

GPS DESIGN: UNDIFFERENCED CARRIER BEAT PHASE OBSERVATIONS AND THE FUNDAMENTAL DIFFERENCING THEOREM

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PREFACE

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PREFACE

An abbreviated version of this technical report appeared as Lindlohr and Wells “GPS design using undifferenced carrier beat phase observations” in *Manuscripta Geodaetica*, 1985, Vol. 10, No. 4, pp. 255-295.

Special aspects of GPS design are additionally dealt with in Grafarend, Lindlohr and Wells [1985] and in Wells, Doucet and Lindlohr [1986].

ABSTRACT

The primary theme of this report is a description of the problems and techniques involved in GPS network design. A secondary theme is that the undifferenced carrier beat phase observation equation is the most appropriate model to use (e.g., Hatch and Larson [1985]), not only for GPS design work but also for final processing of GPS observations in a network adjustment.

The GPS carrier beat phase observation equation proposed by Remondi [1985a] is first reviewed in its undifferenced form. It is then shown that this can be used to describe virtually any GPS observation type, using simple assumptions about what is known a priori and what is not.

GPS design problems are reviewed. Zero-order design describes the datum definition. A three-step approach to first-order design, including redundancy design, configuration design, and logistics design, is defined. The redundancy design tool is developed and applied for several kinds of GPS solutions. Problems in configuration design are introduced, with some suggested approaches to their solution. The basis for logistics design is laid. An approach to second-order design is given.

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1. INTRODUCTION

This report discusses some problems involved in GPS network design. The discussion is based on the undifferenced form of the GPS carrier beat phase observation equation proposed by Remondi [1985a], and used by Hatch and Larson [1985]. In this section we introduce our notation and terminology conventions, and give an outline of the report.

1.1 Notation

$X_r = (x_r, y_r, z_r)^T$ three-dimensional ground station position vector

$X^s = (x^s, y^s, z^s)^T$ three-dimensional satellite position vector

$r = \|X^s - X_r\|$ Euclidean distance between X^s and X_r

ϕ complete GPS L1 carrier beat phase observed quantity

Δ between-receiver phase differencing operator on ϕ

∇ between-satellite phase differencing operator on ϕ

δ between-epoch phase differencing operator on ϕ

example : $\delta \nabla \Delta \phi$ = the triply differenced linear combination of ϕ observed quantities obtained by differencing first between receivers, then satellites, and finally epochs.

α receiver-specific bias in ϕ

β satellite-specific bias in ϕ

γ satellite/receiver pair-specific bias in ϕ

f GPS L1 carrier frequency (154 * 10.23 MHz)

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c	vacuum speed of light (299 792 458 m/s)
n	number of observations
m	maximum number of independent unknowns

quantum numbers:

$g = 1,2,\dots,G$	ground station index (usually a subscript)
$r = 1,2,\dots,R$	receiver index (usually a subscript)
$s = 1,2,\dots,S$	satellite index (usually a superscript)
$t = 1,2,\dots,T$	measurement epochs (usually an argument)
$p = 1,2,\dots,P$	observing session index
$q = 1,2,\dots,Q$	observing campaign index.
$t^s = 1,2,\dots,T^s$	independent satellite coordinate corrections epochs
$t_g = 1,2,\dots,T_g$	independent ground station coordinate corrections epochs
μ	prior information parameter associated with t^s
ν	prior information parameter associated with t_g
λ	observation-to-parameter covariance scale ratio.

1.2 Terminology

- **Range** means the standard distance between two points in a three-dimensional *Euclidean* space. Since real measurements of distance are affected by various disturbances, they are *biased* with respect to the ideal or Euclidean distance. For this reason we call distance observations *biased ranges* [Ashkenazi et al. 1985; Vaníček et al. 1985]. GPS

measurements, be they code pseudo-ranges or carrier beat phase measurements, are always some kind of biased range, as will be shown later.

- A *carrier beat phase observation* is a measurement of the phase of the signal which remains when the incoming Doppler-shifted satellite carrier signal is beat (that is the difference frequency signal is generated) with the nominally-constant reference frequency generated in the receiver [Wells 1985]. This is nothing else than a biased-range observation, where the range units are metres, and the unit of the GPS L1 carrier beat phase observations is cycles. Note that one cycle of carrier beat phase has the same distance dependence (i.e., wavelength) as the original carrier [Remondi 1984; Wells 1985]. These observations can be rescaled from cycles to wavelengths (in metres) by the factor c/f , where f is the L1 carrier frequency, and c is the velocity of light in a vacuum. The result is a biased range. The biases involved in carrier beat phase biased-range observations are a main point of discussion in this report.
- The information in which we are interested (the *signal*) is modelled by *desired parameters* in our mathematical models. Those effects, which must be taken into account but are not of interest in themselves (the *biases* or *noise*), are modelled by *nuisance parameters* in our mathematical models. Here our use of the terms “signal” and “noise” are in the sense of the axiom that “one man’s noise is another man’s signal,” rather than implying any stochastic properties (as in “white noise”).
- The assumption which introduces three unknown coordinates per receiver per observation epoch we will refer to as the *kinematic* or *freely deforming* net approach. In contrast, the *static* net approximation which is often used does not distinguish between different observation epochs.

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- Similarly, the assumption which introduces three unknown parameters per satellite point per observation epoch is often referred to as the *free orbit* or *geometric* approach.

Since we will be considering kinematic and/or free orbit solutions extensively in this report, it is well that we state at the outset why we consider them to be important. The kinematic and/or free orbit assumptions represent the purely geometric considerations in GPS campaign design, but do not take into account all the nongeometric (physical) information we have about orbits and network deformations. It is possible to model this additional nongeometric information in the form of constraints of some kind. We consider one method of applying such constraints later under the topic of second-order design.

However, **for GPS design**, it is very important to consider the kinematic and/or free orbit approaches very carefully. If we are able to select a GPS campaign design for which a kinematic and/or free orbit solution is possible, then the role of the constraints is merely to represent the additional nongeometrical information. In that sense the constraints are optional, and we have more freedom to apply them in whichever way we choose. On the other hand, if we use a campaign design in which no kinematic and/or free orbit solution is possible, then the constraints are no longer optional. They are necessary to keep the normal equations from being singular. This may limit the freedom we have in deciding how to apply the constraints.

1.3 Outline

This report on GPS network design is structured to reflect the various kinds of designs involved in the planning of conventional terrestrial geodetic networks. In the terminology of Grafarend et al. [1979b], for example,

- **Zero-order design** deals with the estimability of the unknown parameters from different types of observations, that is, the datum definition.

- **First-order design** deals with the configuration of the stations relative to each other.
- **Second-order design** deal with problems of assigning the proper covariance matrix to the observations.

We first deal with **GPS zero-order design** (the datum definition problem).

GPS first-order design occupies the main part of this report. In preparation for consideration of GPS first-order design, we first digress to discuss two topics:

- The GPS undifferenced carrier beat phase **observation equation**, which we will use, is introduced and discussed term by term.
- The **fundamental differencing theorem** is introduced, which describes the relationship between our undifferenced equation and various differencing approaches which have been described elsewhere.

We then turn to the three GPS design problems which together we have identified as being the equivalent of classical first-order design. They are:

- **Redundancy design.** We investigate minimal network configurations, using only a Diophantine equation (an equation having only natural number constants and solutions) of the form:

$$n \geq m , \quad (1-1)$$

where n is the number of observations, and m is the maximum number of independent unknowns.

- **Configuration design.** This considers the content of the **first-order design matrix** A , partitioned into the “signal” design matrix A_1 , which contains the geometry of the layout of the network points, and the bias design matrix A_2 , containing the influence of various biases.
- **Logistics design.** The logistics design problem is to select mappings of the available receivers onto the stations which will result in a network determination which is optimal (or at least acceptable) according to some criterion.

Our discussion on **GPS second-order design** considers a strategy for taking into account prior information about satellite orbits and coordinates of deforming networks.

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Finally, a summary of the main aspects of GPS design as well as the resulting conclusions are reviewed.

In **Appendix A**, the validity of the differencing theorem is explicitly demonstrated. The structure of the bias design matrix and corresponding normal equation matrix for the most general case of undifferenced observations is depicted in **Appendix B**. Detailed results for GPS redundancy designs for many solution types are listed in tables in **Appendix C**. The source code for the Apple Macintosh Basic programs used to generate the results presented here are contained in **Appendix D**.

2. GPS DATUM DEFINITION

In this section we deal with **GPS zero-order design** (the datum definition problem), paying particular attention to three questions:

- The datum defect of linear combinations of range measurements.
- Scale datum defect.
- Zero-order design for deformation networks.

A three-dimensional datum (reference system) is defined geometrically by three properties: origin (3 components), orientation (3 components), and scale (1 component).

No single observation type provides complete datum information. Observations which are invariant with respect to scale cannot provide datum scale. Observations which are invariant with respect to translation cannot provide datum origin. Observations which are invariant with respect to rotation cannot provide datum orientation. All geometrical observation types can be thought of as linear combinations of either distances or directions. GPS measurements are exclusively distances or linear combinations thereof (a possible exception is the direction content of measurements made with directional antennas, such as microstrip arrays). Here we will deal only with the distance content of GPS measurements.

2.1 Datum Content of Ranges

Consider a network of points divided into satellite points $s = 1, 2, \dots, S$ and ground points $g = 1, 2, \dots, G$. A single range observation (ignoring biases, and expressing all quantities in metres) is given by

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$$\rho_g^s = \|X^s - X_g\| = \sqrt{\langle X^s - X_g \mid X^s - X_g \rangle} \quad (2-1)$$

where $\langle a \mid b \rangle := a^T b$ denotes the Euclidean inner product between vectors a and b .

Translating the datum by τ we have

$$\bar{X}^s = X^s + \tau$$

$$\bar{X}_g = X_g + \tau.$$

Then

$$\begin{aligned} \bar{\rho}_g^s &= \sqrt{\langle \bar{X}^s - \bar{X}_g \mid \bar{X}^s - \bar{X}_g \rangle} \\ &= \sqrt{\langle X^s + \tau - X_g - \tau \mid X^s + \tau - X_g - \tau \rangle} \\ &= \sqrt{\langle X^s - X_g \mid X^s - X_g \rangle} \\ &= \rho_g^s \end{aligned} \quad (2-2)$$

that is, ranges are invariant with respect to translation.

Rotating the datum, we have

$$\bar{X}^s = R_\theta X^s$$

$$\bar{X}_g = R_\theta X_g$$

where R_θ represents a proper orthogonal transformation ($R^T R = I$). Then

$$\begin{aligned} \bar{\rho}_g^s &= \sqrt{\langle \bar{X}^s - \bar{X}_g \mid \bar{X}^s - \bar{X}_g \rangle} \\ &= \sqrt{(X^s - X_g)^T R^T R (X^s - X_g)} \\ &= \sqrt{R^T R \langle X^s - X_g, X^s - X_g \rangle} \\ &= \sqrt{\langle X^s - X_g \mid X^s - X_g \rangle} \\ &= \rho_g^s \end{aligned} \quad (2-3)$$

that is, ranges are invariant with respect to rotation.

Changing the datum scale by κ , we have

$$\begin{aligned}\bar{X}^s &= \kappa X^s \\ \bar{X}_g &= \kappa X_g .\end{aligned}$$

Then

$$\begin{aligned}\bar{\rho}_g^s &= \sqrt{\langle \bar{X}^s - \bar{X}_g \mid \bar{X}^s - \bar{X}_g \rangle} \\ &= \sqrt{\langle \kappa (X^s - X_g) \mid \kappa (X^s - X_g) \rangle} \\ &= \kappa \sqrt{\langle X^s - X_g \mid X^s - X_g \rangle} \\ &= \kappa \rho_g^s\end{aligned}\tag{2-4}$$

that is, ranges are **not** invariant with respect to scale change.

Therefore, **range observations cannot provide datum origin or orientation**, but only datum scale. The missing origin and orientation components are the datum defect, in this case equal to six. In a network determined only from range measurements, these missing components must be provided by some constraints on the coordinates. These constraints can be in the form of either specified (known) values for some network coordinates, or relationships between some network coordinates, or weights on some or all a priori coordinate values. Here we will assume that specification of these constraints reduces the number of unknown parameters by six.

2.2 Datum Content of Linear Combinations of Ranges

Linear combinations of range observations preserve the same datum information as ranges themselves, that is, linear combinations are also invariant with respect to translation and rotation but not invariant with respect to scale change, that is, for arbitrary coefficients c_i^j

$$\sum_i \sum_j c_i^j \rho_i^j = \sum_i \sum_j c_i^j \sqrt{\langle X^j - X_i \mid X^j - X_i \rangle}. \quad (2-5)$$

Since each inner product term is invariant with respect to translation and rotation, then so is every linear combination. Since all inner product terms are equally rescaled by the same scale change, then so is every linear combination. Hence single, double, triple differences and other linear combinations of range observations contain, at most, the datum information contained in the ranges themselves.

2.3 Datum Content of Biased Ranges

We have so far considered the datum content of range observations which are without biases. We have established that even such idealized range observations cannot provide datum origin or orientation. The actual observations which we wish to consider are not idealized range observations. The remaining question is whether these actual, rather than idealized, range observations can provide scale.

Electromagnetic “range” measurement systems, such as GPS, actually measure either some kind of signal **time delay** (for example, GPS pseudo-range measurements), or some kind of signal **phase comparison** (for example, GPS carrier beat phase measurements). Such measurements are in time and phase units, respectively, and have no scale content without

external information. Thus such measurements, by themselves, have a datum defect of seven, rather than six, for idealized range measurements.

However, we must consider carefully the question of what kind of “external information” might be available and appropriate.

One kind of external information is prior knowledge about the network coordinates, as discussed above. Increasing the number of specified coordinate values from six to seven, or otherwise applying constraints to reduce the number of unknowns by seven rather than six, is one possibility. Another possibility having to do with coordinates is the scale information provided by applying weights to some or all a priori coordinate values, particularly when both some satellite and some ground points are weighted. However, we usually have more appropriate prior knowledge for determining scale; information dealing more directly with the root of the problem, which is propagation.

The scale defect problem is overcome if we can make certain assumptions about propagation. In the case of time delay measurements, scale is provided if the mean electromagnetic velocity over the measured path can be assumed to be known; multiplying the time delays by the velocity yields ranges. In the case of phase comparison measurements, scale is provided if both the mean velocity, and the carrier frequency can be assumed to be known; multiplying the phase measurements by the velocity/frequency ratio yields ranges. There is usually less concern about the validity of the frequency assumption than there is about the velocity assumption.

A slightly different approach to the velocity assumption is to scale the measurements into biased ranges using a conventional value for the speed of light (the vacuum velocity c). Scale is now provided if we can assume that the departures of the actual velocity from the conventional value, due to atmospheric effects, are known. External information is used to provide knowledge of these departures in the form of atmospheric measurements, models for

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the vertical profile of tropospheric refraction, and models for the dispersive character of the ionosphere.

The scale defect question can be reposed: Do electromagnetic range measurements, **together with external information on propagation**, provide scale? This is essentially a philosophical question. One might answer yes, scale is provided, because propagation modelling can be refined to whatever precision is required, although perhaps at great effort and expense. On the other hand, one might answer no, scale is not provided, since we must specify a propagation model in advance, as part of the problem definition, in which case there will always be some limiting precision beyond which the model will fail, and scale, as provided by the range measurements plus propagation model, will be undefined. From the first point of view, electromagnetic range measurements, together with external information on propagation, have a datum defect of six. From the second point of view, they have a datum defect of seven.

Let us discuss an actual GPS example from both points of view. Using single frequency GPS phase measurements, Gervaise et al. [1985] compensated for ionospheric effects with a very simple layer model of the ionosphere, which required only an externally-supplied estimate of the mean total electron content. The adjusted GPS network scale uncertainty in this case turned out to be 0.5 ppm multiplied by the uncertainty in the external estimate of total electron content, expressed in units of 1×10^{17} electrons/m². Taking the first point of view, should it be necessary to meet a certain requirement, the uncertainty in the total electron content estimate can in principle be reduced by drawing on more external information about the state of the ionosphere or, alternatively, a more sophisticated ionospheric model could be adopted. Taking the second view, in this case a particular model was adopted, with particular prior information about the ionosphere (the total electron content uncertainty was felt to be about 1×10^{17} electrons/m²) which yielded a scale uncertainty of about 0.5 ppm. For any requirement demanding better than this, scale is not adequately realized.

Here we assume that GPS carrier beat phase measurements can be scaled from cycles into metres by the ratio cf , and that propagation effects have been sufficiently reduced through preprocessing using external propagation information, that scale can be assumed to be provided. Hence we will assume for the remainder of this report that our measurements have a datum defect of six, rather than seven.

A very different alternative is to reformulate the problem in terms of ratios of biased-range observations [Baarda 1973]. This warrants serious consideration for GPS, but is beyond the scope of the present report.

2.4 Datum Considerations in Deformation Networks

Datum plays a special role in deformation problems.

- If we make a kinematic or freely deforming network assumption (independent coordinates for each receiver for each observation epoch), together with the free-orbit assumption (independent coordinates for each satellite for each observation epoch), then we have a new datum every observation epoch, and the datum defect is no longer 6, but $6T$, where T is the number of observation epochs. A special case, which has the same redundancy design, is when there are six coordinates (three at one station, two at a second, and one at a third) which do not freely deform.
- If we take the view that range measurements, together with external information on propagation, do not provide scale, then the kinematic free-orbit case has a datum defect of 7 for every observation epoch, or a total of $7T$.
- If it is appropriate to make the static assumption (the coordinates deform so slowly that they can be considered constant over the time span of a short GPS campaign), then the deformation trajectory must be determined from GPS campaigns repeated at appropriate intervals. In this case, the “signal” is the deformation and depends critically on accurately relating the datums involved in each campaign.
- Permanent GPS deformation monitoring networks which may well be established in the future, will need a careful and likely complicated datum design, which must combine elements of both the kinematic and static approaches.

3. THE UNDIFFERENCED GPS OBSERVATION EQUATION

Remondi [1984; 1985a, equation 6], Vaníček et al. [1985]; Hofmann-Wellenhof [1985]; Hatch and Larson [1985], and Goad [1985] have proposed a GPS carrier beat phase observation equation of rather simple form. Here we choose to express all terms in the observation equation in units of cycles of the GPS L1 carrier beat phase. For measurements made by code correlation receiver channels, one cycle has a wavelength (c/f) of approximately 19 cm. For squaring type receiver channels, the wavelength is half this distance.

3.1 The Observation Equation

The observation equation for a single (undifferenced beat) phase observation has the following form:

$$\phi_r^s(t) = (f/c) \|X^s(t) - X_r(t)\| + \alpha_r(t) + \beta^s(t) + \gamma_r^s \quad (3-1)$$

where the quantum numbers are

subscripts	$r = 1, 2, \dots, R$	denote ground station receivers
superscripts	$s = 1, 2, \dots, S$	denote satellites
arguments	$t = 1, 2, \dots, T$	denote measurement epochs.

Note that equation (3-1) is linear in the three bias terms, α , β , and γ . While it is nonlinear in the coordinate terms, X^s and X_r , this is irrelevant for our discussion of the differencing theorem in section 4.

We now discuss in some detail the information content of each term in this equation, and how we may go about modelling it.

3.2 The Observed Quantity ϕ

ϕ is the complete GPS L1 carrier beat phase observed quantity,

$$\phi = y - e = E(y) \quad (3-2)$$

where y is the observation, e is the observational error, and $E(y)$ is the expectation of y . ϕ can be interpreted as a biased range. We make **four** assumptions about such observed quantities here.

- There are no unresolved problems of data simultaneity (ensuring that the relationship between measurement epochs at different receivers is taken into account).
- No satellite rises or sets during the observation period.
- There are no receiver malfunctions, that is, all receivers obtain data from each satellite at each epoch.
- The measurements have been preprocessed to remove cycle slips, and (to some extent) ionospheric and tropospheric refraction effects.

Thus during an observation session involving R receivers, S satellites, and T epochs, the number of observed quantities ϕ is

$$n = R S T . \quad (3-3)$$

3.3 Receiver Position X_r

X_r is the three-dimensional receiver position vector. Usually this is the unknown of primary interest, but may not always be so. An example is the case of a network established to track GPS signals in order to determine positions of the GPS satellites. Two basic

assumptions about X_r are possible: that it varies with time (the **kinematic** assumption), or that it is invariant in time (the **static** assumption). We consider both cases here. The static assumption actually involves the following **two** assumptions:

- **noise** effects (for example, earth tides) are accounted for in modelling and preprocessing of the data, and may be ignored;
- **signal** effects, such as network deformations, are so slow with respect to the time span of a GPS observation campaign (hours to days) that the receiver position vectors can be considered constant over this observation time span. In this case the trajectory of the deformations is determined from repeated network solutions at appropriate time intervals.

In this report we consider the following cases for X_r solutions:

<u>Index</u>	<u>Solution</u>	<u>Number of unknowns</u>
4	kinematic, or freely deforming net	$3 R T$
3	linearly deforming net	$6 R$
2	static net with one moving station (rover)	$3 (R - 1) + 3 T$
1	static net	$3 R$
0	known net	0

3.4 Satellite Position X^s

X^s is the three-dimensional satellite position vector at the time of observation. This can be treated as completely known (say, from the broadcast ephemeris); approximately known; or completely unknown. The second case (approximately known) can be considered the most general situation, of which the other two are limiting cases (the degree of approximation being

modelled by the covariance matrix of the a priori satellite parameters). In this case, the uncertainties can be modelled in several ways: at one extreme by a few nuisance parameters (such as one set of tangential, radial and normal translations per satellite pass); and at the other extreme by assuming that the uncertainties take on different and independent values for each observation epoch [Tsimis 1973, Grafarend et al. 1979a]. In this report we consider the following X^s solutions:

TABLE 3.2
SATELLITE POSITION SOLUTION TYPES

<u>Index</u>	<u>Solution</u>	<u>Number of unknowns</u>
3	free orbits	3 $S T$
2	unknown Keplerian corrections	6 S
1	unknown orbit translations	3 S
0	known orbits	0

3.5 Nuisance Parameters α , β , and γ

α are nuisance parameters representing all receiver-specific biases (principally instantaneous receiver clock time offset, but including other effects like refraction). These parameters are dependent on the receiver index and observation epoch, but are independent of the satellite index.

β are nuisance parameters representing all satellite-specific biases (principally instantaneous satellite clock errors, but including other effects like refraction). These parameters are dependent on the satellite index and observation epoch, but are independent of the receiver index.

γ are nuisance parameters representing all satellite/receiver pair-specific biases (principally the initial carrier beat phase integer cycle ambiguity, but including other effects like the temporally constant content of refraction). These parameters are dependent on the satellite and receiver index, but are independent of the observing epoch (at least within each observing session).

A word about modelling these three bias terms. Here we will consider the most general cases, that is, that α takes on a different (and independent) value for each receiver and each observation epoch; that β takes on a different (and independent) value for each satellite and each observation epoch; and that γ takes on a different (and independent) value for each satellite/receiver pair, for each observing session. These models involve the fewest assumptions about the nature of the bias terms. We will show that, after taking measures to avoid rank deficiency in the bias design matrix A_2 , these most general cases are equivalent to various differencing approaches.

Less general models for these biases are possible, however, and will also be considered. For example, some degree of serial correlation could be introduced into the α or β biases (or both) by using either deterministic or stochastic methods, or a combination of the two. A (perhaps extreme) example of a deterministic serial correlation model is the low order time polynomial (for example, quadratic) often used to represent α or β . An example of a stochastic serial correlation model would be the application of weight constraints on the deviation of each new value of α or β from some function of the previous values. It is important to note that these less general models involve more assumptions, are less likely to be generally valid, and cannot be reproduced by differencing the observations. However, for certain applications, they may be appropriate. Three examples are as follows.

- For networks of limited extent (say less than 50 km) the pair-specific biases γ have little atmospheric refraction contamination, and are almost entirely due to the initial integer cycle ambiguity, which can usually be determined accurately in preprocessing, and then held fixed to an integer value in final processing.

- If we speculate that a network of GPS reference receivers is some day established with very accurate clocks (one interesting possibility is to co-locate such receivers with the major master clocks in the world), then the main content of the receiver-specific bias α would be known external to GPS measurements, and one of the less general models could be used.
- For many applications (those not requiring better than metre-level accuracies, for example) less general models for the biases, such as quadratic α and/or β models, may be acceptable. However, we believe this is not an adequate clock model if relative positioning accuracies are desired at the limit (cm to mm) of GPS capabilities (see Figures 3.1 to 3.3).

In this paper we will consider the following cases for α , β , and γ solutions:

<u>Bias type</u>	<u>Index</u>	<u>Solution</u>	<u>Number of unknowns</u>	
receiver-specific (α)	3	free	R	T
	2	quadratic	3	R
	1	offset	R	
	0	known	0	
satellite-specific (β)	3	free		S T
	2	quadratic	3	S
	1	offset		S
	0	known	0	
pair-specific (γ)	1	unknown	R	S
	0	known	0	

3.6 Atmospheric Refraction

Atmospheric refraction is the bias which is least satisfactorily handled (by this equation or by any other). We really can only nibble at the edges of modelling this bias. Preprocessing may take a large bite out of the bias (as noted above):

- for the ionospheric part by using either the dual frequency correction (if the L2 GPS frequency is available), or a suitable ionospheric model (also if available);
- for the troposphere models based on vertical profiles inferred from surface measurements of temperature, pressure, and water vapour pressure may remove much of the bias, particularly for the dry part of tropospheric refraction.

What concerns us here about atmospheric refraction is what role the biases α , β , and γ may play. That part of the atmospheric refraction (or unmodelled residual refraction) which is receiver-specific (possibly due to local tropospheric anomalies, such as weather fronts, which affect the signals at a specific receiver, but coming from all satellites in a very similar way) will be indistinguishable from other receiver-specific biases and will be modelled by α . That part of the atmospheric refraction (or unmodelled residual refraction) which is satellite-specific (possibly due to high altitude ionospheric heterogeneities which affect the signals from a single satellite very similarly at all receivers) will be indistinguishable from other satellite-specific biases and will be modelled by β . That part of the atmospheric refraction (or unmodelled residual refraction) which represents spatial rather than temporal inhomogeneities, and thus is constant in time, will be modelled by γ . Note, however, that the refraction content of γ is unlikely to be great in networks of small size (say, less than 50 or 100 km across), because all receivers will be similarly affected (except rarely by very local tropospheric phenomena), and any inhomogeneities will be mainly satellite-specific and absorbed by β . For larger networks, the refraction content of γ will limit our ability to extract integer values for the initial carrier beat phase ambiguity. Note that in the first two cases (α and β), caution must be used in introducing serial correlation models which ignore the refraction component of each bias. An example is the quadratic polynomial for α or β mentioned above, which for some applications may be suitable to describe clock performance but is unlikely to be helpful in modelling the refraction component. From this point of view, the safest route is to use the most general model for α and β , in which no serial correlation is assumed (different and independent values are estimated for each epoch). This is what will be assumed here.

3.7 Actual Observational Time Series

Figures 3.1 to 3.6 present some time series plots of actual GPS observations. These figures give us some insight into the time dependent biases α and β , and also the time dependent part of refraction. They demonstrate that much lower noise levels are achieved on the differenced data than on the undifferenced data.

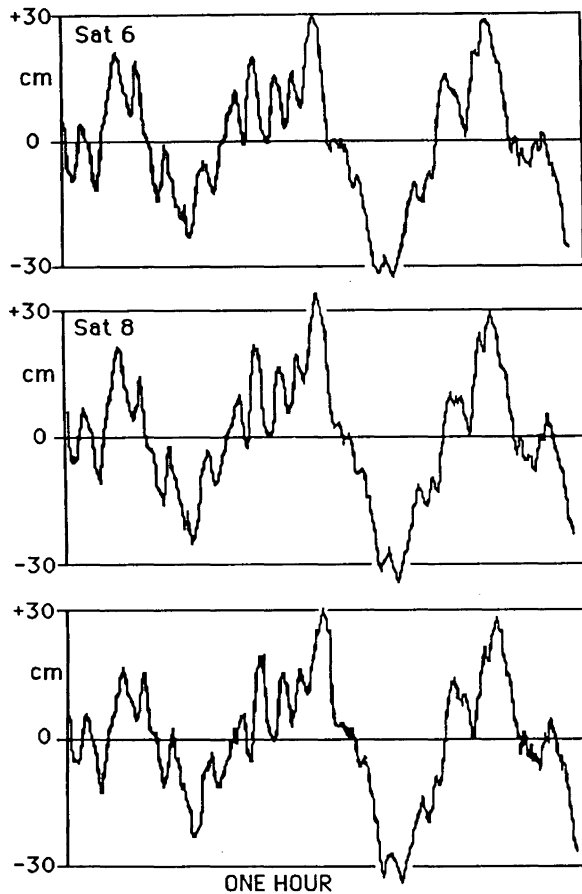
The data were obtained using two Texas Instruments 4100 GPS receivers, located 58 km apart, each tracking the same satellites. The receivers were at stations Metcalfe and Panmure of the Ottawa GPS test network [Vaniček et al., 1985b]. The same one-hour time span (0330-0430 UT, May 23, 1984) is shown in all cases.

Figures 3.1 and 3.2 show residual undifferenced phase time series $\phi(t)$ from the two receivers. Ionospheric refraction effects have been removed by combining the L1 and L2 measurements. Third-order time polynomials were fitted independently to each of the raw satellite-receiver pair one-hour undifferenced phase observation time series $\phi(t)$. This polynomial modelled the following effects:

- $||X^S(t) - X_R(t)||$ term in equation (3-1),
- tropospheric refraction effects, and
- slow variations in receiver and satellite clock noise.

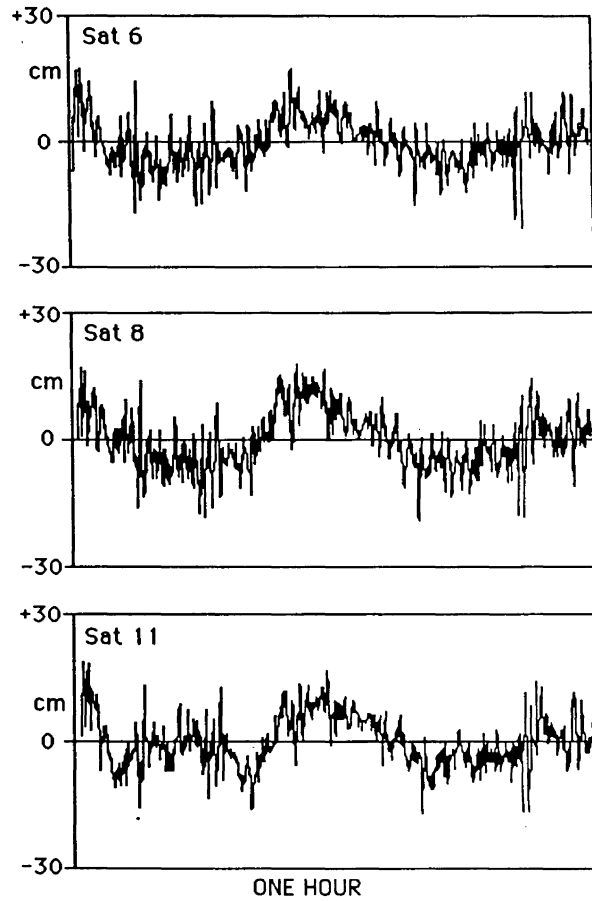
The remaining variations shown in Figures 3.1 and 3.2 are what is left after the polynomial has been removed from the original time series, and represent, mainly the sum of higher frequency receiver clock noise, higher frequency satellite clock noise, and measurement noise. The figures indicate the following.

- The peak-to-peak total noise level of this residual undifferenced data is about 60 cm for the receiver at Metcalfe (Figure 3.1), and about 40 cm for the receiver at Panmure (Figure 3.2).



OBSERVATIONS: undifferenced phase
 RESIDUALS FROM: 3rd degree polynomial
 STATION: Metcalfe
 FREQUENCY: L1/L2 combination

Figure 3.1



OBSERVATIONS: undifferenced phase
 RESIDUALS FROM: 3rd degree polynomial
 STATION: Panmure
 FREQUENCY: L1/L2 combination

Figure 3.2

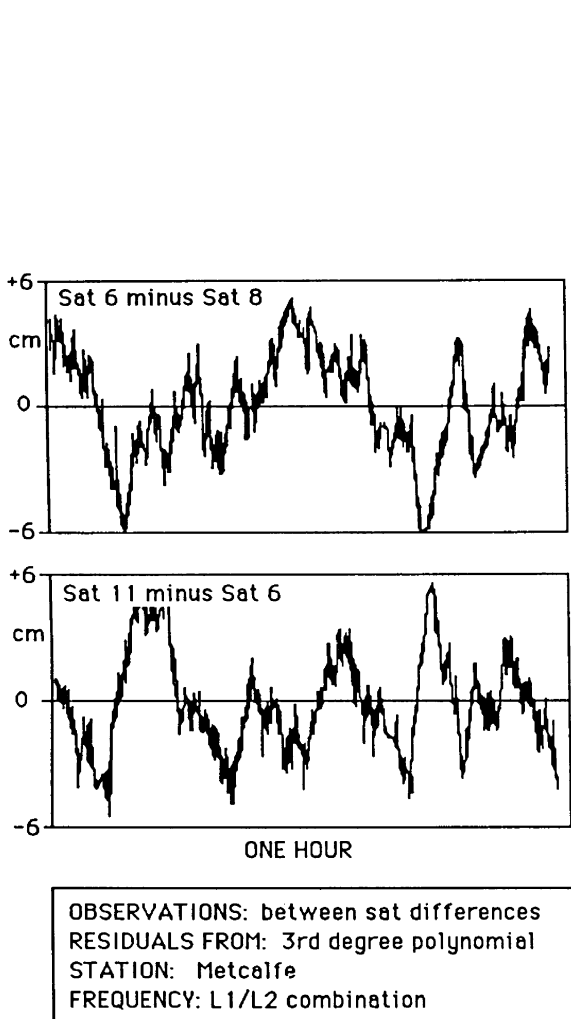


Figure 3.3

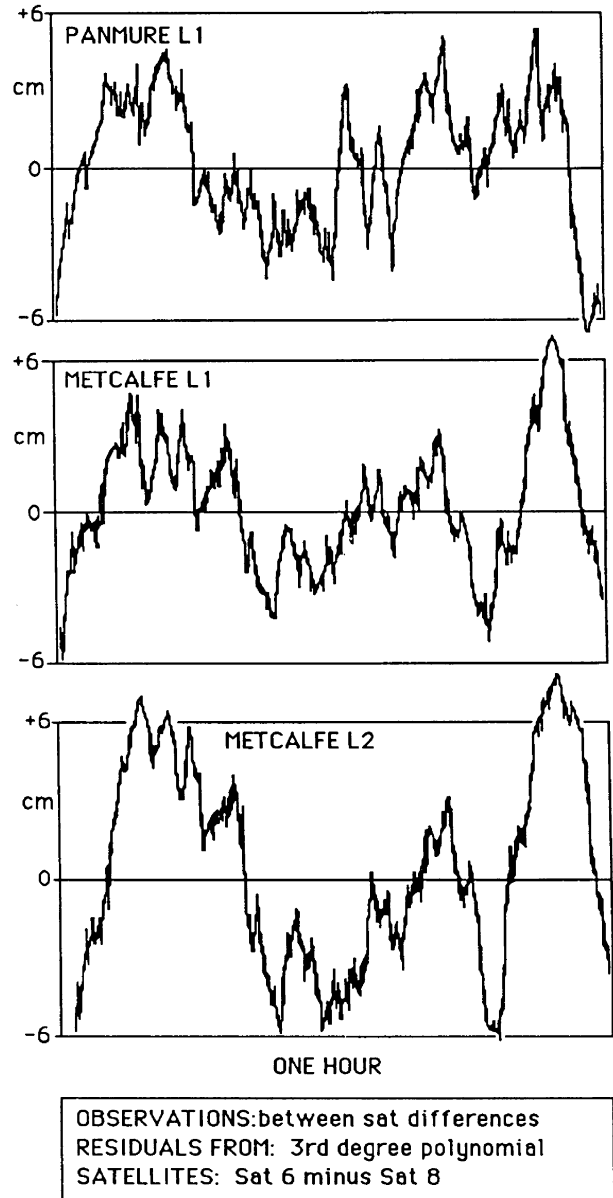
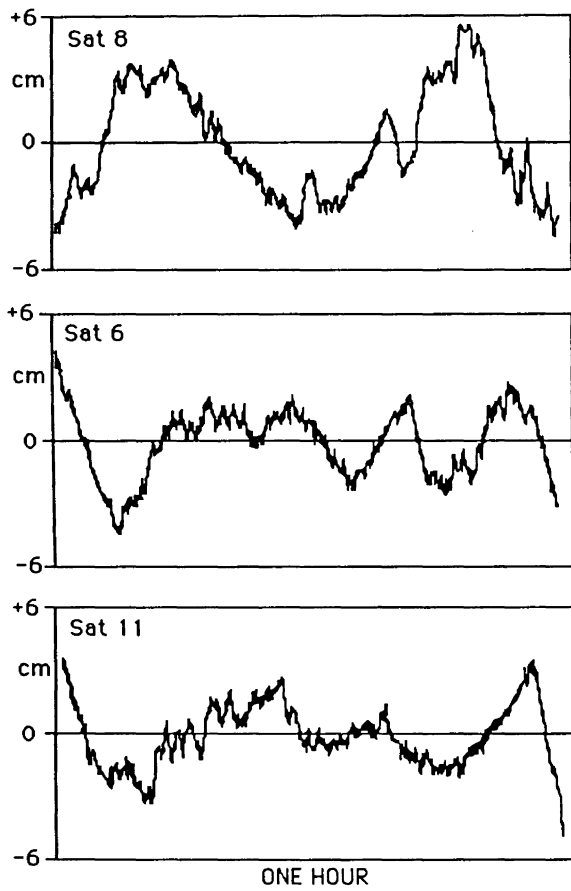
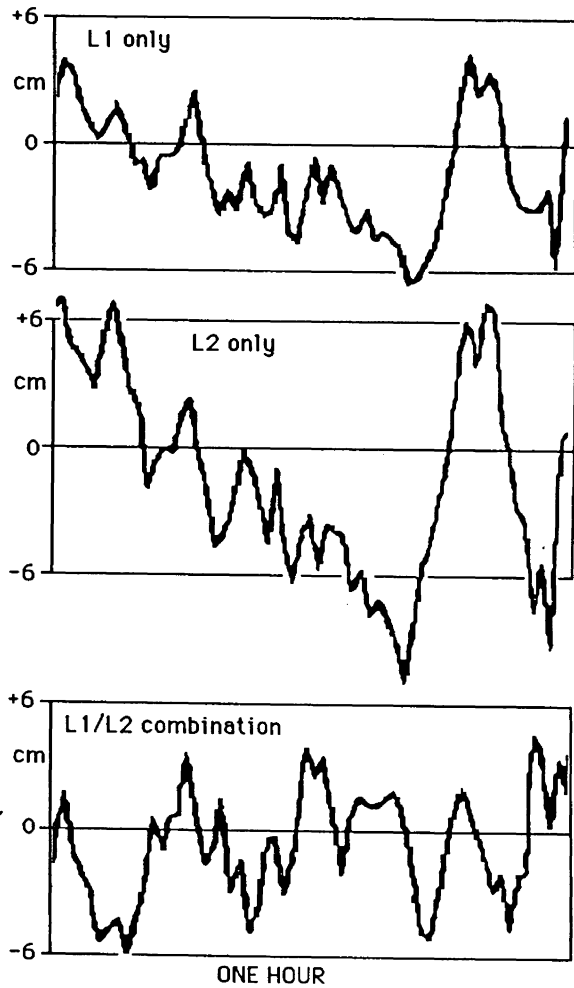


Figure 3.4



OBSERVATIONS: ionospheric correction
 RESIDUALS FROM: 1st degree polynomial
 STATION: Metcalfe

Figure 3.5



OBSERVATIONS: between sat & rcvr diffs
 RESIDUALS FROM: adjustment
 STATIONS: Metcalfe minus Penmure
 SATELLITES: Sat 6 minus Sat 8

Figure 3.6

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- The spectral nature of the time series from the two receivers is drastically different. The Metcalfe data is smoother (more spectral power at longer periods) than the Panmure data. However, such different spectral nature is not typical and is likely due to a combination of the following differences which were present in this case:
 - » different kinds of cesium clocks were used with each receiver,
 - » each receiver was operated by teams from different organizations, which may have led to slightly different operating procedures.
- The time series from different satellites, but at the same receiver, are almost perfectly correlated, as can be seen by comparing the three plots **within** each of Figures 3.1 and 3.2. This correlation is due to receiver-specific effects (α bias), mainly the receiver clock error.
- The time series from different receivers, but from the same satellites, are almost completely uncorrelated (at least before removing the receiver clock errors), as can be seen by comparing plots for the same satellite **between** Figures 3.1 and 3.2. This indicates that satellite-specific effects (β bias), mainly satellite clock errors, are not significant in comparison to receiver-specific effects.
- From these comparisons we see that the first step to reduce the noise level should be to difference between satellites. We consider the results in Figures 3.3 and 3.4.

In Figures 3.3 and 3.4 are shown between-satellite differenced phase time series $\nabla\phi(t)$. In Figure 3.3 are shown two plots: the differences at a single receiver, but differencing between two pairs of satellites, derived from the time series of Figure 3.1. In Figure 3.4 are shown three plots, all of **single-frequency** between-satellite differenced phase time series $\nabla\phi(t)$: the top two compare the differences using the same satellite pair, and the same frequency, but at two receivers; the bottom two compare the differences using the same satellite pair at the same receiver, but at two frequencies, L1 and L2. Note that:

- The top plot in Figure 3.3 is the ionospherically corrected time series corresponding to the combination of the bottom two plots of Figure 3.4.
- The different spectral nature of the time series between Figures 3.1 and 3.2 does not appear between the top two plots of Figure 3.4. This further confirms that these differences were receiver-specific in origin.
- Because single-frequency data were used for Figure 3.4, between-satellite differential ionospheric effects are included.

- For all plots in these two figures, the peak-to-peak noise level has dropped to about 10 cm. This represents the between-satellite differences in satellite clock noise, unmodelled higher-order orbit variations, and measurement noise.

We can extract some information about the dependence of these $\nabla\phi(t)$ time series variations on frequency, receivers, and satellites.

- The $\nabla\phi(t)$ time series from the same satellite pair, and at the same receiver, but using different frequencies (the bottom two plots of Figure 3.4), are the most highly correlated of the three comparisons, but still not as completely correlated as the plots within Figures 3.1 and 3.2.
- The $\nabla\phi(t)$ time series from the same satellite pair, same frequency, but different receivers (the top two plots of Figure 3.4), are slightly less correlated.
- The $\nabla\phi(t)$ time series from the same frequency, same receiver, but different satellite pairs (the top two plots of Figure 3.3), are almost completely uncorrelated.
- From these comparisons we see that we should consider eliminating both ionospheric refraction effects, by combining two-frequency data if available, and eliminating variations in the differential satellite clock bias ($\beta^j - \beta^i$) by differencing between receivers. We consider these in Figures 3.5 and 3.6, respectively.

In Figure 3.5 are shown the ionospheric refraction corrections $\delta\phi(t)$ which were applied to produce the ionospherically corrected undifferenced phase time series $\nabla\phi(t)$ in Figure 3.1. Actually only the residuals to a first-order polynomial fit are shown here, so that bias and linear trend have been removed. The peak-to-peak variations are satellite-dependent and range between 5 and 10 cm.

Figure 3.6 shows between-satellite and receiver double differenced phase residual time series $\Delta\nabla\phi(t)$. These are the residuals from an adjustment, rather than from a polynomial fit, and so it is appropriate to talk about the rms residual values, rather than peak-to-peak variations. The bottom plot in Figure 3.6 is based on ionospherically corrected (two frequency) data, and is equivalent to double differencing the top two plots from Figures 3.1 and 3.2. The top two plots in Figure 3.6 are based on single frequency L1 and L2 data. Each of these one-hour plots is extracted from an adjustment of longer five hour observational time

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series. The rms residual values for the complete five hour adjustments, for the L1, L2, and combined cases, are 3.75 cm, 5.75 cm, and 2.5 cm, respectively. This L2/L1 ratio is expected, since the ionospheric refraction effect on the lower L2 frequency is greater than for L1.

In summary, for these examples,

- the α bias (receiver clocks) is most significant, with peak-to-peak variations of up to 60 cm (3 cycles at the L1 frequency)
- the β bias (satellite clocks) and time varying refraction are about equally significant, with combined peak-to-peak variations of up to 10 cm (0.5 cycle at the L1 frequency)
- the GPS carrier beat phase rms residual noise, having removed the effects of α and β biases, and of first-order time varying refraction, is about 2 cm (0.1 cycle at the L1 frequency).

It would be dangerous, with our present experience and knowledge of GPS, to assume that these conclusions are generally valid beyond the specific cases described. However, they provide some sort of base upon which bias models can be constructed. As the most obvious example, it is clear that, for the observational time series discussed here, a simple quadratic time polynomial model for receiver clocks is not adequate, as shown by the structure of Figures 3.1 and 3.2. However, in other circumstances (other clocks, other receivers, other operational procedures), or for applications where decimetre noise levels can be tolerated, it may be that a low-order polynomial would be adequate.

4. THE FUNDAMENTAL DIFFERENCING THEOREM

The observation equation described above is formulated in terms of undifferenced observations rather than the multiply differenced observations more commonly used for GPS [Bossler et al. 1980; Beutler et al. 1984; Remondi 1984; Goad and Remondi 1984]. It is possible to obtain undifferenced-observation normal equations which are identical to those obtained from differenced observations. In this section we discuss the fundamental differencing theorem which describes the conditions under which this equivalence holds.

The fundamental differencing theorem [Grant 1976; Vaníček et al. 1985], stated informally, is:

Linear biases can be accounted for either by reducing the number of observations so that the biases cancel, or by adding an equal number of unknowns to model the biases. Both approaches give identical results, under certain circumstances.

This theorem is described more formally in Grafarend and Schaffrin [1986], and is proven in detail in Schaffrin and Grafarend [1986]. Here we simply develop some conditions on both the differencing and on the bias modelling, which we will make use of later. The system of observation equations is

$$E\{\varphi\} = A \xi = A_1 \xi_1 + A_2 \xi_2 , \tag{4-1}$$

$$C_\varphi = Q_\varphi \sigma_\varphi^2 = P_\varphi^{-1} \sigma_\varphi^2 = D\{y\}$$

where $\varphi = \phi - \omega$ is the vector of misclosures, ω is a vector whose terms have the form $(f/c)(\rho_r^s)_0$, and $(\rho_r^s)_0$ is an a priori range computed from appropriate $(X_r)_0$ and $(X^s)_0$. Since we

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consider $(X_r)_0$ and $(X^s)_0$ to be nonstochastic, the covariance matrix of the observations C_ϕ is equal to the covariance matrix of the observed quantities C_ϕ . ξ_1 contains the desired parameters, and ξ_2 the nuisance parameters (biases). Similarly, we refer to A_1 as the parameter design matrix, and to A_2 as the bias design matrix. Here we consider the parameters to be corrections to a priori values for X_r and X^s , and the biases to be α, β, γ . Under these circumstances, the bias design is linear, completely characterized by A_2 .

The differencing approach requires finding a differencing operator D such that

$$D A_2 = 0 \quad (4-2)$$

so that operating on equation (4-1) yields

$$D \phi = D A_1 \xi_1, \quad C_{D\phi} = D P_\phi^{-1} D^T \sigma_\phi^2 \quad (4-3)$$

leading to normal equations $N \xi_1 = b$, where, if $\sigma_\phi^2 = 1$

$$N = A_1^T D^T (D P_\phi^{-1} D^T)^{-1} D A_1 \quad (4-4)$$

$$b = A_1^T D^T (D P_\phi^{-1} D^T)^{-1} D \phi.$$

Let us define the matrix product $D^T (D P_\phi^{-1} D^T)^{-1} D$ as the **kernel** of these normal equations.

The undifferenced observation approach (equation (4-1)) leads to the normal equations $N^* \xi = b^*$, where $\sigma_\phi^2 = 1$

$$N^* = \begin{bmatrix} A_1^T P_\phi A_1 & A_1^T P_\phi A_2 \\ A_2^T P_\phi A_1 & A_2^T P_\phi A_2 \end{bmatrix} \quad (4-5)$$

$$b^* = \begin{bmatrix} A_1^T P_\phi \phi \\ A_2^T P_\phi \phi \end{bmatrix}$$

Eliminating ξ_2 we have the normal equations $N \xi_1 = b$ where

$$\begin{aligned}
 N &= A_1^T P_\varphi A_1 - A_1^T P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi A_1 \\
 &= A_1^T \{P_\varphi - P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi\} A_1 \\
 b &= A_1^T P_\varphi \varphi - A_1^T P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi \varphi \\
 &= A_1^T \{P_\varphi - P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi\} \varphi .
 \end{aligned} \tag{4-6}$$

The kernel of the normal equations in this case is $\{P_\varphi - P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi\}$. Note that the two approaches differ only in their kernels. The differencing approach is characterised completely by the differencing operator D , and the undifferenced approach by the bias design matrix A_2 . From the above we see that the two approaches result in identical normal equations when these two matrices satisfy the following conditions:

- D annihilates A_2 , that is $D A_2 = 0$, (a **necessary** but **not sufficient** condition).
- $D P_\varphi^{-1} D^T$ is regular, that is $(D P_\varphi^{-1} D^T)^{-1}$ exists, (**sufficient** but **not necessary**).
- $A_2^T P_\varphi A_2$ is regular, that is $(A_2^T P_\varphi A_2)^{-1}$ exists [Wei 1985], (**sufficient** but **not necessary**).
- The normal equation kernels from the two approaches are equal, that is,

$$D^T (D P_\varphi^{-1} D^T)^{-1} D = P_\varphi - P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi$$

or

$$D^T (D P_\varphi^{-1} D^T)^{-1} D + P_\varphi A_2 (A_2^T P_\varphi A_2)^{-1} A_2^T P_\varphi = P_\varphi . \tag{4-7}$$

Note that only the **fourth** condition is **essential**, from which the first condition follows. Moreover, the second and third conditions can be relaxed [Schaffrin and Grafarend 1986]. Let us now look more closely first at the differencing operator D then at the bias design matrix A_2 .

4.1 Differencing Operator D

If one accepts our model of three bias types α , β , and γ , then there are 15 possible differencing operations, although there may be many different D operators which accomplish each of these. Using the notation ∇ for between-satellite differencing (which eliminates the α bias), Δ for between-receiver differencing (which eliminates the β bias), and δ for between-epoch differencing (which eliminates the γ bias), we have the 15 possibilities:

Single Differencing				Eliminates
1	$\phi \rightarrow \nabla\phi$			α
2	$\phi \rightarrow \Delta\phi$			β
3	$\phi \rightarrow \delta\phi$			γ
Double Differencing				
4	$\phi \rightarrow \nabla\phi \rightarrow \Delta\nabla\phi$			α, β
5	$\phi \rightarrow \Delta\phi \rightarrow \nabla\Delta\phi$			α, β
6	$\phi \rightarrow \nabla\phi \rightarrow \delta\nabla\phi$			α, γ
7	$\phi \rightarrow \delta\phi \rightarrow \nabla\delta\phi$			α, γ
8	$\phi \rightarrow \Delta\phi \rightarrow \delta\Delta\phi$			β, γ
9	$\phi \rightarrow \delta\phi \rightarrow \Delta\delta\phi$			β, γ
Triple Differencing				
10	$\phi \rightarrow \nabla\phi \rightarrow \Delta\nabla\phi \rightarrow \delta\Delta\nabla\phi$			α, β, γ
11	$\phi \rightarrow \nabla\phi \rightarrow \delta\nabla\phi \rightarrow \Delta\delta\nabla\phi$			α, β, γ
12	$\phi \rightarrow \Delta\phi \rightarrow \nabla\Delta\phi \rightarrow \delta\nabla\Delta\phi$			α, β, γ
13	$\phi \rightarrow \Delta\phi \rightarrow \delta\Delta\phi \rightarrow \nabla\delta\Delta\phi$			α, β, γ
14	$\phi \rightarrow \delta\phi \rightarrow \nabla\delta\phi \rightarrow \Delta\nabla\delta\phi$			α, β, γ
15	$\phi \rightarrow \delta\phi \rightarrow \Delta\delta\phi \rightarrow \nabla\Delta\delta\phi$			α, β, γ

In the examples we discuss below, these differencing operations are order-independent, that is, the same total differencing operator D results from double differences 4 or 5, 6 or 7, and 8 or 9. All six triple difference sequences result in the same total differencing operator. There are only seven distinct differencing operations: three single differences, three double differences, and one triple difference.

There are many possibilities for selecting the basis against which the differences are taken by the differencing operator. Here we consider only three. Using the operation $\phi \rightarrow \Delta\phi$ as an example, these are:

- Fixed differencing basis [Remondi 1984]

$$\Delta\phi_r^S(t) = \phi_r^S(t) - \phi_R^S(t) \quad (4-8)$$

- Sequential differencing basis [Beutler et al. 1984]

$$\Delta\phi_r^S(t) = \phi_r^S(t) - \phi_{r+1}^S(t) \quad (4-9)$$

- Orthonormalized differencing basis, using the Gram-Schmidt method [Bock et al. 1985a]

$$\Delta\phi_r^S(t) = [-\left(\frac{1}{r}\right) \sum_{i=1}^r \phi_i^S(t) + \phi_{r+1}^S(t)] \left(\frac{r}{r+1}\right)^{1/2} \quad (4-10)$$

where in all cases, $r = 1, 2, \dots, R-1$. See Appendix A for examples of differencing bases.

Assuming that $P_\phi = I$, our investigations show that:

- D , the differencing operator, depends on the choice of basis.
- $D D^T$, the covariance matrix for the differenced observations, depends on choice of basis. For example, the orthonormalized basis results in $D D^T = I$.
- $D^T (D D^T)^{-1} D$ is invariant under choice of basis.
- For the orthonormalized basis, $D^T (D D^T)^{-1} D$ reduces to $D^T D$.

When we investigated the “common mode” linear combination observable suggested by Bock et al. [1985a]

$$\bar{\Phi}^s(t) = \left(\frac{1}{R}\right) \sum_{i=1}^R \phi_1^s(t)$$

we found that when it was added to the differenced observations, $D^T(D D^T)^{-1} D = I$. In this circumstance, equation (4-7) cannot be satisfied for any A_2 but the null matrix, so the differencing theorem, as we have stated it, cannot be satisfied. The equivalence between our undifferenced approach and this version of differencing, that is, including the common mode observation, is unresolved at present.

4.2 Bias Design Matrix A_2

We start our discussion of A_2 by applying the words “equal number of unknowns” in our statement of the fundamental differencing theorem. According to this theorem, the number of (independent) nuisance parameter unknowns on the right-hand side of the undifferenced observation equation must be equal to the number of observations eliminated from the left-hand side of the observation equation by the differencing operator D . Let the maximum number of independent unknowns (using the undifferenced approach) be $m = m_1 + m_2$, where $m_1 = \text{rank } A_1$ = the maximum number of independent desired, or non-nuisance, parameters in our model; and $m_2 = \text{rank } A_2$ = the maximum number of independent nuisance parameters in our model. Restating the differencing theorem in terms of redundancy Diophantine equations, we have for the **differenced** observations

$$R S T - m_2 \geq m_1 \tag{4-11}$$

and for the **undifferenced** observations

$$RST \geq m_1 + m_2. \quad (4-12)$$

As we saw from Table 4.1, there are only seven distinct differencing operations. In Table 4.2 we list equation (4-11), and m_2 for each of these.

If one considers ξ_2 to be nuisance parameter vectors of maximum possible dimension (for example, $RT+ST+RS$ in the triple difference case), then Schaffrin and Grafarend [1986] have shown that the A_2 matrix will have the rank defects shown in Table 4.2. That is, the amount by which the number of nuisance parameter unknowns must be reduced, in order to equal the number of observations eliminated by differencing, is also just sufficient to make $A_2^T P_\phi A_2$ regular (see also Wei [1985]). The rank defects indicate the **number** of nuisance parameters which must be held fixed in the solution, but do not indicate the **structure** or meaning of the parameters to be held fixed. Let us now consider possible structures. Since the undifferenced solutions which are equivalent to single difference solutions have no rank defects, we can restrict our attention to those solutions equivalent to double and triple differencing. For each of the double difference solutions, there are two simple ways in which the rank defect can be **exactly** satisfied, which means we introduce only the minimum number of conditions necessary to overcome the rank defect. Other less symmetrical ways to exactly satisfy the rank defect, or to more than satisfy the rank defect, are not considered here. For the triple difference solution, there are six simple ways of exactly satisfying the rank defect. Let us consider these 12 possibilities in Table 4.3, and interpret what each implies in Table 4.4. For the sake of completeness, we have added cases 13 and 14 to Table 4.3, which look as simple as the other cases. However, they actually eliminate one more nuisance parameter than is necessary to exactly overcome the A_2 rank deficiency. They are, therefore, not equivalent to the differencing approach, and lead to different normal equations.

TABLE 4.2
BIAS DESIGN RANK DEFECTS

Difference to Eliminate	Differencing Diophantine equation (4-11)	m_2 - add to both sides to get Undifferenced Diophantine eqn. (4-12)	Rank defect of A_2
Single differences			
1 α	$R(S-1) T \geq m_1$	RT	0
2 β	$(R-1) S T \geq m_1$	ST	0
3 γ	$R S (T-1) \geq m_1$	RS	0
Double differences			
4 α, β	$(R-1)(S-1) T \geq m_1$	$RT + ST - T$	T
5 α, γ	$R(S-1)(T-1) \geq m_1$	$RT + RS - R$	R
6 β, γ	$(R-1) S (T-1) \geq m_1$	$ST + RS - S$	S
Triple difference			
7 α, β, γ	$(R-1)(S-1)(T-1) \geq m_1$	$RT + ST + RS - (R + S + T-1)$	$R + S + T-1$

TABLE 4.3
BIAS RANK DEFECT MODELS

Solution	Nuisance unknowns	α model	β model	γ model
Double differences				
1 α, β	$RT + ST - T$	$(R-1) T$	$+ ST$	
2		RT	$+ (S-1) T$	
3 α, γ	$RT + RS - R$	$R (T-1)$		$+ RS$
4		RT		$+ R (S-1)$
5 β, γ	$ST + RS - S$		$S (T-1)$	$+ RS$
6			ST	$+ (R-1) T$
Triple differences				
7 α, β, γ	$RT + ST + RS$	$(R-1)(T-1)$	$+ ST$	$+ (R-1) S$
8	$-(R + S + T-1)$	$(R-1)(T-1)$	$+ S (T-1)$	$+ RS$
9		RT	$+ (S-1)(T-1)$	$+ R(S-1)$
10		$R(T-1)$	$+ (S-1)(T-1)$	$+ RS$
11		$(T-1) T$	$+ ST$	$+ (R-1)(S-1)$
12		RT	$+ (S-1) T$	$+ (R-1)(S-1)$
13*		$(R-1) T$	$+ S (T-1)$	$+ R(S-1)$
14*		$R (T-1)$	$+ (S-1) T$	$+ (R-1) S$

In each column of Table 4.3, there are only four types of models for each group of nuisance parameters. Let us interpret the assumptions lying behind each in Table 4.4.

TABLE 4.4		
PHYSICAL INTERPRETATION OF BIAS RANK DEFECT MODELS		
α models:		
1	RT	all receiver clocks completely unknown
2	$(R-1)T$	one receiver clock completely known, others unknown
3	$R(T-1)$	all receiver clocks synchronized at one epoch
4	$(R-1)(T-1)$	one clock known, all synchronized at one epoch
β models:		
5	ST	all satellite clocks completely unknown
6	$(S-1)T$	one satellite clock completely known, others unknown
7	$S(T-1)$	all satellite clocks synchronized at one epoch
8	$(S-1)(T-1)$	one clock known, all synchronized at one epoch
γ models:		
9	RS	all ambiguities unknown
10	$(R-1)S$	ambiguities to one receiver completely known
11	$R(S-1)$	ambiguities to one satellite completely known
12	$(R-1)(S-1)$	ambiguities to one receiver and one satellite known

Note that when we state here that a bias is “known,” this is intended in the modelling sense only—it forms a “datum,” or reference, relative to which the remaining biases will be determined. For example, for the α , β double difference case, if all receiver clocks and all satellite clocks are left completely free in the solution, there is nothing to provide a “time datum,” which fact is expressed mathematically in the form of a rank defective A_2 matrix. For this example, we see that the two possibilities arising from the rank defect of T are to adopt either one satellite clock or one receiver clock as the “datum” clock.

For the cases we have studied:

- A_2 , the bias design matrix, consists only of elements 0 and 1.
- A_2 and $A_2^T P_\phi A_2$ depend on the ordering of the unknowns and on the choice of rank defect model.
- $P_\phi A_2 (A_2^T P_\phi A_2)^{-1} A_2^T P_\phi$ is invariant with respect to the ordering of the unknowns and on the choice of rank defect model.

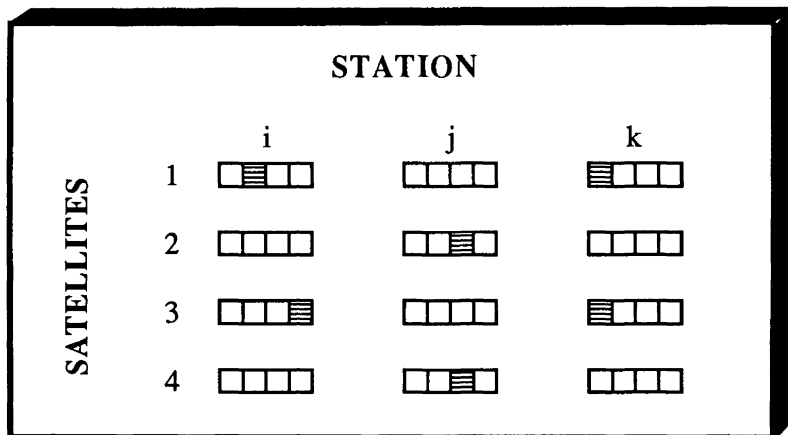
4.3 Advantages of Undifferenced Observations

Some of the advantages in using an undifferenced observation equation, rather than differencing, are as follows.

- Using the differenced-observation approach to eliminate satellite-specific biases involves forming between-receiver differences. This introduces the baseline vector, rather than the position vector, as the basic unknown quantity. This is a new and complex concept, particularly when many receivers are being used, which is unnecessary using the undifferenced observation approach.
- Differencing is completely efficient only when there are no data gaps in any of the data series being differenced. Gappy data requires rejection of perfectly good undifferenced observations, because the “partner” observation in the difference is missing. Figure 4.1 demonstrates this problem. The undifferenced observation approach retains all observations.
- Differencing has been shown above to be equivalent to assuming a very general model for the biases, which is laudably conservative in many cases, but which does not permit the inclusion of serial correlation models for the biases. Such serial correlation models may be justified in some cases, and can be easily (but cautiously!) implemented using the undifferenced observation approach.
- One of the traditional methods of model improvement is careful study of the nature of biases, with resultant refinement of bias modelling. Bias behaviour cannot easily be studied if the biases have been cancelled from the observations by differencing. With the differencing approach, it may be possible to recover bias values (relative to the chosen basis). However, the undifferenced observation approach permits bias solutions as part of the original processing, without the need for back substitution.
- Differencing requires finding some basis system for the observations, as described in section 4.1. In theory the selection is arbitrary, but in practice a poor basis selection may degrade the results. Basis selection requires some care, and adds another step in the analysis and processing of GPS data. It is difficult to ascribe physical meaning to

such bases as the orthonormalized ones (equation (4-10)). With the undifferenced observation approach, a basis selection must also be made to overcome the A_2 rank defect. In this case, however, the physical meanings are evident (Table 4.4). In the future, the differencing basis selection problem will become more acute, and perhaps intractable, with campaigns using several roving receivers tracking the full GPS constellation 24 hours per day.

OBSERVED DATA SERIES



DOUBLE DIFFERENCED DATA SERIES

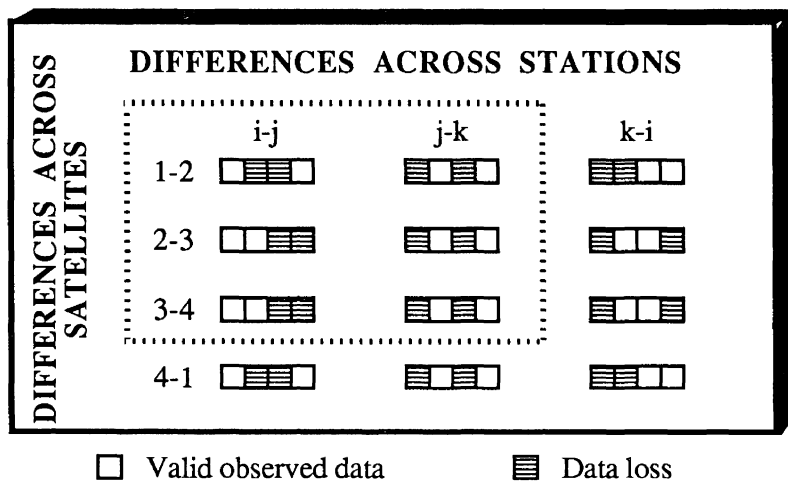


FIGURE 4.1
EFFECTS OF DATA LOSS ON DIFFERENCED DATA
(from Vaníček et al., [1985b])

GPS DESIGN

- Undifferenced observations are closer to the physical observables in GPS receivers, and thus are preferable both for aesthetic reasons and because they permit greater insight into their physical and geometrical meaning.

There are, of course, advantages to using the differenced-observation approach as well. For example, the between-receiver, satellite and epoch triple difference approach, can and has been implemented as an automatic process, including the detection and elimination of cycle slips [Goad and Remondi 1984; Beutler et al. 1984]. In our view this is a valid preprocessing method. However, both for design, and for final network processing, particularly when the limit of GPS capability is required, we feel the above advantages for the undifferenced observation approach tilt the balance overwhelmingly in its favour.

4.4 Commutative Diagram

Relations between the undifferenced observation approach (equations (4-6)), and the more commonly used differencing approach (equations (4-4)) is summarized and more graphically illustrated in the form of a diagram in Figure 4.2. Let us consider the central circuit in this diagram.

The undifferenced approach is described by the top and diagonal lines. The upper horizontal line describes the mapping of φ onto the estimates of both the desired parameters $\bar{\xi}_{1u}$, and the nuisance parameters $\bar{\xi}_{2u}$ (equations (4-5)). The diagonal line describes the reduced normal equations, from which $\bar{\xi}_{2u}$ have been eliminated (equations (4-6)).

Note that we can obtain $\bar{\xi}_{2u}$ equivalently in either of two ways:

- by proceeding to the top right corner, that is, by solution of the full normal equations (4-5), or

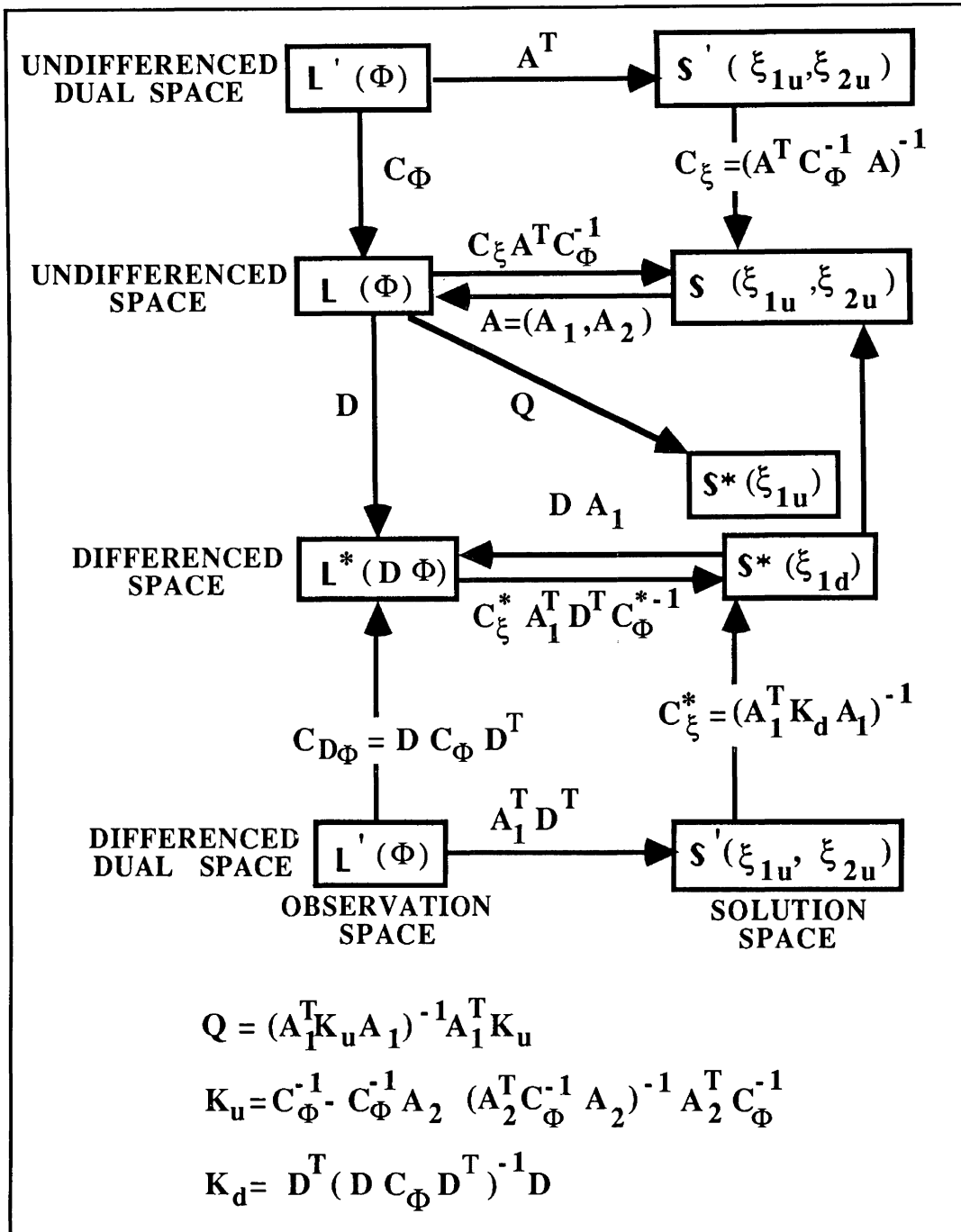


FIGURE 4.2
COMMUTATIVE DIAGRAM

5.1 Assumptions and Criteria for Optimality

We adopt the following assumptions and limitations.

- All receivers are deployed simultaneously and only once (that is, we limit ourselves, for the present, to solutions based on a campaign of one observing session only).
- The campaign involves R receivers, S satellites, and T epochs, and all receivers obtain data from each satellite at each epoch (that is, we always assume the expression $n = R S T$ given by equation (3-3)).
- The expression for m will be limited to various combinations of the expressions for X_g , X^s , α , β , and γ which are listed in Tables 3.1, 3.2, and 3.3.
- We will consider only designs which are minimal in the following sense: given values for any two of R , S , and T , the value chosen for the third is the least value which will satisfy equation (5-1). We limit our consideration to designs which fall within the region $R \in \{1,20\}$, $S \in \{1,10\}$, $T \in \{1,100\}$.

In selecting the most appropriate design from the list of minimal designs for a given type of solution, there are several possible optimality criteria which can be used. We consider some of these.

- **Minimal R .** This criterion has perhaps the strongest economic content, since GPS receivers (at least at present) are very expensive, and their deployment involves manpower costs.
- **Minimal S .** This criterion has implications for both receiver design and satellite constellation design. Actually the goal in either case is not to select minimal S but only values of S which are below the lowest of the two thresholds set by the receiver and constellation.
- **Minimal T .** This has some economic content, since recording and preprocessing expenses may increase slightly with the length of the data series. Note, however, that T represents the number of time epochs at which data is recorded, and not the time span of the observations (which is T multiplied by the interval between data samples). The time span is one of the most important factors in configuration design, to be considered later, and has stronger economic content than T itself.
- **Minimal m .** Such design(s) involve the fewest necessary unknowns, and therefore the smallest set of normal equations, and least computational requirements.

- **Minimal ST , or minimal $R + ST$.** These designs represent the fewest satellite points, and fewest total (ground + satellite) network points, respectively. Particularly for the free orbit solutions, they are a useful indication of what network structures are viable without constraints on the orbits.

5.2 Types of Solutions Considered

The most general type of solution considers all terms (except f and c , when they are assumed to be constants) on the right-hand side of equation (3-1) to be unknown, and all but γ to be time varying with different values for each observation epoch. We refer to this as the **kinematic, free-orbit, triple difference** solution. In total

$$m = 3RT + 3ST + RT + ST + RS - (R+S+T-1) - 6T \quad (5-2)$$

where we must reduce the number of unknowns on the right-hand side by the number of parameters which must be held fixed (or conditions on the parameters) in order to account for the GPS datum defect of six, per epoch in this case.

The Diophantine equation for the kinematic free orbit triple difference solution is, from equations (3-3) and (5-2),

$$RST \geq 3RT + 3ST + RT + ST + RS - (R+S+T-1) - 6T \quad (5-3)$$

Minimal designs for this type of solution are listed in Table C.1. From this list, we can see that (given maximum values for R , S and T of 20, 10, and 100, respectively) the minimum number of receivers R required is 6, the minimum number of satellites S is 5, the minimum number of time epochs T is 2, the minimum number of unknowns m is 300, the minimum number of satellite points (ST) is 20, and the minimum number of total network points ($R+ST$) is 38.

Other types of solutions which might be of interest are listed in Table 5.1. The resulting lists of minimal solutions are given in Appendix C. The minimal values for each of R , S , T , m , ST , and $R+ST$ are given in Table 5.2. A guide to the reasons we selected these particular solutions for study is:

- Solutions in Tables C.1 to C.6 and C.8 are of interest in studying kinematic modelling of deforming networks.
- Solution 40300 (Table C.7) reproduces the well-known GPS four-dimensional navigation minimal configuration of one receiver, four satellites, and one epoch.
- Solutions in Tables C.9 to C.18 are of interest in studying strategies for differential GPS real-time navigation of moving objects, like ships, planes, and other satellites.
- Solutions in Tables C.19 to C.25 are of interest in studying the free orbit approach.
- Solutions in Tables C.26 to C.31 represent the standard methods used at present for processing GPS measurements.
- Solutions in Tables C.32 to C.38 are of interest in studying strategies for establishing networks to determine positions of GPS satellites.

5.3 Comparison with Previous Redundancy Design Studies

Previous GPS redundancy design studies (e.g., Grafarend and Müller [1985]; Delikaraoglou [1985]) lumped all satellite points together, from different satellites and epochs. Structures of the biases β and γ require that we consider points from each satellite separately. Hence the total number of satellite points in these previous studies is represented by the product ST here. The optimality criteria used in previous studies was usually first fewest receivers R , then fewest network points ($R+ST$).

TABLE 5.1
TYPES OF SOLUTION CONSIDERED

SOLUTION $X_g X^s \alpha \beta \gamma X_g$	NUMBER OF UNKNOWNNS					LABEL ON SOLUTION datum	RESULTS
	X^s	α	β	γ	rank		
43331 3RT	+3ST	+RT	+ST	+RS	-(R+S+T-1)	-6T	Kinematic, free orbit, triple difference C. 1
43330 3RT	+3ST	+RT	+ST		-T	-6T	Kinematic, free orbit, double difference C. 2
43030 3RT	+3ST		+ST			-6T	Kinematic, free orbit, single difference C. 3
43001 3RT	+3ST			+RS		-6T	Kinematic, free orbit, Doppler C. 4
43000 3RT	+3ST					-6T	Kinematic, free orbit, unbiased C. 5
40331 3RT		+RT	+ST	+RS	-(R+S+T-1)		Kinematic, known orbit, triple difference C. 6
40300 3RT		+RT					Kinematic, known orbit, 4-D navigation C. 7
33331 6R	+3ST	+RT	+ST	+RS	-(R+S+T-1)	-6	Linear deform, free orbit, triple difference C. 8
23331 3(R-1+T)	+3ST	+RT	+ST	+RS	-(R+S+T-1)	-6	Static (rover), free orbit, triple difference C. 9
23330 3(R-1+T)	+3ST	+RT	+ST		-T	-6	Static (rover), free orbit, double diff. C. 10
23030 3(R-1+T)	+3ST		+ST			-6	Static (rover), free orbit, single diff. C. 11
23001 3(R-1+T)	+3ST			+RS		-6	Static (rover), free orbit, Doppler C. 12
23000 3(R-1+T)	+3ST					-6	Static (rover), free orbit, unbiased C. 13
20331 3(R-1+T)		+RT	+ST	+RS	-(R+S+T-1)		Static (rover), known orbit, triple diff. C. 14
20330 3(R-1+T)		+RT	+ST		-T		Static (rover), known orbit, double diff. C. 15
20030 3(R-1+T)			+ST				Static (rover), known orbit, single diff. C. 16
20001 3(R-1+T)				+RS			Static (rover), known orbit, Doppler C. 17
20000 3(R-1+T)							Static (rover), known orbit, unbiased C. 18
13331 3R	+3ST	+RT	+ST	+RS	-(R+S+T-1)	-6	Static, free orbit, triple difference C. 19
13330 3R	+3ST	+RT	+ST		-T	-6	Static, free orbit, double difference C. 20
13231 3R	+3ST	+3R	+ST	+RS	-S	-6	Static, free orbit, 231 C. 21
13230 3R	+3ST	+3R	+ST			-6	Static, free orbit, 230 C. 22
13030 3R	+3ST		+ST			-6	Static, free orbit, single difference C. 23
13001 3R	+3ST			+RS		-6	Static, free orbit, Doppler C. 24
13000 3R	+3ST					-6	Static, free orbit, unbiased C. 25
10331 3R		+RT	+ST	+RS	-(R+S+T-1)		Static, known orbit, triple difference C. 26
10330 3R		+RT	+ST		-T		Static, known orbit, double difference C. 27
10230 3R		+3R	+ST				Static, known orbit, 230 C. 28
10030 3R			+ST				Static, known orbit, single difference C. 29
10001 3R				+RS			Static, known orbit, Doppler C. 30
10000 3R							Static, known orbit, unbiased C. 31
03331	+3ST	+RT	+ST	+RS	-(R+S+T-1)		Known, free orbit, triple difference C. 32
03330	+3ST	+RT	+ST		-T		Known, free orbit, double difference C. 33
03230	+3ST	+3R	+ST				Known, free orbit, 230 C. 34
03031	+3ST		+ST	+RS	-S		Known, free orbit, 031 C. 35
03030	+3ST		+ST				Known, free orbit, single difference C. 36
03001	+3ST			+RS			Known, free orbit, Doppler C. 37
03000	+3ST						Known, free orbit, unbiased C. 38

TABLE 5.2
MINIMAL VALUES FOR TYPES OF SOLUTIONS CONSIDERED

SOLUTION $X_g X^s \alpha \beta \gamma$	MINIMAL VALUES				m	ST	$R+ST$	TABLE
	R	S	T					
43331	6	5	2		300	20	38	C. 1
43330	6	5	1		49	5	14	C. 2
43030	5	4	1		35	4	12	C. 3
43001	4	4	2		189	16	30	C. 4
43000	4	4	1		24	4	10	C. 5
40331	2	5	2		60	16	22	C. 6
40300	1	4	1		4	4	5	C. 7
33331	5	2	3		216	16	31	C. 8
23331	5	2	2		160	12	26	C. 9
23330	5	2	1		49	5	14	C. 10
23030	5	1	1		35	4	12	C. 11
23001	4	1	2		72	6	17	C. 12
23000	4	1	1		24	4	10	C. 13
20331	2	2	2		40	9	16	C. 14
20330	2	2	1		14	5	9	C. 15
20030	2	1	1		12	4	8	C. 16
20001	1	1	2		12	5	11	C. 17
20000	1	1	1		3	3	4	C. 18
13331	5	2	2		141	12	24	C. 19
13330	5	2	1		49	5	14	C. 20
13231	5	1	2		98	9	21	C. 21
13230	5	1	1		84	8	19	C. 22
13030	5	1	1		35	4	12	C. 23
13001	4	1	2		35	5	12	C. 24
13000	4	1	1		24	4	10	C. 25
10331	2	2	2		24	9	13	C. 26
10330	2	2	1		14	5	9	C. 27
10230	2	1	1		24	7	12	C. 28
10030	2	1	1		12	4	8	C. 29
10001	1	1	2		4	4	5	C. 30
10000	1	1	1		3	3	4	C. 31
03331	5	2	2		52	4	16	C. 32
03330	5	2	1		14	2	9	C. 33
03230	5	1	1		48	4	14	C. 34
03031	5	1	2		14	2	9	C. 35
03030	4	1	1		4	1	5	C. 36
03001	4	1	2		12	2	8	C. 37
03000	3	1	1		3	1	4	C. 38

The specific types of solution considered previously are listed (in our notation) in Table 5.3.

TABLE 5.3 COMPARISON WITH PREVIOUS RESULTS		
SOLN TABLE		
<i>Grafarend and Müller [1985]</i>		
13000	C.25	Static, free orbit, unbiased
13001	C.24	Static, free orbit, Doppler
<i>Delikaraoglou [1985]</i>		
10030	C.29	Static, known orbit, single difference (model G.1)
03030	C.36	Known, free orbit, single difference (model G.2)
13030	C.23	Static, free orbit, single difference (model G.3)
10230	C.28	Static, known orbit, single difference, quadratic receiver clock (model HG.1)
03230	C.34	Known, free orbit, single difference, quadratic receiver clock (model HG.2)
13230	C.22	Static, free orbit, single difference, quadratic receiver clock (model HG.3)
13231	C.21	Static, free orbit, single difference, quadratic receiver clock, ambiguity (model HG.4)

- The 4-6 and 4-10 solutions discussed in Grafarend and Müller [1985] are reproduced as special cases in Tables C.25 and C.24, respectively.
- Models G.1, G.2 and G.3 of Delikaraoglou [1985] are reproduced in our Tables C.29, C.36 and C.23, respectively.
- Models HG.1, HG.2, and HG.3 discussed by Delikaraoglou [1985] differ from our models in Tables C.34, C.35, and C.36, since the Delikaraoglou [1985] models include as unknowns only receiver clock coefficients relative to a reference receiver clock (= $3(R-1)$ unknowns), rather than the independent clock coefficients for each receiver included here (= $3 R$ unknowns). However, the only difference between

Delikaraoglou [1985] and our results is that our values for ST are higher by 3 for minimal R , and higher by 1 for greater than minimal R , in all three cases.

- Model HG.4 in Delikaraoglou [1985] includes only S rather than RS ambiguity bias (γ) unknowns and does not include the A_2 rank defect of S (see Table 4.2, item 6). Therefore, the two sets of results are not comparable.

5.4 Applications of Redundancy Design

Some interesting examples of the application of redundancy design are as follows.

- **Selecting the type of receiver.** One criterion for receiver selection is the number of satellites which can be simultaneously tracked. Various actual receivers are capable of simultaneously tracking one, four, five, six, and nine satellites. If the results of a redundancy design indicate a minimum value for S , then a receiver capable of simultaneously tracking at least S satellites must be selected, if that redundancy design is to be achieved.
- **Selecting satellite systems.** We have limited our study to a maximum of 10 satellites, since this is about the maximum that can be expected to be visible at any time from the GPS constellation. However, larger values of S would be available if, for example, a receiver were to be designed capable of tracking not only GPS satellites but also other satellites from similar systems (for example, the 12 satellites of the U.S.S.R. GLONASS system which are already in orbit). In this case the advantages of larger S values should be studied more carefully.
- **Selecting the number of receivers.** If the results of a redundancy design indicate a minimum required value for R , then only configuration and logistics designs involving at least R receivers can be considered, if that redundancy design is to be achieved.
- **Selecting receiver clocks.** Let us say that the comparison between redundancy designs which involve the most general model for α (index value of 3 in Table 3.3), and redundancy models which involve less general models for α (for example, index values of 2, 1, or 0 in Table 3.3) indicate that the design would be significantly improved (for example, in the sense of fewer R or S or T or m) if the less general model were to be used. The use of more accurate receiver clocks may permit the use of such a less general model. Some caution should be exercised, since α is not necessarily solely due to clock biases.
- **Selecting the processing method.** It may be, for example, that double differences provide significant redundancy design advantages over triple differences. In that case, a preprocessing stage should be considered which will provide integer estimates for γ which can then be held fixed in the final processing.

5.5 Conclusions for Each Optimality Criterion

Many interesting facts from which one can draw conclusions for each optimality criterion have been described in the previous subsection. On the one hand, it is rather simple to get optimal designs if only one criterion is considered. Minimal values of R , S , and T , all equal to one, can only be obtained in a small number of solutions. Solutions which include between-epoch differencing (that is, γ -biases are estimated in the biased-range approach) require a minimum of two epochs. If one is able to allow large enough numbers of receivers and satellites, many minimal redundancy designs can be obtained with only one or two epochs.

On the other hand, for practical reasons, the network designer should consider a combination of some, and perhaps even all, of the optimality criteria mentioned above. For example, if T has been designed as low as possible (one or two), we will end up with rather large values for R and S . This may lead to better redundancy designs for some criteria (say, m and $R+ST$), but will degrade the redundancy design according to other criteria (e.g., R and S). Therefore an appropriate balance between different optimality criteria will often be desired.

6. GPS CONFIGURATION DESIGN

We consider redundancy design to be a simple algebraic first step in the first-order design problem — placing necessary, but not sufficient, conditions on the regularity of the normal equation matrix. Configuration design is the more extensive second step. For GPS, as equation (4-1) implies, there are two types of configuration design, which we label “network configuration design” and “bias configuration design,” referring respectively to the structures of A_1 and A_2 in equation (4-1). Section 4.2 deals with bias configuration (A_2) design. Appendix B contains examples of the structure of A_2 .

6.1 Network Configuration Design

Network configuration design is the GPS first-order design step which is most similar to terrestrial network first-order design. The main design tool for network configuration design is likely to remain computation of the estimated solution covariance matrix under simulated conditions — essentially an iterative approach. However, there may be some aspects that can be tackled in a more satisfying theoretical way.

There are differences between GPS and terrestrial network configuration design. We may consider the entire set of ground and satellite points to be analogous to the set of terrestrial network points in a classical design problem. In terrestrial design, our freedom of selection of station locations is limited by practical considerations, most notably intervisibility. The selection of ground points in GPS configuration design has no such limitations (except for shadowing and reflection of the satellite signal, which is a much more local problem). However, we have much less control over “constellation design,” that is, the configuration of

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the satellite points. They are determined by the designers of the GPS system. We are only free to select the times and time span for which we make observations, and to select which satellites to track (if we do not take the more geodetically reasonable decision to track all available satellites). Therefore the problem of GPS configuration design is mainly concerned with two factors: selecting the observing session times, and the configuration of the ground points. Once full GPS coverage is available after 1989, observations made for equal periods, but at different times in the day, are unlikely to provide significantly different results (apart perhaps from refraction considerations), although the observing time span will still be an important design selection. As regards the ground station configuration, we are freer than in the terrestrial case to select regular geometric patterns of stations which may be more amenable to theoretical analysis than irregular patterns.

One question concerning satellite point selection is the important one of the relative advantages of space and time diversity. Is it better to track as many satellites as possible for a short period, or to track fewer satellite over longer arcs? In our terms, should we prefer to minimize S or T ? For example, for the kinematic, free orbit, triple difference solution (Table C.1), given 14 receivers R , the minimal S -design is 5 satellites for 52 epochs, while the minimal T -design is 8 satellites for 3 epochs. Which is better from the configuration point of view?

For ground networks established to determine the positions of GPS satellites, what distance apart must the stations be to achieve a given accuracy in the satellite coordinates?

6.2 Symmetry of Diophantine Equations

Referring to the contents of the last two paragraphs, let us consider solution types for which the Diophantine equation (5-3) is symmetrical, that is, unchanged under exchange of quantum number values. We observe that:

- Neither R and T nor S and T are symmetric in any of the cases considered in this report.
- R and S are symmetric, that is, all points of a GPS network configuration are equivalent, when they are symmetric for both the non-bias parameters (modelling X_g and X^s) and for the bias parameters (modelling α , β , γ).
- R and S are symmetric for the **non-bias** parameters in the following cases:
 - » kinematic, free orbit
 - » static, unknown orbit translations
 - » known ground stations, known orbits.
- R and S are symmetric for the **bias** parameters in the following cases:
 - » triple difference
 - » between receiver and satellite double difference
 - » between epoch single difference.

7. GPS LOGISTICS DESIGN

Given R receivers and G ground stations (whose locations have already been determined from the configuration design), the logistics design problem is to select P mappings of the receivers onto the stations which will result in a network determination which is optimal (or at least acceptable) according to some criterion.

In the present era of expensive receivers and short daily GPS coverage periods, the logistics design problem degenerates into the rather simple problem of selecting which baselines to observe. Formal logistics design has not generally been undertaken for GPS campaigns so far.

With the advent of inexpensive receivers and continuous coverage, the problem becomes much more complex. Proper logistics design, in addition to proper configuration design, is required to optimize various aspects of network determination, such as accuracy, observation time, and perhaps most important, cost.

In order to deal with the logistics design problem in a more complete way, we extend the number of observing sessions from the single session we have considered so far to multiple sessions. At first we assume that there is a definite break between each of the multiple sessions (as is the case today, with satellite coverage of only a few hours per day). Then we consider the case of continuous satellite coverage (expected after 1989), and suggest that any change whatever in the configuration of either satellites or ground stations (satellite rise or set, receiver move or loss of lock, or even cycle slip) is sufficient reason to declare a new **incremental session** to have begun.

Recently, investigations into logistics design have begun. Hothem et al. [1984] discussed some practical aspects of two-receiver campaigns. Network logistics with three or four

receivers have been treated by Snay [1986]. Design considerations for multi-receiver campaigns are given in Vaníček et al. [1985].

7.1 Extension From Single to Multiple Observing Sessions

So far we have assumed that the number of available and simultaneously tracking receivers equals the number of ground stations to be observed. Now let us use separate quantum numbers to represent receivers and ground stations: for receivers $r = 1, 2, \dots, R$ and for ground stations $g = 1, 2, \dots, G$, where $G > R$. Here, we further assume that during different sessions, different numbers of receivers will be used to observe different numbers of satellites over different time spans, and hence introduce subscripts on R , S and T above. For the p th session we have:

$$r = 1, 2, \dots, R_p$$

$$s = 1, 2, \dots, S_p$$

$$t = 1, 2, \dots, T_p .$$

Each value for s refers to a separate continuously-tracked arc of a specific satellite. If the signal is lost and regained from a satellite, each arc will have a distinct value for s , and S will be incremented by one.

Neglecting receiver failures, in many campaigns R_p will be equal for all sessions. The number of satellites S_p depends on the network location on the earth's surface, the observing time span, and the observing time of day. Depending on the receiver design, T_p may be the same for all sessions (for example, for the Macrometer), or may be completely different from session to session. The number of observations during the p th session now becomes

$$n_p = R_p S_p T_p \tag{7-1}$$

while the total number of observations in the whole campaign assembled from P sessions is

$$n = \sum_p n_p = \sum_p R_p S_p T_p . \quad (7-2)$$

In the most general case, the number of unknowns of various types over the whole campaign, under the modelling assumptions which we have made, are shown in Table 7.1.

TABLE 7.1 MULTIPLE SESSION PARAMETER MODELS					
ground station coordinates	X_g	\Rightarrow	$3 G$	Σ	T_p
satellite position biases	X^s	\Rightarrow	3	Σ	$S_p T_p$
receiver-specific biases	α	\Rightarrow		ΣR_p	T_p
satellite-specific biases	β	\Rightarrow		ΣS_p	T_p
pair-specific biases	γ	\Rightarrow		$\Sigma R_p S_p$	

We must reduce the number of unknowns by the number of parameters required to be held fixed in order to solve for the datum defect, which is $6T_p$ for session p , or a total of $\Sigma 6T_p$, and by the number of bias unknowns required to satisfy the A_2 rank defect, which is $R_p + S_p + T_p - 1$ for session p , or a total of $\Sigma(R_p + S_p + T_p - 1)$. We end up with the total number of unknowns to be:

$$m = \sum_p \{3GT_p + 3S_pT_p + R_pT_p + S_pT_p + R_pS_p - (R_p + S_p + T_p - 1) - 6T_p\} \quad (7-3)$$

In this discussion we have adopted the kinematic model for the ground stations, under which the coordinates for each ground station are assumed to have different and independent values for each time epoch t . This has several implications.

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- During a multiple-session campaign, any particular ground station will be occupied by a receiver for only some of the sessions. During the sessions when a station is not occupied, under the kinematic assumption we have no information whatever about its position or trajectory.
- The kinematic assumption, together with the free orbit assumption, introduces a new datum defect of 6 for each observation epoch t . This introduces the problem of what connects the ground station coordinates between different epochs.
- Even if the known orbit assumption is made, the kinematic assumption still introduces the problem of what connects the ground station coordinates between different sessions. See Figure 7.1 for an example.

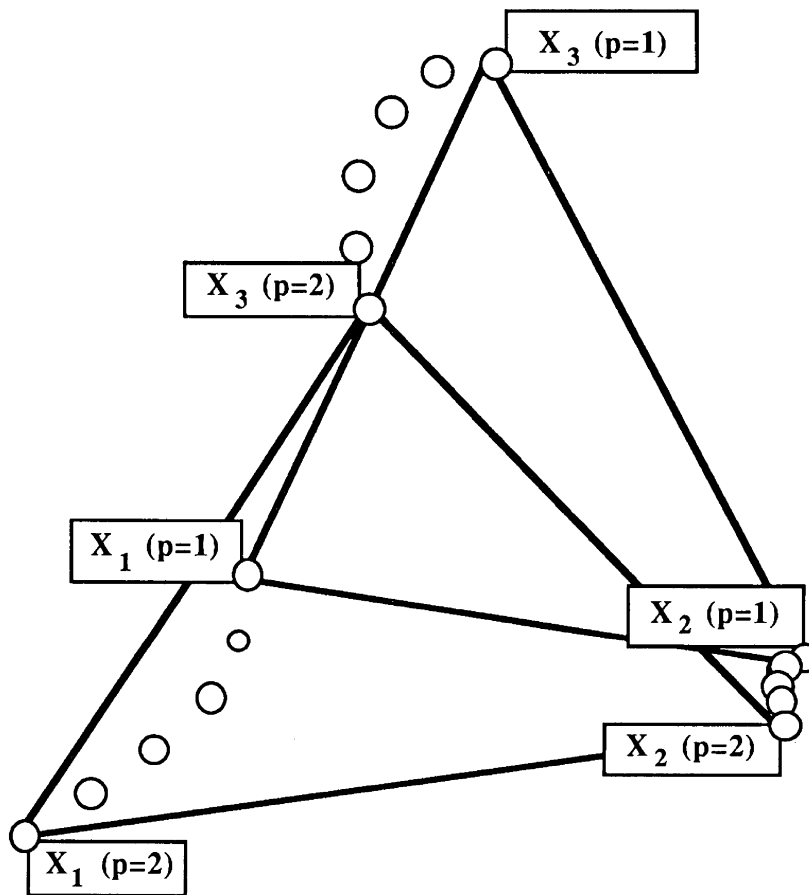


FIGURE 7.1
KINEMATIC NETWORK MOVEMENT

- If significant deformations are expected to occur during the interval between GPS observations (seconds to minutes), then it is likely that GPS is not an appropriate measurement tool for these deformations.
- Due to these implications, we will consider only the **static case** for the remainder of this section. In that case, equation (7-2) becomes

$$m = 3G + \sum_p \{3S_p T_p + R_p T_p + S_p T_p + R_p S_p - (R_p + S_p - 1)\} - 6, \quad (7-4)$$

and the Diophantine equation is

$$\sum_p R_p S_p T_p \geq 3G + \sum_p \{3S_p T_p + R_p T_p + S_p T_p + R_p S_p - (R_p + S_p + T_p - 1)\} - 6. \quad (7-5)$$

7.2 Extension From Multiple to Incremental Observing Sessions

In the above extension to multiple observing sessions, the assumption was still made that during each session each available receiver tracked all available satellites for all epochs (equation (7-1)). This may be a reasonable approximation of the actual situation during a campaign held while the GPS coverage is still limited to a few hours per day. However, once continuous GPS coverage is available, there will no longer be definite breaks between sessions. Then we need to modify the meaning of the session index p . Now p will be incremented whenever the configuration of receivers and satellites changes: that is whenever

- a satellite rises or sets, or
- a receiver loses lock on any satellite, or
- one or more receivers are shut down to be moved. Regarding this last point, the ideas presented by Remondi [1985b] and Hofmann-Wellenhof and Remondi [1985] indicate that maintaining receiver tracking while moving from one station to the next would provide significant advantages.

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We now say that p represents the **incremental session** index. The assumption that all receivers track all satellites for all epochs during a session will now continue to hold. However, the number of incremental sessions may well be many more than under the definition of session accepted so far.

New bias terms may be introduced when p is incremented, depending on the model we use for the bias terms. In the most general model, since new values for α and β are introduced for each time epoch anyway, a change in p has no special effect. In this case, however, new γ values must be introduced, but only for those satellite/receiver pairs which have caused the change in p . Similarly, if we use less general models for α and β , some but not all of the biases will need to have new coefficient terms introduced whenever p changes.

Since, in general, not all but only certain of the bias terms will be affected by incrementing p , we must introduce some factor to indicate which and how many satellites or receivers or receiver/satellite pairs are **unaffected** by the change. The intersection between the new and previous sets performs this task: that is,

$R_p \cap R_{p-1}$ represents the number of unchanged receivers,

$S_p \cap S_{p-1}$ represents the number of unchanged satellites, and

$(R_p \cap R_{p-1})(S_p \cap S_{p-1})$ represents the number of unchanged receiver/satellite pairs.

The total number of observations in the campaign is still given by equation (7-2). However, the total number of unknowns for the most general case is now given by

$$m = 3G + \sum_{p=1}^P \{3S_p T_p + R_p T_p + S_p T_p - (R_p + S_p + T_p - 1)\} - \sum_{p=2}^P (R_p \cap R_{p-1})(S_p \cap S_{p-1}) - 6. \quad (7-6)$$

The Diophantine equation is

$$\sum_{p=1}^p R_p S_p T_p \geq 3G + \sum_{p=1}^p \{3S_p T_p + R_p T_p + S_p T_p - (R_p + S_p + T_p - 1)\} - \sum_{p=2}^p (R_p \cap R_{p-1})(S_p \cap S_{p-1}) - 6. \quad (7-7)$$

There is much yet to be done in investigating GPS logistics design. One must first adopt some particular logistics design strategy, and then study the implications for redundancy design (using equation (7-7)), and for configuration design (both bias and network configuration designs), using the incremental observing session concept we have introduced. Here we conclude by describing some possible logistics design strategies which have been suggested.

7.3 Examples of Logistics Design Strategies

So far, we have dealt mainly with theoretical/analytical considerations of the logistics design problem. Now we give a short review of some concrete strategies for solving logistics design problems.

Some algorithms applicable for GPS surveys are given by Snay [1986], considering three influences.

- The number of receivers is restricted to three or four.
- In order to be able to **detect** setup errors of the receiver antenna with respect to the terrestrial bench mark, **two** independent observed baselines to each ground station are required. In order to be able to **identify** setup errors, **three** independent baselines to each station are required [Hothem et al. 1984]. Therefore, Snay requires that every ground station should be occupied three times (to identify set up errors), or, for a more economical network design, only twice (to merely detect setup errors).
- Finally, Snay distinguishes between one-dimensional loop geometry and a two-dimensional area geometry.

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A couple of examples show the practical procedure for logistics design solutions. However, we have to keep in mind that with a sophisticated observed model, that is, with many unknown bias terms, we very often need five or even more receivers, as can be seen from our redundancy design analyses.

Other important aspects in practical logistics design are considered by Vaníček et al. [1985]. Assuming typical time intervals for observing sessions, as well as equipment setup and teardown times, the total number of receivers required has been computed depending upon the station spacing and network diameter. An idealized structure based on equilateral triangles has been used as the basic figure for the network of ground stations. With increased station spacing, the results differ significantly if the transportation of the receivers is done by land vehicles or helicopters. In small scale applications, however, most of the overhead is due to the setup and teardown delays, rather than travelling time.

As can be seen from these examples, a lot more work remains to be done in the field of GPS logistics design. Logistics design will be a critical component of GPS first-order design when receivers are much less expensive, and 24-hour coverage from the full GPS satellite constellation is provided, both of which are expected to occur in 1990.

8. GPS SECOND-ORDER DESIGN

In this section, we provide some initial hints as to how one specific GPS second-order design problem might be approached. This problem can be stated thus: How may prior information be used to model the “stability” of satellite and ground points? We first discuss what we mean by the term stability. We then discuss some aspects of using prior information that may lead to a solution to our problem.

8.1 Modelling Stability

By stability we mean the time span over which a model is valid. We introduced two basic assumptions (stability models) about the receiver coordinates X_r , which we called the kinematic and static assumptions. Similarly, we considered the free orbit and known orbit assumptions (stability models) regarding the satellite coordinates X^s . In this section, we consider a more generalized approach, of which these are the limiting cases. In doing so, we follow the suggestion of Brouwer [1985] for modelling the stability of VLBI source coordinates (equivalent to our X^s).

To apply the Brouwer approach to GPS, we must distinguish between several sets of time epochs:

- observation epochs t (which may be between seconds and minutes apart)
- epochs t^s at which it is appropriate to estimate new corrections to a priori satellite positions (which may be between seconds and hours apart)
- epochs t_g at which it is appropriate to estimate new corrections to a priori ground station coordinates (which may be between seconds and decades apart).

Let us assume that each of these sets of epochs is equispaced in time, that is, the intervals between epochs are constants δt , δt^s , and δt_g , respectively. We introduce two stability parameters, μ and ν , such that $\mu = \delta t / \delta t^s$, and $\nu = \delta t / \delta t_g$. Then, as $\mu \rightarrow 1$ ($\delta t^s \rightarrow \delta t$, or decreasing stability), the satellite model approaches the free orbit model, and as $\mu \rightarrow 0$ ($\delta t^s \rightarrow \infty$, or increasing stability), the model approaches the known orbit model. Similarly, as $\nu \rightarrow 1$ ($\delta t_g \rightarrow \delta t$, or decreasing stability), the ground station model approaches the kinematic model, and as $\nu \rightarrow 1/T$ ($\delta t_g \rightarrow T \delta t =$ the complete observation period, or increasing stability), the model approaches the static model.

The goal is to have μ and ν stability parameters for groups of network points, or even individual ground points, in the case of deformation studies. The hope is that the stability parameters can be related to some method of taking prior information into account.

8.2 Prior Information Considerations

A very detailed description of the theory and application of stochastic prior information is given in Schaffrin [1985]. We follow some of these ideas here. In general, prior information is taken into account on two levels, that is, by the vector of a priori values and its corresponding covariance matrix. The application of some of these general models for GPS has been presented in Grafarend et al. [1985a].

Let us now consider normal equations $N \xi = b$, which take into account only the covariance matrix of (stochastic) prior information. Then we have

$$\begin{aligned}
 N &= I + C_\xi A^T C_\varphi^{-1} A \\
 b &= C_\xi A^T C_\varphi^{-1} \varphi.
 \end{aligned}
 \tag{8-1}$$

Assuming that the a priori covariance matrix C_ξ is regular, the normal equations can be transformed into

$$N = C_\xi^{-1} + A^T C_\varphi^{-1} A$$

$$b = A^T C_\varphi^{-1} \varphi .$$
(8-2)

From this equation we can see that some a priori weights on the parameters have been added to the normal equation matrix of equation (4-5). Often this is done to regularize the normal equations and make Cayley inversion possible. Here our purpose is different — we wish to use the prior information to model the stability of the solution parameters.

Looking at the problem from the filtering point of view, the solution to the normal equations is

$$\bar{\xi} = N^{-1} b = K \varphi ,$$
(8-3)

where

$$K = \{C_\xi^{-1} + A^T C_\varphi^{-1} A\}^{-1} A^T C_\varphi^{-1} .$$
(8-4)

Using matrix identities given by several authors (e.g., Wells [1974]), we can also write

$$K = C_\xi A^T \{A C_\xi A^T + C_\varphi\}^{-1} .$$
(8-5)

Processing the observations epoch by epoch, using the summation of normal equations technique, we have

$$\bar{\xi}(t) = K(t) \varphi(t)$$
(8-6)

where

$$C_\xi^{-1}(t) = \sum_{k=1}^{t-1} A^T(k) C_\varphi^{-1}(k) A(k) .$$
(8-7)

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Starting with the simplest example, where $C_{\varphi}^{-1} = I \sigma_{\varphi}^{-2}$ and $C_{\xi}^{-1} = I \sigma_{\xi}^{-2}$, we have

$$\begin{aligned} N &= I \sigma_{\xi}^{-2} + \sigma_{\varphi}^{-2} A^T A \\ b &= \sigma_{\varphi}^{-2} A^T \varphi . \end{aligned} \tag{8-8}$$

Introducing $\lambda = (\sigma_{\varphi}/\sigma_{\xi})^2$, which describes the ratio of the a priori variance factors for the observations and the prior information, we end up with the following solution for the parameters:

$$\bar{\xi} = (A^T A + \lambda I)^{-1} A^T \varphi . \tag{8-9}$$

To generalize, rather than the scalar matrix $I \lambda$, we introduce a diagonal matrix so that each point or group of points has associated with it its own value for λ . Still looking at the adjustment from the filtering point of view, we realize that every filter has a time constant associated with it. Two of the challenges remaining to solve our problem are, first, to obtain expressions for such time constants which involve these λ (or other similar) parameters, and, second, to relate these time constants to the stability parameters we introduced earlier.

9. SUMMARY AND CONCLUSIONS

The following statements summarize the important conclusions which can be drawn from the discussions contained in this paper.

- Undifferenced and differenced GPS carrier beat phase observations result in the same normal equations for the estimates of the desired parameters, as long as the models chosen for the undifferenced observation biases satisfy the assumptions of the fundamental differencing theorem.
- Four conditions related to the validity of the fundamental differencing theorem are given and explicitly demonstrated. Only one of them is essential.
- Normal equations for the desired parameters based on undifferenced observations are independent of the choice of model to satisfy the bias rank defect.
- Normal equations based on differenced observations are independent of the choice of differencing basis and of the order of differencing.
- There are many advantages to using the undifferenced rather than the differencing approach.
- As in all geodetic measurements involving electromagnetic propagation, atmospheric refraction is the most difficult effect to model and places the ultimate limitation on the capabilities of GPS.
- GPS first-order design can, in some sense, be defined in terms of three steps: redundancy design, configuration design, and logistics design.
- A linear combination of GPS measurements has the same datum defect as geometrical ranges themselves, that is, origin and orientation (six components). Scale may or may not be provided, depending on the validity of atmospheric refraction modelling.
- Extensive redundancy design results confirm and extend previous work.
- Varying the models for the desired and the nuisance parameters, between the most general case and the simplest one, any possible redundancy design can be evaluated using a convenient computer program.

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- The kinematic and free orbit assumptions are important considerations in design, although all appropriate constraints should be used in actual processing of data.
- Some examples of applications for different optimality criteria for redundancy design are presented.
- Configuration design can be subdivided into bias configuration design, and network configuration design.
- The fundamental differencing theorem is the main tool in bias configuration design.
- Network configuration design consists of selecting the observing session times (constellation design) and the conventional terrestrial configuration of the ground station points.
- GPS logistics design selects a set of mappings of available receivers onto the ground stations selected in the configuration design. This involves extending the concept of a session to include incrementally changing observing conditions.
- One challenge in GPS second-order design is the use of prior information theory to model the stability of satellite and ground points.
- The triple difference solution (whether using the undifferenced approach, the differenced-observation approach, or a combination of the two) accounts for the biases as generally as possible.
- While solutions, other than the triple difference one, involve less complete bias models, for certain applications this may not be a serious problem.

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APPENDIX A

DEMONSTRATION OF THE DIFFERENCING THEOREM

Within this appendix, we intend to demonstrate explicitly the conditions which are sufficient to guarantee the validity of the fundamental differencing theorem.

The demonstration will consist of showing that the following four conditions hold for a specific case. These conditions, already given in Chapter 4, involve only the differencing operator D , and the bias design matrix A_2 :

$$\bullet \quad (D D^T)^{-1} \text{ must exist} \quad (\text{A-1})$$

$$\bullet \quad (A_2^T A_2)^{-1} \text{ must exist} \quad (\text{A-2})$$

$$\bullet \quad D A_2 = 0 \quad (\text{A-3})$$

$$\bullet \quad D^T (D D^T)^{-1} D + A_2 (A_2^T A_2)^{-1} A_2^T = I \quad (\text{A-4})$$

The specific example which we have chosen to use in this demonstration is the sixth design listed in Table C.27: the static, known orbit, double difference solution, using $R = 3$ receivers, $S = 6$ satellites, and $T = 1$ observation epoch.

A.1 Condition (A-1)

The references against which the differences are taken by the differencing operator can be formulated on a fixed basis (case 1), or a sequential basis (case 2), or an orthonormal basis (case 3). In our example, we will use a fixed receiver basis, and a sequential satellite basis. In this case the covariance matrix of the differenced observations ($D D^T$) is

$$\begin{vmatrix} 4 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 4 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 4 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 4 & 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 2 & 4 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 4 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 4 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 4 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 4 \end{vmatrix}$$

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The determinant of $(D D^T)$ is 8748, proving its regularity and satisfying condition (A-1).

For the orthonormalized basis, the differencing operator has been constructed in such a way that the row vectors of D are orthogonal to each other and have a unit norm. This implies that $(D D^T)$ is a unit matrix, with dimension equal to the number of differenced observations. Even if D is composed of three single orthonormal differencing operators, the condition (A-1) holds.

A.2 Condition (A-2)

From Table 4.2, we see that the rank defect of the bias design matrix involving nuisance parameters α and β is equal to T . For our double difference example, A_2 has a rank defect of $T=1$. This can be removed by using either one receiver clock or one satellite clock as a completely known reference (i.e., the master clock). Using α_3 as the master clock, the matrix $A_2^T A_2$ is given by

$$\begin{array}{c}
 \left(\begin{array}{cc|cccccc}
 6 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 0 & 6 & 1 & 1 & 1 & 1 & 1 & 1 \\
 \hline
 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 3 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 3 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 3
 \end{array} \right) \\
 \alpha_1 \quad \alpha_2 \quad \beta^1 \quad \beta^2 \quad \beta^3 \quad \beta^4 \quad \beta^5 \quad \beta^6
 \end{array}$$

The determinant of $A_2^T A_2$ is also equal to 8748, which allows normal Cayley inversion.

A.3 Condition (A-3)

The condition $D A_2 = 0$ can be demonstrated rather simply for the fixed and sequential differencing bases, since then D contains only +1, 0, and -1 values. Here we choose to demonstrate it for the more interesting orthonormalized basis. First we will develop general expressions for the triple differencing case, and then apply them to our simpler double differencing example.

Applying equation (4-10) three times, first we have between-receiver differences

$$\Delta\phi_r^s(t) = \left[-\left(\frac{1}{r}\right) \sum_{i=1}^r \phi_i^s(t) + \phi_{r+1}^s(t) \right] \left(\frac{r}{r+1}\right)^{1/2} . \quad (\text{A-5})$$

Applying the same equation a second time to obtain between-receiver and satellite double differences, we have

$$\nabla\Delta\phi_r^s(t) = \left[-\left(\frac{1}{s}\right) \sum_{j=1}^s \Delta\phi_{r+1}^j(t) + \Delta\phi_{r+1}^{s+1}(t) \right] \left(\frac{s}{s+1}\right)^{1/2} . \quad (\text{A-6})$$

Substituting equation (A-5) twice into (A-6), we have

$$\Delta\phi_r^s(t) = \left[\left(\frac{1}{rs}\right) \sum_{j=1}^s \sum_{i=1}^r \phi_i^j(t) - \left(\frac{1}{s}\right) \sum_{j=1}^s \phi_{r+1}^j(t) - \left(\frac{1}{r}\right) \sum_{i=1}^r \phi_i^{s+1}(t) + \phi_{r+1}^{s+1}(t) \right] \left(\frac{rs}{(r+1)(s+1)}\right)^{1/2} \quad (\text{A-7})$$

Finally we apply the Gram-Schmidt orthonormalization a third time to obtain triple differences

$$\delta\nabla\Delta\phi_r^s(t) = \left[-\left(\frac{1}{t}\right) \sum_{k=1}^t \nabla\Delta\phi_{r+1}^{s+1}(k) + \nabla\Delta\phi_{r+1}^{s+1}(t+1) \right] \left(\frac{t}{t+1}\right)^{1/2} . \quad (\text{A-8})$$

Inserting equation (A-7) into (A-8) twice, we have

$$\delta\nabla\Delta\phi_r^s(t) = \{\Sigma 1 + \Sigma 2 + \Sigma 3 + \Sigma 4 + \Sigma 5 + \Sigma 6 + \Sigma 7 + 8\} \left\{ \frac{rst}{(r+1)(s+1)(t+1)} \right\}^{1/2} \quad (\text{A-9})$$

where

$$\Sigma 1 = -\left(\frac{1}{rst}\right) \sum_{i=1}^r \sum_{j=1}^s \sum_{k=1}^t \phi_i^j(k)$$

$$\Sigma 2 = \left(\frac{1}{st}\right) \sum_{j=1}^s \sum_{k=1}^t \phi_{r+1}^j(k)$$

$$\Sigma 3 = \left(\frac{1}{rt}\right) \sum_{i=1}^r \sum_{k=1}^t \phi_i^{s+1}(k)$$

$$\Sigma 4 = \left(\frac{1}{rs}\right) \sum_{i=1}^r \sum_{j=1}^s \phi_i^j(t+1)$$

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$$\Sigma 5 = - \left(\frac{1}{t}\right) \sum_{k=1}^t \phi_{r+1}^{s+1}(k)$$

$$\Sigma 6 = - \left(\frac{1}{s}\right) \sum_{j=1}^s \phi_{r+1}^j(t+1)$$

$$\Sigma 7 = - \left(\frac{1}{r}\right) \sum_{i=1}^r \phi_i^{s+1}(t+1)$$

$$8 = \phi_{r+1}^{s+1}(t+1)$$

where in all cases $r=1,2,\dots,R-1$; $s=1,2,\dots,S-1$; $t=1,2,\dots,T-1$.

Equation (A-9) is merely a description of the differencing operator D using an orthonormalized basis, where in the sequence of the observation vector, R is the fastest index and T is the slowest one.

From our computations we draw the following conclusions:

- The condition $D A_2 = 0$ holds.
- A_2 is independent of the differencing basis, of course.
- As is shown in Table 4.1, the sequence of the differencing is arbitrary.
- The differencing operator given by equations (A-9) is the most general one, which includes all other differences on a lower level. If one selects terms which are related to only one or two of the quantum numbers r, s, t , every differencing operator using the orthonormalized basis can be obtained.
- If the phase observations are ordered in a receiver-wise manner, that is, t is the fastest index and r the slowest one, then the structure of the D matrix changes, but not its individual elements.
- $D A_2 = 0$ can be more easily verified for the fixed and the sequential bases because only +1, 0, -1 values occur in both the D and A_2 matrices.

The validity of the condition $D A_2 = 0$ is now explicitly demonstrated for the example given earlier in this appendix. For the four possible cases of the double difference observations, it can be shown that:

- (a) $\Sigma 4 + \Sigma 3 = 0$
- (b) $\Sigma 6 + 8 = 0$
- (c) $\Sigma 4 + \Sigma 6 = 0$
- (d) $\Sigma 7 + 8 = 0$.

A.4 Condition (A-4)

This last condition is only explicitly demonstrated for the example used throughout this appendix. For the first part, $D^T(D D^T)^{-1}D$, we have Fig. A.1, and for the second part, $A_2(A_2^T A_2)^{-1}A_2^T$, we have Fig. A.2. The sum of these two matrices is the identity matrix, with dimension equal to the number of undifferenced phase observations.

10	-5	-5	-2	1	1	-2	1	1	-2	1	1	-2	1	1	-2	1	1
-5	10	-5	1	-2	1	1	-2	1	1	-2	1	1	-2	1	1	-2	1
-5	-5	10	1	1	-2	1	1	-2	1	1	-2	1	1	-2	1	1	-2
-2	1	1	10	-5	-5	-2	1	1	-2	1	1	-2	1	1	-2	1	1
1	-2	1	-5	10	-5	1	-2	1	1	-2	1	1	-2	1	1	-2	1
1	1	-2	-5	-5	10	1	1	-2	1	1	-2	1	1	-2	1	1	-2
-2	1	1	-2	1	1	10	-5	-5	-2	1	1	-2	1	1	-2	1	1
1	-2	1	1	-2	1	-5	10	-5	1	-2	1	1	-2	1	1	-2	1
1	1	-2	1	1	-2	-5	-5	10	1	1	-2	1	1	-2	1	1	-2
-2	1	1	-2	1	1	-2	1	1	10	-5	-5	-2	1	1	-2	1	1
1	-2	1	1	-2	1	1	-2	1	-5	10	-5	1	-2	1	1	-2	1
1	1	-2	1	1	-2	1	1	-2	-5	-5	10	1	1	-2	1	1	-2
-2	1	1	-2	1	1	-2	1	1	-2	1	1	10	-5	-5	-2	1	1
1	-2	1	1	-2	1	1	-2	1	1	-2	1	-5	10	-5	1	-2	1
1	1	-2	1	1	-2	1	1	-2	1	1	-2	-5	-5	10	1	1	-2
-2	1	1	-2	1	1	-2	1	1	-2	1	1	-2	1	1	10	-5	-5
1	-2	1	1	-2	1	1	-2	1	1	-2	1	1	-2	1	-5	10	-5
1	1	-2	1	1	-2	1	1	-2	1	1	-2	1	1	-2	-5	-5	10

FIGURE A.1

EXAMPLE FOR $D^T(DP_\phi^{-1}D^T)^{-1}D$

(18 × 18 example. All elements multiplied by 18 to give integers.)

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8	5	5	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1
5	8	5	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1
5	5	8	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2
2	-1	-1	8	5	5	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1
-1	2	-1	5	8	5	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1
-1	-1	2	5	5	8	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2
2	-1	-1	2	-1	-1	8	5	5	2	-1	-1	2	-1	-1	2	-1	-1
-1	2	-1	-1	2	-1	5	8	5	-1	2	-1	-1	2	-1	-1	2	-1
-1	-1	2	-1	-1	2	5	5	8	-1	-1	2	-1	-1	2	-1	-1	2
2	-1	-1	2	-1	-1	2	-1	-1	8	5	5	2	-1	-1	2	-1	-1
-1	2	-1	-1	2	-1	-1	2	-1	5	8	5	-1	2	-1	-1	2	-1
-1	-1	2	-1	-1	2	-1	-1	2	5	5	8	-1	-1	2	-1	-1	2
2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	8	5	5	2	-1	-1
-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	5	8	5	-1	2	-1
-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	5	5	8	-1	-1	2
2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	8	5	5
-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	5	8	5
-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	-1	-1	2	5	5	8

FIGURE A.2

EXAMPLE FOR $P_{\phi}A_2(A_2^T P_{\phi}A_2)^{-1}A_2^T P_{\phi}$

(18 × 18 example. All elements multiplied by 18 to give integers.)

APPENDIX B

STRUCTURE OF BIAS DESIGN MATRIX FOR UNDIFFERENCED OBSERVATIONS

This part of the first-order design, or configuration, matrix, which belongs to the nuisance parameters, is called bias design matrix A_2 . Its complete non-regularized structure is given in Fig. B.1.

The corresponding non-regularized normal equation matrix $N_{22} = A_2^T A_2$, has the structure given in Fig. B.2 for the most general case.

Here we emphasize again that regularity of A_2 or N_{22} , respectively, is a necessary condition for the validity of the fundamental differencing theorem (cf. Appendix A). Therefore, the rank defect of N_{22} has to be overcome by eliminating the number of columns in A_2 and the number of columns and corresponding rows in N_{22} which is equal to the rank defect given in Table 2.

The actual structure of A_2 for the example of Appendix A, after fixing of α_3 as receiver reference clock, is given by Figure B.3:

1	0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0
1	0	0	0	1	0	0	0	0
0	1	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0
1	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1
			β^1	β^2	β^3	β^4	β^5	β^6
	α_1							
	α_2							

FIGURE B.3
EXAMPLE OF A_2 MATRIX STRUCTURE.

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		1			2			T			1			2			T			1			2			..			S								
		1 2 .. R			1 2 .. R			1 2 .. R			1 2 .. S			1 2 .. S			1 2 .. S			1 2 .. R			1 2 .. R			1 2 .. R											
T	S	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..	1	2	..
1	1	1									1									1																	
	2		1								1									1																	
	R			1							1									1																	
2	1	1									1									1																	
	2		1								1									1																	
	R			1							1									1																	
S	1	1									1									1																	
	2		1								1									1																	
	R			1							1									1																	
1	1																			1																	
	2																			1																	
	R																			1																	
2	1																			1																	
	2																			1																	
	R																			1																	
S	1																			1																	
	2																			1																	
	R																			1																	
T	1																			1																	
	2																			1																	
	R																			1																	
S	1																			1																	
	2																			1																	
	R																			1																	

FIGURE B.1
BIAS DESIGN MATRIX A_2

		1			2			T			1			2			T			1			2			...			S		
		1 2 .. R			1 2 .. R			1 2 .. R			1 2 .. S			1 2 .. S			1 2 .. S			1 2 .. R			1 2 .. R			1 2 .. R			1 2 .. R		
1	I	S										1	1	1	1						1			1			1				
	2		S									1	1	1	1						1			1			1				
	R			S								1	1	1	1						1			1			1				
2	I				S										1	1	1	1			1			1			1				
	2					S									1	1	1	1			1			1			1				
	R						S								1	1	1	1			1			1			1				
T	I						S								1	1	1	1			1			1			1				
	2							S							1	1	1	1			1			1			1				
	R								S						1	1	1	1			1			1			1				
1	I									R											1	1	1	1							
	2										R										1	1	1	1							
	S											R									1	1	1	1							
2	I											R									1	1	1	1							
	2											R									1	1	1	1							
	S												R								1	1	1	1							
T	I												R								1	1	1	1							
	2													R							1	1	1	1							
	S														R						1	1	1	1							
1	I																														
	2																														
	R																														
2	I																														
	2																														
	R																														
S	I																														
	2																														
	R																														

FIGURE B.2
NORMAL EQUATION MATRIX N_{22}

APPENDIX C

REDUNDANCY DESIGN RESULTS

In this appendix we list the results of 38 redundancy design studies, out of a possible 640 which could be made, given the solution types listed in Tables 3.1, 3.2, and 3.3. Each study was based on a different solution type.

The characteristics of each solution model are listed at the head of each table in this appendix, and are summarized in Table 5.1. Taking Table C.1 as an example, let us examine the information in the header. In this case our solution code is 43331. These digits are interpreted as follows:

- | | | |
|---|--|-------------------------------|
| 4 | X_g -solution type (from Table 3.1, | 4 = kinematic; 3RT unknowns) |
| 3 | X^S -solution type (from Table 3.2, | 3 = free orbit; 3ST unknowns) |
| 3 | α -solution type (from Table 3.3, | 3 = free; RT unknowns) |
| 3 | β -solution type (from Table 3.3, | 3 = free; ST unknowns) |
| 1 | γ -solution type (from Table 3.3, | 1 = unknown; RS unknowns). |

For these studies we always assume that R receivers track S satellites for T epochs, so that the total number of observations, n , is always RST . The total number of unknowns, m , is

- the sum of the unknowns resulting from the solution type tables as above (in this case $3RT + 3ST + RT + ST + RS$),
- reduced by the A_2 rank defect from Table 4.2 (in this case $R+S+T-1$),
- also reduced by the datum defect, if neither X_g nor X^S is assumed known (in this case $6T$).

Combining these terms we have $m = 1 - R - S - 7T + 4RT + 4ST + RS$ for this example, or a coefficient vector of (1 -1 -1 -7 4 4 1) on the basis (1, R , S , T , RT , ST , RS).

Within the body of the table are listed all those R,S,T combinations (i.e., redundancy designs) which are minimal in the following sense: given values for any two of R , S , and T , the value chosen for the third is the least value for which $n \geq m$. In addition to R , S , and T , are listed three additional parameters which might be used as optimality criteria: m , the number of unknowns; ST , the number of satellite points; and $R+ST$, the total number of network points. The actual redundancy, $n-m$, is also listed. We have limited our consideration to redundancy designs which fall within the region $RM \{1,20\}$, $SM \{1,10\}$, $TM \{1,100\}$. At the bottom of each table, the minimum values in each of the columns is listed. These minimum values are summarized in Table 5.2.

TABLE C.1

Kinematic, free orbit, triple difference

 $(4 \ 3 \ 3 \ 3 \ 1)$ $n = RST,$ $m = 3RT + 3ST + RT + ST + RS - (R + S + T - 1) - 6T$ $(1 \ -1 \ -1 \ -7 \ 4 \ 4 \ 1)$

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
6	9	40	2160	360	366	0
6	10	15	900	150	156	0
7	8	14	784	112	119	0
7	9	8	504	72	79	0
7	10	6	420	60	67	0
8	7	14	784	98	106	0
8	8	7	448	56	64	0
8	9	6	422	54	62	10
8	10	5	388	50	58	12
9	6	40	2160	240	249	0
9	7	8	504	56	65	0
9	8	6	422	48	57	10
9	9	4	324	36	45	0
10	6	15	900	90	100	0
10	7	6	420	42	52	0
10	8	5	388	40	50	12
10	10	3	300	30	40	0
11	6	10	660	60	71	0
11	7	5	385	35	46	0
11	8	4	346	32	43	6
12	6	8	575	48	60	1
12	9	3	319	27	39	5
13	6	7	543	42	55	3
13	7	4	364	28	41	0
14	5	52	3640	260	274	0
14	6	6	503	36	50	1
14	8	3	334	24	38	2
15	5	28	2100	140	155	0
16	5	20	1600	100	116	0
16	6	5	480	30	46	0
17	5	16	1360	80	97	0
18	5	14	1258	70	88	2
19	5	12	1140	60	79	0
19	7	3	399	21	40	0
19	10	2	380	20	39	0
20	5	11	1099	55	75	1
minimum values	6	5	300	20	38	

TABLE C.2

Kinematic, free orbit, double difference
 $(4\ 3\ 3\ 3\ 0)$
 $n = RST,$
 $m = 3RT + 3ST + RT + ST - T - 6T$
 $(0\ 0\ 0\ -7\ 4\ 4\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	6	9	1	53	9	15	1
	7	7	1	49	7	14	0
	9	6	1	53	6	15	1
	13	5	1	65	5	18	0
minimum values	6	5	1	49	5	14	

TABLE C.3

Kinematic, free orbit, single difference
 $(4\ 3\ 0\ 3\ 0)$
 $n = RST,$
 $m = 3RT + 3ST + ST - 6T$
 $(0\ 0\ 0\ -6\ 3\ 4\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	5	9	1	45	9	14	0
	6	6	1	36	6	12	0
	7	5	1	35	5	12	0
	10	4	1	40	4	14	0
minimum values	5	4	1	35	4	12	

TABLE C.4
Kinematic, free orbit, Doppler
 $(4\ 3\ 0\ 0\ 1)$
 $n = RST, m = 3RT + 3ST + RS - 6T$
 $(0\ 0\ 0\ -6\ 3\ 3\ 1)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	7	28	784	196	200	0
	4	8	16	512	128	132	0
	4	9	12	432	108	112	0
	4	10	10	400	100	104	0
	5	5	25	625	125	130	0
	5	6	10	300	60	65	0
	5	7	7	245	49	54	0
	5	8	6	238	48	53	2
	5	9	5	225	45	50	0
	6	5	10	300	50	56	0
	6	6	6	216	36	42	0
	6	7	5	207	35	41	3
	6	8	4	192	32	38	0
	7	4	28	784	112	119	0
	7	5	7	245	35	42	0
	7	6	5	207	30	37	3
	7	7	4	193	28	35	3
	7	9	3	189	27	34	0
	8	4	16	512	64	72	0
	8	5	6	238	30	38	2
	8	6	4	192	24	32	0
	8	8	3	190	24	32	2
	9	4	12	432	48	57	0
	9	5	5	225	25	34	0
	9	7	3	189	21	30	0
	10	4	10	400	40	50	0
	11	4	9	395	36	47	1
	12	4	8	384	32	44	0
	12	5	4	240	20	32	0
	12	6	3	216	18	30	0
	12	10	2	240	20	32	0
	14	4	7	392	28	42	0
	14	9	2	252	18	32	0
	18	4	6	432	24	42	0
	18	8	2	288	16	34	0
minimum values	4	4	2	189	16	30	

TABLE C.5
Kinematic, free orbit, unbiased
 $(4\ 3\ 0\ 0\ 0)$
 $n = RST, m = 3RT + 3ST - 6T$
 $(0\ 0\ 0\ -6\ 3\ 3\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	6	1	24	6	10	0
	5	5	1	24	5	10	1
	6	4	1	24	4	10	0
minimum values	4	4	1	24	4	10	

TABLE C.6

Kinematic, known orbit, triple difference
 $(4 \ 0 \ 3 \ 3 \ 1)$
 $n = RST, m = 3RT + RT + ST + RS - (R+S+T-1)$
 $(1 \ -1 \ -1 \ -1 \ 4 \ 1 \ 1)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	8	7	112	56	58	0
	2	9	4	72	36	38	0
	2	10	3	60	30	32	0
	3	6	10	180	60	63	0
	3	7	4	84	28	31	0
	3	8	3	71	24	27	1
	3	10	2	60	20	23	0
	4	6	5	120	30	34	0
	4	7	3	84	21	25	0
	4	9	2	72	18	22	0
	5	5	16	400	80	85	0
	5	6	4	120	24	29	0
	6	5	10	300	50	56	0
	7	5	8	280	40	47	0
	7	8	2	112	16	23	0
	8	5	7	280	35	43	0
	10	5	6	300	30	40	0
	10	6	3	180	18	28	0
	16	5	5	400	25	41	0
minimum values	2	5	2	60	16	22	

TABLE C.7

Kinematic, known orbit, 4D navigation
 $(4 \ 0 \ 3 \ 0 \ 0)$
 $n = RST,$
 $m = 3RT + RT$
 $(0 \ 0 \ 0 \ 0 \ 4 \ 0 \ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
minimum values	1	4	1	4	4	5	0
	1	4	1	4	4	5	

TABLE C.8
Linear deform, free orbit, triple difference
(3 3 3 3 1)
 $n = RST,$
 $m = 6R$ $+3ST+RT+ST+RS-(R+S+T-1)-6$
(-5 5 -1 -1 1 4 1)

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	5	40	1000	200	205	0
5	6	22	660	132	137	0
5	7	16	560	112	117	0
5	8	13	520	104	109	0
5	9	12	536	108	113	4
5	10	10	500	100	105	0
6	3	40	720	120	126	0
6	4	15	360	60	66	0
6	5	10	300	50	56	0
6	6	8	287	48	54	1
6	7	7	291	49	55	3
6	8	6	287	48	54	1
6	10	5	300	50	56	0
7	3	16	336	48	55	0
7	4	9	252	36	43	0
7	5	7	242	35	42	3
7	6	6	246	36	43	6
7	7	5	242	35	42	3
7	9	4	252	36	43	0
8	2	49	784	98	106	0
8	3	12	284	36	44	4
8	4	7	224	28	36	0
8	5	6	232	30	38	8
8	6	5	232	30	38	8
8	7	4	224	28	36	0
9	2	28	504	56	65	0
9	3	10	264	30	39	6
9	4	6	216	24	33	0
9	5	5	220	25	34	5
9	6	4	216	24	33	0
9	10	3	264	30	39	6
10	2	21	420	42	52	0
10	3	8	240	24	34	0
10	8	3	240	24	34	0
11	2	18	394	36	47	2
11	4	5	220	20	31	0
11	5	4	220	20	31	0
12	2	16	381	32	44	3
12	3	7	249	21	33	3
12	7	3	249	21	33	3
13	2	14	364	28	41	0
14	2	13	364	26	40	0
15	3	6	268	18	33	2
15	6	3	268	18	33	2
16	2	12	381	24	40	3
17	4	4	272	16	33	0
18	2	11	394	22	40	2
minimum values	5	2	3	216	16	31

TABLE C.9
 Static (rover), free orbit, triple difference
 (2 3 3 3 1)

$$n = RST,$$

$$m = 3(R-1+T)+3ST+RT+ST+RS-(R+S+T-1)-6$$

(-8 2 -1 2 1 4 1)

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	8	34	1360	272	277	0
5	9	19	855	171	176	0
5	10	14	700	140	145	0
6	5	15	449	75	81	1
6	6	9	322	54	60	2
6	7	7	291	49	55	3
6	8	6	284	48	54	4
6	9	5	269	45	51	1
7	4	10	280	40	47	0
7	5	6	210	30	37	0
7	6	5	207	30	37	3
7	7	4	196	28	35	0
8	3	15	359	45	53	1
8	4	6	192	24	32	0
8	5	5	193	25	33	7
8	6	4	186	24	32	6
8	8	3	190	24	32	2
9	3	9	241	27	36	2
9	4	5	177	20	29	3
9	5	4	174	20	29	6
9	7	3	183	21	30	6
10	3	7	207	21	31	3
10	4	4	160	16	26	0
10	6	3	174	18	28	6
11	2	34	748	68	79	0
11	3	6	194	18	29	4
11	5	3	163	15	26	2
11	10	2	220	20	31	0
12	2	19	456	38	50	0
12	3	5	179	15	27	1
12	9	2	215	18	30	1
13	2	14	364	28	41	0
13	8	2	208	16	29	0
14	2	12	334	24	38	2
14	4	3	168	12	26	0
15	2	10	300	20	35	0
15	3	4	180	12	27	0
15	7	2	210	14	29	0
16	2	9	288	18	34	0
18	2	8	286	16	34	2
19	6	2	228	12	31	0
20	2	7	280	14	34	0
minimum values	5	2	160	12	26	

TABLE C.10

Static (rover), free orbit, double difference

 $(2\ 3\ 3\ 3\ 0)$

$$n = RST, \quad m = 3(R-1+T)+3ST+RT+ST \quad -T-6$$

 $(-9\ 3\ 0\ 2\ 1\ 4\ 0)$

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	8	6	240	48	53	0
5	9	3	135	27	32	0
5	10	2	100	20	25	0
6	5	5	149	25	31	1
6	6	3	105	18	24	3
6	7	2	81	14	20	3
6	9	1	53	9	15	1
7	4	4	112	16	23	0
7	5	2	70	10	17	0
7	7	1	49	7	14	0
8	3	8	191	24	32	1
8	4	3	93	12	20	3
9	3	5	133	15	24	2
9	4	2	72	8	17	0
9	6	1	53	6	15	1
10	3	4	117	12	22	3
11	2	24	528	48	59	0
11	3	3	99	9	20	0
12	2	14	335	28	40	1
13	2	10	260	20	33	0
13	5	1	65	5	18	0
14	2	9	249	18	32	3
15	2	8	236	16	31	4
16	2	7	221	14	30	3
17	2	6	204	12	29	0
19	3	2	114	6	25	0
Minimum values	5	2	49	5	14	

TABLE C.11

Static (rover), free orbit, single difference

 $(2\ 3\ 0\ 3\ 0)$

$$n = RST, \quad m = 3(R-1+T)+3ST \quad +ST \quad -6$$

 $(-9\ 3\ 0\ 3\ 0\ 4\ 0)$

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	4	6	120	24	29	0
5	5	3	75	15	20	0
5	6	2	60	12	17	0
5	9	1	45	9	14	0
6	2	9	108	18	24	0
6	3	3	54	9	15	0
6	4	2	47	8	14	1
6	6	1	36	6	12	0

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Table C.11 cont'd

	7	2	4	56	8	15	0
	7	3	2	42	6	13	0
	7	5	1	35	5	12	0
	8	1	15	120	15	23	0
	8	2	3	48	6	14	0
	9	1	9	81	9	18	0
	10	1	7	70	7	17	0
	10	4	1	40	4	14	0
	11	1	6	66	6	17	0
	13	1	5	65	5	18	0
	13	2	2	52	4	17	0
	19	1	4	76	4	23	0
minimum values	5	1	1	35	4	12	

TABLE C.12

Static (rover), free orbit, Doppler

$$n = RST, \quad m = \begin{pmatrix} 2 & 3 & 0 & 0 & 1 \\ -9 & 3 & 0 & 3 & 0 & 3 & 1 \end{pmatrix} + RS \quad -6$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	4	19	304	76	80	0
	4	5	12	239	60	64	1
	4	6	9	216	54	58	0
	4	7	8	223	56	60	1
	4	8	7	224	56	60	0
	5	2	16	160	32	37	0
	5	3	7	105	21	26	0
	5	4	6	116	24	29	4
	5	5	5	121	25	30	4
	5	6	4	120	24	29	0
	6	2	7	84	14	20	0
	6	3	5	87	15	21	3
	6	4	4	93	16	22	3
	6	6	3	108	18	24	0
	7	1	19	133	19	26	0
	7	2	6	80	12	19	4
	7	3	4	81	12	19	3
	7	5	3	101	15	22	4
	8	1	12	95	12	20	1
	8	2	5	76	10	18	4
	8	4	3	92	12	20	4
	9	1	9	81	9	18	0
	9	2	4	72	8	17	0
	9	3	3	81	9	18	0
	9	8	2	144	16	25	0
	10	1	8	79	8	18	1
	10	7	2	139	14	24	1
	11	1	7	77	7	18	0
	11	6	2	132	12	23	0
	14	1	6	83	6	20	1
	14	5	2	139	10	24	1
	18	2	3	108	6	24	0
minimum values	4	1	2	72	6	17	

TABLE C.13
Static (rover), free orbit, unbiased

$$(2 \ 3 \ 0 \ 0 \ 0)$$

$$n = RST,$$

$$m = 3(R-1+T)+3ST \quad -6$$

$$(-9 \ 3 \ 0 \ 3 \ 0 \ 3 \ 0)$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	4	3	48	12	16	0
	4	5	2	39	10	14	1
	4	6	1	24	6	10	0
	5	2	6	60	12	17	0
	5	3	2	30	6	11	0
	5	5	1	24	5	10	1
	6	2	3	36	6	12	0
	6	4	1	24	4	10	0
	7	1	12	84	12	19	0
	8	1	8	63	8	16	1
	9	1	6	54	6	15	0
	9	2	2	36	4	13	0
	11	1	5	54	5	16	1
	15	1	4	60	4	19	0
minimum values	4	1	1	24	4	10	

TABLE C.14
Static (rover), known orbit, triple difference

$$(2 \ 0 \ 3 \ 3 \ 1)$$

$$n = RST,$$

$$m = 3(R-1+T) + RT+ST+RS-(R+S+T-1)$$

$$(-2 \ 2 \ -1 \ 2 \ 1 \ 1 \ 1)$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	5	7	70	35	37	0
	2	6	4	48	24	26	0
	2	7	3	42	21	23	0
	2	10	2	40	20	22	0
	3	3	10	90	30	33	0
	3	4	4	48	16	19	0
	3	5	3	44	15	18	1
	3	7	2	42	14	17	0
	4	3	5	60	15	19	0
	4	4	3	48	12	16	0
	4	6	2	48	12	16	0
	5	2	16	160	32	37	0
	5	3	4	60	12	17	0
	6	2	10	120	20	26	0
	7	2	8	112	16	23	0
	7	5	2	70	10	17	0
	8	2	7	112	14	22	0
	10	2	6	120	12	22	0
	10	3	3	90	9	19	0
	16	2	5	160	10	26	0
minimum values	2	2	2	40	9	16	

TABLE C.15

Static (rover), known orbit, double difference
 $(2\ 0\ 3\ 3\ 0)$
 $n = RST,$
 $m = 3(R-1+T) \quad +RT+ST \quad -T$
 $(-3\ 3\ 0\ 2\ 1\ 1\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	5	3	30	15	17	0
	2	6	2	23	12	14	1
	2	7	1	14	7	9	0
	3	3	6	54	18	21	0
	3	4	2	24	8	11	0
	3	6	1	17	6	9	1
	4	3	3	36	9	13	0
	4	5	1	20	5	9	0
	5	2	12	120	24	29	0
	6	2	8	95	16	22	1
	7	2	6	84	12	19	0
	7	3	2	42	6	13	0
	9	2	5	89	10	19	1
	13	2	4	104	8	21	0
minimum values	2	2	1	14	5	9	

TABLE C.16

Static (rover), known orbit, single difference
 $(2\ 0\ 0\ 3\ 0)$
 $n = RST,$
 $m = 3(R-1+T) \quad +ST$
 $(-3\ 3\ 0\ 3\ 0\ 1\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	4	3	24	12	14	0
	2	5	2	19	10	12	1
	2	6	1	12	6	8	0
	3	2	6	36	12	15	0
	3	3	2	18	6	9	0
	3	5	1	14	5	8	1
	4	2	3	24	6	10	0
	4	4	1	16	4	8	0
	5	1	12	60	12	17	0
	6	1	8	47	8	14	1
	7	1	6	42	6	13	0
	7	2	2	28	4	11	0
	9	1	5	44	5	14	1
	13	1	4	52	4	17	0
minimum values	2	1	1	12	4	8	

TABLE C.17

Static (rover), known orbit, Doppler
 $(2 \ 0 \ 0 \ 0 \ 1)$
 $n = RST,$
 $m = 3(R-1+T) \quad +RS$
 $(-3 \ 3 \ 0 \ 3 \ 0 \ 0 \ 1)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	1	4	4	16	16	17	0
	1	5	3	14	15	16	1
	1	6	2	12	12	13	0
	2	2	7	28	14	16	0
	2	3	3	18	9	11	0
	2	5	2	19	10	12	1
	3	2	4	24	8	11	0
	3	4	2	24	8	11	0
	4	1	13	52	13	17	0
	5	1	9	44	9	14	1
	6	1	7	42	7	13	0
	6	2	3	36	6	12	0
	8	1	6	47	6	14	1
	12	1	5	60	5	17	0
minimum values	1	1	2	12	5	11	

TABLE C.18

Static (rover), known orbit, unbiased
 $(2 \ 0 \ 0 \ 0 \ 0)$
 $n = RST,$
 $m = 3(R-1+T)$
 $(-3 \ 3 \ 0 \ 3 \ 0 \ 0 \ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	1	3	1	3	3	4	0
	2	2	3	12	6	8	0
	3	2	2	12	4	7	0
	4	1	9	36	9	13	0
	5	1	6	30	6	11	0
	6	1	5	30	5	11	0
	9	1	4	36	4	13	0
minimum values	1	1	1	3	3	4	

TABLE C.19

Static, free orbit, triple difference

 $(1 \ 3 \ 3 \ 3 \ 1)$ $n = RST,$ $m = 3R + 3ST + RT + ST + RS - (R + S + T - 1) - 6$ $(-5 \ 2 \ -1 \ -1 \ 1 \ 4 \ 1)$

R	S	T	m	ST	$R+ST$	$n-m$
5	5	25	625	125	130	0
5	6	15	449	90	95	1
5	7	11	385	77	82	0
5	8	10	397	80	85	3
5	9	9	401	81	86	4
5	10	8	397	80	85	3
6	3	22	396	66	72	0
6	4	9	216	36	42	0
6	5	7	207	35	41	3
6	6	6	211	36	42	5
6	7	5	207	35	41	3
6	9	4	216	36	42	0
7	3	9	189	27	34	0
7	4	6	165	24	31	3
7	5	5	169	25	32	6
7	6	4	165	24	31	3
7	9	3	189	27	34	0
8	2	25	400	50	58	0
8	3	7	165	21	29	3
8	4	5	154	20	28	6
8	5	4	154	20	28	6
8	7	3	165	21	29	3
9	2	15	269	30	39	1
9	3	6	157	18	27	5
9	4	4	141	16	25	3
9	6	3	157	18	27	5
10	2	11	220	22	32	0
10	3	5	147	15	25	3
10	5	3	147	15	25	3
11	2	10	217	20	31	3
11	10	2	217	20	31	3
12	2	9	212	18	30	4
12	3	4	144	12	24	0
12	4	3	144	12	24	0
12	9	2	212	18	30	4
13	2	8	205	16	29	3
13	8	2	205	16	29	3
14	2	7	196	14	28	0
14	7	2	196	14	28	0
18	2	6	215	12	30	1
18	6	2	215	12	30	1
minimum values	5	2	141	12	24	

TABLE C.20

Static, free orbit, double difference

$$(1 \ 3 \ 3 \ 3 \ 0)$$

$$n = RST,$$

$$m = 3R + 3ST + RT + ST - T - 6$$

$$(-6 \ 3 \ 0 \ -1 \ 1 \ 4 \ 0)$$

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	5	9	225	45	50	0
5	6	5	149	30	35	1
5	7	3	105	21	26	0
5	9	2	89	18	23	1
6	3	12	216	36	42	0
6	4	4	96	16	22	0
6	5	3	87	15	21	3
6	6	2	70	12	18	2
6	9	1	53	9	15	1
7	3	5	105	15	22	0
7	4	3	81	12	19	3
7	5	2	67	10	17	3
7	7	1	49	7	14	0
8	2	18	288	36	44	0
8	3	4	94	12	20	2
8	4	2	64	8	16	0
9	2	11	197	22	31	1
9	3	3	81	9	18	0
9	6	1	53	6	15	1
10	2	8	160	16	26	0
11	2	7	153	14	25	1
12	2	6	144	12	24	0
13	5	1	65	5	18	0
15	2	5	149	10	25	1
16	3	2	96	6	22	0
minimum values	5	2	49	5	14	

TABLE C.21

Static, free orbit, 231

$$(1 \ 3 \ 2 \ 3 \ 1)$$

$$n = RST,$$

$$m = 3R + 3ST + 3R + ST + RS - S - 6$$

$$(-6 \ 6 \ -1 \ 0 \ 0 \ 4 \ 1)$$

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	1	28	140	28	33	0
5	2	16	160	32	37	0
5	3	12	180	36	41	0
5	4	10	200	40	45	0
5	5	9	224	45	50	1
5	6	8	240	48	53	0
5	8	7	280	56	61	0
6	1	18	107	18	24	1
6	2	10	120	20	26	0

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Table C.21 cont'd

6	3	8	141	24	30	3
6	4	7	162	28	34	6
6	5	6	175	30	36	5
6	6	5	180	30	36	0
6	10	4	240	40	46	0
7	1	14	98	14	21	0
7	2	8	112	16	23	0
7	3	6	126	18	25	0
7	4	5	140	20	27	0
7	6	4	168	24	31	0
8	1	13	101	13	21	3
8	2	7	112	14	22	0
8	5	4	157	20	28	3
8	9	3	213	27	35	3
9	1	12	104	12	21	4
9	3	5	132	15	24	3
9	4	4	144	16	25	0
9	7	3	188	21	30	1
10	1	11	107	11	21	3
10	2	6	120	12	22	0
10	6	3	180	18	28	0
11	1	10	110	10	21	0
13	3	4	156	12	25	0
13	5	3	192	15	28	3
15	1	9	134	9	24	1
16	2	5	160	10	26	0
16	10	2	320	20	36	0
19	4	3	228	12	31	0
19	9	2	342	18	37	0
minimum values	5	1	2	98	9	21

TABLE C.22

Static, free orbit, 230

(1 3 2 3 0)

$n = RST,$

$$m = 3R \quad +3ST+3R+ST \quad -6$$

(-6 6 0 0 0 4 0)

<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
5	1	24	120	24	29	0
5	2	12	120	24	29	0
5	3	8	120	24	29	0
5	4	6	120	24	29	0
5	5	5	124	25	30	1
5	6	4	120	24	29	0
5	8	3	120	24	29	0
6	1	15	90	15	21	0
6	2	8	94	16	22	2
6	3	5	90	15	21	0
6	4	4	94	16	22	2
6	5	3	90	15	21	0
6	8	2	94	16	22	2
7	1	12	84	12	19	0

Table C.22 cont'd

	7	2	6	84	12	19	0
	7	3	4	84	12	19	0
	7	4	3	84	12	19	0
	7	6	2	84	12	19	0
	8	1	11	86	11	19	2
	9	1	10	88	10	19	2
	9	2	5	88	10	19	2
	9	5	2	88	10	19	2
	9	10	1	88	10	19	2
	10	1	9	90	9	19	0
	10	3	3	90	9	19	0
	10	9	1	90	9	19	0
	13	1	8	104	8	21	0
	13	2	4	104	8	21	0
	13	4	2	104	8	21	0
	13	8	1	104	8	21	0
minimum values	5	1	1	84	8	19	0

TABLE C.23

Static, free orbit, single difference

$$(1 \ 3 \ 0 \ 3 \ 0)$$

$$n = RST,$$

$$m = 3R \quad +3ST \quad +ST \quad -6$$

$$(-6 \ 3 \ 0 \ 0 \ 0 \ 4 \ 0)$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	5	1	9	45	9	14	0
	5	2	5	49	10	15	1
	5	3	3	45	9	14	0
	5	5	2	49	10	15	1
	5	9	1	45	9	14	0
	6	1	6	36	6	12	0
	6	2	3	36	6	12	0
	6	3	2	36	6	12	0
	6	6	1	36	6	12	0
	7	1	5	35	5	12	0
	7	5	1	35	5	12	0
	10	1	4	40	4	14	0
	10	2	2	40	4	14	0
	10	4	1	40	4	14	0
minimum values	5	1	1	35	4	12	0

TABLE C.24

Static, free orbit, Doppler
 $(1\ 3\ 0\ 0\ 1)$
 $n = RST,$
 $m = 3R + 3ST + RS - 6$
 $(-6\ 3\ 0\ 0\ 0\ 3\ 1)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	1	10	40	10	14	0
	4	2	7	56	14	18	0
	4	3	6	72	18	22	0
	4	6	5	120	30	34	0
	5	1	7	35	7	12	0
	5	2	5	49	10	15	1
	5	3	4	60	12	17	0
	5	9	3	135	27	32	0
	6	1	6	36	6	12	0
	6	2	4	48	8	14	0
	6	4	3	72	12	18	0
	7	3	3	63	9	16	0
	8	9	2	144	18	26	0
	9	1	5	45	5	14	0
	9	7	2	126	14	23	0
	10	6	2	120	12	22	0
	12	2	3	72	6	18	0
	12	5	2	120	10	22	0
	18	4	2	144	8	26	0
minimum values	4	1	2	35	5	12	

TABLE C.25

Static, free orbit, unbiased
 $(1\ 3\ 0\ 0\ 0)$
 $n = RST,$
 $m = 3R + 3ST - 6$
 $(-6\ 3\ 0\ 0\ 0\ 3\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	1	6	24	6	10	0
	4	2	3	24	6	10	0
	4	3	2	24	6	10	0
	4	6	1	24	6	10	0
	5	1	5	24	5	10	1
	5	5	1	24	5	10	1
	6	1	4	24	4	10	0
	6	2	2	24	4	10	0
	6	4	1	24	4	10	0
minimum values	4	1	1	24	4	10	

TABLE C.26

Static, known orbit, triple difference
 $(1\ 0\ 3\ 3\ 1)$
 $n = RST,$
 $m = 3R + RT + ST + RS - (R + S + T - 1)$
 $(1\ 2\ -1\ -1\ 1\ 1\ 1)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	2	7	28	14	16	0
	2	3	4	24	12	14	0
	2	4	3	24	12	14	0
	2	7	2	28	14	16	0
	3	2	6	35	12	15	1
	3	6	2	35	12	15	1
	4	2	5	40	10	14	0
	4	3	3	36	9	13	0
	4	5	2	40	10	14	0
minimum values	2	2	2	24	9	13	

TABLE C.27

Static, known orbit, double difference
 $(1\ 0\ 3\ 3\ 0)$
 $n = RST,$
 $m = 3R + RT + ST - T$
 $(0\ 3\ 0\ -1\ 1\ 1\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	2	6	24	12	14	0
	2	3	3	18	9	11	0
	2	4	2	16	8	10	0
	2	7	1	14	7	9	0
	3	2	5	29	10	13	1
	3	6	1	17	6	9	1
	4	2	4	32	8	12	0
	4	3	2	24	6	10	0
	4	5	1	20	5	9	0
minimum values	2	2	1	14	5	9	

TABLE C.28

Static, known orbit, 230
 $(1\ 0\ 2\ 3\ 0)$
 $n = RST,$
 $m = 3R \quad +3R+ST$
 $(0\ 6\ 0\ 0\ 0\ 1\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	1	12	24	12	14	0
	2	2	6	24	12	14	0
	2	3	4	24	12	14	0
	2	4	3	24	12	14	0
	2	6	2	24	12	14	0
	3	1	9	27	9	12	0
	3	2	5	28	10	13	2
	3	3	3	27	9	12	0
	3	5	2	28	10	13	2
	3	9	1	27	9	12	0
	4	1	8	32	8	12	0
	4	2	4	32	8	12	0
	4	4	2	32	8	12	0
	4	8	1	32	8	12	0
	7	1	7	49	7	14	0
	7	7	1	49	7	14	0
minimum values	2	1	1	24	7	12	

TABLE C.29

Static, known orbit, single difference
 $(1\ 0\ 0\ 3\ 0)$
 $n = RST,$
 $m = 3R \quad +ST$
 $(0\ 3\ 0\ 0\ 0\ 1\ 0)$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	2	1	6	12	6	8	0
	2	2	3	12	6	8	0
	2	3	2	12	6	8	0
	2	6	1	12	6	8	0
	3	1	5	14	5	8	1
	3	5	1	14	5	8	1
	4	1	4	16	4	8	0
	4	2	2	16	4	8	0
	4	4	1	16	4	8	0
minimum values	2	1	1	12	4	8	

TABLE C.30
Static, known orbit, Doppler

$$(1 \ 0 \ 0 \ 0 \ 1)$$

$$n = RST,$$

$$m = 3R \quad +RS$$

$$(0 \ 3 \ 0 \ 0 \ 0 \ 0 \ 1)$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	1	1	4	4	4	5	0
	1	2	3	5	6	7	1
	1	3	2	6	6	7	0
minimum values	1	1	2	4	4	5	

TABLE C.31
Static, known orbit, unbiased

$$(1 \ 0 \ 0 \ 0 \ 0)$$

$$n = RST$$

$$m = 3R$$

$$(0 \ 3 \ 0 \ 0 \ 0 \ 0 \ 0)$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	1	1	3	3	3	4	0
	1	2	2	3	4	5	1
	1	3	1	3	3	4	0
minimum values	1	1	1	3	3	4	

TABLE C.32
Known, free orbit, triple difference

$$(0 \ 3 \ 3 \ 3 \ 1)$$

$$n = RST,$$

$$m = +3ST + RT + ST + RS - (R + S + T - 1)$$

$$(1 \ -1 \ -1 \ -1 \ 1 \ 4 \ 1)$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	5	5	16	400	80	85	0
	5	6	10	300	60	65	0
	5	7	8	280	56	61	0
	5	8	7	280	56	61	0
	5	10	6	300	60	65	0
	6	3	10	180	30	36	0
	6	4	5	120	20	26	0
	6	5	4	120	20	26	0
	6	10	3	180	30	36	0
	7	3	4	84	12	19	0
	7	4	3	84	12	19	0
	8	2	7	112	14	22	0
	8	3	3	71	9	17	1
	8	7	2	112	14	22	0
	9	2	4	72	8	17	0
	9	4	2	72	8	17	0
	10	2	3	60	6	16	0
	10	3	2	60	6	16	0
	13	2	2	52	4	17	0
minimum values	5	2	2	52	4	16	

TABLE C.33

Known, free orbit, double difference
 $(0\ 3\ 3\ 3\ 0)$
 $n = RST,$
 $m = \begin{matrix} +3ST+RT+ST & -T \\ (0\ 0\ 0\ -1\ 1\ 4\ 0) \end{matrix}$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	5	4	1	20	4	9	0
	6	3	1	17	3	9	1
	7	2	1	14	2	9	0
minimum values	5	2	1	14	2	9	

TABLE C.34

Known, free orbit, 230
 $(0\ 3\ 2\ 3\ 0)$
 $n = RST,$
 $m = \begin{matrix} +3ST+3R+ST \\ (0\ 3\ 0\ 0\ 0\ 4\ 0) \end{matrix}$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	5	1	15	75	15	20	0
	5	2	8	79	16	21	1
	5	3	5	75	15	20	0
	5	4	4	79	16	21	1
	5	5	3	75	15	20	0
	5	8	2	79	16	21	1
	6	1	9	54	9	15	0
	6	2	5	58	10	16	2
	6	3	3	54	9	15	0
	6	5	2	58	10	16	2
	6	9	1	54	9	15	0
	7	1	7	49	7	14	0
	7	2	4	53	8	15	3
	7	4	2	53	8	15	3
	7	7	1	49	7	14	0
	8	1	6	48	6	14	0
	8	2	3	48	6	14	0
	8	3	2	48	6	14	0
	8	6	1	48	6	14	0
	10	1	5	50	5	15	0
	10	5	1	50	5	15	0
	16	1	4	64	4	20	0
	16	2	2	64	4	20	0
	16	4	1	64	4	20	0
minimum values	5	1	1	48	4	14	

TABLE C.35

$$\begin{aligned} &\text{Known, free orbit, } 031 \\ &\quad (0 \ 3 \ 0 \ 3 \ 1) \\ &\quad n = RST, \\ m = &\quad +3ST \quad +ST+RS-S \\ &\quad (0 \ 0 \ -1 \ 0 \ 0 \ 4 \ 1) \end{aligned}$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	5	1	4	20	4	9	0
	6	1	3	17	3	9	1
	7	1	2	14	2	9	0
minimum values	5	1	2	14	2	9	

TABLE C.36

$$\begin{aligned} &\text{Known, free orbit, single difference} \\ &\quad (0 \ 3 \ 0 \ 3 \ 0) \\ &\quad n = RST, \\ m = &\quad +3ST \quad +ST \\ &\quad (0 \ 0 \ 0 \ 0 \ 0 \ 4 \ 0) \end{aligned}$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	1	1	4	1	5	0
minimum values	4	1	1	4	1	5	

TABLE C.37

$$\begin{aligned} &\text{Known, free orbit, Doppler} \\ &\quad (0 \ 3 \ 0 \ 0 \ 1) \\ &\quad n = RST, \\ m = &\quad +3ST \quad +RS \\ &\quad (0 \ 0 \ 0 \ 0 \ 0 \ 3 \ 1) \end{aligned}$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	4	1	4	16	4	8	0
	5	1	3	14	3	8	1
	6	1	2	12	2	8	0
minimum values	4	1	2	12	2	8	

TABLE C.38

$$\begin{aligned} &\text{Known, free orbit, unbiased} \\ &\quad (0 \ 3 \ 0 \ 0 \ 0) \\ &\quad n = RST, \\ m = &\quad +3ST \\ &\quad (0 \ 0 \ 0 \ 0 \ 0 \ 3 \ 0) \end{aligned}$$

	<i>R</i>	<i>S</i>	<i>T</i>	<i>m</i>	<i>ST</i>	<i>R+ST</i>	<i>n-m</i>
	3	1	1	3	1	4	0
minimum values	3	1	1	3	1	4	

APPENDIX D

SOURCE CODE OF COMPUTER PROGRAMS USED

This appendix is a compendium of source codes of four Apple Macintosh Basic programs which were written in August 1985 by D. Wells. These programs were used to generate the results presented in this report.

The redundancy design tables in Appendix C were created by the 'Diophantine Equation Calculations' program listed in Appendix D.1. Program 'GPS Differencing Design' (Appendix D.2) allows the user to evaluate all matrix products which involve the differencing operator D using fixed and sequential differencing bases as described in section 4.1. Program 'GPS Undifferenced Design' (Appendix D.3) offers the same possibilities for the bias design matrix A_2 as given in section 4.2. Finally, in Appendix D.4, program 'GPS Annihilation' checks whether the matrix product of D and A_2 , in fact, results in a zero matrix, as it is demanded by condition (A-1).

Note that, within all four programs, the variable 'g', instead of 'r', is sometimes used to represent specific receiver related indices.

Appendix D.1

'DIOPHANTINE EQUATION CALCULATIONS - D. Wells 85-08-09

'this version has rank defects built in

```
'open output
'device1$ = "scrn:"
'device1$ = "lpt1:direct"
device1$ = "DesignSoft:AppendC"
```

```
'device2$ = "scrn:"
'device2$ = "lpt1:direct"
device2$ = "DesignSoft:Table 5.1"
```

```
'device3$ = "scrn:"
'device3$ = "lpt1:direct"
device3$ = "DesignSoft:Table 5.2"
```

```
OPEN device1$ FOR OUTPUT AS #1
OPEN device2$ FOR OUTPUT AS #2
OPEN device3$ FOR OUTPUT AS #3
starttime = TIMER
```

```
'GOTO jumpstuff
table% = 0
CALL FindDesign (4, 3, 3, 3, 1, table%)
CALL FindDesign (4, 3, 3, 3, 0, table%)
CALL FindDesign (4, 3, 0, 3, 0, table%)
CALL FindDesign (4, 3, 0, 0, 1, table%)
CALL FindDesign (4, 3, 0, 0, 0, table%)
CALL FindDesign (4, 0, 3, 3, 1, table%)
CALL FindDesign (4, 0, 3, 0, 0, table%)
CALL FindDesign (3, 3, 3, 3, 1, table%)
CALL FindDesign (2, 3, 3, 3, 1, table%)
CALL FindDesign (2, 3, 3, 3, 0, table%)
CALL FindDesign (2, 3, 0, 3, 0, table%)
CALL FindDesign (2, 3, 0, 0, 1, table%)
CALL FindDesign (2, 3, 0, 0, 0, table%)
CALL FindDesign (2, 0, 3, 3, 1, table%)
CALL FindDesign (2, 0, 3, 3, 0, table%)
CALL FindDesign (2, 0, 0, 3, 0, table%)
CALL FindDesign (2, 0, 0, 0, 1, table%)
CALL FindDesign (2, 0, 0, 0, 0, table%)
CALL FindDesign (1, 3, 3, 3, 1, table%)
CALL FindDesign (1, 3, 3, 3, 0, table%)
CALL FindDesign (1, 3, 2, 3, 1, table%)
CALL FindDesign (1, 3, 2, 3, 0, table%)
CALL FindDesign (1, 3, 0, 3, 0, table%)
CALL FindDesign (1, 3, 0, 0, 1, table%)
CALL FindDesign (1, 3, 0, 0, 0, table%)
```

GPS DESIGN

```
CALL FindDesign (1, 0, 3, 3, 1, table%)
CALL FindDesign (1, 0, 3, 3, 0, table%)
CALL FindDesign (1, 0, 2, 3, 0, table%)
CALL FindDesign (1, 0, 0, 3, 0, table%)
CALL FindDesign (1, 0, 0, 0, 1, table%)
CALL FindDesign (1, 0, 0, 0, 0, table%)
CALL FindDesign (0, 3, 3, 3, 1, table%)
CALL FindDesign (0, 3, 3, 3, 0, table%)
CALL FindDesign (0, 3, 2, 3, 0, table%)
CALL FindDesign (0, 3, 0, 3, 1, table%)
CALL FindDesign (0, 3, 0, 3, 0, table%)
CALL FindDesign (0, 3, 0, 0, 1, table%)
CALL FindDesign (0, 3, 0, 0, 0, table%)
```

```
GOTO endstuff
```

```
endstuff:
```

```
stoptime = TIMER
elapsedtime = stoptime - starttime
```

```
PRINT "start=",starttime
PRINT "stop=",stoptime
PRINT "elapsed=",elapsedtime
CLOSE #1
CLOSE #2
CLOSE #3
CALL OKWindow
```

```
END
```

```
'-----
SUB FindDesign (ng%,ns%,na%,nb%,nn%,table%) STATIC
```

```
table% = table% + 1
PRINT USING "###";ng%;ns%;na%;nb%;nn%,table%

CALL coeff (ng%,ns%,na%,nb%,nn%,k0,kg,ks,kt,kgt,kst,kgs)
```

```
maxg% = 20
maxs% = 10
maxt% = 100
mint% = 1
```

```
' get least g, given maxs%,maxt%
CALL GMinValue (k0,kg,ks,kt,kgt,kst,kgs,ming%,maxs%,maxt%)
' get least s, given maxg%,maxt%
CALL SMinValue (k0,kg,ks,kt,kgt,kst,kgs,maxg%,mins%,maxt%)
' get least t, given maxg%,maxs%
CALL TMinValue (k0,kg,ks,kt,kgt,kst,kgs,maxg%,maxs%,mint%)
```

```
' get max t (expected to occur either at ming%, or mins%)
CALL TMaxValue (k0,kg,ks,kt,kgt,kst,kgs,ming%,mins%,firsts%,maxt%,maxt2%)
```

```
'PRINT #1, "min/max";ming%;maxg%,mins%,maxs%,mint%;maxt%;maxt2%
```

```

maxt% = maxt2%

CALL Header (ng%,ns%,na%,nb%,nn%,table%,k0,kg,ks,kt,kgt,kst,kgs)

'GOTO EndOfSub

    FOR s = 1 TO maxs%
        minit(s) = 999
    NEXT s

        minunk = 9999
        minst = 9999
        minpoints = 9999

FOR g% = ming% TO maxg%
    lastt = 9999
    g = g%
    CALL SMinValue (k0,kg,ks,kt,kgt,kst,kgs,g%,mins%,maxt%)
    FOR s% = mins% TO maxs%
        s = s%
        CALL TMinValue (k0,kg,ks,kt,kgt,kst,kgs,g%,s%,mint%)
        FOR t% = mint% TO maxt%
            t = t%
            obs% = g * s * t
            unkn% = k0 + kg*g + ks*s + kt*t + kgt*g*t + kst*s*t + kgs*g*s

            'PRINT USING "###";3, k0;kg;ks;kt;kgt;kst;kgs
            'PRINT USING "####";4, g;s;t;obs%;unkn%

            IF obs% < unkn% THEN GOTO nextt      'Diophantine equation not satisfied
            IF t => lastt THEN GOTO nexts      'not minimal in s, for given g,t
            lastt = t
            IF t => minit(s) THEN GOTO nexts   'not minimal in g, for given s,t
            minit(s) = t
            st = s * t
            points = g + st
            redund = obs% - unkn%
            PRINT #1,
CHR$(9);g;CHR$(9);s;CHR$(9);t;CHR$(9);unkn%;CHR$(9);st;CHR$(9);points;CHR$(9);
redund
            IF unkn% < minunk THEN minunk = unkn%
            IF st < minst THEN minst = st
            IF points < minpoints THEN minpoints = points

            GOTO nexts                          'keep only minimal t, for given g,s
nextt:
    NEXT t%
nexts:
    NEXT s%
NEXT g%
    PRINT #1, "minimum values"

```

GPS DESIGN

```

PRINT #1,
CHR$(9);ming%;CHR$(9);mins%;CHR$(9);mint%;CHR$(9);minunk;CHR$(9);minst;CHR$(
9);minpoints

```

```
'print Table 5.2 data
```

```

PRINT #3, USING "#";ng%;ns%;na%;nb%;nn%;
PRINT #3,
CHR$(9);ming%;CHR$(9);mins%;CHR$(9);mint%;CHR$(9);minunk;CHR$(9);minst;CHR$(
9);minpoints;
PRINT #3, CHR$(9);gnd$;sat$;bias$;
PRINT #3, CHR$(9);" C.";table%

```

```

EndOfSub:
END SUB

```

```

'-----
SUB Header (ng%,ns%,na%,nb%,nn%,table%,k0,kg,ks,kt,kgt,kst,kgs) STATIC

```

```

gnd$ = " "
IF ng% = 4 THEN gnd$ = "Kinematic, "
IF ng% = 3 THEN gnd$ = "Linear deform, "
IF ng% = 2 THEN gnd$ = "Static (rover), "
IF ng% = 1 THEN gnd$ = "Static, "
IF ng% = 0 THEN gnd$ = "Known, "

sat$ = " "
IF ns% = 3 THEN sat$ = "free orbit, "
IF ns% = 2 THEN sat$ = "Keplerian orbit, "
IF ns% = 1 THEN sat$ = "translated orbit, "
IF ns% = 0 THEN sat$ = "known orbit, "

bias$ = STR$(na%*100 + nb%*10 + nn%)
IF na% = 3 AND nb% = 3 AND nn% = 1 THEN bias$ = "triple difference"
IF na% = 3 AND nb% = 3 AND nn% = 0 THEN bias$ = "double difference"
IF na% = 0 AND nb% = 3 AND nn% = 0 THEN bias$ = "single difference"
IF na% = 0 AND nb% = 0 AND nn% = 1 THEN bias$ = "Doppler"
IF na% = 0 AND nb% = 0 AND nn% = 0 THEN bias$ = "unbiased"
IF ng% = 4 AND na% = 3 AND nb% = 0 AND nn% = 0 THEN bias$ = "4D navigation"

ug$ = " "
IF ng%=4 THEN ug$ = "3GT "
IF ng%=3 THEN ug$ = "6G "
IF ng%=2 THEN ug$ = "3(G-1+T)"
IF ng%=1 THEN ug$ = "3G "
us$ = " "
IF ns%=3 THEN us$="+3ST"
IF ns%=2 THEN us$="+6S"
IF ns% = 1 THEN us$="+3S"
ua$ = " "
IF na%=3 THEN ua$="+GT"
IF na%=2 THEN ua$="+3G"

```

```

IF na%=1 THEN ua$="+G "
ub$ = " "
IF nb%=3 THEN ub$="+ST"
IF nb%=2 THEN ub$="+3S"
IF nb%=1 THEN ub$="+S "
un$ = " "
IF nn%=1 THEN un$="+GS"
bd$ = " "
IF na% = 3 AND nb% = 3 THEN bd$ = "-T"
IF na% = 3 AND nn% = 1 THEN bd$ = "-G"
IF nb% = 3 AND nn% = 1 THEN bd$ = "-S"
IF na% = 3 AND nb% = 3 AND nn% = 1 THEN bd$ = "-(G+S+T-1)"
dd$ = " "
IF ng%*ns% <> 0 THEN dd$="-6 "
IF ng%=4 AND ns% >0 THEN dd$="-6T"

'print appendix tables header
PRINT #1 , " "
PRINT #1 ,"TABLE C .";table%
PRINT #1 ,gnd$;sat$;bias$;" (";ng%;ns%;na%;nb%;nn%;)"
PRINT #1 ,"n = RST, m = ";ug$;us$;ua$;ub$;un$;bd$;dd$;"(";k0;kg;ks;kt;kgt;kst;kgs;)"
PRINT #1 , " "
PRINT #1 ,CHR$(9);"R";CHR$(9);"S";CHR$(9);"T";CHR$(9);"m ";CHR$(9);"ST";
CHR$(9);"R+ST";CHR$(9);"n-m"
PRINT #1 , " "

'print Table 5.1 data
PRINT #2, USING "#";ng%;ns%;na%;nb%;nn%;
PRINT #2,CHR$(9);ug$;us$;ua$;ub$;un$;bd$;dd$;
PRINT #2,CHR$(9);gnd$;sat$;bias$;
PRINT #2,CHR$(9);" C.";table%

END SUB

'-----
SUB coeff (ng%,ns%,na%,nb%,nn%,k0,kg,ks,kt,kgt,kst,kgs) STATIC

' get coefficients for general equation
' n = cg*g+cg1*(t-1)+cgt*g*t +          ground coord model
'   + cs*s+cst*s*t +                   satellite model
'   + ca*g+cat*g*t + cb*s+cbt*s*t + cn*g*s + bias model
'   + bd + bdt*t + bds*s + bdt*t +     bias rank defect
'   + dd+ddt*t                          datum defect

IF ng%=3 THEN cg=6 ELSE IF ng%=2 OR ng%=1 THEN cg=3 ELSE cg=0
IF ng%=2 THEN cg1=3 ELSE cg1=0
IF ng%=4 THEN cgt=3 ELSE cgt=0

IF ns%=2 THEN cs=6 ELSE IF ns% = 1 THEN cs = 3 ELSE cs = 0
IF ns%=3 THEN cst=3 ELSE cst = 0

```

GPS DESIGN

```

IF na%=2 THEN ca=3 ELSE IF na%=1 THEN ca=1 ELSE ca=0
IF na%=3 THEN cat=1 ELSE cat=0

IF nb%=2 THEN cb=3 ELSE IF nb%=1 THEN cb=1 ELSE cb=0
IF nb%=3 THEN cbt=1 ELSE cbt=0

IF nn%=1 THEN cn=1 ELSE cn=0

IF na% = 3 AND nb% = 3 AND nn% = 1 THEN bd = 1 ELSE bd = 0
IF na% = 3 AND nb% = 3 THEN bdt = -1 ELSE bdt = 0
IF na% = 3 AND nn% = 1 THEN bdg = -1 ELSE bds = 0
IF nb% = 3 AND nn% = 1 THEN bds = -1 ELSE bdg = 0

IF ng% * ns%=0 THEN dd=0 ELSE IF ng%=4 AND ns% >0 THEN dd=0 ELSE dd=-6
IF ng%=4 AND ns% >0 THEN ddt=-6 ELSE ddt=0

'PRINT #1, "c-coeffs";cg;cg1;cgt;cs;cst;"bias";ca;cat;cb;cbt;cn;"defect";bd;bdt;bds;bdg;dd;ddt

'compute coefficients to
'n = k0 + kg*g + ks*s + kt*t + kgt*g*t + kst*s*t + kgs*g*s

k0 = -cg1 + dd + bd
kg = cg + ca + bdg
ks = cs + cb + bds
kt = cg1 + ddt + bdt
kgt = cgt + cat
kst = cst + cbt
kgs = cn

'PRINT #1, "k-coeffs";k0;kg;ks;kt;kgt;kst;kgs

END SUB

'-----
SUB GMinValue (k0,kg,ks,kt,kgt,kst,kgs,outg%,inps%,inpt%) STATIC
' find outg% = least g to satisfy m=>n, given inps%, inpt%
' where m = g * s * t
' n = k0 + kg*g + ks*s + kt*t + kgt*g*t + kst*s*t + kgs*g*s

s = inps%
t = inpt%
numer% = k0 + ks*s + kt*t + kst*s*t
denom% = s*t - kg - kgt*t - kgs*s
value% = 1
IF denom% =0 THEN GOTO zeroDenom:
value% = INT(numer% / denom%)
IF INT(numer% MOD denom%) <> 0 THEN value% = value% + 1
IF value% < 1 THEN value% = 1
zeroDenom:
outg% = value%
'PRINT #1, "Gmin-numer,denom,outg,inps,inpt",numer%;denom%;outg%;s;t

```

END SUB

```

'-----
SUB SMinValue (k0,kg,ks,kt,kgt,kst,kgs,inpg%,outs%,inpt%) STATIC
' find outs% = least s to satisfy m=>n, given inpg%, inpt%
' where m = g * s * t
'      n = k0 + kg*g + ks*s + kt*t + kgt*g*t + kst*s*t + kgs*g*s

      g = inpg%
      t = inpt%
      numer% = k0 + kg*g + kt*t + kgt*g*t
      denom% = g*t - ks - kst*t - kgs*g
      value% = 1
      IF denom% =0 THEN GOTO zeroDenomS:
      value% = INT(numer% / denom%)
      IF INT(numer% MOD denom%) <> 0 THEN value% = value% + 1
      IF value% < 1 THEN value% = 1
      zeroDenomS:
      outs% = value%
'PRINT #1, "Smin-numer,denom,inpg,outs,inpt",numer%;denom%;g;outs%;t

END SUB

```

```

'-----
SUB TMinValue (k0,kg,ks,kt,kgt,kst,kgs,inpg%,inps%,outt%) STATIC
' outt% = the least t to satisfy m=>n, given inpg%,inps%
' where m = g * s * t
'      n = k0 + kg*g + ks*s + kt*t + kgt*g*t + kst*s*t + kgs*g*s

      g = inpg%
      s = inps%
      numer% = k0 + kg*g + ks*s + kgs*g*s
      denom% = g*s - kt - kgt*g - kst*s
      value% = 1
      IF denom% =0 THEN GOTO zeroDenomT:
      value% = INT(numer% / denom%)
      IF INT(numer% MOD denom%) <> 0 THEN value% = value% + 1
      IF value% < 1 THEN value% = 1
      zeroDenomT:
      outt% = value%
'PRINT #1, "Tmin-numer,denom,inpg,inps,outt",numer%;denom%;g;s;outt%

END SUB

```

```

'-----
SUB TMax Value (k0,kg,ks,kt,kgt,kst,kgs,ming%,mins%,firsts%,maxt%,maxt2%) STATIC
'find firsts% = the least s to satisfy m=>n, given ming%,maxt%

```


GPS DESIGN

```
CALL SMinValue (k0,kg,ks,kt,kgt,kst,kgs,ming%,firsts%,maxt%)
```

```
' firstt% = the least t to satisfy m=>n, given ming%,firsts%
```

```
CALL TMinValue (k0,kg,ks,kt,kgt,kst,kgs,ming%,firsts%,firstt%)
```

```
' find laterg% = the least g to satisfy m=>n, given mins%,maxt%
```

```
CALL GMinValue (k0,kg,ks,kt,kgt,kst,kgs,laterg%,mins%,maxt%)
```

```
' find latert% = least t to satisfy m=>n, given laterg%,mins%
```

```
CALL TMinValue (k0,kg,ks,kt,kgt,kst,kgs,laterg%,mins%,latert%)
```

```
' select greatest of firstt%, latert% as new maxt2%
```

```
    maxt2% = firstt%
```

```
    IF latert% > maxt2% THEN maxt2% = latert%
```

```
'PRINT #1, "Tmax-firstt, later, maxt2", firstt%;latert%;maxt2%
```

```
END SUB
```

```
'-----
```

```
SUB OKWindow STATIC
```

```
    WINDOW 3, "", (485,320)-(505,335),2
```

```
    TEXTMODE 1
```

```
    BUTTON 1,1,"OK", (0,0)-(20,15),1
```

```
    DialogActive = 1
```

```
    WHILE DialogActive
```

```
        EventType = DIALOG(0)
```

```
        IF EventType = 1 THEN GOTO endsub
```

```
    WEND
```

```
    endsub:
```

```
    WINDOW CLOSE 3
```

```
END SUB
```

Appendix D.2

'GPS DIFFERENCING DESIGN (D operator) - D. Wells 85-08-15

DEFDBL A-z

```
device$ = "scm:"
device$ = "lpt1:direct"
'device$ = "GPSDesign:DOutput"
OPEN device$ FOR OUTPUT AS #1
starttime = TIMER
```

```
R% = 3
S% = 6
T% = 1
```

```
' D operator design parameters
' differencing modes
```

```
inmode1$ = "receiver"
inmode2$ = "satellite"
inmode3$ = "none"
```

```
' referencing for differences
seqR$ = "fix" 'use "fix" or "seq"
seqS$ = "fix"
seqT$ = "fix"
```

CALL Doperator (R%,S%,T%,inmode1\$,inmode2\$,inmode3\$,seqR\$,seqS\$,seqT\$)

```
stoptime = TIMER
elapsedtime = stoptime - starttime
PRINT #1,
PRINT #1, "elapsed secs=";elapsedtime
CLOSE #1
CALL OKWindow
```

END

```
'-----
SUB Doperator (R%,S%,T%,inmode1$,inmode2$,inmode3$,seqR$,seqS$,seqT$) STATIC
' compute differencing operator D, DDT, (DDT)(-1), DT(DDT)(-1)D
```

DEFDBL A-z

```
DIM totalD%(100,100)
DIM nextD%(100,100)
DIM tempD%(100,100)
DIM COV(100,100)
DIM ker(100,100)
DIM inmode$(3)
```

GPS DESIGN

```
DIM karr%(3,3)
```

```
mm% = 100
```

```
inmode$(1) = inmode1$
```

```
inmode$(2) = inmode2$
```

```
inmode$(3) = inmode3$
```

```
RST% = R% * S% * T%
```

```
PRINT #1, "GPS Differencing Design"
```

```
PRINT #1, "Design (#rcvrs, #sats, #epochs) = ";R%;S%;T%
```

```
PRINT #1, "D: Differencing references are: " ;seqR$;" rec ref, " ;seqS$;" sat ref, " ;seqT$;" time  
ref"
```

```
PRINT #1, "Differencing between ";inmode$(1);", then ";inmode$(2);", then ";inmode$(3)
```

```
GOSUB initialk
```

```
GOSUB initialD
```

```
FOR idiff% = 1 TO 3
```

```
IF inmode$(idiff%) = "none" THEN GOTO nomorediffs
```

```
IF inmode$(idiff%) = "receiver" THEN GOSUB betweenReceiver
```

```
IF inmode$(idiff%) = "satellite" THEN GOSUB betweenSatellite
```

```
IF inmode$(idiff%) = "time" THEN GOSUB betweenTime
```

```
GOSUB productD
```

```
NEXT idiff%
```

```
nomorediffs:
```

```
GOSUB covariance
```

```
GOSUB inverse
```

```
GOSUB kernel
```

```
EXIT SUB
```

```
-----  
initialD:
```

```
' set totalD = Identity(dim=rst) to start
```

```
rowtotalD% = RST%
```

```
coltotalD% = RST%
```

```
FOR i% = 1 TO rowtotalD%
```

```
FOR j% = 1 TO coltotalD%
```

```
totalD%(i%,j%) = 0
```

```
NEXT j%
```

```
totalD%(i%,i%) = 1
```

```
NEXT i%
```

```
RETURN
```

```
-----  
initialk:
```

```
'set up k array for inmode$ sequence
```

'this sets up the following 15 cases

'inmode			kxR%			kxS%			kxT%			
'1	'2	'3	r	s	t	r	s	t	r	s	t	
'r	-	-	1	0	0	0	0	0	0	0	0	single differencing modes
's	-	-	0	0	0	0	1	0	0	0	0	
't	-	-	0	0	0	0	0	0	0	0	1	
'r	s	-	1	0	0	1	1	0	0	0	0	double differencing modes
's	r	-	1	1	0	0	1	0	0	0	0	
'r	t	-	1	0	0	0	0	0	1	0	1	
't	r	-	1	0	1	0	0	0	0	0	1	
's	t	-	0	0	0	0	1	0	0	1	1	
't	s	-	0	0	0	0	1	1	0	0	1	
'r	s	t	1	0	0	1	1	0	1	1	1	triple differencing sequences
'r	t	s	1	0	0	1	1	1	1	0	1	
's	r	t	1	1	0	0	1	0	1	1	1	
't	r	s	1	0	1	1	1	1	0	0	1	
's	t	r	1	1	1	0	1	0	0	1	1	
't	s	r	1	1	1	0	1	1	0	0	1	

```
FOR i = 1 TO 3
FOR j = 1 TO 3
  karr%(i,j) = 0
NEXT j
NEXT i
```

```
IF inmode$(1) = "receiver" THEN karr%(1,1) = 1: col2% = 1
IF inmode$(1) = "satellite" THEN karr%(2,2) = 1: col2% = 2
IF inmode$(1) = "time" THEN karr%(3,3) = 1: col2% = 3
```

```
IF inmode$(2) = "receiver" THEN karr%(1,1) = 1: karr%(1,col2%) = 1
IF inmode$(2) = "satellite" THEN karr%(2,2) = 1: karr%(2,col2%) = 1
IF inmode$(2) = "time" THEN karr%(3,3) = 1: karr%(3,col2%) = 1
```

```
IF inmode$(3) = "receiver" THEN karr%(1,1) = 1: karr%(1,2) = 1: karr%(1,3) = 1
IF inmode$(3) = "satellite" THEN karr%(2,1) = 1: karr%(2,2) = 1: karr%(2,3) = 1
IF inmode$(3) = "time" THEN karr%(3,1) = 1: karr%(3,2) = 1: karr%(3,3) = 1
```

```
krR% = karr%(1,1): ksR% = karr%(1,2): ktR% = karr%(1,3)
krS% = karr%(2,1): ksS% = karr%(2,2): ktS% = karr%(2,3)
krT% = karr%(3,1): ksT% = karr%(3,2): ktT% = karr%(3,3)
```

```
PRINT #1, "Differencing sequence array: for R S T"
PRINT #1, " between-receiver differencing: ";krR%;ksR%;ktR%
PRINT #1, " between-satellite differencing: ";krS%;ksS%;ktS%
PRINT #1, " between-epoch differencing: ";krT%;ksT%;ktT%
```

```
RETURN
```

```
'-----
```

GPS DESIGN

betweenReceiver:

'build GPS between-receiver differencing transformation matrix R

maxR% = R% - krR%

maxS% = S% - ksR%

maxT% = T% - ktR%

rownextD% = maxR% * maxS% * maxT%

colnextD% = R% * maxS% * maxT%

FOR i% = 1 TO rownextD%

FOR j% = 1 TO colnextD%

nextD%(i%,j%) = 0

NEXT j%

NEXT i%

FOR it% = 1 TO maxT%

FOR is% = 1 TO maxS%

FOR ir% = 1 TO maxR%

factor% = (is%-1) + (it%-1)*maxS%

i% = ir% + factor% * maxR%

jp% = ir% + factor% * R%

jn% = R% + factor% * R% 'fixed rcvr diffs

IF seqR\$ = "seq" THEN jn% = 1 + ir% + factor% * R% 'seq rcvr diffs

nextD%(i%,jp%) = 1

nextD%(i%,jn%) = -1

NEXT ir%

NEXT is%

NEXT it%

PRINT #1,

PRINT #1, "between-receiver differencing operator: Dr(";rownextD%;";";colnextD%;")"

GOSUB nextDprinter

RETURN

'-----

betweenSatellite:

'build GPS between-satellite differencing transformation matrix S

maxR% = R%-krS%

maxS% = S%-ksS%

maxT% = T%-ktS%

rownextD% = maxR%*maxS%*maxT%

colnextD% = maxR%* S% *maxT%

FOR i% = 1 TO rownextD%

FOR j% = 1 TO colnextD%

nextD%(i%,j%) = 0

NEXT j%

NEXT i%

FOR it% = 1 TO maxT%

FOR is% = 1 TO maxS%

FOR ir% = 1 TO maxR%

```

    i% = ir% + ((is%-1)+(it%-1)*maxS%)*maxR%
    jp% = ir% + ((is%-1)+(it%-1)* S%)*maxR%
    jn% = ir% + ((S%-1)+(it%-1)* S% )*maxR%
IF seqS$ = "seq" THEN jn% = ir% + (is% + (it%-1)* S% )*maxR%
    nextD%(i%,jp%) = 1
    nextD%(i%,jn%) = -1
NEXT ir%
NEXT is%
NEXT it%

PRINT #1,
PRINT #1, "between-satellite differencing operator: Ds(";rownextD%;";";colnextD%;")"
GOSUB nextDprinter

RETURN

'-----
betweenTime:
'build GPS between-epoch differencing transformation matrix TT
    maxR% = R%-krT%
    maxS% = S%-ksT%
    maxT% = T%-ktT%
    rownextD% = maxR%*maxS%*maxT%
    colnextD% = maxR%*maxS%* T%

FOR i% = 1 TO rownextD%
FOR j% = 1 TO colnextD%
    nextD%(i%,j%) = 0
NEXT j%
NEXT i%

FOR it% = 1 TO maxT%
FOR is% = 1 TO maxS%
FOR ir% = 1 TO maxR%
    factor1% = ir% + (is% - 1) * maxR%
    factor2% = maxR% * maxS%
    i% = factor1% + (it%-1)*factor2%
    jp% = factor1% + (it%-1)*factor2%
    jn% = factor1% + ( T%-1)*factor2%
IF seqT$ = "seq" THEN jn% = factor1% + it% *factor2%
    nextD%(i%,jp%) = 1
    nextD%(i%,jn%) = -1
NEXT ir%
NEXT is%
NEXT it%

PRINT #1,
PRINT #1, "between-epoch differencing operator: Dt(";rownextD%;";";colnextD%;")"
GOSUB nextDprinter

RETURN

```

GPS DESIGN

```
'-----  
productD:  
  'form totalD = nextD * totalD  
  IF colnextD% <> rowtotalD% THEN GOTO error1  
  rowtotalD% = rownextD%  
  
  FOR i = 1 TO rowtotalD%  
  FOR j = 1 TO coltotalD%  
    tempD%(i,j) = 0  
  FOR k = 1 TO colnextD%  
    tempD%(i,j) = tempD%(i,j) + nextD%(i,k)*totalD%(k,j)  
  NEXT k  
  NEXT j  
  NEXT i  
  
  FOR i = 1 TO rowtotalD%  
  FOR j = 1 TO coltotalD%  
    totalD%(i,j) = tempD%(i,j)  
  NEXT j  
  NEXT i  
  
PRINT #1,  
PRINT #1, "total differencing operator: D(";rowtotalD%;",";coltotalD%;")"  
GOSUB totalDprinter  
  
RETURN  
  
error1:  
  PRINT #1, "nonconformable matrices";colnextD%;rowtotalD%  
STOP  
  
'-----  
covariance:  
  'form COV = totalD * totalD(T)  
  n% = rowtotalD%  
  FOR i = 1 TO n%  
  FOR j = 1 TO i  
    COV(i,j) = 0  
  FOR k = 1 TO coltotalD%  
    COV(i,j) = COV(i,j) + totalD%(i,k)*totalD%(j,k)  
  NEXT k  
  COV(j,i) = COV(i,j)  
  NEXT j  
  NEXT i  
  
PRINT #1,  
PRINT #1, "covariance: D*DT(";n%;",";n%;")"  
GOSUB COVprinter  
  
RETURN  
  
'-----
```

```

inverse:
' matrix inversion for symmetric positive-definite matrices
'   Matrix to be inverted is upper n% x n% portion of COV
'   COV is dimensioned mm% x mm% in calling routine
' parameters (I = input, O = output)
' I/O COV = matrix to be inverted (dim = mm% x mm%)
'   on output, upper left n% x n% submatrix contains inverse
'   of input upper left n% x n% submatrix
' I n% = dimension of actual part (upper left corner) of COV to be inverted (n% < mm%)
' I mm% = dimensioned size of COV in calling routine
' O DET = non-exponent portion of determinant of input n% x n% matrix
' O idexp% = exponent (of 10) part of the determinant.
'   The full determinant = DET * 10 ^ idexp%

test = 9.999999999999999D-21
DET = 0#

' Choleski decomposition COV = U' U
' u(i,i) = sqr[COV(i,i) - sum{k=1,i-1} u(k,i)*u(k,i)]
' u(i,j>i) = [COV(i,j) - sum{k=1,i-1} u(k,i)*u(k,j)] / u(i,i)
' det(COV) = det(U)^2    det(U) = prod{i=1,n} u(i,i)

FOR j% = 1 TO n%
FOR i% = 1 TO j%
  IF i% = 1 THEN GOTO firstrow
  sum1 = 0#
  FOR k% = 1 TO i%-1
    sum1 = sum1 + COV(k%,i%) * COV(k%,j%)
  NEXT k%
  COV(i%,j%) = COV(i%,j%) - sum1
firstrow:
  IF i% <> j% THEN COV(i%,j%) = COV(i%,j%) / COV(i%,i%): GOTO firstnex
  DET = DET + LOG(COV(i%,i%))/LOG(10#)
  COV(i%,i%) = SQR (COV(i%,i%))
firstnex:
  NEXT i%
NEXT j%

  idexp% = DET
  rpart = DET - idexp%
  IF ABS(rpart) < test THEN DET = 1# ELSE DET = 10# ^ rpart

' inverse B = U(-1)
' b(i,i) = 1 / u(i,i)
' b(i,j>i) = - [sum{k=i,j-1} b(i,k)*u(k,j)] / u(j,j)

FOR j% = 1 TO n%
FOR i% = 1 TO j%
  IF i% => j% THEN COV(j%,j%) = 1# / COV(j%,j%): GOTO nexti
  sum2 = 0#
  FOR k% = i% TO j%-1
    sum2 = sum2 + COV(i%,k%) * COV(k%,j%)

```


GPS DESIGN

```

NEXT k%
  COV(i%,j%) = - sum2 / COV(j%,j%)
nexti:
  NEXT i%
  NEXT j%

      ' inverse P = COV(-1)
      ' p(i,j=>i) = sum{k=j,n} b(i,k)*b(j,k)
      ' p(i,j<i) = p(j,i)

FOR j% = 1 TO n%
FOR i% = 1 TO j%
  sum3 = 0#
FOR k% = j% TO n%
  sum3 = sum3 + COV(i%,k%) * COV(j%,k%)
NEXT k%
  COV(i%,j%) = sum3
IF i% <> j% THEN COV(j%,i%) = sum3
NEXT i%
NEXT j%

PRINT #1,
PRINT #1, " inverse of D*DT(",n%,";",n%,"), multiplied by ";RST%
GOSUB inverseprinter

RETURN

'-----
kernel:
'compute Ker = totalD(T) * (totalD * totalD(T))(inverse) * totalD
  FOR i = 1 TO coltotalD%
  FOR j = 1 TO coltotalD%
    ker(i,j) = 0
  FOR k = 1 TO rowtotalD%
  FOR l = 1 TO rowtotalD%
    ker(i,j) = ker(i,j) + totalD%(k,i)*COV(k,l)*totalD%(l,j)
  NEXT l
  NEXT k
  NEXT j
  NEXT i

PRINT #1,
PRINT #1, " DT* inv (D*DT) * D(",RST%,";",RST%,"), multiplied by ";RST%
GOSUB Kernelprinter

RETURN

'-----
nextDprinter:
  FOR i% = 1 TO rownextD%
  FOR j% = 1 TO colnextD%-1
PRINT #1, USING "###";nextD%(i%,j%);

```

```

NEXT j%
PRINT #1, USING "###";nextD%(i%,colnextD%)
NEXT i%
RETURN

'-----
totalDprinter:
FOR i% = 1 TO rowtotalD%
FOR j% = 1 TO coltotalD%-1
PRINT #1, USING "###";totalD%(i%,j%);
NEXT j%
PRINT #1, USING "###";totalD%(i%,coltotalD%)
NEXT i%
RETURN

'-----
COVprinter:
FOR i% = 1 TO n%
FOR j% = 1 TO n%-1
PRINT #1, USING "###";COV(i%,j%);
NEXT j%
PRINT #1, USING "###";COV(i%,n%)
NEXT i%
RETURN

'-----
inverseprinter:
FOR i% = 1 TO n%
FOR j% = 1 TO n%-1
QQ = COV(i%,j%) * RST%
CALL ratio(QQ,ia%,ib%)
PRINT #1, USING "###";ia%;
IF ib% <> 1 THEN PRINT #1, "/";
IF ib% <> 1 THEN PRINT #1, USING "##";ib%;
NEXT j%
QQ = COV(i%,n%) * RST%
CALL ratio(QQ,ia%,ib%)
IF ib% <> 1 THEN PRINT #1, USING "###";ia%;
IF ib% <> 1 THEN PRINT #1, "/";
IF ib% <> 1 THEN PRINT #1, USING "##";ib%
IF ib% = 1 THEN PRINT #1, USING "###";ia%
NEXT i%

nomatrix:
FullDet = DET * 10# ^ idexp%
PRINT #1, "det of D*DT = ";
PRINT #1, USING "#####.##";FullDet

RETURN

'-----
Kernelprinter:

```

GPS DESIGN

```

    FOR i% = 1 TO RST%
    FOR j% = 1 TO RST%-1
        QQ = ker(i%,j%) * RST%
    CALL ratio(QQ,ia%,ib%)
PRINT #1, USING "###";ia%;
    IF ib% <> 1 THEN PRINT#1, "/";
    IF ib% <> 1 THEN PRINT #1, USING "###";ib%;
    NEXT j%
        QQ = ker(i%,RST%) * RST%
    CALL ratio(QQ,ia%,ib%)
    IF ib% <> 1 THEN PRINT #1, USING "###";ia%;
    IF ib% <> 1 THEN PRINT #1, "/";
    IF ib% <> 1 THEN PRINT #1, USING "###";ib%
    IF ib% = 1 THEN PRINT #1, USING "###";ia%
    NEXT i%
RETURN

'-----
endofmainsub:
    END SUB

'-----
SUB ratio(x,ia%,ib%) STATIC
    ' given x, find ia%/ib% = x
    DEFDBL A-z
    eps = .0000001#
    ia% = 0
    ib% = 1
tryagain:
    test = x * ib% - ia%
    IF ABS(test) < eps THEN EXIT SUB
    IF test * x > 0 THEN ia% = ia% + 1 * SGN(x)
    IF test * x < 0 THEN ib% = ib% + 1
    GOTO tryagain

END SUB

'-----
SUB OKWindow STATIC
    WINDOW 3,"", (485,320)-(505,335),2
    TEXTMODE 1
    BUTTON 1,1,"OK", (0,0)-(20,15),1
    DialogActive = 1
    WHILE DialogActive
        EventType = DIALOG(0)
        IF EventType = 1 THEN GOTO endsub
    WEND
    endsub:
    WINDOW CLOSE 3
END SUB

```

Appendix D.3

'GPS UNDIFFERENCED DESIGN (A₂ matrix) - D. Wells 85-08-17

```
DEFDBL A-z
  device$ = "scm:"
  device$ = "lpt1:direct"
  'device$ = "GPSDesign:A2Output"
  OPEN device$ FOR OUTPUT AS #1
  starttime = TIMER
```

'select which quantum number is to be fast for each bias parameter

```
afast$ = "t"
'afast$ = "g"
```

```
bfast$ = "t"
'bfast$ = "s"
```

```
Nfast$ = "g"
'Nfast$ = "s"
```

'between receiver and satellite (alpha/beta) double difference example

```
R% = 3
'S% = 6
'T% = 1
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,1,0,0,0,R%,S%)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,0,0,1,0,R%,S%)
```

'triple difference example

```
R% = 2
S% = 3
T% = 4
CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,1,1,0,0,1,0)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,1,1,0,1,0,0)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,0,0,1,1,0,1)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,0,1,1,1,0,0)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,1,0,0,0,1,1)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,0,0,1,0,1,1)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,1,0,0,1,0,1)
'CALL A2Design (R%,S%,T%,afast$,bfast$,Nfast$,0,1,1,0,1,0)
```

endstuff:

```
  stoptime = TIMER
  elapsedtime = stoptime - starttime
  PRINT #1,
  PRINT #1, "elapsed secs=";elapsedtime
  CLOSE #1
  CALL OKWindow
```

END

GPS DESIGN

```

-----
'comments on A2Design arguments

' A2 rank defect design parameters

'kra% = number of known receiver clocks
'kta% = number of receiver clock sync epochs

'ksb% = number of known sat clocks
'ktb% = number of sat clock sync epochs

'krN% = number of rcvrs with known ambig TO ALL sats
'ksN% = number of sats with known ambig TO ALL rcvrs

'How to select values for rank defect parameters

'biases est      bias unknowns   kra  kta   ksb  ktb   krN  ksN
-----
'alpha           RT                0   0     S   T     R   S   single differences
'beta            ST                R   T     0   0     R   S
'N               RS                R   T     S   T     0   0

'alpha,beta      RT+ST-T          1   0     0   0     R   S   double differences
'               0   0     1   0     R   S
'alpha,N         RT+RS-R          0   1     S   T     0   0
'               0   0     S   T     0   1
'beta,N          ST+RS-S          R   T     0   1     0   0
'               R   T     0   0     1   0

'alpha,beta,N    RT+ST+RS         1   1     0   0     1   0   triple differences
'               -(R+S+T-1)  1   1     0   1     0   0
'               0   0     1   1     0   1
'               0   1     1   1     0   0
'               1   0     0   0     1   1
'               0   0     1   0     1   1
'               1   0     0   1     0   1 (+1)
'               0   1     1   0     1   0 (+1)

-----
SUB A2Design (R%,S%,T%,afast$,bfast$,Nfast$,kra%,kta%,ksb%,ktb%,krN%,ksN% )
STATIC

DEFDBL A-z

DIM A2%(100,100)
DIM BB(100,100)
DIM CC(100,100)
mm% = 100

'total number of observations
RST% = R% * S% * T%

```

```

'total number of bias unknowns for undifferenced approach (A2)
n% = (R%-kra%)*(T%-kta%) + (S%-ksb%)*(T%-ktb%) + (R%-krN%)*(S%-ksN%)

PRINT #1, "GPS Undifferenced Design (A2 matrix)"
PRINT #1, "Design (#rcvrs, #sats, #epochs) = ";R%;S%;T%
PRINT #1, "Total #obs =";RST%;" Total #undifferenced bias unknowns =";n%
PRINT #1, "A2: Rank defect parameters for alpha: ";kra%;" ";kta%
PRINT #1, "          for beta: ";ksb%;ktb%
PRINT #1, "          for N: ";krN%;ksN%
PRINT #1, "fast quantum number for alpha unknowns = ";afast$
PRINT #1, "          beta unknowns = ";bfast$
PRINT #1, "          N unknowns = ";Nfast$

    begintime = TIMER
    GOSUB design
    endtime = TIMER
    timetaken = endtime - begintime
    PRINT #1, "calculated in ";timetaken;" seconds"
    begintime = TIMER
    GOSUB normal
    endtime = TIMER
    timetaken = endtime - begintime
    PRINT #1, "calculated in ";timetaken;" seconds"
    begintime = TIMER
    GOSUB inverse
    endtime = TIMER
    timetaken = endtime - begintime
    PRINT #1, "calculated in ";timetaken;" seconds"
    begintime = TIMER
    GOSUB kernel
    endtime = TIMER
    timetaken = endtime - begintime
    PRINT #1, "calculated in ";timetaken;" seconds"
EXIT SUB

'-----
design:
'build GPS design matrix elements for biases
  FOR i% = 1 TO RST%
    FOR j% = 1 TO n%
      A2%(i%,j%) = 0
    NEXT j%
  NEXT i%

  betaOffset = (R%-kra%)*(T%-kta%)
  NOffset = betaOffset + (S%-ksb%)*(T%-ktb%)

  FOR it% = 1 TO T%
    FOR is% = 1 TO S%
      FOR ig% = 1 TO R%
        i% = ig% + (is%-1)*R% + (it%-1)*R%*S%
        'alpha bias

```

GPS DESIGN

```

ja% = ig% + (it%-1)*(R%-kra%) 't slow, g fast
IF afast$ = "t" THEN ja% = it% + (ig%-1)*(T%-kta%) 'g slow, t fast
IF ig% =< (R%-kra%) AND it% =< (T%-kta%) THEN A2%(i%,ja%) = 1
'beta bias
jb% = betaOffset + is% + (it%-1)*(S%-ksb%) 't slow, s fast
IF bfast$ = "t" THEN jb% = betaOffset + it% + (is%-1)*(T%-ktb%) 's slow, t fast
IF is% =< (S%-ksb%) AND it% =< (T%-ktb%) THEN A2%(i%,jb%) = 1
'N bias
jN% = NOffset + ig% + (is%-1)*(R%-krN%) 's slow, g fast
IF Nfast$ = "s" THEN jN% = NOffset + is% + (ig%-1)*(S%-ksN%) 'g slow, s fast
IF ig% =< (R%-krN%) AND is% =< (S%-ksN%) THEN A2%(i%,jN%) = 1
NEXT ig%
NEXT is%
NEXT it%

```

```

PRINT #1,
PRINT #1, "Undifferenced bias design matrix: A2(";RST%";";n%;"")"
GOSUB A2printer

```

RETURN

```

'-----
normal:
'form N22 = A2' A2
FOR j = 1 TO n%
FOR i = 1 TO j
BB(i,j) = 0
FOR k = 1 TO RST%
BB(i,j) = BB(i,j) + A2%(k,i)*A2%(k,j)
NEXT k
BB(j,i) = BB(i,j)
NEXT i
NEXT j

```

```

PRINT #1,
PRINT #1, "Normal equation matrix bias partition: A2T*A2=N22(";n%";";n%;"")"
GOSUB BBprinter

```

RETURN

```

'-----
inverse:
'matrix inversion for symmetric positive-definite matrices
' Matrix to be inverted is upper n% x n% portion of BB
' BB is dimensioned mm% x mm% in calling routine
'parameters (I = input, O = output)
' I/O BB = matrix to be inverted (dim = mm% x mm%)
' on output, upper left n% x n% submatrix contains inverse
' of input upper left n% x n% submatrix
' I n% = dimension of actual part (upper left corner) of BB to be inverted (n% < mm%)
' I mm% = dimensioned size of BB in calling routine
' O DET = non-exponent portion of determinant of input n% x n% matrix

```

```

' O idexp% = exponent (of 10) part of the determinant.
'   The full determinant = DET * 10 ^ idexp%

test = 9.9999999999999999D-21
DET = 0#

' Choleski decomposition N22 = U' U
' u(i,i) = sqr[N22(i,i) - sum{k=1,i-1} u(k,i)*u(k,i)]
' u(i,j>i) = [N22(i,j) - sum{k=1,i-1} u(k,i)*u(k,j)] / u(i,i)
' det(N22) = det(U)^2    det(U) = prod{i=1,n} u(i,i)

FOR j% = 1 TO n%
FOR i% = 1 TO j%
  IF i% = 1 THEN GOTO firstrow
  sum1 = 0#
FOR k% = 1 TO i%-1
  sum1 = sum1 + BB(k%,i%) * BB(k%,j%)
NEXT k%
  BB(i%,j%) = BB(i%,j%) - sum1
firstrow:
  IF i% <> j% THEN BB(i%,j%) = BB(i%,j%) / BB(i%,i%): GOTO firstnex
  DET = DET + LOG(BB(i%,i%))/LOG(10#)
  BB(i%,i%) = SQR (BB(i%,i%))
firstnex:
  NEXT i%
  NEXT j%

  idexp% = DET
  rpart = DET - idexp%
  IF ABS(rpart) < test THEN DET = 1# ELSE DET = 10# ^ rpart

  FullDet = DET * 10# ^ idexp%
  PRINT #1, "det of N22 = ";
  PRINT #1, USING "#####.##";FullDet

' inverse B = U(-1)
' b(i,i) = 1 / u(i,i)
' b(i,j>i) = - [sum{k=i,j-1} b(i,k)*u(k,j)] / u(j,j)

FOR j% = 1 TO n%
FOR i% = 1 TO j%
  IF i% => j% THEN BB(j%,j%) = 1# / BB(j%,j%): GOTO nexti
  sum2 = 0#
FOR k% = i% TO j%-1
  sum2 = sum2 + BB(i%,k%) * BB(k%,j%)
NEXT k%
  BB(i%,j%) = - sum2 / BB(j%,j%)
  BB(j%,i%) = 0
nexti:
  NEXT i%
  NEXT j%

```


GPS DESIGN

```
'PRINT #1,
'PRINT #1, "Choleski root of N22=A2T*A2(";n%";";n%") multiplied by ";RST%
'GOSUB rootprinter
```

```
' inverse P = N22(-1)
' p(i,j=>i) = sum{k=j,n} b(i,k)*b(j,k)
' p(i,j<i) = p(j,i)
```

```
FOR j% = 1 TO n%
FOR i% = 1 TO j%
    sum3 = 0#
FOR k% = j% TO n%
    sum3 = sum3 + BB(i%,k%) * BB(j%,k%)
NEXT k%
CC(i%,j%) = sum3
IF i% <> j% THEN CC(j%,i%) = sum3
NEXT i%
NEXT j%
```

```
PRINT #1,
PRINT #1, "inverse of N22=A2T*A2(";n%";";n%";"), multiplied by";RST%
GOSUB CCprinter
```

RETURN

```
'-----
kernel:
'compute Ker = CC * CCT, where CC = A2 * Root
FOR i = 1 TO RST%
FOR j = 1 TO n%
    CC(i,j) = 0
FOR k = 1 TO j
    CC(i,j) = CC(i,j) + A2%(i,k)*BB(k,j)
NEXT k
NEXT j
NEXT i
```

```
FOR i = 1 TO RST%
FOR j = 1 TO i
    BB(i,j) = 0
FOR k = 1 TO n%
    BB(i,j) = BB(i,j) + CC(i,k)*CC(j,k)
NEXT k
IF i <> j THEN BB(j,i) = BB(i,j)
NEXT j
NEXT i
```

```
PRINT #1,
PRINT #1, " A2 * inv N22 * A2T (";RST%";";RST%";"), multiplied by";RST%
GOSUB Kernelprinter
```

RETURN

```

'-----
A2printer:
  FOR i% = 1 TO RST%
  FOR j% = 1 TO n%-1
  PRINT #1, USING "###";A2%(i%,j%);
  NEXT j%
  PRINT #1, USING "###";A2%(i%,n%)
  NEXT i%
RETURN

'-----
BBprinter:
  FOR i% = 1 TO n%
  FOR j% = 1 TO n%-1
  PRINT #1, USING "###";BB(i%,j%);
  NEXT j%
  PRINT #1, USING "###";BB(i%,n%)
  NEXT i%
RETURN

'-----
CCprinter:
  FOR i% = 1 TO n%
  FOR j% = 1 TO n%-1
  QQ = CC(i%,j%) * RST%
  CALL ratio (QQ,ia%,ib%)
  PRINT #1, USING "###";ia%;
  IF ib% <> 1 THEN PRINT#1, "/";
  IF ib% <> 1 THEN PRINT #1, USING "##";ib%;
  NEXT j%
  QQ = CC(i%,n%) * RST%
  CALL ratio (QQ,ia%,ib%)
  IF ib% <> 1 THEN PRINT #1, USING "###";ia%;
  IF ib% <> 1 THEN PRINT#1, "/";
  IF ib% <> 1 THEN PRINT #1, USING "##";ib%
  IF ib% = 1 THEN PRINT #1, USING "###";ia%
  NEXT i%
RETURN

'-----
rootprinter:
  FOR i% = 1 TO n%
  FOR j% = 1 TO n%-1
  QQ = BB(i%,j%) * RST%
  PRINT #1, USING "###.#####";QQ;
  NEXT j%
  QQ = BB(i%,n%) * RST%
  PRINT #1, USING "###.#####";QQ
  NEXT i%
RETURN

```

GPS DESIGN

```
'-----
Kernelprinter:
  FOR i% = 1 TO RST%
    FOR j% = 1 TO RST% -1
      QQ = BB(i%,j%) * RST%
      CALL ratio (QQ,ia%,ib%)
    PRINT #1, USING "###";ia%;
      IF ib% <> 1 THEN PRINT#1, "/";
      IF ib% <> 1 THEN PRINT #1, USING "###";ib%;
    NEXT j%
      QQ = BB(i%,RST%) * RST%
      CALL ratio (QQ,ia%,ib%)
      IF ib% <> 1 THEN PRINT #1, USING "###";ia%;
      IF ib% <> 1 THEN PRINT #1, "/";
      IF ib% <> 1 THEN PRINT #1, USING "###";ib%
      IF ib% = 1 THEN PRINT #1, USING "###";ia%
    NEXT i%
RETURN

'-----
endofmainsub:
  END SUB

'-----
SUB ratio(x,ia%,ib%) STATIC
  'given x, find ia%/ib% = x
  DEFDBL A-z
  eps = .0000001#
  ia% = 0
  ib% = 1
tryagain:
  test = x * ib% - ia%
  IF ABS(test) < eps THEN EXIT SUB
  IF test * x > 0 THEN ia% = ia% + 1 * SGN(x)
  IF test * x < 0 THEN ib% = ib% + 1
  GOTO tryagain
END SUB

'-----
SUB OKWindow STATIC
  WINDOW 3,"", (485,320)-(505,335),2
  TEXTMODE 1
  BUTTON 1,1,"OK", (0,0)-(20,15),1
  DialogActive = 1
  WHILE DialogActive
    EventType = DIALOG(0)
    IF EventType = 1 THEN GOTO endsub
  WEND
  endsub:
  WINDOW CLOSE 3
END SUB
```

Appendix D.4

'GPS ANNIHILATION OF A₂ BY D — D. Wells 85-08-17

DEFDBL A-z

```

device$ = "scrn:"
device$ = "lpt1:direct"
'device$ = "GPSDesign:AnnOutput"
OPEN device$ FOR OUTPUT AS #1
starttime = TIMER

```

R% = 3

S% = 6

T% = 1

' D design parameters

```

inmode1$ = "receiver"
inmode2$ = "satellite"
inmode3$ = "none"

```

```

seqR$ = "fix"
seqS$ = "fix"
seqT$ = "fix"

```

' A2 rank defect design parameters

'biases est	bias unknowns	kra	kta	ksb	ktb	krN	ksN	
'alpha	RT	0	0	S	T	R	S	single differences
'beta	ST	R	T	0	0	R	S	
'N	RS	R	T	S	T	0	0	
'alpha,beta	RT+ST-T	1	0	0	0	R	S	double differences
'alpha,N	RT+RS-R	0	1	S	T	0	0	
'beta,N	ST+RS-S	0	0	S	T	0	1	
		R	T	0	0	1	0	
'alpha,beta,N	RT+ST+RS	1	1	0	0	1	0	triple differences
	-(R+S+T-1)	1	1	0	1	0	0	
		0	0	1	1	0	1	
		0	1	1	1	0	0	
		1	0	0	0	1	1	
		0	0	1	0	1	1	
		1	0	0	1	0	1 (+1)	
		0	1	1	0	1	0 (+1)	

GPS DESIGN

kra% = 1 'number of known receiver clocks
 kta% = 0 'number of receiver clock sync epochs

ksb% = 0 'number of known sat clocks
 ktb% = 0 'number of sat clock sync epochs

krN% = R% 'number of rcvrs with known ambig to all sats
 ksN% = S% 'number of sats with known ambig to all rcvrs

CALL

Danda2(R%,S%,T%,inmode1\$,inmode2\$,inmode3\$,seqR\$,seqS\$,seqT\$,kra%,kta%,ksb%,
 ktb%,krN%,ksN%)

```

      stoptime = TIMER
      elapsedtime = stoptime - starttime
PRINT #1,
PRINT #1, "elapsed secs=";elapsedtime
      CLOSE #1
      CALL OKWindow
END

```

'-----
 SUB Danda2
 (R%,S%,T%,inmode1\$,inmode2\$,inmode3\$,seqR\$,seqS\$,seqT\$,kra%,kta%,ksb%,ktb%,kr
 N%,ksN%) STATIC

DEFDBL A-z

```

      DIM A2%(100,100)
      DIM totalD%(100,100)
      DIM nextD%(100,100)
      DIM tempD%(100,100)
      DIM inmode$(3)
      DIM karr%(3,3)
      mm% = 100

```

'total number of observations
 RST% = R% * S% * T%

'total number of unknowns for undifferenced approach (A2)
 $n\% = (R\% - kra\%)*(T\% - kta\%) + (S\% - ksb\%)*(T\% - ktb\%) + (R\% - krN\%)*(S\% - ksN\%)$

'differencing modes
 inmode\$(1) = inmode1\$
 inmode\$(2) = inmode2\$
 inmode\$(3) = inmode3\$

```

PRINT #1, "GPS Annihilation of A2 by D"
PRINT #1, "Design (#rcvrs, #sats, #epochs) = ";R%;S%;T%
PRINT #1, "Total #obs =";RST%;"      Total #undifferenced bias unknowns =";n%
PRINT #1,
PRINT #1, "A2: Rank defect parameters for alpha: ";kra%;" ";kta%

```

```

PRINT #1, "                for beta: ";ksb%;ktb%
PRINT #1, "                for N:   ";krN%;ksN%
PRINT #1,
PRINT #1, "D: Differencing references are: ";seqR$;" rec ref, ";seqS$;" sat ref, ";seqT$;" time
ref"
PRINT #1, "Differencing between ";inmode$(1);", then ";inmode$(2);", then ";inmode$(3)

```

```
GOSUB initialk
```

```
'compute A2
GOSUB design
```

```
'compute D
GOSUB initialD
```

```

FOR idiff% = 1 TO 3
  IF inmode$(idiff%) = "none" THEN GOTO nomorediffs
  IF inmode$(idiff%) = "receiver" THEN GOSUB betweenReceiver
  IF inmode$(idiff%) = "satellite" THEN GOSUB betweenSatellite
  IF inmode$(idiff%) = "time" THEN GOSUB betweenTime
GOSUB productD
NEXT idiff%

```

```

nomorediffs:
' try out annihilation
GOSUB annihil
EXIT SUB

```

```
'-----
initialk:
```

```
'set up k array for inmode$ sequence
'this sets up the following 15 cases
```

'inmode	kxR%			kxS%			kxT%			
' 1 2 3	r	s	t	r	s	t	r	s	t	
' r - -	1	0	0	0	0	0	0	0	0	single differencing modes
' s - -	0	0	0	0	1	0	0	0	0	
' t - -	0	0	0	0	0	0	0	0	1	
' r s -	1	0	0	1	1	0	0	0	0	double differencing modes
' s r -	1	1	0	0	1	0	0	0	0	
' r t -	1	0	0	0	0	0	1	0	1	
' t r -	1	0	1	0	0	0	0	0	1	
' s t -	0	0	0	0	1	0	0	1	1	
' t s -	0	0	0	0	1	1	0	0	1	
' r s t	1	0	0	1	1	0	1	1	1	triple differencing sequences
' r t s	1	0	0	1	1	1	1	0	1	
' s r t	1	1	0	0	1	0	1	1	1	
' t r s	1	0	1	1	1	1	0	0	1	
' s t r	1	1	1	0	1	0	0	1	1	
' t s r	1	1	1	0	1	1	0	0	1	

GPS DESIGN

```

FOR i = 1 TO 3
FOR j = 1 TO 3
    karr%(i,j) = 0
NEXT j
NEXT i

IF inmode$(1) = "receiver" THEN karr%(1,1) = 1: col2% = 1
IF inmode$(1) = "satellite" THEN karr%(2,2) = 1: col2% = 2
IF inmode$(1) = "time" THEN karr%(3,3) = 1: col2% = 3

IF inmode$(2) = "receiver" THEN karr%(1,1) = 1: karr%(1,col2%) = 1
IF inmode$(2) = "satellite" THEN karr%(2,2) = 1: karr%(2,col2%) = 1
IF inmode$(2) = "time" THEN karr%(3,3) = 1: karr%(3,col2%) = 1

IF inmode$(3) = "receiver" THEN karr%(1,1) = 1: karr%(1,2) = 1: karr%(1,3) = 1
IF inmode$(3) = "satellite" THEN karr%(2,1) = 1: karr%(2,2) = 1: karr%(2,3) = 1
IF inmode$(3) = "time" THEN karr%(3,1) = 1: karr%(3,2) = 1: karr%(3,3) = 1

krR% = karr%(1,1):  ksR% = karr%(1,2):  ktR% = karr%(1,3)
krS% = karr%(2,1):  ksS% = karr%(2,2):  ktS% = karr%(2,3)
krT% = karr%(3,1):  ksT% = karr%(3,2):  ktT% = karr%(3,3)

```

```

PRINT #1, "Differencing sequence array: for R S T"
PRINT #1, " between-receiver differencing: ";krR%;ksR%;ktR%
PRINT #1, " between-satellite differencing: ";krS%;ksS%;ktS%
PRINT #1, " between-epoch differencing: ";krT%;ksT%;ktT%

```

RETURN

'-----
design:

'build GPS design matrix elements for biases

```

FOR i% = 1 TO RST%
FOR j% = 1 TO n%
    A2%(i%,j%) = 0
NEXT j%
NEXT i%

```

```

betaOffset = (R%-kra%)*(T%-kta%)
NOffset = betaOffset + (S%-ksb%)*(T%-ktb%)

```

```

FOR it% = 1 TO T%
FOR is% = 1 TO S%
FOR ig% = 1 TO R%
    i% = ig% + (is%-1)*R% + (it%-1)*R%*S%
    'alpha bias
    ja% = ig% + (it%-1)*(R%-kra%) 't slow, g fast
    ja% = it% + (ig%-1)*(T%-kta%) 'g slow, t fast
    IF ig% =< (R%-kra%) AND it% =< (T%-kta%) THEN A2%(i%,ja%) = 1
    'beta bias
    jb% = betaOffset + is% + (it%-1)*(S%-ksb%) 't slow, s fast

```

```

jb% = betaOffset + it% + (is%-1)*(T%-ktb%) 's slow, t fast
IF is% =< (S%-ksb%) AND it% =< (T%-ktb%) THEN A2%(i%,jb%) = 1
  'N bias
jN% = NOffset + ig% + (is%-1)*(R%-krN%) 's slow, g fast
jN% = NOffset + is% + (ig%-1)*(S%-ksN%) 'g slow, s fast
IF ig% =< (R%-krN%) AND is% =< (S%-ksN%) THEN A2%(i%,jN%) = 1
NEXT ig%
NEXT is%
NEXT it%

PRINT #1,
PRINT #1, "Undifferenced bias design matrix: A2(";RST%;";";n%;")"
GOSUB A2printer

RETURN
'-----
initialD:
' set totalD = Identity(dim=rst) to start
  rowtotalD% = RST%
  coltotalD% = RST%
  FOR i% = 1 TO rowtotalD%
  FOR j% = 1 TO coltotalD%
    totalD%(i%,j%) = 0
  NEXT j%
  totalD%(i%,i%) = 1
  NEXT i%

RETURN
'-----
betweenReceiver:
'build GPS between-receiver differencing transformation matrix R
  maxR% = R% - krR%
  maxS% = S% - ksR%
  maxT% = T% - ktR%
  rownextD% = maxR% * maxS% * maxT%
  colnextD% = R% * maxS% * maxT%

  FOR i% = 1 TO rownextD%
  FOR j% = 1 TO colnextD%
    nextD%(i%,j%) = 0
  NEXT j%
  NEXT i%

  FOR it% = 1 TO maxT%
  FOR is% = 1 TO maxS%
  FOR ir% = 1 TO maxR%
    factor% = (is%-1) + (it%-1)*maxS%
    i% = ir% + factor% * maxR%
    jp% = ir% + factor% * R%
    jN% = R% + factor% * R%
    IF seqR$ = "seq" THEN jN% = 1 + ir% + factor% * R%

```

'fixed rcvr diffs
'seq rcvr diffs

GPS DESIGN

```

        nextD%(i%,jp%) = 1
        nextD%(i%,jN%) = -1
    NEXT ir%
    NEXT is%
    NEXT it%

PRINT #1,
PRINT #1, "between-receiver differencing operator: Dr(";rownextD%;",";colnextD%;")"
GOSUB nextDprinter

RETURN

'-----
betweenSatellite:
    'build GPS between-satellite differencing transformation matrix S
    maxR% = R%-krS%
    maxS% = S%-ksS%
    maxT% = T%-ktS%
    rownextD% = maxR%*maxS%*maxT%
    colnextD% = maxR%* S% *maxT%

    FOR i% = 1 TO rownextD%
    FOR j% = 1 TO colnextD%
        nextD%(i%,j%) = 0
    NEXT j%
    NEXT i%

    FOR it% = 1 TO maxT%
    FOR is% = 1 TO maxS%
    FOR ir% = 1 TO maxR%
        i% = ir% + ((is%-1)+ (it%-1)*maxS%)*maxR%
        jp% = ir% + ((is%-1)+ (it%-1)* S%)*maxR%
        jN% = ir% + ((S%-1)+ (it%-1)* S% )*maxR%
        IF seqS$ = "seq" THEN jN% = ir% + (is% + (it%-1)* S% )*maxR% 'fixed sat diffs
        nextD%(i%,jp%) = 1
        nextD%(i%,jN%) = -1
    NEXT ir%
    NEXT is%
    NEXT it%

PRINT #1,
PRINT #1, "between-satellite differencing operator: Ds(";rownextD%;",";colnextD%;")"
GOSUB nextDprinter

RETURN

'-----
betweenTime:
    'build GPS between-epoch differencing transformation matrix TT
    maxR% = R%-krT%
    maxS% = S%-ksT%
    maxT% = T%-ktT%
```

```

rownextD% = maxR%*maxS%*maxT%
colnextD% = maxR%*maxS%*  T%

FOR i% = 1 TO rownextD%
FOR j% = 1 TO colnextD%
    nextD%(i%,j%) = 0
NEXT j%
NEXT i%

FOR it% = 1 TO maxT%
FOR is% = 1 TO maxS%
FOR ir% = 1 TO maxR%
    factor1% = ir% + (is% - 1) * maxR%
    factor2% = maxR% * maxS%
    i% = factor1% + (it%-1)*factor2%
    jp% = factor1% + (it%-1)*factor2%
    jN% = factor1% + ( T%-1)*factor2%
    IF seqT$ = "seq" THEN jN% = factor1% + it% *factor2%
    nextD%(i%,jp%) = 1
    nextD%(i%,jN%) = -1
NEXT ir%
NEXT is%
NEXT it%

PRINT #1,
PRINT #1, "between-epoch differencing operator: Dt(";rownextD%;";";colnextD%;")"
GOSUB nextDprinter

RETURN

'-----
productD:
'form totalD = nextD * totalD
    IF colnextD% <> rowtotalD% THEN GOTO error1
    rowtotalD% = rownextD%

    FOR i = 1 TO rowtotalD%
    FOR j = 1 TO coltotalD%
        tempD%(i,j) = 0
    FOR k = 1 TO colnextD%
        tempD%(i,j) = tempD%(i,j) + nextD%(i,k)*totalD%(k,j)
    NEXT k
    NEXT j
    NEXT i

    FOR i = 1 TO rowtotalD%
    FOR j = 1 TO coltotalD%
        totalD%(i,j) = tempD%(i,j)
    NEXT j
    NEXT i

PRINT #1,

```

GPS DESIGN

```

PRINT #1, "total differencing operator: D(";rowtotalD%";";coltotalD%";)"
GOSUB totalDprinter

RETURN

error1:
  PRINT #1, "nonconformable matrices";colnextD%;rowtotalD%
STOP

'-----
annihil:
  'check that D * A2 = 0
PRINT #1,
PRINT #1, "checking that D (";rowtotalD%";";RST%";) * A2(";RST%";";n%";) = null
matrix"

  FailFlag = 0
  FOR i = 1 TO rowtotalD%
  FOR j = 1 TO n%
    DxA2ij% = 0
  FOR k = 1 TO RST%
    DxA2ij% =DxA2ij% + totalD%(i,k)*A2%(k,j)
  NEXT k
  IF DxA2ij% <> 0 THEN FailFlag = 1
  IF DxA2ij% <> 0 THEN PRINT #1, "row ";i;" of D fails to annihilate col";j;" of A2.
(D*A2)ij =";DxA2ij
  NEXT j
  NEXT i

PRINT #1,
  IF FailFlag = 0 THEN PRINT #1, "D completely annihilates A2"

RETURN

'-----
A2printer:
  FOR i% = 1 TO RST%
  FOR j% = 1 TO n%-1
  PRINT #1, USING "###";A2%(i%,j%);
  NEXT j%
  PRINT #1, USING "###";A2%(i%,n%)
  NEXT i%
RETURN

'-----
nextDprinter:
  FOR i% = 1 TO rownextD%
  FOR j% = 1 TO colnextD%-1
  PRINT #1, USING "###";nextD%(i%,j%);
  NEXT j%
  PRINT #1, USING "###";nextD%(i%,colnextD%)
  NEXT i%

```

RETURN

```
'-----
totalDprinter:
  FOR i% = 1 TO rowtotalD%
    FOR j% = 1 TO coltotalD%-1
      PRINT #1, USING "###";totalD%(i%,j%);
    NEXT j%
  PRINT #1, USING "###";totalD%(i%,coltotalD%)
  NEXT i%
RETURN
```

```
'-----
endofmainsub:
  END SUB
```

```
'-----
SUB OKWindow STATIC
  WINDOW 3,"", (485,320)-(505,335),2
  TEXTMODE 1
  BUTTON 1,1,"OK", (0,0)-(20,15),1
  DialogActive = 1
  WHILE DialogActive
    EventType = DIALOG(0)
    IF EventType = 1 THEN GOTO endsub
  WEND
  endsub:
  WINDOW CLOSE 3
END SUB
```