EVALUATION OF MATHEMATICAL MODELS FOR GYROCOMPASS BEHAVIOUR: ERROR MODELLING AND APPLICATIONS

N. T. CHRISTOU

April 1983

TECHNICAL REPORT
NO. 100

PREFACE

In order to make our extensive series of technical reports more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

EVALUATION OF MATHEMATICAL MODELS FOR GYROCOMPASS BEHAVIOUR: ERROR MODELLING AND APPLICATIONS

Nikolaos T. Christou

Department of Surveying Engineering
University of New Brunswick
P.O. Box 4400
Fredericton, N.B.
Canada
E3B 5A3

April 1983

Latest Reprinting September 1992

PREFACE

This report is an unaltered printing of the author's M.Sc.E. thesis, entitled "Evaluation of Mathematical Models for Gyrocompass Behaviour: Error Modelling and Applications", submitted to this Department in April, 1983.

The principle thesis advisor for this work was Dr. D.E. Wells, whose assistance is highly appreciated. Details of financial support and other assistance rendered are given in the acknowledgements.

ABSTRACT

Heading information is a fundamental parameter in ship's navigation. Traditionally a gyrocompass is used as the primary sensor to provide heading reference on board ship. However, gyrocompass indicated headings are subject to a number of errors, which are functions of the ship's motion and of the latitude of operation.

The objective of this thesis is to investigate the gyrocompass behaviour, study its deviations under different conditions of operation and develop suitable algorithms for the software compensation of these deviations. To meet this objective, mathematical models describing the gyrocompass behaviour are developed using different dynamic considerations. In particular, the gyrocompass equations of motion and their solutions are developed for the cases of a stationary, uniformly moving, and manoeuvering ship. A general discrete-time model as well as a special model to represent a manoeuvering ship are developed. Specific attention is drawn to the problem of high latitude behaviour of the gyrocompass.

Simulation studies of the gyrocompass dynamic response are carried out using the mathematical models developed in this study. The simulation results indicate that transient errors of 1° are expected at latitudes of 30°, while errors in excess of 10° are likely to occur at latitudes beyond 70°. These errors may degrade considerably not only the gyrocompass performance, but also the performance of a multi-sensor integrated navigation system (e.g. introducing as much as 0.5 nautical

miles error in a satellite fix), or they may introduce an error of as much as 2 mgals in real-time Eötvös correction calculations in precise sea gravimetry.

An open-loop software compensation procedure of gyrocompass errors is proposed as an alternative to manual mechanical compensation traditionally used, to improve the gyrocompass performance. The algorithm developed in this thesis is a function of the gyrocompass design parameters and of the particular dynamics of the ship's motion.

Finally, recommendations for future work include sea-trials of the developed software compensation algorithm, extension of the mathematical models to incorporate random disturbing forces, and evaluation of the dynamic response of modern marine gyrocompasses, such as, the Sperry MK 37 Gyrocompass Equipment.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONTINUED)

TABLE OF CONTENTS (CONTINUED)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (CONTINUED)

ACKNOWLEDGEMENTS

This research was supported in part by an operating grant from the Natural Sciences and Engineering Council of Canada held by Dr. Wells. Financial assistance has also been provided by the Surveying Engineering Department, through graduate teaching and research assistantships.

I wish to thank my supervisor, Dr. D.E. Wells, for his helpful comments throughout this investigation.

I have benefited greatly from discussions with Dr. Donald Small from the Department of Mathematics and Statistics, Dr. Petr Vaníček and Dr. Richard Langley from the Surveying Engineering Department.

In addition, I would like to extend my thanks and my appreciation to my colleagues in room E-6, Head Hall, Surveying Engineering Department and particularly to my friends Spiros Pagiatakis, Tom Inzinga and Hal Janes. They took the time to discuss with me many problems during this research. My association with them will be unforgetable.

The assistance of Mrs. Debbie Smith was invaluable. Her faultless typing is especially acknowledged.

I thank my parents for their financial support at the beginning of my graduate studies. Without their moral and material support this research would have never become reality.

Above all, I want to express my thanks to my wife Vassoula, whose patience and support, both while being overseas and after she joined me here at U.N.B., was the greatest help in completing this work.

CHAPTER 1

INTRODUCTION

Navigation is the art of finding the position of a ship at sea, and conducting it safely from place to place [*Admiralty Manual of Navigation 1964*]. The process of navigation, in general, consists of defining the route, conducting the craft along it, and finding the vessel's position from time to time to check its progress [*Encyclopaedia Brittanica 1970*].

The above definition addresses navigation from the traditional viewpoint. Modern navigation relies more and more upon mechanical and electronic devices. This framework is supplemented by more and more sophisticated high-speed digital computers.

The essential sensors in modern navigation may be summarized as: ship's log and gyrocompass (representing the *classical dead-reckoning function*); electronic aids to navigation, such as, radionavigation systems (Omega, Loran, VLF and VHF systems, etc.); satellite navigation systems such as, the Navy Navigation Satellite System (NNSS) and the Global Positioning System (GPS); inertial navigation systems, and acoustic navigation systems.

The multiplicity and diversity of the navigation systems available today open a new era in navigation. It is no longer purely an "art", but also a definite scientific function of applied research and technology. This new era calls for evaluation and use of the full potential of any navigation component sensor on board ships, leading to what is known as *multisensor integrated navigation system*.

In this thesis we develop methods for software compensation of gyrocompass errors. These methods are useful in integrated navigation systems, for the real-time calculation of the Eötvös correction in marine gravimetry, etc. In this chapter we describe the problem, outline the treatment of the problem, and summarize the main contributions made in this thesis.

1.1 Problem Description

Heading of the ship is a basic navigation parameter, and is used in manual, automatic and computer-oriented applications.

The gyrocompass is the primary instrument used to provide heading reference on board ship. Alternatives might be to measure the azimuth of the ship's head by astronomical means (a time consuming, weather dependent, and less accurate technique); to use two radiopositioning receiving antennae (along the fore-aft axis of the ship) interferometrically; or to use two acoustic transducers along the keel interferometrically. These last two alternatives are not self-contained, as the gyrocompass is, requiring radio or acoustic reference beacons. Such systems have been proposed, but none is presently in wide use.

Characteristic of the classical dead-reckoning function (i.e., the estimation of ship's position and velocity from observations of ship's speed and heading) is the monotonically increasing magnitude of the position error with time [*Grant* 1976]. The contribution of the gyrocompass errors to this position error can be significant, especially during ship's manoeuvres and/or high latitude operations.

In many practical applications, the approach to gyrocompass-error compensation methods appears to be oversimplified. The provisions made by the manufacturer for manual compensation procedures are often used as the only means of the system's reliable performance. For example, in *Grant* [1976] it is stated that over a short time interval (e.g., less than 10 minutes) the ship's log and gyrocompass provide smoother estimates of ship's velocity than estimates derived from Loran-C. Therefore, the classical dead-reckoning function was used to provide information during ship's manoeuvres to reduce the influence of the Loran-C measurement noise on Loran-C positions [*Grnat* 1976]. But, gyrocompass observations are in error, this being especially true during ship's turns, when the gyrocompass can exhibit undesirable oscillations. Hence, the gyrocompass information may be "worthless" in evaluating another system's performance, since by itself it is unreliable if its behaviour is not adequately modelled and its deviations properly accounted for. Another example is an actual, measured gyrocompass error in Lancaster Sound in 1972 [*Eaton* 1982]. A maximum error of 6° in CSS "Baffin" gyrocompass was measured after 180° turns at 13.5 knots. Such gyrocompass errors might also give trouble in running sounding lines on a survey.

In computing the real-time Eötvös correction for precise marine gravimetry, gyrocompass deviations may introduce errors larger by a factor of two than the current gravimeter measurement accuracies. When ship's log and gyrocompass provide velocity information for calculating a satellite navigation fix, gyrocompass errors are important. It is also noted here that the performance capabilities of the current commercial marine gyrocompasses approach their operational limits as latitude increases. The reasons are increased instability of the gyrocompass (long natural period of free oscillations, no Schuler tuning) and increased bias errors. These reasons will be examined in the subsequent chapters in more detail.

In view of the above stated problems, our objective is to develop mathematical models that make the best use of the strenghts of the gyrocompass, and at the same time compensate for its weaknesses in order to minimize the influence of the gyrocompass errors on the indicated headings. Specifically, in this study we examine the gyrocompass performance as a function of ship's motion and as a function of latitude. The particular problem addressed in this thesis is to specify an algorithm to compensate for errors in gyrocompass indicated headings under the following conditions:

i. the gyrocompass has a manual speed and latitude compensator,

ii. the gyrocompass must continue to operate normally (but not as well compensated) when the software compensation is not used, and

iii. the software compensation continues to be useful at high latitudes.

1.2 Outline of Treatment

In Chapter 2 the basic definitions related to the fundamental principles of gyroscopic theory are given, along with a description of the reference frames which will be used in this study. A brief introduction to gyroscopic theory and its numerous applications is included. The particular application of the gyroscope as a gyrocompass is outlined.

Chapter 3 presents the principles of gyrocompass operation as well as its history and evolution to the present. A short description of some current systems is presented.

The gyrocompass equations of motion and their solutions are developed for a stationary, uniformly moving, and manoeuvring vessel, and at high latitudes, in Chapters 4 through 7, respectively.

The performance of these various mathematical models is evaluated using a computer simulation of the performance of a typical gyrocompass. To evaluate the effect of certain inputs and approximations on the output error in indicated heading of the gyrocompass, a computer program was developed and the numerical results obtained are illustrated diagramatically in Chapter 8. The simulation study enables us to determine the gyrocompass response under different dynamic conditions.

In Chapter 9 the software compensation of gyrocompass errors is described and possible alternatives for the high latitude behaviour are proposed. The relative advantages and disadvantages of the open-loop software compensation procedure are examined.

The last chapter assesses the results obtained and discusses their importance to the navigational problem. Conclusions are drawn and recommendations are made for continuing the present work. Alternatives and extensions to this work are discussed.

Appendices contain all the lengthy, but necessary, mathematical derivations used to arrive at the final expressions presented in the main body of the text. Also supplementary reference and explanatory material is given.

1.3 Contributions Made in This Thesis

The main contributions made in this study are:

i. the development of an open-loop software compensation algorithm to account for the gyrocompass errors, both transient and steady-state,

ii. the application of this algorithm to the high latitude behaviour of the gyrocompass problem, thus improving its performance considerably,

iii. the formulation and solution of the gyrocompass equations of motion for any arbitrary track of the ship using a discrete-time model,

iv. the formulation and solution of the gyrocompass equations of motion for a circular path of the manoeuvring ship.

The above contributions are the direct result of the application of the theory of linear dynamic systems in a simple, straightforward way. The clear, concise, and consistent formulation of the equations of motion of the gyrocompass is due to the use of the Lagrangian approach. The uniform notation followed through the whole study helps to avoid misunderstanding and misinterpretations. Finally, an extensive bibliography was compiled.

CHAPTER 2

BASIC DEFINITIONS - INTRODUCTION TO GYROSCOPIC THEORY AND ITS APPLICATIONS

In this chapter the basic terms used in gyroscopic theory are defined. Applications of the gyroscope are presented briefly. The application of the gyroscope as a gyrocompass is discussed.

2.1 Definitions

Dynamics relate the motion of a physical body to its interactions with its surroundings, (i.e., the response of the physical body in its environment).

Galileo showed that there are preferred reference systems in which the deviation of a body from uniform motion (or rest) is always attributable to external influences. These preferred reference systems are called *Inertial* or *Galilean Systems*. In such a reference system we can speak of absolute acceleration and absolute angular velocity, but neither velocity nor position can be considered absolute.

Inertial Space is a reference space in which Newton's laws of motion are valid. It is considered to be non-rotating with respect to the "*fixed stars*", whose positions for navigational purposes appear to be fixed in space.

The following reference frames may be defined:

a. Inertial frame; it is earth-centred, non-rotating with respect to inertial space,

b. Earth frame; it is geocentric, non-rotating with respect to the earth,

c. Navigational frame; centred at any point on the earth's surface (topocentric), non-rotating with respect to the local vertical,

d. Body frame; fixed relative to the body in a preferable manner.

The frames defined above have been identified by their lack of rotation, but they have not been specifically oriented to the direction of certain individual axes. This allows the choice of a specific coordinate frame later to suit the problem treated.

The peculiar motions of spinning bodies have always fascinated mathematicians, physicists, and engineers [*Magnus 1974*]. In the broad literature relating to problems of spinning bodies the term *gyro* is used to describe, quite generally, a rotating rigid body.

A very common definition of a *gyroscope* specifies a rotating rigid body at a large angular velocity about an instantaneous axis, which always passes through a fixed point. This fixed point may be the centre of gravity of the body, or it may be any other point. This broad definition can be made more specific under the following simplifications:

- the axis of rotation is both a principal axis and an axis of symmetry,
- the ratio between angular speeds along the spin-axis and the transverse axes is infinitely large.

Therefore, a gyroscope is a rigid body that rotates at high angular velocity about one of its principal axes of inertia, and of which the rotations about axes perpendicular to the gyro-axis (spin-axis) are very slow compared to the main rotation. The following two definitions are coming as an immediate result of the theory of rotating rigid bodies.

Angular Momentum (or *moment of momentum*) is a vector property of any physical body that is spinning with respect to inertial space about an axis.

Torque is the rotational effect of an applied force about an axis. In the absence of an applied torque an angular momentum vector maintains a fixed orientation in an inertial space, thereby providing a *directional reference.* By applying a calibrated torque to a spinning body one can command the angular momentum vector to rotate relative to inertial space in a known and prescribed manner.

2.2 Brief Introduction to Gyroscopic Theory and Its Applications

The device which has proved most suitable to indicate a reference direction is the gyroscope. Two gyroscopic principles are the direct consequence of the preceded definitions namely, *gyroscopic inertia* and *gyroscopic precession.* Gyroscopic inertia is that property of the gyroscope which makes it try to keep the spin-axis parallel to its original position.

Gyroscopic precession is that property of the gyroscope that causes the spin-axis to change direction when a torque is applied to it.

For an angular reference, it would be sufficient to have a device which was held fixed in angular position in inertial space in spite of any angular or linear acceleration, or velocity of the support structure.

Free gyro is any gyroscope on which no external moments act to change its motion's character. The angular momentum and the kinetic energy of rotation of a free gyro remain constant.

The overall objective in the design of an angular-reference device is to create an instrument which will respond to angular-rotation inputs. The gyroscope serves the function of an instrument that will respond to angular-rate-inputs, i.e., angular velocity. Depending upon its own internal characteristics (or those arising from external circuits) and equipment coupled to the gyroscope, it can respond in such a way as to [*Wrigley et al. 1969*]:

a. measure the input angular velocity (providing a signal proportional to it), or
b. maintain a reference angular attitude (independent of the input angular velocity), or
c. measure the integral of the angular velocity input.

Although the apparent effect of the earth's rotation on gyroscopes was first shown by *Léon Foucault* in 1852, the ability to construct sufficiently accurate units did not exist until the beginning of the twentieth century.

For many applications in guidance and control it is necessary to have available certain directional references. These references, which serve as the basis for obtaining *navigational data*, or for *stabilization* of a vehicle, or some of its equipment, must be maintained despite various interferences.

Specific applications of the gyroscope include the gyrocompass, rate-measuring gyroscopes, direction-indicators for aircrafts, artificial-horizons, autopilots, inertial navigation units, ship's motion stabilizers, gyroscopic vibration absorbers, etc.

2.3 Gyroscope on Gimbals - The Gyrocompass

In the previous section two important principles of gyroscopic theory were defined, i.e., gyroscopic inertia and gyroscopic precession.

Gyroscopic inertia depends upon angular velocity, mass, and radius of gyration, i.e., upon angular momentum.

Gyroscopic precession can be caused only by a force attempting to tilt or turn the spin-axis about another axis. A torque about the spin-axis cannot cause precession. Any torque about either one of the other two transverse axes will cause the gyroscope to precess about an axis at right angles to that about which the torque acts. Precession will continue as torque acts, but will cease when the torque is removed. If the plane in which the torque is acting remains unchanged, the gyroscope will precess until the plane of the spin is in the plane of the torque. Analytically, it is represented by

$$\vec{\omega}_{prec} \times \vec{H} = \vec{M} \qquad (2.1)$$

where;

$\vec{\omega}_{prec}$: is the angular velocity of precession of the gyroscope angular momentum with respect to inertial space,

$\vec{H}$: is the angular momentum along the gyroscope spin-axis, and

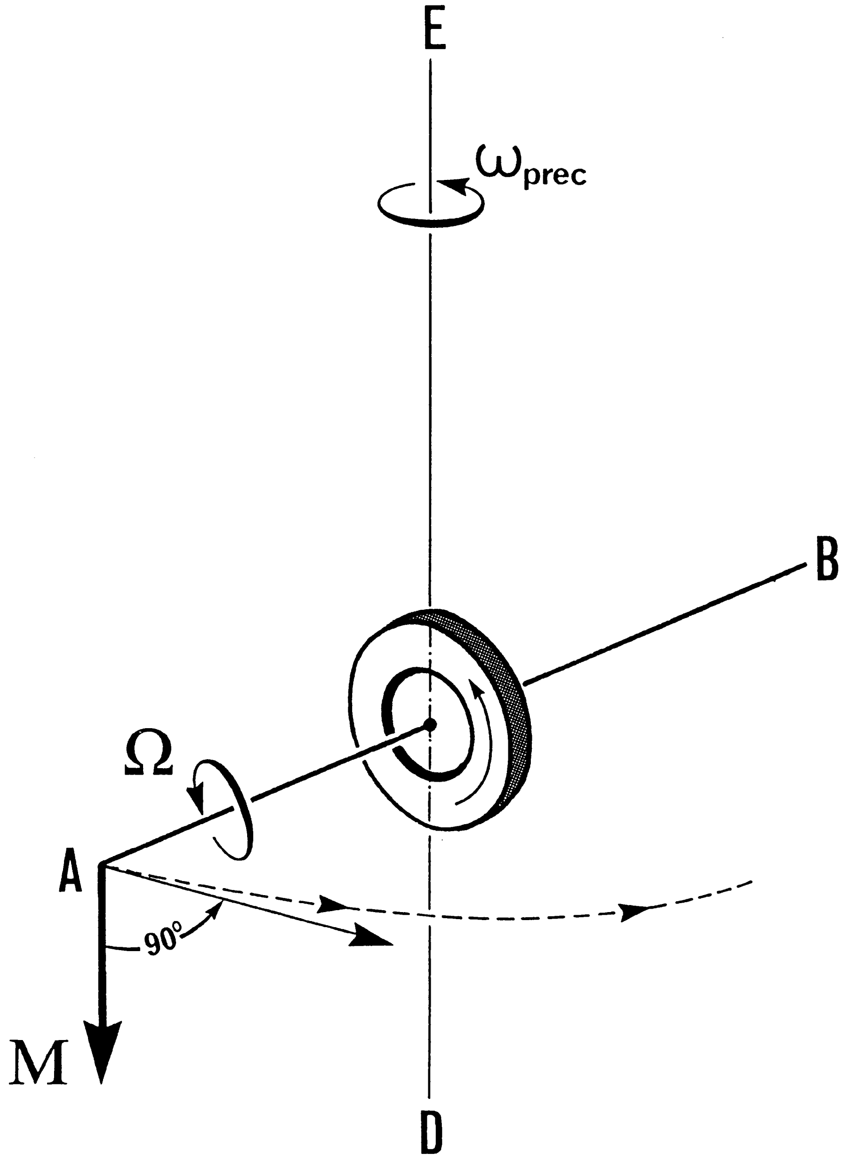
$\vec{M}$: is the applied torque.

Physically, this equation means that the gyro-axis angular momentum vector $\vec{H}$, precesses relative to inertial space in an attempt to align itself with the applied torque $\vec{M}$.

The *gyrocompass* is a navigational instrument which accurately seeks the direction of *true north* under the combined effect of gravity and the earth's daily rotation [*Wrigley et al. 1969*].

True north is the direction represented by a horizontal line in the plane of the meridian, or, the intersection of the horizontal plane and the local meridian.

To make a gyroscope into a gyrocompass the gyroscope has to seek and maintain the true north direction. A gyrocompass is a *gimballed spinning wheel*. The gyroscope is so mounted that the wheel-axle (gyro-axis) has freedom of angular motion. The number of gimbal rings, or the nature of the support determines the type of the gyroscope. A two-degree-of-freedom gyroscope has one gimbal ring (or equivalent support) in addition to the gyro-element gimbal ring. (The gyro-element consists of the spinning rotor, the drive mechanism and the spin-axis support.) It should be noted here that the term "two-degree-of-freedom" gyroscope does not account for the freedom in spin of the gyro-wheel itself, which provides it with



Precession of the Gyroscope

Fig. 2.1

one more degree-of-freedom that is usually omitted in the literature. We will keep this convention in here, and we will talk about a "two-degree-of-freedom" gyroscope referring actually to the degrees-of-freedom that the support structure provides.

As originally constructed the gyrocompass had a two-degree-of-freedom gyroscope with a mass attached to it, that gave the gyrocompass a pendulocity, and therefore providing by some means of vertical stabilization.

In conclusion, the gyrocompass tracks true north by attempting to align the gyro-axis with the horizontal component of the earth's angular velocity. In the next chapter the history and development of the gyrocompass will be presented together with a brief description of some current gyrocompass designs.

CHAPTER 3

BASIC PRINCIPLES OF GYROCOMPASS OPERATION

This chapter is devoted to the particular application of the gyroscope, the gyrocompass. A historical review of the gyrocompass development is presented. A short description of some current systems in commercial use is then given.

The principles of operation of a Sperry-type gyrocompass are introduced since this is the system in which we are interested in the present analysis.

An outline of the errors associated with the gyrocompass is given. The main sources of errors are identified in an attempt to examine their influence on gyrocompass readings.

3.1 The Gyrocompass as a Heading Indication Sensor - Historical Review and Development

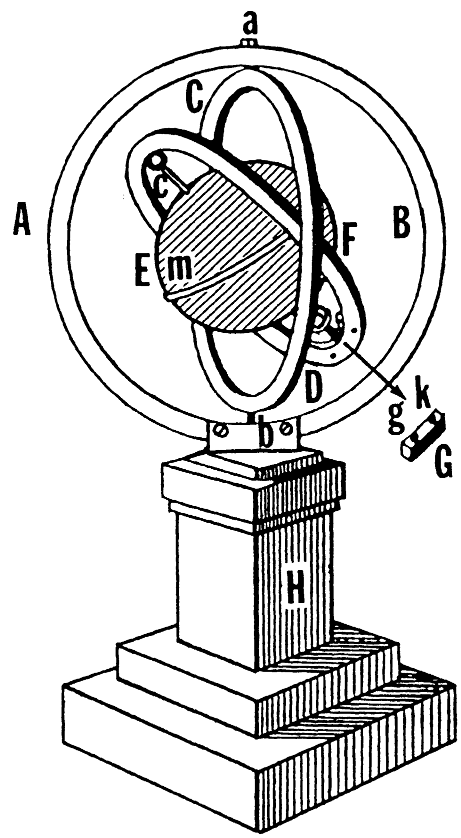
The history and development of the gyrocompass are closely related to the history of this unique device, the gyroscope. It is in this respect that the use of the gyroscope as a heading indication sensor is examined to provide historical information about the evolution of the gyrocompass.

In 1752 the first written statement on a gyroscopic device was published in the *London Philosophical Transactions* [*Sorg 1976*]. Sorg, studying the history of the gyroscope, gives a fascinating list of literature on the subject. This is the source from where most of the material appearing in this section is drawn.

The first scientists who tried to apply the theory of spinning bodies in directional instruments were *Serson* and *Lomonossow*. Their efforts were concentrated on the design of an *artificial-horizon* by employing a spinning top. In a lecture given at the Russian Academy of Sciences in 1759 entitled *"Investigations about better accuracy of the sea-routes"*, Lomonossow proposed a spinning top to create an artificial-horizon device on a rocking ship.

Serson's interests inclined mostly towards the design of an artificial-horizon device for use in sextant observations at sea at times when there was fog around the sea-horizon. Such an instrument was ultimately tested onboard a British Admiralty yacht in 1743 and its function was favourably reported upon.

In 1817 there is a publication in the *"Tuebingen Blaetter fuer Naturwissenschaft und Arzneikunde"*, (translation: "Tübingen Letters for Natural Sciences and Medicine"), by *Bohnenberger* from the University of Tübingen, Germany. In this publication, the first gyro with a *Cardan-suspension* was shown, (Fig. 3.1). With this model Bohnenberger could demonstrate the laws of the gyroscope and he also could show that the spin-axis, acted upon no external forces, does not change its direction in space. However Bonhenberger did not know anything about the value of this principle as being used in direction indicating devices. In



Gyroscope by Bohnenberger

Fig. 3.1

the same publication it was stated that the mathematical treatment of this type of device was done by the French *Poisson* in 1813.

The man who created the word *gyroscope* was the French scientist *Léon Foucault*. In his memoir read before the Academy of Sciences in Paris (1852) Foucault describes his experiments relating to the movement of the earth and he concludes:

> *"Comme tous ces phénomènes dependent du movement de la Terre et en sont des manifestations variées, je propose de nominer gyroscope l'instrument unique qui m'a servi à les constanter."*

In this manner the word "gyroscope" was first introduced. Its etymology from Greek means an *apparatus allowing to view rotation.* Today it denotes a variety of mechanisms used to measure angles, angular velocities, accelerations, or to indicate north.

In one of his experiments, Foucault found that with a gyroscope one can find north, using proper gimbal structure and damping so that the spin-axis will settle to a direction which coincides with the direction of the horizontal component of the earth's-rotation vector. The idea of the gyrocompass was born. But Foucault had no great success with his device mainly due to lack of technical means to provide gyro-wheel spinning for a long period of time with high speed.

It was *Trouvé* who designed in 1865 the gyro-wheel as the rotor of an electric motor.

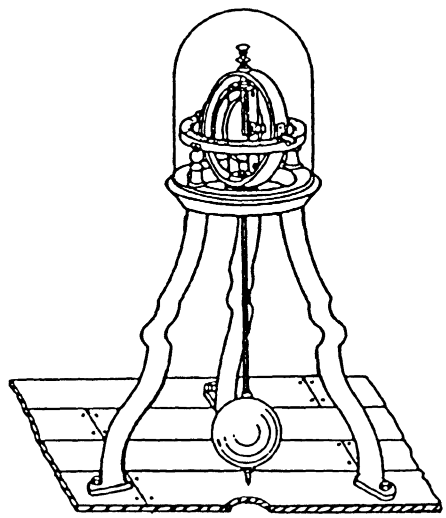
Two improtant improvements were made by Trouvé; an electric motor to drive the gyro-wheel fast enough and the constraint of the spin-axis to the horizontal plane by using the force of gravity. The first practical gyrocompass had been developed. Trouvé's gyrocompass, developed for correcting magnetic compasses onboard ships, but only in harbors, is shown in Figure 3.2.

Similar devices to Trouvé's gyrocompass were built by the American physicist *Hopkins* in 1878 and by the Frenchman *Dubois*.

The next step in the gyrocompass development was made by *Lord Kelvin* (Sir William Thomson) in 1884. He proposed, for avoiding the friction of the bearings on the gimbals, to suspend the gyro by a *torsion-free wire*, or if possible to use a *floated-suspension* instead of the Cardan-suspension. Lord Kelvin's second proposal was applied by the Dutch scientist *Van den Bos* in 1885 for his gyrocompass. This patent was bought by the German company Siemens & Halske and some devices were built.

At the beginning of the twentieth century the gyrocompass development was forced along by the German *Herman Anschütz-Kaempfe*. In 1900 he was planning a trip to the north pole in a submarine, but he was frustrated by the total absence of reliable navigation equipment. His idea was to develop a direction-keeping instrument, but the trials were not successful. This led him to undertake the development of a north-seeking instrument. The result of his efforts was the famous Anschütz gyrocompass, patented in 1904 [*Sorg 1976*].

Although Anschütz is acknowledged as the inventor of the first sea-worthy gyrocompass the date of its actual production is somewhat vague. From a brief review of the literature, *Wrigley et al.* [*1969*] dates the first ever produced gyrocompass by Anschütz as 1908. Pearson [*Gyros, 1965, Paper 5, pp. 1-11*] dates it as 1910. *Sorg* [*1976*] states implicitly



Trouvé's Gyrocompass

Fig. 3.2

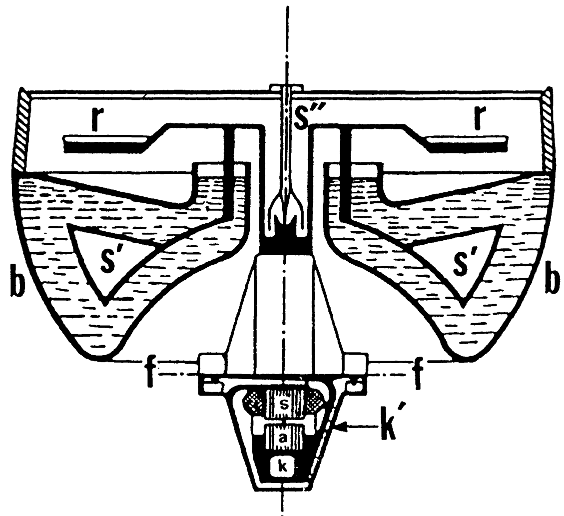
that already in 1910 the German Navy was equipped with Anschütz gyrocompasses undergoing extended sea-trials. *Grant* [*1967*] says that about 1914 Anschütz-Kaempfe and Sperry simultaneously developed "what might be termed the first good gyros suitable for navigation purposes".

In 1906 the young German scientist *Maximilian Schuler* saw the work of Anschütz and he also started to work in the field of gyroscopes. His first proposal resulted in a gyrocompass which had a rotor driven by an a-c current at high speed [*Sorg 1976*], and is shown in Figure 3.3.

During the second decade of the twentieth century several designs of gyrocompasses existed. In 1911 *Elmer Sperry* in the United States produced a gyrocompass that was easier to manufacture [*Wrigley et al. 1969*]. In 1912 a third type of gyrocompass appeared, built by *S.G. Brown* and *John Perry* in London [*Sorg 1976-77*]. Anschütz-Kaempfe and his staff, including Schuler, were working in Kiel and they came up with a new design, a three-gyroscope sensitive element. It was later followed by a two-rotor gyrocompass, a system in use now for more than fifty years.

But the most significant advancement was made by Maximilian Schuler in his paper written in 1923, where he showed that a pendulous system of the proper frequency stays vertical when moving around the earth.

In this paper, entitled "*The Disturbance of Pendulum and Gyroscopic Apparatus by the Acceleration of the Vehicle*", Schuler stated in the introduction: ..."I asked myself the question: would this sort of acceleration error be capable of elimination by an appropriate construction?"..."The answer is yes. And the solution is almost trivial."...



First Anschütz single-rotor
Gyrocompass

Fig.3.3

Finally, Schuler concluded in his paper:

> ..."An oscillatory mechanical system, on whose centre of gravity a central force acts, will not be forced into oscillations by any arbitrary movement over a spherical surface about the centre of force, if its period of oscillation is equal to that of a pendulum of the length of the sphere's radius in the applied force field" [*Schuler 1967*].

This general law, due to Schuler, was the most important progression development not only in the gyrocompass theory and design, but also in today's inertial technology.

Henceforth, the development of the gyrocompass was only a matter of expanding technology and not a matter of developing new principles. However, a great number of ingenious engineers further-developed the existing gyrocompass mechanizations. In *Ferry* [*1932*] and *Rawlings* [*1944*] one can find half a dozen gyrocompass designs existing by the 1940's.

To complete this survey of gyrocompass evolution some other names of devoted scientists should be mentioned, such as those of *Martienssen* and *Geckeler*. Martienssen, as early as 1906, computed the N-S acceleration error of a gyrocompass, thus inspiring Schuler later on to arrive at his unique contribution, Schuler's period of 84 minutes.

Geckeler came up with a modified Anschütz-type gyrocompass design, which has received particular attention in soviet literature, as can be seen from the extended list of references provided in the bibliography at the end of this thesis.

To conclude this section it is necessary to refer to the recent advancements in gyrocompass development. The new technology attempts to substitute the conventional gyroscopes by *laser-gyros*. The same general principles of rotational motion are used, but the mechanization is completely different. It has been proven feasible to construct laser-gyros and to use them as heading indication sensors, but they are still in the trial process. Until a low-cost, reliable laser-gyro for marine use becomes available on the market, the conventional gyrocompass design will be perhaps the primary instrumentation for a heading indication sensor onboard surface ships.

Recapitulating, today the most common types of commercial gyrocompasses are those of Sperry, Anschütz and Arma-Brown. Descriptions of these systems are given briefly in the next section. Particular attention is given to the Sperry-type gyrocompass.

3.2 Description of Some Present Systems

The previous section contains the history of the development of the gyrocompass. We now turn to consideration of the actual instruments which are to be found in service on the world's navy and merchant ships. Only the most common types will be presented here namely, the Anschütz, the Arma-Brown and the Sperry designs. The description of these systems will be intentionally limited. However, for details the interested reader can refer to the operational manuals of the systems. The major sources of information used here are; *Rawlings* [*1944*], *Arnold and Maunder* [*1961*], *Gyros* [*1965*], *Klinkert* [*1964*], *Operation and Service Manual of Sperry MK 37 Mod 1 Gyrocompass Equipment* [*1975*].

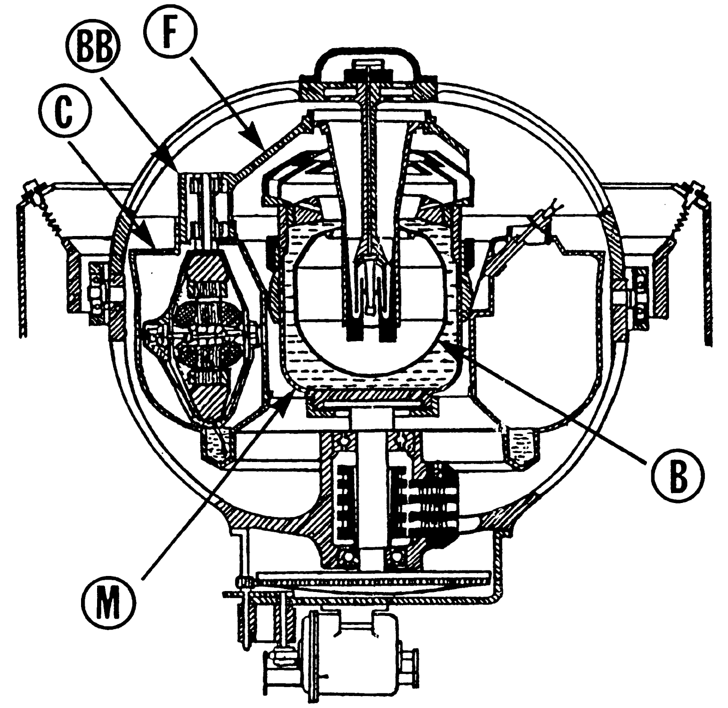
The first Anschütz gyrocompass design (due to Max Schuler in collaboration with the Anschütz firm) presented in the previous section (Figure 3.3), uses a single gyroscope hanging from a hollow ring-shaped steel float resting on mercury. Before many of these single-rotor Anschütz compasses had been made, it was discovered that they were subject to large errors due to the rolling motion of the ship especially on intercardinal directions [*Rawlings 1944*].

The next design, intended to overcome the above imperfections, employed a triple-rotor gyrocompass. A schematic diagram [after *Rawlings*] is shown in Figure 3.4.

We will discuss this design because it is the predecessor of today's Anschütz compasses, which are a modification of this triple-rotor system.

The compass has three separate and similar gyroscopes suspended from a frame F, carrying the compass card (or dial) C. The frame is supported from a hollow steel-ball B, floated in a bowl of mercury M. The gyroscopes are situated at the corners of an equilateral triangle whose apex is under the 180° mark of the compass card. The whole arrangement is pendulous and north-seeking. The gyroscope at the south corner, which is the principal meridian-seeking element, is fixed so that the north-south (N-S) line of the card lays parallel to its spin-axis. The other two gyro-casings have their vertical axes mounted in ball-bearings BB in the frame. They are free to move in azimuth independently of the compass card, except for a pair of light springs which keep their spin-axes normally at an angle of 30° with the meridian.

These two gyroscopes are linked together in such a manner as to ensure that the intersection of their spin-axes always lies under the N-S diameter of the compass card. These two gyroscopes therefore



Anschütz triple-rotor Gyrocompass

Fig. 3.4

contribute something to the north-seeking effect of the south-gyroscope and at the same time exert a stabilizing effect which resists displacement of the east-west (E-W) diameter of the compass card from the horizontal position.

Although the apparatus just described, enjoyed the reputation of being one of the most successful gyrocompasses on the market for twenty years, mechanical defects occurred, which prevented the very high degree of accuracy which its designer Anschütz had set as his ideal.

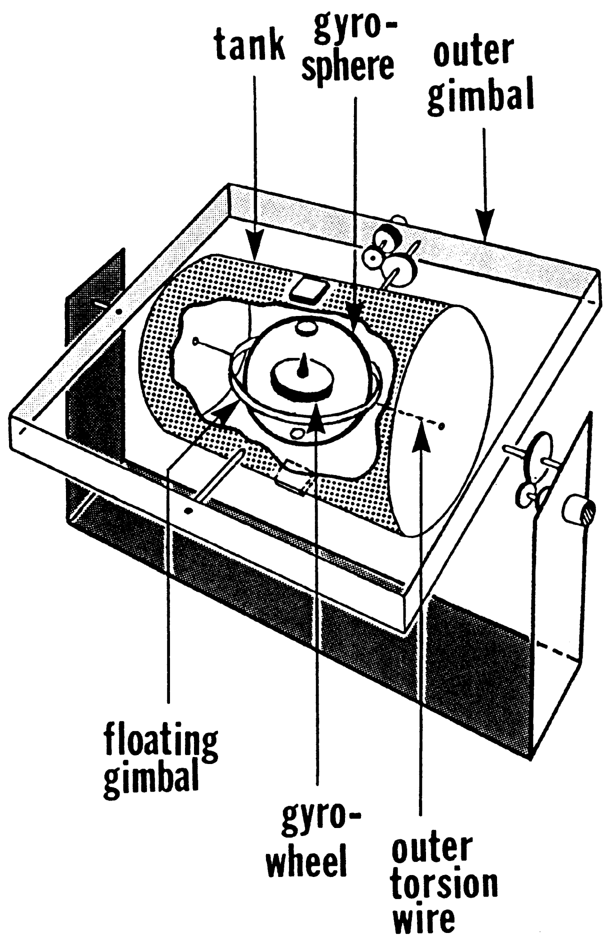
Anschütz sought to remedy all the drawbacks of previous design in one stroke by redesigning his compass. The principal innovation consists of enlarging the float so, as to make it large enough to include everything in a gyro-sphere (i.e. the gyroscopes, the damping trough, etc.).

The gyro-sphere is entirely submerged in liquid. The whole assembly is centralized in the outer sphere by a system of coils producing an alternating magnetic field, thus generating Foucault currents, thereby producing a repulsion effect which centralizes the ball both laterally and vertically. The triple-rotor arrangement is now substituted by a two-rotor meridian-seeking component. The south-gyroscope is not necessary any more, and the two *oblique* gyroscopes are set at a smaller, but equal angle, with the meridian.

The Arma-Brown gyrocompass system combines what is called a directional gyro with a gyrocompass [*Klinkert 1964*]. The Arma gyrocompass is a modification of a double-rotor Anschütz gyrocompass. The Brown gyrocompass is a single-rotor Sperry-type gyrocompass system. The Arma-Brown design is a completely different mechanization. It is a floating two-degree-of-freedom gyroscope, supported at neutral buoyancy, free of mechanical gimbal pivots or ball-bearings. In Figure 3.5 the structure of an Arma-Brown gyrocompass equipment is shown, [after D. Barnett *Gyros 1965, Paper 12, pp. 159-165*].

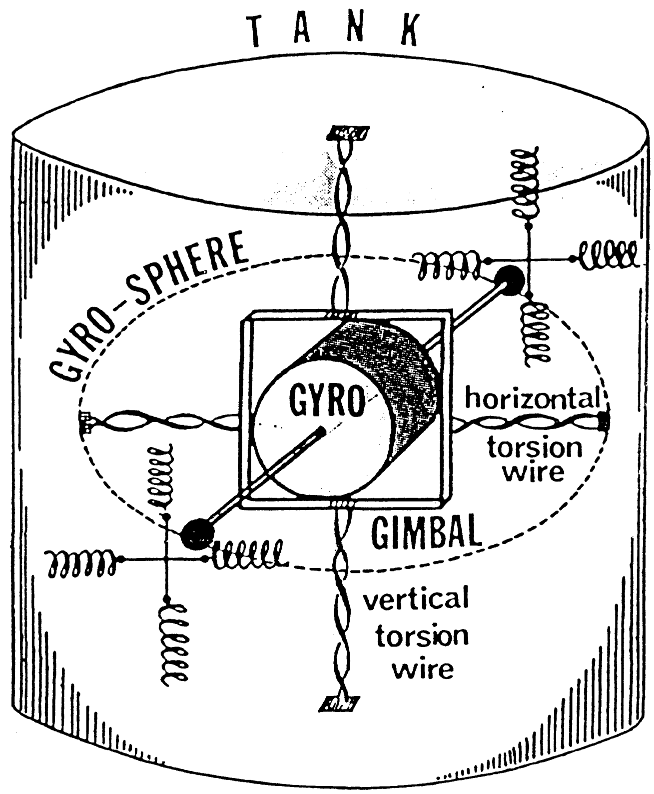
The gyro-wheel is mounted in an hermetically sealed container which is substantially spherical, but has a deep circular recess to accommodate a floating gimbal-ring. The gyro-sphere, containing the gyro-rotor assembly, is completely supported by the floatation fluid of the outer tank. It is centred by two successive pairs of fine wire filaments, referred to as *torsion wires*. One pair of these wires, diametrically opposed, connects the gyro-sphere to the floating gimbal-ring, whose plane is at right angles to the spin-axis, hence permitting the gyro-sphere to tilt about one gimbal axis. The second pair of these torsion wires, at right angles to the first pair, connects the gimbal-ring to opposite points inside the tank that holds the floatation fluid, providing the gyroscope with one more degree-of-freedom. The torsion wires and associated gimbal mounting are more clearly shown in the following Figure 3.6 [after *Klinkert 1964*]. Gravity reference is obtained by a small pendulum which is fixed to the outer tank, which is supported in a set of gimbals which are connected to the binnacle housing.

The third and last gyrocompass design treated here is the recent Sperry-type gyrocompass system. This system deserves our attention since it is the one that is used in the whole analysis of the present work.



The Arma - Brown Gyrocompass

Fig. 3.5



Gimbal Mounting of Arma-Brown Gyrocompass Equipment

Fig. 3.6

The main parts are the Gyro-sphere, the Phantom Yoke and the Binnacle Assembly. The gyro-sphere containing the gyro-rotor is immersed in silicone fluid and it is designed and adjusted to have neutral buoyancy. Its essential features are illustrated diagramatically in Figure 3.7, [after *Arnold and Maunder 1961*].

The complete instrument is supported through the gimbal *a*, and is pivoted about axes $O'x'$, $O'y'$. It is therefore free to assume a vertical position irrespective of the motion of the supports. The compass assembly rests on bearing *b*, and consists of the compass card *c* and phantom ring *d*, together with inner ring *e*, in which the rotor and casing are mounted.

The inner ring assembly is carried by a wire suspension *f*, which passes through a tube and is fixed at the upper end to *c*. Due to the directive force on the rotor the ring *e* tends to move relative to *d* in order to align its axis with the meridian. Any such movement, however, is sensed by a servo-system which immediately rotates the phantom ring by means of the azimuth motor *g* to keep both rings *d* and *e* coincident.

The gravity reference is obtained by using a mercury ballistic *h* instead of a pendulous mass as in the elementary gyrocompass design. The mercury ballistic consists of a frame pivoted about the E-W axis of the phantom ring, to which are attached two pairs of bottles *i*, containing mercury. Each N-S pair is interconnected by a pipe *j* of small bore which allows the mercury to flow from one to the other.

A link-arm *k* attached to the frame engages with the rotor casing (gyro-sphere) *l* through a pin *m*, which is offset to the east by an angle γ.

Diagrammatic illustration of Sperry Gyrocompass

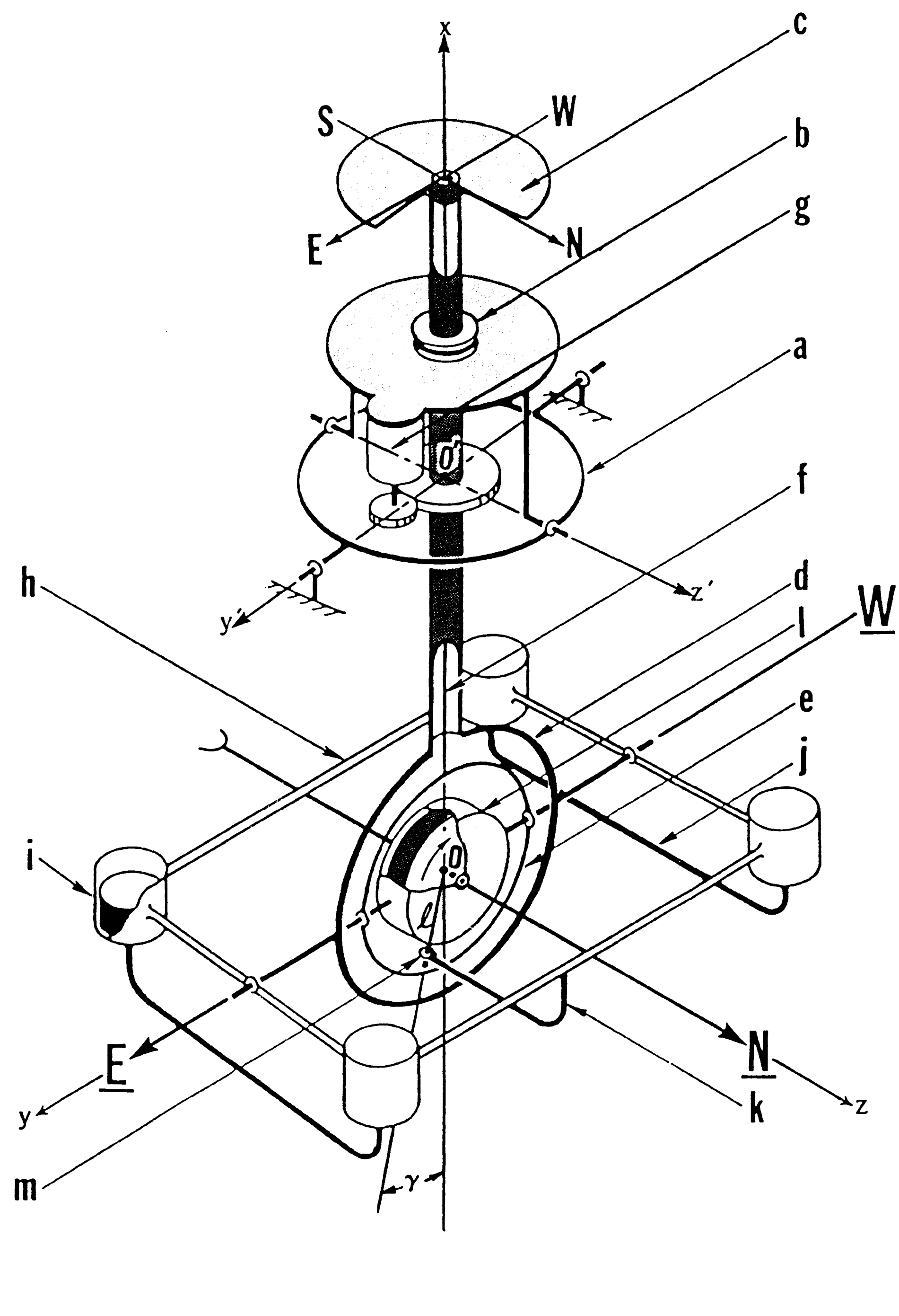


Fig. 3.7

NOMENCLATURE:

a : gimbal suspension

b : bearing

c : compass card

d : phantom ring

e : inner ring

f : wire suspension

g : azimuth motor

h : mercury ballistic

i : -"- bottle

j : connecting tube

k : link arm

l : rotor casing

m : connecting pin

ℓ : metacentric height

γ : offset angle

Fig. 3.8

The gyro-sphere ℓ is the north-seeking part of the gyrocompass. Inside the gyro-sphere is the gyro-wheel.

The basic design of three modern gyrocompasses has been described. Figure 3.9 summarizes the principal differences among the mechanical arrangements employed to seek north.

3.3 Principles of Gyrocompass Operation and Associated Errors

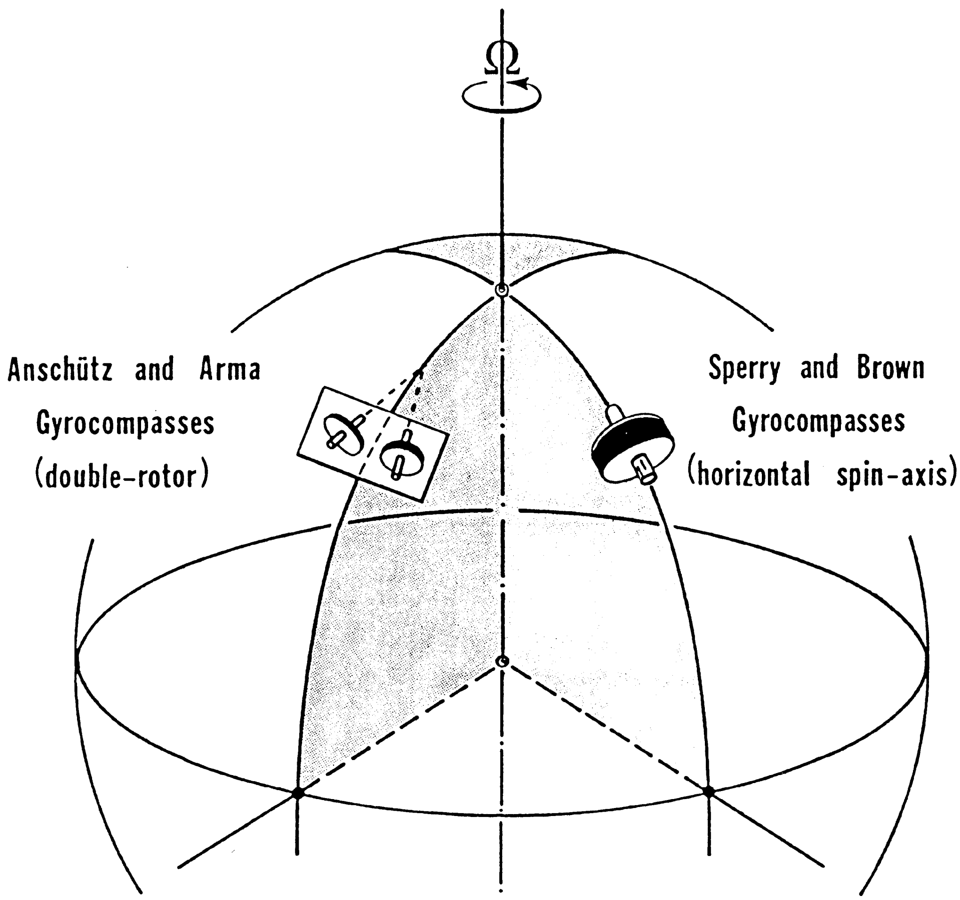
In the following we shall attempt to describe the underlying theory of the gyrocompass and give a summary of the errors associated with its operation.

The physical behaviour of a gyrocompass, which consists essentially of a gyroscope whose motion is controlled by the combined action of the *earth's rotation* and the moment produced by a *gravitational force*, is examined.

The gyroscopic principles outlined in the previous chapter are used to demonstrate the gyrocompass application.

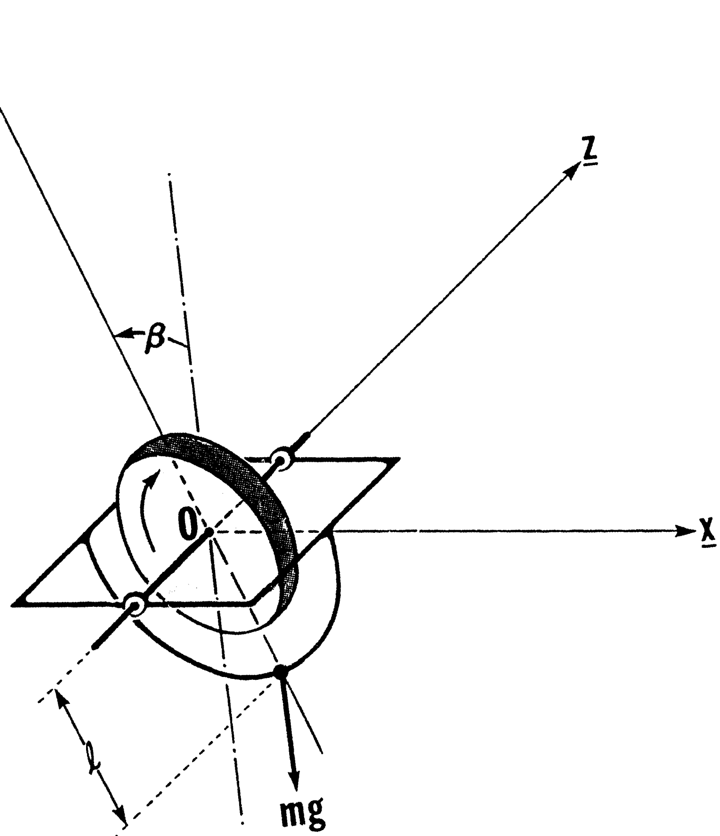
Lets consider a gyro-wheel suspended at its centre of mass and free to adopt any position in space (Figure 3.10). Let also the rotor, spinning around its axis of symmetry, be placed initially at the equator with its spin-axis horizontal and pointing a few degrees east of north. In the course of a day the spin-axis would remain motionless relative to inertial space (i.e., gyroscopic inertia, or equivalently, a property known as *rigidity of the gyroscope*).

But for an observer on the earth, the spin-axis would appear to rise in the east and set in the west. For example, suppose that the spin-axis were set pointing East at 12 o'clock midnight, and the earth-



Gyroscope arrangements for azimuth determination

Fig. 3.9



Essential Features of a Gyrocompass

Fig. 3.10

observer were left to study its motion during the succeeding 24 hours. At 6 a.m. he would find the spin-axis pointing North with its tip tilted upward above the horizontal plane. At 12 o'clock noon the spin-axis would point West and laying on the horizontal plain again, while at 6 p.m. it would point North, but tilted downward below the horizontal plane. Finally, at midnight the spin-axis would have returned to its original position. The spin-axis would thus appear to the earth-observer to be describing a cone at an angular velocity equal to that of the earth but in an opposite direction. The phenomenon just described is the second most important gyroscopic property, the *gyroscopic precession.*

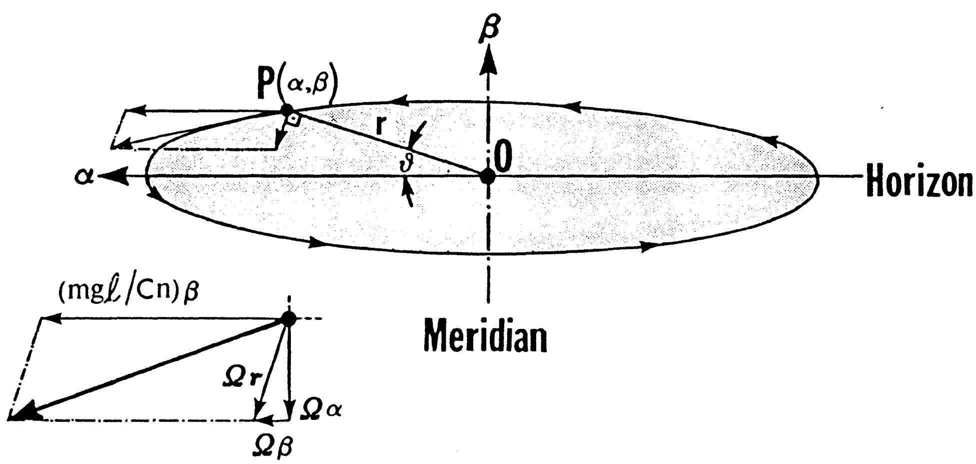
The next step in making a gyroscope into a gyrocompass is to make the gyro-wheel seek the meridian. To do this, a weight mg is added to the bottom of the vertical gimbal, which causes the gimbal to be pendulous about the horizontal axis Oy.

To find what actually occurs we allow the earth to rotate and we combine the two actions: *the earth's rotation* and *the gravitational torque.* We trace the path of the spin-axis as we did before, letting the spin-axis to point initially east of the meridian. While the earth rotates, the spin-axis tilts up, but now there is a horizontal torque directed westward due to the pull of gravity on the pendulous mass. The spin-axis precesses about the vertical axis toward the meridian, continuing to rise because of the earth's rotation, until finally the meridian is reached. At this point the pendulous torque is maximum. This resulting path is the superposition of the two motions the spin-axis performs. One is the precession due to the earth's rotation, the other being the precessional motion imposed by the applied pendulous torque.

As the spin-axis continues to precess through the meridian the earth's rotation causes it to set, thus reducing the amount of tilt and consequently the pendulous torque. Since tilt becomes less, the speed of precession in azimuth decreases and finally the spin-axis becomes horizontal. The pendulous weight causes no torque. At this point the gyro-axis has precessed as far west of the meridian as it was east originally. While the earth continues to rotate, the gyro-axis continues to set. This causes it to dip below the horizon and the gravitational torque produced due to the tilt has now the opposite direction than previously (i.e. an eastward direction). Hence, the spin-axis precesses toward the meridian again. Eventually, the spin-axis precesses past the meridian and back to its starting position, where this whole process is repeated. Because the precessional speed is directly proportional to the amount of tilt, the spin-axis is tracing out an ellipse about the meridian and the horizon, (Figure 3.11).

The rotor and pendulous weight described above form the essential elements of a gyrocompass. For the gyrocompass to operate properly, it is necessary that the oscillation be damped out so that the gyro spin-axis can settle on the meridian and not keep passing through it. Damping an oscillator involves changing its energy states, one way to do this being the change of its velocity.

There are several ways to illustrate the damping action on the oscillations of the gyrocompass. One way is to add a small weight to the east of the vertical gimbal as it is described in the *Operational and Service Manual of Sperry MK 37 Gyrocompass Equipment 1975, pp. 1-15 to 1-16*]. A second approach is to employ some kind of mechanical



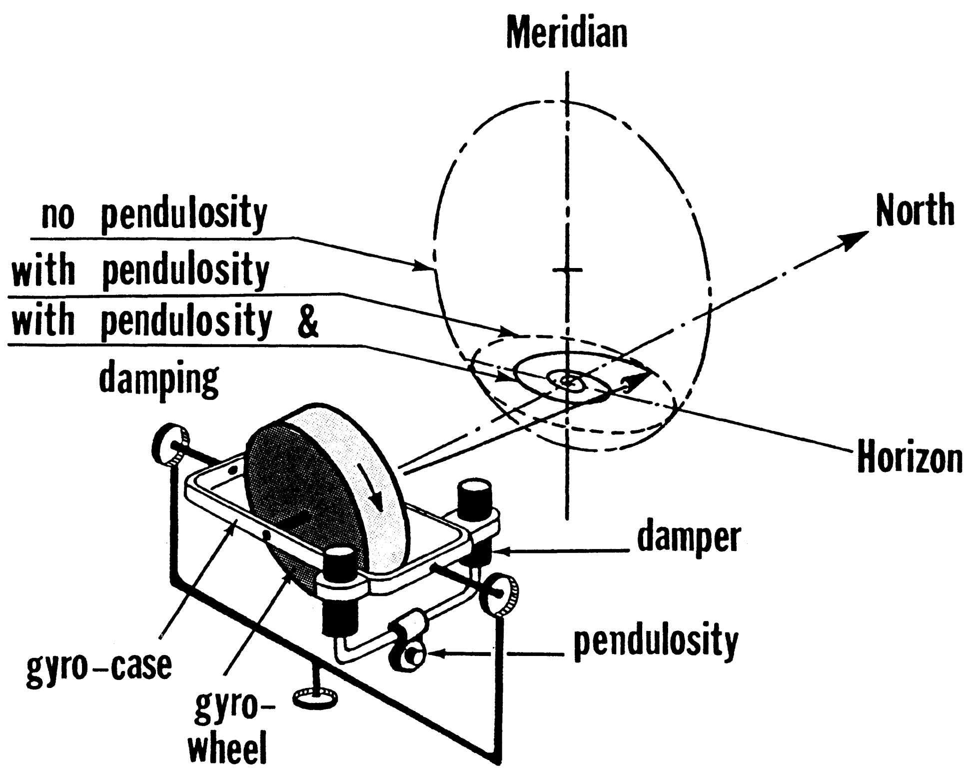
Path followed by the gyro spin-axis

Fig. 3.11

arrangement of interconnected tanks filled with a viscous fluid and attached to the vertical gimbal as it is described in *Wrigley et al.* [*1969*, *pp. 187*]. Both ways result in anti-pendulous action while the spin-axis is tilted, therefore reducing the azimuthal precessional motion in every successive oscillation of the gyrocompass. Thus the elliptical path followed by the spin-axis is changed into a spiralling-in motion toward the meridian, where finally it settles. The same action (damping) can be illustrated by offsetting the pendulous mass to the east by a small angle γ, a configuration which produces an identical effect (spiral path) as the two previously mentioned procedures. Figure 3.12 summarizes in an illustrative way what has been described so far, [after *Wrigley et al. 1969*].

Although the universal use the gyrocompass at sea is a testimony of its unique ability to provide directional reference with respect to true north, it is also subject to several errors. Some of these are persistent while others are temporal. Using somewhat different terminology, they can be characterized as a steady-state and transient errors. Gyrocompass errors may be systematic or non-systematic. Some of these can be eliminated or offset in the design of the compass, while others require *manual* or *software* adjustment for their correction.

The *total combined error* (i.e., the resultant error) at any time is called *gyro error* (*GE*) and is expressed in degrees east or west of the meridian to indicate the direction in which the spin-axis is offset from true north.



Paths of Angular Momentum

Fig. 3.12

The gyrocompass associated errors are of three kinds: those associated with the way damping is accomplished; those associated with the motion of the transporting vehicle; and those associated with the design of the mechanical suspension.

The damping error applies only to those gyrocompasses in which damping is achieved by offsetting the point of application of the gravity force. It depends on latitude, increasing as tan ϕ.

The errors introduced by the motion of the transporting vehicle are related to velocity and acceleration inputs. Velocity introduced errors occur to all compasses that use the earth's rotation as a directional datum. They are independent of the instrument's design and they are predictable. Acceleration-induced errors on the other hand, belong in part to the way in which the instrument is constructed and to the dynamic response of the gyrocompass. In general, they are less predictable and not so easy to compensate for. In most of the cases, they introduce temporary (transient) errors in the compass readings.

CHAPTER 4

EQUATIONS OF MOTION OF A STATIONARY GYROCOMPASS

In the previous chapters the basic definitions and operational principles related to the gyrocompass were outlined. In this chapter we develop the equations of motion of the gyrocompass. In particular, we are interested in laying out the equations of motion of a stationary gyrocompass, that is, a gyrocompass on a fixed base on the earth's surface.

The specific gyrocompass design we deal with in this study is the Sperry gyrocompass. As it was described in the previous chapter the modern Sperry gyrocompasses make use of the ballistic-mercury design to produce the necessary gravitational torques. However, the mechanization of the equations of motion in the present analysis refers to a more elementary design, the pendulous-mass gyrocompass.

Two main reasons led us to this choice. The first is that, in terms of analysis, both designs - i.e. the mercury-ballistic and the pendulous gyrocompass - are equivalent. The second reason is that, there is not available information for the exact actual design of the Sperry gyrocompass. Furthermore, the analysis in terms of the mercury-ballistic design introduces additional theoretical complications. Our concern is not to develop something new in the field of gyrocompass theory and design, but, conversely, to try to make the best use of the existing information in the most efficient and beneficial way.

It is instructive to mention here two basic assumptions that will be used in the whole course of the present work. A spherical earth is assumed, rotating with constant angular velocity $\Omega = 7.29 \times 10^{-5}$ $\mathrm{rad \cdot sec^{-1}}$. Also constant acceleration of gravity $g = 9.81\ \mathrm{m \cdot sec^{-2}}$ is supposed. Other approximations that will be used to a certain extent are approximations in physical modelling leading to mathematical simplifications, such as those presented in Table 4.1.

The approach chosen for the mathematical analysis is that of physical dynamic system analysis, since the objective of the investigation is to understand and predict the dynamic behaviour of the given system.

Whatever the particular physical system under study is, the procedure for analytical investigation usually incorporates each of the following stages [*Cannon, 1967*]:

I. - specify the system to be studied and assign to it a simple physical model whose behaviour will be sufficiently close to the behaviour of the actual system,

II. - derive a mathematical model to represent the physical model, i.e. evaluate the differential equations of motion of the physical model, and

III. - study the dynamic behaviour of the mathematical model, by solving the differential equations of motion.

Table 4.1.

APPROXIMATION	MATHEMATICAL SIMPLIFICATION
neglect small effects	reduces the number and complexity of the differential equations
assume linear relationships	makes equations linear, allows superposition of solutions
assume constant parameters	leads to differential equations with constant coefficients
neglect uncertainty and noise	all quantities have definite values that are known precisely thus leading to a deterministic approach, it simplifies the analysis by avoiding the need for statistical treatment, therefore dynamical effects of uncertainty and response to random disturbances are ignored.

A fourth possible stage can be the selection of the physical parameters of the system so that it will behave as desired, but that goes beyond the scope of the present work since our aim is not to improve the actual design of the gyrocompass.

In this work, we shall present the Lagrangian approach to the formulation of the equations of motion because this method circumvents, to some extent, the difficulties found in the direct application of Newton's laws of motion. The reasoning behind this, is that the Lagrangian approach involves scalar quantities, while Newton's laws of motion involve vectorial treatment. Furthermore, the use of Lagrange's equations presents the equations of motion in a standard, convenient form.

Another important concept in the description of a dynamic system is that of *degrees of freedom*. In general, the number of degrees of freedom is equal to the number of coordinates which are used to specify the configuration of the system minus the number of independent equations of constraint [*Greenwood 1965*]. In the case of the gyrocompass two coordinates are necessary to specify at any time the position of the spin-axis; the tilt angle β with respect to the horizontal plane and the azimuth deviation α with respect to the meridian.

In summary, the equations of motion of a pendulous gyrocompass design are developed. In a further step, the combined pendulocity and damping action is formulated and the dynamic response of a stationary gyrocompass is evaluated. Finally, the initial conditions of the motion are examined.

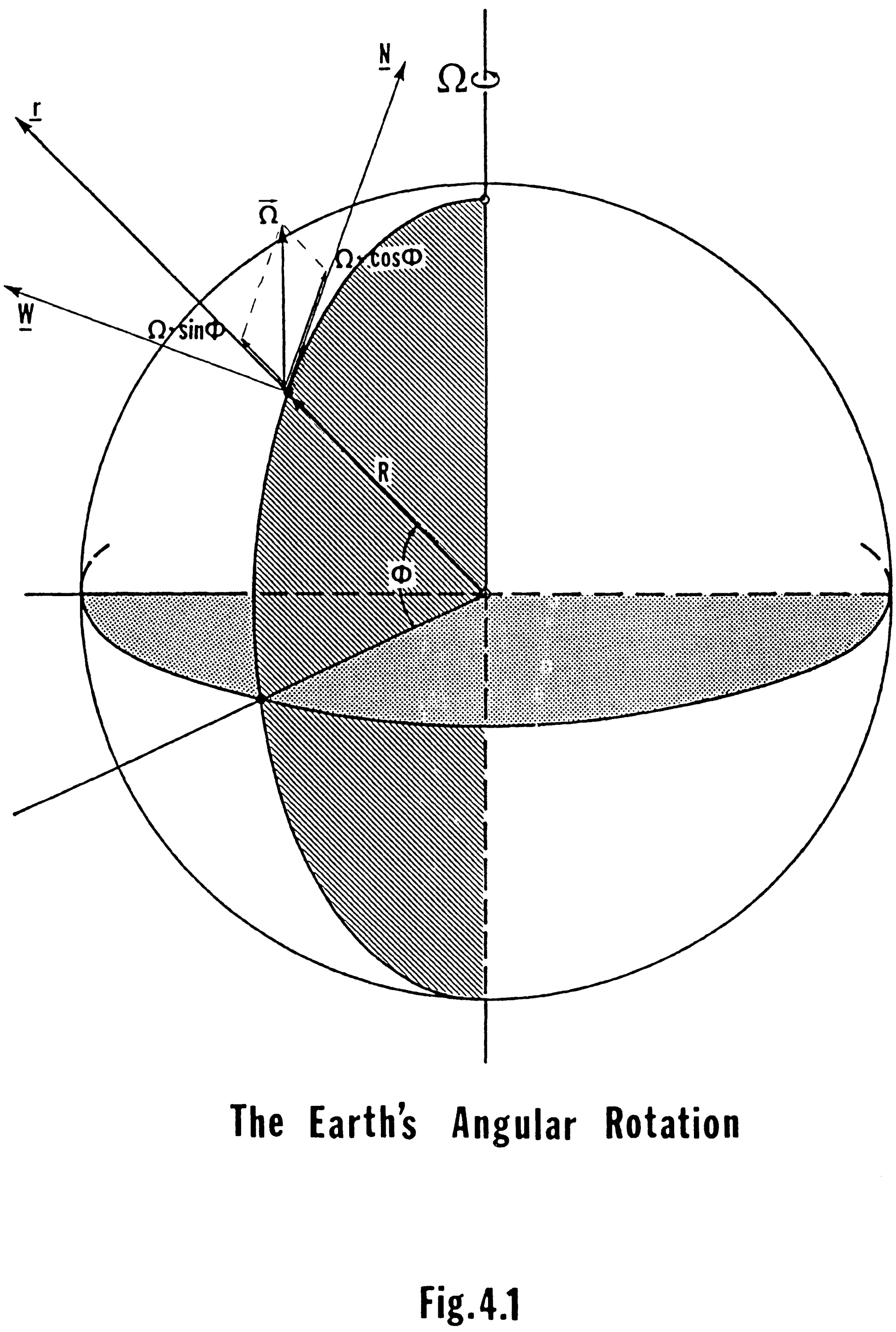
4.1 Equations of Motion with Pendulocity

We now proceed to analyze the motion of a gyrocompass, which consists essentially of a spinning rotor with a horizontal axis supported in a frame free to turn about a vertical axis.

Figure 4.1 illustrates the earth which rotates about its polar axis at angular velocity Ω in a direction from west (W) to east (E).

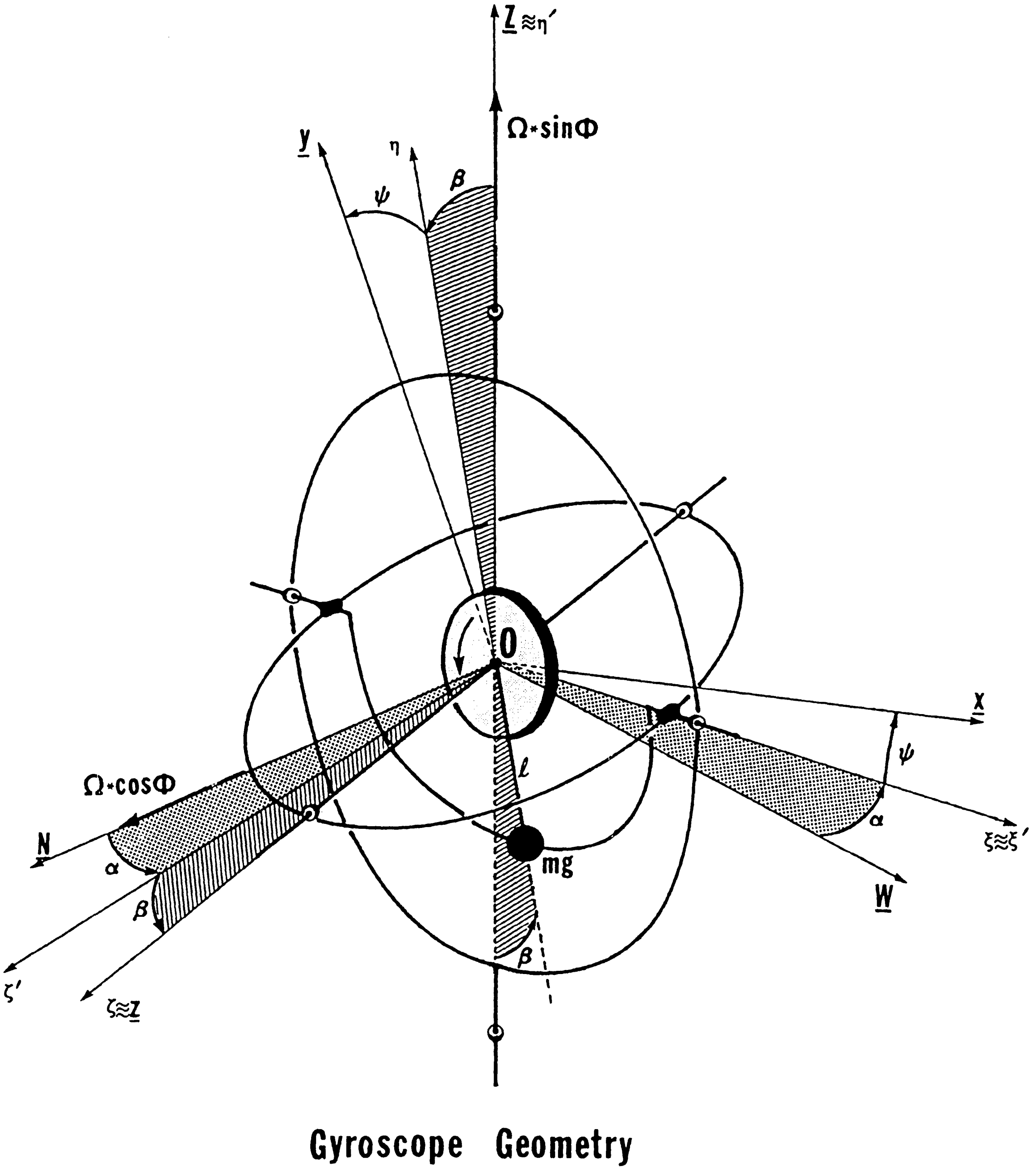
Figure 4.2 shows the geometry of the gyroscope assembly as well as the components of angular displacement of the gyrocompass.

The gyroscope assembly is fitted with a pendulous mass m. For the rotor, the principal axes are chosen to be $O\zeta\xi\eta$ and the principal moments of inertia C, B, A. For a symmetrical rotor $A = B$. Axes $O\zeta\xi\eta$ may also be arranged to be the principal axes of the rotor casing whose principal moments of inertia are C', A', B'. The inner ring is assumed to have principal moments of inertia A'', B'', C'' about axes $O\zeta'\xi'\eta'$, where $O\eta'$ is vertical and $O\zeta'$ is horizontal. Let us consider the gyroscope of Figure 4.2 at a latitude ϕ on the earth's surface, as shown in Figure 4.1, (also consult Figure 3.8 for the gyrocompass arrangement and nomenclature). The Z-axis in this case is not an inertial axis, but coincides with the local vertical at all times. The direction of the gyrocompass spin-axis $O\zeta$ is defined by a rotation α about the Z-axis and a rotation β about the ξ'-axis. A rotation ψ about the ζ-axis results to the final Ozxy frame shown in Figure 4.2. By inspection it is easy to find the relation between the rotation angles α, β, ψ and the *Eulerian angles*. The earth's angular velocity Ω is resolved into two components about axes N and Z, respectively, so that the gyroscope precession consists of the components $\Omega \sin \phi$ and $\dot{\alpha}$. To provide the torque about the horizontal axis $O\xi$ necessary for



The Earth's Angular Rotation

Fig.4.1



Gyroscope Geometry

Fig. 4.2

producing the precession $\Omega \sin \phi$, a pendulous weight mg is attached at the point $(\zeta, \xi, \eta) = (0, 0, -\ell)$ of the rotor casing.

In view of the above definitions the angular velocity components of the inner ring are

$$\begin{bmatrix} \omega_{\zeta'} \\ \omega_{\xi'} \\ \omega_{\eta'} \end{bmatrix} = \begin{bmatrix} \Omega \cos \phi \cos \alpha \\ -\Omega \cos \phi \sin \alpha \\ \Omega \sin \phi + \dot{\alpha} \end{bmatrix} \tag{4.1}$$

whereas the angular velocity components of the rotor casing are

$$\begin{bmatrix} \omega_{\zeta} \\ \omega_{\xi} \\ \omega_{\eta} \end{bmatrix} = \begin{bmatrix} \Omega \cos \phi \cos \alpha \cos \beta - (\Omega \sin \phi + \dot{\alpha}) \sin \beta \\ -\Omega \cos \phi \sin \alpha + \dot{\beta} \\ (\Omega \sin \phi + \dot{\alpha}) \cos \beta + \Omega \cos \phi \cos \alpha \sin \beta \end{bmatrix} \tag{4.2}$$

Finally, the angular velocity components of the rotor (spinning wheel) about system $O\zeta\xi\eta$ are

$$\begin{bmatrix} \Omega_{\zeta} \\ \Omega_{\xi} \\ \Omega_{\eta} \end{bmatrix} = \begin{bmatrix} \omega_{\zeta} + \dot{\psi} \\ \omega_{\xi} \\ \omega_{\eta} \end{bmatrix} \tag{4.3}$$

So far, following the procedure outlined in the introduction of this chapter, we have specified the system to be studied by assigning the simple physical model shown in Figure 4.2, and now we are ready to evaluate the mathematical model to represent it. Again, it is pointed out that the approach to evaluate the equations of motion is Lagrange's equations, which involve the kinetic and potential energies of the body (system) at some chosen instant.

The kinetic energy of a system may be expressed in terms of the motion of the centre of mass, and of the particles relative to the centre of mass. In the general case, when both translational and rotational motion are present, we have, for the kinetic energy, the well known expression

$$T = \frac{1}{2} Mv_G^2 + \frac{1}{2} \cdot I_G \cdot \omega^2 \tag{4.4}$$

where G is the centre of mass of the system, M is the mass of the system, v_G the linear velocity of the centre of mass, ω the instantaneous angular velocity about G, and I_G is the moment of inertia about the axis of ω. If, however, we stipulate that the axes of rotation are fixed to the body and their origin coincides with the centre of mass G, and in addition they are the principal axes of the body, then eqn. (4.4) becomes

$$T = \frac{1}{2} Mv_G^2 + \frac{1}{2} (A\omega_i^2 + B\omega_j^2 + C\omega_k^2) \tag{4.5}$$

where A, B, and C are the principal moments of inertia at G, and ω_i, ω_j, ω_k are the angular velocity components along the directions $\vec{i}$, $\vec{j}$, $\vec{k}$, (i.e. along the principal axes of the body).

Since here we examine a stationary gyrocompass with respect to the earth's surface, then its centre of mass, (point 0 in Fig. 4.2), does not have translational motion. That is, the first term in eqn. (4.5) drops out because $v_G = 0$. Hence, the kinetic energy of our physical model (gyrocompass) assumes the form

$$T = \frac{1}{2}\{[C(\Omega_\zeta)^2 + B(\Omega_\xi)^2 + A(\Omega_\eta)^2] +$$

$$+ [C'(\omega_\zeta)^2 + A'(\omega_\xi)^2 + B'(\omega_\eta)^2] +$$

$$+ [A''(\omega_{\zeta'})^2 + B''(\omega_{\xi'})^2 + C''(\omega_{\eta'})^2]\} \quad (4.6)$$

The potential energy of our physical model is simply

$$U = mg\ell(1-\cos\beta) \quad (4.7)$$

Now we define the Lagrangian function $\mathcal{L}$ as follows [*Greenwood 1965*]:

$$\mathcal{L} = T - U \quad (4.8)$$

Then Lagrange's equations of motion assume the form [*Landau and Lifshitz 1976*]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2, \ldots, n) \quad (4.9)$$

where the symbol ∂ denotes partial differentiation, $\dot{q}_i$ and q_i are the generalized velocities and coordinates respectively, and $i = 1, 2, \ldots, n$ the degrees of freedom of the system.

Lagrange's equations (4.9) are the equations of motion of the system and they constitute a set of n second-order equations for n unknown functions $q_i(t)$. The general solution of these equations contains *2n arbitrary constants* [*Landau and Lifshitz 1976*]. In order to determine these constants and thereby to define uniquely the motion of the system, it is necessary to know the initial conditions which specify the state of the system at some given instant, for example the initial values of all the coordinates and velocities.

From the analysis of our physical model, namely eqns. (4.6) and (4.7), it is obvious that except the gravitational force, mg, no other force is acting on the system. Furthermore, the potential function U is only a function of position, i.e., $U = U(q_i)$. Therefore, equations (4.9) reduce to the expression

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} \tag{4.10}$$

Equations (4.10) were used to evaluate the analytical expressions for the differential equations which describe the two modes of motion of the gyrocompass namely, the motion in azimuth α and the motion in tilt β. The assumptions listed in Table 4.1 were used and the lengthy mathematical derivations are presented in *Appendix I*. The final expressions for the equations of motion are:

$$D_1\ddot{\alpha} + E_1\dot{\beta} + G_1\alpha = 0 \tag{4.11}$$

and

$$D_2\ddot{\beta} + E_2\dot{\alpha} + G_2\beta = F_2 \tag{4.12}$$

where the parameters D_1, E_1, G_1, D_2, E_2, G_2 and F_2 are given in their explicit form in Appendix I.

In summary, the equations of motion of a stationary pendulous gyrocompass have been developed using well known principles of mechanics and postulated mathematical assumptions. Once the Lagrangian function is found, the procedure for obtaining the equations of motion is straightforward.

4.2 Equations of Motion with Pendulocity and Damping

As it was described in section 3.3 and shown in Figures 3.12 and 3.13, the pendulocity causes the gyrocompass to oscillate, the spin-axis following an elliptical path.

The oscillation of the gyrocompass is an undesirable effect, as the instrument is expected to indicate true north. This oscillation can be damped by displacing the pendulous mass at an angle γ to the east. This configuration is also illustrated in Figure 3.8.

The kinetic energy associated with the system is given by the same equation (4.6). The potential energy is again given by eqn. (4.7). But the displaced mass m has an additional effect. It produces a torque about the η-axis (Fig. 4.2).

The new equations of motion of the stationary gyrocompass with pendulocity and damping are derived in *Appendix II*. The final expressions are

$$D_1\ddot{\alpha} + E_1\dot{\beta} + G_1\alpha + F_1\beta = 0 \tag{4.13}$$

and

$$D_2\ddot{\beta} + E_2\dot{\alpha} + G_2\beta = F_2 \tag{4.14}$$

where the coefficients D_1, E_1, G_1, D_2, E_2, G_2, F_2 and F_1 have the explicit forms given in Appendix II. The motion that the spin-axis is now performing is a spiraling-in motion toward the meridian, as was pointed out in section 3.3.

4.3 Dynamic Response of the Gyrocompass

So far we have examined the first two of the three stages of a dynamic investigation. We have derived the mathematical model, i.e., a set of equations of motion, for the physical model.

We now come to our principal concern, stage III, to determine how the physical model will behave and what motions it will have. In general, this is done by solving the differential equations of motion.

In the previous section we found that the equations (4.15) and (4.16) are *linear differential equations with constant coefficients*, an important fact, which allows us to study their solutions in view of the theory of ordinary linear differential equations.

Specifically, we shall find that when a linear, constant-coefficient dynamic system is disturbed by some *forcing function* the resulting motion is the sum of two distinct components:

(i) a *forced response* which resembles in character the forcing function, and

(ii) a *natural motion* whose character depends only on the physical characteristics of the system itself and not upon the forcing function.

In formal mathematical language the above are known as

(i) the particular solution, and

(ii) the homogeneous or complementary solution.

Further, it will be found that the *natural motion* of a linear, constant-coefficient system is made up of some combination of two elementary motion patterns, an exponential decay and a sinusoidal motion.

Investigation of the above basic elements forms the core of almost all of our future study of the gyrocompass behaviour because, as we shall see in subsequent chapters, all of the possible motions can always be computed by *superposing the responses* of our dynamic system to several dynamic inputs. In Appendix III the *Superposition Principle* (or Superposition Theorem) is presented in detail.

In our investigation we will also make use of some complex-number algebra to ease computations.

A function of the form e^{st} will be used to describe mathematically the types of motion by letting, in general, s be a complex number.

The *Laplace technique* is, of course, also a convenient method for solving differential equations. It constitutes a powerful alternative to the procedure of assuming a solution of the form e^{st}. However, it will not be used in here.

The following concepts concerning the dynamic response of physical systems which are represented by linear, ordinary differential equations with constant coefficients are introduced [*Cannon 1967*]:

a. superposition of time responses is valid,

b. the total response will consist of two distinct parts, the natural motion and the forced motion,

c. the forced motion will have the same character as the forcing function, and its magnitude will be proportional to the magnitude of the forcing function,

d. the natural motion will be always of the form ke^{st} where s depends only on the physical system, and k is a constant,

e. the *characteristic equation* is an algebraic equation in s, whose roots are the values of s which make the expression ke^{st} a correct solution to the homogeneous differential equation,

f. the solution of the *homogeneous* equation (unforced motion) is usually designated as the *complementary function*, while the solution of the *forced motion* is called the *particular integral*. Both solutions constitute the *complete solution* of the differential equation.

4.4 Natural Motion Alone - Transient Response

The gyrocompass can be considered as a *linear, damped, second-order system*. The equations of motion are given by

$$D_1\ddot{\alpha} + E_1\dot{\beta} + G_1\alpha + F_1\beta = 0 \tag{4.15}$$

$$D_2\ddot{\beta} + E_2\dot{\alpha} + G_2\beta = F_2 \tag{4.16}$$

These equations can be rewritten, ignoring the terms $D_1\ddot{\alpha}$, $D_2\ddot{\beta}$, and $E_1\dot{\beta}$ and using the explicit forms of the rest of the coefficients, as:

$$-Cn\dot{\beta} + Cn\Omega \cos\phi\ \alpha - mg\ell\gamma\beta = 0 \tag{4.17}$$

and

$$Cn\dot{\alpha} + mg\ell\beta = -Cn\Omega \sin\phi \tag{4.18}$$

The assumption that the terms $D_1\ddot{\alpha}$, $D_2\ddot{\beta}$ are small and therefore can be ignored is based on:

i. the two motions, namely, the motion in azimuth and the motion in tilt are small enough and therefore their second derivatives assume even smaller values,

ii. these derivatives combined with the coefficients D_1, D_2 are small terms as compared to the remaining ones, especially since D_1, D_2

are functions of A, A', B' and C" (see Appendix I or II) which are by definition much smaller than C (C is the moment of inertia of the gyro-wheel),

iii. the motion in tilt β is much smaller than the motion in azimuth α, and in addition the term $(2A+A'+B'-C')\Omega \cos \phi$ is much smaller than Cn,

iv. since the two motions are not independent, there is need to further simplify the governing equations of motion to be able to uncouple the two motions.

Combining eqns. (4.17) and (4.18) we have the following expressions

$$\ddot{\alpha} + \left(\frac{mg\ell\gamma}{Cn}\right)\dot{\alpha} + \left(\frac{mg\ell\Omega \cos \phi}{Cn}\right)\alpha = -\frac{mg\ell\gamma\Omega \sin \phi}{Cn} \tag{4.19a}$$

$$\ddot{\beta} - \left(\frac{mg\ell\gamma}{Cn}\right)\dot{\beta} + \left(\frac{mg\ell\Omega \cos \phi}{Cn}\right)\beta = -\frac{\Omega^2}{2} \sin 2\phi \tag{4.19b}$$

Equations (4.19a) and (4.19b) are the equations of motion of the gyrocompass that will be used in succeeding sections. Since the motion in azimuth is our major interest, the rest of the analysis will be confined to eqn. (4.19a)

If we designate the following short-hand definitions

$$2\sigma = \frac{mg\ell\gamma}{Cn}, \quad f_o^2 = \frac{mg\ell\ \Omega \cos \phi}{Cn} \tag{4.20}$$

then the governing differential equation of motion (4.19a) assumes the form

$$\ddot{\alpha} + 2\sigma\ \dot{\alpha} + f_o^2\alpha = -\frac{mg\ell\gamma\ \Omega \sin \phi}{Cn} \tag{4.21}$$

Since in this section we only consider the case of unforced motion (natural motion) the right-hand side of eqn. (4.21) is set to zero. Thus, we examine the response of the gyrocompass using the *homogeneous part* of the governing equation, namely

$$(Cn)\ddot{\alpha} + (mg\ell\gamma)\dot{\alpha} + (mg\ell\ \Omega \cos\phi)\alpha = 0 \tag{4.22}$$

We assume a solution to the above equation of the form

$$\alpha = \mathcal{A}\, e^{st} \tag{4.23}$$

Substituting eqn. (4.23) in eqn. (4.22) we have

$$Cn[\mathcal{A}\, s^2 e^{st}] + mg\ell\gamma\ [\mathcal{A}\, se^{st}] + mg\ell\ \Omega \cos\phi [\mathcal{A}\, e^{st}] = 0$$

Provided that

$$[\mathcal{A}\, e^{st}] \neq 0$$

we obtain the *characteristic equation* of the system

$$(Cn)s^2 + (mg\ell\gamma)s + (mg\ell\ \Omega \cos\phi) = 0 \tag{4.24}$$

The characteristic equation contains s but not $\mathcal{A}$, and thereby represents completely the dynamic characteristics of the system.

The roots of the characteristic equation are

$$s_1 = -(\frac{mg\ell\gamma}{2Cn}) \pm \sqrt{(\frac{mg\ell\gamma}{2Cn})^2 - (\frac{mg\ell\ \Omega \cos\phi}{Cn})} \tag{4.25a}$$

or

$$s_1 = -(\frac{mg\ell\gamma}{2Cn}) \pm \sqrt{\Delta_1}, \quad \text{for} \quad \Delta_1 \geq 0 \quad \text{and}$$

$$s_2 = -(\frac{mg\ell\gamma}{2Cn}) \pm j\sqrt{(\frac{mg\ell\ \Omega \cos\phi}{Cn}) - (\frac{mg\ell\gamma}{2Cn})^2} \tag{4.25b}$$

or

$$s_2 = -(\frac{mg\ell\gamma}{2Cn}) \pm j\sqrt{\Delta_2}, \quad \text{for} \quad \Delta_2 \geq 0 \ .$$

The roots of the system's characteristic equation are called *eigenvalues or characteristics* of the system, and are given in two forms depending on the relative magnitude of the coefficients (Cn), ($mg\ell\gamma$), and ($mg\ell\ \Omega \cos\phi$).

Because we will use the above expressions frequently in the future we extend the short-hand definitions to simplify the algebraic operations introducing the terms

$$f^2 = f_o^2 - \sigma^2 \quad \text{and} \quad \mu = \frac{\sigma}{f_o}$$

Hence, the roots of eqn. (4.24) assume the forms

$$s_1 = (-\sigma_1, -\sigma_2) = -\sigma\left(1 \pm \sqrt{1 - \frac{1}{\mu^2}}\right), \quad \text{for} \quad \mu > 1 , \tag{4.26a}$$

and

$$s_2 = -\sigma \pm jf \qquad , \text{for } \mu < 1, \tag{4.26b}$$

Substitution of these eigenvalues into the assumed solution (eqn. (4.23)) yields the following expressions for the motion in azimuth:

$$\alpha = C_1 e^{-\sigma_1 t} + C_2\, e^{-\sigma_2 t}, \text{ for } \mu > 1 \tag{4.27a}$$

and

$$\alpha = e^{-\sigma t}\,(C_1\, e^{jft} + C_2\, e^{-jft})$$

or

$$\alpha = C_3\, e^{-\sigma t}\cos(ft-\psi), \text{ for } \mu < 1 \tag{4.27b}$$

Before proceeding any futher, the dynamic characteristics implied by the above equations and a brief explanation of all the mathematical symbols will be given.

The sumbol j represents the factor $\sqrt{-1}$. Equation (4.27b) is derived from the preceeding equation using what is known as *Euler's equation*, i.e.,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The C's in eqns. (4.27a) and (4.27b) are constants. Formally, the homogeneous solution to a linear differential equation of order r must contain r constants C. Therefore, in the present case we have, for a second-order differential equation, two integration constants namely, C_1 and C_2, or C_3 and ψ.

The quantity f_o is called the *undumped natural frequency* and is the frequency at which the same gyrocompass would oscillate if damping were absent (i.e., the frequency at which the gyrocompass would oscillate if only pendulocity were considered). The quantity f is called the *damped natural frequency*.

The constant σ is called the *damping coefficient*, whereas the quantity $1/\sigma$ is called the *damping time constant* of the system and indicates the time required for the motion to damp to (1/e)th its original value.

The parameter μ implies the relative damping of the system, i.e., the rate of damping with respect to the rate of oscillation and is commonly called the *damping ratio*. In particular, if μ is negative the system is unstable (the motion grows without bound); if μ is zero, the system is just neutrally stable (the motion neither grows nor decays); if μ is increased toward 1, the relative damping increases and the system becomes asymtotically stable (the system, once disturbed, overshoots but as time increases it tends to become stable) [*Cannon 1967*].

Finally, we refer to equations (4.27) as representing the *natural response* of the system. The solution to the homogeneous equation is also known as the *transient solution* and the response of the system (i.e., the natural response) as the *transient response* [*Greenwood 1965*].

It is also noted here that the constants C_1 and C_2, or C_3 and ψ are evaluated from the conditions at time t = 0, but not until the forced motion solution is known [*Greenwood 1965*].

It is noted that in the case of the gyrocompass the quantity μ is, in general, greater than zero and less than one (in extremely special cases it also assumes the values of 1 and infinity). Hence, the suitable equation to represent the gyrocompass transient response is eqn. (4.27b) and we will use from now on only this expression. In the *time domain* this equation represents a *damped sinusoidal motion*.

4.5 Forced Motion Alone - Steady-State Response

To find the dynamic response of the gyrocompass we begin with eqn. (4.19). For convenience we rewrite it as

$$\ddot{\alpha} + D\dot{\alpha} + F^2\alpha = \mathscr{M}(t) \qquad (4.28)$$

where $\mathscr{M}(t)$, in general, is the *forcing function*, or in particular, is the *applied torque* to the gyrocompass. The forcing function $\mathscr{M}(t)$ may assume the general form

$$\mathscr{M}(t) = M\,e^{st} \qquad (4.29)$$

where M and s may be, in general, complex numbers. This allows us to investigate many types of possible forcing functions depending on the values that M and s assume. For example for M being real we have

$$
\begin{array}{llll}
\text{constant} & \mathcal{M} = M & (s = 0)\ , & \\
\text{exponential} & \mathcal{M} = M\,e^{-\sigma t} & (s = -\sigma)\ , & (4.30) \\
\text{sinusoid} & \mathcal{M} \overset{Re}{=} M\,e^{j\omega t} & (s = j\omega)\ , & \\
\text{damped sinusoid} & \mathcal{M} \overset{Re}{=} M\,e^{-\sigma+j\omega t} & (s = -\sigma+j\omega)\ , &
\end{array}
$$

where Re means *"the real part of"*.

Also the forcing function may be a ramp, a parabola, a periodic random function, or a non-periodic random function, but these cases are not actually considered in here. We state again that the choice of the exponential function (e^{st}) is a very convenient one because the output of our system will be of the same form since it is a linear dynamic system (e^{st} retains the same variable part upon differentiation).

Another very useful procedure to obtain the solution of the forced motion is the method of *undetermined coefficients* (or *Lagrange's multipliers*) which is described in detail in any standard textbook on differential equations. This method will be used in later sections.

Returning to eqn. (4.28) and assuming that the forced response has the form of eqn. (4.29), substitution leads to

$$\alpha_f = \mathcal{A} e^{st}$$

and

$$Cn\ [\mathcal{A} s^2 e^{st}] + mg\ell\gamma [\mathcal{A} s\ e^{st}] + mg\ell\ \Omega \cos\phi [\mathcal{A} e^{st}] = Me^{st}$$

or

$$\{(Cn)s^2 + (mg\ell\gamma)s + (mg\ell\ \Omega \cos\phi)\}\mathcal{A} = M$$

or

$$\mathcal{A} = \frac{M}{(Cn)s^2 + (mg\ell\gamma)s + (mg\ell\ \Omega \cos \phi)}$$

and finally

$$\alpha_f = \frac{Me^{st}}{(Cn)s^2 + (mg\ell\gamma)s + (mg\ell\ \Omega \cos \phi)} \tag{4.31}$$

But from eqn. (4.19) we have

$$\mathcal{M}(t) = -mg\ell\gamma\ \Omega \sin \phi = M = \text{constant}$$

hence from eqns. (4.30) follows that s = 0.

Ultimately the forced response becomes

$$\alpha_f = -\gamma \tan \phi \tag{4.32}$$

The solution of the forced motion is also known as the *steady-state solution* and the corresponding response as the *steady-state response*, and in the above case it persists with undiminished magnitude. This implies that the gyrocompass after the oscillations have ceased (transient response), points to a direction α_f from the true north, thus introducing a systematic error commonly known as the *damping (or latitude) error*. It is obvious from eqn. (4.32) that the damping error is latitude dependent and is also a function of the offset angle γ which is used for introducing the damping action on the gyrocompass. The direct dependence of the damping coefficient σ on γ provides a useful property (alteration of the damping percentage) which will be discussed in Chapter 7 in more detail.

In the next section the total dynamic response of the gyrocompass will be examined in view of the initial conditions of the motion.

4.6 Initial Conditions

We now turn to the problem of finding the two integration constants of the natural motion. We want to find C_1 and C_2, or C_3 and ψ in eqn. (4.27b) in terms of the initial conditions $\alpha(0)$ and $\dot{\alpha}(0)$. We recall that in the case of the gyrocompass the parameter μ (damping ratio) is less than 1, which implies that the gyrocompass is a *subcritically damped* or *underdamped system* [*Cannon 1967; Greenwood 1965*].

We consider $\alpha(t)$ and $\dot{\alpha}(t)$ at $t = 0$ and the alternative form of eqn. (4.27b) is

$$\alpha(t) = C_1 \; e^{(-\sigma+jf)t} + C_2 \; e^{(-\sigma-jf)t} \tag{4.33}$$

and

$$\dot{\alpha}(t) = C_1(-\sigma+jf)e^{(-\sigma+jf)t} + C_2(-\sigma-jf)e^{(-\sigma-jf)t}$$

or for $t = 0$,

$$\begin{aligned} \alpha(0) &= C_1 + C_2 \\ \dot{\alpha}(0) &= C_1(-\sigma+jf) + C_2(-\sigma-jf) \end{aligned} \tag{4.34}$$

Simultaneous solution of eqns. (4.34) yields

$$\begin{aligned} C_1 &= \alpha(0) \quad \left[\frac{\sigma+ jf}{2jf}\right] + \dot{\alpha}(0) \quad \left[\frac{1}{2jf}\right] \\ C_2 &= \alpha(0) \quad \left[\frac{-\sigma+jf}{2jf}\right] + \dot{\alpha}(0) \quad \left[\frac{-1}{2jf}\right] \end{aligned} \tag{4.35}$$

We rewrite eqn. (4.33) in the form

$$\alpha(t) = e^{-\sigma t} \; [C_1 \; e^{jft} + C_2 \; e^{-jft}]$$

and using Euler's equation ($e^{\pm j\theta} = \cos\theta \pm j \sin\theta$) we have

$$\alpha(t) = e^{-\sigma t}[(C_1+C_2)\cos ft + j(C_1-C_2)\sin ft],$$

and finally using equations (4.34) we obtain

$$\alpha(t) = e^{-\sigma t}\{\alpha(0)\cos ft + \frac{1}{f}[\sigma\,\alpha(0) + \dot{\alpha}(0)]\sin ft\} \qquad (4.36)$$

or, alternatively, the transient response has the form

$$\alpha(t) = \alpha_m\, e^{-\sigma t}\cos(ft-\psi) \qquad (4.37a)$$

where

$$\alpha_m = \{[\alpha_{(0)}]^2 + [\frac{\sigma}{f}\,\alpha(0) + \frac{1}{f}\,\dot{\alpha}(0)]^2\}^{1/2} \qquad (4.37b)$$

$$\psi = \tan^{-1}\{[\frac{\sigma}{f}\,\alpha(0) + \frac{1}{f}\,\dot{\alpha}(0)]/\alpha(0)\} \qquad (4.37c)$$

Using the principle of superposition the total dynamic response of the gyrocompass is the sum of the transient and steady-state responses:

$$\alpha(t) = \alpha_T + \alpha_{SS} \qquad (4.38)$$

where

α_T is given by eqns. (4.37), and

α_{SS} is given by eqn. (4.32).

Recapitulating, the equations of motion of a stationary gyrocompass with pendulocity and damping have been developed. The solutions of the equations of motion were found and they are expressed in transient and steady-state terms. Once the transient motion ceases the gyrocompass indicates in a direction off true north computed by eqn. (4.32). This systematic error is called damping error and varies as $\tan\phi$.

CHAPTER 5

EQUATIONS OF MOTION OF A UNIFORMLY MOVING GYROCOMPASS

The last equations in the previous chapter namely, eqns. (4.32) and (4.37), or equivalently eqn. (4.38), define the gyrocompass behaviour in an insufficient way, since until now we have not considered at all the motion of the transporting vehicle. Everything we discussed referred to a stationary gyrocompass with respect to the earth's surface.

In this chapter we discuss the gyrocompass equations of motion and its dynamic response when it is mounted on a moving platform. In particular, the dynamic analysis of a gyrocompass moving on the earth's surface at a latitude ϕ with constant speed v in a certain direction H with respect to true north is examined. The analysis follows the same procedure as in the previous chapter. The transient and steady-state response is evaluated as previously. The only difference is the equations of motion. In the present case the gyrocompass directive force is altered because of the motion of the ship on the earth's surface. This motion has an additional effect on the settling position of the spin-axis, thus introducing an additional error in the gyrocompass indicated headings.

Another important topic examined at the end of this chapter is the necessary and sufficient conditions for a gyrocompass to be *Schuler* *tuned*, that is, the conditions for which the motion of a gyrocompass is not affected by accelerations introduced by the ship which is circum-navigating the earth. Finally, a summary of the gyrocompass errors is given.

5.1 Equations of Motion of a Gyrocompass Mounted on a Moving Vehicle

Before we proceed in the details of the dynamic analysis in this section a few important definitions are stated.

On a moving vehicle on the earth's surface, the inertial angular velocity of the local navigational frame consists of the sum of the earth's angular velocity and the angular velocity of the local navigational frame relative to the earth.

The horizontal component of the inertial angular velocity of the local navigational frame defines the direction of *dynamic north.*

The local navigational frame is a reference frame whose axes are oriented toward the true north and east directions, the third axis being along the local vertical and positive such that the coordinate system is left-handed.

If the ship is steaming at constant speed v and on a course making an angle H with the meridian, the northerly and easterly velocity components are respectively (in magnitude)

$$v_N = v \cos H \quad \text{and} \quad v_E = v \sin H \tag{5.1}$$

The inertial angular velocity of the ship is

$$\Omega^* = \Omega + \dot{\lambda} \tag{5.2}$$

where Ω is the inertial angular velocity of the earth and $\dot{\lambda}$ is the angular velocity of the ship with respect to the earth's surface, i.e., it is the time rate of change in longitude at a latitude $\phi°$ and is given by the formula:

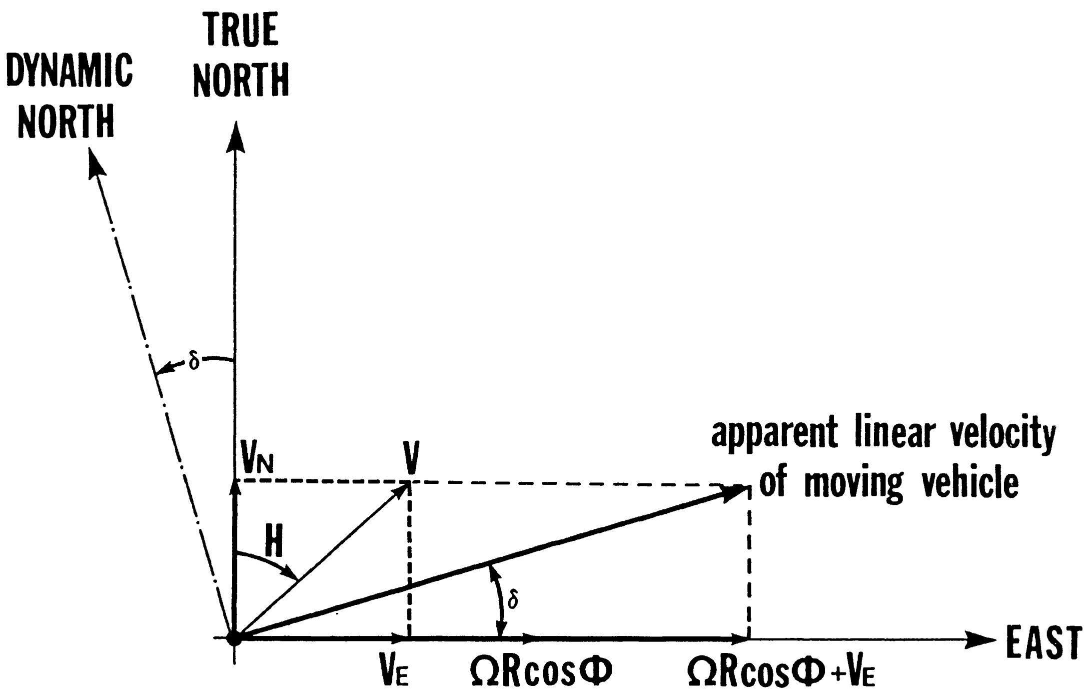
$$\dot{\lambda} = \frac{v_E}{R \cos \phi} = \frac{v \sin H}{R \cos \phi} \tag{5.3}$$

Figure 5.1 illustrates the above described situation.

The gyrocompass is thus, in effect, mounted on a horizontal plane which has an inertial angular velocity Ω^* given by eqn. (5.2). It is obvious that the gyrocompass is incapable of distinguishing the sources that produce Ω^*. It only senses the resultant inertial angular velocity. As a consequence of the above the angular velocity components of the rotor (eqn. (4.3)) about system $O\zeta\xi\eta$ (Fig. 4.2) are changed. Therefore the equations of motion developed in Chapter 4 are not valid. They have to be re-evaluated to include the new situation.

Equations (4.1), (4.2), and (4.3) can be rewritten by substituting Ω with Ω^* from eqn. (5.2). In that case, the kinetic energy of the physical model in hand assumes the form

$$\begin{aligned} T = \frac{1}{2} \{ &(A+A') \, [\dot{\beta}-\Omega^*\cos \phi \sin \alpha]^2 + \\ &(A+B')[(\Omega^*\sin \phi+\dot{\alpha})\cos \beta + \Omega^*\cos \phi \cos \alpha \sin \beta]^2 + \\ &C[\Omega^*\cos \phi \cos \alpha \cos \beta - (\Omega^*\sin \phi+\dot{\alpha})\sin \beta + \dot{\psi}]^2 + \\ &C'[\Omega^*\cos \phi \cos \alpha \cos \beta - (\Omega^*\sin \phi+\dot{\alpha})\sin \beta]^2 + \\ &A''[\Omega^*\cos \phi \cos \alpha]^2 + \\ &B''[\Omega^*\cos \phi \sin \alpha]^2 + C''[\Omega^*\sin \phi+\dot{\alpha}]^2 \} \end{aligned} \tag{5.4}$$



Linear Velocity Diagram

Fig. 5.1

The potential energy is given by

$$U = mg^{*}\ell(1-\cos\beta) \tag{5.5}$$

where g^* is the apparent gravity sensed by the gyrocompass, and is given in detail in *Appendix IV*. This makes a small change in the directive force on the compass, but not in its angular position. In addition to the above we have also assumed constant gravitational acceleration g (which in reality varies with latitude). Therefore, the equations of motion are only affected by the change of the inertial angular velocity Ω into Ω^*.

Pursuing the Lagrangian formulation as before, the final forms of the equations of motion are given here:

$$Cn\dot{\alpha} + mg\ell\beta + mu\ell = -Cn\Omega^{*}\sin\phi \tag{5.6a}$$

and

$$-Cn(\dot{\beta}+\dot{\phi}) + Cn\Omega^{*}\cos\phi\ \alpha - mg\ell\gamma\beta - mu\ell\gamma = 0 \tag{5.6b}$$

where u is the resultant absolute acceleration along the rotor axis resulting from the motion of the vehicle, and Ω^* is the apparent angular velocity of the earth as judged by the gyrocompass. The term $(\dot{\beta}+\dot{\phi})$ indicates the actual tilt of the gyroscope spin-axis due to the motion of the ship, and $\dot{\phi}$ is the time rate of change in latitude, given by the expression

$$\dot{\phi} = \frac{v_N}{R} = \frac{v\cos H}{R} \tag{5.7}$$

After some algebraic operations the equation of motion in azimuth becomes

$$\ddot{\alpha} + \left(\frac{mg\ell\gamma}{Cn}\right)\dot{\alpha} + \left(\frac{mg\ell\Omega^*\cos\phi}{Cn}\right)\alpha = \frac{mg\ell}{Cn}\dot{\phi} - \frac{mg\ell\gamma\Omega^*\sin\phi}{Cn} - \dot{\Omega}^*\sin\phi - \Omega^*\dot{\phi}\cos\phi \tag{5.8}$$

It is shown from eqn. (5.8) that the inclusion of the acceleration terms along the rotor spin-axis in the equations of motion does not affect the final result.

Also, it is obvious from eqn. (5.7) that for ship's speeds not exceeding, for instance, 15 knots the terms $\dot{\phi}$ and $\dot{\Omega}^*$ are both small quantities, and for uniform motion of the ship on the earth's surface $\dot{\phi}$ and $\dot{\lambda}$ can be considered constant.

Equation (5.8) describes the motion of the gyrocompass in a complete and sufficient way. It constitutes the governing differential equation of the gyrocompass mounted on a uniformly moving ship. We seek now the solution of eqn. (5.8) in the same way as we did in Chapter 4.

5.2 Dynamic Response of the Gyrocompass

We will examine the dynamic response of the gyrocompass again in terms of the natural (unforced), and forced motion. It is noted here that, since the motion of the ship is a uniform motion, the coefficients of the differential equation (5.8) are considered constant, thus allowing us to use the theory of ordinary linear differential equations with constant coefficients. The total dynamic response will be the sum of the transient and steady-state responses.

The procedures described in Chapter 4, namely sections 4.4, 4.5, and 4.6 will be used for that purpose. Only the final equations will be given.

Also in the proceeding solution the last two terms in eqn. (5.8) are neglected. The reason is that, $\dot{\Omega}^*$ is small compared with $\Omega^*\dot{\phi}$, and since $(mg\ell/Cn)$ is many times larger than Ω^* we may safely neglect those two terms.

5.3 Natural Motion Alone - Transient Response

The homogeneous equation of motion is deduced from eqn. (5.8) and has the form:

$$(Cn)\ddot{\alpha} + (mg\ell\gamma)\dot{\alpha} + (mg\ell\Omega^*\cos\phi)\alpha = 0 \qquad (5.9)$$

Assuming that the solution is of the form

$$\alpha = \mathcal{A}\, e^{st}$$

the transient response of the gyrocompass is given by

$$\alpha_T = C_3\, e^{-\sigma t}\cos\,(ft-\psi) \qquad (4.27b) \text{ repeated}$$

The constants C_3 and ψ are evaluated in their analytic form later.

The above solution was obtained under the "loose" assumption that $[\Omega^*\cos\,\phi]$ or $[(\Omega+\dot{\lambda})\cos\,\phi]$ is constant. This assumption enables us to obtain an "approximate" straightforward solution for the transient response of the gyrocompass, otherwise the solution would be extremely difficult.

5.4 Forced Motion Alone - Steady-State Response

Now we proceed to examine the resulting response of the gyrocompass to the forced motion alone. Recalling the right-hand side terms of eqn. (5.8) we note that under the assumptions of very slowly varying latitude and longitude and uniform motion, the following are valid:

$$\frac{mg\ell}{Cn}\dot{\phi} \qquad \text{is constant,}$$

$$\frac{mg\ell\gamma\Omega^*\sin\phi}{Cn} \qquad \text{is constant,}$$

$$\dot{\Omega}^* < \Omega^*\dot{\phi} \quad \text{and} \quad \Omega^*\dot{\phi} << \left(\frac{mg\ell}{Cn}\right) ,$$

$$v = ct \quad \text{and} \quad H = ct.$$

Using the procedure outlined in section 4.5, the steady-state response of the gyrocompass is

$$\alpha_{SS} = -\gamma \tan\phi + \frac{\dot{\phi}}{\Omega^* \cos\phi} \qquad (5.10)$$

The first term on the right-hand side of eqn. (5.10) is recognized as the damping error. The second term is a function of the ship's velocity since it contains the quantities $\dot{\phi}$ and $\dot{\lambda}$. It is commonly known as the *speed-and-course error* and it is a universal error for any type of gyrocompass. It is obvious that the speed-and-course error assumes its smallest value (zero degrees) when sailing in an easterly course (H=90°) and its largest value when sailing in a northerly direction. Its significance and its great importance is examined in later sections.

5.5 Initial Conditions

The initial conditions for the gyrocompass transient response are evaluated following the procedure in section 4.6. The total dynamic response of a gyrocompass mounted on a uniformly moving platform is

$$\alpha(t) = \alpha_m e^{-\sigma t}\cos(ft-\psi) - \gamma \tan\phi + \tan\delta \tag{5.11}$$

where;

$$\alpha_m = \{[\alpha(0)]^2 + [\tfrac{\sigma}{f}\alpha(0) + \tfrac{1}{f}\dot{\alpha}(0)]^2\}^{1/2}, \tag{a}$$

$$\psi = \tan^{-1}\{[\tfrac{\sigma}{f}\alpha(0) + \tfrac{1}{f}\dot{\alpha}(0)]/\alpha(0)\}, \tag{b}$$

$$\sigma = \frac{mg\ell\gamma}{Cn}, \quad f_o = \{\frac{mg\ell\Omega^*\cos\phi}{Cn}\}^{1/2}, \tag{c}$$

$$f = \{f_o^2 - \sigma^2\}^{1/2}, \tag{d}$$

$$\tan\delta = \frac{\dot{\phi}}{\Omega^*\cos\phi} = \frac{v\cos H}{\Omega R\cos\phi + v\sin H}, \tag{e}$$

A clear look at eqn. (5.11e) shows what was already pointed in Figure 5.1. Angle δ is the angle that the direction of dynamic north makes with the direction of true north. In other words, the gyrocompass mounted on a moving vehicle tracks the dynamic north instead of true north. The question now arises, is the gyrocompass capable of following the changing dynamic north as the vehicle moves on the surface of the earth?

5.6 Conditions for a Schuler Tuned Gyrocompass

When gyroscopic instruments are mounted on moving vehicles they are liable to be disturbed by the motions of the vehicle.

In the previous section the velocity-induced error δ was found to affect the equilibrium (steady-state) position of the gyrocompass, thus causing it to indicate dynamic north instead of true north (in the absence of the damping error). Errors are also introduced by the acceleration of the vehicle. In fact, even for a uniform motion on the earth's surface, the vehicle is subject to an acceleration (*Appendix V*) due to the earth's curvature. Thus, the equilibrium position of the gyrocompass, once reached, will be disturbed at any time because of vehicle's motion.

Equation (5.11e) gives the equilibrium position of the gyro-axis for any speed and course of the ship when the damping error does not occur (e.g., at the equator). To ease the analysis in this section we will disregard the damping error as affecting the equilibrium position (steady-state) of the gyrocompass, since it does not play any role in the investigation which follows.

The rate of change of δ from equilibrium (under the assumption of small angles $\tan\delta \simeq \sin\delta \simeq \delta$, and regarding the fact that $(\Omega R\cos\phi) \ll (v\sin H)$ for moderate latitudes) is given by

$$\dot{\delta} = \left(\frac{d\delta}{dt}\right) = \frac{\dot{v}_N}{\Omega R\cos\phi} = \frac{u_N}{\Omega R\cos\phi} \qquad (5.12)$$

When the point of suspension of the gyrocompass is given a northward acceleration u_N, the inertia of the pendulous weight mg resists such acceleration by the inertia force mu_N and this exerts a moment $m\ell u_N$

about the point of suspension and in the plane of the meridian.

Since the vector representing this moment points toward the west, the gyroscope spin-axis precesses toward the west with an angular velocity ω_{prec} in the plane elevated at an angle β above the horizon. Hence from eqn. (2.1) we have

$$\omega_{prec} Cn = m\ell u_N \tag{5.13}$$

If there is to be no disturbance in the indication of the gyrocompass, the precessional angular velocity of the angular momentum vector must be equal to the rate of change of the speed-and-course error $\dot{\delta}$. Hence from eqn. (5.13) using eqn. (5.12), we have

$$\frac{u_N}{\Omega R \cos \phi} Cn = m\ell u_N$$

or

$$m\ell\Omega R \cos \phi = Cn$$

or

$$mg\ell\Omega R \cos \phi = g\ Cn$$

and finally

$$\frac{mg\ell\Omega \ \cos\phi}{Cn} = \frac{g}{R} \tag{5.14}$$

We recall two important facts at this stage. First, that the period of oscillation of a pendulum whose length is equal to the earth radius R is equal to

$$T = 2\pi \sqrt{\frac{R}{g}}$$

and second, that the period of the free oscillations of the gyrocompass (the period of the natural undamped oscillation) is, by virtue of the second equation of (4.20) and the definition of page 61,

$$T_o = 2\pi \sqrt{\frac{Cn}{mg\ell\Omega \cos \phi}} \qquad (5.15)$$

Then condition (5.14) indicates that the natural undamped period of the gyrocompass T_o is equal to the period of a pendulum having a length equal to the earth's radius. This period is, for nominal values of R = 6371 km ang g = 9.8 m sec^{-2}, equal to *84.4 minutes*.

A pendulum having a period of oscillation equal to 84.4 minutes will remain vertical over the earth's surface under any arbitrary acceleration of the carrier vehicle.

Equivalently, a gyroscope which is held in its equilibrium position through the force of gravity, will not move out of its equilibrium position under any arbitrary movement over the earth's surface if it possesses the period of 84.4 minutes. Therefore, the gyrocompass would indicate dynamic north regardless of the motion of the ship on which it is mounted.

This is the most important contribution made by Schuler, after whom the period of 84.4 minutes is called *Schuler period.* Any instrument possessing such a natural period of oscillation is to be said *Schuler tuned.* Similarly, *Schuler tuning* is the process of assigning the appropriate values to the physical parameters of a system to meet the necessary and sufficient condition (5.14), if the system is required to function independently of acceleration disturbancies.

From eqn. (5.15) it can be seen that in the case of the gyrocompass Schuler's period is *latitude dependent.* For ϕ = 90° (or very close to 90°) the period of precession of the gyrocompass becomes infinite. That means, once the instrument is disturbed it precesses with a practically infinite period. Thus the operation of the gyrocompass becomes critical as latitude approaches 90°.

In connection with the damping ratio μ, it means that μ tends to infinity, which indicates that, once the gyrocompass is disturbed, several hours are needed for the spin-axis to come to rest at the north direction.

5.7 Error Budget of The Gyrocompass

Until now we have seen the development of the equations of motion of the gyrocompass, their solution using linear system dynamics analysis, and two major sources of errors, namely the damping action and the motion of the transporting vehicle. We have examined the two response motions of the gyrocompass, i.e., the transient response and the steady-state response. The nature of the transient response is a damped sinusoidal motion in the time domain, (or, a spiraling-in motion toward the meridian along the meridian-horizon axes), with starting initial conditions $\alpha(0)$ and $\dot{\alpha}(0)$. The steady-state response implies that the gyrocompass in its equilibrium position (transient motion has ceased) is offset from the meridian by an amount equal to the algebraic sum of the damping and speed-and-course error, (see eqn. (5.11)). Also, we have shown that if the gyrocompass is Schuler-tuned then only velocity-induced errors occur, accelerations not affecting the equilibrium position, and thus, resulting to a gyrocompass which exhibits negligible dynamics in the measurement of the heading of the vessel with respect to true north.

The gyrocompass spin-axis remains at equilibrium unless it is subject to other external disturbance torques about its horizontal and vertical axes. In the following we will examine the possible sources of additional errors instead of their analysis because, they are closely related to the actual design of the gyrocompass, and the means by which the manufacturer seeks their elimination during the design.

Possible sources of gyrocompass errors include the rolling and pitching motion of the vessel, the gimbal suspension, the offset of the centre of gravity of the gyrocompass assembly from the proper position, and finally random disturbances. The above, result in what are known as the: *rolling error*, *gimballing error*, *quadrantal error*, and *random error* respectively, [*American Practical Navigator 1977*, *Arnold and Maunder 1961*, *Rawlings 1944*, *Manual of the Admiralty Gyro-Compass 1953*]. Most of these errors are treated in details in the references listed above and we will not discuss them any further in this work. However, it is our belief that most of them (except of the random occurring errors) have been eliminated through the appropriate mechanical design as that was implicitely stated in the personal communication with the chief engineer of the Sperry firm.

In the previous section we stated the necessary and sufficient condition for a gyrocompass to be Schuler tuned (eqn. (5.14)). Also we found that Schuler's period (eqn. (5.15)) is latitude dependent. The careful mechanical design of the gyrocompass requires the appropriate selection of the gyrocompass design characteristics namely, the angular momentum of the spin-axis Cn and the pendulocity moment $mg\ell$, such that the gyrocompass natural undamped period T_o has the designated value of

84.4 minutes at a certain latitude $\phi°$. Since the operational latitude of the gyrocompass is varying, the above nominal value of 84.4 min. has to be maintained for all possible latitudes. Therefore, the design characteristics should vary in order to fulfill the Schuler condition. In actual compass designs, Schuler tuning is maintained by either adjusting the pendulous weight mg (i.e., varying the mass), or by adjusting the angular momentum Cn (i.e., varying the rotational speed of the gyro-wheel). The first way is followed by the Sperry firm [*Rawlings, 1944*], where the use of the mercury-ballistic is easily providing the necessary effective pendulous weight. The second procedure is followed in the Arma and Anschütz designs.

If, however, the gyrocompass is mistuned (i.e., the natural undamped period of the gyrocompass is not the Schuler period, or in practice, is not close enough to the Schuler period), then the accelerations due to the vessel's motion affect the gyrocompass indicated headings. These accelerations may result either from the uniform ship's motion over the spherical earth (earth's curvature $\frac{1}{R}$), or, from changes in speed and/or course of the vessel. Then, the precessional angular velocity of the disturbed spin-axis will not equal to the rate of change of dynamic north and therefore the gyrocompass will indicate false headings. This situation is especially critical during ship's manoeuvres and is discussed in detail in the next chapter.

The acceleration errors occurring are commonly known as the *ballistic deflection* and *ballistic tilt* errors, referring to the azimuthal and tilt motion of the gyrocompass, respectively. They may exhibit large values and degrade considerably the performance of the gyrocompass, introducing temporary deviations of few degrees. They depend upon both, the dynamics of the problem and the response of the gyrocompass, i.e., the kind and persistance of the acceleration inputs and the dynamic characteristics of the gyrocompass. Table 5.1 summarizes the gyrocompass errors, their sources and possible compensation techniques used to account for their influence on gyrocompass indicated headings.

In conclusion, gyrocompass errors may severely degrade its performance, unless they are accounted for. These errors might be systematic or random. In this study only the systematic effects are considered. Some of the systematic errors are easier to compensate for (e.g. damping and speed-and-course errors), while others are less predictable (e.g. acceleration-induced errors), and they may vary rapidly with time.

Table 5.1.: Sperry-type, single-rotor Gyrocompass Error Budget

Gyrocompass error	Source of error	Compensation Technique
• damping error	damping action	(a) mechanical (preset latitude, ϕ_c^o) (b) software compensation
• speed-and-course error	ship's northerly velocity	(a) mechanical (preset speed, u_c) (b) software compensation
• ballistic deflection error	acceleration	(a) Schuler tuning for a specific latitude (b) software compensation
• roll-and-pitch errors	roll and pitch motion of the ship	(a) eliminated through the mechanical design
• gimballing errors	suspension type	(a) eliminated through designing
• random errors	random torques	(a) software compensation only

CHAPTER 6

EQUATIONS OF MOTION OF A GYROCOMPASS UNDER SHIP'S MANOEUVRES

In the previous chapter we investigated the dynamic response of the gyrocompass when mounted on a ship sailing under uniform motion, i.e., v = const. and H = const. But, this is hardly the real situation. The ship often performs changes in course and speed and manoeuvres. Our objective is to model the gyrocompass performance by modelling the ship's motion in a realistic and adequate way. Thus, errors introduced to the gyrocompass due to ship's arbitrary motion can be predicted and, if properly assessed, provide correct heading information.

Under these circumstances the inertial angular velocity of the local navigational frame Ω^* is varying with time, since it depends on $\dot{\lambda}$ which is now a function of time. Therefore, the equations of motion resume the same form as before, but now the solution is completely different. A special model devised to represent the ship's track helps to overcome the difficulties encountered, due to the fact that the differential equation of motion has now time-dependent coefficients. The dynamic response of the gyrocompass is again computed using techniques of linear dynamic systems analysis. In view of the above, the transient response of the gyrocompass assumes particular interest. The transient response is examined by finding the appropriate initial conditions.

A special case is investigated under the assumption that the ship's track is a circular one. The time-dependent coefficient in the differential equation of motion takes a particular form. It becomes periodic and the solution involves elements of the general perturbation theory.

In this chapter the most useful and important properties of linear systems' analysis are used. The general method of modelling the ship's track can be applied to any particular path configuration and allows the gyrocompass response to be evaluated under the most complicated situations.

6.1 Equations of Motion Under Ship's Manoeuvres

When a ship travels along an arbitrary track the apparent angular velocity of the earth Ω^*, which the gyrocompass senses, is given by the expression

$$\Omega^*(t) = \Omega + \frac{v(t) \sin [H(t)]}{R \cos [\phi(t)]} \tag{6.1}$$

where $v(t)$ is the instantaneous speed of the ship and $H(t)$ the instantaneous course. From eqn. (6.1) we see that the inertial angular velocity of the local navigational frame is time-dependent.

The equation of motion of the gyrocompass is the same as eqn. (5.8) but now we substitute Ω^* by $\Omega^*(t)$, which yields

$$D\ddot{\alpha} + E\dot{\alpha} + G(t)\alpha = \sum_{i=1}^{4} F_i(t) \tag{6.2}$$

where $D = Cn,$ $\quad F_1(t) = mg\ell\,\dot{\phi}(t),$

$$E = mg\ell\gamma, \qquad F_2(t) = -mg\ell\gamma\Omega^*(t)\,\sin[\phi(t)]\ ,$$

$$G(t) = mg\ell\Omega^*(t)\,\cos\phi, \qquad F_3(t) = -\dot{\Omega}^*(t)\,\sin[\phi(t)]\ ,$$

$$F_4(t) = -\Omega^*(t)\,\dot{\phi}(t)\,\cos[\phi(t)]\ .$$

Equation (6.2) is a linear, second-order differential equation with a time-dependent coefficient and time-dependent forcing functions.

6.2 General Modelling of Ship's Track During a Manoeuvre

When a ship starts to manoeuvre we need to model its motion in an expressed functional way, so the time-dependent coefficient G(t) in eqn. (6.2) assumes a form which is easier to cope with.

We will examine here the most general case which involves both change in speed and change in heading. During a turn the ship's track becomes uncertain owing to the characteristics of the vessel, the load, wind conditions, currents, etc. When a vessel moves in a curved path there is an acceleration towards the centre of path, due to the centripetal force. In a ship the centripetal force is the resultant of all the lateral forces acting on the hull and rudder. From the literature related to the study of ship's motions, the following conclusions are drawn:

i. each vessel has its own turning characteristics,

ii. its behaviour during manoeuvres is unique, and usually it is obtained after sea-trials,

iii. there is not a general model which can provide valuable information to model the longitude and latitude rates of change, $\dot{\lambda}$ and $\dot{\phi}$ respectively, as the vessel performs course and speed changes, and

iv. it is extremely difficult, if not impossible, to calculate disturbing torques applied to the gyrocompass due to ship's manoeuvring.

In *Appendix VI* useful information related to the above problem is provided. It is difficult though to make use of this information. The reasons are stated in the same Appendix. A different approach is suggested as an alternative to cure the problem of modelling the ship's track. Since on board ship there are speed and course sensors, the motion of the vessel can be sampled at any desirable, discrete time instant. Thus, the time dependent-terms in eqn. (6.2) can be evaluated from these observations, namely from $v(t_i)$ and $H(t_i)$. In this case, we can assume that we are sampling the motion at small time intervals Δt_k, thus the velocity vector is supposed to have a discrete change from time t_i to time t_j, and in addition to remain constant during the interval Δt_k. The apparent angular velocity of the earth $\Omega^*(t)$ is again given by eqn. (6.1), but now it can be evaluated at the discrete instants t_i and t_j, remaining constant through the interval Δt_k. Therefore, the differential equation which governs the motion in azimuth of the gyro spin-axis takes the form

$$D\ddot{\alpha} + E\dot{\alpha} + G(t_j)\alpha = \sum_{i=1}^{4} F_i(t_j) \qquad (6.3)$$

where $G(t_j)$ and $F_i(t_j)$ are not time-dependent, but assume different values at the discrete instants t_j. Hence at each time Δt_k we have one complete solution of equation (6.3).

6.3 Gyrocompass Dynamic Response

The solution of eqn. (6.3) is given in the same form as the solution of eqn. (5.8) with the only difference being that it should be evaluated at every sampling interval Δt_k. The complete solution is given by the expression

$$\alpha(t_j) = \alpha_m\Big|_{t_j} e^{-\sigma tj}\cos(ft_j+\psi(t_j)) - \gamma \tan \phi_j + \tan \delta_j \qquad (6.4)$$

or, in the simple form

$$\alpha(t_j) = \alpha_T(t_j) + \alpha_{SS}(t_j)$$

where $\alpha_T(t_j)$ is the transient response of the gyrocompass at the sampling instant t_j and valid over the interval Δt_k until the next sample is obtained, and $\alpha_{SS}(t_j)$ is the steady-state response of gyrocompass during the same interval Δt_k. The amplitude $\alpha_m\Big|_{t_j}$ and the phase angle $\psi(t_j)$ of the transient response are again evaluated from the initial conditions $\alpha(0)$ and $\dot{\alpha}(0)$ at each discrete sampling instant t_j and we assume that they remain valid until the next sampling instant t_{j+1}, i.e., they remain valid over the interval Δt_k.

Before proceeding in the investigation of the initial conditions $\alpha(0)$, $\dot{\alpha}(0)$ at each sampling instant, we will spend some time on the particulars of the dynamic response.

A dynamic system is said to be in the *steady-state* when the variables describing its behaviour are either *invariant with time*, or are (sections of) *periodic functions of time*, [*Gardner and Barnes 1942*]. From a physical point of view, it may be said that a *transient state* exists in a physical system while the energy conditions of one steady-state are being changed to those of a second steady-state, [*Gardner and Barnes 1942*].

We define the *unit step function* as a forcing function of unit magnitude applied to a dynamic second-order linear system at time $t = 0$ [*Greenwood 1965*]

$$F(t) = u(t) \tag{6.6}$$

where the unit step function $u(t)$ is shown in Figure 6.1.

If we write the differential equation of a linear, damped, second-order dynamic system as

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{6.7}$$

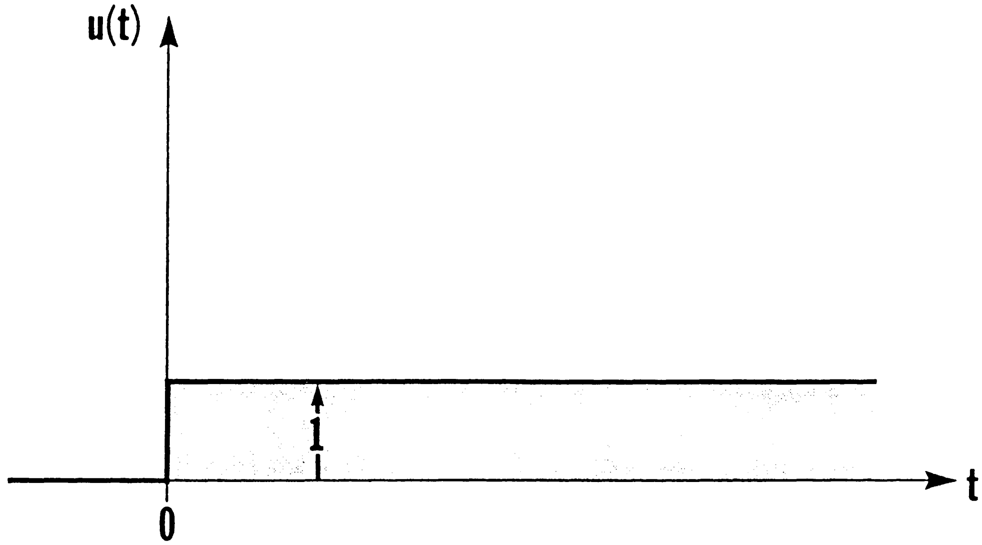
then the steady state solution for $F(t) = u(t)$ is simply

$$x_{SS} = \frac{1}{k} \tag{6.8}$$

Assuming that the initial velocity and displacement of the above mass-spring damper system (eqn. (6.7)) are zero for the complete solution, we can immediately solve for the initial conditions on the transient portion of the solution.

$$\begin{aligned} x_T(0) &= -x_{SS}(0) = -\frac{1}{k} \\ \dot{x}_T(0) &= -\dot{x}_{SS}(0) = 0 \end{aligned} \tag{6.9}$$

It is seen that they are just the negative of the steady-state values at $t = 0$, for this particular example of the unit step forcing function. So, we can write the transient solution directly from the results obtained previously as



Unit Step Function

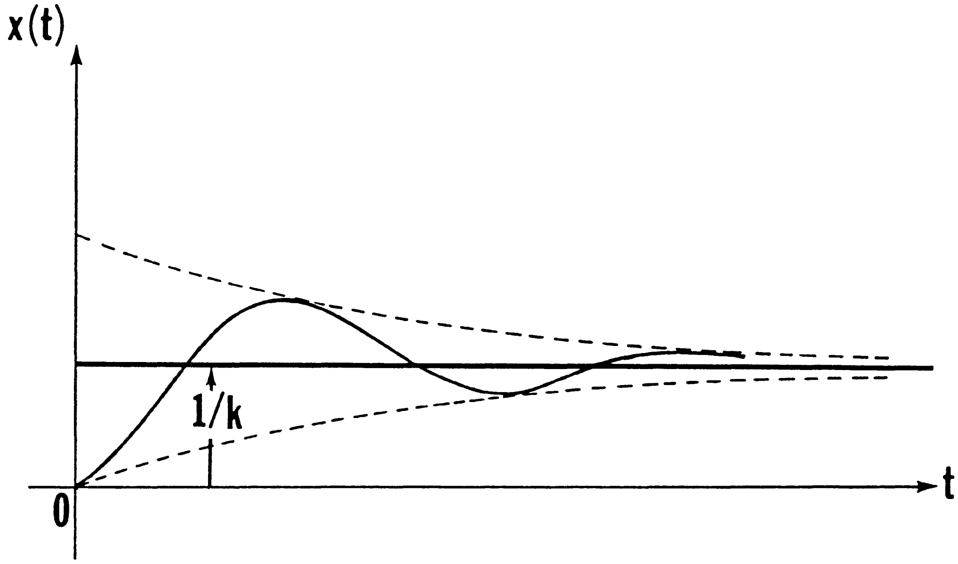
Fig. 6.1

$$x_T(t) = C_3\, e^{-st} \cos\,(ft-\psi) \qquad (6.10)$$

where C_3 and ψ are evaluated, as it was shown in the previous chapters, from $x_T(0)$ and $\dot{x}_T(0)$. Figure 6.2 shows the corresponding response of an underdamped system to a unit step input.

Concluding, the response of a linear, underdamped, second-order dynamic system to a step function looks very similar to the transient response except that the initial slope (i.e., $\dot{x}(0)$) is zero. Actually, the application of a constant force can be thought of as simply changing the static equilibrium position of the system [*Halfman 1962*]. With a redefined *displacement coordinate*, the response to the step function becomes merely the transient response to a negative initial displacement [*Halfman 1962*]. If we have a series of step functions, since we are dealing with a linear system, we can find the response to each step separately, and then apply the superposition principle to get the total response [*Halfman 1962*].

Bearing in mind the definitions and properties just described above we treat the gyrocompass case in a similar fashion. At each sampling instant the steady-state of the gyrocompass is changing. The difference of the steady-state values is viewed as a step forcing function resulting in a response similar to the transient one. The starting initial conditions at the beginning of a manoeuvre are as described in eqn. (6.9). The series of the step functions is resulting in the total gyrocompass dynamic response, which is a damped sinusoidal oscillation about the continuously changing equilibrium position. Since the gyrocompass is a mechanical system and as such it cannot exhibit abrupt changes either in "displacement" (α) or in "velocity" ($\dot{\alpha}$) by any



Response of an Underdamped System
to a Unit Step Function

Fig. 6.2

finite torque input [*Cannon 1967, pp. 201-202*], the following constraints are imposed on the initial conditions (after the disturbance has started) for two time instants t_1 and t_2

$$\alpha_{t_2}(0^+) = \alpha_{t_2}(0^-) + \{\gamma(\tan\phi_2 - \tan\phi_1) - (\tan\delta_2 - \tan\delta_1)\} \tag{6.11}$$

where

$$\alpha_{t_2}(0^-) = \alpha(t_1 = \Delta t_1)$$

and

$$\dot{\alpha}_{t_2}(0^+) = \dot{\alpha}_{t_2}(0^-) = \dot{\alpha}(t_1 = \Delta t_1) \tag{6.12}$$

where

$$\Delta t_1 = t_2 - t_1 .$$

In the preceding analysis, the last two terms in the right-hand side of equation (6.2) (or, eqn. (6.3)) have been neglected because they are very small.

6.4 Special Case - Circular Arc Approximation

In addition to the general case of modelling the ship's motion in discrete time intervals described in the previous section, a special case is investigated. The assumption we adopt is that the turn of the ship can be treated as a circular arc, which is probably reasonable for favourable sea conditions [*Rose 1974*].

Equation (6.1) is rewritten in the form

$$\Omega^*(t) = \Omega + \frac{v \sin[H(t)]}{R \cos\phi} \tag{6.13}$$

where

$$v = \text{constant} \quad \text{and} \quad H(t) = H_o - \omega t \tag{6.14}$$

H_o is the initial heading at the beginning of the turn

ω is the circular frequency of the turn.

In order to ease calculations we start from a northern course ($H_o = 0°$) and for a left circulation we have

$$H(t) = -\omega t \quad \text{and} \quad \Omega^*(t) = \Omega - \frac{v \sin \omega t}{R \cos \phi} \tag{6.15}$$

Thus, the term $(mg\Omega^*\cos\phi)/Cn$ from eqn. (5.8) becomes

$$\frac{mg\ell\Omega^*\cos\phi}{Cn} = \frac{mg\ell\Omega\cos\phi}{Cn} - \frac{mg\ell v \sin \omega t}{R\,Cn}$$

but

$$\frac{mg\ell\Omega\cos\phi}{Cn} = f_o^2 \quad \text{and} \quad \frac{mg\ell}{Cn} = \frac{2\sigma}{\gamma}$$

and therefore eqn. (5.8) becomes

$$\ddot{\alpha} + \left(\frac{mg\ell\gamma}{Cn}\right)\dot{\alpha} + \left(f_o^2 - \frac{2\sigma v}{R\gamma}\sin\omega t\right)\alpha =$$

$$= \frac{mg\ell}{Cn}\dot{\phi} - \frac{mg\ell\gamma\Omega^*\sin\phi}{Cn} \tag{6.16}$$

if we ignore the last two terms in the right-hand side of eqn. (5.8).

As we did before, we will examine the natural and forced motions separately and we will combine the results, since our system is linear and the superposition principle is applied.

We write the homogeneous part of eqn. (6.16) as

$$\ddot{\alpha} + \left(\frac{mg\ell\gamma}{Cn}\right)\dot{\alpha} + \left(f_o^2 - \frac{2\sigma v}{R\gamma}\sin\omega t\right)\alpha = 0 \tag{6.17}$$

This is a linear differential equation of second-order with a periodic coefficient. The theory of ordinary linear differential equations which we followed in the analysis previously is not applicable any more. We will seek the solution of the above equation in a different way. Thus we rewrite eqn. (6.17) in a more handy form:

$$\ddot{\alpha} + 2\sigma\dot{\alpha} + (f_o^2 - \frac{2\sigma v}{R\gamma}\sin\omega t)\alpha = 0 \tag{6.18}$$

We proceed further by changing the dependent variable α using the linear transformation

$$\alpha = e^{-\sigma t}y \tag{6.19}$$

where y is the new dependent variable. From eqn. (6.19) we have upon differentiation with respect to t the expressions

$$\dot{\alpha} = -\sigma e^{-\sigma t}y + e^{-\sigma t}\dot{y}$$

and

$$\ddot{\alpha} = \sigma^2 e^{-\sigma t}y - 2\sigma e^{-\sigma t}\dot{y} + e^{-\sigma t}\ddot{y}$$

which back substituted in eqn. (6.18) give the expression

$$\ddot{y} + [(f_o^2 - \sigma^2) - \frac{2\sigma v}{R\gamma}\sin\omega t]y = 0 \tag{6.20}$$

A careful look at eqn. (6.20) indicates that it is an equation of the *Mathieu type*. We perform some rearrangement of terms of the above equation which leads to

$$\ddot{y} + k^2 y = (\frac{2\sigma v}{R\gamma}\sin\omega t)y \tag{6.21}$$

where $k^2 = f_o^2 - \sigma^2 = f^2$, and now we change the independent variable through a new linear transformation

$$\omega t = \tau \tag{6.22}$$

Let

$$\frac{d}{d\tau} = \frac{1}{\omega}\frac{d}{dt} \quad \text{and} \quad \frac{d^2}{d\tau^2} = \frac{1}{\omega^2}\frac{d^2}{dt^2},$$

then the differential equation (6.21) assumes the form

$$\frac{d^2y}{d\tau^2} + \frac{k^2}{\omega^2}y = \frac{2\sigma v}{R\gamma\omega^2}\sin\tau\ y$$

or

$$\frac{d^2y}{d\tau^2} + n^2y = \varepsilon \sin\tau\ y \tag{6.23}$$

where

$$n^2 = \left(\frac{k}{\omega}\right)^2 \quad \text{and} \quad \varepsilon = \frac{2\sigma v}{R\gamma\omega^2}$$

Equation (6.23) is more easily recognised as the Mathieu equation. The magnitude of the parameters n^2 and ε is approximately 2.2×10^{-3} and 5.5×10^{-5} respectively.

Without any further details of the problem we seek the solution of eqn. (6.23) using a *general perturbational solution* in the form [*Struble and Fletcher 1962*]

$$y = Y\cos(n\tau-\rho) + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon^N y_N$$

where each of Y, ρ, y_1, y_2, ..., y_N is a variable. Then the first order solution of (6.23) is given by

$$y(\tau) = Y\cos(n\tau-\rho) - \varepsilon\left\{\frac{Y}{2(2n+1)}\cos[(n+1)\tau-\rho] - \frac{Y}{2(2n-1)}\cos[(n-1)\tau-\rho]\right\} \tag{6.24}$$

The details and the necessary conditions for the above solution can be found in *Struble and Fletcher (1962)*. By applying the inverse transformation using eqn. (6.22) we arrive at

$$y(t) = Y\cos(ft-\rho) - \varepsilon\left\{\frac{Y}{2(2\frac{f}{\omega}+1)}\cos\left[\left(\frac{f}{\omega}+1\right)\omega t-\rho\right] - \frac{Y}{2(2\frac{f}{\omega}-1)}\cos\left[\left(\frac{f}{\omega}-1\right)\omega t-\rho\right]\right\} \qquad (6.25)$$

and

$$\alpha_T(t) = e^{-\sigma t}y(t) \qquad (6.26)$$

Questions such as of *stability of motion* have been already answered in *Struble and Fletcher (1962)*, but also are obvious from eqn. (6.26).

Assuming that the initial conditions are $\alpha(0)$ and $\dot{\alpha}(0)$, the final form of the gyrocompass transient response is

$$\alpha_T(t) = A_m e^{-\sigma t}\{\cos(ft-\rho_m) - \qquad (6.27)$$

$$- \frac{\varepsilon}{2}\left(\frac{\omega}{(2f-\omega)}\cos[(f+\omega)t-\rho_m] - \frac{\omega}{(2f-\omega)}\cos[(f-\omega)t-\rho_m]\right)\}$$

where;

$$A_m = \left\{\left[\frac{\alpha(0)}{\left(1+\frac{\varepsilon\omega^2}{4f^2-\omega^2}\right)}\right]^2 + \left[\frac{\left(\frac{\sigma}{f}\alpha(0) + \frac{1}{f}\dot{\alpha}(0)\right.}{\left(1-\frac{\varepsilon\omega^2}{4f^2-\omega^2}\right)}\right]^2\right\}^{1/2} \qquad (a)$$

$$\rho_m = \tan^{-1}\left(\left\{\frac{\left(1+\frac{\varepsilon\omega^2}{4f^2-\omega^2}\right)}{\left(1-\frac{\varepsilon\omega^2}{4f^2-\omega^2}\right)}\right\}\left[\frac{\sigma}{f} + \frac{1}{f}\frac{\dot{\alpha}(0)}{\alpha(0)}\right]\right) \qquad (b)$$

and

$$\varepsilon = \frac{2\sigma v}{R\gamma\omega^2}, \quad f^2 = f_o^2 - \sigma^2, \quad \omega = \frac{2\pi}{T} .$$

Some remarks are necessary at this point. First of all, the gyrocompass transient response is a function of the circular frequency ω with which the ship is circulating. Secondly, if the circular motion ceases (i.e., $\omega = 0$ and $H = H_o$ = constant) the transient response will assume the same expression as in eqns. (5.11), which is the expression of the transient response of the gyrocompass mounted on a uniformly moving platform, a fact that is expected. Thirdly, the effect of the circular ship's motion is reflected in the transient response as a slight modification of the magnitudes of the amplitude and the phase angle of the response, and as a slight modification of the sinusoidal part of the motion of the spin-axis.

The steady-state response of the gyrocompass is given by

$$\alpha_{SS} = -\gamma \tan \phi_j + \tan \delta_j \tag{6.28}$$

evaluated at any specific time instant j.

The complete dynamic response of the gyrocompass is the sum of the responses namely,

$$\alpha(t) = \alpha_T(t) + \alpha_{SS} \tag{6.29}$$

In conclusion, the dynamic response of the gyrocompass was found in the case of an arbitrary motion of the transporting vessel over the earth's surface. The general model employed to represent the motion of the vessel indicates potential superiority over the simpler model of the circular arc approximation. The equations of motion of the gyrocompass do not change form from the previously used equations, but their solution involves a considerable amount of mathematical complication due to the presence of the time-dependent coefficient.

It is important to note here that the dynamic response of the gyrocompass using the general model of the ship's track is valid for any case, irrespective of the fact that the ship is either executing a manoeuvre or is sailing along a straight line. This fact indicates that even the smallest changes in course and/or speed can be modelled in a straightforward way and, hence, correct gyrocompass heading information is obtainable.

CHAPTER 7

HIGH LATITUDE BEHAVIOUR OF THE GYROCOMPASS

Three basic questions are asked and answered in this chapter. First, what is meant by the term high latitude? Second, why are we concerned with high latitude navigation? And, finally, how does the gyrocompass perform at high latitudes?

High latitude in this study is defined as the geographic location beyond the parallel of 70°. In recent years, the arctic areas of the world have been of tremendous interest for both, the scientific and economic society. Understanding the physical processes and phenomena of the arctic areas guides the scientific society. National priorities in energy resources exploration leads the interests of the economic society to new virginal areas of the planet, such as the Arctic.

The increased demand for exploration activities in these areas is the reason for *high latitude navigation*. The unique environmental conditions of the arctic constitutes the primary limitation in accurate and reliable navigation. The environmental factors imposing limitations in navigation are remoteness and isolation, severe weather conditions, ice-covered seas and low-lying ice-covered coast-lines [*American Practical Navigator 1977*].

In addition, the high latitude places restrictions on the performance of navigation sensors. The operation of north-seeking gyroscopes is severely affected because the directive force (earth's horizontal component of angular velocity $\Omega \cos \phi$) is greatly reduced. The present satellite navigation system (NNSS) also has degraded performance at high latitudes [*Wells and Grant 1981*]. Coverage by either traditional or modern electronic aids to navigation is limited [*American Practical Navigator 1977*]. Ice-covered seas and lands severely attenuate radio navigation signals [*Wells and Grant 1981*]. Celestial navigation is impaired by the arctic weather. Under these conditions it is important to use the full strength and potential of all available navigation sensors, including the gyrocompass, to provide acceptable navigational accuracy and reliability [*Wells and Grant 1981*].

The performance of the gyrocompass degrades with latitude. The damping error is increased as $\tan \phi$. Speed-and-course error increases too. In Chapter 5 we discussed the Schuler-tuning as the appropriate function for reducing the acceleration-induced errors, and also it was shown that the undamped natural period of the gyrocompass (Schuler-period) is latitude dependent. It was stated there, that as latitude increases the period T_o increases, thus, instability of the gyrocompass becomes the critical factor.

All the above statements make necessary the study of the gyrocompass behaviour at navigable high latitude waters (70° to 80° latitude). In this chapter the equations of motion of the gyrocompass are investigated in view of high latitude operations. The problems associated with high latitude and the functional limitations of the gyrocompass are explored. The mathematical models developed in the previous sections are examined under the new conditions. Finally, considerations are given to the possible mechanical adjustments and/or compensation techniques required to reduce the problems due to high latitude degraded performance of the gyrocompass.

7.1 Equations of Motion

The gyrocompass depends, for its directive property, on the horizontal component of the earth's angular velocity $\Omega \cos \phi$ and, thus, it becomes less satisfactory in high latitudes where this component is greatly reduced.

The equations of motion developed in Chapter 5, namely eqn. (5.8), are still valid. The proposed general modelling of the ship's track is applicable and the gyrocompass response is as given in eqns. (5.11).

7.2 Problems and Limitations Imposed by High Latitude - Mechanical Adjustments and Compensation Techniques

Recall eqn. (5.15) which gives the period of undamped natural oscillations of the gyrocompass

$$T_o = 2\pi \sqrt{\left(\frac{Cn}{mg\ell\Omega \cos \phi}\right)} \quad .$$

An increase in latitude increases the undamped natural period of free oscillations of the gyrocompass. However, we have seen in section 5.6, that in order to avoid acceleration effects it is necessary to retain the value of 84.4 minutes for T_o, either by adjusting the pendulous moment $mg\ell$, or by changing the angular momentum Cn of the gyro-wheel varying its rotational speed n. In the Sperry instruments this is achieved by varying $mg\ell$ as $\sec \phi$. But practical difficulties are involved since the value of $mg\ell$ at latitude $\phi = 85°$ requires to be 11.5 times its value at the equator. To overcome this problem the manufacturer has devised a small adjustable weight, called *latitude rider*, which is added to the gyro-casing to assist the normal precession at high latitudes (Sperry-type gyrocompass) [*Arnold and Maunder 1961, Manual of The Admiralty Gyrocompass 1953*]. If this additional weight $m'g$ is situated at a distance ℓ' along the horizontal axis (northern-axis), then it will provide a torque which results in a precession in azimuth at right angles to it, and by adjusting ℓ' it is possible to make the precession rate $(m'g\ell')/Cn \simeq \Omega \sin \phi$, which is the required precession at latitude ϕ [*Arnold and Maunder 1961*]. Although the above procedure results in a definite improvement of the gyrocompass performance at high latitudes, its practicality and adaptability is somewhat questionable for the following reasons:

i. the latitude rider is set for a specific latitude, and in order to adjust it in other latitudes manual mechanical intervening by experienced personnel is required,

ii. the maximum setting of the latitude rider is for latitude of 70° approximately [*Manual of the Admiralty Gyrocompass 1953*],

iii. specific knowledge of the gyrocompass design is required. Therefore, possible mathematical analysis is limited due to lack of particular information (m', ℓ', location of $m'g$),

iv. manuals of recent gyrocompass models (e.g. *Operation and Service Manual of the Sperry MK37 Gyrocompass Equipment* [1977]) do not include any information about the subject,

v. the commercial gyrocompass designs are intended for operation up to 65°-70° latitudes. Appropriate functioning beyond these latitudes being subject to special adjustments, or special gyrocompass systems.

Under these circumstances, the possiblity of examining the new conditions, in view of high latitude modified equations of motion of the gyrocompass, becomes severely minimal.

So, in effect, in the analysis carried out here, no such mechanical adjustments were considered and the undamped natural period of oscillations of the gyrocompass T_o (and consequently the damped period T) were considered in the most general case.

A further effect to be examined is what happens with the relative damping, i.e., the damping ratio μ. If μ assumes the value of 1 then $\sigma = f_o$ and the period of damped oscillations T becomes infinite. Thus, if T_o is to be retained at reasonable levels not differing from 84.4 minutes by much, the damping ratio must be kept less than 1. This can be achieved only by reducing the magnitude of σ, (recall that $\sigma = (mg\ell\gamma/2Cn)$). So a trade off must be found between the relative magnitudes of mg, γ, and f_o. Thereby, the offset angle γ is usually decreased to allow reasonable operation of the gyrocompass at high latitudes. The latter (smaller γ) has a positive effect on gyrocompass error budget. Smaller γ values reduce the damping error (i.e., $-\gamma \tan \phi$) which is expected to be large at high latitudes.

However, reducing the value of γ involves once again human intervening and mechanical adjustment. Thus, when a gyrocompass is to be used in latitudes higher than 70° the necessary mechanical adjustments have to be performed, if possible, before sailing. On return to lower latitudes the adjusted quantities and parameters should return to their normal values. This creates a complex operating procedure. The situation becomes even more complex if the ship is supposed to sail back and forth in high and lower latitudes for some time, thus continually adjusting the corresponding parameters.

Finally, we have to note that at high latitudes the speed-and-course error and the ballistic deflection error are considerable. It is obvious that the ballistic deflection (which is the movement of the gyro spin-axis resulting from vehicle acceleration), is receiving most of our attention because Schuler tuning is not likely to exist at high latitudes. This source of error (i.e., ship's acceleration) becomes even more critical when manoeuvring of the vessel is involved. Then the only means by which acceleration-induced errors can be predicted and accounted for is computational. In addition velocity-induced errors, although predictable, they become larger and larger as latitude increases because the corresponding linear velocity component of the earth $\Omega R \cos \phi$ decreases. Therefore, the errors encountered at high latitudes become excessively large and the gyrocompass performance degrades rapidly.

In summary, there are two main reasons that cause the performance of a gyrocompass to degrade, the increased bias errors and the increased instability due to lack of Schuler tuning.

CHAPTER 8

SIMULATION STUDIES AND RESULTS

In order to test the mathematical models developed in the previous chapters and evaluate the dynamic response of the gyrocompass, simulation studies were carried out.

These simulation studies include the assumptions made for the physical model, the evaluation of the system's behaviour, the performance characteristics, and the results obtained.

The goals to be achieved are two-fold. The first is to apply the modelling methods of the ship's track developed in the previous sections and test their validity. The second is to obtain the dynamic response of the gyrocompass as a function of time, and compare it with existing information to assure the ability in predicting the gyrocompass deviations from true north under different dynamic inputs.

The simulations are referred to a particular gyrocompass, the Sperry MK V gyrocompass, for which the following constants are the design parameters [from *Arnold and Maunder 1961*:

$Cn = 207$ ft lb sec , (angular momentum of the gyro-wheel)

$mg\ell = 6.3$ ft lb , (the pendulous moment)

$\gamma = 1^\circ\!.55$ (or 0.02705 radians), (the offset angle).

A FORTRAN program is designed to perform the computations. It is based on the mathematical models presented in Chapters 5 and 6.

The different dynamic inputs include various types of manoeuvres (90°, 180°, 270° and 360°), at different speeds (5, 10, and 15 knots), and latitudes (30°, 60°, 75°, and 80°).

The program computes the transient response of the gyrocompass based on the appropriate initial conditions and the steady-state values. The ballistic deflection is incorporated into the program computations (Appendix V).

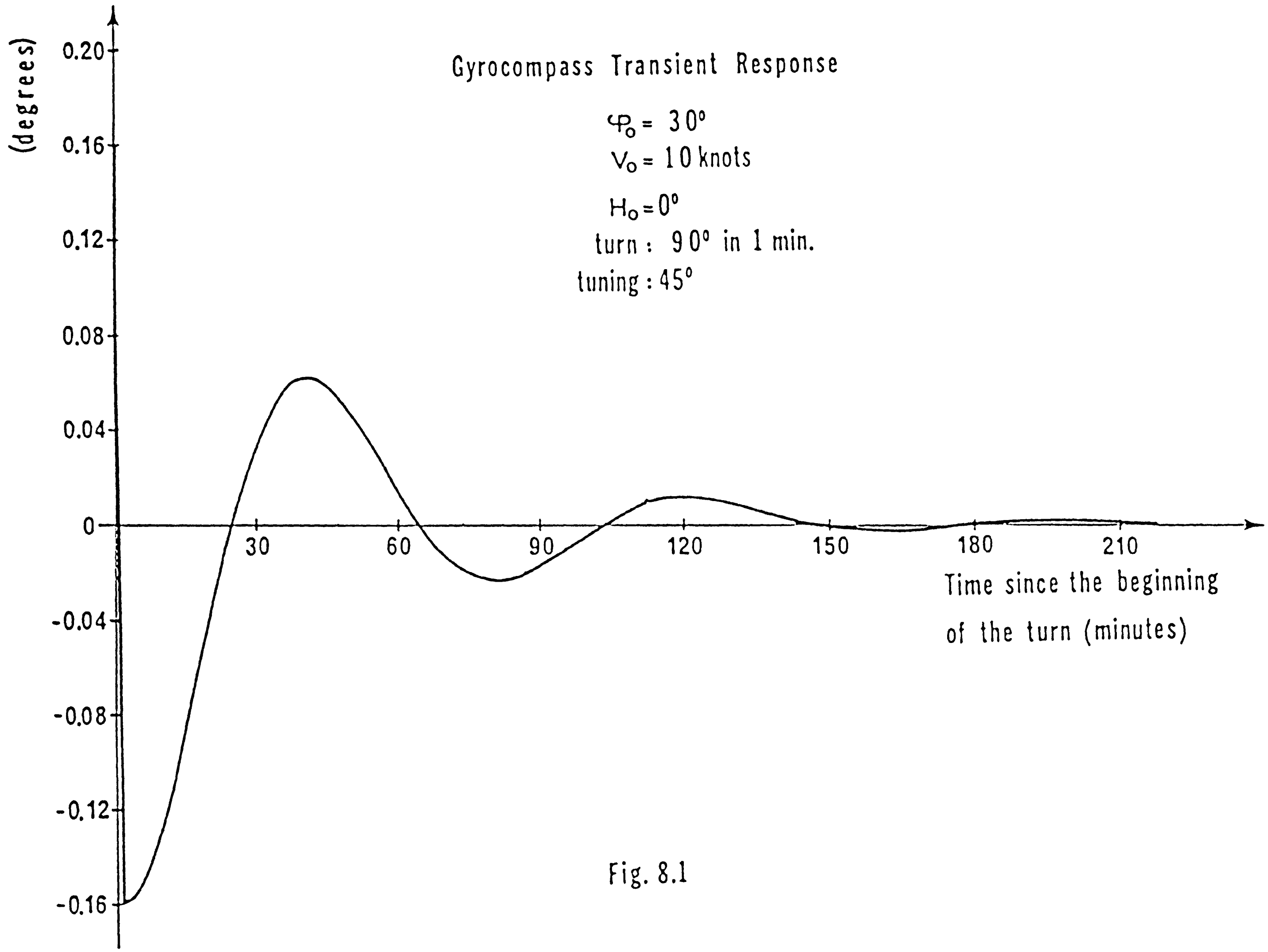
8.1 Simulation Results Using the General Modelling of the Ship's Track

In this part of the simulation study eqns. (6.4), (6.5), (6.11) and (6.12) are used and the transient response of the gyrocompass is computed.

The first concern is to examine the effect of the dynamic inputs (speed, manoeuvre characteristics) on the magnitude of the initial amplitude of the oscillations and the corresponding time to damp out their influence on the indicated headings. The second goal is to examine the above as a function of latitude.

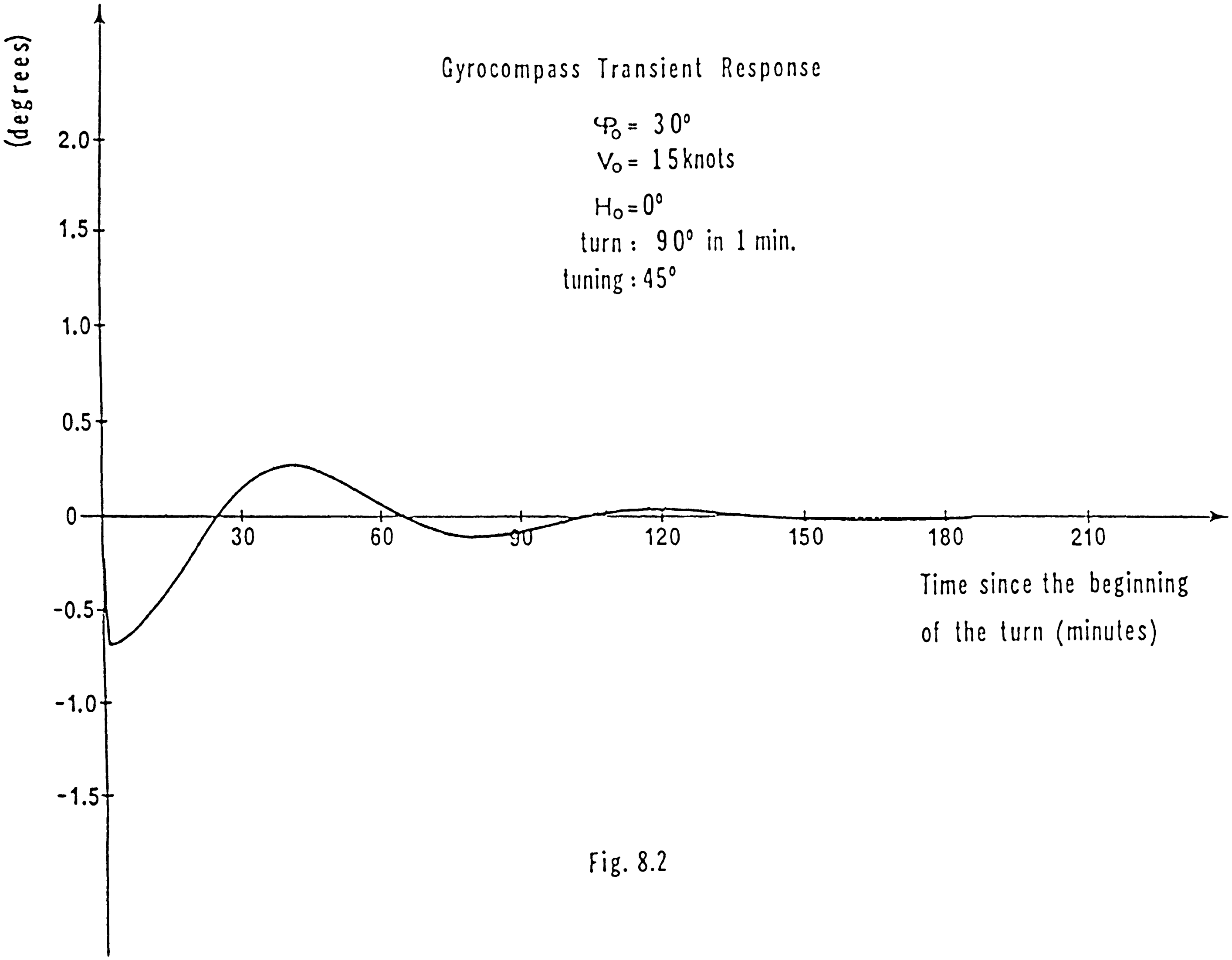
Various sampling time intervals Δt_k were tested. It is noted here that the sampling interval must be less than (or, equal to) the time within the manoeuvre is completed, otherwise spurious results may be obtained. The interval Δt_k can be as small as the sampling interval for v and H (ship's speed and course indication). In this manner, more accurate estimates of the changing equilibrium (steady-state) position of the gyrocompass can be determined. Consequently, more accurate estimates of the transient errors of the compass can be computed. However, the Δt_k interval may assume various values depending on the particular application. For general navigation purposes that interval may be as long as the intervals associated with speed and/or course changes. For precise sea-gravimetry using *real-time Eötvös correction* that interval should be as small as one second (1^s) [*Wells and Grant 1981*].

Figures 8.1 to 8.10 show the gyrocompass transient response for two different latitudes 30° and 60° and for different manoeuvring characteristics. The gyrocompass is tuned for 45° latitude and the initial heading at the beginning of the manoeuvre is $H_o = 0°$. Several rates of turn are also used. From the results obtained it is concluded that the transient errors are significant immediately after the completion of the manoeuvre. In general, the magnitude of the transient errors tapers off an hour after the manoeuvre has ended. During this interval their influence on the knowledge of the actual ship's heading is obviously prominent.



(degrees)
0.20
0.16
0.12
0.08
0.04
0
-0.04
-0.08
-0.12
-0.16
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)
Gyrocompass Transient Response
$\varphi_o = 30°$
$V_o = 10$ knots
$H_o = 0°$
turn : 90° in 1 min.
tuning : 45°

Fig. 8.1



Gyrocompass Transient Response
$\varphi_0 = 30°$
$V_0 = 15$ knots
$H_0 = 0°$
turn : 90° in 1 min.
tuning : 45°
(degrees)
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.2

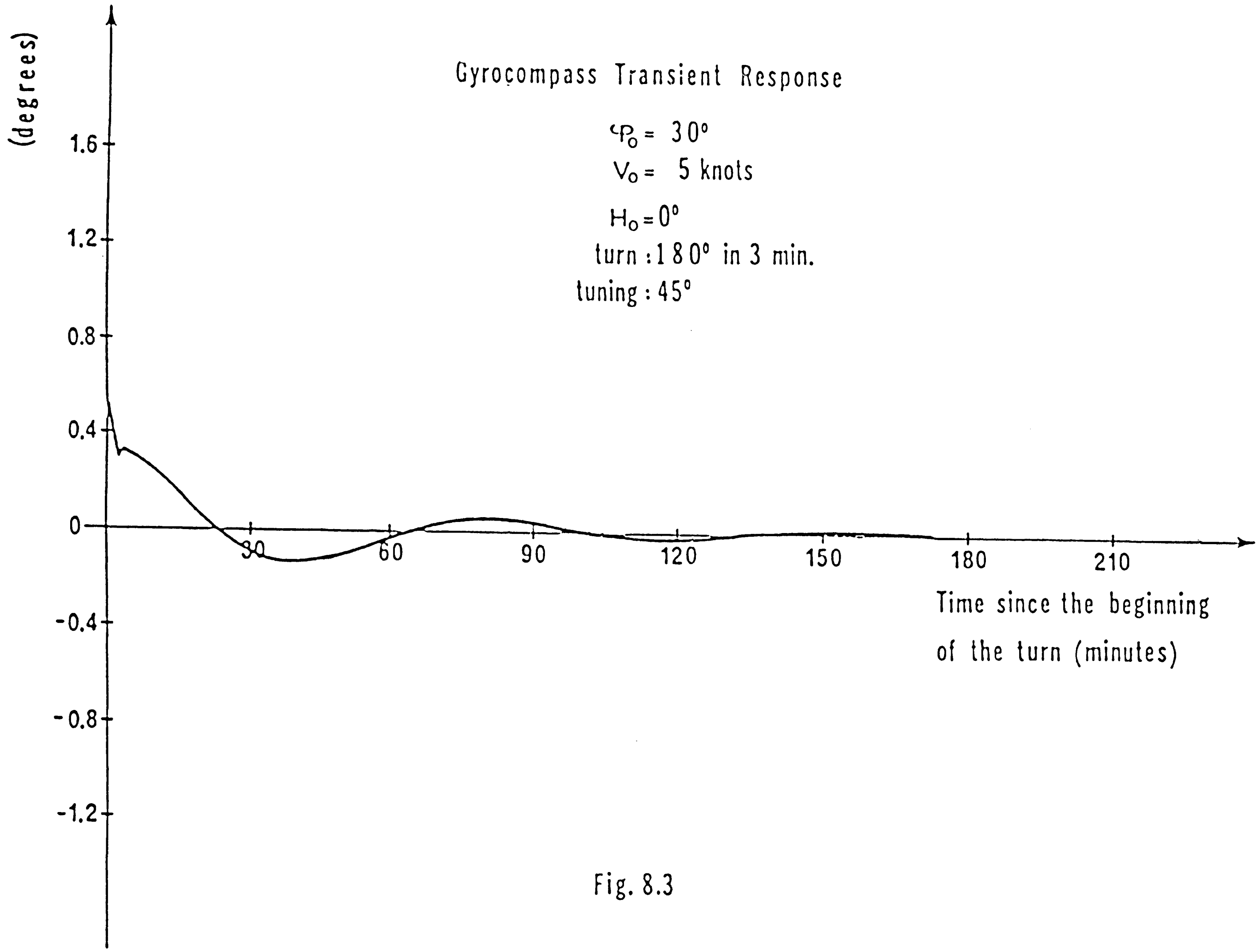

Gyrocompass Transient Response
$\varphi_0 = 30°$
$V_0 = 5$ knots
$H_0 = 0°$
turn : 180° in 3 min.
tuning : 45°
(degrees)
1.6
1.2
0.8
0.4
0
-0.4
-0.8
-1.2
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.3

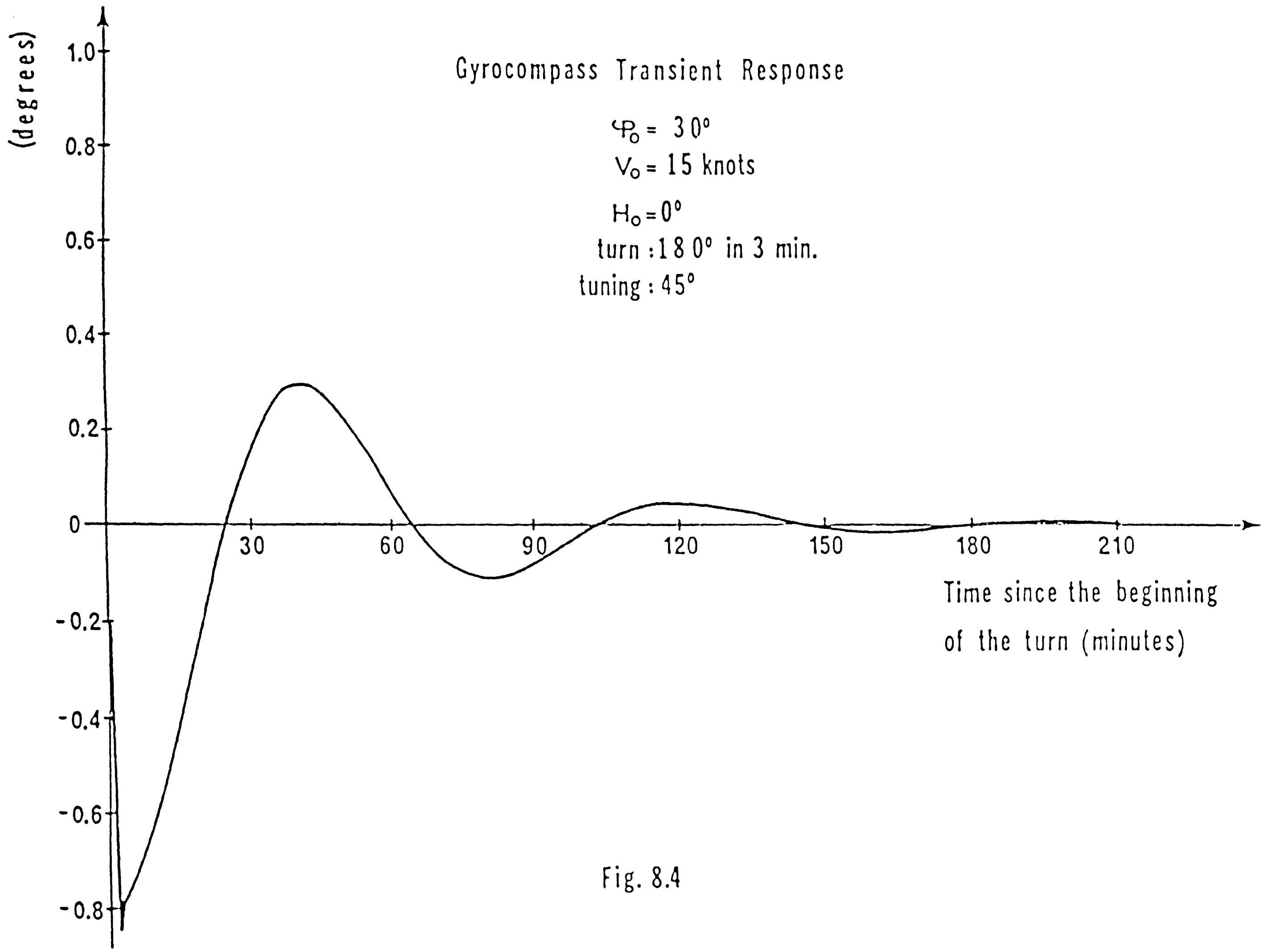

(degrees)
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Gyrocompass Transient Response
φ_o = 30°
V_o = 15 knots
H_o = 0°
turn : 180° in 3 min.
tuning : 45°
30
60
90
120
150
180
210
Time since the beginning
of the turn (minutes)

Fig. 8.4

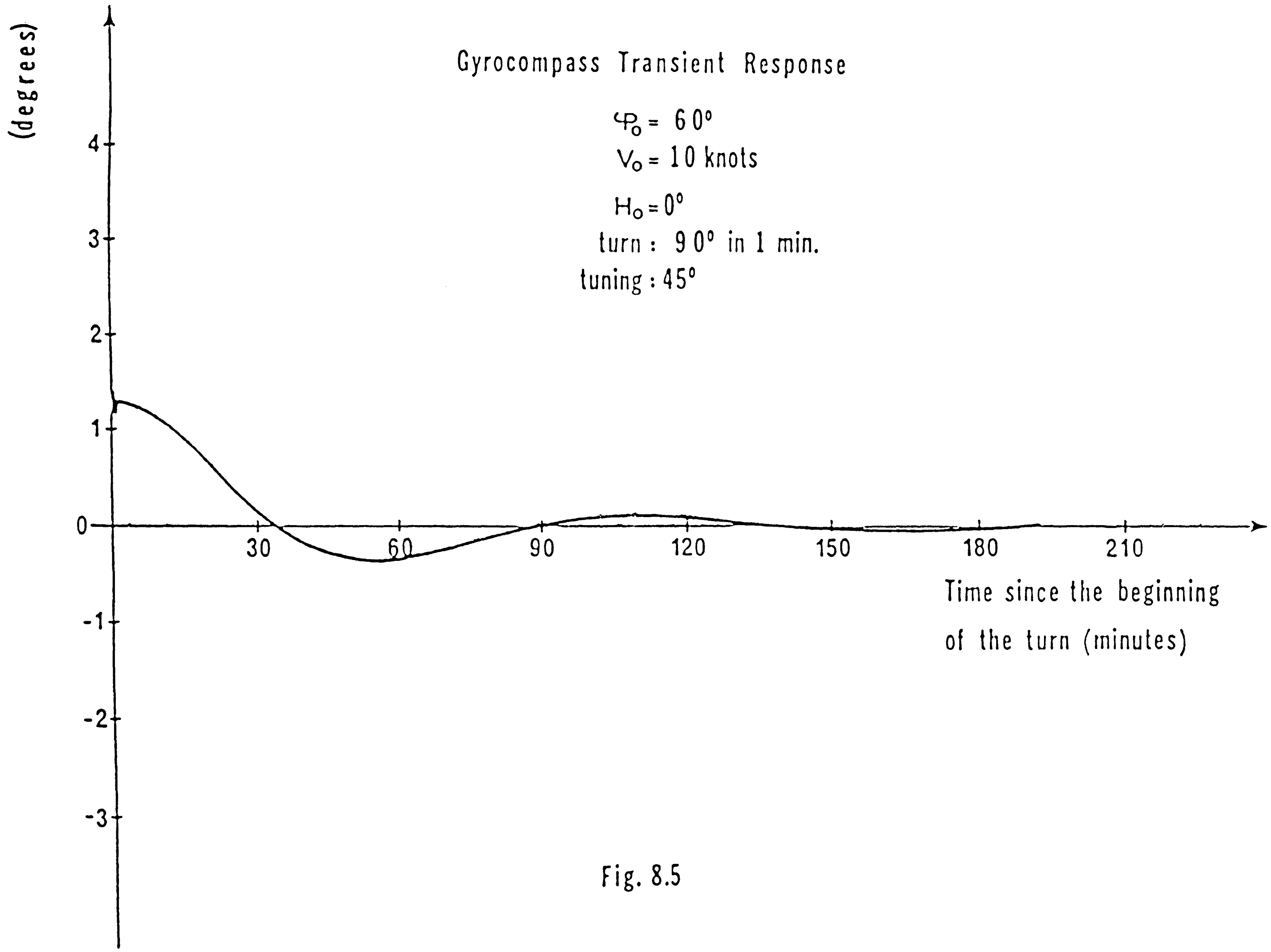

Gyrocompass Transient Response
$\varphi_o = 60°$
$V_o = 10$ knots
$H_o = 0°$
turn : 90° in 1 min.
tuning : 45°
(degrees)
4
3
2
1
0
-1
-2
-3
30
60
90
120
150
180
210
Time since the beginning
of the turn (minutes)

Fig. 8.5

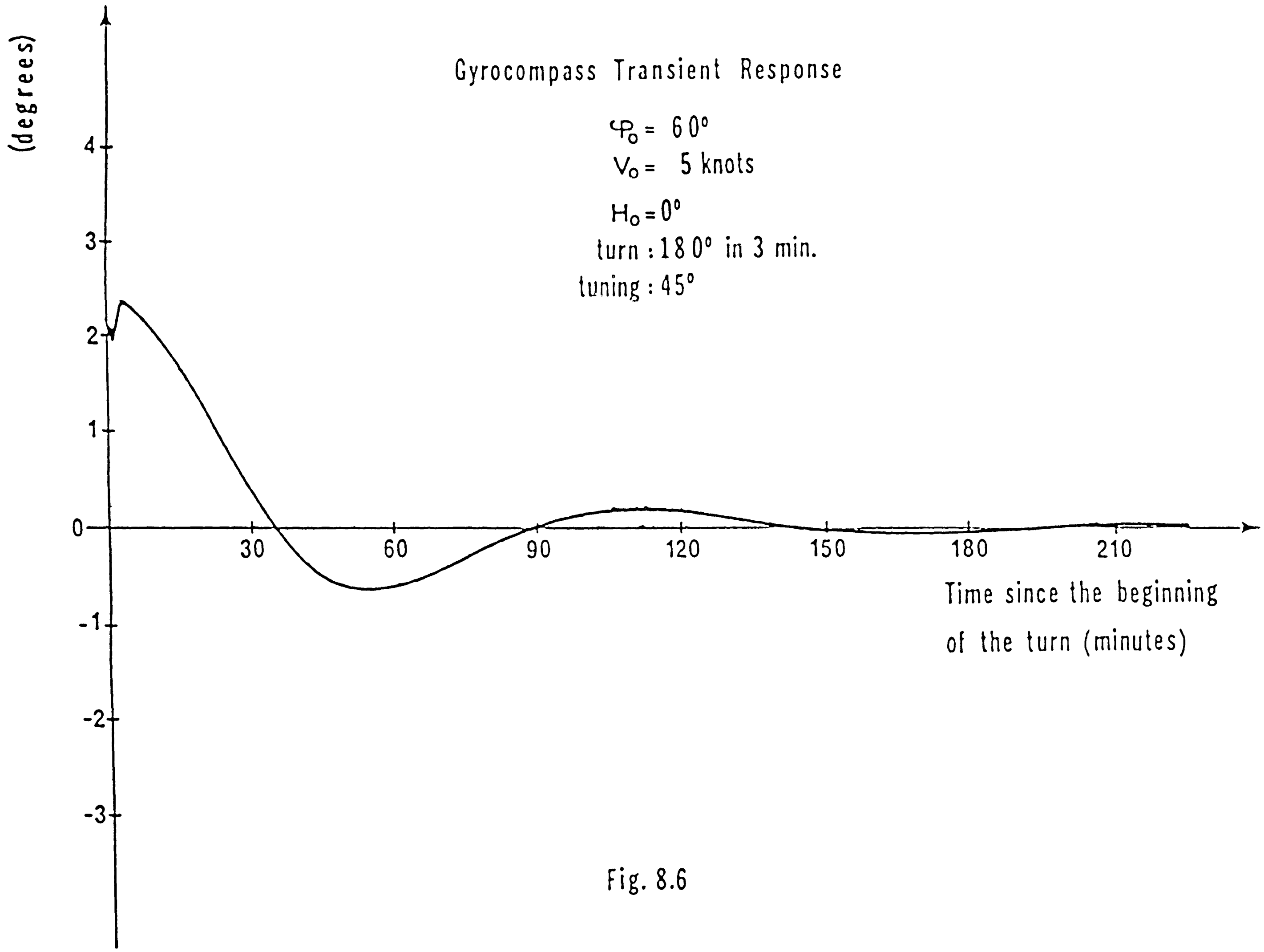

Gyrocompass Transient Response
$\varphi_o = 60°$
$V_o = 5$ knots
$H_o = 0°$
turn : 180° in 3 min.
tuning : 45°
(degrees)
4
3
2
1
0
-1
-2
-3
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.6

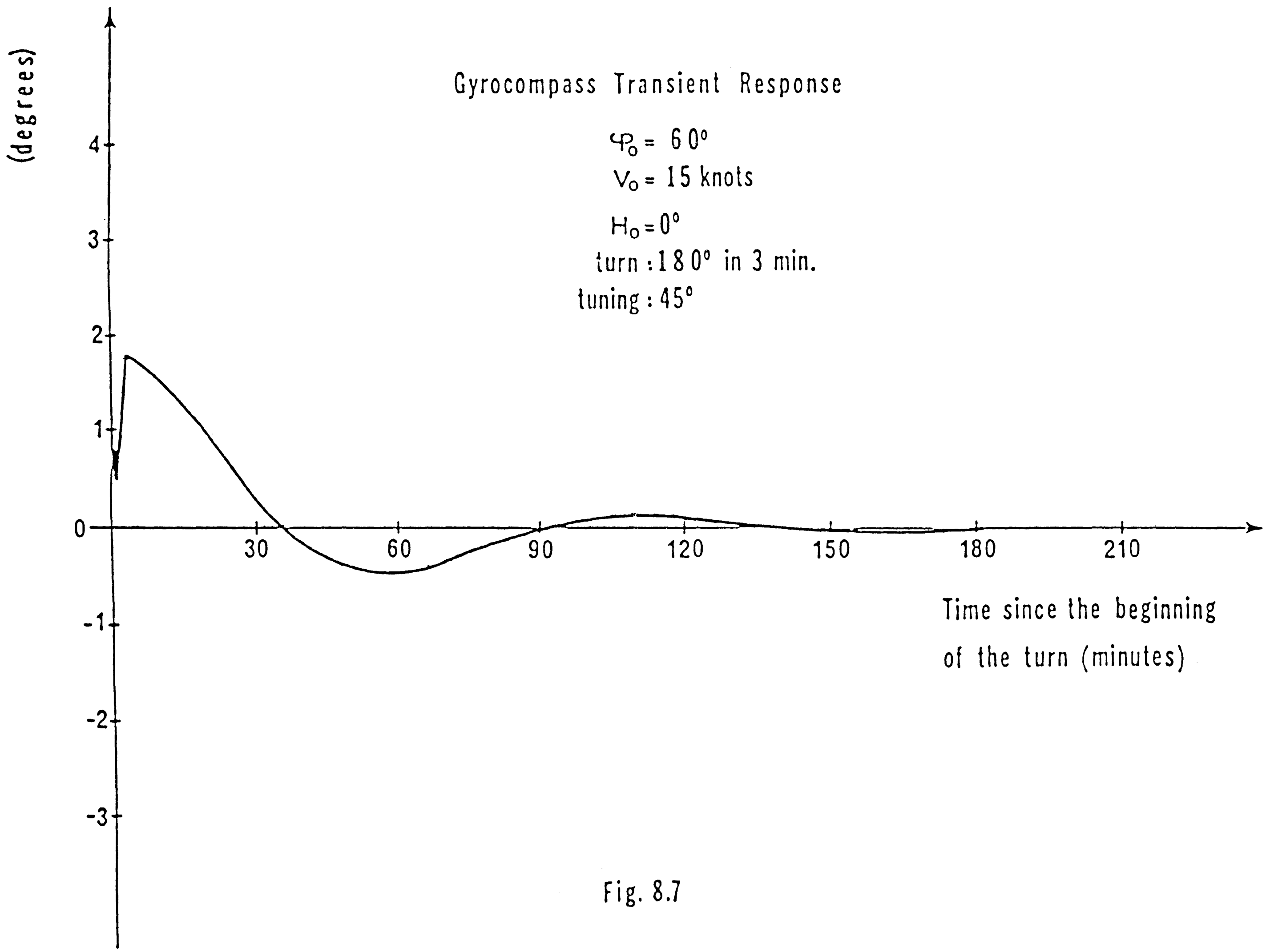
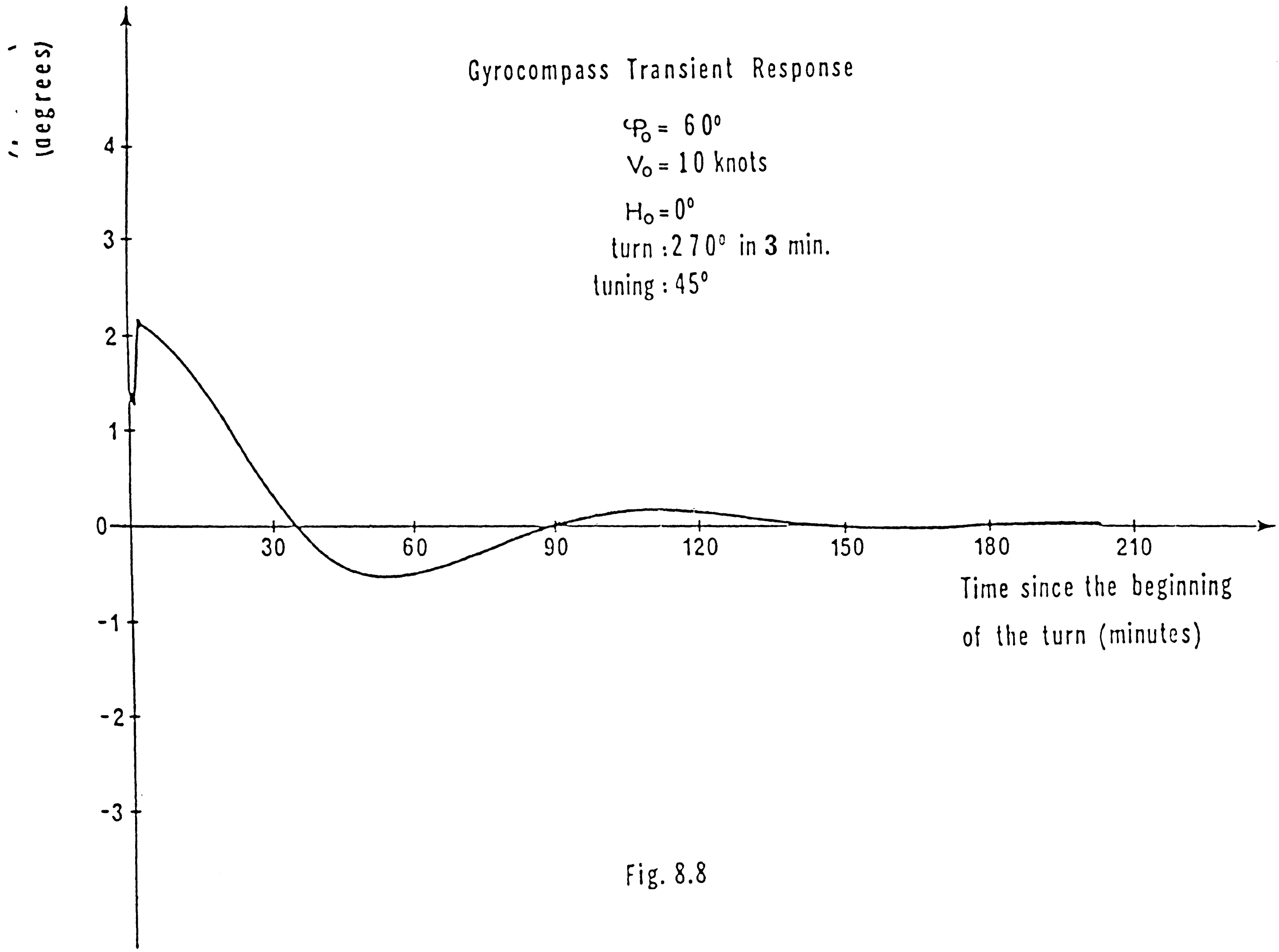
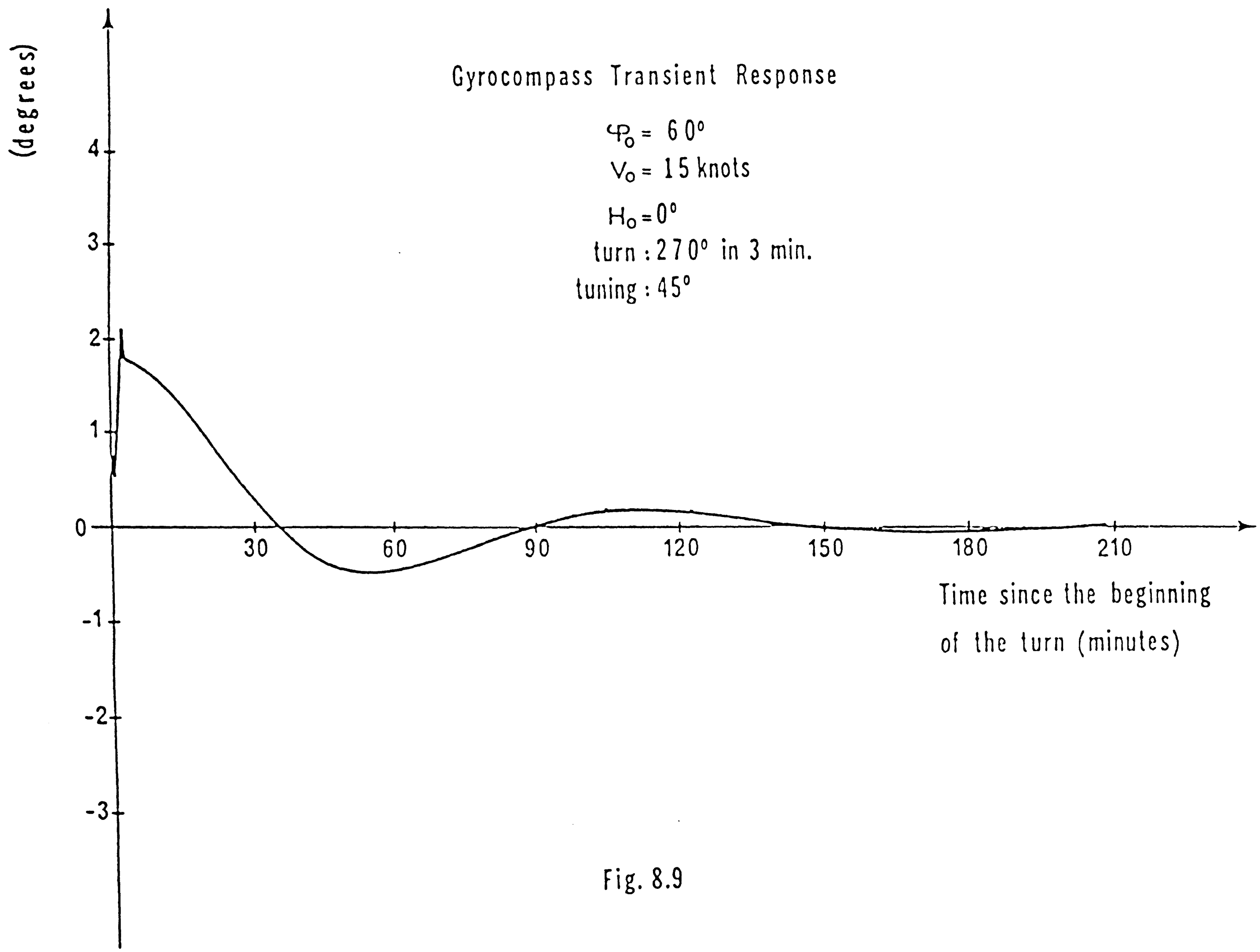

Gyrocompass Transient Response
φ₀ = 60°
V₀ = 15 knots
H₀ = 0°
turn : 180° in 3 min.
tuning : 45°
(degrees)
4
3
2
1
0
-1
-2
-3
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.7



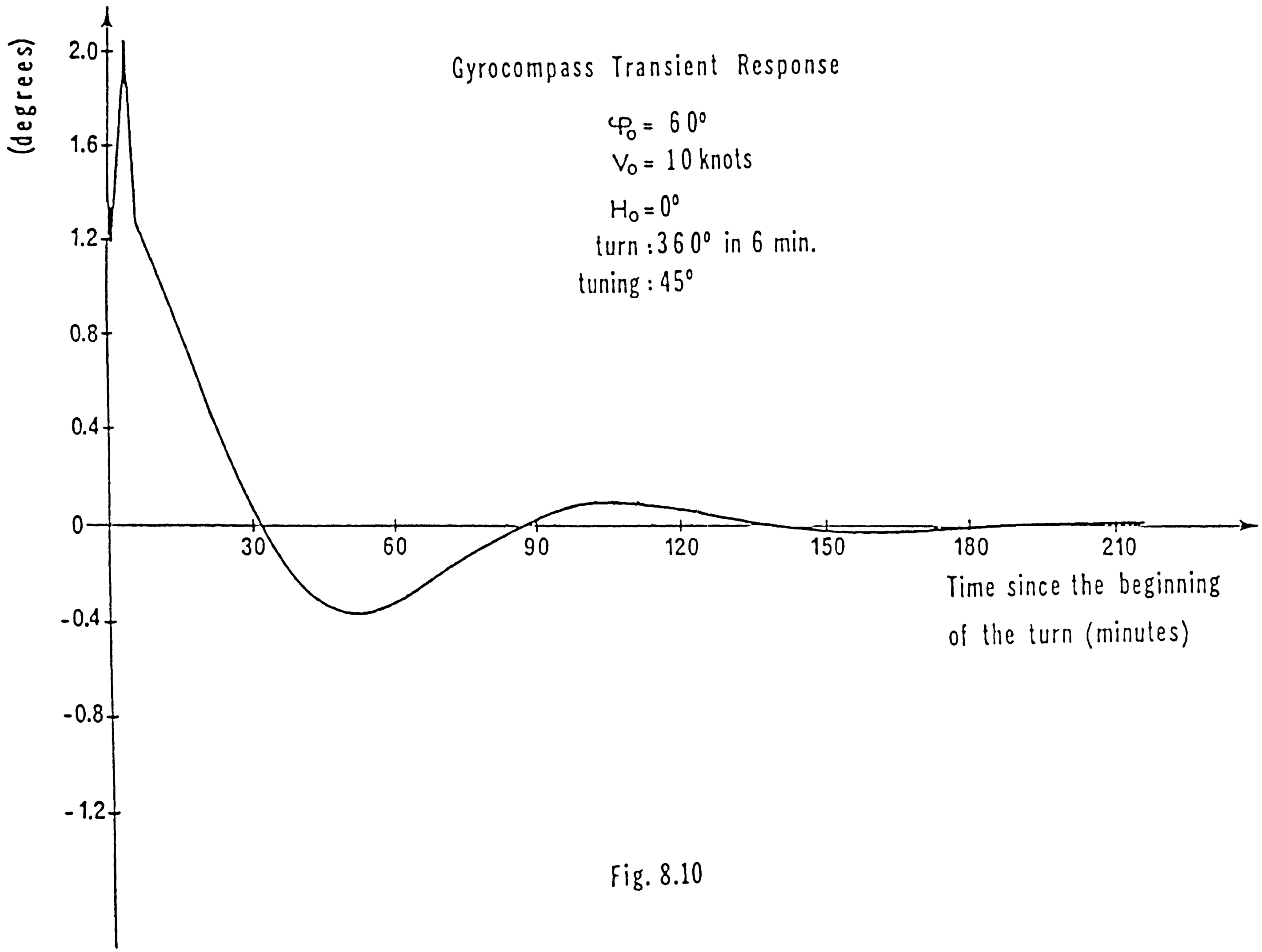
Gyrocompass Transient Response
$\varphi_0 = 60°$
$V_0 = 10$ knots
$H_0 = 0°$
turn : 270° in 3 min.
tuning : 45°
(degrees)
4
3
2
1
0
-1
-2
-3
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.8



Gyrocompass Transient Response
$\varphi_o = 60°$
$V_o = 15$ knots
$H_o = 0°$
turn : 270° in 3 min.
tuning : 45°
(degrees)
4
3
2
1
0
-1
-2
-3
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.9



Gyrocompass Transient Response
φ_o = 60°
V_o = 10 knots
H_o = 0°
turn : 360° in 6 min.
tuning : 45°
(degrees)
2.0
1.6
1.2
0.8
0.4
0
-0.4
-0.8
-1.2
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.10

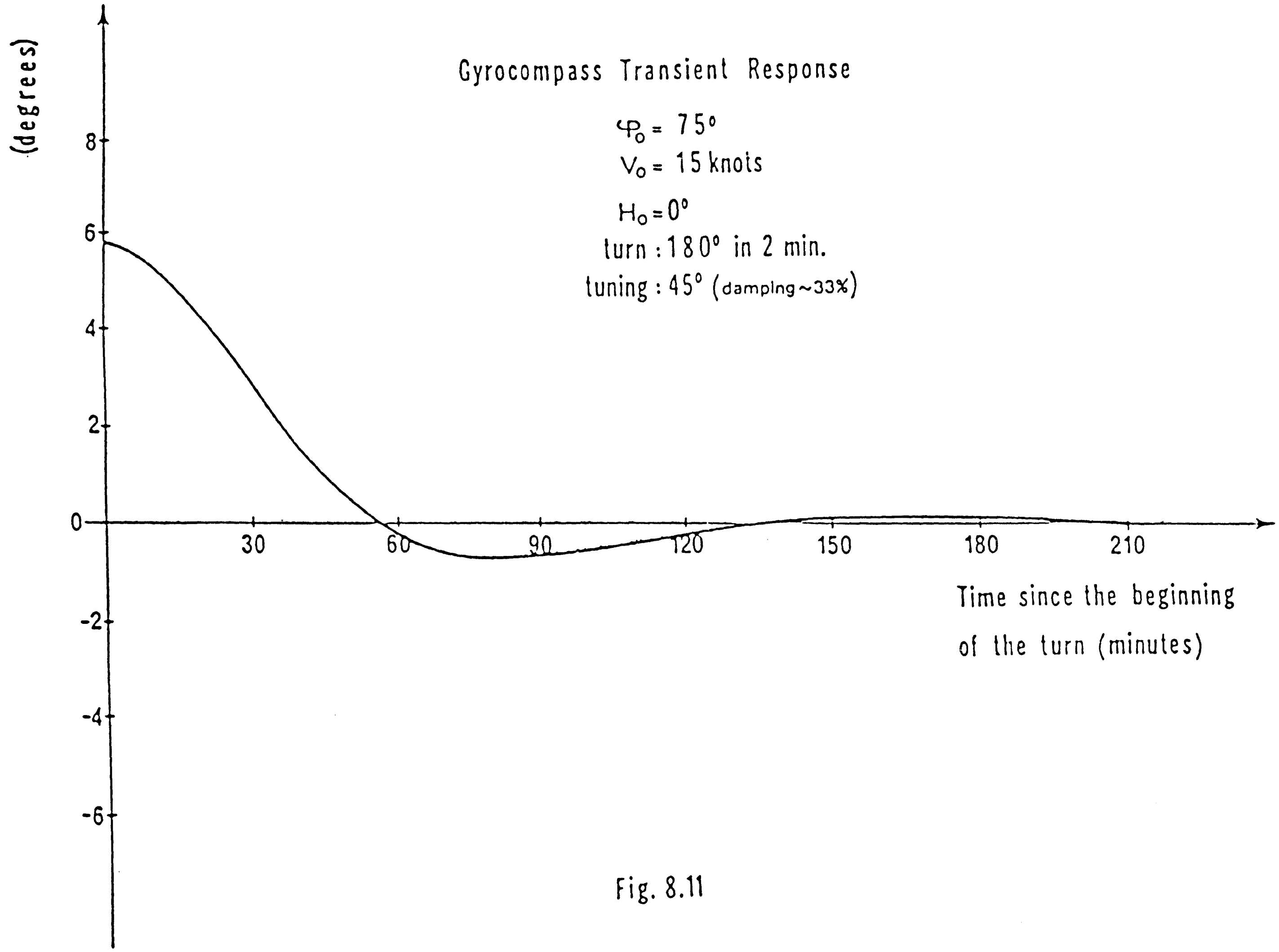
8.2 Simulation Results for a Special Case of The Ship's Track - The Circular-Arc Approximation

In this part of the simulation study the circular-arc approximation of the ship's track is examined. Equations (6.27), (6.28) and (6.29) were programmed and the transient response of the gyrocompass was computed. Two different cases are examined, and are given in Figures 8.11 and 8.12. The gyrocompass is tuned for latitude of 45° and different amount of damping is used. The manoeuvre characteristics are included in the legends of the figures.

8.3 Simulation Results Using The General Modelling of The Ship's Track in High Latitudes

Particular attention is devoted in the simulation studies carried out for high latitudes, because of the increased instability of the gyrocompass and the large magnitude of the steady-state errors.

The gyrocompass is tuned for 45° latitude. Different manoeuvres are simulated at different rates of turn. From Figures 8.13 to 8.19 it is seen that the period of oscillations of the gyrocompass is increasing as latitude increases. The transient errors assume values larger than 10° immediately after the end of the manoeuvre. An error of 1° after 90 minutes is still present.



Gyrocompass Transient Response
$\varphi_o = 75°$
$V_o = 15$ knots
$H_o = 0°$
turn : 180° in 2 min.
tuning : 45° (damping ~33%)
(degrees)
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.11

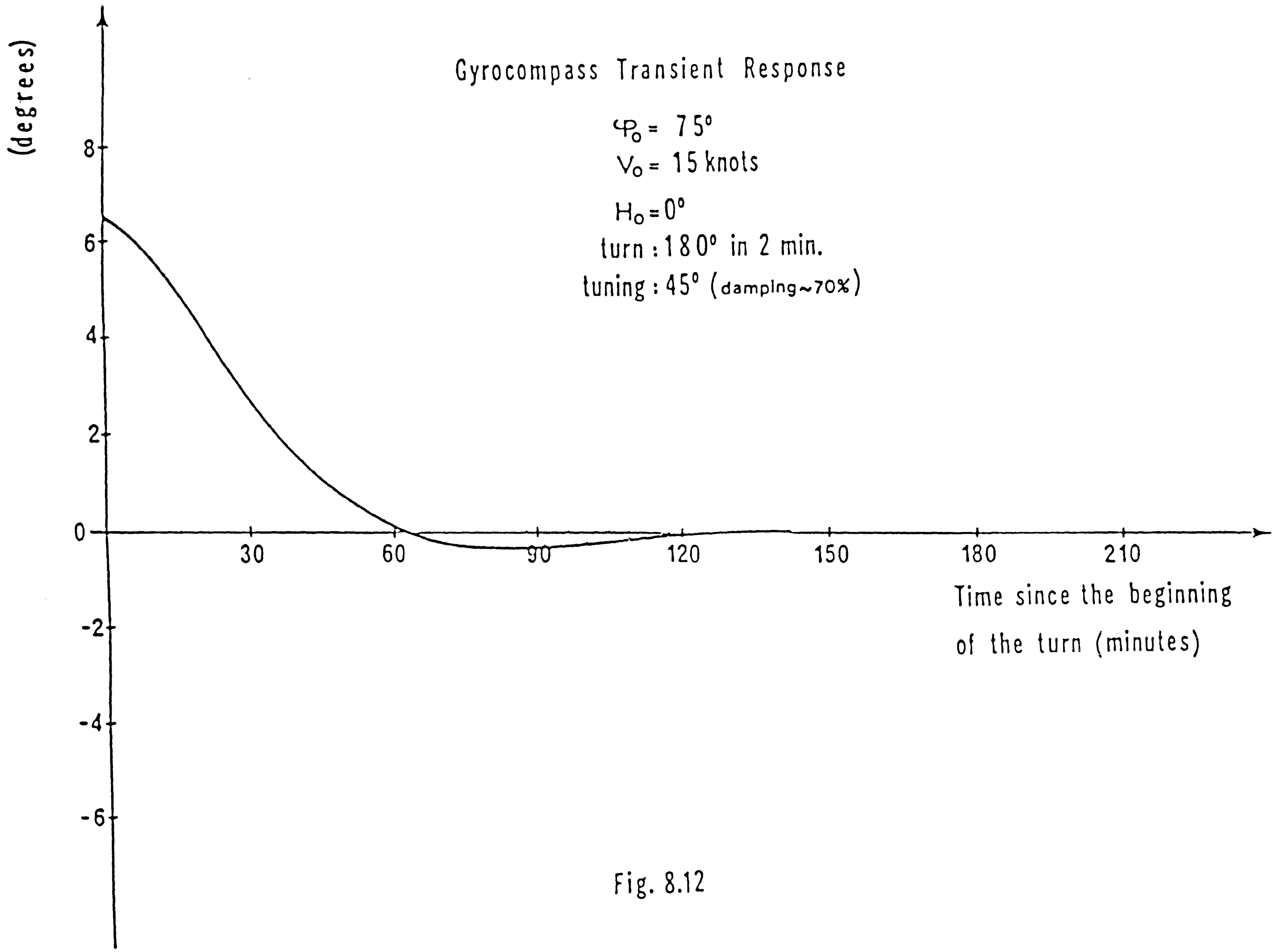

(degrees)
8
6
4
2
0
-2
-4
-6
Gyrocompass Transient Response
$\varphi_o = 75°$
$V_o = 15$ knots
$H_o = 0°$
turn : 180° in 2 min.
tuning : 45° (damping~70%)
30
60
90
120
150
180
210
Time since the beginning
of the turn (minutes)

Fig. 8.12

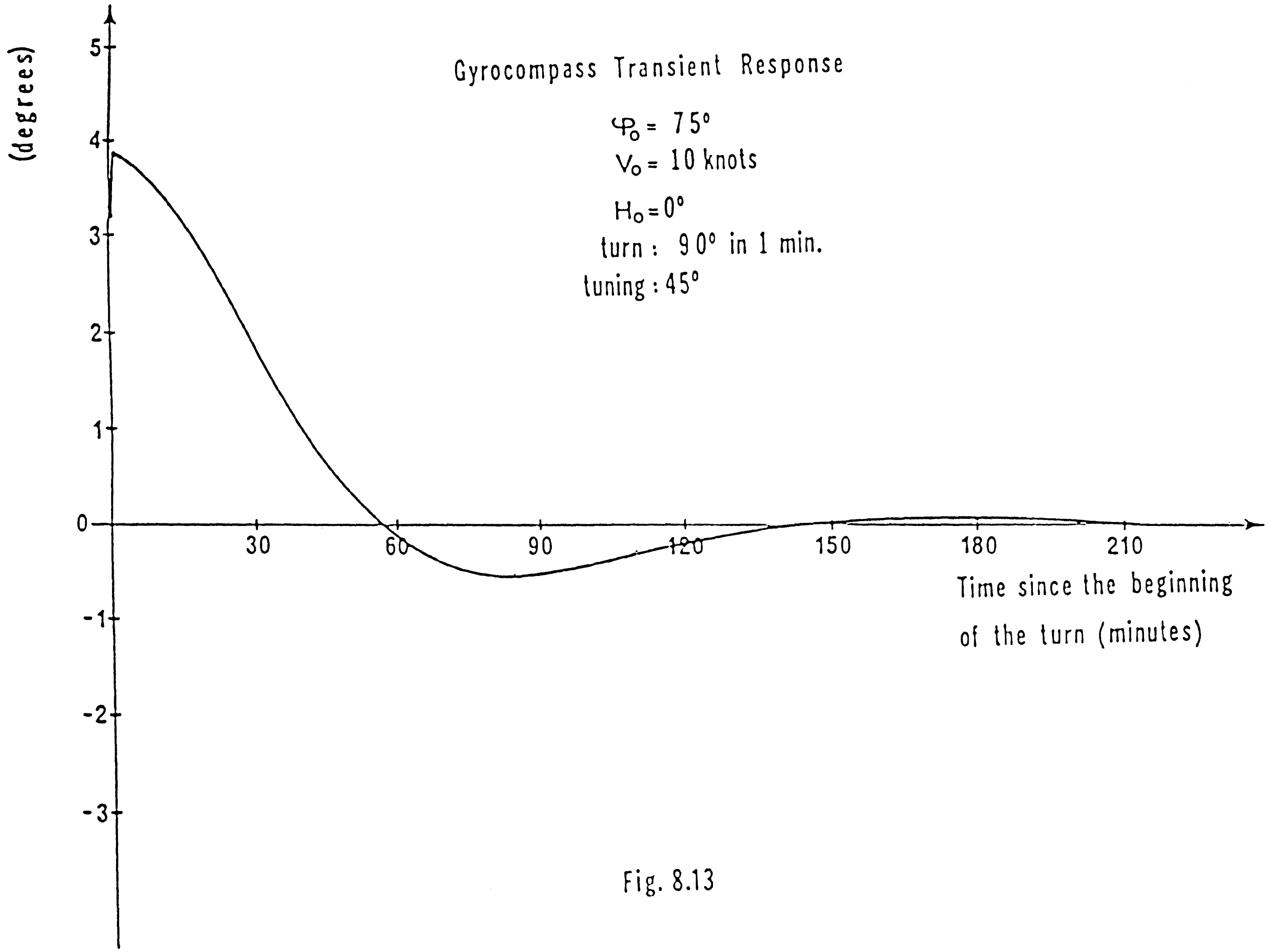
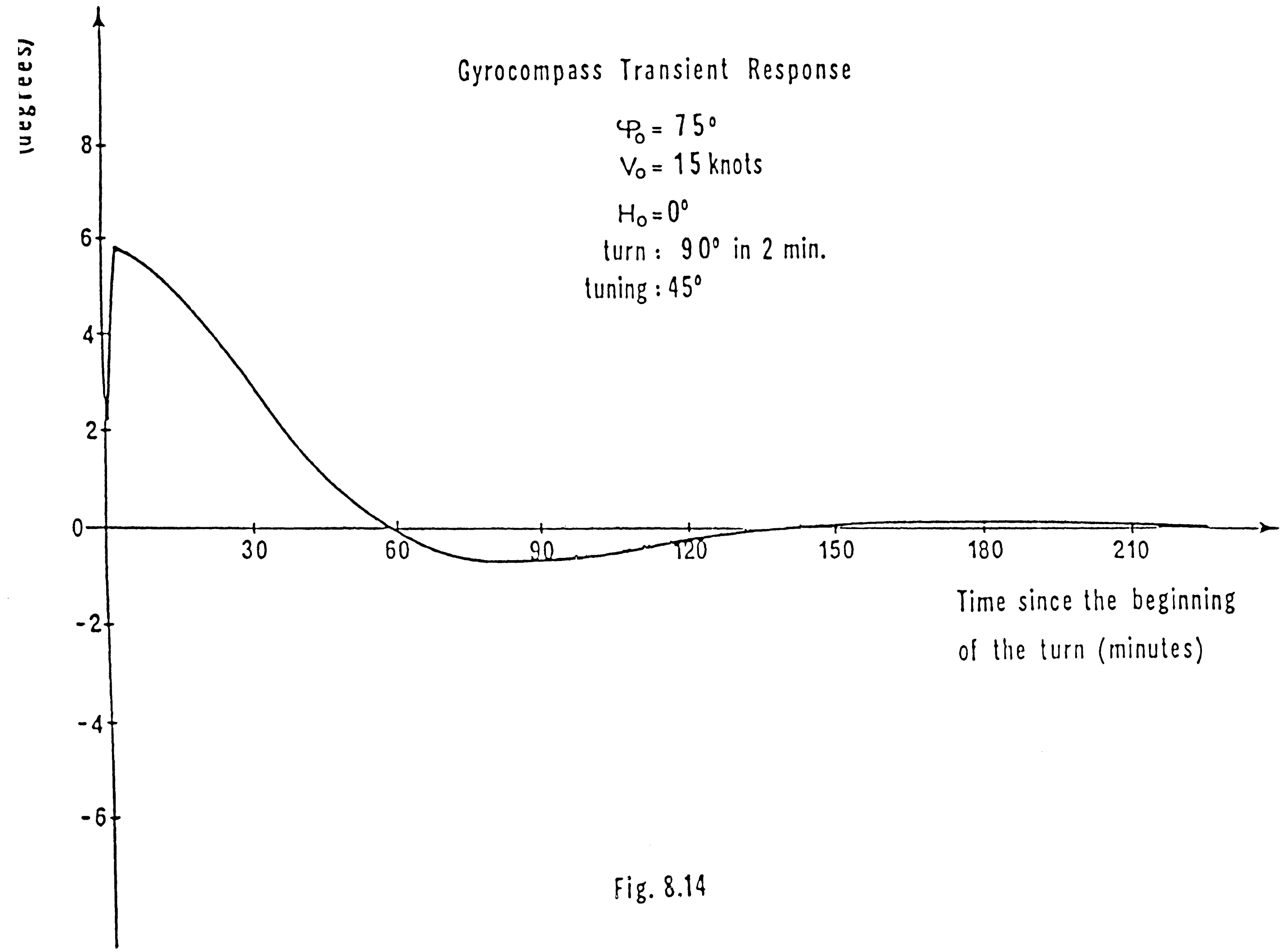
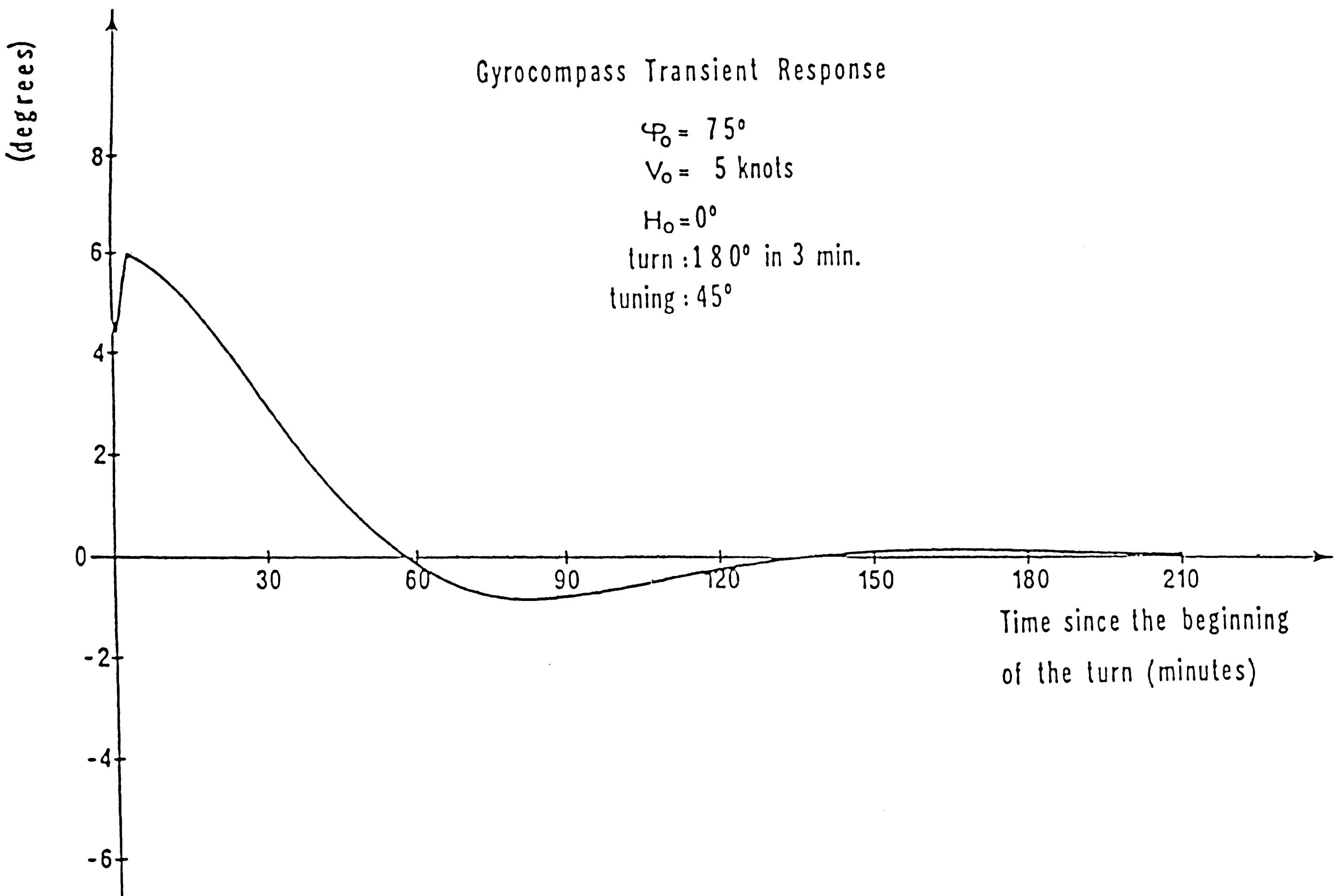

Gyrocompass Transient Response
$\varphi_o = 75°$
$V_o = 10$ knots
$H_o = 0°$
turn : 90° in 1 min.
tuning : 45°
(degrees)
5
4
3
2
1
0
-1
-2
-3
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.13



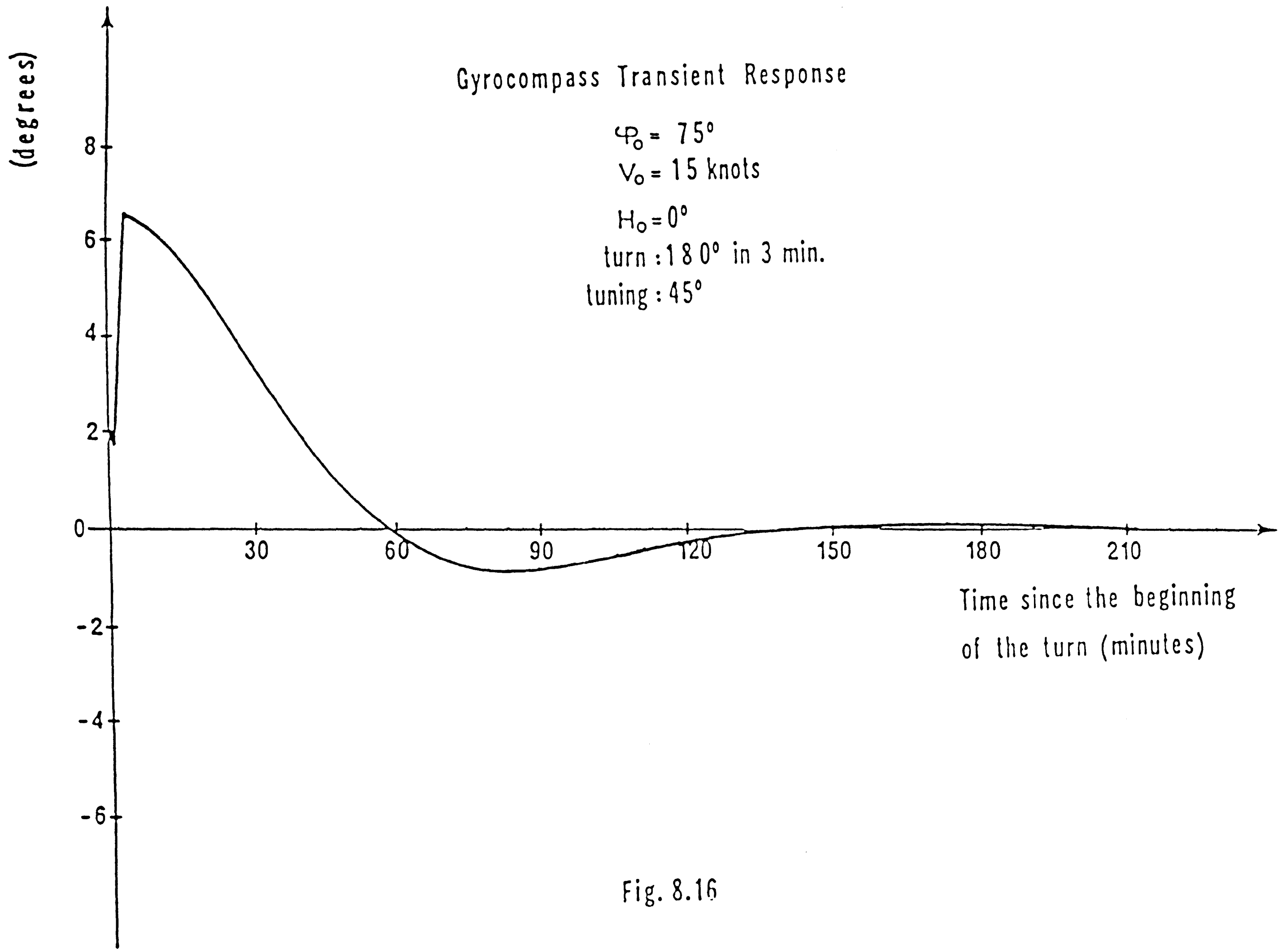
Gyrocompass Transient Response
$\varphi_0 = 75°$
$V_0 = 15$ knots
$H_0 = 0°$
turn : 90° in 2 min.
tuning : 45°
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning
of the turn (minutes)

Fig. 8.14



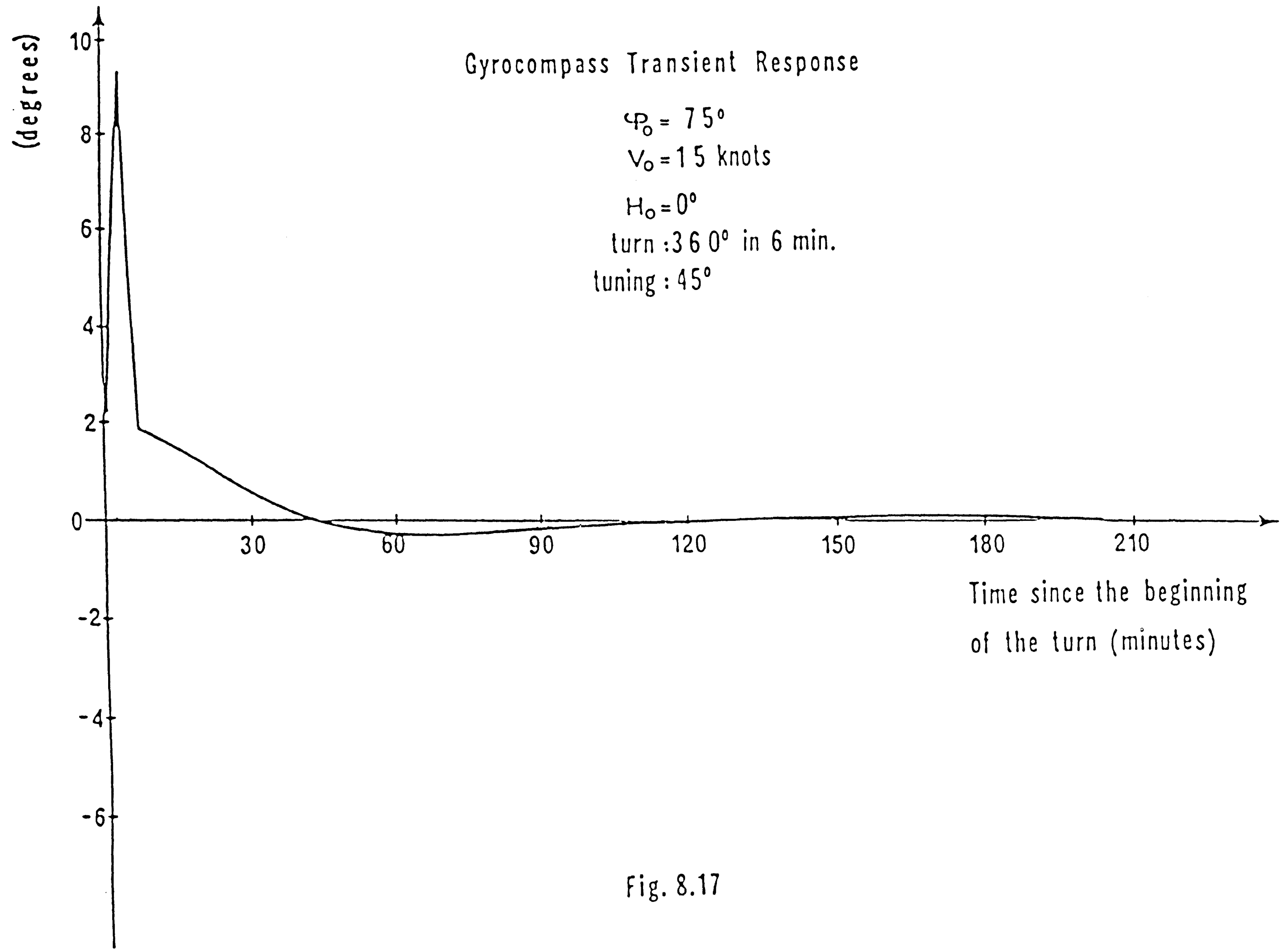
Gyrocompass Transient Response
φ_o = 75°
V_o = 5 knots
H_o = 0°
turn : 180° in 3 min.
tuning : 45°
(degrees)
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.15



(degrees)
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Gyrocompass Transient Response
$\varphi_0 = 75°$
$V_0 = 15$ knots
$H_0 = 0°$
turn : 180° in 3 min.
tuning : 45°
Time since the beginning of the turn (minutes)

Fig. 8.16



Gyrocompass Transient Response
$\varphi_o = 75°$
$V_o = 15$ knots
$H_o = 0°$
turn : 360° in 6 min.
tuning : 45°
(degrees)
10
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.17

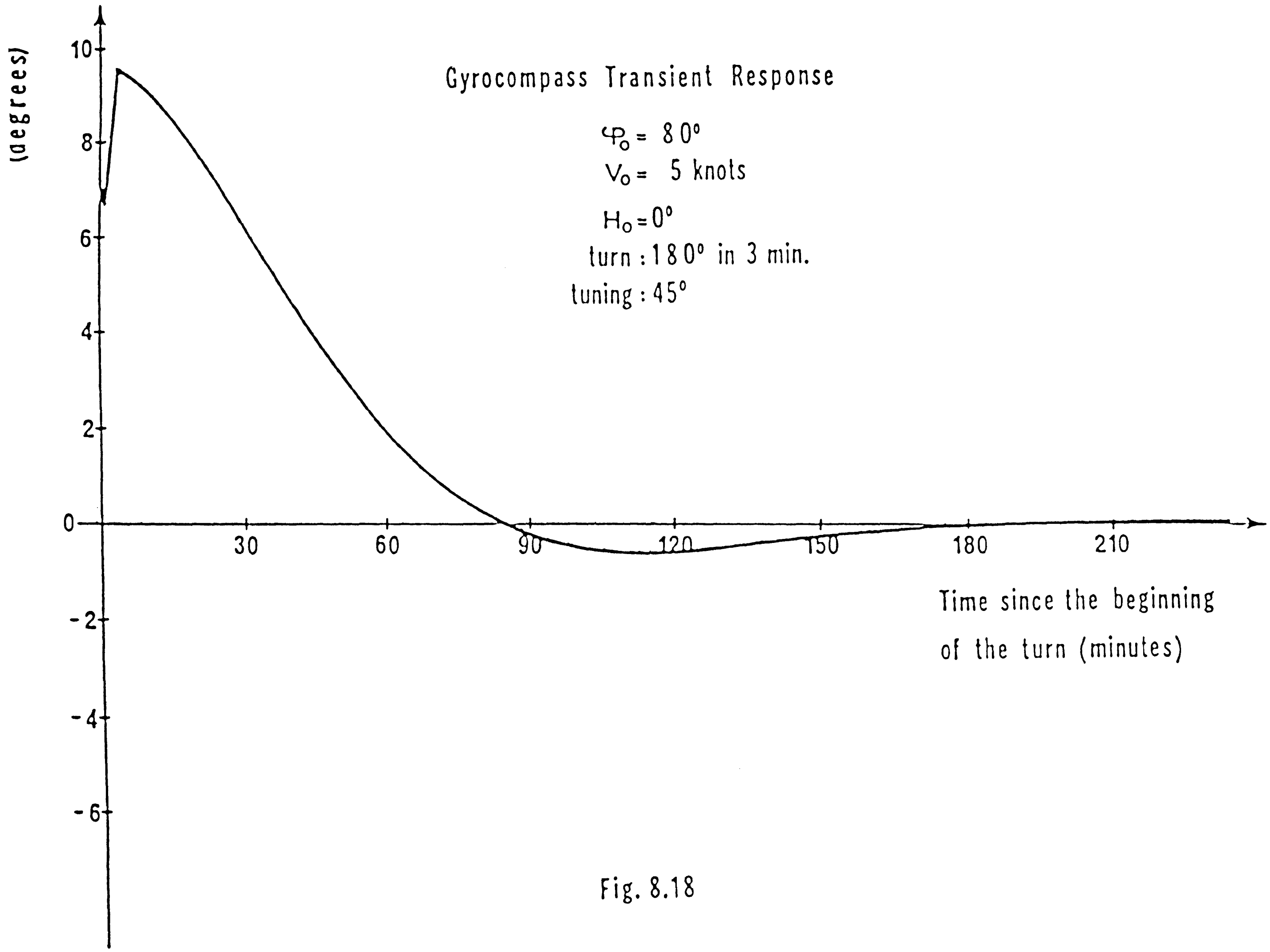

Gyrocompass Transient Response
φ_o = 80°
V_o = 5 knots
H_o = 0°
turn : 180° in 3 min.
tuning : 45°
(degrees)
10
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 8.18

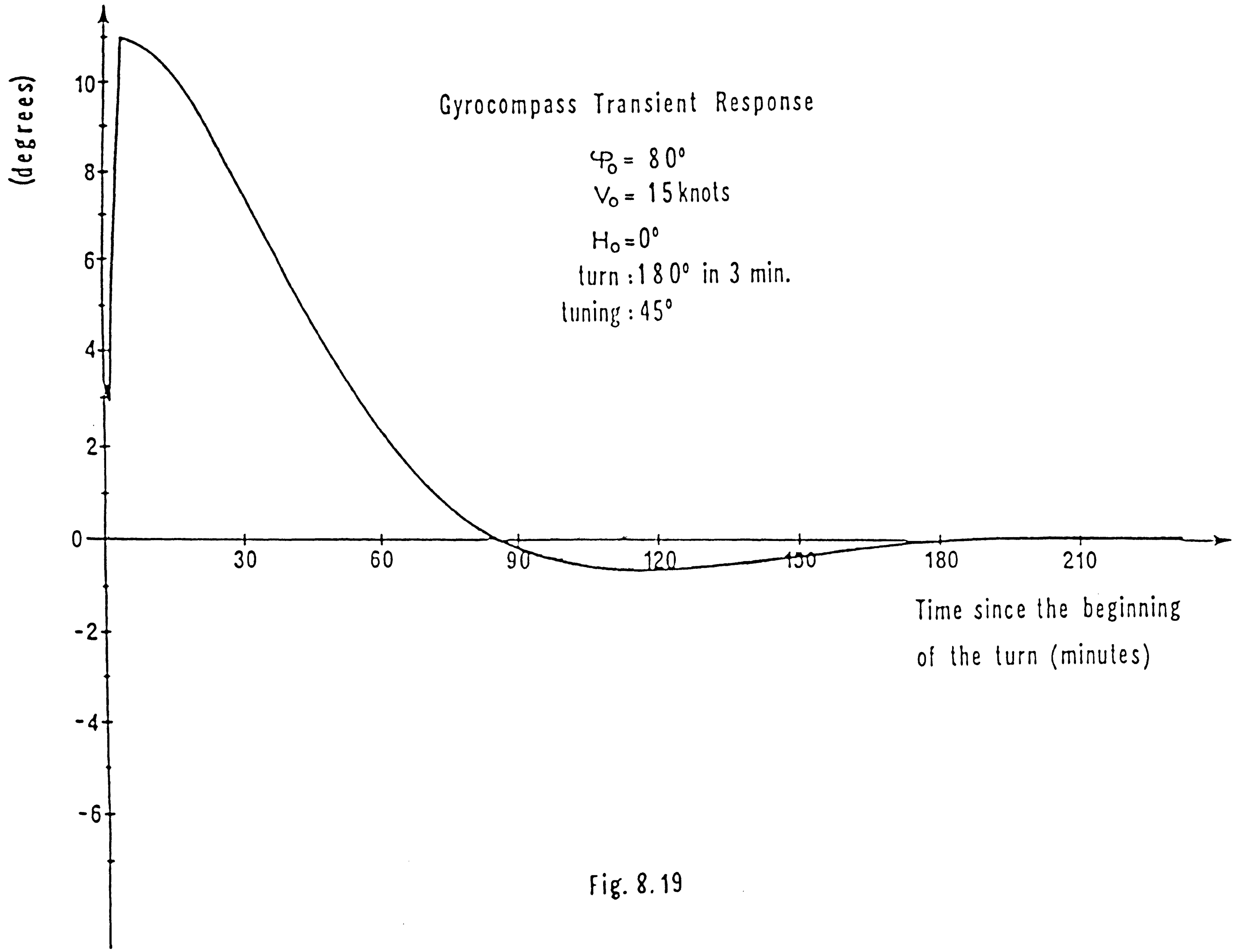

Gyrocompass Transient Response
$\varphi_0 = 80°$
$V_0 = 15$ knots
$H_0 = 0°$
turn : 180° in 3 min.
tuning : 45°
(degrees)
10
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning
of the turn (minutes)

Fig. 8.19

CHAPTER 9

SOFTWARE COMPENSATION OF GYROCOMPASS DEVIATIONS

In the previous chapters the major elements of dynamic analysis of the gyrocompass were discussed from the standpoint of their characteristics and contribution to the overall system's behaviour.

We have established that the dynamic characteristics of the physical system can be described by mathematical equations, although, as always, such mathematical models are only an approximation of the dynamic behaviour of a physical system.

In this chapter use of these mathematical models for software compensation of the gyrocompass deviations is treated as an example of a control problem.

Control systems can be classified as either open-loop or closed-loop. In open-loop control, the output has no influence on the system dynamics. In closed-loop control the output does influence the system dynamics [Morgan 1978].

Open-loop control is only effective when the response of the system is known. In a closed-loop control system feedback of the output is implied.

In this thesis the open-loop software compansation of gyrocompass errors is proposed. The reasons are:

i. the problem of open-loop compensation becomes a simple software processing problem,

ii. no specialized hardware is needed, and

iii. no interference with the actual gyrocompass operation is necessary.

Therefore, the gyrocompass will physically still deviate from true north, but the software compensation algorithm computes the correct heading for use as a directional reference.

Inputs for the software compensation are the indicated heading H from the gyrocompass together with data from other devices (such as ship's speed log) sampled at discrete time intervals. The sampling time interval depends on the particular requirements of the navigation function. The speed and heading information, together with latitude and longitude information are necessary to evaluate the gyrocompass dynamic response at any time instant t_j, as is seen from equations (6.4), (6.5), (6.11), (6.12) and equation (V-20) from Appendix V. The design parameters of the gyrocompass of interest have to be known, namely the angular momentum of the spinning wheel (Cn), the pendulous torque ($mg\ell$) and the offset angle (γ) which is used to introduce the damping action. Once again it is noted that all the above are applicable for the Sperry-type gyrocompass only. Then the following equations are programmed to give the gyrocompass response:

$$\alpha(t_j) = \alpha_m\Big|_{t_j} e^{-\sigma t_j}\cos(ft_j+\psi(t_j)-\gamma \tan \phi_j + \tan \delta_j \tag{9.1}$$

where the parameters α_m and ψ are given from eqns. (5.11) with initial conditions

$$\alpha_{t_{j+1}}(0^+) = \alpha_{t_{j+1}}(0^-) + \{\gamma(\tan\phi_{j+1}-\tan\phi_j)-(\tan\delta_{j+1}-\tan\delta_j)\} \quad (9.2)$$

where

$$\alpha_{t_{j+1}}(0^-) = \alpha(t_j = \Delta t_{j-1}) \quad (9.3)$$

and

$$\dot{\alpha}_{t_{j+1}}(0^+) = \dot{\alpha}_{t_{j+1}}(0^-) = \dot{\alpha}(t_j = \Delta t_{j-1}) \quad (9.4)$$

where

$$\Delta t_j = t_j - t_{j-1} \quad (9.5)$$

The term $\{\gamma(\tan\phi_{j+1}-\tan\phi_j)-(\tan\delta_{j+1}-\tan\delta_j)\}$ represents the changing steady-state error of the gyrocompass between instants t_{j+1}, t_j. In the general case of a gyrocompass which is not Schuler tuned equation (V-20) is used to compute the ballistic deflection error (Appendix V).

The true heading (H_{TRUE})of the vessel is the indicated heading (H_{IND}) plus the gyrocompass response (i.e., the *gyro-error* GER) as computed from equation (9.1)

$$H_{TRUE}(t_j) = H_{IND}(t_j) + GER(t_j) \quad (9.6)$$

The above described gyrocompass response corresponds to the discrete-time general model of the ship's track developed in section 6.2. The above presented analysis is better depicted in the following block diagram (Figure 9.1).

Block-Diagram of Gyrocompass Open-Loop Software Compensation

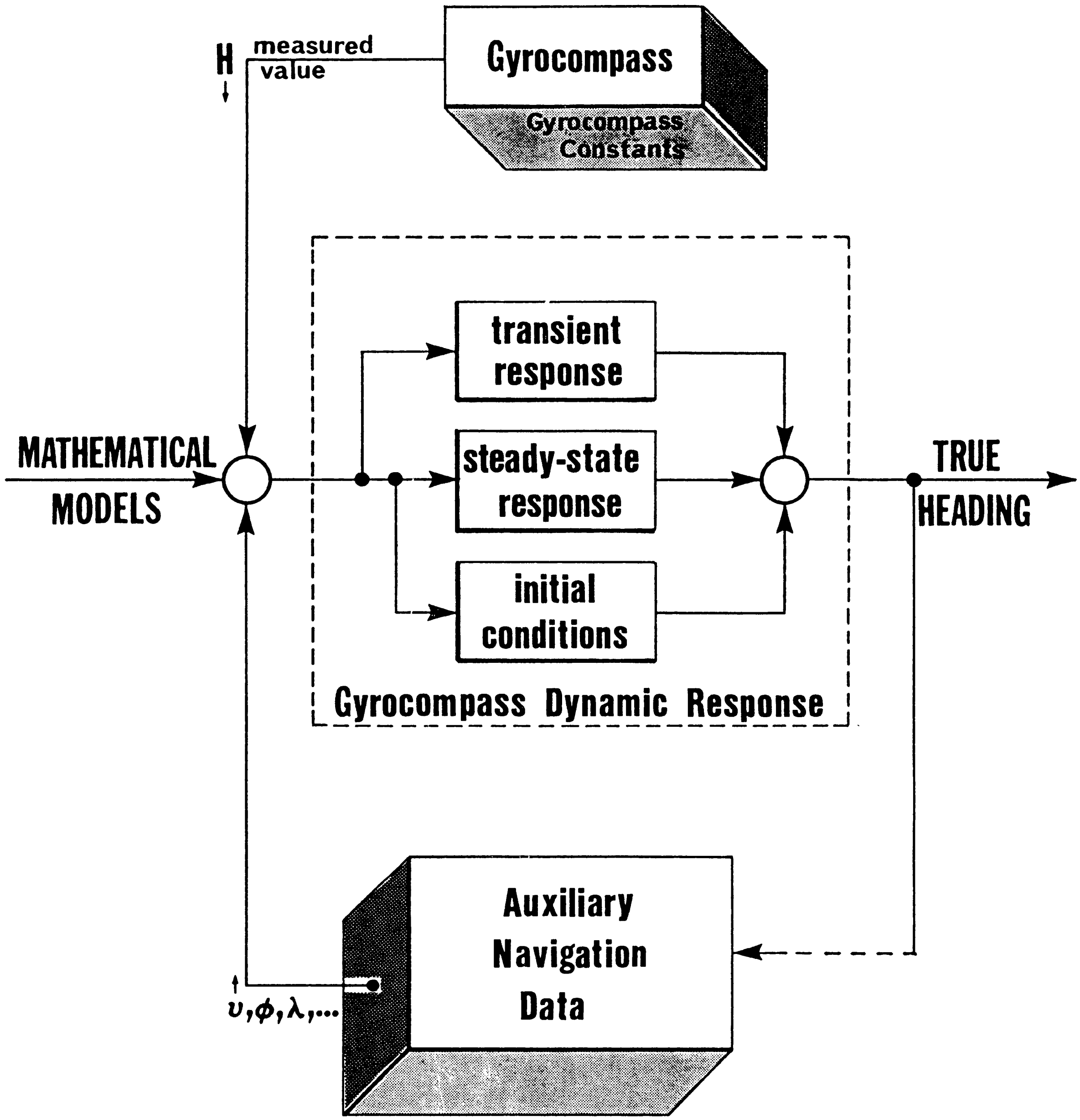


Fig. 9.1

There are relative advantages and disadvantages of an open-loop compensation technique. They are summarized in a point form below.

Advantages:

i. Software compensation is superior than the mechanical compensation provided by the manufacturer. The compensation units (settings for latitude ϕ_c° and speed v_c) have limited capabilities. Latitude can be preset for up to 70°. The gyrocompass room, where the compensation units are installed on the ship, is not easily accessible. Thus, the mechanical compensation settings are not appropriately and frequently updated as it is required. Usually the navigator adjusts the compensation settings every 3-5 degrees of latitude and retains the speed setting in the normal cruising speed. Changes therefore, either in latitude or speed, are not incorporated in the mechanical compensation procedure. This is not the situation with the software approach. All the changing dynamic inputs are taken into account and the accuracy of the heading information is improved considerably.

ii. Software compensation (open-loop control) is easily implemented in a digital computer and it does not require complex interfaces and additional electronics, as it does in the case of a closed-loop control structure.

iii. Open-loop software compensation does not influence the dynamics of the system. Thus it eliminates the risk of a spurius feedback input which can create considerable problems in the behaviour and performance of the dynamic system (e.g., a wrongly computed torque input in the feedback loop can cause the gyrocompass to exhibit dynamic instability, and affect its settling for a considerable time before the system is again capable in indicating true north).

<u>Disadvantages:</u>

i. Using the open-loop control structure the gyrocompass may exhibit large deviations which, although they are accounted for through the software program, may degrade the performance of the system by making it more susceptible to external disturbances. This case is better understood when manoeuvring at high latitudes, where northerly accelerations can not be accommodated by the Schuler tuning (mistuned gyrocompass, excessive natural undamped period), and large gyrocompass deviations occur. In the above case (manoeuvres at high latitudes) the open-loop control may improve the gyrocompass performance, but its behaviour still remains critical (increased bias errors, increased instability).

A trade-off between the open-loop control (i.e., software compensation) and manual mechanical compensation is the most feasible approach for marine gyrocompasses operating at high latitudes. The approach proposed here includes:

(a) manually set the mechanical compensation units of the gyrocompass to the maximum value for latitude and at a nominal speed that the ship is liekly to retain,

(b) use the software compensation program, modified as shown below to incorporate the partial mechanical compensation,

(c) compute the "*residual*" *gyro error* (RGER) to obtain the true heading of the vessel.

In this case, if the manual mechanical compensation settings are ϕ_c and v_c, then the residual gyrocompass error (RGER) is computed by

$$\text{RGER} = \alpha(t_j) - \{-\gamma \tan \phi_c + \tan \delta_c\} \tag{9.7}$$

where $\alpha(t_j)$ is given by equation (9.1) and

$$\tan \delta_c = \frac{v_c \cos H_{IND}}{\Omega R \cos \phi_c} \tag{9.8}$$

This approach proposed in here is illustrated in Figure 9.2. From the diagram it is seen that the residual transient gyrocompass error immediately after the manoeuvre is of the order of 1° at a latitude of 75° as opposed to approximately 6° when no manual mechanical compensation is used. The particular characteristics of the manoeuvre and the mechanical compensation settings are given in the legend of Figure 9.2. The gyrocompass is tuned for a latitude of 45°, and the simulation is referred to the Sperry MK V design.

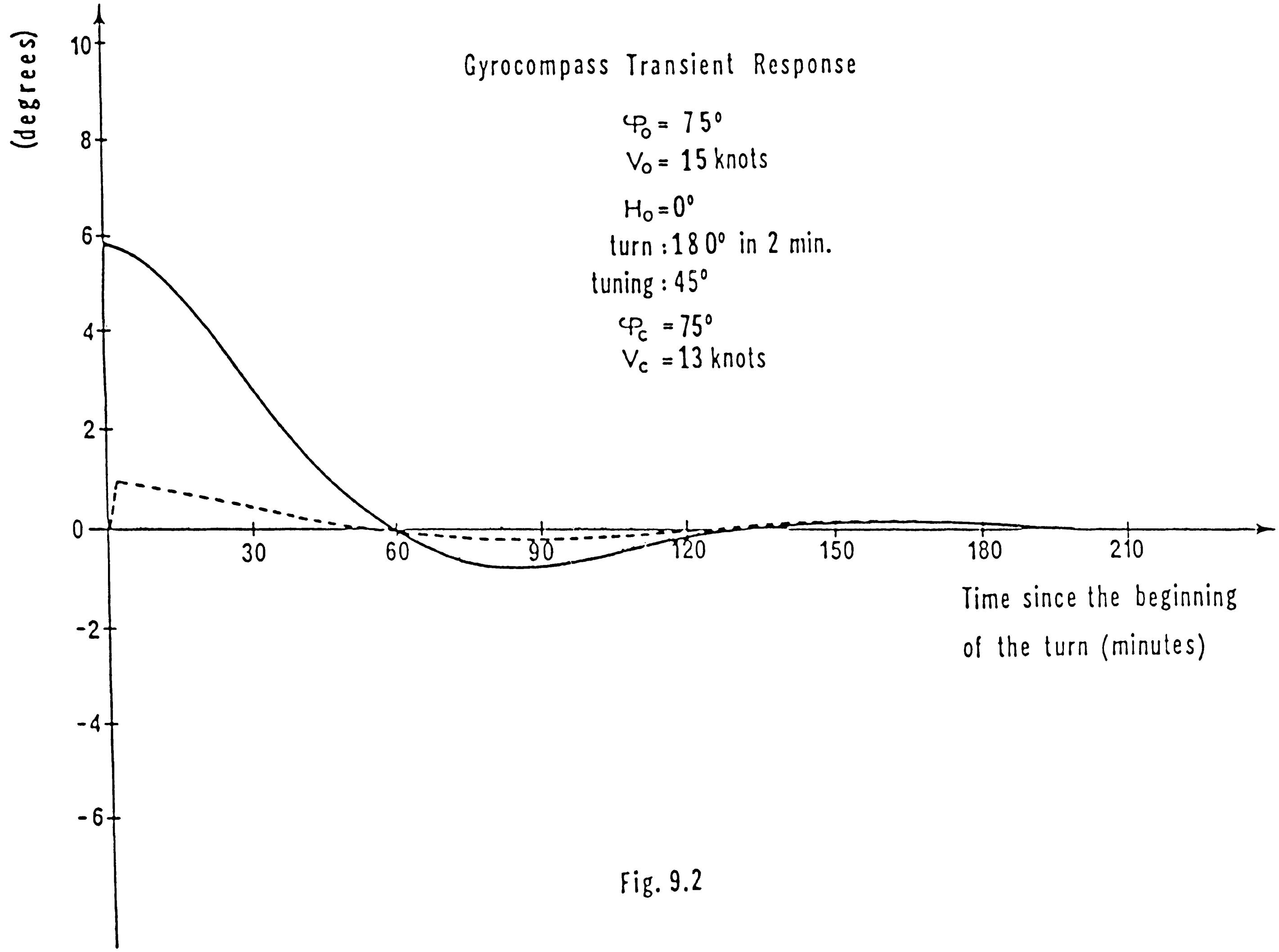

Gyrocompass Transient Response
$\varphi_o = 75°$
$V_o = 15$ knots
$H_o = 0°$
turn : 180° in 2 min.
tuning : 45°
$\varphi_c = 75°$
$V_c = 13$ knots
(degrees)
10
8
6
4
2
0
-2
-4
-6
30
60
90
120
150
180
210
Time since the beginning of the turn (minutes)

Fig. 9.2

CHAPTER 10

CONCLUSIONS AND RECOMMENDATIONS

The objectives of this research are summarized into the following: develop mathematical models for a gyrocompass; evaluate these models to obtain the gyrocompass behaviour; model as many of the gyrocompass errors as possible, thus making it possible for predicting what kind of response should be expected and of what magnitude; consider particular applications where the above modelling may contribute in reducing the heading errors involved, hence augmenting the performance of several other navigation functions.

The equations of motion of the gyrocompass were derived using Lagrange's method.

The theory of linear dynamic systems analysis is a powerful tool in obtaining the response functions of a dynamic system. It was presented here in its simplest form.

To test the derived response functions, simulations involving a real gyrocompass were performed.

Software algorithms were developed to compensate for the gyrocompass deviations.

The high latitude behaviour of the gyrocompass was considered as a special case of the software compensation procedure developed here.

The general discrete-time model of ship's track, devised here is proven a realistic and workable procedure.

Before proceeding to the interpretation and significance of the simulation results a few more conclusions are drawn.

If the gyrocompass is supposed to assist specific and precise navigation functions, an understanding of its dynamic characteristics is required. More specific information about its design characteristics and parameters should be known. This will lead to the important fact that software compensation procedures will recover completely all the gyrocompass bias errors (e.g., damping error, speed-and-course error) and at least most of the not so easily predictable errors (e.g. transient errors due to ballistic deflection).

Software compensation procedures are by far better and more accurate than the mechanical compensation settings provided by the manufacturer. The latter are probably sufficient for general navigation when cruising in constant speed and heading for long periods of time in low latitudes, but they are inadequate for special navigation requirements.

10.1 Interpretation of The Results and Their Significance

The gyrocompass behaviour is described by two response functions. The steady-state response is of less importance than the transient response, because it is well known and its magnitude varies according to a prescribed way. On the contrary, the transient response is more crucial. It introduces temporal errors, the magnitude of which is a function of the particular dynamics of the problem and it can be predicted only if mathematical modelling is available. As far as transient errors are concerned, there is not other means but software procedures in correcting compass indicated headings.

In real situations when frequent changes in course and/or speed occur, the gyrocompass exhibits oscillations which are undesirable, because they introduce errors in the compass readings thus providing false information about the direction of travel with respect to true north.

If continuous disturbances occur in the gyrocompass, it is more likely that the oscillations will become critical for the system's performance. These oscillations will require a very long time for the gyrocompass to settle to an equilibrium position again, hence degrading its effectiveness as a valuable heading-sensor.

The results from the simulation study are consistent with an actual gyrocompass deviation reported in 1972 in Lancaster Sound [*Eaton 1982*]. An error of 6° was measured in the gyrocompass (Sperry MK 37) after a 180° turn at 13.5 knots at a latitude approximately of 72°.

The significance of the gyrocompass computed transient response (Chapter 8) is very important. Transient errors of the order of few degrees occur. Such large transient errors might result in velocity estimates of the vessel with an error of close to 1 knot in the north-south component. This in turn could introduce an error in a transit satellite fix of about 0.5 nautical miles [*Stansell 1978*].

Offshore hydrographic cruises are in fact multidisciplinary in nature, combining bathymetric, seismic, gravity and magnetic measurements. Navigation requirements are extremely demanding in this respect. The new generation of sea-gravimeters is capable of providing marine gravity accuracies of the order of ± 1 mgal. The major drawback in obtaining this accuracy is the knowledge of the Eötvös correction. Navigation uncertainties can produce Eötvös correction errors from 1 to 10 mgals [*Rose 1974*].

Gyrocompass errors of the order of few degrees may introduce as much as 2 mgals in the determination of the Eötvös correction. The situation becomes more critical if a ship, having similar errors in heading information, performs the crossing-tracks for the gravity survey. The two errors, if combined, could easily show an error of more than 3-4 mgals in the Eötvös correction just due to the gyrocompass deviations.

The situation is of paramount importance when the real-time Eötvös correction is calculated and fed back to the observed gravity. In this case software compensation of the gyrocompass errors is absolutely necessary.

Maximum transient gyrocompass errors can be as large as 10 to 15 degrees at high latitudes. This limits any of the navigation functions that require directional information. These errors certainly affect the navigation accuracy requirements. Since integration of navigation sensors is meant to improve reliability, accuracy, and consistency of navigation [*Wells and Grant 1981*], gyrocompass errors might degrade the performance of such a multisensor system considerably.

10.2 Further Developments and Recommendations

This work can be extended to include more recent gyrocompass designs, such as Sperry MK 37 Gyrocompass equipment.

Mathematical models that involve random disturbances in the gyrocompass will augment the performance of the system, since more error sources can be included not necessarily having an explicit functional form. The mathematical treatment can be merged with the statistical one.

The mathematical models and response functions developed here need an independent check at sea to evaluate their effectiveness. These independent checks might also help in the improvement of these models. Ways to provide external means of ship's azimuth (azimuth of the fore-aft axis) may include underwater acoustic navigation systems, or precise radio navigation systems, both requiring two sensitive elements positioned along the ship's longitudinal axis.

A very important and necessary element in evaluating the gyrocompass response is the knowledge of the design characteristics. This allows the development of the necessary mathematical models and the response functions. A close collaboration with the manufacturer will certainly improve the knowledge of the particular gyrocompass behaviour and performance.

This is especially true when the high latitude performance of a gyrocompass is investigated. For example, in the case of ice-breaking operations when navigating in the arctic, what is the effect of sudden disturbances, for a prolonged time, on the gyrocompass indications [*Canadian Coast Guard, 1981*]?

How can the stability conditions (Schuler tuning) be improved at high latitudes with less mechanical adjustments?

BIBLIOGRAPHY

Admiralty Manual of Navigation (1964). B.R. (45): I, Her Majesty's Stationery Office, London.

Admiralty Manual of Navigation (1964). B.R. (67): III, Her Majesty's Stationery Office, London.

American Practical Navigator, An Epitome of Navigation (1977). Vol. I, Defence Mapping Agency Hydrographic Center, Pub. No. 9.

Arnold, R.N. and L. Maunder (1961). Gyrodynamics, Its Engineering Applications. Academic Press, London.

Atkin, R.H. (1959). Classical Dynamics. John Wiley & Sons, New York.

Attwood, E.L. and H.S. Pengelly (1967). Theoretical Naval Architecture. Revised by J. Sims, Longmans, London.

Bartlett, J.H. (1975). Classical and Modern Mechanics. The University of Alabama Press, Alabama.

Bogoliubov, N.N. and Y.A. Mitropolsky (1961). Asymptotic Methods in The Theory of Non-Linear Oscillations. Hindustan Publish. Corp., Delhi.

Burg, K.E. (1966). "Marine Geodetic Problems of Industry and Commerce." Proceedings of First Marine Geodesy Symposium, September 28-30, Columbus, Ohio.

Canadian Coast Guard (1981). Arctic Pilot Project (Northern Component). Termpal Code Assessment of the Arctic Pilot Project, August.

Cannon, R.H. (1967). Dynamics of Physical Systems. McGraw-Hill, New York.

Caughey, T.K. (1967). "Effect of Vibration on the Accuracy of a Vertical Reference Pendulum." S.I.A.M., Journal of Applied Mathematics, 15 (5): 1199-1208.

Christou, N.T. (1981). "The Gyrocompass as a Navigational Aid." Unpublished Report on the C.S.S. Hudson Cruise Nov. 1981, Department of Surveying Engineering, University of New Brunswick, Fredericton.

Crane, L.C. (1979). "Manoeuvring Trials of a 278000-DWT Tanker in Shallow and Deep Waters." The Society of Naval Architects and Marine Engineers Transactions, Vol. 87, pp. 251-283.

Deimel, R.F. (1950). Mechanics of the Gyroscope: The Dynamics of Rotation. Dover, U.S.A.

Eaton, R.M. (1982) Personal Communication. Bedford Institute of Oceanography, Dartmouth, N.S.

Encyclopaedia Brittanica, 1970 Edition.

Eschelbach, R.E. (1979). "State of the Art Development of Integrated Navigation Systems." Proceedings of the First International Hydrographic Technical Conference, May 14-18, Ottawa.

Ferry, E.S. (1932). Applied Gyrodynamics. John Wiley & Sons, New York.

Fish, S. (1951). Transient Analysis in Electrical Engineering. Prentice-Hall, New York.

Gardner, M.F. and J.L. Barnes (1942). Transients in Linear Systems. Vol. 1, John Wiley & Sons, New York.

Garnell, P. and D.J. East (1977). Guided Weapon Control Systems. Pergamon Press, Oxford.

Gill, A.D. (1979). "Mathematical Modelling of Ship Manoeuvring". Proceedings of the Conference on Mathematical Aspects of Marine Traffic, pp. 193-227, Editor: S.H. Hollingdale, Academic Press, London.

Goldstein, H. (1950). Classical Mechanics. Addison-Wesley, Reading, Massachusetts.

Gossick, B.R. (1967). Hamilton's Principle and Physical Systems. Academic Press, New York.

Grant, A.J. (1967). "Introduction to the Translation of Schuler's Original Paper." NAVIGATION: Journal of the Institute of Navigation, 14 (1): 22-25.

Grant, S.T. (1976). Integration of Passive Ranging Loran-C Satellite Navigation, Ship's Log, and Ship's Gyrocompass. Master of Science in Engineering Thesis, University of New Brunswick, Dept. of Surveying Engineering, Fredericton.

Grant, S.T. and D.E. Wells (1982). "Interactions among Integrated Navigation System Components." Marine Geodesy (in press).

Gray, A. (1959). A Treatise on Gyrostatics and Rotational Motion. Dover, New York.

Greenwood, D.T. (1965). Principles of Dynamics. Prentice-Hall, New Jersey.

Gunderson, H., H. Rigas, and F.S. VanVleck (1974). "A Technique for Determining Stability Regions for the Damped Mathieu Equation." S.I.A.M., Journal of Applied Mathematics, 26 (2): 345-349.

GYROS (1965). A Symposium arranged by the Automatic Control Group, The Institution of Mechanical Engineers, Vol. 179 Part 3E.

Haaland, T. (1976). "Marine Heading Reference System." Proceedings of the Institute of Navigation, National Navigation Meeting, 4-5 November, San Diego, California.

Hagedorn, P. (1981). Non-Linear Oscillations. Translated and Edited by W. Stadler, Clarendon Press, Oxford.

Halfman, R.L. (1962). Dynamics, Particles, Rigid Bodies and Systems. Addison-Wesley, Massachusetts.

Huston, R.L. and L.F. Doty (1966). "Note on the Krylov-Bogoliubov Method Applied to Linear Differential Equations." S.I.A.M., Journal of Applied Mathematics, 14 (3): 424-428.

Inter-Governmental Maritime Consultative Organization (1976). Operational Performance Standards for Shipborne Navigational Equipment. I.M.C.O. Publication, London.

Ishlinskii, A.Yu. (1965). Mechanics of Gyroscopic Systems. Israel Program for Scientific Translations, Jerusalem.

Killingbeck, J. and G.H.A. Cole (1971). Mathematical Techniques and Physical Applications. Academic Press, New York.

Klinkert, J. (1964). The Free Gyroscope and the Arma Brown Compass. Kandy Publications.

Koshliakov, V.N. (1959). "The Theory of a Gyrocompass." P.M.M., Journal of Applied Mathematics and Mechanics, 23 (5): 810-817 (translation: pp. 1164-1173).

Koshliakov, V.N. (1961). "On the reduction of the Equations of Motion of a Gyrohorizon-compass. P.M.M., Journal of Applied Mathematics and Mechanics, 25 (5): 801-805, (translation: pp. 1205-1211).

Koshliakov, V.N. (1962). "On the Stability of Motion of a Horizontal Gyrocompass in the Presence of Dissipative Forces." P.M.M., Journal of Applied Mathematics and Mechanics, 26 (3): 412-417, (translation: pp. 615-622.

Koshliakov, V.N. (1964). "On Geckeler's Equations in the Theory of Gyrocompass." P.M.M., Journal of Applied Mathematics and Mechanics, 28 (4): 708-715 (translation: pp. 867-875.

Koshliakov, V.N. (1964). "On Equations for the Position Determination of a Moving Object." P.M.M., Journal of Applied Mathematics and Mechanics, 28 (6): 1135-1137 (translation: pp. 1362-1365).

Kroschel, K. (1974). Statistische Nachrichten Theorie. Zweiter Teil: Signalschätzung. Springer-Verlag, Berlin/Heidelberg.

Kroschel, K. (1975). Digitale Filter. Manuskript Zur Vorlesung, Universität Karlsruhe, Institut Für Nachrichtensysteme.

Kryloff, N. and N. Bogoliuboff (1947). Introduction to Non-Linear Mechanics. Princeton University Press, Reprinted by Kraus Reprint Co., 1970, New York.

Landaw, L.D. and E.M. Lifshitz (1976). Mechanics. Third Edition, Pergamon Press.

Leonhard, W. (1972). Diskrete Regelsysteme. Hochschultaschenbücher, 523/5232, Bibliographisches Institut, Mahnheim.

Levadi, V.S. (1959). Calculation of Transient Response from Frequency Response. Engineering Experiment Station Bulletin Vol. XXVIII, No. 5, The Ohio State University, Columbus, Ohio.

Liashenko, V.F. (1962). "On the Reduction of the Equations of Motion of a Gyrohorizoncompass and Two-Gyroscope Vertical." P.M.M., Journal of Applied Mathematics and Mechanics, 26 (2): 369-372 (translation: pp. 537-542.

Magnus, K. (1974). Gyrodynamics. Springer-Verlag, Wien.

Manual of the Admiralty Gyro-Compass (Sperry Type) (1953). B.R. (53), Her Majesty's Stationery Office, London.

Markey, W.R. and J. Hovorka (1961). The Mechanics of Inertial Position and Heading Indication. Methuen & Co. Ltd., London.

Meirovitch, L. (1970). Methods of Analytical Dynamics. McGraw-Hill, New York.

Miller, N. (1951). A First Course in Differential Equations. Oxford University Press, London.

Minorsky, N. (1947). Introduction to Non-Linear Mechanics. Eduards Brothers, Ann Arbor, Michigan.

Morgan, M.J. (1978). Dynamic Positioning of Offshore Vessels. The Petroleum Publishing Co., Oklahoma.

Nash, R.A. (1968). "Effect of Vertical Deflections and Ocean Currents on a Manoeuvring Ship." IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-4:5, New York.

Nicholson, H . (1980. Modelling of Dynamical Systems. Vol. 1, The Institution of Electrical Engineers, London.

Operation and Service Manual of Sperry MK 37 Mod 1 Gyrocompass Equipment (1975).

Page, L. (1935). Introduction to Theoretical Physics. Second Edition, D. Van Nostrand Co., New York.

Potter, B. (1967). Stability Criteria for Linear Dynamical Systems. Oliver & Boyd, Edinburgh.

Price, W.G. and R.E.D. Bishop (1974). Probabilistic Theory of Ship Dynamics. Chapman and Hall, London.

Rawlings, A.L. (1944). The Theory of the Gyroscopic Compass and Its Applications. Second Edition, MacMillan, New York.

Roitenberg, Ia.N. (1960). "On the Motion of Gyroscopic Devices Under the Influence of Random Forces." P.M.M., Journal of Applied Mathematics and Mechanics, 24 (3): 463-472, (Translation: pp. 677-690).

Roitenberg, Ia.N. (1964). "On the Theory of a Gyrocompass." P.M.M., Journal of Applied Mathematics and Mechanics, 28 (5): 812-828, (Translation: pp. 991-1009).

Roitenberg, Ia.L. (1965). "On the Motion of a Gyrocompass with Random Displacements of Its Point of Support." P.M.M., Journal of Applied Mathematics and Mechanics, 29 (1): 165-172, (Translation: pp. 179-186).

Rose, J.C. (1974). "Open Ocean Navigation Accuracy Considerations for Marine Gravity Measurements." Proceedings of the International Symposium on Applications of Marine Geodesy, June 3-5, Columbus, Ohio.

Rosenberg, R.M. (1977). Analytical Dynamics of Discrete Systems. Plemum Press, New York.

Rosser, J.B. (1966). Space Mathematics. Part 1. Vol. 5, Lectures in Applied Mathematics, American Mathematical Society, Rhode Island.

Sandberg, I.W. (1964). "On the Stability of Solutions of Linear Differential Equations with Periodic Coefficients." S.I.A.M., Journal of Applied Mathematics, 12 (2): 487-496.

Sandberg, W. (1979). "The Estimation of Ship Motion-Induced Forces." Proceedings of the Fourth Ship Technology and Research (S.T.A.R.) Symposium, Huston, Texas, April 25-28.

Savet, P.H. (1961). Gyroscopes: Theory and Design. McGraw-Hill, New York.

Scarborough, J.B. (1958). The Gyroscope: Theory and Applications. Interscience Publishers, New York.

Schuler, M. (Translated by J.M. Slater) (1967). "The Disturbance of Pendulum and Gyroscopic Apparatus by the Acceleration of the Vehicle." NAVIGATION: Journal of the Institute of Navigation, 14 (1): 26-32.

Shimanov, S.N. (1963). "On the Theory of Linear Differential Equations with Periodic Coefficients and Time Lag." P.M.M., Journal of Applied Mathematics and Mechanics, 27 (3): 450-458 (Translation pp. 674-687).

Soong, T.T. (1965). "On Perturbation Solutions to Linear System Dynamics - I. Convergence." S.I.A.M., Journal of Applied Mathematics, 13 (4): 977-987.

Sorg, H.W. (1976). "From Serson to Draper - Two Centuries of Gyroscopic Development." NAVIGATION: Journal of the Institute of Navigation, 23 (1): 313-324.

Stancell, T.A. (1978). "The Transit Navigation Satellite System." Magnavox Publication R-5933.

Stevens, K.K. (1966). "On Linear Ordinary Differential Equations with Periodic Coefficients." S.I.A.M., Journal of Applied Mathematics, 14 (4): 782-795.

Struble, R.A. and J.E. Fletcher (1962). "General Perturbational Solution of the Mathieu Equation." S.I.A.M., Journal of Applied Mathematics, 10 (2): 314-328.

Sveshnikov, A.A. (1962). "On the Motion of a Gyroscopic Pendulum with Random Displacements of Its Point of Suspension." P.M.M., Journal of Applied Mathematics and Mechanics, 26 (3): 402-411 (Translation: pp. 601-614).

Taylor, M. (1971). "A Simple Gyrocompass for Marine Navigation." Proceedings of the Institute of Navigation, National Marine Meeting, Washington, Seattle, October 13-14.

Thomson, W.T. (1961). Introduction to Space Dynamics. John Wiley & Sons, New York.

Waters, D.C. (1962). "An Application of Vector Angles to Gyroscopic Analysis." I.R.E. Transactions on Aerospace and Navigational Electronics, Vol. ANE-9 (4): 235-240.

Weber, E. (1954). Linear Transient Analysis. Vol. 1, John Wiley & Sons, New York.

Wells, D.E. and S.T. Grant (1981). "An Adaptable Integrated Navigation System: BIONAV". Paper Presented at the Canadian Petroleum Association, Colloquium III, October 14-16, Alberta.

Wrigley, W., W.M. Hollister and W.G. Denhard (1969). Gyroscopic Theory, Design, and Instrumentation. The MIT Press, Cambridge, Massachusetts.

Ziegler, H. (1963). Gyrodynamics. International Union of Theoretical and Applied Mechanics, Symposium Celerina, August 20-23, Springer-Verlag, Berlin.

APPENDIX I

LAGRANGIAN FORMULATION OF THE EQUATIONS OF MOTION OF AN UNDAMPED STATIONARY GYROCOMPASS

In this Appendix Lagrange's equations of motion are used to derive the differential equations which describe the two modes of motion of the gyrocompass, namely, the motion in azimuth and the motion in tilt.

Recall equations (4.6) and (4.7)

$$T = \frac{1}{2}\{[C(\Omega_\zeta)^2 + B(\Omega_\xi)^2 + A(\Omega_\eta)^2] +$$
$$[C'(\omega_\zeta)^2 + B'(\omega_\xi)^2 + A'(\omega_\eta)^2 + \qquad \text{(I-1)}$$
$$[A''(\omega_{\zeta'})^2 + B''(\omega_{\xi'})^2 + C''(\omega_{\eta'})^2]\}$$

and

$$U = mg\ell(1 - \cos\beta) \qquad \text{(I-2)}$$

We rewrite the above equation (I-1) using equations (4.1), (4.2) and (4.3), and after some rearrangement of terms we have

$$2T = (A+A')\ [\dot{\beta} - \Omega \cos\phi \sin\alpha]^2 +$$
$$(A+B')\ [(\Omega \sin\phi+\dot{\alpha})\cos\beta + (\Omega \cos\phi \cos\alpha)\sin\beta]^2 +$$
$$C\ [(\Omega \cos\phi \cos\alpha)\cos\beta - (\Omega \sin\phi+\dot{\alpha})\sin\beta+\dot{\psi}]^2 +$$
$$C'[(\Omega \cos\phi \cos\alpha)\cos\beta - (\Omega \sin\phi+\dot{\alpha})\sin\beta]^2 +$$
$$A''[\Omega \cos\phi \cos\alpha]^2 +$$
$$B''[\Omega \cos\phi \sin\alpha]^2 +$$
$$C''[(\Omega \sin\phi+\dot{\alpha})]^2 \qquad \text{(I-3)}$$

Recall equations (4.10)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = - \frac{\partial U}{\partial q_i} \qquad \text{(I-4)}$$

Using equations (I-4) we have

$$2\frac{\partial T}{\partial \dot{\alpha}} = 2(A+B')\{(\Omega \sin\phi+\dot{\alpha})\cos\beta + (\Omega \cos\phi \cos\alpha)\sin\beta\}\cos\beta -$$
$$2C\{(\Omega \cos\phi \cos\alpha)\cos\beta - (\Omega \sin\phi+\dot{\alpha})\sin\beta + \dot{\psi}\}\sin\beta -$$
$$2C'\{(\Omega \cos\phi \cos\alpha)\cos\beta - (\Omega \sin\phi+\dot{\alpha})\sin\beta\}\sin\beta +$$
$$2C''\{(\Omega \sin\phi+\dot{\alpha})\} \qquad \text{(I-5)}$$

or

$$\frac{\partial T}{\partial \dot{\alpha}} = (A+B')\{(\Omega \sin\phi+\dot{\alpha})\cos^2\beta + \Omega \cos\phi \cos\alpha \sin\beta \cos\beta\} -$$
$$C\ \{\Omega \cos\phi \cos\alpha \sin\beta \cos\beta - (\Omega \sin\phi+\dot{\alpha})\sin^2\beta + \dot{\psi} \sin\beta\} -$$
$$C'\{\Omega \cos\phi \cos\alpha \sin\beta \cos\beta - (\Omega \sin\phi+\dot{\alpha})\sin^2\beta\} +$$
$$C''\{(\Omega \sin\phi + \dot{\alpha})\} \qquad \text{(I-6)}$$

By applying the mathematical simplifications listed in Table 4.1 we have, for small angle approximation,

$$\cos \varepsilon \doteq 1 \quad \text{and} \quad \sin \varepsilon \doteq \varepsilon \tag{I-7}$$

thus

$$\begin{aligned} \frac{\partial T}{\partial \dot{\alpha}} = {} & (A+B')\{(\Omega \sin \phi + \dot{\alpha}) + (\Omega \cos \phi)\beta\} - \\ & C\ \{(\Omega \cos \phi)\beta - (\Omega \sin \phi + \dot{\alpha})\beta^2 + \dot{\psi}\beta\} - \\ & C'\{(\Omega \cos \phi)\beta - (\Omega \sin \phi + \dot{\alpha})\beta^2\} + \\ & C''\{(\Omega \sin \phi + \dot{\alpha})\} \end{aligned} \tag{I-8}$$

but β is small and β^2 is even smaller, thus $\Omega\beta^2$ << and $\dot{\alpha}\beta^2$ << (because they are second order terms). These terms can be safely neglected and, after some rearrangements, we have

$$\frac{\partial T}{\partial \dot{\alpha}} = (A+B'+C'')(\Omega \sin \phi+\dot{\alpha})+[(A+B'-C')\Omega \cos \phi-C(\Omega \cos \phi+\dot{\psi})]\beta \tag{I-9}$$

but

$$(\Omega \cos \phi + \dot{\psi}) \doteq \dot{\psi} = n \tag{I-10}$$

where $\dot{\psi}$ is the rotational speed of the gyro-wheel which is many orders of magnitude larger than the earth's horizontal rate ($\Omega \cos \phi$). Thus

$$C(\Omega \cos \phi + \dot{\psi}) \doteq Cn = \text{constant} \tag{I-11}$$

within the assumptions postulated. The term (Cn) expresses the angular momentum of the gyro-wheel, which can be considered constant for all purposes in this analysis.

Finally, we have

$$\frac{\partial T}{\partial \dot{\alpha}} = (A+B'+C')(\Omega \sin \phi + \dot{\alpha}) + [A+B'-C')\ \Omega \cos \phi - Cn]\beta \qquad (I-12)$$

and

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) = (A+B'+C'')\ddot{\alpha} + [(A+B'-C')\Omega \cos \phi - Cn]\dot{\beta}} \quad . \qquad (I-13)$$

The term $\partial T/\partial \alpha$ is evaluated now

$$2\,\frac{\partial T}{\partial \alpha} = 2(A+A')\{[\dot{\beta} - \Omega \cos \phi \sin \alpha][-\Omega \cos \phi \cos \alpha]\} +$$

$$2(A+B')\{[(\Omega \sin \phi + \dot{\alpha})\cos \beta + (\Omega \cos \phi \cos \alpha)\sin \beta]$$

$$[-\Omega \cos \phi \sin \alpha \sin \beta]\} +$$

$$2C\ \{[(\Omega \cos \phi \cos \alpha)\cos \beta - (\Omega \sin \phi + \dot{\alpha})\sin \beta + \dot{\psi}]$$

$$[-\Omega \cos \phi \sin \alpha \cos \beta]\} +$$

$$2C'\{[(\Omega \cos \phi \cos \alpha \cos \beta - (\Omega \sin \phi + \dot{\alpha})\sin \beta]$$

$$[-\Omega \cos \phi \sin \alpha \cos \beta]\}+$$

$$2A''\{(\Omega \cos \phi \cos \alpha)(-\Omega \cos \phi \sin \alpha)\} +$$

$$2B''\{(\Omega \cos \phi \sin \alpha)(\Omega \cos \phi \cos \alpha)\} \qquad (I-14)$$

and using the assumptions of eqn. (I-7) and eqn. (I-10)

$$\frac{\partial T}{\partial \alpha} = -(A+A')\{\dot{\beta}-(\Omega \cos \phi)\alpha\}\Omega \cos \phi$$

$$-(A+B')\{(\Omega \sin \phi + \dot{\alpha}) + (\Omega \cos \phi)\beta\}(\Omega \cos \phi)\alpha\beta$$

$$-Cn\{(\Omega \cos \phi)\alpha\}$$

$$-\ C'\{\Omega \cos \phi - (\Omega \sin \phi + \dot{\alpha})\beta\}(\Omega \cos \phi)\alpha$$

$$-A''(\Omega \cos \phi)(\Omega \cos \phi)\alpha$$

$$+B''(\Omega \cos \phi)(\Omega \cos \phi)\alpha \qquad (I-15)$$

or,

$$-\frac{\partial T}{\partial \alpha} = (A+A')\Omega \cos\phi\{\dot{\beta}-(\Omega\cos\phi)\alpha\} +$$

$$(A+B')\Omega\cos\phi\{(\Omega\sin\phi+\dot{\alpha})+\Omega\cos\phi\}\alpha\beta +$$

$$Cn\Omega\cos\phi\ \alpha +$$

$$C'\Omega\cos\phi\{\Omega\cos\phi-(\Omega\sin\phi+\dot{\alpha})\beta\}\alpha +$$

$$(A''+B'')\ (\Omega\cos\phi)^2\alpha \tag{I-16}$$

again second order terms, like $\Omega^2\alpha$, $\Omega^2\alpha\beta$, $\Omega\dot{\alpha}\alpha\beta$, $\Omega^2\alpha\beta^2$, and $\Omega\dot{\alpha}\alpha$, are neglected and the final expression is

$$\boxed{-\frac{\partial T}{\partial \alpha} = [(A+A')\Omega\cos\phi]\dot{\beta} + [Cn\Omega\cos\phi]\alpha} \tag{I-17}$$

Using the same procedure as above we have

$$\frac{\partial T}{\partial \dot{\beta}} = (A+A')[\dot{\beta}-(\Omega\cos\phi)\alpha] \tag{I-18}$$

and

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) = (A+A')\ddot{\beta} - [(A+A')\Omega\cos\phi]\dot{\alpha}}\quad , \tag{I-19}$$

and

$$2\frac{\partial T}{\partial \beta} = 2(A+B')\{[(\Omega\sin\phi+\dot{\alpha})\cos\beta+(\Omega\cos\phi\cos\alpha)\sin\beta]$$

$$[\Omega\cos\phi\cos\alpha\cos\beta-(\Omega\sin\phi+\dot{\alpha})\sin\beta\ \} +$$

$$2C\{[(\Omega\cos\phi\cos\alpha)\cos\beta-(\Omega\sin\phi+\dot{\alpha})\sin\beta+\dot{\psi}]$$

$$[-\Omega\cos\phi\cos\alpha\sin\beta-(\Omega\sin\phi+\dot{\alpha})\cos\beta]\} +$$

$$2C'\{(\Omega\cos\phi\cos\alpha)\cos\beta-(\Omega\sin\phi+\dot{\alpha})\sin\beta]$$

$$[-\Omega\cos\phi\cos\alpha\sin\beta-(\Omega\sin\phi+\dot{\alpha})\cos\beta]\} \tag{I-20}$$

or,

$$\frac{\partial T}{\partial \beta} = (A+B')\{[(\Omega \sin \phi + \dot{\alpha}) + (\Omega \cos \phi)\beta][(\Omega \cos \phi) - (\Omega \sin \phi + \dot{\alpha})\beta]\} -$$

$$Cn\{(\Omega \cos \phi)\beta + (\Omega \sin \phi + \dot{\alpha})\} -$$

$$C'\{[(\Omega \cos \phi)-(\Omega \sin \phi + \dot{\alpha})\beta][(\Omega \cos \phi)\beta + (\Omega \sin \phi + \dot{\alpha})]\} \quad (I-21)$$

or,

$$-\frac{\partial T}{\partial \beta} = -(A+B'-C')\{[(\Omega \cos \phi) - (\Omega \sin \phi + \dot{\alpha})\beta]$$

$$[(\Omega \cos \phi)\beta + (\Omega \sin \phi + \dot{\alpha})]\} +$$

$$+ Cn\dot{\alpha} + Cn\Omega \sin \phi + [Cn\Omega \cos \phi]\beta \qquad (I-22)$$

or,

$$-\frac{\partial T}{\partial \beta} = -(A+B'-C')\{(\Omega \cos \phi)^2\beta + (\Omega \cos \phi)(\Omega \sin \phi + \dot{\alpha}) -$$

$$-(\Omega \cos \phi)(\Omega \sin \phi + \dot{\alpha})\beta^2 - (\Omega \sin \phi + \dot{\alpha})^2\beta\} +$$

$$+ Cn\dot{\alpha} + Cn\Omega \sin \phi + [Cn\Omega \cos \phi]\beta \qquad (I-23)$$

or,

$$-\frac{\partial T}{\partial \beta} =-(A+B'-C') \{\Omega^2\cos^2\phi\beta + \Omega^2\sin \phi \cos \phi + \Omega \cos \phi \dot{\alpha} -$$

$$-\Omega^2\sin \phi \cos \phi\beta^2 - \Omega \cos \phi \dot{\alpha}\beta^2 - \Omega^2\sin^2\phi\beta -$$

$$-\dot{\alpha}^2\beta - 2\Omega \sin \phi\dot{\alpha}\beta\} + Cn\dot{\alpha} + Cn\Omega \sin \phi +$$

$$+[Cn\Omega \cos \phi]\beta \qquad (I-24)$$

or,

$$-\frac{\partial T}{\partial \beta} = -(A+B'-C')\{\Omega \cos \phi\dot{\alpha}\} + Cn\dot{\alpha} + Cn\Omega \sin \phi +$$

$$-(A+B'-C')\Omega^2\sin \phi \cos \phi + [Cn\Omega \cos \phi]\beta \qquad (I-25)$$

after neglecting the second order terms, such as, $\Omega^2\beta$, $\Omega^2\beta^2$, $\Omega\dot{\alpha}\beta^2$, $\dot{\alpha}^2\beta$, $\Omega\dot{\alpha}\beta$,

or,

$$-\frac{\partial T}{\partial \beta} = -[(A+B'-C')\Omega \cos \phi - Cn]\dot{\alpha} + [Cn\Omega \cos \phi]\beta +$$

$$-[(A+B'-C')\Omega \cos \phi - Cn]\Omega \sin \phi \quad . \qquad \text{(I-26)}$$

But

$$(A+B'-C')\Omega \cos \phi << Cn \quad .$$

Therefore,

$$\boxed{-\frac{\partial T}{\partial \beta} = Cn\dot{\alpha} + [Cn\Omega \cos \phi]\beta + Cn\Omega \sin \phi} \qquad \text{(I-27)}$$

From eqn. (I-2) and eqn. (I-4) we have

$$\boxed{\frac{\partial U}{\partial \alpha} = 0} \qquad \text{(I-28)}$$

$$\frac{\partial U}{\partial \beta} = -mg\ell \sin \beta \qquad \text{(I-29)}$$

or,

$$\boxed{-\frac{\partial U}{\partial \beta} = mg\ell\beta} \qquad \text{(I-30)}$$

The equations of motion are written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} - \frac{\partial U}{\partial \alpha} = 0} \qquad \text{(I-31)}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} - \frac{\partial U}{\partial \beta} = 0} \qquad \text{(I-32)}$$

and using the expressions from eqns. (I-13), (I-17), (I-28) and (I-19), (I-27), (I-30) we have

$$(A+B'+C")\ddot{\alpha} + [(A+B'-C')\Omega \cos \phi - Cn]\dot{\beta} + [(A+A')\Omega \cos \phi]\dot{\beta} +$$

$$[Cn\Omega \cos \phi]\alpha = 0 \qquad \text{(I-33)}$$

and

$$(A+A')\ddot{\beta} - [(A+A')\Omega \cos \phi]\dot{\alpha} + Cn\dot{\alpha} + [Cn\Omega \cos \phi]\beta +$$

$$+ Cn\Omega \sin \phi + mg\ell\beta = 0 \qquad \text{(I-34)}$$

which reduce to

$$(A+B'+C'')\ddot{\alpha} + [(2A+A'+B'-C')\Omega \cos \phi - Cn]\dot{\beta} + [Cn\Omega \cos \phi]\alpha = 0 \qquad \text{(I-35)}$$

and

$$(A+A')\ddot{\beta} - [(A+A')\Omega \cos\phi - Cn]\dot{\alpha} + [Cn\Omega \cos \phi + mg\ell]\beta = -Cn\Omega \sin \phi \qquad \text{(I-36)}$$

In eqn. (I-36) the term ($Cn\Omega \cos \phi$) is approximately three orders of magnitude smaller than ($mg\ell$) thus, it can be neglected without any significant loss.

The final expressions for the equations of motion of an undamped, stationary gyrocompass are

$$D_1\ddot{\alpha} + E_1\dot{\beta} + G_1\alpha = 0 \qquad \text{(I-37)}$$

$$D_2\ddot{\beta} + E_2\dot{\alpha} + G_2\beta = F_2 \qquad \text{(I-38)}$$

where:

$$D_1 = A+B'+C''$$

$$E_1 = (2A+A'+B'-C')\Omega \cos \phi - Cn$$

$$G_1 = Cn\Omega \cos \phi$$

$$D_2 = (A+A') \qquad \text{(I-39)}$$

$$E_2 = -(A+A')\Omega \ \text{os} \ \phi - Cn$$

$$G_2 = mg\ell$$

$$F_2 = Cn\Omega \sin \phi.$$

APPENDIX II

LAGRANGIAN FORMULATION OF THE EQUATIONS OF MOTION OF A DAMPED STATIONARY GYROCOMPASS

In the analysis presented in Appendix I the system analyzed (undamped gyrocompass) was a conservative mechanical system (i.e., a system in which the total mechanical energy - potential plus kinetic energy - remains constant). That is, no work is done on the system by external forces, and no mechanical energy is "*dissipated*". Lagrange's equations of motion, namely eqn. (4.10) or eqn. (I-4) in Appendix I, are the equations of motion derived for a conservative system.

In this Appendix the equations of motion of a damped gyrocompass are developed. Damping the oscillations of the gyrocompass involves dissipation of energy, and the mechanical system in turn becomes non-conservative. Therefore, Lagrange's equations of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{II-1}$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \frac{\partial U}{q_i} = Q_i \tag{II-2}$$

where, assuming that all the forces derivable from the potential function are accounted for in the Lagrangian, we denote the nonpotential forces simply by Q_i. In our case, the potential function U does not depend explicitly on time, and Q_i coincides with the nonconservative forces associated with q_i.

In the case of the damped gyrocompass the pendulous weight mg is displaced at an angle γ to the east. This would provide a torque opposing the motion in azimuth α, thus decreasing the azimuthal precession of the spin-axis in each consecutive swing about the equilibrium position. Therefore, the generalized force Q_i in eqn. (II-2) is associated with the motion in azimuth. Then Lagrange's equations have the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} - \frac{\partial U}{\partial \alpha} = Q_\alpha \qquad \text{(II-3)}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} - \frac{\partial U}{\partial \beta} = 0 \qquad \text{(II-4)}$$

Equation (II-4) is the same as eqn. (I-32) in Appendix I. Equation (II-3) is the same as eqn. (I-31) in Appendix I, except the term Q_α on the right-hand side.

The generalized force Q_α is nothing else but the additional torque applied to the gyrocompass about the η-axis due to the displaced mass m. It is given by the expression

$$Q_\alpha = M_\alpha = mg\ell\gamma\beta \qquad \text{(II-5)}$$

The equations of motion in azimuth and tilt are written as

$$D_1\ddot{\alpha} + E_1\dot{\beta} + G_1\alpha + F_1\beta = 0 \qquad \text{(II-6)}$$

$$D_2\ddot{\beta} + E_2\dot{\alpha} + G_2\beta = F_2 \qquad \text{(II-7)}$$

where:

$$D_1 = A+B'+C''$$

$$E_1 = (2A+A'+B'-C')\Omega \cos \phi - Cn$$

$$G_1 = Cn\Omega \cos \phi$$

$$F_1 = -mg\ell\gamma \qquad \text{(II-8)}$$

$$D_2 = (A+A')$$

$$E_2 = -(A+A')\Omega \cos \phi - Cn$$

$$G_2 = mg\ell$$

$$F_2 = Cn\Omega \sin \phi \quad .$$

APPENDIX III

THE SUPERPOSITION PRINCIPLE

The *principle of superposition* can be stated as follows [*Greenwood 1965*]:

> "If $x_1(t)$ is the response of a linear system to an input $F_1(t)$ for initial conditions $x_1(0)$, $\dot{x}_1(0)$, and so on, and if $x_2(t)$ is the response of the same system to an input $F_2(t)$ for initial conditions $x_2(0)$, $\dot{x}_2(0)$, and so on, then $x_1(t) + x_2(t)$ is the response of that system to the input $F_1(t) + F_2(t)$, assuming the initial conditions are $x_1(0) + x_2(0)$, $\dot{x}_1(0) + \dot{x}_2(0)$, and so on."

The superposition principle can be extended to more than two inputs or forcing functions.

The superposition principle applies to any system which is described by *linear* differential equations. It encourages the process of finding general solutions. It is because of the additive property of the superposition principle that linear systems are so easily analyzed. For nonlinear systems this principle does not apply and it is not possible to talk in terms of general solutions, even in relatively simple cases.

A very important property of the superposition principle is that a general input or output function can be considered to be composed of a sequence of small superimposed functions. Also it implies that the differentiation of a general input to a linear system results in the differentiation of the output. Such an example is the unit impulse function which is the time derivative of the unit step function.

Finally, it is noted that the principle of superposition allows us to obtain the complete solution to a differential equation as the sum of the transient and steady-state solutions.

APPENDIX IV

MOTION AROUND A ROTATING SPHERE

In this Appendix the linear accelerations of a point P moving around a rotating sphere are derived in the local navigational frame.

Let the point P move around a rotating sphere of radius R. At any instant the position of P may be defined by its radial distance h from the surface of the sphere, the angle of latitude ϕ and the angle of longitude λ.

We define the coordinate system OXYZ tc be fixed in space and the sphere to rotate around the OZ-axis with angular velocity Ω = constant (Figure 1).

The cylindrical coordinates of P are

$$r = (R+h)\cos\phi \tag{IV-1}$$

$$\theta = \lambda + \Omega t \tag{IV-2}$$

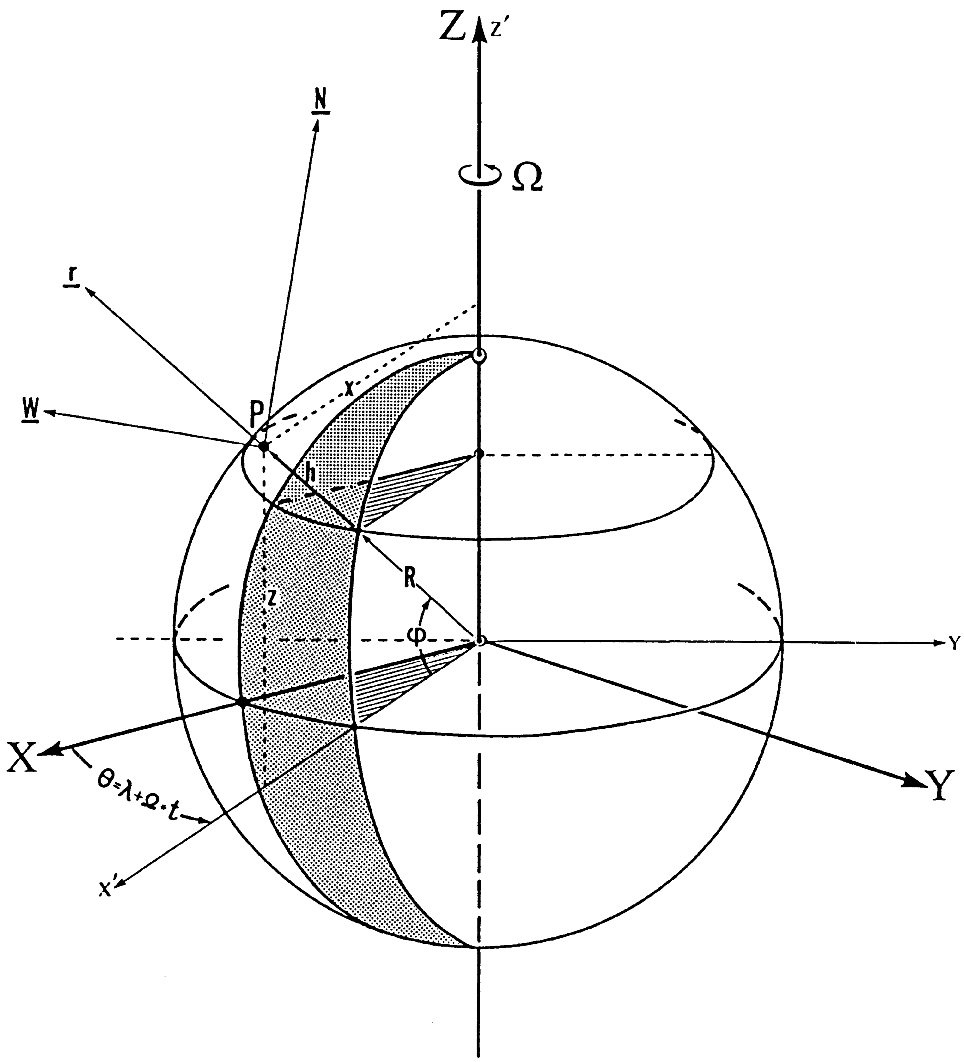
$$z = (R+h)\sin\phi \tag{IV-3}$$

The acceleration components in the instantaneous coordinate system Oxyz are (P lies on the Oxz plane)

$$a_x = \ddot{r} + \dot{\theta}^2 \tag{IV-4}$$

$$a_y = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{IV-5}$$

$$a_z = \ddot{z} \tag{IV-6}$$



Motion Around a Rotating Sphere

Fig.1

But,

$$\dot{r} = \dot{h}\cos\phi - (R+h)\dot{\phi}\sin\phi$$

$$\ddot{r} = \ddot{h}\cos\phi - \dot{h}\dot{\phi}\cdot\sin\phi - \dot{h}\dot{\phi}\sin\phi - (R+h)[\dot{\phi}^2\cos\phi+\ddot{\phi}\sin\phi]$$

$$\dot{\theta} = \dot{\lambda} + \Omega$$

$$\ddot{\theta} = \ddot{\lambda}$$

$$\dot{z} = \dot{h}\sin\phi + (R+h)\dot{\phi}\cos\phi$$

$$\ddot{z} = \ddot{h}\sin\phi + \dot{h}\dot{\phi}\cos\phi + \dot{h}\dot{\phi}\cos\phi-(R+h)[\dot{\phi}^2\cos\phi-\ddot{\phi}\sin\phi].$$

Thus,

$$a_x = -(R+h)\{[\dot{\phi}^2+(\dot{\lambda}+\Omega)^2]\cos\phi + \ddot{\phi}\sin\phi\}-2\dot{h}\dot{\phi}\sin\phi + \ddot{h}\cos\phi \qquad \text{(IV-7)}$$

$$a_y = (R+h)\ddot{\lambda}\cos\phi + 2(\dot{\lambda}+\Omega)\{\dot{h}\cos\phi - (R+h)\dot{\phi}\sin\phi\} \qquad \text{(IV-8)}$$

$$a_z = (R+h)\{\ddot{\phi}\cos\phi-\dot{\phi}^2\sin\phi\} + 2\dot{h}\dot{\phi}\cos\phi + \ddot{h}\sin\phi \qquad \text{(IV-9)}$$

At point P, the corresponding acceleration components in the local navigational frame are

$$a_{E-W} = a_y \qquad \text{(IV-10)}$$

$$a_{N-S} = a_z\cos\phi - a_x\sin\phi \qquad \text{(IV-11)}$$

$$a_r = a_z\sin\phi + a_x\cos\phi \qquad \text{(IV-12)}$$

After some algebraic operations the final expressions are

$$a_{E-W} = (R+h)\ddot{\lambda}\cos\phi + 2(\dot{\lambda}+\Omega)\{\dot{h}\cos\phi - (R+h)\dot{\phi}\sin\phi\} \qquad \text{(IV-13)}$$

$$a_{N-S} = (R+h)\{\ddot{\phi} + (\dot{\lambda}+\Omega)^2\cos\phi\sin\phi\} + 2\dot{h}\dot{\phi} \qquad \text{(IV-14)}$$

$$a_r = -(R+h)\{\dot{\phi}^2 + (\dot{\lambda}+\Omega)^2\cos^2\phi\} + \ddot{h} \qquad \text{(IV-15)}$$

Equations (IV-13), (IV-14), (IV-15) are rewritten in the following form for

$$h = 0 \qquad \text{(IV-16)}$$

$$a_{E-W} = R\{\ddot{\lambda} \cos \phi - 2\dot{\phi}(\dot{\lambda}+\Omega)\sin \phi\} \tag{IV-17}$$

$$a_{N-S} = R\{\ddot{\phi} + (\dot{\lambda}+\Omega)^2 \cos \phi \sin \phi\} \tag{IV-18}$$

$$a_r = -R\{\dot{\phi}^2 + (\dot{\lambda}+\Omega)^2 \cos^2\phi\} \tag{IV-19}$$

The velocity components of P in the same respective directions are

$$v_{E-W} = (R+h)\dot{\lambda} \cos \phi \tag{IV-20}$$

$$v_{N-S} = (R+h)\dot{\phi} \tag{IV-21}$$

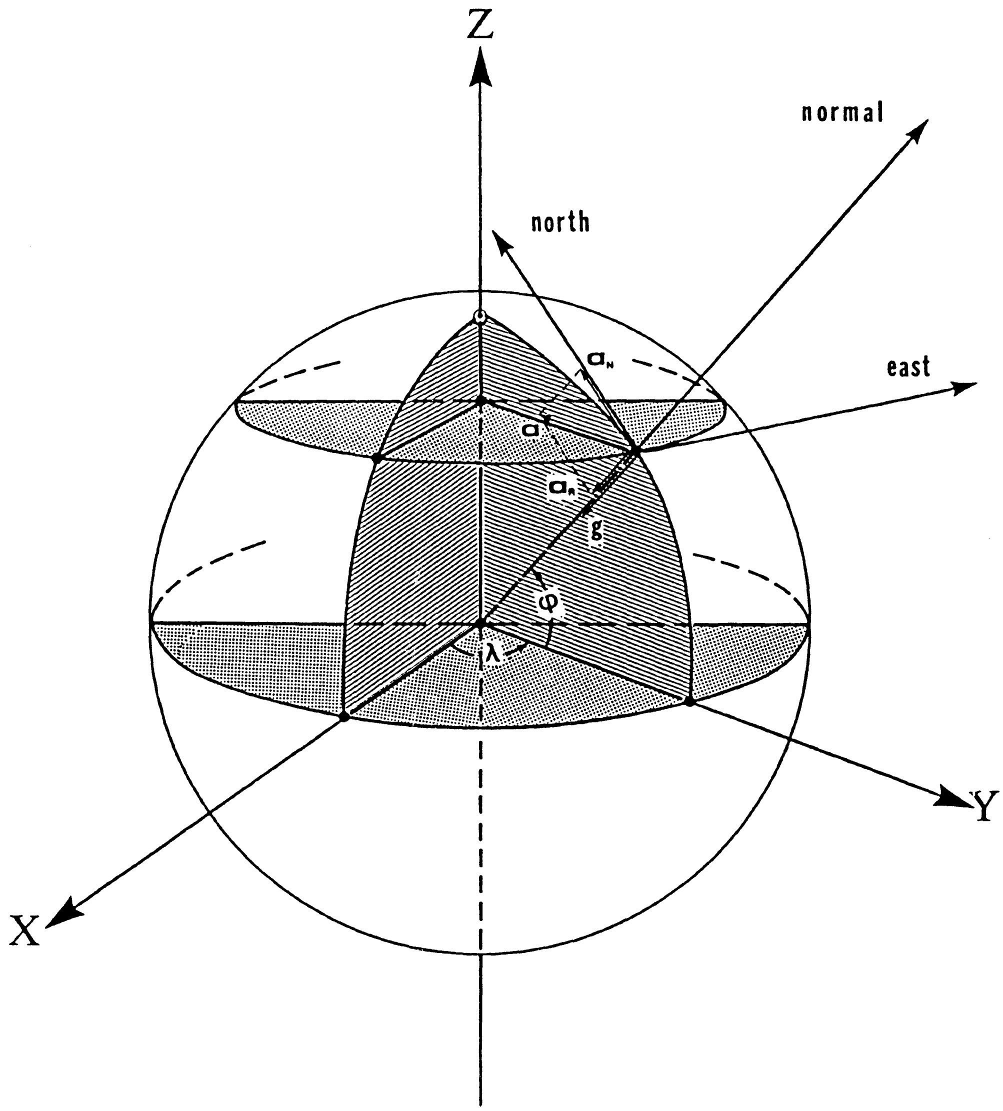
$$v_r = \dot{h} \tag{IV-22}$$

Special Case: Navigation Along a Parallel of latitude $\phi°$.

When navigating on the surface of the earth along a parallel of latitude $\phi°$ (Figure 2), the apparent gravitational acceleration sensed is

$$g^* = (g + a_r) = g + a \cos \phi \tag{IV-23}$$

where a is the centripetal acceleration.



Navigation along a Parallel

Fig.2

APPENDIX V

BALLISTIC DEFLECTION

The movement of the gyrocompass spin-axis resulting from the vessel's acceleration is called *ballistic deflection.* Acceleration induced errors are less predictable than those due to velocity, since they may vary rapidly with time. Acceleration errors are especially likely to occur during ship's manoeuvres. However, it is essential for a gyrocompass to continue operating during manoeuvres or alterations of ship's course and/or speed.

If we suppose that the ship is at a point P on the earth's surface and it moves with respect to the earth, then the acceleration components of the ship in the local navigational frame (defined by the east, north, and radial directions and earth fixed) will be (from Appendix IV, eqns. (IV-17), (IV-18), (IV-19))

$$a_{E-W} = R\{\ddot{\lambda}\cos\phi - 2\dot{\phi}(\dot{\lambda}+\Omega)\sin\phi\} \tag{V-1}$$

$$a_{N-S} = R\{\ddot{\phi} + (\dot{\lambda}+\Omega)^2\cos\phi\sin\phi\} \tag{V-2}$$

$$a_r = -R\{\dot{\phi}^2 + (\dot{\lambda}+\Omega)^2\cos^2\phi\} \tag{V-3}$$

The latitude and longitude rates of change of the ship's position as a function of the course and speed are obtained from Appendix IV, eqns. (IV-20), (IV-21), (IV-22) and (IV-16)

$$\dot{\phi} = \frac{U_{N-S}}{R}, \quad \text{or,} \quad \dot{\phi} = \frac{v\cos H}{R} \tag{V-4}$$

$$\dot{\lambda} = \frac{U_{E-W}}{R\cos\phi}, \quad \text{or,} \quad \dot{\lambda} = \frac{v\sin H}{R\cos\phi} \tag{V-5}$$

and

$$\ddot{\phi} = \{\frac{\dot{v} \cos H}{R} - \frac{v\dot{H} \sin H}{R}\} \tag{V-6}$$

$$\ddot{\lambda} = \{\frac{\dot{v} \sin H + v\dot{H} \cos H}{R \cos \phi} + \frac{v^2}{R} \frac{\tan \phi}{\cos \phi} \sin H \cos H\} . \tag{V-7}$$

Substitution of the expressions for $\dot{\phi}$, $\dot{\lambda}$, $\ddot{\phi}$, $\ddot{\lambda}$ in eqns. (V-1), (V-2), and (V-3) yields

$$a_{E-W} = \dot{v} \sin H + v\dot{H} \cos H - \frac{v^2}{R} \tan \phi \sin H \cos H - 2\Omega \sin \phi v \cos H \tag{V-8}$$

$$a_{N-S} = \dot{v} \cos H - v\dot{H} \sin H + \frac{v^2}{R} \tan \phi \sin^2 H + \Omega^2 R \cos \phi \sin \phi +$$

$$+ 2\Omega \sin \phi \; v \sin H \tag{V-9}$$

$$a_r = 0 . \tag{V-10}$$

If we consider the result of the added acceleration, we need only consider those terms which contain $\dot{v}$ and $\dot{H}$ because the terms which contain v and H have already been accounted for in the gyrocompass response analysis. Thus, the resulting residual acceleration would cause the gyro spin-axis to precess during any alteration of speed and/or course, introducing temporal (transient) errors in the compass readings. If the resulting precessional velocity were of the proper magnitude, it would cause the compass to precess during the time the velocity of the ship is changing from the resting position proper to the speed-and-course and latitude error at the beginning of the acceleration, to the resting position proper to the speed-and-course and latitude error at the end of the acceleration. Under this condition, there would be zero ballistic deflection error. This condition is fulfilled when the gyrocompass is Schuler tuned. The spin-axis of the gyrocompass will then move without oscillation to the resting position appropriate to the new speed, course, and latitude. In this case the gyrocompass is said to be *aperiodic* or *dead-beat*.

The ballistic deflection depends only on the compass constants and the linear acceleration of the ship. The ballistic deflection error and the speed-and-course error are in the same direction.

Below the dynamic response of a gyrocompass, which is not Schuler tuned, is investigated under acceleration inputs.

At the beginning of the acceleration the compass is pointing in a direction ON' (Figure 1),

$$\varepsilon = -\gamma \tan \phi + \tan \delta \qquad \text{(V-11)}$$

where; ε is the gyrocompass total error

$-\gamma \tan \phi$ is the latitude error

$\tan \delta$ is the speed-and-course error.

The residual accelerations are

$$(a_{E-W})^{res} = \dot{v} \sin H + v\dot{H} \cos H \qquad \text{(V-12)}$$

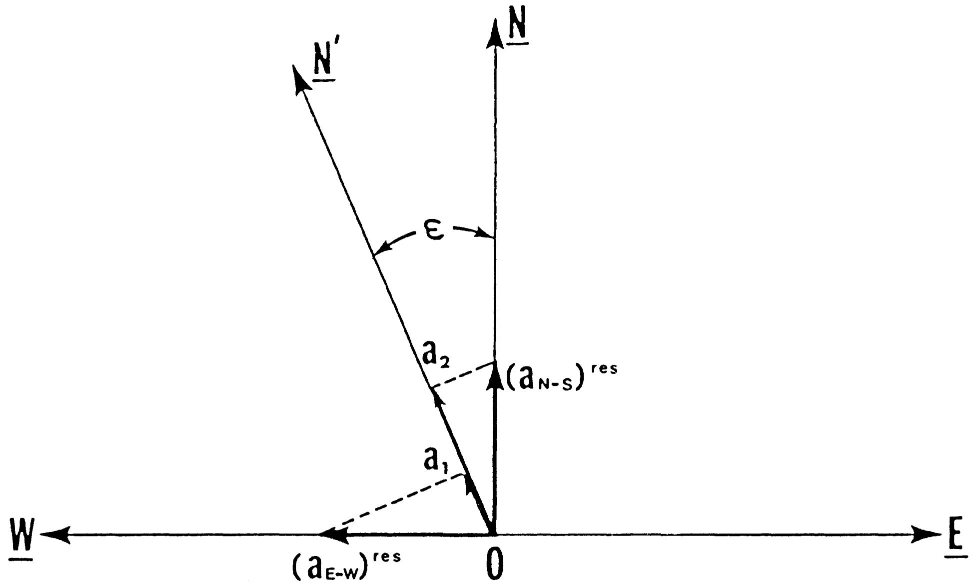
$$(a_{N-S})^{res} = \dot{v} \cos H - v\dot{H} \sin H \; . \qquad \text{(V-13)}$$

But only forces due to acceleration perpendicular to the phantom ring can be transmitted to the gyrocompass, thus we only deal with the N-S component of the induced acceleration.

Analyzing $(a_{N-S})^{res}$ and $(a_{E-W})^{res}$ into two components along the ON'-axis and perpendicular to it, we obtain

$$a_{N'-S'} = a_1+a_2 = (\dot{v} \cos H-v\dot{H} \sin H)\cos \varepsilon + (\dot{v} \sin H+v\dot{H} \cos H)\sin \varepsilon \; . \qquad \text{(V-14)}$$

This acceleration $a_{N'-S'}$ causes the spin axis to precess in azimuth with a rate given by



Residual acceleration components

Fig.1

$$\dot{\alpha}_a = \frac{m\ell}{Cn} a_{N'-S'} \qquad \text{(V-15)}$$

The changing resting postion of the spin-axis is

$$\dot{\varepsilon} = \frac{d}{dt}\{\tan^{-1}(\frac{v \cos H}{\Omega R \cos \phi + v \sin H}) - \gamma \tan \phi\} \qquad \text{(V-16)}$$

and the ballistic deflection error is given by

$$d = \int_{t_1}^{t_2} (\dot{\varepsilon} - \dot{\alpha}_a) dt \qquad \text{(V-17)}$$

where; d is the ballistic deflection error

t_1, t_2 are the time instants at the beginning and end of the acceleration.

From eqn. (V-16) it follows that

$$\dot{\varepsilon} = \frac{d}{dt}\{\tan^{-1}(\frac{v \cos H}{\Omega R \cos \phi + v \sin H}\} - \gamma \sec^2\phi\, \dot{\phi}$$

or,

$$\dot{\varepsilon} = \frac{1}{1 + (\frac{v \cos H}{\Omega R \cos \phi + v \sin H})^2} \frac{d}{dt}\{\frac{v \cos H}{\Omega R \cos \phi + v \sin H}\} - \gamma \sec^2\phi\, \dot{\phi}$$

or

$$\dot{\varepsilon} = \frac{(\Omega R \cos \phi + v \sin H)^2}{(\Omega R \cos \phi + v \sin H)^2 + (v \cos H)^2} \frac{d}{dt}\{\frac{v \cos H}{\Omega R \cos \phi + v \sin H}\} - \gamma \sec^2\phi\, \dot{\phi}$$

but

$$\frac{d}{dt}\{\frac{v \cos H}{\Omega R \cos \phi + \sin H}\} = \frac{\dot{v} \cos H - v\dot{H} \sin H}{\Omega R \cos \phi + v \sin H} -$$

$$\frac{v \cos H}{(\Omega R \cos \phi + v \sin H)^2} [\dot{v} \sin H + v\dot{H} \cos H - \Omega R\, \dot{\phi} \sin \phi] .$$

Thus,

$$\dot{\varepsilon} = \frac{(\Omega R\cos\phi+v\sin H)^2}{(\Omega R\cos\phi+v\sin H)^2+(v\cos H)^2}\left\{\frac{(a_{N-S})^{res}}{(\Omega R\cos\phi+v\sin H)} - \right.$$

$$\left. - \frac{v\cos H}{(\Omega R\cos\phi+v\sin H)^2}\left[(a_{E-W})^{res} - \Omega R\dot{\phi}\sin\phi\right]\right\} - \gamma\sec^2\phi\,\dot{\phi}$$

or finally, after some rearrangement of terms,

$$\dot{\varepsilon} = \frac{(a_{N-S})^{res}+\delta(a_{E-W})^{res}+\Omega\delta\sin\phi\, v\cos H}{(\Omega R\cos\phi+v\sin H)+\delta v\cos H} - \gamma\sec^2\phi\,\frac{v\cos H}{R} \qquad \text{(V-18)}$$

and

$$(\dot{\varepsilon}-\dot{\alpha}_a) = \frac{(1-k)\left[(a_{N-S})^{res}-\delta(a_{E-W})^{res}\right] + k(a_{E-W})^{res}\,\gamma\tan\phi}{(\Omega R\cos\phi + v\sin H)} \qquad \text{(V-19)}$$

where;

$$k = \frac{mg\ell\,(\Omega R\cos\phi+v\sin H)}{g\,Cn}$$

$$(a_{N-S})^{res} = \dot{v}\cos H - v\dot{H}\sin H$$

$$(a_{E-W})^{res} = \dot{v}\sin H + v\dot{H}\cos H$$

$$\delta = \frac{v\cos H}{(\Omega R\cos\phi+v\sin H)} \quad .$$

The ballistic deflection error is then obtained from equations (V-17) and (V-19). The expression is

$$d = \int_{t_1}^{t_2} (\dot{\varepsilon}-\dot{\alpha}_a)dt$$

or,

$$d = \frac{(1-k)}{\Omega^* R\cos\phi}\{v_2\cos H_2 - v_1\cos H_1\} +$$

$$+ \left[2\sigma\tan\phi - \frac{\delta(1-k)}{\Omega^* R\cos\phi}\right]\{v_2\sin H_2 - v_1\sin H_1\} \qquad \text{(V-20)}$$

where;

$$2\sigma = \frac{mg\ell\gamma}{Cn} \quad \text{and} \quad \Omega^* R\cos\phi \doteq \Omega R\cos\phi_2 + v_2\sin H_2 \ .$$

APPENDIX VI

MODELLING METHODS OF SHIP'S TRACK

Modelling the track of a manoeuvring ship presents difficulties. There is no unified or general approach. However, there are several ways to treat this problem. They are briefly listed below.

The advance and transfer method. When the rudder is first put over, a resultant force acts on the hull and the rudder of the ship. This is a centripetal force (the result of all the lateral forces acting on the ship), thus causing the ship to follow a curved path. The centre of gravity of the ship will move in a spiral path which is known as the turning circle. In Figure 1 the instantaneous position of the ship during a turn is shown.

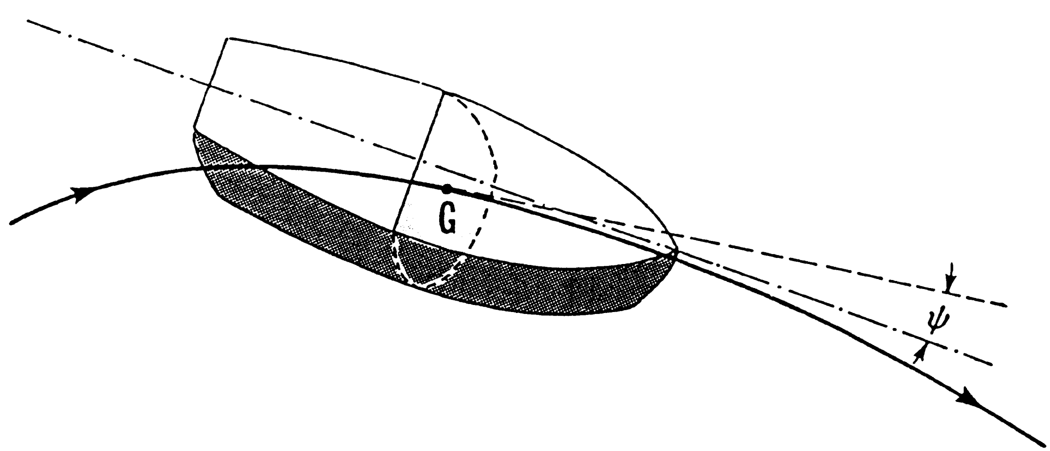


Fig.1

The angle ψ between the fore-aft axis and the tangent is known as the drift angle. The point where the drift angle is zero is called the pivoting point. This point is moving along the fore-aft axis at any given instant. The gyrocompass platform is usually not far from the pivoting point when the ship is moving ahead.

Figure 2 shows a typical turning circle. The advance, A, is the maximum distance the centre of gravity G of the ship travels in the direction of the original course from the time the rudder was put over [*Attwood and Pengelly 1967*]. The transfer,T,(or, sometimes called tactical diameter) is the maximum distance the centre of gravity G moves at right angles to the original course [*Attwood and Pengelly 1967*].

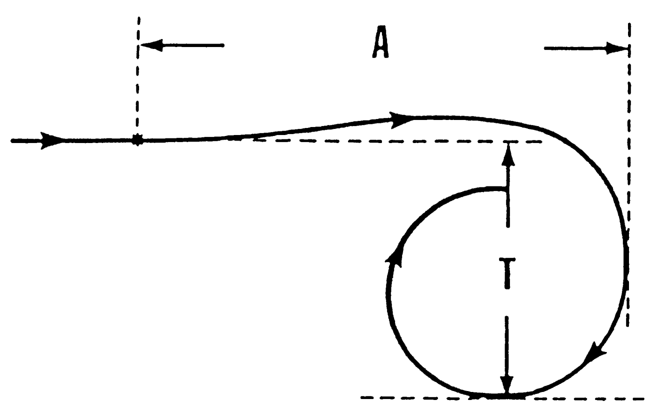


Fig. 2: Turning Circle

The advance and transfer characteristics are not greatly influenced by speed except at high speeds when the transfer increases considerably. They are determined usually by carring out systematic turning trials at sea in order to provide manoeuvring information.

Rose [1974] proposes that the proper way to treat a turn would be to use the advance and transfer characteristics of the ship.

The circular arc approximation. The turn of a ship can be approximated by a circular arc. This is reasonable because in calm and moderate seas the ship's turn is not strongly affected by the waves. For a circular arc, the radius of turn ,R, in nautical miles is given by [*Rose 1974*]:

$$R = \frac{\bar{V}\,\Delta tm}{\Delta\theta}\,\frac{6}{2\pi} \qquad \text{(VI-1)}$$

where; $\bar{V}$ is the mean speed during the turn,

Δtm is the duration of the turn in minutes, and

$\Delta\theta$ is the angle of turn in degrees.

Mathematical Modelling of Ship Manoeuvring. In Gill [1979] a general and a special mathematical model describing the performance of a manoeuvring ship are discussed. The equations of motion are developed and the response is computed. Most of the external forces acting on the ship are accounted for. This mathematical modelling may help in providing useful information for modelling the ship's track during manoeuvres.

Random Manoeuvring. In some instances, a ship may manoeuvre in a "random" manner, Figure 3. Consider the following ship manoeuvre models:

a. The ship maintains a constant speed v_o and changes heading at times t_1, t_2, t_3, ... Between these times, changes in the heading are held constant. The discrete values of heading at times t_1, t_2, t_3, ... namely, H_1, H_2, H_3, ..., are known.

b. The ship changes speed and heading simultaneously at times t_1, t_2, t_3, ..., both of which remain constant between the time changes. Their discrete values are known at any instant.

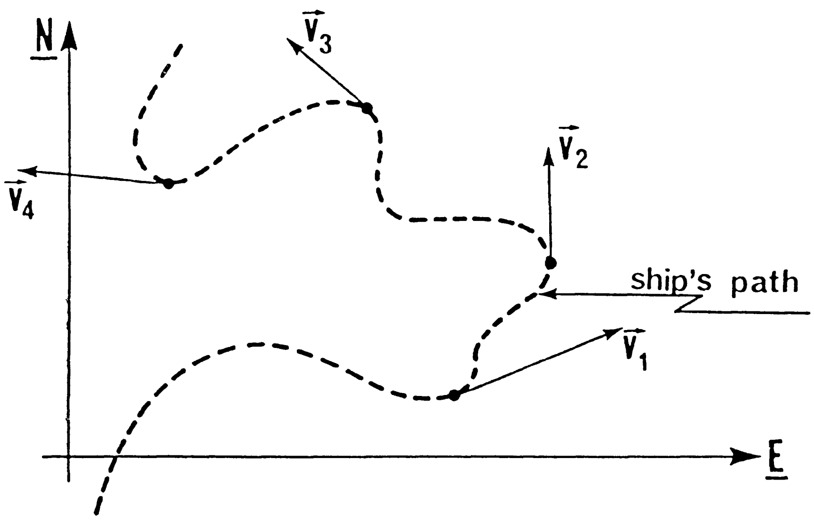


Fig.3

c. In the absence of exact knowledge of the ship's path, both speed and heading can be treated as random processes with associated probability density functions. They can be correlated or not in time and space.

APPENDIX VII

RECOMMENDATION ON

PERFORMANCE STANDARDS FOR GYROCOMPASSES

This is an extract from the *Inter-Governmental Maritime Consultative Organisation*,(I.M.C.O.), publication under the title "*Operational Performance Standards for Shipborne Navigational Equipment*".

The Foreword of the above publication states:

> "Following the adoption by the IMCO Assembly of amendments to Chapter V of the International Convention for the Safety of Life at Sea, 1960, related to the mandatory carriage of radar, radio-direction finder, gyro-compass and echo-sounding devices, and to the use of automatic pilots, the Maritime Safety Committee of the Organisation decided that international performance standards for shipborne navigational equipment should be established."...
>
> ..."Subsequently, the Sub-Committee on Safety of Navigation prepared the following Recommendations which were approved by the Maritime Safety Committee and adopted by the Assembly:"...
>
> ..."(e) Performance standards for gyrocompasses (Resolution A.280 (viii));"...

CHAPTER V - GYRO COMPASSES

RECOMMENDATION ON PERFORMANCE STANDARDS FOR GYRO-COMPASSES

1. Introduction

1.1 The gyro-compass required by Regulation 12 of Chapter V, as amended, should determine the direction of the ship's head in relation to geographic (true) north.

1.2 In addition to the general requirements contained in Chapter I of this publication, the gyro-compass should comply with the following minimum performance requirements:

2. Definitions

For the purpose of this Recommendation, the following definitions apply:

2.1 The term "gyro-compass" comprises the complete equipment and includes all essential elements of the complete design.

2.2 The "true heading" is the horizontal angle between the vertical plane passing through the true meridian and the vertical plane passing through the ship's fore and aft datum line. It is measured from True North (000°) clockwise through 360°.

2.3 The compass is said to be "settled" if any three readings taken at intervals of 30 minutes (when the compass is on a stationary base) are within a band of 0.7 degrees.

2.4 The "settle point heading" is the average value of three readings taken at 30 minute intervals after the compass has settled.

2.5 The "settle point error" is the difference between settle point heading and true heading.

2.6 The errors to which the gyro-compass is subject are considered to have a probability of 68.3 per cent, where the errors are taken as differences between the observed values and their mean value.

The "maximum error" is understood as triple the above error and has a probability of 99.7 per cent.

3. Method of presentation

The compass card should be graduated in equal intervals of one degree or a fraction thereof. A numerical indication should be provided at least at every ten degrees, starting from 000° clockwise through 360°.

4. Accuracy

4.1 *Setting time of equipment*

The compass should settle within six hours of switching on in latitudes of up to 70°.

4.2 *Performance under operational conditions*

(a) The maximum value of one settle point error of the master compass should not exceed $\pm$ 2° in the general conditions mentioned in paragraphs 3.1 and 4 of Chapter 1 and including variations in magnetic field likely to be experienced in the ship in which it is installed.

(b) The maximum error of the master compass in latitudes up to 70° should not exceed:

(i) $\pm$ 1° when the ship is travelling on a straight course at a constant speed in conditions of calm sea;

(ii) $\pm$ 2.5° due to a rapid alteration of course of 180° at speeds up to 20 knots;

(iii) $\pm$ 2° due to a fast alteration of speed of 20 knots;

(iv) $\pm$ 3° when rolling and pitching with any period between 3 and 15 seconds, a maximum angle of 22.5° and a maximum horizontal acceleration of 3 m/s^2.

(c) The maximum divergence in reading between the master compass and repeaters should not exceed $\pm$ 0.3° under the conditions mentioned in sub-paragraph 4.2(a).

Note: When the compass is used for purposes other than steering and bearing, a higher accuracy might be necessary.

To ensure that the maximum error referred to in sub-paragraph 4.2 (b) (iv) is not exceeded in practice, it will be necessary to pay particular attention to the siting of the master compass.

5. Construction and installation

5.1 The master compass and any repeaters used for taking visual bearings should be installed in a ship with their fore and aft datum lines parallel to the ship's fore and aft datum line to within ± 0.5°. The lubber line should be in the same vertical plane as the centre of the card of the compass and should be aligned accurately in the fore and aft direction.

5.2 Means should be provided for correcting the errors induced by speed and latitude.

5.3 An automatic alarm should be provided to indicate a major fault in the compass system.

5.4 The system should be designed to enable heading information to be provided to other navigational aids such as radar, radio direction-finder and automatic pilot.

VITA

Candidate's full name:	Nikolaos Theodoros Christou
Place and date of birth:	Athens, Greece May 6, 1956
Permanent address:	17 Olympou Street Kifissia, Athens, GREECE
Schools attended:	Kifissia Public High School 1968-1974
University attended:	National Technical University of Athens, 1974-1979 Faculty of Rural and Surveying Engineering
Publications:	"High Latitude Behaviour of The Gyrocompass: Its Performance and Its Deviations." Proceedings of the National Marine Meeting, The Institute of Navigation, Cambridge, Massachusetts, October 27-29, 1982.