N)^{-1}$	10^{12}		10^{12}	
$N^{-1} - N^{-1} \Delta N^* N^{-1}$	1.0202×10^{10}	9.89798×10^{11}	1.22×10^{11}	8.78×10^{11}
$N^{-1} - N^{-1} \Delta NN^{-1}$	1.01×10^{10}	1.02×10^8	1.001×10^{11}	2.19×10^{10}

APPENDIX III
RESIDUALS IN A RIGOROUS DENSIFICATION
NETWORK ADJUSTMENT

APPENDIX III

RESIDUALS IN A RIGOROUS DENSIFICATION NETWORK ADJUSTMENT

In general, residuals can be expressed as;

$$\hat{\mathbf{r}} = \mathbf{A}\hat{\delta} + \omega \quad \text{III.1}$$

where,

$$\begin{aligned} \omega &= f(x^{(0)}) - \ell \\ \hat{\delta} &= -(\mathbf{N} + \mathbf{P}\mathbf{X})^{-1} \mathbf{A}^T \mathbf{P}\omega \end{aligned} \quad \text{III.2}$$

Substituting (2) into (1) gives

$$\hat{\mathbf{r}} = (\mathbf{I} - \mathbf{A}(\mathbf{N} + \mathbf{P}\mathbf{X})^{-1} \mathbf{A}^T \mathbf{P})\omega$$

or

$$\hat{\mathbf{r}} = (\mathbf{I} - \mathbf{M})\omega \quad \text{III.3}$$

where,

$$\mathbf{M} = \mathbf{A}(\mathbf{N} + \mathbf{P}\mathbf{X})^{-1} \mathbf{A}^T \mathbf{P} \quad \text{III.4}$$

$$\dim(\mathbf{I} - \mathbf{M}) = \dim(\mathbf{M}) = (n, n)$$

$$n = \dim(\ell)$$

The author has proved that in fact:

The matrix \mathbf{M} is not an idempotent matrix. It

becomes idempotent when $\mathbf{P}\mathbf{X} \approx 0$.

The Covariance Matrix $\hat{\mathbf{C}}_{\mathbf{r}}$

$\hat{\mathbf{C}}_{\mathbf{r}}$ can be derived by applying the covariance law to equation (III.3), i.e.,

$$\hat{\mathbf{C}}_{\mathbf{r}} = \mathbf{C}_{\omega} + \mathbf{M}\mathbf{C}_{\omega}\mathbf{M}^T - \mathbf{C}_{\omega}\mathbf{M}^T - \mathbf{M}\mathbf{C}_{\omega} \quad \text{III.5}$$

The covariance matrix \mathbf{C}_{ω} is derived considering the stochasticity of the initial positions as;

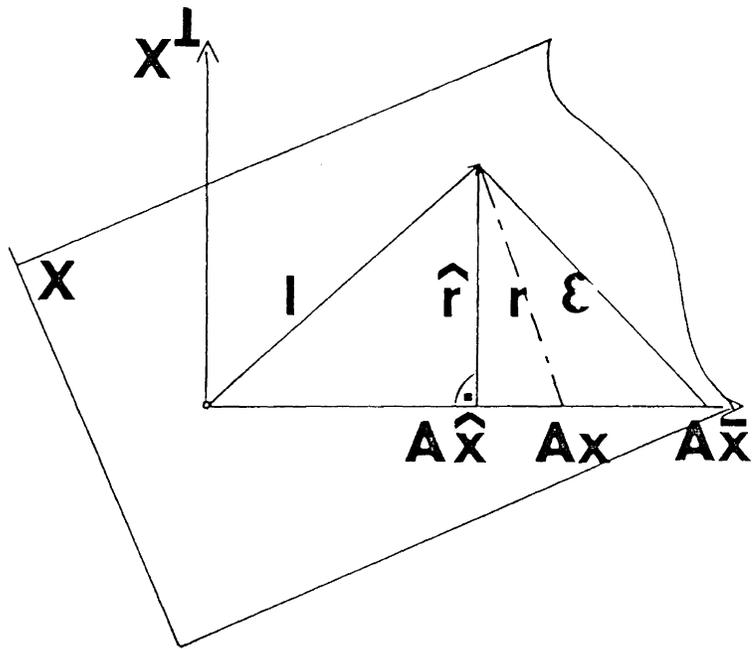


Figure III.1

The Geometry of the Least-Squares Residuals.

X - A subspace in a Hilbert space H .

X^\perp - The orthogonal complement of X .

\hat{r} - The least-squares residual vector $(\hat{Ax} - \ell)$.

r - The residual $Ax - \ell$

ϵ - The expected error $(A\bar{x} - \bar{\ell})$

$$C_{\omega} = P^{-1} + AP_X^{-1}A^T \quad \text{III.6}$$

where,

$$P^{-1} = C_{\rho}$$

$$P_X^{-1} = C_{x(0)}$$

The matrix product MC_{ω} is obtained from equations (III.4) and (III.6) as;

$$MC_{\omega} = A(A^T PA + P_X)^{-1} A^T P (P^{-1} + AP_X^{-1} A^T) \quad \text{III.7}$$

But [Liebelt, 1967];

$$(A^T PA + P_X)^{-1} A^T P = P_X^{-1} A (P^{-1} + AP_X^{-1} A^T) \quad \text{III.8}$$

Then,

$$MC_{\omega} = AP_X^{-1} A^T \quad \text{III.9}$$

Further,

$$C_{\omega} M^T = (MC_{\omega})^T \quad \text{III.10}$$

and

$$MC_{\omega} M^T = A(N + P_X)^{-1} A^T P A P_X^{-1} A^T \quad \text{III.11}$$

Substituting equations (III.6), (III.9), (III.10) and (III.11) into (III.5) gives;

$$C_{\rho}^{\wedge} = P^{-1} + AP_X^{-1} A^T + A(N + P_X)^{-1} A^T P A P_X^{-1} A^T - AP_X^{-1} A^T - A(N + P_X)^{-1} A^T P (P^{-1} + AP_X^{-1} A^T)$$

which can be simplified to;

$$C_{\rho}^{\wedge} = P^{-1} - A(N + P_X)^{-1} A^T \quad \text{III.12}$$

Equation (III.12) is the covariance matrix of the residuals in the P_X -adjustment.

The a posteriori variance factor

The expected value of the quadratic sum of the residuals

can be evaluated in terms of the matrix $M = A(N+Px)^{-1}A^T P$. Let
[Grafarend, 1981];

$$E(\hat{r}^T \hat{P} \hat{r}) = \text{tr } E(\hat{P} \hat{r} \hat{r}^T) \quad \text{III.13}$$

where,

$$\hat{r} = (I-M)\omega$$

$$\text{tr } E(\hat{P} \hat{r} \hat{r}^T) = \text{tr } [P(I-M)C_\omega(I-M)^T]$$

or

$$E(\hat{r}^T \hat{P} \hat{r}) = \text{tr } [PC_\omega - PMC_\omega - PC_\omega M^T + PMC_\omega M^T] \quad \text{III.14}$$

where; $C_\omega = E(\omega\omega^T)$

or (see equation (III.6) and introduce a scale factor σ_0^2

$$C_\omega = \sigma_0^2 [P^{-1} + APX^{-1}A^T] \quad \text{III.15}$$

Equation (III.14) can be evaluated further by substituting for the matrices C_ω and M . The following are the expressions of the matrix products involved.

$$PC_\omega = \sigma_0^2 I + \sigma_0^2 PAPX^{-1}A^T \quad \text{III.16}$$

$$PMC_\omega = \sigma_0^2 PA(N+PX)^{-1}A^T P(P^{-1} + APX^{-1}A^T)$$

Recalling the identity (III.8),

$$PMC_\omega = \sigma_0^2 PAPX^{-1}A^T \quad \text{III.17}$$

Further,

$$PC_\omega M^T = \sigma_0^2 PA(N+PX)^{-1}A^T + \sigma_0^2 PAPX^{-1}A^T PA(N+PX)^{-1}A^T \quad \text{III.18}$$

$$PMC_\omega M^T = \sigma_0^2 PA(N+PX)^{-1}A^T P(P^{-1} + APX^{-1}A^T) PA(N+PX)^{-1}A^T$$

again, using the identity (III.7) we obtain;

$$PMC_\omega M^T = \sigma_0^2 PAPX^{-1}A^T PA(N+PX)^{-1}A^T \quad \text{III.19}$$

When the expressions (III.16) - (III.19) are substituted into

(III.14) we get;

$$E(\hat{r}^T \hat{P} \hat{r}) = \sigma_0^2 \text{tr } (I) - \sigma_0^2 \text{tr } [PA(N+PX)^{-1}A^T] \quad \text{III.20}$$

We recall the properties of the trace [cf., Mikhail, 1976], in particular that;

$$\text{tr} (AB) = \text{tr} (BA)$$

then,

$$\text{tr} [PA(N+Px)^{-1}A^T] = \text{tr} [A(N+Px)^{-1}A^TP]$$

recalling equation (III.4) we obtain further that,

$$\text{tr} [PA(N+Px)^{-1}A^T] = \text{tr} (M) \quad \text{III.21}$$

Equation (III.20) is simplified by (III.21) to;

$$E(\hat{r}^T P \hat{r}) = \sigma_0^2 \text{tr} (I) - \sigma_0^2 \text{tr} (M) \quad \text{III.22}$$

The identity matrix in equation (III.22) has the following dimensions

$$\text{dim}(I) = (n,n)$$

The trace of I therefore equals to n, i.e.,

$$\text{tr} (I) = n$$

Let the trace of M be u' . Then;

$$E(\hat{r}^T P \hat{r}) = \sigma_0^2 (n - u') \quad \text{III.23}$$

Equation (III.23) gives the expected quadratic sum of the residuals when σ_0^2 is a priori known. The variance factor can also be

estimated using equation (III.23), when the expected quadratic sum of the residuals $\hat{r}^T P \hat{r}$ has been computed, i.e.,

$$\sigma_0^2 = \frac{\hat{r}^T P \hat{r}}{n - u'} \quad \text{III.24}$$

The denominator in (III.24) is the number of degrees of freedom in the Px-adjustment.

APPENDIX IV
SOLUTION OF TRIPLE PARTITIONED
SYSTEM OF NORMAL EQUATIONS

APPENDIX IV

TRIPLE PARTITIONING OF THE DENSIFICATION NETWORK

The Mathematical Model

The Mathematical Model in linear form for the weighted constraint adjustment

$$\tilde{A}_f \tilde{\delta}_j^f + \tilde{A}_j \tilde{\delta}_j^{nf} + A_n \delta_n - r + W = 0 \quad : C_2 \quad \text{IV.1}$$

$$\tilde{\delta}_j^f = r_f \quad : P_X^f \quad \text{IV.2}$$

$$\tilde{\delta}_j^{nf} = r_{nf} \quad : P_X^{nf} \quad \text{IV.5}$$

The Least-Squares Criterion

$$\min (r^T P_r r + r_f^T P_X^f r_f + r_{nf}^T P_X^{nf} r_{nf}) \quad \text{IV.4}$$

The Variation Function

$$\begin{aligned} \phi = & r^T P_r r + r_f^T P_X^f r_f + r_{nf}^T P_X^{nf} r_{nf} + 2K^T (\tilde{A}_f \tilde{\delta}_j^f + \tilde{A}_j \tilde{\delta}_j^{nf} + A_n \delta_n - r + W) + \\ & + 2K_f^T (\tilde{\delta}_j^f - r_f) + 2K_j^T (\tilde{\delta}_j^{nf} - r_{nf}) \end{aligned} \quad \text{IV.5}$$

The Normal Equations in Nine Unknowns

The Normal equations matrix for the combined adjustment in hypermatrix form becomes:

$$\left\{ \begin{array}{cccccccccc} P & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_X^f & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_X^{nf} & 0 & 0 & B & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 0 & \tilde{A}_f & \tilde{A}_j & A_n & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & \tilde{A}_f^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}_j^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_n^T & 0 & B & 0 & 0 & 0 & 0 \end{array} \right\} \left\{ \begin{array}{c} \hat{r} \\ \hat{r}_f \\ \hat{r}_{nf} \\ \hat{K} \\ \hat{K}_f \\ \hat{K}_j \\ \hat{\delta}_n \\ \hat{\delta}_f \\ \hat{\delta}_j \\ \hat{\delta}_j^{nf} \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ W \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} = 0 \quad \text{IV.6}$$

Eliminating \hat{r} , \hat{r}_f and \hat{r}_{nf} reduces the hypermatrix to:

$$\begin{pmatrix} P^{-1} & 0 & 0 & \tilde{A}_f & \tilde{A}_j & A_n \\ 0 & (Px^f)^{-1} & 0 & 0 & B & 0 \\ 0 & 0 & (Px^{nf})^{-1} & 0 & 0 & B \\ \tilde{A}_f^T & 0 & 0 & 0 & 0 & 0 \\ \tilde{A}_j^T & B & 0 & 0 & 0 & 0 \\ \tilde{A}_n^T & 0 & B & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{K} \\ \hat{K}_f \\ \hat{K}_j \\ \hat{\delta}_n \\ \hat{\delta}_j^f \\ \hat{\delta}_j^{nf} \end{pmatrix} + \begin{pmatrix} W \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \emptyset \quad \text{IV.7}$$

The Normal Equations in Three Unknowns (setting B = -I)

Eliminating \hat{K} , \hat{K}_f and \hat{K}_j leads to:

$$\begin{pmatrix} A_n^T P A_n & A_n^T \tilde{P} A_f & A_n^T \tilde{P} A_j \\ \tilde{A}_f^T P A_n & (\tilde{A}_f^T \tilde{P} A_f + Px^f) & A_f^T P A_j \\ \tilde{A}_j^T P A_n & \tilde{A}_j^T \tilde{P} A_f & (\tilde{A}_j^T \tilde{P} A_j + Px^{nf}) \end{pmatrix} \begin{pmatrix} \hat{\delta}_n \\ \hat{\delta}_j^f \\ \hat{\delta}_j^{nf} \end{pmatrix} + \begin{pmatrix} A_n^T P W \\ A_f^T P W \\ A_j^T P W \end{pmatrix} = \emptyset \quad \text{IV.8}$$

or

$$\begin{pmatrix} (\tilde{A}_f^T \tilde{P} A_f + Px^f) & \tilde{A}_f^T \tilde{P} A_j & \tilde{A}_j^T P A_n \\ \tilde{A}_j^T P A_n & (\tilde{A}_j^T \tilde{P} A_j + Px^{nf}) & A_n^T P A_n \\ A_n^T P A_f & A_n^T P A_j & A_n^T P A_n \end{pmatrix} \begin{pmatrix} \hat{\delta}_j^f \\ \hat{\delta}_j^{nf} \\ \hat{\delta}_n \end{pmatrix} + \begin{pmatrix} \tilde{A}_f^T P W \\ \tilde{A}_j^T P W \\ A_n^T P W \end{pmatrix} = \emptyset \quad \text{IV.9}$$

or

$$\begin{pmatrix} \hat{\delta}_j^f \\ \hat{\delta}_j^{nf} \\ \hat{\delta}_n \end{pmatrix} = - \begin{pmatrix} (\tilde{A}_j^T \tilde{P} A_j + Px^{nf}) & \tilde{A}_f^T \tilde{P} A_j & \tilde{A}_f^T P A_n \\ \tilde{A}_j^T P A_n & A_n^T P A_f & A_n^T P A_n \\ A_n^T P A_f & A_n^T P A_j & A_n^T P A_n \end{pmatrix}^{-1} \begin{pmatrix} \tilde{A}_f^T P W \\ \tilde{A}_j^T P W \\ A_n^T P W \end{pmatrix} \quad \text{IV.10}$$

The Normal Equations in Two Unknowns

From (IV.9) we obtain;

$$\begin{bmatrix} (\tilde{A}_j^T \tilde{P} \tilde{A}_j + P_x^{nf}) - \tilde{A}_j^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_j & \tilde{A}_j^T \tilde{P} \tilde{A}_n - \tilde{A}_j^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_n \\ \tilde{A}_n^T \tilde{P} \tilde{A}_j - \tilde{A}_n^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_j & \tilde{A}_n^T \tilde{P} \tilde{A}_n - \tilde{A}_n^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_n \end{bmatrix} \begin{bmatrix} \hat{\delta}_j^{nf} \\ \hat{\delta}_n \end{bmatrix} + \begin{bmatrix} \tilde{A}_j^T \tilde{P} \tilde{W} - \tilde{A}_j^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{W} \\ \tilde{A}_n^T \tilde{P} \tilde{W} - \tilde{A}_n^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{W} \end{bmatrix} = \emptyset \quad \text{IV.11}$$

Let,

$$N_{jj} = (\tilde{A}_j^T \tilde{P} \tilde{A}_j + P_x^{nf}) - \tilde{A}_j^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_j \quad \text{IV.12}$$

$$N_{jn} = \tilde{A}_j^T \tilde{P} \tilde{A}_n - \tilde{A}_j^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_n \quad \text{IV.13}$$

$$N_{nj} = \tilde{A}_n^T \tilde{P} \tilde{A}_j - \tilde{A}_n^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_j \quad \text{IV.14}$$

$$N_{nn} = \tilde{A}_n^T \tilde{P} \tilde{A}_n - \tilde{A}_n^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{A}_n \quad \text{IV.15}$$

$$u_j = \tilde{A}_j^T \tilde{P} \tilde{W} - \tilde{A}_j^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{W} \quad \text{IV.16}$$

$$u_n = \tilde{A}_n^T \tilde{P} \tilde{W} - \tilde{A}_n^T \tilde{P} \tilde{A}_f (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_x^f)^{-1} \tilde{A}_f^T \tilde{P} \tilde{W} \quad \text{IV.17}$$

Then,

$$\begin{bmatrix} \hat{\delta}_j^{nf} \\ \hat{\delta}_n \end{bmatrix} = \begin{bmatrix} N_{jj} & N_{jn} \\ N_{nj} & N_{nn} \end{bmatrix}^{-1} \begin{bmatrix} u_j \\ u_n \end{bmatrix} \quad \text{IV.18}$$

or

$$\begin{bmatrix} \hat{\delta}_j^{nf} \\ \hat{\delta}_n \end{bmatrix} = - \begin{bmatrix} H_{jj} & H_{jn} \\ H_{nj} & H_{nn} \end{bmatrix} \begin{bmatrix} u_j \\ u_n \end{bmatrix} \quad \text{IV.19}$$

where,

$$H_{jj} = (N_{jj} - N_{jn} N_{nn}^{-1} N_{nj})^{-1} \quad \text{IV.20}$$

$$H_{nn} = (N_{nn} - N_{nj} N_{jj}^{-1} N_{jn})^{-1} \quad \text{IV.21}$$

$$H_{jn} = -N_{jj}^{-1} N_{jn} H_{nn} = -H_{jj} N_{jn} N_{nn}^{-1} \quad \text{IV.22}$$

$$H_{nj} = -N_{nn}^{-1} N_{nj} H_{jj} = -H_{nn} N_{nj} N_{jj}^{-1} \quad \text{IV.23}$$

The Solution Vectors

The solution vector, $\hat{\delta}_j^{nf}$

$$\hat{\delta}_j^{nf} = -H_{jj} u_j - H_{jn} u_n \quad \text{IV.24}$$

or

$$\hat{\delta}_j^{nf} = H_{jj} (N_{jn} N_{nn}^{-1} u_n - u_j) \quad \text{IV.25}$$

The solution vector, $\hat{\delta}_n$

$$\hat{\delta}_n = -H_{nj} u_j - H_{nn} u_n \quad \text{IV.26}$$

or

$$\hat{\delta}_n = H_{nn} (N_{nj} N_{jj}^{-1} u_j - u_n) \quad \text{IV.27}$$

Eliminating $\hat{\delta}_n$ from (IV.8) leads to;

$$\begin{bmatrix} (P_X^{nf} + \tilde{A}_j^T \tilde{P} \tilde{A}_j) - \tilde{A}_j^T \tilde{P} \tilde{A}_n (A_n^T \tilde{P} A_n)^{-1} A_n^T \tilde{P} \tilde{A}_j & \tilde{A}_j^T \tilde{P} \tilde{A}_f - \tilde{A}_j^T \tilde{P} \tilde{A}_n (A_n^T \tilde{P} A_n)^{-1} A_n^T \tilde{P} \tilde{A}_f \\ \tilde{A}_f^T \tilde{P} \tilde{A}_j - \tilde{A}_f^T \tilde{P} \tilde{A}_n (A_n^T \tilde{P} A_n)^{-1} A_n^T \tilde{P} \tilde{A}_j & (P_X^f + \tilde{A}_f^T \tilde{P} \tilde{A}_f) - \tilde{A}_f^T \tilde{P} \tilde{A}_n (A_n^T \tilde{P} A_n)^{-1} A_n^T \tilde{P} \tilde{A}_f \end{bmatrix} \begin{bmatrix} \hat{\delta}_j^{nf} \\ \hat{\delta}_j^f \end{bmatrix} + \begin{bmatrix} \tilde{A}_j^T \tilde{P} \tilde{W} - \tilde{A}_j^T \tilde{P} \tilde{A}_n (A_n^T \tilde{P} A_n)^{-1} A_n^T \tilde{P} \tilde{W} \\ \tilde{A}_f^T \tilde{P} \tilde{W} - \tilde{A}_f^T \tilde{P} \tilde{A}_n (A_n^T \tilde{P} A_n)^{-1} A_n^T \tilde{P} \tilde{W} \end{bmatrix} = \emptyset \quad \text{IV.28}$$

$$\text{Let, } N_{jj}^* = (\tilde{A}_j^T \tilde{P} \tilde{A}_j + P_X n_f) - \tilde{A}_j^T \tilde{P} A_n (A_n^T \tilde{P} A_n)^{-1} \tilde{A}_n^T \tilde{P} A_j \quad \text{IV-29}$$

$$N_{ff} = (\tilde{A}_f^T \tilde{P} \tilde{A}_f + P_X f) - \tilde{A}_f^T \tilde{P} A_n (A_n^T \tilde{P} A_n)^{-1} \tilde{A}_n^T \tilde{P} \tilde{A}_f \quad \text{IV-30}$$

$$N_{jf} = \tilde{A}_j^T \tilde{P} \tilde{A}_f - \tilde{A}_j^T \tilde{P} A_n (A_n^T \tilde{P} A_n)^{-1} \tilde{A}_n^T \tilde{P} \tilde{A}_f \quad \text{IV-31}$$

$$N_{fj} = \tilde{A}_f^T \tilde{P} \tilde{A}_j - \tilde{A}_f^T \tilde{P} A_n (A_n^T \tilde{P} A_n)^{-1} \tilde{A}_n^T \tilde{P} \tilde{A}_j \quad \text{IV-32}$$

$$u_j = \tilde{A}_j^T \tilde{P} W - \tilde{A}_j^T \tilde{P} A_n (A_n^T \tilde{P} A_n)^{-1} \tilde{A}_n^T \tilde{P} W \quad \text{IV-33}$$

$$u_f = \tilde{A}_f^T \tilde{P} W - \tilde{A}_f^T \tilde{P} A_n (A_n^T \tilde{P} A_n)^{-1} \tilde{A}_n^T \tilde{P} W \quad \text{IV-34}$$

Then,

$$\begin{bmatrix} \hat{\delta}_j^{nf} \\ \hat{\delta}_j^f \end{bmatrix} = - \begin{bmatrix} N_{jj}^* & N_{jf} \\ N_{fj} & N_{ff} \end{bmatrix}^{-1} \begin{bmatrix} u_j^* \\ u_f \end{bmatrix} \quad \text{IV-35}$$

or

$$\begin{bmatrix} \hat{\delta}_j^{nf} \\ \hat{\delta}_j^f \end{bmatrix} = - \begin{bmatrix} H_{jj}^* & H_{jf} \\ H_{fj} & H_{ff} \end{bmatrix} \begin{bmatrix} u_j^* \\ u_f \end{bmatrix} \quad \text{IV-36}$$

$$\text{where, } H_{jj}^* = (N_{jj}^* - N_{jf} N_{ff}^{-1} N_{fj})^{-1}$$

$$H_{ff} = (N_{ff} - N_{fj} N_{jj}^{*-1} N_{jf})^{-1}$$

$$H_{fj} = -H_{ff} N_{fj} N_{jj}^{-1} \quad \text{IV-37}$$

$$H_{jf} = -H_{jj}^* N_{jf} N_{ff}^{-1} \quad \text{IV-38}$$

The solution vectors $\hat{\delta}_j^{nf}$, $\hat{\delta}_j^f$

$$\hat{\delta}_j^{nf} = -H_{jj}^* u_j^* - H_{jf} u_f$$

$$\hat{\delta}_j^{nf} = H_{jj}^* (N_{jf} N_{ff}^{-1} u_f - u_j^*) \quad \text{IV-39}$$

Similarly,

$$\hat{\delta}_j^f = -H_{ff} u_f - H_{fj} u_j^*$$

$$\hat{\delta}_j^f = H_{ff} (N_{fj} N_{jj}^{-1} u_j^* - u_f) \quad \text{IV-40}$$

APPENDIX V
COMPATIBILITY TEST SOFTWARE
(Computer Program CTEST)

```
1 //AABKGP51 JOB '6703,RAGO',LUGOE,MSGCLASS=A,NOTIFY=6767
  ***PASSWORD SACF1770 PASSWORD(S) SUCCESSFULLY SCANNED
  ***JOBPARM M=8,L=50,R=3072
2 // EXEC FORTYCLG,RQ=3072K,PARM.FORT='LANGLVL(66),KOMAP'
11 //FORT.SYSIN DD *
19 //LKED.USERLIB DD DSN=A.M12129.SELIBOBJ,DISP=SHR
29 //GO.FT01F001 DD DSN=A.M67034.EXISTJNC.DATA,DISP=SHR
30 //GO.FT02F001 DD DSN=A.M67034.CVMTB.DATA,DISP=SHR
31 //GO.FT03F001 DD DSN=A.M67034.CVMTA.DATA,DISP=SHR
32 //GO.FT15F001 DD DSN=A.M67034.DENSJNC.DATA,DISP=SHR
33 //GO.SYSIN DD *
//
```

JOB 2779

.....1.....2.....3.....4.....5.....6.....7.....8

C
C *****

C C T E S T
C THIS PROGRAM IS PART OF THE DENSIFICATION PACKAGE
C PROGRAM TESTS THE COMPATIBILITY OF ADJUSTED COORDINATES
C GROUPS OF STATIONS ARE ALSO TESTED ON REQUEST
C COVARIANCES ARE RIGOROUSLY CONSIDERED
C BY F.N.LUGOE - SEPTEMBER 26,1984

C *****

C THIS PROGRAM IS ADAPTED TO ACCEPT THE COVARIANCE MATRIX
C IN THE FORMAT GIVEN BY OUTPUT FROM PROGRAM GEOPAN
C FOR ANY OTHER FORMAT SUBROUTINE READ MUST BE MODIFIED

C *****

ISN 1 C IMPLICIT REAL*8(A-H,O-Z)
ISN 2 C SET DIMENSIONS EQUAL TO EXPECTED TOTAL NUMBER OF STATIONS
C DIMENSION N(100),N3(100),X(100),Y(100),X2(100),Y2(100)
ISN 3 C DIMENSION VECTORS EQUAL ROW DIMENSION OF OUTPUT GEOPAN MATRIX
C DIMENSION N1(1014),N2(1014)
ISN 4 C CHANGE DIMENSIONS BELOW TO SUIT THE NETWORK TO BE TESTED
ISN 5 C DIMENSION CDX(78,78),CX2(78,78),CY2(78,78),PCX(78),YDC(78,78)
C DIMENSION CDY(78,78),DCX(78,78),YDY(78)
ISN 6 C DIMENSIONS TO EQUAL TOTAL NUMBER OF EXPECTED NETWORK STATIONS
C DIMENSION DX1(100),DDX(100),DX(100),DY(100)
ISN 7 C DIMENSION BELOW TO EQUAL NUMBER OF POINTS IN THE NETWORK
C INTEGER IW1(78),IW2(78)
C READ ENTITIES FROM A CARD
ISN 8 C READ(5,*)IQ1,NQ,IFAC,IE,IH,ICODE,VAR3,VAR2,ALPHA
C DIMENSION OF THE POSITION VECTOR=TWICE THE NUMBER OF STATIONS
C IS IQ1.
C COMPATIBILITY TESTING OF NQ-POINT SUBNETWORKS CONSIDERED.
C THE FACTOR FOR SCALING COORDINATE DIFFERENCES AND COVARIANCE
C MATRIX TO AVOID UNDERFLOW IS IFAC.
C THE COLUMN DIMENSION OF EACH GEOPAN PRINTED SUBMATRIX IS IE.
C THE TOTAL NUMBER OF COLUMNS IS IH.
C ICODE=0 IF CROSS-COVARIANCE BETWEEN SOLUTIONS IS ZERO AND
C ICODE=1 IF CROSS-COVARIANCE IS NON-ZERO.
C THE A POSTERIORI VARIANCE FACTORS IF COVARIANCE MATRICES WERE
C NOT PROPERLY SCALED ARE VAR3 AND VAR2 OR 1 IF ALREADY SCALED.

```

      *.....1.....2.....3.....4.....5.....6.....7.....8
C     THE SIGNIFICANCE LEVEL OF THE TEST IS ALPHA.
C
ISN   9      IQQ=MQ*2
ISN  10      JSUM=1
ISN  11      ALPH=(1.0-ALPHA/100.0)*(IQ1/IQQ)
ISN  12      IF(ALPH.GE.1.0)GOTO 9
ISN  13      ALP=(1.0-ALPH)*100
ISN  14      IF(ALPH.LT.1.0)GOTO 4
ISN  15      9 ALP=0.0
C     CHANGE CYCLES IN LOOP WHEN NETWORK HAS >100 STATIONS
ISN  16      4 DO 1 I=1,100
C
C     READ CO-ORDINATES FROM EXISTING SOLUTION
C     AND CORRESPONDING STATION NUMBERS
ISN  17      READ(1,200,END=2)N(I),X(I),Y(I)
C
C     READ THE COORDINATES FROM THE DENSIFICATION SOLUTION
C     AND CORRESPONDING STATION NUMBERS
ISN  18      1 READ(16,200)N3(I),X2(I),Y2(I)
ISN  19      2 ICOUNT=I-1
ISN  20      IQ=ICOUNT*2
ISN  21      IK=IQ**2
ISN  22      IQ=IQ-IE
C
C     COMPUTE DISPLACEMENT VECTORS
C
ISN  23      DO 3 I=1,ICOUNT
ISN  24          DX(I)=X2(I)-X(I)
ISN  25          DY(I)=Y2(I)-Y(I)
ISN  26      3 CONTINUE
ISN  27      I=0
ISN  28      J=0
ISN  29      DO 5 II=1,ICOUNT
ISN  30          I=I+1
ISN  31          J=I+1
ISN  32          DDX(I)=DX(II)
ISN  33          DDX(J)=DY(II)
ISN  34          I=I+1
ISN  35      5 CONTINUE
C
ISN  36      SUM1=0.0
ISN  37      SUM2=0.0
ISN  38      K=0
ISN  39      ISUM=0
C
ISN  40      WRITE(6,552)
C     READ COVARIANCE MATRIX OF THE EXISTING AND DENSIFICATION SOLUTIONS
C
ISN  41      652 FORMAT('-', 'THE NUMBER OF EXCLUDED POINTS IS:',I4/)
ISN  42      IRDA=IQ1
ISN  43      NA=IQ
ISN  44      MA=IQ
ISN  45      CALL READ(2,IQ,IE,IH,6,CY2,M1)
ISN  46      CALL READ(3,IQ,IE,IH,6,CX2,M2)

```

.....1.....2.....3.....4.....5.....6.....7.....8

```
C
C   SCALE THE INPUT COVARIANCE MATRICES
C
ISM  47      WRITE(6,454)
C      CALL MOUTD(CY2,IQ1,IQ1,IQ1)
ISM  48      DO 21 I=1,IQ1
ISM  49      DO 20 J=1,IQ1
ISM  50          CDY(I,J)=0.000
ISM  51      20 CONTINUE
ISM  52      21 CONTINUE
ISM  53      DO 31 I=1,IQ1
ISM  54      DO 30 J=1,IQ1
ISM  55          CDY(I,J)=CDY(I,J)+CY2(I,J)*VAR2*IFAC
ISM  56      30 CONTINUE
ISM  57      31 CONTINUE
ISM  58      CALL MOUTD(CDY,IQ1,IQ1,IQ1)
ISM  59      47 FORMAT('-', 'THE DETERMINANT IS:', F25.15)
ISM  60      WRITE(6,554)
C      CALL MOUTD(CX2,IQ1,IQ1,IQ1)
ISM  61      DO 23 I=1,IQ1
ISM  62      DO 22 J=1,IQ1
ISM  63          CDX(I,J)=0.000
ISM  64      22 CONTINUE
ISM  65      23 CONTINUE
ISM  66      DO 39 I=1,IQ1
ISM  67      DO 38 J=1,IQ1
ISM  68          CDX(I,J)=CDX(I,J)+CX2(I,J)*VAR3*IFAC
ISM  69      38 CONTINUE
ISM  70      39 CONTINUE
ISM  71      CALL MOUTD(CDX,IQ1,IQ1,IQ1)
C
C   DERIVE COVARIANCE MATRIX OF DISPLACEMENTS
C
ISM  72      IF(ICODE.NE.0)GOTO 12
ISM  73      CALL MADDD(DCX,IQ1,CDX,IQ1,CDY,IQ1,IQ1,IQ1)
ISM  74      IF(ICODE.EQ.0)GOTO 13
ISM  75      12 CALL MSUBD(DCX,IQ1,CDX,IQ1,CDY,IQ1,IQ1,IQ1)
ISM  76      13 WRITE(6,555)
ISM  77      CALL MOUTD(DCX,IQ1,IQ1,IQ1)
C
C   INVERT COVARIANCE MATRIX OF DISPLACEMENTS
C
ISM  78      CALL MINVD(DCX,IRDA,NA,DETA,IW1,IW2)
ISM  79      WRITE(6,47)DETA
ISM  80      DO 26 I=1,IQ
ISM  81      DO 25 J=1,IQ
ISM  82          DCX(I,J)=DCX(I,J)*IFAC
ISM  83      25 CONTINUE
ISM  84      26 CONTINUE
ISM  85      WRITE(6,556)
ISM  86      CALL MOUTD(DCX,IRDA,NA,NA)
C
C   DERIVE THE WEIGHTED SUM OF THE SQUARES OF DISPLACEMENTS
C
ISM  87      CALL MTVMUL(DCX,IQ1,DDX,IQ,PCX,IQ,NA,NA)
ISM  88      CDDX=0.0
```

.....1.....2.....3.....4.....5.....6.....7......8

```
ISM      89      DD 60 I=1,IQ
ISM      90      CDDX=CDDX+PCX(I)*DDX(I)
ISM      91      60      CONTINUE
ISM      92      454  FORMAT('-', 'THE EXISTING COVARIANCE MATRIX '/')
ISM      93      466  FORMAT('-', 'THE WEIGHT MATRIX OF DISPLACEMENTS'/)
ISM      94      554  FORMAT('-', 'THE DENSIFICATION COVARIANCE MATRIX '/')
ISM      95      555  FORMAT('-', 'THE COVARIANCE MATRIX OF DISPLACEMENTS'/)
ISM      96      556  FORMAT('-', 'THE WEIGHT MATRIX OF DISPLACEMENTS'/)

C
C      PERFORM COMPATIBILITY TEST FOR INDIVIDUAL SUBNETWORKS
C
ISM      97      IF(IQQ.EQ.IQ)GOTO 80
ISM      98      IJ=1
ISM      99      K=0
ISM     100      120  ADX=0.0
ISM     101      WRITE(6,754)
ISM     102      754  FORMAT('1', 'TESTING INDIVIDUAL SUBNETWORKS'/)
ISM     103      WRITE(6,756)
ISM     104      756  FORMAT('-', 'THE SUBNETWORK POINTS ARE: '/')
ISM     105      JR=IQ*JSUM
ISM     106      758  FORMAT(7X,I4)
ISM     107      DO 121 I=1,IQQ
ISM     108      K=K+1
ISM     109      IF(IJ.GT.JR)GOTO 11
ISM     110      WRITE(6,758)N(IJ)
ISM     111      IJ=IJ+1
ISM     112      11  CONTINUE
ISM     113      DX1(I)=DDX(K)
ISM     114      121  CONTINUE
ISM     115      ADX=ADX+CHISQ(ALP,IQQ)
ISM     116      10  FORMAT('-', 'CONFIDENCE LEVEL AND DEGREES OF FREEDOM'/,
C      @'-', 'ON WHICH NETWORKS ARE TESTED ARE:',F9.2,I4/)
C      COMPILE APPROPRIATE WEIGHT MATRIX FOR EACH SUBNETWORK
C
ISM     117      CALL PMAT(DCX,IQ1,YDC,IQQ,IQ,JSUM)
ISM     118      WRITE(6,456)
ISM     119      CALL MOUTD(YDC,IQ1,IQQ,IQQ)

C
C      DERIVE THE SUM OF WEIGHTED DISPLACEMENTS FOR EACH SUBNETWORK
C
ISM     120      CALL MTVMUL(YDC,IQ1,DX1,IQ,YDY,IQ,IQ,IQ1)
ISM     121      CYY=0.0
ISM     122      DO 122 I=1,IQQ
ISM     123      CYY=CYY+YDY(I)*DX1(I)
ISM     124      122  CONTINUE
ISM     125      WRITE(6,10)ALP,IQQ
ISM     126      WRITE(6,209)ADX
ISM     127      WRITE(6,191)CYY
ISM     128      IF(CYY.LE.ADX)WRITE(6,762)
ISM     129      IF(CYY.GT.ADX)WRITE(6,752)
ISM     130      KRR=IQQ*JSUM
ISM     131      JSUM=JSUM+1
ISM     132      IRR=(IQ-KRR)/2
ISM     133      IF(IRR.GE.0)GOTO 14
ISM     134      IRR=0
ISM     135      14  WRITE(6,652)IRR
```

*.....1.....2.....3.....4.....5.....6.....7.....8

```
ISN 138      IF(KRR.LT.IQ)GOTO 120
ISN 139      80 CONTINUE
ISN 140      WRITE(6,125)
ISN 141      125 FORMAT('1','TESTING ALL POINTS TOGETHER'//)
C
C      COMPUTE THE CHI-SQUARE CRITICAL VALUE FOR THE WHOLE NETWORK
C
ISN 142      DO 56 I=1,ICOUNT
ISN 143      DDPX=X2(I)-X(I)
ISN 144      DDPY=Y2(I)-Y(I)
ISN 145      XYP=DSQRT(DDPX**2+DDPY**2)
ISN 146      WRITE(6,717)N3(I),DDPX,DDPY,XYP
ISN 147      717 FORMAT('-','DIFFERENCES IN COORDINATES AND POSITIONS ARE:',//,
      @7X,I4,3F15.6/)
ISN 148      56 CONTINUE
ISN 149      QX=CHISQ(ALPHA,IQ)
ISN 150      ISUM=ISUM+1
ISN 151      ID=IQ*ISUM
ISN 152      IR=(IQ-ID)/2
ISN 153      WRITE(6,652)IR
C
C      TEST THE COMPATIBILITY OF NETWORK POINTS TOGETHER
ISN 154      WRITE(6,750)
ISN 155      WRITE(6,10)ALPHA,IQ
ISN 156      WRITE(6,209)QX
ISN 157      WRITE(6,191)CDDX
ISN 158      IF(CDDX.LE.QX)WRITE(6,762)
ISN 160      IF(CDDX.GT.QX)WRITE(6,752)
ISN 162      750 FORMAT('-','TEST R E S U L T S FOR THE WHOLE N E T W O R K'//)
ISN 163      752 FORMAT('-','***WARNING***THE TWO SETS ARE STATISTICALLY N O T
      @COMPATIBLE'//)
ISN 164      762 FORMAT('-','THE TWO SETS ARE STATISTICALLY COMPATIBLE'//,
      @'-','*** T H E   D E N S I F I C A T I O N   I S   R I G O R O U S ***'//)
**
ISN 165      552 FORMAT('-','I M P U T   C O V A R I A N C E   M A T R I C E S'//)
ISN 166      191 FORMAT('-','THE SUM OF WEIGHTED DISPLACEMENTS EQUAL TO: ',7X,2F14.8//)
**
ISN 167      209 FORMAT('-','F O R   C O M P A T I B I L I T Y, S U M O F W E I G H T E D D I S P L A C E M E N T S'//,
      @7X,'MUST BE S M A L L E R   O R   E Q U A L   T O: ',7X,2F10.6//)
ISN 168      200 FORMAT(I8,2F15.4)
ISN 169      STOP
ISN 170      END
```

*.....1.....2.....3.....4.....5.....6.....7.....8

```
C
C *****
C
C      EXTRACTING SUBMATRICES ABOUT THE DIAGONAL
C
C      SUBROUTINE PMAT
C
C      BY F.M.LUGOE AUGUST,1984
C
C *****
C
C SUBROUTINE TO EXTRACT APPROPRIATE SUBMATRIX OF THE WEIGHT
C MATRIX REQUIRED IN TESTING INDIVIDUAL SUBNETWORKS
C
```

```
ISN      1      SUBROUTINE PMAT(DCY,MS,YDC,IQQ,NM,N)
ISN      2      IMPLICIT REAL*8(A-H,O-Z)
ISN      3      DIMENSION DCY(MS,NM),YDC(MS,MS)
C
C      N IS NUMBER OF SUBNETWORKS IN THE NETWORK
C      IQQ IN THE NUMBER OF POINTS IN A SUBNETWORK
C      MS EQUALS TO NM THE TOTAL NUMBER OF POINTS IN NETWORK
ISN      4      DO 2 IM=1,MS
ISN      5      DO 1 JM=1,MS
ISN      6          YDC(IM,JM)=0.00
ISN      7      1 CONTINUE
ISN      8      2 CONTINUE
ISN      9      I=N*IQQ-IQQ+1
ISN     10      J=I
ISN     11      IJ=I
ISN     12      DO 6 II=1,IQQ
ISN     13      DO 6 JJ=1,IQQ
ISN     14          YDC(II,JJ)=YDC(II,JJ)+DCY(I,J)
ISN     15      5      J=J+1
ISN     16          J=IJ
ISN     17          I=I+1
ISN     18          IF(I.GT.MS)GOTO 7
ISN     19      6 CONTINUE
ISN     20      7 CONTINUE
ISN     21      RETURN
ISN     22      END
```



```

      *.....1.....2.....3.....4.....5.....6.....7.*.....8
ISN   37      DO 9 I=1,7
ISN   38      9 READ(M,6,END=10)
ISN   39      IK=IK+1
ISN   40      II=IC
ISN   41      J1=J3+1
ISN   42      IF(IK.LE.ID)GOTO 7
ISN   43      10 CONTINUE
      C      WRITE(6,11)
      C      11 FORMAT('-','THE MATRIX FROM SUBROUTINE READ'/)
      C      CALL MOUTD(CX,IQ,IQ,IQ)
ISN   44      12 CONTINUE
ISN   45      RETURN
ISN   46      END

```

*.....1.....2.....3.....4.....5.....6.....7.....8

C
C
C
C
C

MULTIPLYING A MATRIX BY A VECTOR

```
ISN 1      SUBROUTINE MTVMUL(A,IA,B,IB,C,IC,M,N)
ISN 2      REAL*8 A(IA,M),B(IB),C(IC)
ISN 3      DO 10 I=1,M
ISN 4      C(I)=0
ISN 5      DO 10 J=1,N
ISN 6  10    C(I)=A(I,J)*B(J)+C(I)
ISN 7      RETURN
ISN 8      END
```

APPENDIX V-2
PROGRAM TO COMPUTE DISPLACEMENT GRADIENTS AND
CURVATURES AT NETWORK STATIONS
(Computer Program STRAIN1)

```
1 //AABKQP50 JOB '6703,RAGO',LUGOE,MSGCLASS=A,NOTIFY=6767 JOB 2715
  ***PASSWORD SACF1770 PASSWORD(S) SUCCESSFULLY SCANNED
  ***JOBPARM M=5,L=50,R=2048
2 // EXEC FORTVCG,RC=1024K,RG=1024K,PARM.FORT='LANGLYL(66),NOMAP,NOLIST'
11 //FORT.SYSIN DD *
25 //GO.FT03F001 DD DSN=A.M67034.STROW2.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
26 //GO.FT01F001 DD DSN=A.M67034.EXISTJNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
27 //GO.FT15F001 DD DSN=A.M67034.DENSJNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
28 //GO.FT17F001 DD DSN=A.M67034.MTX2JNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=OLD
29 //GO.FT09F001 DD DSN=A.M67034.DXDY2JNC.OMEGA,
  // UNIT=P3350,VOL=SER=USER11,DISP=OLD
30 //GO.SYSIN DD *
  //
```

.....1.....2.....3.....4.....5.....6.....7.....8

```
C *****
C
C           S T R A I N 1
C
C   A MODIFICATION OF THE STRAIN PROGRAM BY K. THAPA(1980)
C
C   PACKAGE TO COMPUTE DISPLACEMENT GRADIENTS AND CURVATURES
C
C           B Y
C
C           F U R A H A M. L U G O E
C
C           S E P T., 1 9 8 3
C
C   THIS PROGRAM IS PART OF THE DENSIFICATION PACKAGE
C *****
```

```
ISN      1      IMPLICIT REAL*8(A-H,O-Z)
ISN      2      DIMENSION R(20,3),Q(3,20),S(3,3),EE(3),FF(3),CDX(20),CDY(20)
ISN      3      DIMENSION CG(20),CH(20),TS(3,3),TE(3),TF(3),EO(3),FO(3)
ISN      4      DIMENSION A(20,3),B(20),C(3,3),D(20),E(3),F(3),T(3,20),P(3,3)
ISN      5      DIMENSION N(800),M3(800),X(800),Y(800),X2(800),Y2(800)
ISN      6      DIMENSION DX(20),DY(20),ST(20)
ISN      7      DIMENSION G(20),H(20),IW1(3),IW2(3)
ISN      8      DIMENSION S1(3,3),S2(3,3),SP(3,3),SS(3,3),SSS(3,3),PS(3,3)
ISN      9      DIMENSION SH(3),SG(3),GG(3),HH(3)
ISN     10      IRDA=3
ISN     11      NA=3
ISN     12      MA=3
ISN     13      PI=DARCOS(-1.00)
ISN     14      N(1)=0
ISN     15      DO 1 I=2,800
C          READ CO-ORDINATES OF FIRST ADJUSTMENT AND STATION NUMBERS
C
ISN     16      READ(1,200,END=2)N(I),X(I),Y(I)
C
C          READ CO-ORDINATES OF SECOND ADJUSTMENT AND STATION NUMBERS
C
ISN     17      1 READ(15,200)M3(I),X2(I),Y2(I)
ISN     18      2 ICOUNT=I-1
ISN     19      K=1
C
C          READ THE STATION NUMBERS ARRANGED IN TWO COLUMNS GIVING
C          THE CONNECTIVITY AT EACH STATION
C
ISN     20      30 READ(3,100,END=50)M1,M2
ISN     21      100 FORMAT(2X,I8,2X,I8)
ISN     22      IF(M1.EQ.N(K))GOTO 4
ISN     23      IF(K.EQ.1)GOTO 6
```

♦.....♦.....1.....2.....3.....4.....5.....6.....7.....8

```
ISM 24 IF(M1.EQ.6910170)GOTO 30
ISM 25 50 WRITE(6,44)N(K)
ISM 26 44 FORMAT(' THE DESIGN MATRIX FOR THIS STATION:',I4)
ISM 27 CALL MOUTD(A,20,J,NA)
C
C BY CALLING THESE SUBROUTINES LS-SOLUTION FOR DISPLACEMENT
C GRADIENTS ARE OBTAINED
C TRANSPOSE DESIGN MATRIX 'A' TO OBTAIN MATRIX 'T'
ISM 28 CALL MATTP(A,20,T,3,J,3)
C MULTIPLY THE MATRICES 'A' AND 'T' TO GET 'P'
ISM 29 CALL MATMUL(T,3,A,20,P,3,3,J,3)
C PRINT MATRIX 'P' FOR EACH STATION
ISM 30 WRITE(6,444)N(K)
ISM 31 444 FORMAT(' THE MATRIX OF NORMAL EQUATIONS FOR STATION:',I4)
ISM 32 CALL MOUTD(P,IRDA,NA,MA)
C FIND THE NORMAL EQUATIONS INVERSE 'P'
ISM 33 CALL MINVD(P,IRDA,NA,DETA,IW1,IW2)
C PRINT INVERSE 'P' AND ITS DETERMINANT FOR EACH STATION
ISM 34 WRITE(6,555)N(K)
ISM 35 555 FORMAT(' THE NORMAL EQUATIONS INVERSE FOR STATION:',I4)
ISM 36 CALL MOUTD(P,IRDA,MA,NA)
ISM 37 WRITE(6,52)DETA
ISM 38 52 FORMAT('0',' DETA IS:',2F20.6)
C MULTIPLY MATRIX 'T' BY DISPLACEMENT VECTORS 'DX','DY'
ISM 39 CALL MTVMUL(T,3,DX,20,G,20,3,J)
ISM 40 CALL MTVMUL(T,3,DY,20,H,20,3,J)
C COMPUTE THE DIAGONAL ELEMENTS OF THE STRAIN TENSOR
ISM 41 CALL MTVMUL(P,3,G,20,E0,3,3,3,3)
ISM 42 CALL MTVMUL(P,3,H,20,F0,3,3,3,3)
ISM 43 WRITE(6,44)N(K)
ISM 44 CALL MOUTD(R,20,J,NA)
C
C BY CALLING THESE SUBROUTINES LS-SOLUTION FOR DISPLACEMENT
C CURVATURE ARE OBTAINED
C TRANSPOSE DESIGN MATRIX 'R' TO OBTAIN MATRIX 'Q'
ISM 45 CALL MATTP(R,20,Q,3,J,3)
C MULTIPLY THE MATRICES 'Q' AND 'R' TO GET 'S'
ISM 46 CALL MATMUL(Q,3,R,20,S,3,3,J,3)
C PRINT MATRIX 'S' FOR EACH STATION
ISM 47 WRITE(6,444)N(K)
ISM 48 CALL MOUTD(S,IRDA,NA,MA)
C FIND THE NORMAL EQUATIONS INVERSE OF 'S'
ISM 49 CALL MINVD(S,IRDA,NA,DETA,IW1,IW2)
C PRINT INVERSE OF 'S' AND ITS DETERMINANT FOR EACH STATION
ISM 50 WRITE(6,555)N(K)
ISM 51 CALL MOUTD(S,IRDA,MA,NA)
ISM 52 WRITE(6,52)DETA
C MULTIPLY MATRIX 'Q' BY CONVERTED DISP. VECTORS 'CDX','CDY'
ISM 53 CALL MTVMUL(Q,3,CDX,20,CG,20,3,J)
ISM 54 CALL MTVMUL(Q,3,CDY,20,CH,20,3,J)
C PERFORM MATRIX OPERATIONS FOR THE SOLUTION OF STRAIN-RATES
C
C OBTAIN THE PRODUCT OF THE TRANSPOSE OF R AND A
ISM 55 CALL MATMUL(Q,3,A,20,S1,3,3,J,3)
C OBTAIN THE PRODUCT OF THE TRANSPOSE OF R AND R
ISM 56 CALL MATMUL(Q,3,R,20,S3,3,3,J,3)
```

.....1.....2.....3.....4.....5.....6.....7.....8

```
ISN 57 C OBTAIN THE PRODUCT OF THE TRANSPOSE OF A AND R
      CALL MATMUL(T,3,R,20,S2,3,3,J,3)
ISN 58 C OBTAIN THE PRODUCT OF S1 AND INVERSE P
      CALL MATMUL(S1,3,P,3,SP,3,3,3,3)
ISN 59 C OBTAIN THE PRODUCT OF SP AND S2
      CALL MATMUL(SP,3,S2,3,SS,3,3,3,3)
ISN 60 C TAKE THE DIFFERENCE OF SS FROM S3
      CALL MSUBCS(SSS,3,S3,3,SS,3,3,3)
ISN 61 C OBTAIN THE PRODUCT OF SP WITH VECTORS G AND H
      CALL MTVMUL(SP,3,G,20,SG,3,3,J,3)
ISN 62 C CALL MTVMUL(SP,3,H,20,SH,3,3,J,3)
      C TAKE THE VECTOR DIFFERENCES
      CALL MSUBCS(QQ,1,CG,1,SG,1,J,1)
ISN 63 C CALL MSUBCS(HH,1,CH,1,SH,1,J,1)
ISN 64 C TAKE THE INVERSE OF THE MATRIX SSS
      CALL MINVD(SSS,IRDA,NA,DETA,IW1,IW2)
ISN 65 C MULTIPLY THE INVERSE WITH A VECTOR
      CALL MTVMUL(SSS,3,GG,3,EE,3,3,3,3)
ISN 66 C CALL MTVMUL(SSS,3,HH,3,FF,3,3,3,3)
ISN 67 C
      C COMPUTE THE DISPLACEMENT GRADIENTS FOR THE NON LINEAR APPROXIMATION
      C
      C TAKE THE TRANSPOSE OF S1
      C
ISN 68 C CALL MATMUL(S1,3,TS,3,3,3)
ISN 69 C MULTIPLY INVERSE P WITH TRANSPOSE TS
      CALL MATMUL(P,3,TS,3,PS,3,3,3,3)
      C MULTIPLY THIS PRODUCT WITH CURVATURE VECTORS
      C
ISN 70 C CALL MTVMUL(PS,3,EE,3,TE,3,3,3,3)
      C
ISN 71 C CALL MTVMUL(PS,3,FF,3,TF,3,3,3,3)
      C SUBTRACT TO OBTAIN THE GRADIENT VECTORS
ISN 72 C CALL MSUBCS(E,1,E0,1,TE,1,3,1)
      C
ISN 73 C CALL MSUBCS(F,1,F0,1,TF,1,3,1)
      C
      C PRINT DISPLACEMENT GRADIENTS FOR EACH STATION
ISN 74 C WRITE(6,128)M(K)
ISN 75 C 128 FORMAT(7X,'THE VALUES OF DISPLACEMENT GRADIENTS',/,
      @ ' FOR STATION NUMBER',I8/)
      C
      C COMPUTE OFF-DIAGONAL ELEMENT OF THE SYMMETRIC STRAIN TENSOR
      C
ISN 76 C EP12=0.5*(E(2)+F(1))
ISN 77 C WRITE(6,451)E(1),EP12,F(2)
ISN 78 C 451 FORMAT(4X,3(3X,D14.8)/)
ISN 79 C 455 FORMAT(1X,3F18.10)
      C
      C WRITE THE ELEMENTS OF SYMMETRIC STRAIN TENSOR IN A FILE
      C WITH UNIT 17
      C
ISN 80 C WRITE(17,455)E(1),EP12,F(2)
ISN 81 C COX=E(3)*1.0D+4
ISN 82 C COY=F(3)*1.0D+4
ISN 83 C WRITE(6,410)COX,COY
```

*.....1.....2.....3.....4.....5.....6.....7.....8

C
C PRINT THE VALUES OF CONSTANTS
C

ISN 84 410 FORMAT(7X,'THE C O S T A N T S ARE RESPECTIVELY',3X,
@2(4X,D14.8)/)

C
C COMPUTE AND PRINT VARIOUS OTHER COMPONENTS OF STRAIN TENSOR
C THESE ARE ONLY USEFUL FOR CRUSTAL ANALYSIS
C

ISN 85 GM1=E(1)-F(2)
ISN 86 GM2=2*E(2)
ISN 87 GM=DSQRT(GM1*GM1+GM2*GM2)
ISN 88 OMEGA=0.5*(E(2)-F(1))
ISN 89 WRITE(6,360)GM1,GM2,GM,OMEGA
ISN 90 360 FORMAT(7X,'THE VALUES OF GM1,GM2,GM AND OMEGA ARE:/'
@4X,2(3X,D14.8)/,4X,2(3X,D14.8)/)
ISN 91 DX1=X2(K)-X(K)
ISN 92 DY1=Y2(K)-Y(K)
ISN 93 WRITE(6,138)N(K),DX1,DY1,OMEGA

C
C WRITE STATION NUMBERS DISPLACEMENTS AND VALUES OF
C AVERAGE DIFFERENTIAL ROTATION ON A FILE I N UNIT 9
C

ISN 94 WRITE(9,135)N(K),DX1,DY1,OMEGA
ISN 95 135 FORMAT(I8,2F16.8,F16.8)
ISN 96 138 FORMAT(7X,'THE STATION NAME & DISPLACEMENTS & OMEGA ARE/'
@,3X,I8,2F16.8,3X,D14.8/)

ISN 97 SHR=2*EP12
ISN 98 DILAT=E(1)+F(2)
ISN 99 SIGMA=DSQRT(GM1*GM1+SHR*SHR)
ISN 100 EPSLN=SHR+SIGMA
ISN 101 IF(DILAT.EQ.0.0) GOTO 23
ISN 102 TNTH=(-GM1+SIGMA)/DILAT
ISN 103 DIRETN=DATAN(TNTH)*180.DO/PI
ISN 104 WRITE(6,500)SHR,DILAT,SIGMA,EPSLN,DIRETN

23 CONTINUE
500 FORMAT(7X,'THE VALUES OF SHEAR DILATION,MAX.SHEAR STRAIN',
@7X,'AND ITS DIRECTION ARE',7X,3D14.6/,7X,2D14.6/)

C COMPUTE EXTENSION AND CONTRACTION RATES FOR EACH STATION
ERX=2*EE(1)
ERY=2*EE(2)
CRX=2*FF(1)
CRY=2*FF(2)

C PRINT STRAIN-RATES, ROTATION-RATES AND SHEAR-RATES

ISN 111 WRITE(6,90)ERX
ISN 112 WRITE(6,91)EE(3)
ISN 113 WRITE(6,92)ERY
ISN 114 WRITE(6,95)CRX
ISN 115 WRITE(6,96)FF(3)
ISN 116 WRITE(6,97)CRY
ISN 117 XSHEA=0.5*EE(3)+FF(1)
ISN 118 YSHEA=EE(2)+0.5*FF(3)
ISN 119 WRITE(6,81)XSHEA,YSHEA
ISN 120 XROT=0.5*EE(3)-FF(1)
ISN 121 YROT=EE(2)-0.5*FF(3)
ISN 122 WRITE(6,82)XROT,YROT

```

*.....1.....2.....3.....4.....5.....6.....7.....8

ISN 123 90 FORMAT(7X,'THE EXTENSION-RATE ON X IS;'.D14.6)
ISN 124 91 FORMAT(7X,'THE EXTENSION-RATE ON XY IS;'.D14.6)
ISN 126 92 FORMAT(7X,'THE EXTENSION-RATE ON Y IS;'.D14.6)
ISN 126 95 FORMAT(7X,'THE CONTRACTION-RATE ON X IS;'.D14.6)
ISN 127 96 FORMAT(7X,'THE CONTRACTION-RATE ON XY IS;'.D14.6)
ISN 128 97 FORMAT(7X,'THE CONTRACTION-RATE ON Y IS;'.D14.6)
ISN 129 81 FORMAT(7X,'THE SHEAR-RATES ON X AND Y ARE;'.2D14.6)
ISN 130 82 FORMAT(7X,'THE ROTATION-RATES ON X AND Y ARE;'.2D14.6)
C CHECK VALUES
C
ISN 131 WRITE(6,11)K,ICOUNT
ISN 132 11 FORMAT(7X,'THE CURRENT VALUES OF K AND ICOUNT ARE;'.2I6)
ISN 133 IF(K.EQ.ICOUNT)GOTO 20
ISN 134 6 CONTINUE
ISN 135 K=K+1
ISN 136 JJ=1
ISN 137 ST(1)=M1
ISN 138 SX=X(K)
ISN 139 SY=Y(K)
C
C COMPUTE THE DISPLACEMENTS AND ELEMENTS OF MATRICES 'A' & 'R'
C FOR THE STATION EACH STATION
C
ISN 140 DX(1)=(X2(K)-X(K))*1.0D-4
ISN 141 DY(1)=(Y2(K)-Y(K))*1.0D-4
ISN 142 A(1,1)=0.0
ISN 143 A(1,2)=0.0
ISN 144 A(1,3)=1.0
ISN 145 R(1,1)=0.0
ISN 146 R(1,2)=0.0
ISN 147 R(1,3)=0.0
ISN 148 CDX(1)=(X2(K)-X(K))*1.0D-4
ISN 149 CDY(1)=(Y2(K)-Y(K))*1.0D-4
ISN 150 4 CONTINUE
ISN 151 DO 10 I=1,JJ
ISN 152 10 IF(N2.EQ.ST(I))GO TO 30
ISN 153 JJ=I
ISN 154 J=JJ
ISN 155 ST(I)=M2
ISN 156 DO 5 I=1,ICOUNT
ISN 157 IF(N2.NE.M3(I))GOTO 5
C COMPUTE THE ELEMENTS OF MATRICES 'A' & 'R' AND THE DISPLACEMENTS
C AT STATIONS WHICH ARE CONNECTED BY OBSERVATIONS TO THE POINT
C AT WHICH DISPLACEMENT GRADIENTS AND DISPLACEMENT CURVATURES
C ARE BEING COMPUTED
C
ISN 158 DX(J)=(X2(I)-X(I))*1.0D-4
ISN 159 DY(J)=(Y2(I)-Y(I))*1.0D-4
ISN 160 A(J,1)=(X(I)-SX)*1.0D-4
ISN 161 A(J,2)=(Y(I)-SY)*1.0D-4
ISN 162 A(J,3)=1.0
ISN 163 R(J,1)=(X(I)-SX)**2*1.0D-4
ISN 164 R(J,2)=(Y(I)-SY)**2*1.0D-4
ISN 165 R(J,3)=(X(I)-SX)*(Y(I)-SY)*1.0D-4
ISN 166 CDX(J)=(X2(I)-X(I))*1.0D-4
ISN 167 CDY(J)=(Y2(I)-Y(I))*1.0D-4

```

*.....1.....2.....3.....4.....5.....6.....7.....8

ISM
ISM
ISM
ISM
ISM

168
169
170
171
172

5 CONTINUE
200 FORMAT(I8,2F15.4)
GOTO 30
20 STOP
END

```

C          MULTIPLYING A MATRIX BY ANOTHER SAY, AB=C
C
ISM      1      SUBROUTINE MATMUL(A,IA,B,IB,C,IC,L,M,N)
ISM      2      IMPLICIT REAL*8(A-H,O-Z)
ISM      3      DIMENSION A(IA,M),B(IB,N),C(IC,N)
ISM      4      DO 2 I=1,L
ISM      5      DO 2 J=1,N
ISM      6      C(I,J)=0.000
ISM      7      DO 1 K=1,M
ISM      8      C(I,J)=C(I,J)+A(I,K)*B(K,J)
ISM      9      1 CONTINUE
ISM     10      2 CONTINUE
ISM     11      RETURN
ISM     12      END

```

```

C          MATRIX TRANSPOSITION
C
ISM      1      SUBROUTINE MATTMP(A,IDA,B,IDB,M,N)
ISM      2      IMPLICIT REAL*8(A-H,O-Z)
ISM      3      DIMENSION A(IDA,M),B(IDB,M)
ISM      4      DO 1 I=1,M
ISM      5      DO 2 J=1,N
ISM      6      B(J,I)=A(I,J)
ISM      7      2 CONTINUE
ISM      8      1 CONTINUE
ISM      9      RETURN
ISM     10      END

```

```

C          MULTIPLYING A MATRIX BY A VECTOR
C
ISM      1      SUBROUTINE MTYMUL(A,IA,B,IB,C,IC,M,N)
ISM      2      REAL*8 A(IA,N),B(IB),C(IC)
ISM      3      DO 10 I=1,M
ISM      4      C(I)=0
ISM      5      DO 10 J=1,N
ISM     10      C(I)=A(I,J)*B(J)+C(I)
ISM      7      RETURN
ISM      8      END

```

APPENDIX V-3
PROGRAM TO COMPUTE
THE PARAMETERS OF STRAIN ELLIPSES
(Computer Program PASTEL)

```
1 //AABKGP62 JOB '6703,RAG0',LUGOE,MSGCLASS=A,NOTIFY=6767
  ***PASSWORD $ACF1770 PASSWORD(S) SUCCESSFULLY SCANNED
  ***JOBPARM M=5,L=50,R=1024
2 // EXEC FORTVCLQ,LIB=LIBD,RG=1024K,PARM='LANGLYL(66)'
11 //FORT.SYSIN DD *
29 //QO.FT01F001 DD DSN=A.M67034.EXISTJNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
30 //QO.FT15F001 DD DSN=A.M67034.MTX2JNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
31 //QO.FT31F001 DD DSN=A.M67034.CHET2.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=OLD
32 //QO.SYSIN DD *
  //
```

JOB 2781

.....1.....2.....3.....4.....5.....6.....7.....8

C
C P A S T E L
C (PARAMETERS OF STRAIN ELLIPSE)
C PROGRAM COMPUTES ELEMENTS OF THE STRAIN ELLIPSE AT EACH STATION
C AND DETERMINES STATIONS WITH BIASED OBSERVATIONS
C ORIGINAL PROGRAM 'EVALUE' BY K.THAPA(1980)
C EXPANDED AND MODIFIED BY F.N.LUGOE - JUNE,1983
C

ISM
ISM
ISM
ISM
ISM
ISM

1
2
3
4
5
6

IMPLICIT REAL*8(A-H,O-Z)
REAL*4 PHI
DIMENSION A(3),D(2),Z(2,2),WK(5),P(2)
N=2
IZ=2
IJOB=2

C
C READ NUMBER OF STATION NUMBERS

ISM
ISM
ISM

7
8
9

READ(5,126)MS
126 FORMAT(I4)
DO 77 I=1,MS

C
C READ STATION NUMBER AND CO-ORDINATES OBTAINED FROM
C EITHER ONE OF THE TWO SEPARATE ADJUSTMENTS

ISM

10

READ(1,100)M1,X,Y

C
C READ ELEMENTS OF STRAIN TENSOR MATRIX IN UNIT 15
C THIS IS AN OUTPUT FROM PROGRAM STRAIN

ISM
ISM

11
12

READ(15,455)A
455 FORMAT(1X,3F18.10)

C
C CALL SUBROUTINE EIGRS TO COMPUTE THE EIGEN VALUES
C AND EIGENVECTORS OF THE STRAIN TENSOR

ISM
ISM
ISM
ISM
ISM

13
14
15
16
17

CALL EIGRS(A,N,IJOB,D,Z,IZ,WK,IER)
100 FORMAT(I8,2F16.4)
P(1)=DABS(D(1))
P(2)=DABS(D(2))
25 FORMAT('-', 'THE EIGENVALUES AND ORIENTATION ANGLE PHI'//,
@7X, 'FOR STATION NUMBER',I4//)

ISM
ISM

18
19

PI=DARCOS(-1.00)
IF(P(1).EQ.0.0.OR.P(2).EQ.0.0)QOTO 99

C
C DECIDE WHICH OF THE TWO EIGEN VALUES IS GREATER

.....1.....2.....3.....4.....5.....6.....7.....8

```
C
ISN 20      IF(P(1).GT.P(2)) GOTO 29
C
C COMPUTE ORIENTATION ANGLE FOR THE SEMI-MAJOR AXIS
C OF THE STRAIN ELLIPSE
C
ISN 21      ANGL=-(A(1)-D(2))/A(2)
ISN 22      PHI=DATAN(ANGL)
ISN 23      GOTO 69
ISN 24      29 ANGL=-(A(1)-D(1))/A(2)
ISN 25      PHI=DATAN(ANGL)
ISN 26      GOTO 69
ISN 27      99 PHI=0.0
ISN 28      69 CONTINUE
ISN 29      D(1)=D(1)*1.005
ISN 30      D(2)=D(2)*1.005
ISN 31      300 FORMAT('-', 'THE RATIO OF THE AXES OF THE STRAIN ELLIPSE AT STATION
          @ NUMBER:', I4)
ISN 32      301 FORMAT('-', 'T H E   R A T I O   I S:', F18.8)
ISN 33      302 FORMAT('-', 'THE FLATTENING OF THE STRAIN ELLIPSE IS:', F10.6)
ISN 34      IF(P(2).GT.P(1))GOTO 177
ISN 35      DD=D(1)/D(2)
ISN 36      DD1=(D(1)-D(2))/D(1)
ISN 37      WRITE(6,300)M1
ISN 38      WRITE(6,301)DD
ISN 39      WRITE(6,302)DD1
ISN 40      WRITE(6,26)M1
C
C WRITE THE VALUES OF AXES OF STRAIN ELLIPSE AND ANGLE OF
C ORIENTATION ON FILE IN UNIT 31 AS IN INPUT FOR PROGRAM NETPLOT
C
ISN 41      WRITE(31,140)(D(J),J=1,2),PHI
ISN 42      WRITE(6,141)(D(J),J=1,2),PHI
ISN 43      GOTO 48
ISN 44      177 DD=D(2)/D(1)
ISN 45      DD1=(D(2)-D(1))/D(2)
ISN 46      WRITE(6,300)M1
ISN 47      WRITE(6,301)DD
ISN 48      WRITE(6,302)DD1
ISN 49      WRITE(6,26)M1
ISN 50      WRITE(31,140)D(2),D(1),PHI
ISN 51      WRITE(6,141)D(2),D(1),PHI
ISN 52      48 CONTINUE
ISN 53      140 FORMAT(1X,2F18.8,F15.6)
ISN 54      WRITE(6,123)
ISN 55      141 FORMAT(7X,2D14.8,3X,F10.6/)
ISN 56      123 FORMAT(7X, 'PRINT THE NUMBERS WHICH INDICATE THE RELIABILITY',
          @ 7X, 'OF THE COMPUTATION OF EIGENVALUES AND EIGENVECTORS'/)
ISN 57      WRITE(6,136)IER,WK(1)
ISN 58      136 FORMAT(7X,I8,F10.6/)
ISN 59      WRITE(6,121)M1
ISN 60      121 FORMAT(7X, 'THE EIGENVECTORS FOR STATION #', I8/)
ISN 61      WRITE(6,145)((Z(J,K),K=1,2),J=1,2)
ISN 62      145 FORMAT(7X,2F12.8,7X,F12.8/)
ISN 63      77 CONTINUE
ISN 64      STOP
ISN 65      END
```

APPENDIX V-4
PROGRAM TO PLOT STRAIN
PATTERNS AT NETWORK STATIONS
(Computer Program NETPLOT1)

```
1 //AABKQP53 JOB '6703,RAGO',LUGDE,MSGCLASS=A,NOTIFY=6767 JOB 2783
  ***PASSWORD $ACF1770 PASSWORD(S) SUCCESSFULLY SCANNED
  ***JOBPARM M=4,L=50,R=1024
2 // EXEC FORTPLOT,TYPE=FILEPLOT
12 //FORT.SYSIN DD *
33 //GO.FT08F001 DD DSN=A.M67034.DXDY2JNC.OMEGA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
34 //GO.FT04F001 DD DSN=A.M67034.MTX2JNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
35 //GO.FT03F001 DD DSN=A.M67034.CHET2.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
36 //GO.FT02F001 DD DSN=A.M67034.EXISTJNC.DATA,
  // UNIT=P3350,VOL=SER=USER11,DISP=SHR
37 //GO.SYSIN DD *
  //
```

```

.....1.....2.....3.....4.....5.....6.....7.....
C .....
C
C
C           M E T P L O T 1
C
C   P L O T T I N G   O F   S T R A I N   P A T T E R N S   ( E L L I P S E S   A N D   A R C S )   U S I N G
C
C           O U T P U T   F R O M   P R O G R A M S   S T R A I N   A N D   P A S T E L
C
C .....
ISN      1      I M P L I C I T   R E A L * 4 ( A - H , O - Z )
ISN      2      R E A L * 8   W , Z , R M A J , R M I N , A N G L , A
ISN      3      R E A L * 8   D X 1 , D Y 1 , D O M E G A , P X , P Y
ISN      4      D I M E N S I O N   A ( 3 )
ISN      5      L O G I C A L * 1   I S T M ( 8 ) , I T O M ( 8 )
ISN      6      D A T A   I S T M ( 8 ) / ' \ ' /

ISN      7      D A T A   I T O M ( 8 ) / ' \ ' /
C
C   R E A D   T H E   N U M B E R   O F   S T A T I O N   N U M B E R S
C

ISN      8      R E A D ( 5 , 2 0 1 ) N S
ISN      9      2 0 1   F O R M A T ( I 4 )
C
C   S U B R O U T I N E   S H I F T   I S   U S E D   T O   S H I F T   T H E   P A P E R   T O   P R O P E R   S E T T I N G
C

ISN     10      C A L L   S H I F T ( 1 . 0 , 1 . 0 )
C
C   S U B R O U T I N E   A R E A   A L L O W S   U S   T O   F I X   T H E   S I Z E   O F   T H E   P L O T
C

ISN     11      C A L L   A R E A ( 8 . 5 9 3 1 , 7 . 0 )
C
C   S U B R O U T I N E   S E T P L O T   I S   U S E D   T O   F I X   T H E   S C A L E   O F   T H E   P L O T
C   T H A T   I S ,   T O   F I X   T H E   C O R N E R S   O F   T H E   R E C T A N G L E   A S S I G N E D   B Y   T H E
C   S U B R O U T I N E   A R E A
C

ISN     12      C A L L   S E T P L T ( 1 8 0 0 0 0 . 0 , 3 0 0 0 0 0 . 0 , 3 5 0 0 0 0 . 0 , 4 3 8 0 0 0 . 0 )
ISN     13      D O   9 9   J = 1 , N S
C
C   R E A D   S T A T I O N   N U M B E R S   A N D   C O - O R D I N A T E S   O F   S T A T I O N S
C

ISN     14      R E A D ( 2 , 4 5 0 ) ( I S T M ( I ) , I = 1 , 7 ) , W , Z
ISN     15      4 5 0   F O R M A T ( 1 X , 7 A 1 , 2 F 1 5 . 4 )
C
C   R E A D   M A J O R   A N D   M I N O R   A X E S   O F   S T R A I N   E L L I P S E S   A N D   T H E
C   O R I E N T A T I O N   A N G L E   O F   T H E   S E M I - M A J O R   A X E S
C

```

```

      *.....1.....2.....3.....4.....5.....6.....7.....8
ISN   16      READ(3,150)RMAJ,RMIN,PHI
ISN   17      150 FORMAT(1X,2F18.8,F15.6)
ISN   18      X=W
ISN   19      Y=Z
ISN   20      RMAJOR=RMAJ
ISN   21      RMINOR=RMIN
      C
      C READ STATION NUMBER DISPLACEMENTS OF POINTS AND
      C THE VALUES OF AVERAGE DIFFERENTIAL ROTATION
      C
ISN   22      READ(8,415)(ITON(I),I=1,7),DX1,DY1,DOMEGA
ISN   23      415 FORMAT(1X,7A1,2F16.8,F16.8)
      C
      C MAKE BOTH AXES NEGATIVE TO PLOT THEM IN USER UNITS
      C
ISN   24      RMAJOR=-ABS(RMAJOR)*1.0D4
ISN   25      RMINOR=-ABS(RMINOR)*1.0D4
      C
      C READ THE ELEMENTS OF THE SYMMETRIC STRAIN TENSOR MATRIX
      C
ISN   26      READ(4,485)A
ISN   27      455 FORMAT(1X,3F18.10)
ISN   28      COSPHI=COS(PHI)
ISN   29      SIMPHI=SIN(PHI)
      C
      C COMPUTE THE CO-ORDINATES AT EACH ENDS OF THE AXES
      C OF THE STRAIN ELLIPSE
      C
ISN   30      X1=X+RMAJOR*COSPHI
ISN   31      Y1=Y+RMAJOR*SIMPHI
ISN   32      X2=X-RMAJOR*COSPHI
ISN   33      Y2=Y-RMAJOR*SIMPHI
ISN   34      X3=X-RMINOR*SIMPHI
ISN   35      X4=X+RMINOR*SIMPHI
ISN   36      Y3=Y+RMINOR*COSPHI
ISN   37      Y4=Y-RMINOR*COSPHI
      C
      C SPECIFY THE RADIUS OF THE SEGMENT OF THE CIRCLE FOR PLOTTING
      C THE VALUES OF OMEGA
      C
ISN   38      RADIUS=5000.0
ISN   39      SX=X
ISN   40      SY=Y
ISN   41      DX=DX1
ISN   42      DY=DY1
ISN   43      OMEGA=DOMEGA
ISN   44      OMEGA=OMEGA*1.0E5
      C
      C COMPUTE THE CO-ORDINATES OF THE CORNERS OF THE SEGMENT OF
      C THE CIRCLE
      C
ISN   45      X5=SX+RADIUS
ISN   46      Y5=SY
ISN   47      X6=SX+RADIUS*COS(OMEGA)
ISN   48      Y6=SY+RADIUS*SIN(OMEGA)
ISN   49      X7=X

```

.....1.....2.....3.....4.....5.....6.....7.....8

```
C
C SUBROUTINE 'NOWPLOT' ENABLES ONE TO DRAW POINTS AND ST. LINES
C
ISN 50 CALL NOWPLT(0,X1,Y1)
C
C SUBROUTINE DASHLINES ENABLES ONE TO PLOT DASHLINES
C
C PLOT DASH LINES FOR NEGATIVE VALUES OF STRAIN ELLIPSES
C
ISN 51 IF(RMAJ.LT.0.0)CALL DSHLMS(X2,Y2,(0.04,0.04))
C
C PLOT SOLID LINES FOR POSITIVE VALUES OF AXES OF STRAIN ELLIPSES
C
ISN 53 IF(RMAJ.GE.0.0)CALL NOWPLT(2,X2,Y2)
ISN 55 CALL NOWPLT(0,X4,Y4)
ISN 56 IF(RMIN.LT.0.0)CALL DSHLMS(X3,Y3,(0.04,0.04))
ISN 58 IF(RMIN.GE.0.0)CALL NOWPLT(2,X3,Y3)
C
C PLOT THE STRAIN ELLIPSE BY CALLING THE SUBROUTINE 'ELLIPS'
C
ISN 60 CALL ELLIPS(X,Y,RMAJOR,RMINOR,PHI)
ISN 61 IF(OMEGA.EQ.0.0)GOTO 88
ISN 62 IF(OMEGA.LT.0.0) GOTO 73
C
C PLOT THE ARC OF THE CIRCLE BY CALLING THE SUBROUTINE 'ARC'
C
ISN 63 CALL ARC(SX,SY,X5,Y5,X6,Y6)
C
C PLOT SOLID LINES FOR THE RADII OF THE SEGMENT OF THE
C FOR POSITIVE VALUES OF OMEGA AND DRAW THE SEGMENT OF THE CIRCLE
C
ISN 64 CALL NOWPLT(0,X5,Y5)
ISN 65 CALL NOWPLT(2,SX,SY)
ISN 66 CALL NOWPLT(0,X6,Y6)
ISN 67 CALL NOWPLT(2,SX,SY)
ISN 68 GOTO 88
ISN 69 73 CALL ARC(SX,SY,X6,Y6,X5,Y5)
C
C PLOT DASH LINES FOR THE RADII OF THE SEGMENT OF THE CIRCLE
C FOR NEGATIVE VALUES OF OMEGA AND DRAW THE SEGMENT OF THE CIRCLE
C
ISN 70 CALL NOWPLT(0,X5,Y5)
ISN 71 CALL DSHLMS(SX,SY,(0.04,0.04))
ISN 72 CALL NOWPLT(0,X6,Y6)
ISN 73 CALL DSHLMS(SX,SY,(0.04,0.04))
ISN 74 88 CONTINUE
C
C PLOT STATION NUMBERS BY CALLING SUBROUTINE 'CHARS'
C
ISN 75 CALL CHARS(X7,Y,ISTN,0...15)
C
ISN 76 WRITE(6,75)(ITON(I),I=1,7),OMEGA,PHI
ISN 77 75 FORMAT(7X,'THE STATION NAMES AND VALUES OF OMEGA AND PHI ARE'//,
@ 7X,'GIVEN BELOW'/7X,7A1,2F16.8/)
ISN 78 99 CONTINUE
C
```

.....1.....2.....3.....4.....5.....6.....7.....8

C SUBROUTINE ENDPLT IS CALLED TO INDICATE THE END OF ALL PLOTTING

C
CALL ENDPLT
STOP
END

ISM 79
ISM 80
ISM 81

APPENDIX V-5
COMPUTER PROGRAM RANDOM
(A Random Number Generator)

```

.....
1  $JOB WATFIY LUQOE/Q,PAGES=40,TIME=30
2  IMPLICIT REAL*8(A-H,O-Z)
3  IX=6681
4  DO 10 I=1,24
5  CALL GAUSS(IX,1.00,0.00,R)
6  Q=1.98*R
7  WRITE(6,1)Q
8  1 FORMAT(' ',5X,F6.2)
9  10 CONTINUE
10 STOP
11 END

12 SUBROUTINE GAUSS(IX,SD,AY,R)
13 IMPLICIT REAL*8(A-H,O-Z)
14 A=0.00
15 DO 20 I=1,12
16 CALL RANDU(IX,IY,Y)
17 IX=IY
18 20 A=A+Y
19 R=(A-6.00)*SD+AY
20 RETURN
21 END

22 SUBROUTINE RANDU(IX,IY,YFL)
23 IMPLICIT REAL*8(A-H,O-Z)
24 INTEGER SEED
25 DATA SEED /66639/
26 IY=IX*SEED
27 IF(IY) 3,6,6
28 8 IY=IY+2147483647+1
29 6 YFL=IY
30 YFL=YFL*0.4686613D-9
31 RETURN
32 END

33 STOP
34 END
$ ENTRY

```

APPENDIX V-6

LEAST-SQUARES COMPUTATION OF TRANSFORMATION
PARAMETERS IN TWO DIMENSIONS

(Computer Program SMTRA)

```
1 //AABKQP54 JOB '6703,RAGO',LUGOE,MSGCLASS=A,NOTIFY=6767          JOB 2790
  ***PASSWORD $ACF1770 PASSWORD(S) SUCCESSFULLY SCANNED
  ***JOBPARM M=1,L=2,R=2048
2 // EXEC FORTVCLG,RG=2048K,RL=2048K,PARM='NOMAP,NOLIST'
11 //FORT.SYSIN DD *
29 //GO.FT01F001 DD DSN=A.M67034.EXISTJNC.DATA,
  // UMIT=P3350,VOL=SER=USER11,DISP=SHR
30 //GO.FT15F001 DD DSN=A.M67034.DENSJNC.DATA,
  // UMIT=P3350,VOL=SER=USER11,DISP=SHR
31 //GO.FT16F001 DD DSN=A.M67034.SIGMA1.DATA,
  // UMIT=P3350,VOL=SER=USER11,DISP=SHR
32 //GO.SYSIN DD *
  //
```

```

.....1.....2.....3.....4.....5.....6.....7.....8
C
C *****
C
C           S   M   T   R   A
C
C   LEAST-SQUARES COMPUTATION OF TRANSFORMATION PARAMETERS IN 2D
C
C           BY   F.M.LUGOE - FEB 25,1984
C *****
C
1   IMPLICIT REAL*8(A-H,O-Z)
2   DIMENSION H(50),N3(50),X(50),Y(50),X2(50),Y2(50)
3   DIMENSION S3(50),S4(50),IW1(4),N2(50),CX(4,4)
4   DIMENSION H(50,50),T(4,50),TM(4,50),TA(4,4),Q(50,50),XY(100)
5   DIMENSION IW2(4),AT(4,50),HT(4,50),ZX(4),C(50,4)
6   DIMENSION GX(50),VX(50),P(50,50)
7   SUM=0.00
8   SUM1=0.00
9   IRDA=4
10  NA=4
C
C 'NUMB' MUST BE MADE EQUAL TO THE NUMBER OF COMMON STATIONS
C
11  READ(5,*)NUMB
12  DO 280 I=1,50
C
C   READ CO-ORDINATES TO BE TRANSFORMED AS PROVIDED IN A SECONDARY
C   COORDINATE SYSTEM AND CORRESPONDING STATION NUMBERS
C
13  READ(1,440,END=2)N(I),X(I),Y(I)
C
C   READ THE VARIANCES OF COORDINATES GIVEN IN THE
C   PRIMARY COORDINATE SYSTEM
C
14  READ(16,772)N2(I),S3(I),S4(I)
15  440 FORMAT(I8,2F15.4)
16  772 FORMAT(I8,2F14.6)
C
C   READ THE COORDINATES AS PROVIDED IN THE PRIMARY COORDINATE
C   SYSTEM AND CORRESPONDING STATION NUMBERS
C
17  280 READ(15,440)N3(I),X2(I),Y2(I)
18  2 ICOUNT=I-1
C   COMPILE THE DESIGN MATRIX FOR THE TRANSFORMATION
19  IC=ICOUNT*2
20  X=0
21  J=1
22  DO 28 JJ=1,ICOUNT
23  C(J,1)=X(JJ)
24  C(J,2)=-Y(JJ)
25  C(J,3)=0.101

```

.....1.....2.....3.....4.....5.....6.....7.....8

```
26      C(J,4)=0.000
27      J=J+1
28      K=K+1
29      C(J,1)=-C(K,2)
30      C(J,2)=C(K,1)
31      C(J,3)=C(K,4)
32      C(J,4)=C(K,3)
33      J=J+1
34      K=K+1
35      28 CONTINUE
36      44 FORMAT(7X,4F15.4)
C
C      COMPILE THE WEIGHT MATRIX OF THE OBSERVABLES 'X' AND 'Y'
C
37      DO 303 K=1,ICOUNT
38      DO 310 J=1,IC
39      IF(J.EQ.K)GOTO 11
40      H(K,J)=0.000
41      GOTO 12
42      11      H(K,K)=0.1/S3(K)
43      12      CONTINUE
44      310 CONTINUE
45      303 CONTINUE
46      DO 403 K=1,ICOUNT
47      DO 410 J=1,IC
48      IF(J.EQ.K)GOTO 411
49      P(K,J)=0.000
50      GOTO 13
51      411      P(K,K)=0.1/S4(K)
52      13      CONTINUE
53      410 CONTINUE
54      403 CONTINUE
55      I=1
56      II=1
57      JJ=1
58      DO 20 JI=1,ICOUNT
59      Q(I,I)=H(II,II)
60      I=I+1
61      Q(I,I)=P(JJ,JJ)
62      II=II+1
63      JJ=JJ+1
64      I=I+1
65      20 CONTINUE
66      I=0
67      J=0
68      DO 35 IK=1,ICOUNT
69      I=I+1
70      J=J+1
71      XY(I)=X2(IK)
72      XY(J)=Y2(IK)
73      I=I+1
74      35 CONTINUE
75      WRITE(6,*) (XY(I),I=1,IC)
76      CALL MOUTD(C,50,IC,4)
77      CALL MOUTD(Q,50,IC,IC)
```

```

      .....1.....2.....3.....4.....5.....6.....7.....8
C   COMPUTE THE NORMAL EQUATIONS
C   TRANSPOSE THE DESIGN MATRIX
78   CALL MATTMP(C,50,T,4,IC,4)
C   MULTIPLY TRANSPOSE BY WEIGHT MATRX
79   CALL MATMUL(T,4,Q,50,TM,4,4,IC,IC)
C   MULTIPLY FURTHER
80   CALL MATMUL(TM,4,C,50,TA,4,4,IC,4)
C   INVERT THE NORMAL EQUATIONS MATRIX 'TA'
81   CALL MINVD(TA,IRDA,NA,DETA,IW1,IW2)
82   WRITE(6,25)DETA
83   CALL MOUTD(TA,4,4,4)
84   25 FORMAT(7X,'THE NORMAL EQUATIONS INVERSE: ',2F20.12/)
C   MULTIPLY THE INVERSE BY THE TRANSPOSE OF 'A'
85   CALL MATMUL(TA,4,T,4,AT,4,4,4,IC)
C   MULTIPLY FURTHER
86   CALL MATMUL(AT,4,Q,50,HT,4,4,IC,IC)
C
C   OBTAIN THE VECTOR OF TRANSFORMATION PARAMETERS
87   CALL MTVMUL(HT,4,XY,IC,ZX,4,4,IC)
C   PRINT THE TRANSFORMATION PARAMETERS
88   WRITE(6,30)(ZX(J),J=1,2)
89   WRITE(6,40)(ZX(J),J=3,4)
90   30 FORMAT('- ', 'K * COS W AND K * SIN W ARE: ',2F12.8)
91   40 FORMAT('- ', 'TRANSLATION PARAMETERS ARE: ',2F18.8)
C   COMPUTE THE RESIDUALS
92   CALL MTVMUL(C,50,ZX,4,GX,50,IC,4)
C   PRINT TRANSFORMED PARAMETERS AND THEIR RESIDUALS
93   DO 50 I=1,IC
94     YX(I)=GX(I)-XY(I)
95     SUM1=SUM1+YX(I)**2
96   50 CONTINUE
C   COMPUTE THE A PRIORI VARIANCE FACTOR
97   SIGMA1=DSQRT(SUM1/(IC-4))
C
C   DERIVE THE COVARIANCE MATRIX OF THE TRANSFORMATION PARAMETERS
C
C   THE NUMBER OF PARAMETERS TO BE PRINTED USING FORMAT STATEMENTS
C   320-326 MUST BE EQUAL TO 'NUMB'*2 PRIOR TO RUNNING THE PROGRAM
C
98   320 FFORMAT('- ', 'THE TRANSFORMED COORDINATES ARE: '/')
99   321 FFORMAT('- ', 'THE TRANSFORMATION RESIDUALS ARE: '/')
100  DO 117 I=1,4
101    DO 117 J=1,4
102      CX(I,J)=SIGMA1*TA(I,J)
103  117 CONTINUE
104    WRITE(6,80)SIGMA1
105    CALL MOUTD(CX,4,4,4)
106    80 FFORMAT('- ', 'COVARIANCE MATRIX OF TRANSFORMATION PARAMETERS: ',
      @7X,' WITH A PRIORI VARIANCE FACTOR: ',F15.6/)
107    IM=0
108    DO 600 I=1,NUMB
109      IN=IM+1
110      IM=IN+1
111    600 WRITE(6,3)N2(I),GX(IN),GX(IM)
112      WRITE(6,321)
113      IM=0

```

.....1.....2.....3.....4.....5.....6.....7.....8

```
114 DO 601 I=1,NUMB  
115   IM=IM+1  
116   IM=IM+1  
117   601 WRITE(6,3)M2(I),VX(IM),VX(IM)  
118     3 FORMAT(7X,I4,2F15.4)  
C  
119 STOP  
120 END
```

```

C          MULTIPLYING A MATRIX BY ANOTHER SAY, AB=C
C
ISM      1      SUBROUTINE MATMUL(A,IA,B,IB,C,IC,L,M,N)
ISM      2      IMPLICIT REAL*8(A-H,O-Z)
ISM      3      DIMENSION A(IA,M),B(IB,N),C(IC,N)
ISM      4      DO 2 I=1,L
ISM      5      DO 2 J=1,M
ISM      6      C(I,J)=0.000
ISM      7      DO 1 K=1,N
ISM      8      C(I,J)=C(I,J)+A(I,K)*B(K,J)
ISM      9      1 CONTINUE
ISM     10      2 CONTINUE
ISM     11      RETURN
ISM     12      END

```

```

C          MATRIX TRANSPOSITION
C
ISM      1      SUBROUTINE MATTMP(A,IDA,B,IDB,M,N)
ISM      2      IMPLICIT REAL*8(A-H,O-Z)
ISM      3      DIMENSION A(IDA,M),B(IDB,M)
ISM      4      DO 1 I=1,M
ISM      5      DO 2 J=1,M
ISM      6      B(J,I)=A(I,J)
ISM      7      2 CONTINUE
ISM      8      1 CONTINUE
ISM      9      RETURN
ISM     10      END

```

```

C          MULTIPLYING A MATRIX BY A VECTOR
C
ISM      1      SUBROUTINE MTVMUL(A,IA,B,IB,C,IC,M,N)
ISM      2      REAL*8 A(IA,M),B(IB),C(IC)
ISM      3      DO 10 I=1,M
ISM      4      C(I)=0
ISM      5      DO 10 J=1,N
ISM     10      C(I)=A(I,J)*B(J)+C(I)
ISM      7      RETURN
ISM      8      END

```

APPENDIX V-7
COORDINATE TRANSFORMATION IN 2D
(Computer Program SIMFRA)

```
1 //AABKQP55 JOB '6703, RAG0', LUGOE, MSGCLASS=A, NOTIFY=6767 JOB 2791
  ***PASSWORD $ACF1770 PASSWORD(S) SUCCESSFULLY SCANNED
  ***JOBPARM M=1, L=2, R=1024
2 // EXEC FORTVCLG, RG=1024K, RL=1024K, PARM='NOMAP, NOLIST'
11 //FORT.SYSIM DD *
29 //GO.FT31F001 DD DSM=A.M67034.TRMSF1.DATA,
  // UNIT=P3350, VOL=SER=USER11, DISP=SHR
30 //GO.SYSIM DD *
  //
```

.....1.....2.....3.....4.....5.....6.....7.....8

C
C *****
C
C S I M T R A
C
C PROGRAM PERFORMS TRANSFORMATION OF COORDINATES IN 2D
C
C BY F.N.LUGDE - FEB 26,1984
C *****
C

ISN 1 IMPLICIT REAL*8(A-H,O-Z)
ISN 2 DIMENSION N(100),X(100),Y(100)
ISN 3 DIMENSION A(100,4),B(100,4)
ISN 4 DIMENSION ZX(4),GX(100),GY(100),ZY(4)
ISN 5 DO 1 I=1,100

C
C READ CO-ORDINATES TO BE TRANSFORMED AS PROVIDED IN A SECONDARY
C COORDINATE SYSTEM AND CORRESPONDING STATION NUMBERS
C

ISN 6 READ(31,200,END=2)N(I),X(I),Y(I)
ISN 7 200 FORMAT(I8,2F15.4)
ISN 8 1 CONTINUE
ISN 9 2 ICOUNT=I-1

C
C THE TRANSFORMATION PARAMETERS BELOW MUST BE REPLACED EACH TIME
C BY THE APPROPRIATE PARAMETERS OF THE NETWORK TO BE TRANSFORMED
C ZX(1)=KX*CDOS(W); ZX(2)=KX*SIN(W); ZX(3)=DX; ZX(4)=0.0
C ZY(1)=KY*CDOS(W); ZY(2)=KY*SIN(W); ZY(3)=0.0; ZY(4)=DY
C KX AND KY ARE SCALE VALUES

ISN 10 ZX(1)=1.00000068
ISN 11 ZX(2)=-0.00000789
ISN 12 ZX(3)=-2.73206367
ISN 13 ZX(4)=0.000
ISN 14 ZY(1)=1.00000119
ISN 15 ZY(2)=-0.00001131
ISN 16 ZY(3)=0.000
ISN 17 ZY(4)=2.91819465
ISN 18 IQ=ICOUNT+1
ISN 19 DO 280 I=1,ICOUNT

C
C COMPILE THE DESIGN MATRIX FOR THE TRANSFORMATION

ISN 20 A(I,1)=X(I)
ISN 21 A(I,2)=-Y(I)
ISN 22 A(I,3)=0.101
ISN 23 A(I,4)=0.00
ISN 24 280 CONTINUE
ISN 25 DO 300 I=1,ICOUNT
ISN 26 B(I,1)=Y(I)
ISN 27 B(I,2)=X(I)
ISN 28 B(I,3)=0.00
ISN 29 B(I,4)=0.101
ISN 30 300 CONTINUE

.....1.....2.....3.....4.....5.....6.....7.....8

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ISN 31      C PRINT THE TRANSFORMATION PARAMETERS
ISN 32      WRITE(6,30)(ZX(J),J=1,4)
ISN 33      WRITE(6,40)(ZY(J),J=1,4)
ISN 34      30 FORMAT(7X,'THE X-TRANSFORMATION PARAMETERS ARE:',4F12.8)
ISN 34      40 FORMAT(7X,'THE Y-TRANSFORMATION PARAMETERS ARE:',4F12.8)
ISN 35      C PERFORM THE TRANSFORMATION
ISN 36      CALL MTVMUL(A,100,ZX,4,GX,100,ICOUNT,4)
ISN 36      CALL MTVMUL(B,100,ZY,4,GY,100,ICOUNT,4)
ISN 37      C
ISN 37      C PRINT THE TRANSFORMED COORDINATES
ISN 38      WRITE(6,320)
ISN 38      321 FORMAT(7X,I4,2F15.4)
ISN 39      DO 296 I=1,ICOUNT
ISN 40      296 WRITE (6,321)M(I),GX(I),GY(I)
ISN 41      320 FORMAT(' ','THE TRANSFORMED COORDINATES ARE:')
ISN 42      STOP
ISN 43      END
```

........1.....2.....3.....4.....5.....6.....7.*.....8

C MULTIPLYING A MATRIX BY A VECTOR
C

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ISN 1            SUBROUTINE MTVMUL(A,IA,B,IB,C,IC,M,N)  
ISN 2            REAL*8 A(IA,N),B(IB),C(IC)  
ISN 3            DO 10 I=1,M  
ISN 4            C(I)=0  
ISN 5            DO 10 J=1,N  
ISN 6            10 C(I)=A(I,J)*B(J)+C(I)  
ISN 7            RETURN  
ISN 8            END
```