AEROTRIANGULATION

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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

FORWARD

These lecture notes have been written to support the lectures in aerotriangulation at UNB. They should be regarded just as such and can by no means be considered complete. For most approaches examples have been selected while many other formulations are mentioned or even neglected. This selection was based partly on the availability of programmes to our students and is not intended as classification.

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W. Faig
Summer 1976

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I. Introduction

1.1 Purpose and Definition of Aerotriangulation

There are basically two definitions, a classical one and a more modern one.

Classical: Aerotriangulation means the determination of ground control coordinates by photogrammetric means thereby reducing the terrestrial survey work for photo-control.

With the advent of electronic computers and sophisticated numerical procedures this definition is somewhat enlarged and does not only apply for providing control for photogrammetric mapping.

Modern: Photogrammetric methods of determining the coordinates of points covering larger areas.

This means large point nets.

1.2 Applications of Aerotriangulation

1) Provide control for photogrammetric purposes for both small scale and large scale maps.

   a) small scale mapping:

      whole countries scales 1:50,000 or so

      required accuracy 1 ÷ 5 m

   b) large scale mapping 1:1,000 ÷ 1:10,000

      required accuracy 0.1 ÷ 1 m

2) Point densification for geodetic, surveying and cadastral purposes (3rd & 4th order nets)

3) Satellite photogrammetry; determination of satellite orbits; world net (H.H. Schmidt)

1.3 <u>Some Remarks Concerning the History of Aerotriangulation</u>.

Approximately in 1905 the idea of radial triangulation appeared.

The first numerical attempts have to be credited to Sebastian
Finsterwalder at around 1910. After World War I spacial aerotriangulation
was introduced (Nistri had a patent in 1919). Up until 1935 analogue
experiments continue (Multiplex!) From then on the idea of base-
change in plotters was utilized for larger projects, mainly in Germany
(Zeiss) & Holland.

The real break through for aerotriangulation came with the
computer (1950's). Some of the key-names associated with aerotriangulation
are: Gotthardt, Schut, Schmidt, Brown, Ackermann.

Although I shall present to you the graphical mechanical approach
spanning graphical radial triangulation to Jerie's Analogue Computer
very briefly, it should be stated now, that these methods are mainly
historical, although still used.

1.4 <u>Overview of Methods of Aerotriangulation</u>

It is obvious, that every step means loss of accuracy. Therefore
the bundle adjustment with simultaneous solution is theoretically the
approach which gives the highest accuracy. However, the block adjustment
with independent models follows closely.

FIGURE 1.4 Overview of Aerotriangulation
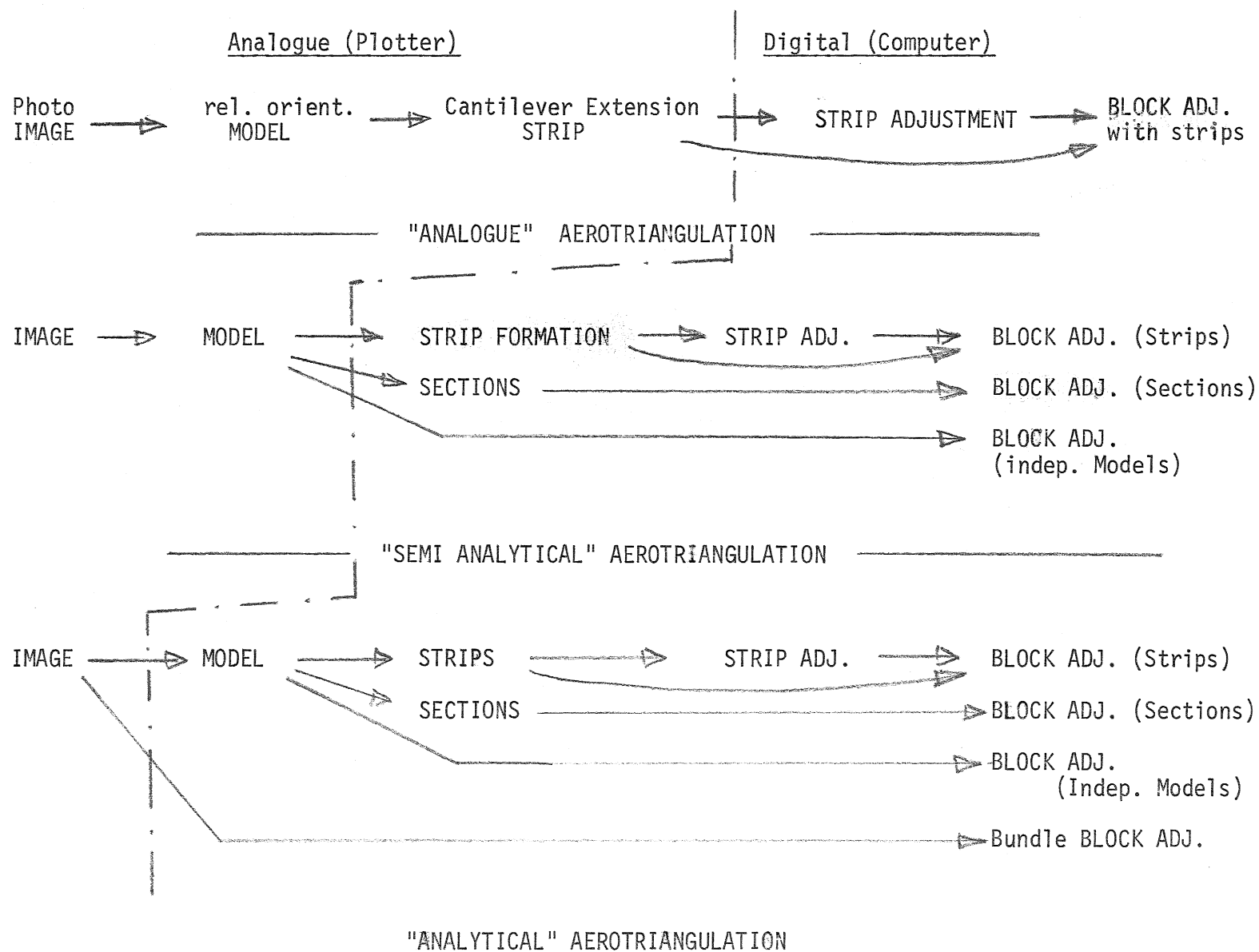
1.5 Some Reviewing Remarks on Basic Analytical Photogrammetry.

Analytical Photogrammetry means numerical evaluation of the content of photographic images. Being a point by point approach, it does not replace analogue photogrammetry but rather complements it. Although aerotriangulation is its main application, it is not the only one.

Unlike in analogue work, where it is possible to reproduce the physical situation (Porro-Koppe-Principle),the physical situation would have to be modelled mathematically. The quality of any numerical evaluation depends on how well the mathematical model describes the physical situation.

In the case of photography, the mathematical model is the concept of central projection which has the following characteristics:

projection centre is a point

light propagates according to geometric optics (straight rays)

the image plane is perpendicular to the axis of the system

We all know that these assumptions are not fulfilled.

- no matter how small the projection centre is (e.g. diaphragm closed), it consists of an infinite number of points.

- light propagates dually in electromagnetic waves and photons, therefore changes in the space to be penetrated (e.g. density changes) result in directional changes. (e.g. light passing through glass (lenses) changes direction).

- the image plane is neither plane nor perpendicular to the system axis.

.˙. There goes our mathematical model. A better model would mean that each camera has to be modelled separately, which would result in a special evaluation method for each type of photography - which is ridiculous.

Therefore we keep the model as it is and adjust our photography such that it fits the model. This process is called "IMAGE REFINEMENT". Togethere with the basic data for central projection (principal point in terms of image coordinates & camera constant) the image refinement parameters are included with the data of interior orientation.

These data are:
- camera constant (linear dependency with radial lens distortion)
- a newly defined principal point
- radial lens distortion (symmetric)
- decentering lens distortion, usually broken down into a symmetric radial and tangential components
- film distortion & image plane deformation
- refraction

Contrary to other beliefs, the earth curvature correction is not part of image refinement, in fact it is non existent if geocentric coordinates are used.

The first four data of interior orientation are obtained by camera calibration (goniometer measurements or multicollimators etc.) in the laboratory.

Film distortion can best be detected with the aid of a reseau grid, and refraction is a function of flying height, temperature, pressure and

humidity at the time of photography. The phenomenon of refraction has been thoroughly investigated in connection with geodetic astronomy and is presented there.

If, for one reason or another some information for the image refinement is not available, nothing can be done, but one has to be aware that less accurate results are to be expected.

After the photogrammetric data are prepared to suit our mathematical model, computations can proceed. Considering the light rays in vector form (p_i and πp_i) we come to the most fundamental and most important equation, the COLLINEARITY EQUATION. With it any photographic situation can be described and it is the base of single photo orientation (often called space resection) which may or may not include the parameters of interior orientation as unknowns.

If we want to combine two photographic images to a stereomodel, the concept of relative orientation is needed, which mathematically is described by a very important condition, the COPLANARITY CONDITION.

This condition just states, that the two sets of collinearity equations describing image rays of the same point and different photographs span one plane, with other words, the rays _intersect_ (space intersection).

The formation of model coordinates is then just the application of the coplanarity condition to all points in question.

This leads to a model which is similar to the actual situation but located somewhere in space. The absolute orientation has the purpose to bring it down to earth, which means application of

- a scale factor

- a space shift (usually in terms of the three components of the three axes of our 3D-space)

- a space rotation (also in terms of three rotations around the same coordinate axes)

For aerotriangulation we often have relatively oriented models in space usually transformed into one common system, e.g. the system of 1st model (u', v', w' instead of x', y', f).

To form a strip, a scale transfer is performed by comparison of one or more common distances. Having a common projection centre, it is theoretically sufficient to compare one elevation only. After the strip is formed, it can be absolutely oriented as a unit.

Of course, it is not necessary to mathematically follow the analogue steps of forming models etc. In this case the coplanarity condition is not explicitly needed, however the intersection of rays is still to be maintained (optimization). Modern solutions using the photographic image as a unit perform a total transformation to the ground. (simultaneous adj., bundle approach).

This again indicates the need for ground control, which can be in different coordinate systems. Finally, the need and concept of interior orientation remains and cannot be separated from exterior orientation unless in the laboratory.

2. Graphical and Mechanical Methods of Aerotriangulation

2.1 Basic Idea of Radial Triangulation

Consider vertical photographs, then the principal point of the photograph is also the image of the nadir point. A set of angles from this point would exactly correspond to actually measured angles on the ground. Using several such sets from continuous photographs a triangulation chain can be established with the principal points as radial centres and freely chosen points in the overlap area of 3 pictures.
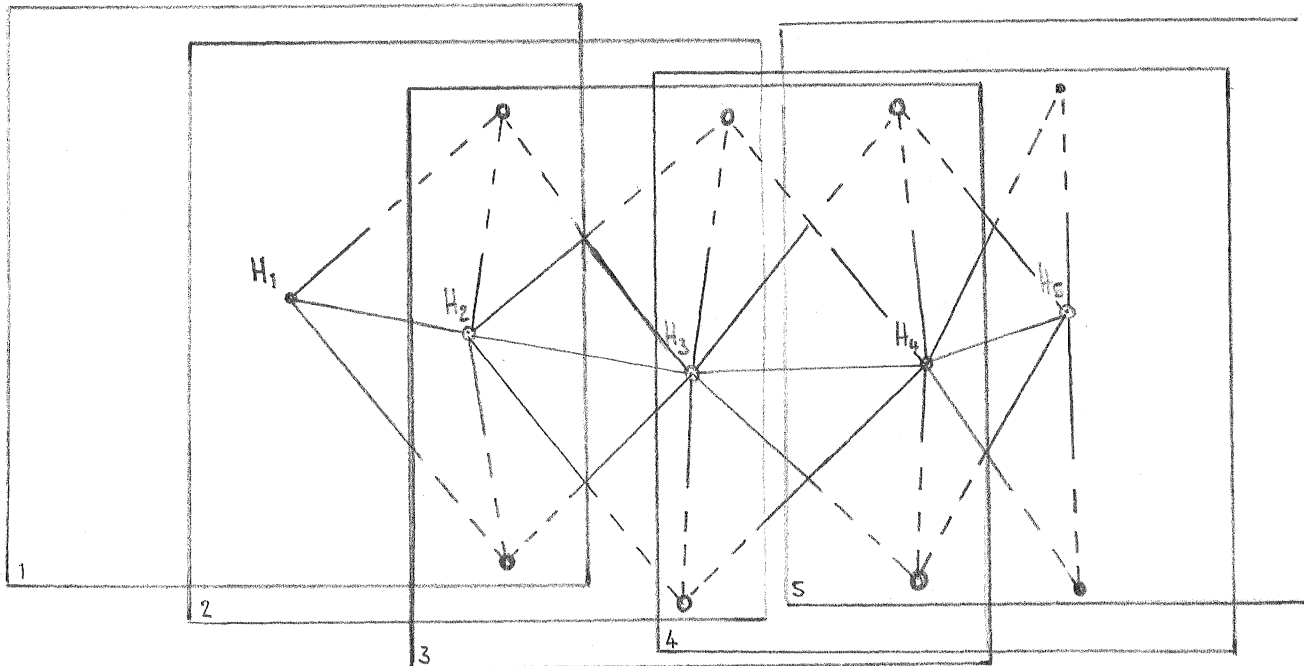
Practically this is done by transforming the principal point of the neighbouring pictures onto the photograph. It is important to be quite careful, because the exact identification of the points influences directly the accuracy of the result. Then the sets are usually drawn onto transparent paper for each photograph and placed such, that the directions between principal points coincide.
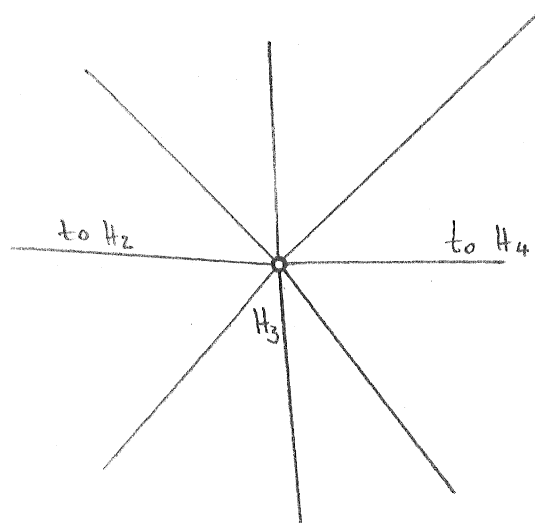
At the beginning the distance between the first two centres can be chosen randomly. However, from then on it is fixed, since always the rays of 3 pictures intersect in one point.

If for the whole strip 2 control points are known, the scale of the lay-out and the direction can absolutely be determined, and with it we have all the corner points as points in the terrestrial system. If we have several adjacent strips with side lap, the solution becomes quite good, since a fair amount of overdetermination occurs. Besides the principal point, also the nadir point or the isocentre can be used as radial centres. Usually we do not have exact vertical photography. The selection of one of the other points is based on the height differences in the terrain and on the tilt of the photography. Usually approximations for these points are obtained from the photographic picture of a level bubble at the time of exposure. A more exact determination (e.g. tilt analysis) is not economical for this purpose.

2.2 Methods of Radial Triangulation

2.2.1 Graphical Method

The directions are directly plotted from the aerial photographs onto transparent paper. Then the system is laid out onto a base map which includes the coordinate axes and the control points. The method is quite cumbersome for large nets, especially due to difficulties in adjusting the error indicating figures. The accuracy is not very high.

2.2.2 Numerical Methods of Radial Triangulation

Depending on whether the principal point (or the fiducial centre), the nadir point or the isocentre are used, several methods can be distinguished. The adjustment is based on the side condition, as used in terrestrial triangulation.

Although the directions can be measured to $\pm 1^c$ using a radial triangulator , e.g. Wild RT1 which utilizes angular measurements with stereo viewing, the directions have remaining errors of several ^{c} due to tilt of the photography. It is left to you to determine the accuracy.

2.2.3 Slotted Templets

This is a purely mechanical adjustment. Using heavy cardboard or similar material, the radial centres are punched as circular holes, the directions as slots. The directions to existing control are also cut as slots. On a base plan, onto which the control points were plotted, the templets are laid out and connected with studs. The control points stay fixed (pricked or nailed down) while the other studs can move such as to minimize the stress in the lay-out. Their fixed position is marked through the centre of the stud (pricked).
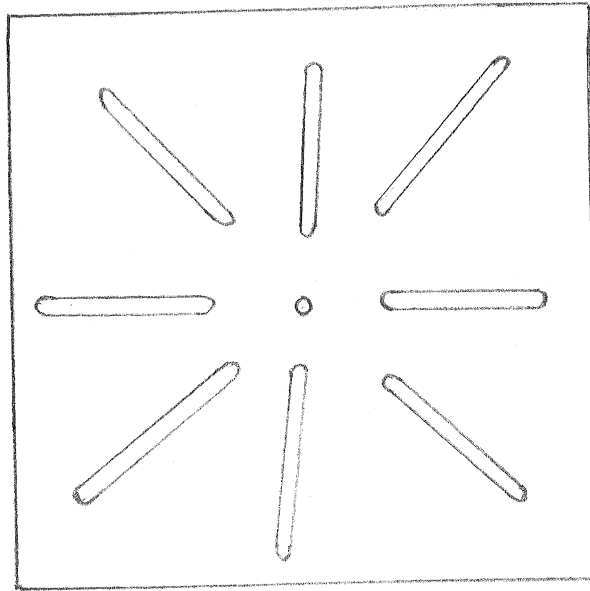
Since this is the mechanical equivalent to a least squares adjustment, it shows that strips have a rather weak stability (towards the side) in the centre unless they are controlled in this area. By using parallel strips, the centre becomes strengthened, and less control is necessary to support the block.

The instrument used for centering is called Radialsecator (e.g. RSI - Zeiss which allows scales between 1:2 and 2:1 and tilt correction up to 30%). The slots are 50 mm long and 4 mm wide.

With good material an accuracy of 1% of the horizontal distances can be obtained, which is sufficient for rectification. The advantage of this method is its simplicity. Limitations are the border of the working space.

2.2.4 Stereo Templets

A further step leads from slotted templets to stereo templets. A stereo templet consists of two slotted templets of the _same_ stereo model, which includes the same selected four corner points. However opposite corners are chosen for centre points.
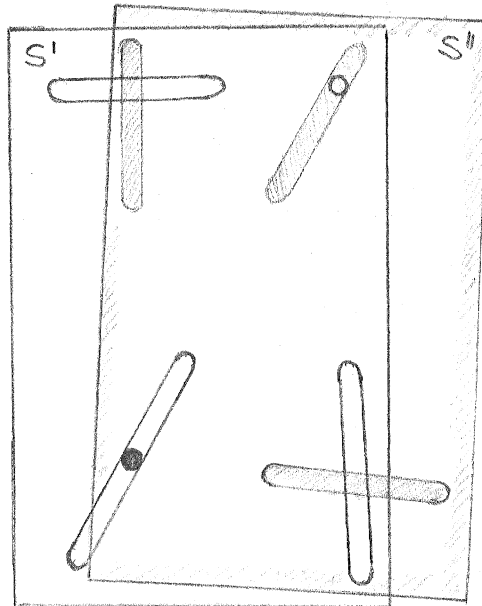
The stereo model has to be relatively oriented (otherwise it would not be a stereo model) and approx. absolutely oriented. The scale is still free for small changes. The layout of stereo templets follows the one of slotted templets.

Stereo templets can also cover 2-3 models if they are analogue and obtained together, e.g. at multiplex.

2.2.5 Jerie's ITC-Analogue Computer

The mechanical analogue block adjustment, as developed at the ITC in Delft is a further development of the stereo templet method. In this case, a photogrammetric block is divided into near square sections. For each section double templets are cut, which represent the panimetry of four tie points with the other sections and possibly additional points within the section.

The templets are not directly connected but by the use of multiplets, which are 4 springloaded buttons on a carrier which can be shifted in direction of the coordinate axes. Therefore, the carrier can always move to such a position where the resultant vector of all forces is equal to zero. After fixing a double templet onto the buttons, the carrier will move to the adjusted point position. At the same time the double templets will shift such that they fit best to the conditions (due to resultant forces). In the final phase all sections of the block will be at a position which represents the results of a rigorous adjustment.

In order to determine the transformation elements required for the connection of the sections, Jerie uses the residuals to the preliminary orientation with high magnification. Therefore all transformation parameters are obtained with magnification and can be graphically obtained. Then a new numerical orientation is performed. By repeating the procedure with again a magnification the accuracy can be increased without limits.

Due to uncertaincies in the longitudinal tilt, the same principle cannot be applied to vertical adjustment. Therefore a three dimensional arrangement has to be used in order to simulate elastic deformations of a body in space. At the IGN (Institute Geographique National) in Paris such a system utilizing plastic supports has been developed.

3. Methods of Strip Triangulation and Instrumentation

3.1 The Principle of Continuous Cantilever Extension with Scale Transfer

The idea is to reconstruct the exposure situation.
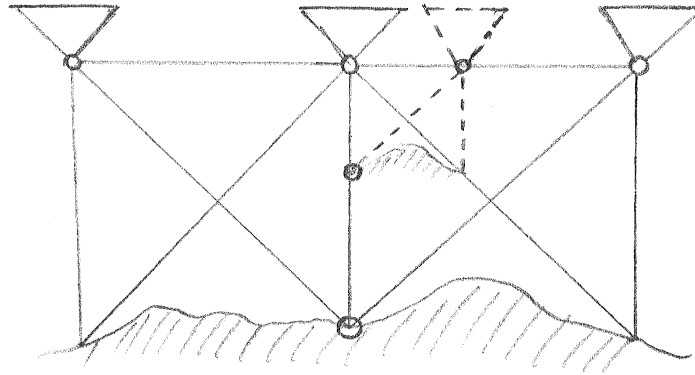
1st model: rel. & abs. orientation

2nd model: dependent pair relative orientation

using κ_{III}, by_{III}

ϕ_{III}, bz_{III}

ω_{III}

The problem is b_x. This is initially randomly chosen (... in figure), then the point or points common to models 1 and 2 are set to have the same elevation by changing b_x.

Therefore : Strip triangulation is nothing else but a continuous application of cantilever extension (dependent pair relative orientation) and scale transfer (base components).

If there were no error propagation, this would be just perfect (e.g. Multiplex).

Scale transfer is basically the comparison of a distance in both models with changing of the base length until the distance is the same in both models.

Any two points in the common overlap area of the two models
can be chosen. By selecting one rather unique common point (the
projection centre) only one other point is necessary because the z-
values are directly comparable distances.

The base is shifted until the elevations are the same.
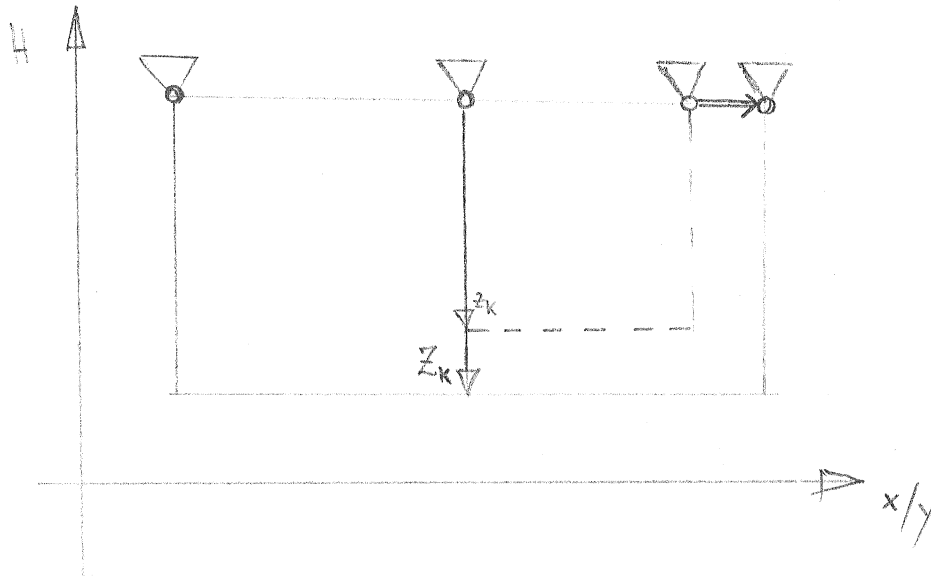
Since the base extends primarily in x-direction, this means basically a bx-
strip. However, the other base components have to be changed in the same
ratio.

Example: Shift 3rd multiplex projector until there is no more x-parallax
on the measuring table, which has its elevation from the previous model.
This parallax can be either objective or subjective:
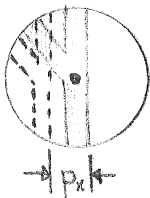
Objective: Subjective:
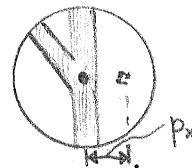
The eye recognizes the object

in stereo, but sees two floating

marks.

If you use elevations for scale transfer, make sure that the counter remains connected and the same!

If a plane-distance is used, well defined points have to be utilized.

3.2 Triangulation Instruments

1934 Multiplex was the first triangulation instrument. It is however, a low order instrument, as you all know from previous experience. "1st order" plotters for aerotriangulation use in effect the same principle but with only two projectors. This means base change and image change.

left   right
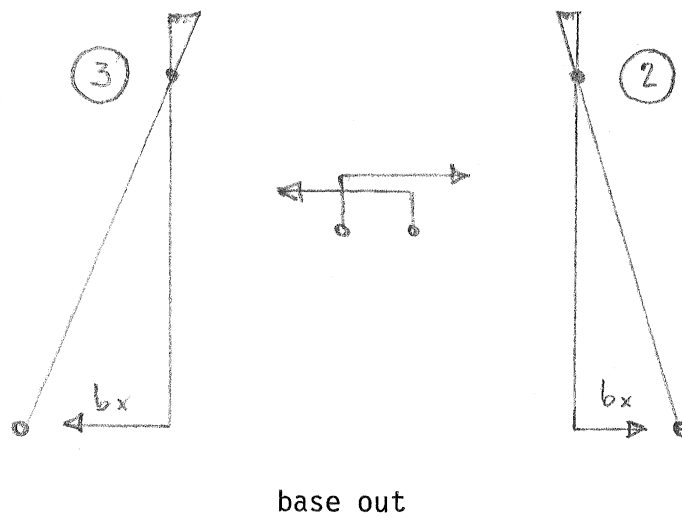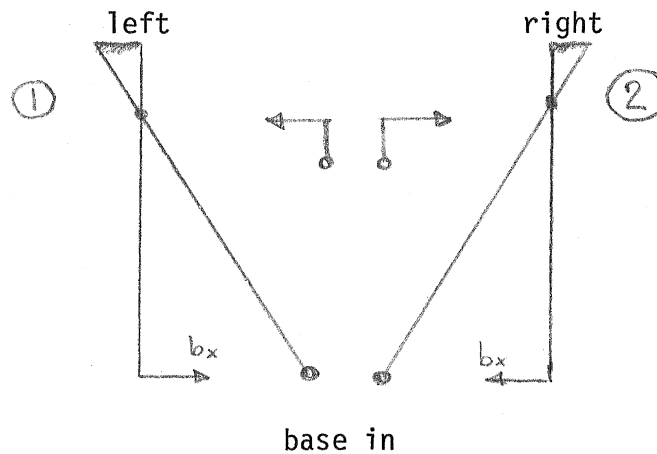
base in

base out

change of viewing

Then

        ③   ④

base in (normal viewing)

        ⑤   ④

base out (changed viewing)

Since z is fixed, the same reference height is maintained for scale transfer. The problem lies in x/y-coordinates.

to x: Measure the point in 1st model and record value, then disconnect x-counter, orient the 2nd model with elements of photo 3, perform scale transfer. Then set measuring mark onto point and connect x-counter.

to y: There are two opinions:

a) do not disconnect y-counter

b) disconnect the y-counter in same manner as the x-counter.

Both ways are correct, if the base component b_x is exactly parallel to the x-axis of the plotter. Otherwise secondary errors are introduced if opinion a) is taken. It also can be used as instrumental check.

There are certain problems due to instrumental ranges, a) b_z-range

geoid (ref. for elevations)
Ellipsoid (ref. for planimetry)
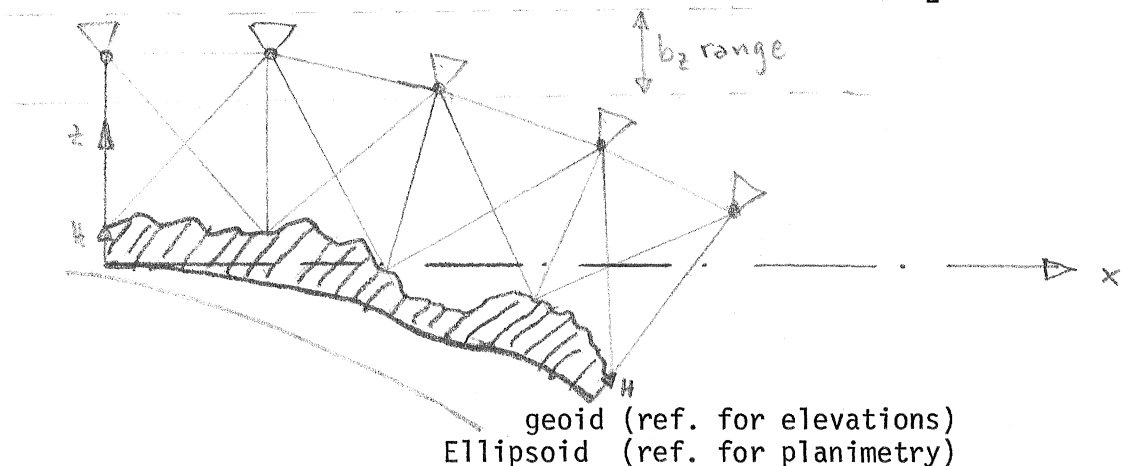
for longer strips:

$$z \neq H$$

therefore, earth curvature is considered an error. It is quite obvious, that for a longer strip the b_z-range is soon insufficient.

A similar problem occurs, if the 1st model was not or incorrectly absolutely oriented, then the b_z range is also quickly insufficient. The b_z limitation is therefore the main reason for an absolute orientation of the first model.

What can be done? ⟶ b_z shifts
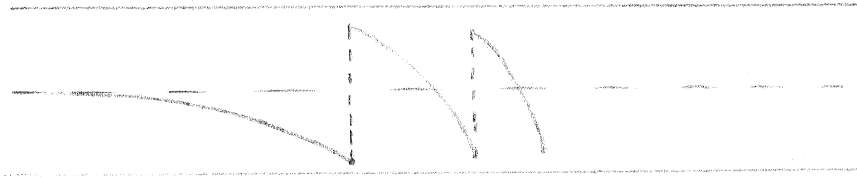
b_z shift means also change in height reading.

change height counter after scale transfer (change b_z and z!)

⟶ Start with a high b_z-value
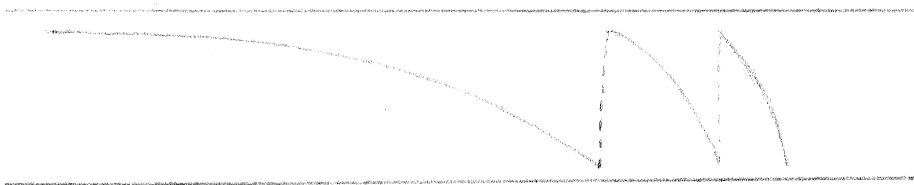
⟶ Start with an initial ϕ-rotation (then dep. pair rel. or.)

This will cause wrong height readings and lead to projection corrections.

Another possibility is to break the strip
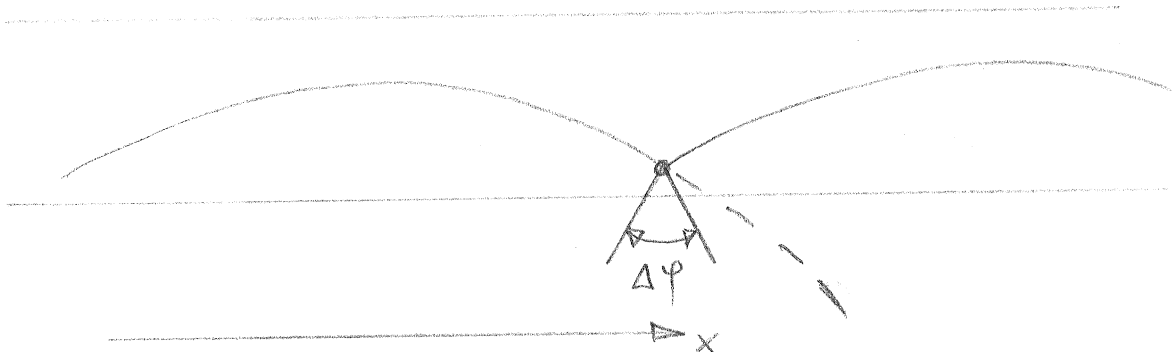
$\Delta\phi$ has to be measured resp. set in the instrument.

e.g. Turn a 2g ϕ-rotation, but the dials are not always that reliable.

Better compute x-shift using ϕ and elevation. Then shift x by

introducing Δx and turn with ϕ screw until points coincide.

The best way is aero-levelling, which means working with a

fixed and constant z.
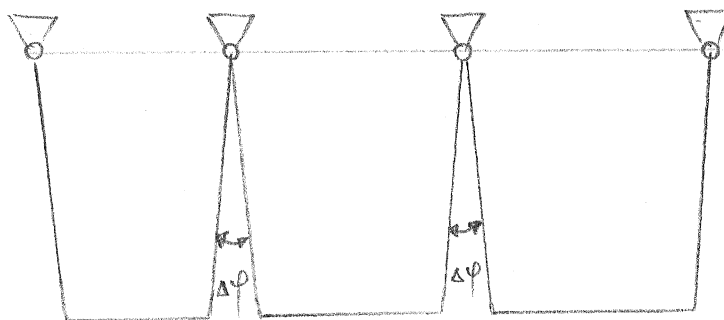
$\Delta\phi$ is the convergence due to earth curvature ($\approx$ 1 c/km base length)

Similar considerations might be necessary for the b_y range

(e.g. initial κ-setting or by-shift!)

3.3 Strip Triangulation with Plotters Without Base Change
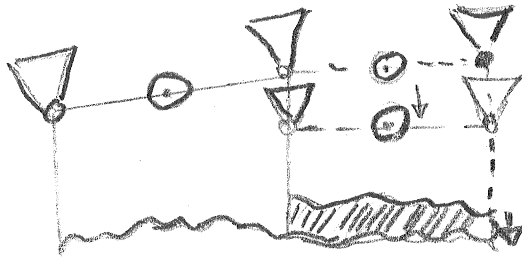
Since the rotational centre is fixed, only a $\parallel$ movement is possible.

Problem: The absolute orientation of the right projector has to be reconstructed in the left projector.

This would be not much of a problem if there were base components.

Just use a cross-level, and in this case level plate with Φ and Ω or $(w_L$ and $W_R)$

Since the rotational centre is a fixed instrumental point, a ΔZ_0-value has to be introduced to compensate the parallel shift.

$$Z_0 = \frac{b_1}{2} \, \Phi_1 - \frac{b_2}{2} \, \Phi_2$$

This is an iteration, since b_2 will only be obtained after scale transfer, when using Z. If the scale transfer utilized a planimetric distance, then ΔZ_0 can be directly computed.

Further to this cumbersome approach: Additional instrumental errors have to be considered (both projectors are not exactly the same).

Since the A-8 does not have base components the instrumental base cannot be rotated but is a straight line along the strip. Therefore if the flight line looks like this:
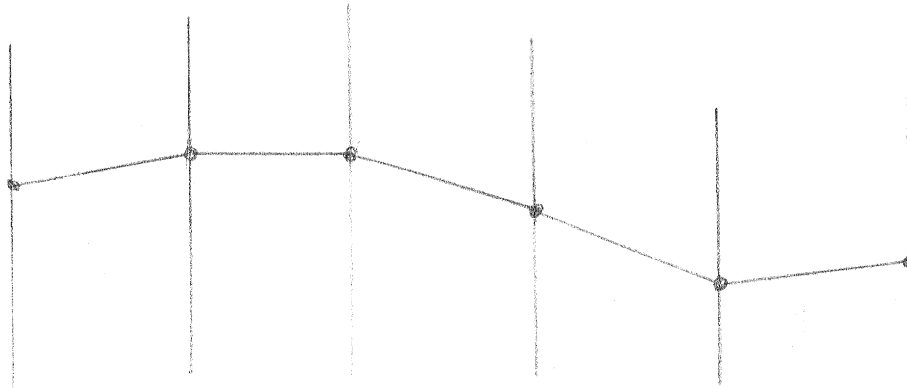
the instrumental strip would not be continuous, but rather like
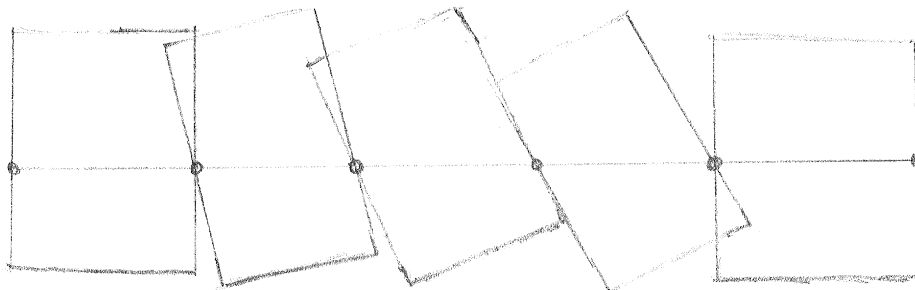
and the machine coordinates have to be transformed into a common

system by a series of Helmert transformations.

Following this, it is better to use <u>independent</u> models

and completely compute the strips, even in elevation.

3.4 Precision Comparators

Jena 1818 Stereo Comparator

Accuracy: $\pm$ 10 cm for x' and y'

$\pm$ 3 cm for px, py

(several observations)

Working sequence:

1) placing and clamping of photo plates

2) fix eye base obtain a parallax free image of the floating marks
    in the measuring plane

3) Make fiducial lines parallel to instrument axes (κ - rotation)

4) Read or set the numerical values for fiducial centre (representing principal pt.)

5) Drive to image point in left photo, using x' and y' motions

6) Obtain stereo coverage by moving right photo with p_x and p_y

7) Read or register x', y', px, py.

Although comparators are very easy in principle and use, they demand a great mechanical effort. The following conditions have to be fulfilled.

1) Straight and easy movement of all carriers

2) Parallelity of axes for left and right photos

3) Perpendicularity of x and y axes

4) Image plane has to be parallel to axes

5) Parallelity of lead screws and measuring spindles to image motions

6) Precision division of scales and of spindles

Comparators are calibrated with the use of grid plates, which have to be more accurate!

Most stereo comparators follow the same construction principle with higher magnification etc. such as

Wild STK-1

OMI comparator

SOM comparator

Hilger and Watts comparator

The Zeiss Oberkochen PSK has newer construction principles and is much more compact

The photoplates (1) are clamped unto precision glass-grid plates (2) which are vertical.

The grid enables the coordination of image points in units of its grid, which is 5 mm.
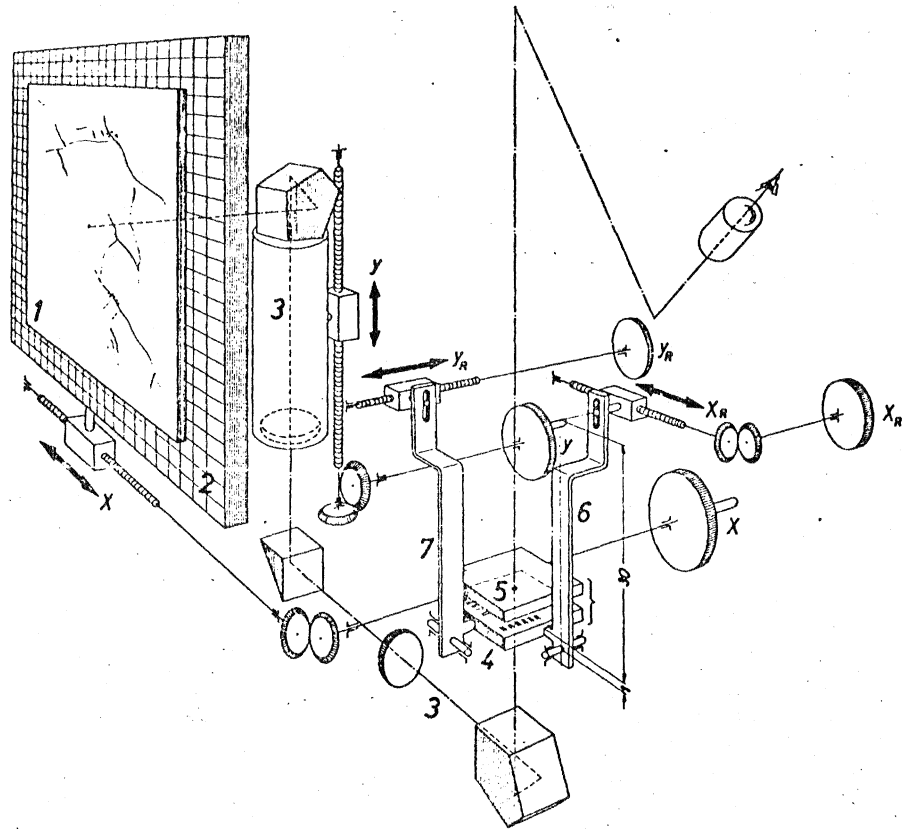
Abb. 10. Schema der linken Hälfte des Präzisions-Stereokomparators von Zeiss *(Werk-zeichnung Carl Zeiss, Oberkochen)*.

(from: Schwidefsky "Photogrammetrie")

Since both measuring diapositive and grid plate are made from glass, temperature changes do not effect the accuracy (illumination!!) The fine measurement is done with the aid of a fine scale (4) onto which both photo and grid is imaged through the optical system (3). In the same optical plane the floating mark (better measuring mark, since the comparator can be used mono and stereo) is positioned.

Two measuring spindles via levers (6)and (7) can move the measuring scale in x and y directions until grid line and measuring scale march coincide. Large gear transmissions permit reading to 1 μm, which is mechanically possible since the maximal way is 5 μm.
8 to 16 x magnification is possible.

There are also a series of mono comparators, most of which again follow the basic carrier principle as discussed.

Just a few words to the

NRC - mono Comparator

(built by Space-Optics and sold by Wild)

The Model 102 Monocomparator accepts either glass plate diapositives or film in up to 9" x 9" format.

Measurement of image points in X and Y coordinates are made with respect to precision scribed measuring marks on a glass plate. The measuring marks are spaced at 20 mm intervals in a 12" x 12" matrix. The photo is positioned with respect to a specific measuring mark location by slewing the grid plate and photo on an air supported carriage. The precision coordinates are measured with respect to the referenced measuring mark by two short precision lead screws with 20 mm of travel. A measurement consists of a macro movement of the grid plate and a micro adjustment of the lead screw. Air supply to the carriage is controlled by a foot pedal. The lead screws are moved by finger touch measuring disks. The X and Y coordinates are continuously displayed and are recorded on any digital storage device such as punched card or paper tape. Identification numbers may be inserted along with the X and Y coordinates.

The design concept of the Model 102 Monocomparator uniquely combines high accuracy with high productivity.

Those factors which contribute to accuracy are:

. Short lead screws (20 m)

.. Precision glass measuring mark plate

. Selected materials with uniform thermal coefficient of expansion and high thermal conductivity

. Adherence to Abbe's principle

The Monocomparator has an overall accuracy of $\pm$ 2.5 micrometres and a repeatability of measurement in the order of $\pm$ micrometre.

A special construction is the DBA- Monocomparator which utilizes the measurement of distances.
(Multilaterative Comparator)
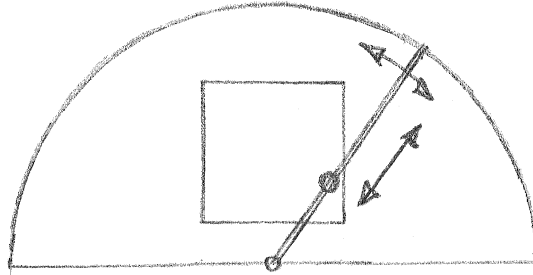
By changing the plate into all 4 possible positions, the point is determined by 4 distances. The overdetermination permits self-calibration (det. of comparator parameters by LS-adjustment). A program for obtaining photo coordinates (and necessary comp. parameters) is supplied by DBA.

Theory of Operation (according to DBA)

The theoretical basis for the comparator is best illustrated with the aid of Fig. 1 which shows the four measurements r_{1j}, r_{2j}, r_{3j}, r_{4j} of a point x_j, y_j. It will be noted that in the actual measuring process, the pivot of the measuring arm remains stationary while the plate is measured in four different positions. This is precisely geometrically equivalent to a process in which the plate itself remains stationary while the pivot assumes four different positions. The actual measurements r_{ij} are from the zero mark of the scale to the point rather than from the pivot to the point. To convert the measurements to radial distances from pivots, one must specify the radial and tangential offsets (a, β) of the pivot relative to the zero mark. If x_i^c, y_i^c denote the coordinates of the pivot corresponding to position i of the plate

(i = 1, 2, 3, 4), one can write the following four observational

equations relating the measured values r_{ij} and the desired coordinates

x_j, y_j:

$$(r_{1j} + a)^2 + \beta^2 = (x_j - x_1^c)^2 + (y_j - y_1^c)^2$$

$$(r_{2j} + a)^2 + \beta^2 = (x_j - x_2^c)^2 + (y_j - y_2^c)^2$$

$$(r_{3j} + a)^2 + \beta^2 = (x_j - x_3^c)^2 + (y_j - y_4^c)^2$$

$$(r_{4j} + a)^2 + \beta^2 = (x_j - x_4^c)^2 + (y_j - y_4^c)^2$$

These equations recognize that since there is in reality only one pivot

and measuring arm, a common a and β apply to all four positions of the

plate. If the ten parameters of the comparator (i.e. a and β plus four

sets of x_i^c, y_i^c were exactly known, we could regard the above system as

involving four equations in the two unknowns x_j, y_j. Accordingly, the

process of coordinate determination in this case owuld reduce to a

straightforward, four station, two-dimensional least squares trilateration

In practice, the parameters of the comparator are not known to

sufficient accuracy to warrant their enforcement. It follows that

they must be determined as part of the overall reduction. This becomes

possible if one resorts to a solution that recovers the parameters of the

comparator while simultaneously executing the trilateration of all

measured points. Inasmuch as one is free to enforce any set of parameters

that is sufficient to define uniquely the coordinate system being employed

three of the eight coordinates of the pivots can be eliminated through

the exercise of this prerogative. In Figure 1 we have elected to define

the y axis as the line passing through pivots 1 and 3 thereby making

$x_1^C = x_3^C = 0$. Similarly, the x axis (and hence also the origin) is established by the particular line perpendicular to the y axis that renders the y coordinates of pivots 2 and 4 of equal magnitude but opposite sign (thus $y_4^C = -y_2^C$). This choice of coordinate system has the merit of placing the origin near the center of the plate.

By virtue of the definition of the coordinate system, only seven independent parameters of the comparator need be recovered. If n distinct points are measured on the plate, the above equations may be considered to constitute a system of 4n equations involving as unknowns the seven parameters of the comparator plus the 2n coordinates of the measured points. When n > 4, there will exist more observational equations than unknowns, and a least squares adjustment leading to a 2n + 7 by 2n + 7 system of normal equations can be performed. Although the size of the normal equations increases linearly with the number n of measured points, their solution presents no difficulties, even on a small computer. This is because they possess a patterned coefficient matrix that can be exploited to collapse the system to one of order 7 x 7, involving only the parameters of the comparator. Once these have been determined, as independent, four station, least squares trilateration can be performed to establish the coordinates of each point. Details of the data reduction are to be found in the reference cited below.*

---

* Reference: D. Brown, "Computational Tradeoffs in the Design of a One Micron Plate Comparator", presented to 1967 Semi-Annual Convention of American Society of Photogrammetry, St. Louis, Mo., Oct. 2-5. Available upon request from DBA systems, Inc.

Suffice it to say here, that the internal contradiction resulting from the redundancy of the measuring process can be exploited to effect an accurate calibration of the parameters of the comparator for the particular plate being measured. Hence, the designation of the instrument as a self-calibrating multilaterative comparator.

The Computer Program

A fortran source program is provided with the comparator to reduce raw measurements to comparator coordinates. The program is designed so that a minimum of modification is required to adapt it to almost any computer. One version is designed specifically to run on a minimal computer configuration such as an IBM 1130 with card input. Another version is designed for medium to large scale computers. Both feature automatic editing and rigorous error propagation. Typical running time on an IBM 360/50 for the reduction of a plate containing 25 measured images is well under 30 seconds.

In addition to producing the final coordinates and standard deviations of the measured points, the program generates the four measuring residuals for each point and the rms closure of trilateration. The residuals provide a truly meaningful indication of total measuring accuracy, and the rms error of the residuals, representing as it does a rms error of closure, provides a particularly suitable criterion for quality control.
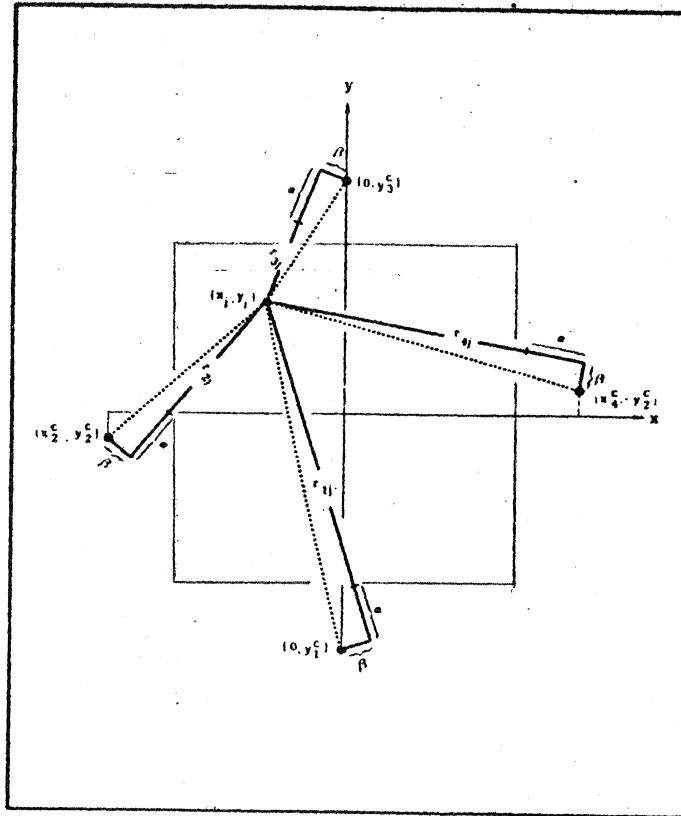
Figure 1: Illustrating Geometrical Equivalent of Multilaterative Comparator.

3.5 Point Selection, Transfer, Marking, Targetting etc.

The adequate preparation of the available photography for aerotriangulation purposes is very important. The prime requirement for aerotriangulation is that the photographs overlap. Usually a 60% overlap is available (often 90% in flight and then every 2nd photo is discarded). The overlap between strips is 20 - 30%, sometimes 60%.

1) Layout

Contact prints of the whole strip (or block) are laid out and examined for overlap areas (thinning of 90% overlap, gaps?) deformations (large ones will show), general information, e.g. scale, topography etc.

2) <u>Selection of triangulation points in the overlap areas</u>

- pass points = common points in three or more consecutive photographs

  within the strip (pass from model to model)

- tie points = points located in common overlap area between strips.

       Usually the tie points are also pass points. This however,

requires a regular strip and block pattern.

       The pass points are selected such that they are located in

the vicinity of the V. Gruber points, used commonly for relative orien-

tation. This means 6 points per model.

side lap area
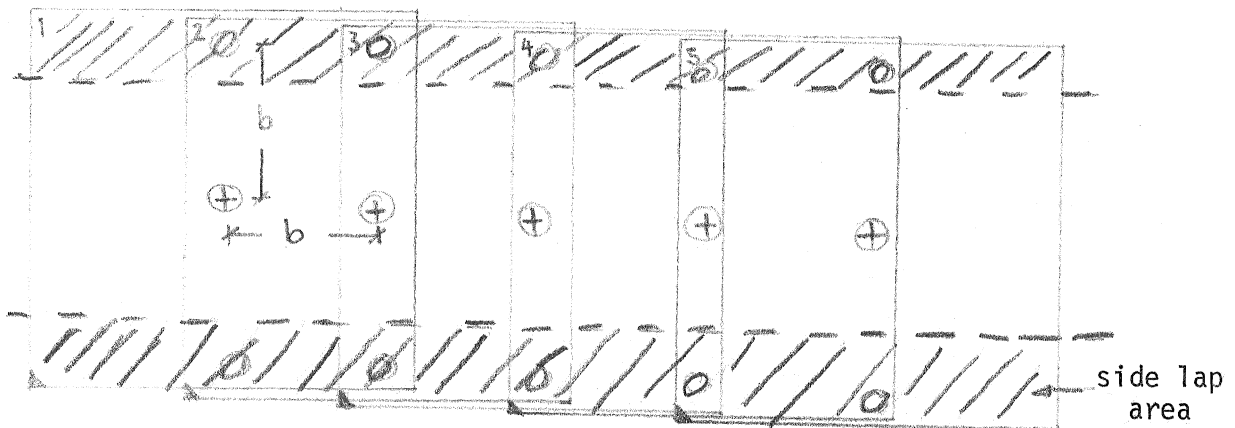
If the strip is measured in stereo (e.g. analogue triang., indep.

models on plotter, stereo comparator), then only three points per

photo have to be marked, whereas for mono comparator measurements,

nine points per photo are required (except for 1st and last in strip,

which req. 6)

Stereo   Mono

The tie points, which are often passpoints, should be selected in the centre of the common overlap area (relief displacement etc!)

Since the stereo models are <u>not</u> created across the strips it is necessary to mark tie points in <u>both</u> strips!

More than 30% side lap strengthens the vertical behaviour of a block, and with 60% the principal points are included as tie points also.

The points selected are usually natural terrain points, because they provide the highest accuracy. However, the following has to be kept in mind:

- <u>do not</u> choose a point on top of houses, trees etc., if not absolutely necessary. It is easier to:

- measure ground points, such as

  road and/or railroad intersections or junctions, junctions of ditches or other characteristic line features, Detail points, e.g. rocks, small bushes, high contrasts edges or corners of shadows (again high light/dark contrast in photography → be aware of time lapse between strips!)

3) <u>Marking of points</u>

a) Manual

- Circle the area on the contact print with easily recognizable marker

- Provide a sketch of the details of the area surrounding the selected point (often on back of so-called "control" print)

- describe point, if necessary (e.g. "top step")

- mark point number beside point
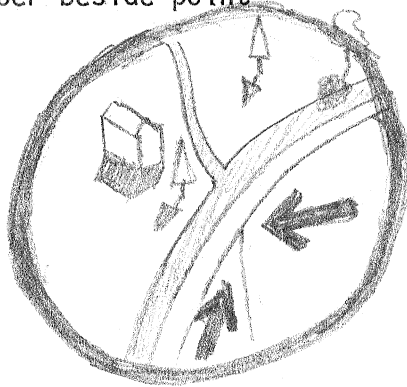
4321

Junction of Ditches
South of Road
Near old Barn

It is important for the operator, that all pass and tie points are marked as well as existing ground control. The latter is often specially coded according to its nature (e.g. hor. or vertical or both).

b) Mechanical

There are several instruments of point marking, e.g.

> Wild PUG
>
> Zeiss (Oberkochen) Snap Marker
>
> Kern PMG1
>
> Zeiss (Jena) TRANSMARK

They range from quite simple and punching a hole into the emulsion to quite sophisticated and burning a hole using laser light.

Stereo markers such as the PUG have to be used in order to correctly identify points for mono-comparator work. In this case the hole is drilled.

c) Targetting (Pre-signalization)

For highest accuracy, all photogrammetric triangulation points should be targetted on the ground. This eliminates the error contributed by point transfer which can be several μm. Targetting

is usually done for those points for which coordinates are desired (e.g. control densification by photogrammetric means, cadastral surveys, construction projects, urban mapping) whereas pass- and tie points are usually not pretargetted. Pretargetting requires field work and has to be done only a few days before photography in order to keep the loss of targets down (people interference). The important thing is to create a good light/dark contrast.

The simplest way is using paint (crosses on highways, or direct painting of survey monuments). Card board, plastic or fabric targets are also in use.

Examples for targets
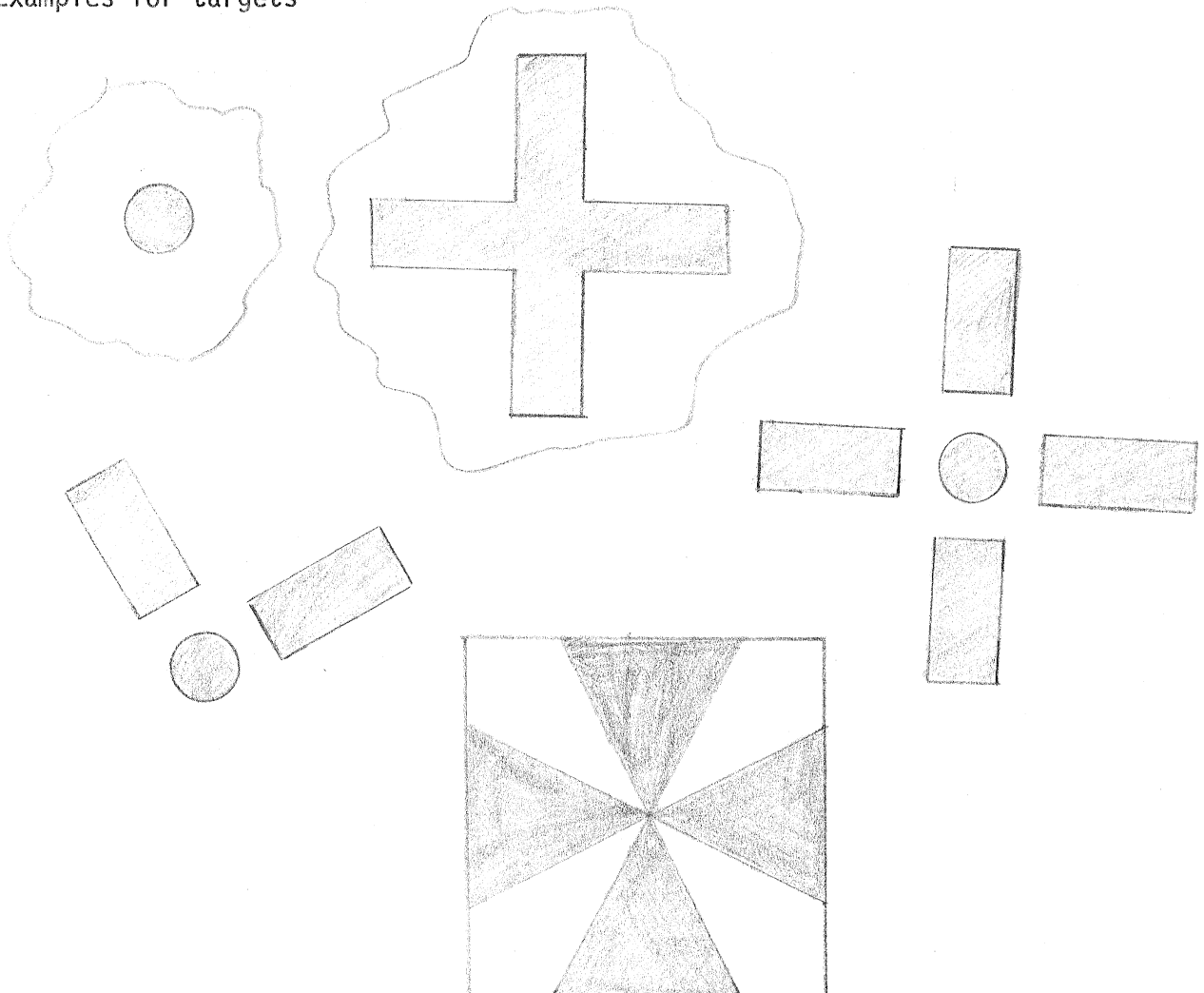
It is quite obvious, that the target size has to match the scale of photography (resolution, identif.) Target should be 50 - 100 µm on photograph with the central part being at least 25 - 30 µm in diameter.

Except when there are not many characteristic surface features, pretargetting is not used for topographic mapping.

Sometimes a lower reconnaissance flight and 35 mm film is used to obtain detail pictures of the surroundings of points. These pictures are then used to identify the points in the small scale compilation photography. (Correlation with Zoon- Transfer Scope [Bausch and Lomb]).

4. Error Theory of Strip Triangulation

4.1 General Assumptions

Error theory is always the base of adjustment. Although each model is deformed due to errors in interior and exterior orientation this simplified theory propagates that all errors are caused by transfer from model to model. The assumption is justified because the actual model deformations for aerial photogrammetry are in the magnitude of 1 - 10% of the transfer errors as empirically determined.

This means basically, that the 7 elements of transfer are associated with errors, namely:

Scale transfer error $\quad\quad\quad\quad\quad \Delta s_i$ $(i = 1, 2 \ldots (n - 1))$

Azimuth - transfer error $\quad\quad \Delta a_i$ $\quad\quad\quad$ because of (n-1)

Longitudinal transfer error $\Delta \phi_i$ $\quad\quad\quad\quad$ connection for

Lateral tilt transfer error $\quad \Delta \omega_i$ $\quad\quad\quad\quad$ n - models

x-shift error $\quad\quad\quad\quad\quad\quad \Delta x_i$

y-shift error $\quad\quad\quad\quad\quad\quad \Delta y_i$

z-shift error $\quad\quad\quad\quad\quad\quad \Delta z_i$

4.2 Vermeir's Simplified Theory of Transfer Errors

Vermeir (ITC) considers only the errors in Δs, $\Delta \phi$, $\Delta \omega$ Δa and neglects the shift errors, since they are small. The strip axis is used! Considering azimuth errors, the deformation of the strip axis is as follows:
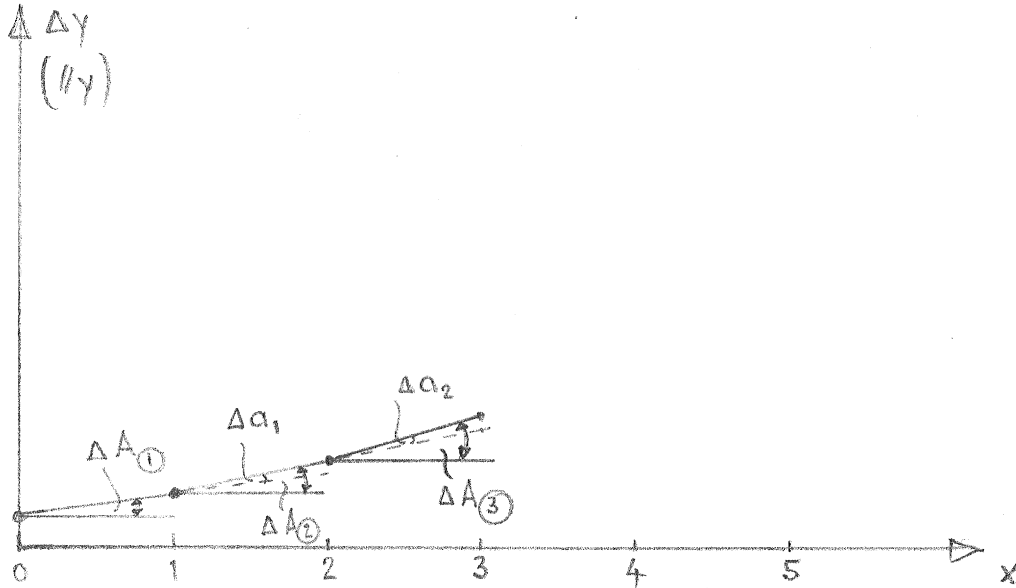
numerically:

$$\Delta A \text{①} = \Delta A \text{①} = \Delta A \text{①}$$

$$\Delta A \text{②} = \Delta A \text{①} + \Delta a_1 = \Delta A \text{①} + \Delta a_1$$

$$\Delta A \text{③} = \Delta A \text{②} + \Delta a_2 = \Delta A \text{①} + \Delta a_1 + \Delta a_2$$

- - - - - - - - - - - - - - - - -

$$\Delta A_i = \Delta A \text{(i-1)} + \Delta a_{i-1} = \Delta A \text{①} + \Delta a_1 + \Delta a_2 + \ldots + \Delta a_{i-1}$$

therefore:

$$\Delta A \text{①} = \Delta A \text{①} + \sum_{\nu=1}^{i=1} \Delta a_\nu$$

Similarly for ϕ- tilt:

this leads to:

$$\Delta\Phi_i = \Delta\Phi_{\textcircled{1}} + \sum_{\nu=1}^{i-1} \Delta\phi_\nu$$
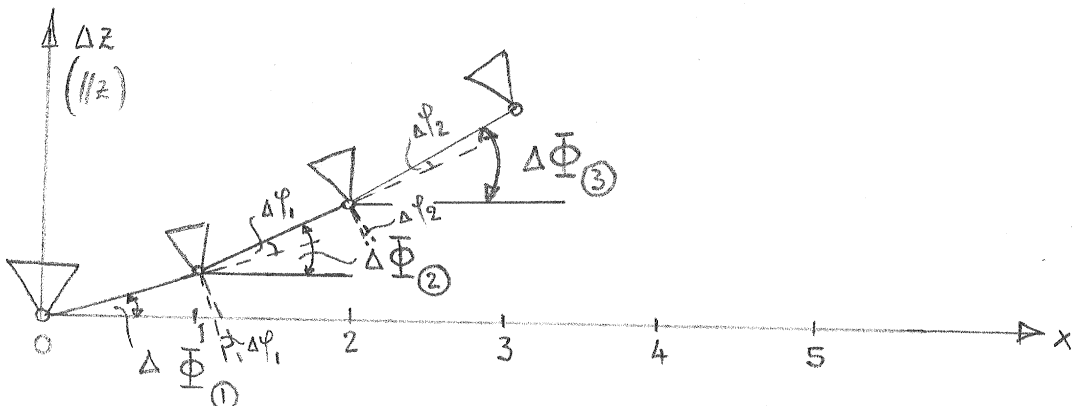
and also for ω - tilt:

$$\Delta\Omega_i = \Delta\Omega_{\textcircled{1}} + \sum_{\nu=1}^{i-1} \Delta\omega_\nu$$
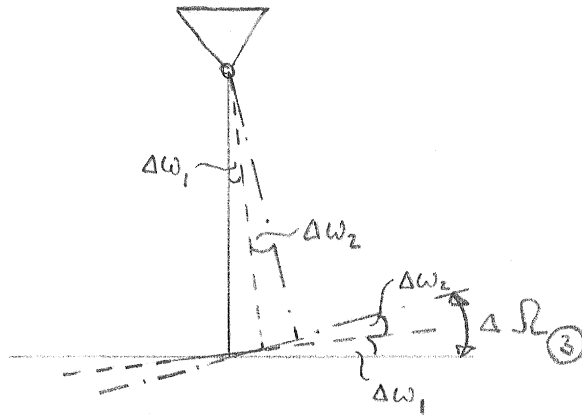
Scale transfer errors:

therefore

$$\Delta S_{(i)} = \Delta S_{(1)} + \sum_{\nu=1}^{i-1} \Delta s_{\nu}$$
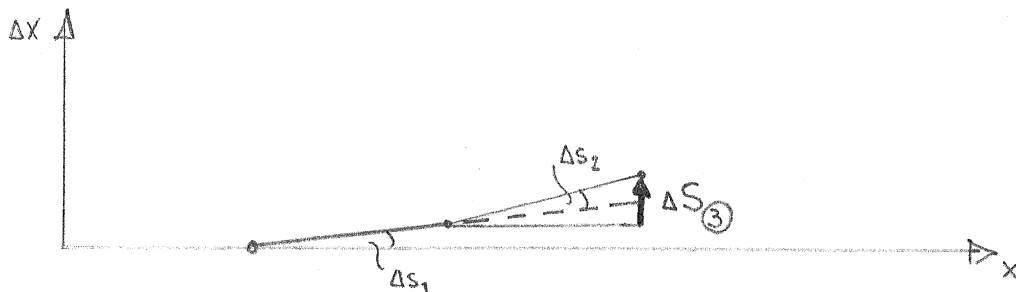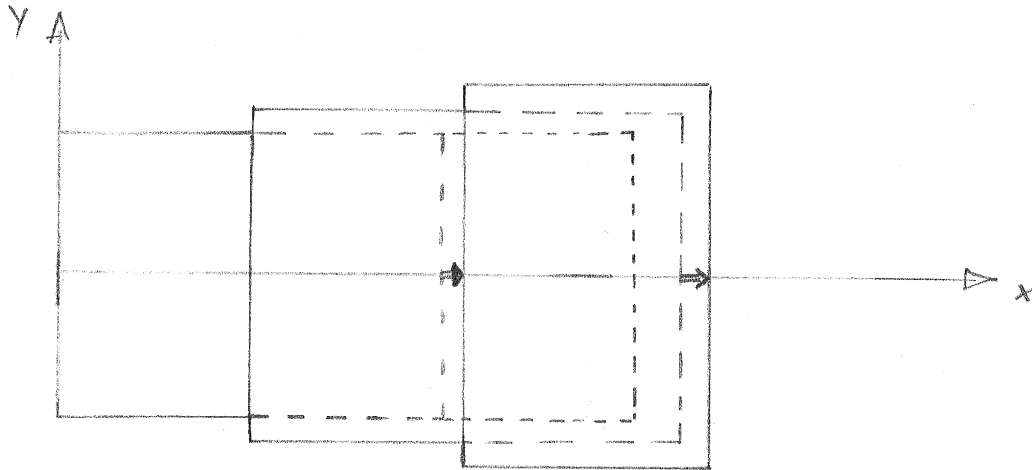
4.3 Coordinate Errors in Strip Axis

Now we can consider coordinate errors:
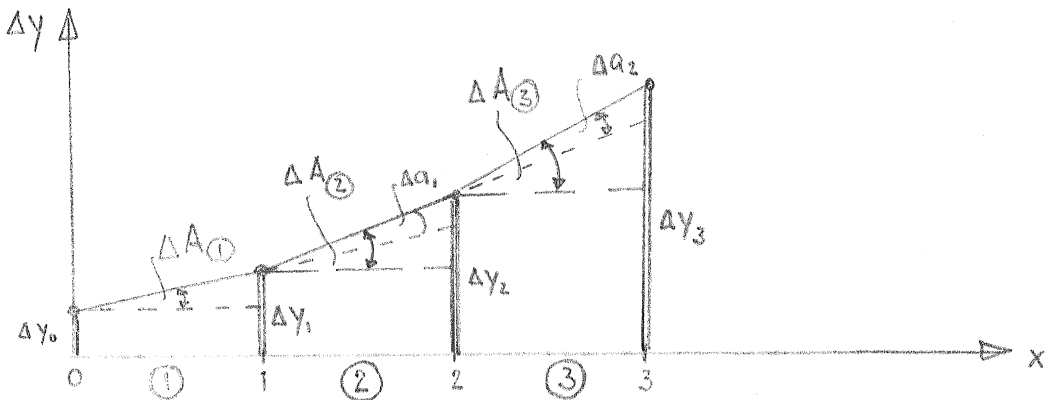
$$\Delta y_0 = \Delta y_0 \qquad\qquad = \Delta y_0$$

$$\Delta y_1 = \Delta y_0 + b_{(1)}\Delta A_{(1)} = \Delta y_0 + b_{(1)}\Delta A_{(1)}$$

$$\Delta y_2 = \Delta y_1 + b_{(2)}\Delta A_{(2)} = \Delta y_0 + b_{(1)}\Delta A_{(1)} + b_{(2)}\Delta A_{(2)}$$

$$\Delta A_{(1)} + \Delta a_1$$

$$\Delta y_i = \Delta y_{i-1} + b_{(i)}\Delta A_{(i)} = \Delta y_0 + b_{(1)}\Delta A_{(1)} + b_{(2)}\Delta A_{(2)} + \dots + b_{(i)}\Delta A_{(i)}$$

initial errors   transfer errors

$$\therefore \quad \Delta y_i = \Delta y_0 + \sum_{\mu=1}^{i} b_{(\mu)}\Delta A_{(\mu)}$$

$$= \Delta y_0 \sum_{\mu=1}^{i} b_{(\mu)}(\Delta A_{(1)} + \sum_{\nu=1}^{\mu-1} a_{\nu})$$

$b \approx$ constant!

$b_{(\mu)} \approx b$

$\Sigma b = x$

$$\Delta y_i = \Delta y_0 + X_i \, \Delta A \, \textcircled{1} + b \sum_{\mu=1}^{i} \sum_{\nu=1}^{\mu-1} \Delta a_\nu$$

This is the famous double summation of random errors, which leads to a systematic behaviour, as will be shown later.

There is another way of representing the same error:
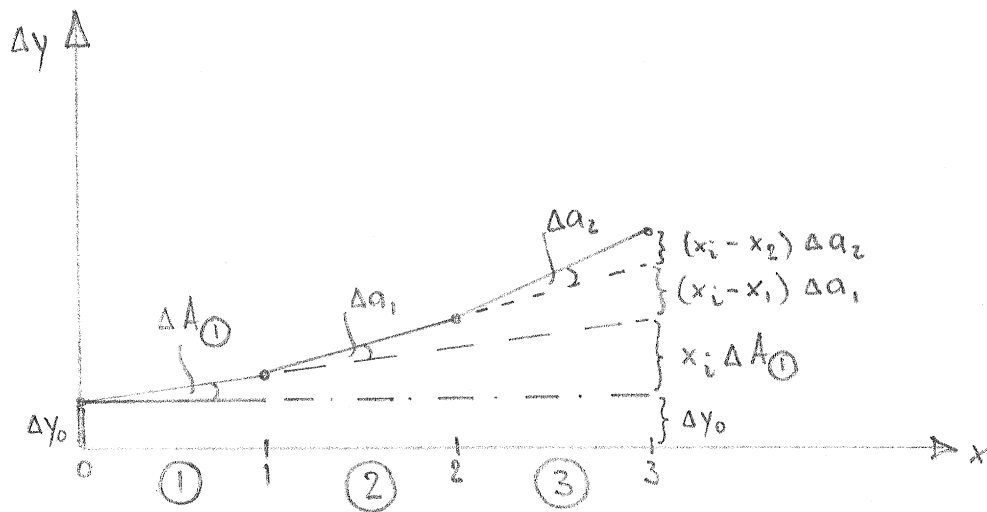
$$\Delta y_i = \Delta y_0 + x_i \, \Delta A \, \textcircled{1} + \sum_{\nu=1}^{i-1} (x_i - x_\nu) \, \Delta a_\nu$$

Since this formula represents directly the same as the previous one, the double summation characteristics are still there, only hidden!
Similarly:

$$\Delta x_i = \Delta x_0 + \sum_{\mu=1}^{i} b_{\textcircled{\mu}} \, \Delta S_{\textcircled{\mu}}$$

$$= \Delta x_0 + x_i \, \Delta S_{\textcircled{1}} + b \sum_{\mu=1}^{i} \sum_{\nu=1}^{\mu-1} \Delta s_\nu$$

$$= \Delta x_0 + x_i \, \Delta S_{\textcircled{1}} + \sum_{\nu=1}^{i-1} (x_i - x_\nu) \, \Delta s_\nu$$

and

$$\Delta \dot{z}_i = \Delta z_0 + \sum_{\mu=1}^{i} b_{\textcircled{μ}} \, \Delta\Phi_{\textcircled{μ}}$$

$$= \Delta z_0 + x_i \, \Delta\Phi_{\textcircled{1}} + \Big|\, b \sum_{\mu=1}^{i} \sum_{\nu=1}^{\mu-1} \Delta\phi_\nu$$

$$= \Delta z_0 + x_i \, \Delta\Phi_{\textcircled{1}} + \Big|\, \sum_{\nu=1}^{i-1} (x_i - x_\nu)\Delta\phi_\nu$$

initial | deformation

error | due to transfer error

Before covering the off axis errors, I would like to say a few words about double summation.

4.4 Double Summation

If one considers, for example the azimuth error Δa as a random error E_a, then their accumulated effect in the μth model will be:

$$E_{a_\mu} = E_{a1} + E_{a2} + \ldots + E_{a_{\mu-1}} = \sum_{\nu=1}^{\mu-1} E_{a_\nu}$$

These single errors E_{a_i} in turn, create a lateral error.

$$E_{y_i} = b\,(E_{y1} + E_{y2} + \ldots + E_{yi}) = b \sum_{\mu=1}^{i} E_{y\mu}$$

$$= b \sum_{\mu=1}^{i} \sum_{\nu=1}^{\mu-1} E_{a\nu}$$

We have errors which are obtained by double summation of a series of random errors. Generally:

$$\alpha_i = \sum_{\mu=1}^{i} \sum_{1}^{\mu-1} \varepsilon = (i-1)\,\varepsilon_1 + (i-2)\,\varepsilon_2 + \ldots + \varepsilon_{i-1}\,)$$

and this series has certain systematic characteristics. This was

first discovered by Gotthardt (1944 - rolling dice) and Roelofs (1949

drawing lots) with the aid of statistical experiments. Moritz (1960)

gives a theoretical explanation based on the fact, that all x_i are

strongly correlated by the ε_i.

Here are three practical examples taken from Finsterwalder

Hoffmann Photogrammetrie, p. 370.

i	Series 1			Series 11			Series 111		
	ε_i	$[\varepsilon_i]$	$[[\varepsilon_i]]$	ε_i	$[\varepsilon_i]$	$[[\varepsilon_i]]$	ε_i	$[\varepsilon_i]$	$[[\varepsilon_i]]$
1	+ 3	+ 3	+ 3	− 9	− 9	− 9	− 7	−7	− 7
2	+ 1	+ 4	+ 7	− 6	−15	− 24	+ 4	−3	−10
3	−12	− 8	− 1	+ 8	− 7	− 31	+ 8	+5	− 5
4	+ 3	− 5	− 6	− 2	− 9	− 40	+ 2	+7	+ 2
5	+ 3	− 2	− 8	+15	+ 6	− 34	− 3	+4	+ 6
6	+ 3	+ 1	− 7	+ 3	+ 9	− 25	− 9	−5	+ 1
7	+ 4	+ 5	− 2	− 3	+ 6	− 19	+ 1	−4	− 3
8	− 7	− 2	− 4	+ 6	+12	− 7	+ 5	+1	− 2
9	− 7	− 9	− 13	+11	+23	+ 16	+ 3	+4	+ 2
10	− 8	−17	− 30	0	+23	+ 39	− 4	0	+ 2
11	0	−17	− 47	− 6	+17	+ 56	− 2	−2	0
12	− 4	−21	− 68	+ 3	+20	+ 76	+ 6	+4	+ 4
13	+ 1	−20	− 88	+10	+30	+106	− 9	−5	− 1
14	+ 5	−15	−103	− 2	+28	+134	+ 3	−2	− 3
15	0	−15	−118	−12	+16	+150	+10	+8	+ 5
16	− 5	−20	−138	−10	+ 6	+156	− 7	+1	+ 6
17	+ 5	−15	−153	− 8	− 2	+154	−10	−9	− 3
18	+ 3	−12	−165	+11	+ 9	+163	+ 5	−4	− 7
19	− 9	−21	−186	− 2	+ 7	+170	+10	+6	− 1
20	− 1	−22	−208	− 7	0	+170	− 5	+1	0
21	− 1	−23	−231	− 3	− 3	+167			
22	− 3	−26	−257	− 2	− 5	+162			
23	− 7	−33	−290	+ 2	− 3	+159			
24	− 9	−42	−332	− 7	−10	+149			
25	+ 2	−40	−372	+ 7	− 3	+146			

It is evident, that the two error series are random, but show systematic character after double summation.

This is a statistical statement, as can be shown with a 3rd series, which remains random!

4.5 Off Axis Points and Their Coordinate Errors

a) Scale Error:

Errors in y and z (the x-error has already been considered for the nadir points)

$$\Delta y_i = y_i \ \Delta S_{(i)} = y_i (\Delta S_{(1)} + \sum_{\nu=1}^{i-1} \Delta s_\nu)$$

$$\Delta z_i = z_i \ \Delta S_{(i)} = z_i (\Delta S_{(1)} + \sum_{\nu=1}^{i-1} \Delta s_\nu)$$

b) Azimuth error:

$$\Delta x_i - y_i \ \Delta A_{(i)} = -y_i (\Delta A_{(1)} + \sum_{\nu=1}^{i-1} \Delta a_\nu)$$

$$\Delta z_i = 0 .$$

c) Longitudinal error:

$$\Delta x_i = z_i \ \Delta \Phi_{(i)} = z_i (\Delta \Phi_{(1)} + \sum_{\nu=1}^{i-1} \Delta \phi_\nu)$$

$$\Delta y_i = 0 .$$

d) Lateral error:

$$\Delta y_i = -z_i \ \Delta\Omega_{(i)} = -z_i \left(\Delta\Omega_{(1)} + \sum_{\nu=1}^{i-1} \Delta\omega_\nu \right)$$

$$\Delta z_i = y_i \ \Delta\Omega_{(i)} = y_i \left(\Delta\Omega_{(1)} + \sum_{\nu=1}^{i-1} \Delta\omega_\nu \right)$$

All these influences combined give the following errors in x, y, z

for a point in model i, when the origin of the coordinate system is

close to the 1st projection centre and the x-axis follows the direction

of flight:

$$\Delta x = \Delta x_0 + x\Delta S_{(1)} - y\Delta A_{(1)} + z\Delta\Phi_{(1)} + \sum_{\nu=1}^{i-1} (x-x_\nu)\Delta s_\nu - y \sum_{\nu=1}^{i-1} \Delta a_\nu + z \sum_{\nu=1}^{i-1} \Delta\phi_\nu + r_x$$

$$\Delta y = \Delta y_0 + y\Delta S_{(1)} + x\Delta A_{(1)} - z\Delta\Omega_{(1)} + y \sum_{\nu=1}^{i-1} \Delta s_\nu + \sum_{\nu=1}^{i-1} (x-x_\nu)\Delta a_\nu - z \sum_{\nu=1}^{i-1} \Delta\omega_\nu + r_y$$

$$\Delta z = \Delta z_0 + z\Delta S_{(1)} - x\Delta\Phi_{(1)} + y\Delta\Omega_{(1)} + z \sum_{\nu=1}^{i-1} \Delta s_\nu - \sum_{\nu=1}^{i-1} (x-x_\nu)\Delta\phi_\nu + y \sum_{\nu=1}^{i-1} \Delta\omega_\nu + r_z$$

initial errors   transfer errors   pointing error during measurement

Now we simplify again:

$$\Delta s_1 = \Delta s_2 = \ldots\ldots = \Delta s$$

$$\Delta a_1 = \Delta a_2 = \ldots\ldots = \Delta a$$

Therefore

$$\sum_{\nu=1}^{i-1} (x-x_i) \ \Delta s_\nu \ \text{for a constant } \Delta s \text{ becomes:}$$

$$\sum_{\nu=1}^{i-1} (x-x_i)\Delta s_\nu = (x-x_1)\Delta s_1 + (x-x_2)\Delta s_2 + \ldots\ldots + (x-x_{i-1})\Delta s_{i-1}$$

$$= [(i-1)x - (x_1 + x_2 + \ldots\ldots + x_{i-1})]\Delta s$$

and if we consider $b_\mu \approx b$:

$$= [(i-1)x - \frac{x_i(i-1)}{2}]\Delta s = (i-1)(x - \frac{x_i}{2})\,\Delta s$$

Now, if we just consider the nadir points, then $x \approx x_i$

$$\sum_{\nu=1}^{i-1} (x-x_i)\,\Delta s = \frac{i-1}{2}\,x\,\Delta s$$

However: $i \approx \frac{x_i}{b} \approx \frac{x}{b}$ (i = number of models) and for longer strips we can set $(i-1) \approx i$, therefore:

$$\sum_{\nu=1}^{i-1} (x-x_i)\,\Delta s = \frac{\Delta s}{2b}\,x^2 = \Delta x \text{ (major influence!)}$$

This means, that if we have a constant scale error, the influence is in form of a parabola.
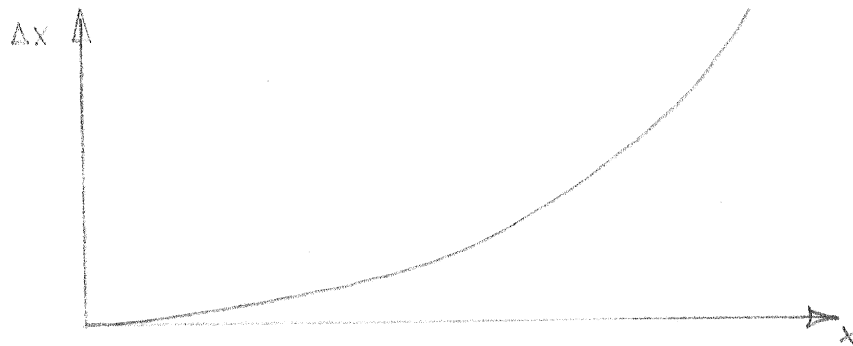
The single summation will give:

$$\sum_{\nu=1}^{i-1} \Delta a_\nu = \Delta a_1 + \Delta a_2 + \ldots\ldots\ldots + \Delta a_{i-1}$$

with $\Delta a_1 = \Delta a_2 = \ldots = \Delta a$

$$= (i-1) \, \Delta a$$

$$\approx \frac{\Delta a}{b} \, x$$

The single summation results in a linear behaviour. Therefore:

$$\Delta x = \Delta x_0 + x \Delta S_1 - y \Delta A_1 + z \Delta \Phi_1 + \frac{\Delta s}{2b} x^2 - \frac{\Delta a}{b} xy + \frac{\Delta \phi}{b} xz + r_x$$

$$\Delta y = \Delta y_0 + y \Delta S_1 + x \Delta A_1 - z \Delta \Omega_1 + \frac{\Delta a}{2b} x^2 + \frac{\Delta s}{b} xy - \frac{\Delta \omega}{b} xz + r_y$$

$$\Delta z = \Delta z_0 + z \Delta S_1 - x \Delta \Phi_1 + y \Delta \Omega_1 - \frac{\Delta \phi}{2b} x^2 - \frac{\Delta \omega}{b} xy + \frac{\Delta s}{b} xz + r_z.$$

If we want to establish a general equation, we can set:

$$\Delta x_0 = a_0 \qquad \Delta y_0 = b_0 \qquad \Delta z_0 = c_0$$

$$\Delta S_1 = a_1 \qquad \Delta A_1 = b_1 \qquad \Delta \Phi_1 = c_1 \qquad \Delta \Omega_1 = d_1$$

$$\frac{\Delta s}{2b} = a_2 \qquad \frac{\Delta a}{2b} = b_2 \qquad \frac{\Delta \phi}{2b} = c_2 \qquad \frac{\Delta \omega}{2b} = d_2$$

And now we have:

$$\Delta x = a_0 + a_1 x - b_1 y + c_1 z + a_2 x^2 - 2b_2 xy + 2c_2 xz$$

$$\Delta y = b_0 + b_1 x + a_1 y - d_1 z + b_2 x^2 + 2a_2 xy - 2d_2 xz$$

$$\Delta z = c_0 - c_1 x + d_1 y + a_1 z - c_2 x^2 - 2d_2 xy + 2a_2 xz$$

These functions are not independent, except for the axis where $y = z = 0$.
However, we have now some equations which are independent of the models.
Before, the strip deformations were functions of x, y, z <u>and</u> i!!

For flat terrain, <u>z = const.:</u>

$$\Delta x = a_0 + a_1 x - b_1 y + a_2 x^2 - 2b_2 xy = f(x, y)$$

these are different

$$\Delta y = b_0 + b_1 x + a_1 y + b_2 x^2 + 2a_2 xy$$

$$\Delta z = c_0 + c_1 + d_1 y + c_2 x^2 + d_2 xy$$

Now we make them all plus and omit the 2 since the general coefficients are different.

Transfer errors   1st summation   2nd summation

$\Delta s, \Delta a, \Delta \phi, \Delta \omega \longrightarrow \Delta S_i \; \Delta A_i \; \Delta \Phi_i \; \Delta \Omega_i \longrightarrow \Delta x, \Delta y, \Delta z$

for $b \rightarrow 0$ $\int$ $\int\int$
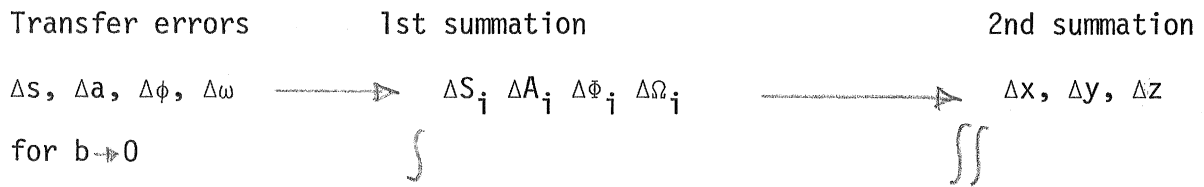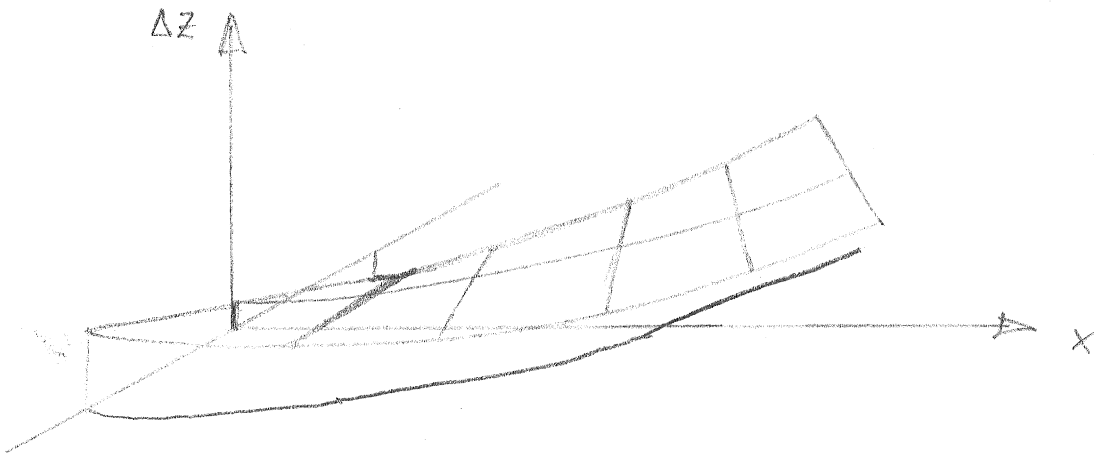
Now we can reverse the whole procedure

           1st differentiation   2nd derivative

$\Delta x, \Delta y, \Delta z \longrightarrow \Delta S \; \Delta A \; \Delta \Phi \; \Delta \Omega \longrightarrow \Delta s, \Delta a, \Delta \phi, \Delta \omega$

We had:

$$\Delta x = a_0 + a_1 x + a_2 x^2 + \ldots - y(b_1 + b_2 x + \ldots) + z(c_1 + 2c_2 x + \ldots)$$

$$\Delta y = b_0 + b_1 x + b_2 x^2 + \ldots + y(a_1 + 2a_2 x + \ldots) - z(d_1 + 2d_2 x + \ldots)$$

$$\Delta z = c_0 + c_1 x + c_2 x^2 + \ldots + y(d_1 + (2)d_2 x + \ldots) + z(a_1 + 2a_2 x + \ldots)$$

$$\Delta x = \Delta x_{axis} - y\left(\frac{d\Delta y_{axis}}{dx}\right) + z\left(\frac{d\Delta z_{axis}}{dx}\right)$$

$$\Delta y = \Delta y_{axis} + y\left(\frac{d\Delta x_{axis}}{dx}\right) - z(\Delta\Omega_{(x)})$$

$$\Delta z = \Delta z_{axis} + y(\Delta\Omega_{(x)}) + z\left(\frac{d\Delta x_{axis}}{dx}\right)$$

Compare this with original coefficients:

$$\Delta x = \Delta x_A - y\Delta A_{(x)} \quad + z\Delta\Phi_{(x)}$$

$$\Delta y = \Delta y_A + y\Delta S_{(x)} \quad - z\Delta\Omega_{(x)}$$

$$\Delta z = \Delta z_A + y\Delta\Omega_{(x)} \quad + z\Delta S_{(x)}$$

$$= 0 \text{ for } z = \text{const.!}$$

(up to 10% of flying height).

These are independent polynomials, the others are derivatives.

So far we have only talked about random errors. There are also systematic ones, caused by instrument errors, distortions, etc. as well as by earth curvature, refraction, etc.

Historically systematic errors were approximated by a 2nd order polynomial. Since the strip deformations appeared to have similar behaviour, it was assumed that the main error sources were systematic. This lead to intensiv e instrument checks. Today it is obvious that systematic errors and random errors with systematic characteristics due to double summation are superimposed. The initial doubting came, when

triangulation with the C8 and radial triangulation showed similar behaviour, although they are the result of extremely different procedures and instruments.

Before going to actual strip adjustments, a few words on the magnitude of the errors:

$$\sigma_{\Delta s} \approx 0.1 - 0.2\ \text{‰}$$

$$\sigma_{\Delta a} = \sigma_{\Delta \phi} = \sigma_{\Delta \omega} \approx 1^C \ \text{(same claim } 0.5^C)$$

$$\left.\begin{array}{c} \sigma_{\Delta x} \\ \sigma_{\Delta y} \\ \sigma_{\Delta z} \end{array}\right\} \quad \text{increase proportional with } \sqrt{x}^3$$

while

$$\left.\begin{array}{c} \sigma_{\Delta S} \\ \sigma_{\Delta A} \\ \sigma_{\Delta \Omega} \\ \sigma_{\Delta \Phi} \end{array}\right\} \quad \text{increase proportional with } \sqrt{x}$$

According to Ackermann:

$$Q_{xx} = \frac{b^2}{6} Q_{\Delta s \Delta s} \bar{x}(\bar{x}-1)(2\bar{x}-1) + b^2 Q_{\Delta a \Delta a}(\bar{x}-1)\bar{y}^2 + b^2 Q_{\Delta \phi \Delta \phi}(\bar{x}-1)\bar{z}^2$$

where

$$\bar{x} = \frac{x}{b}; \quad \bar{y} = \frac{y}{b}; \quad \bar{z} = \frac{z}{b} \quad \text{(normed quantities)}$$

There are strong correlations between x_i, y_i, and z_i as well as with other points. In addition a measuring error of $\approx 10\ \mu m$ in the image scale has to be considered.

5. Strip Adjustment with Polynomials

5.1 General Remarks

Originally, strip adjustment was a non linear interpolation procedure. This depends strongly on the number and distribution of control points. The problem is, to approximate the real errors as good as possible.

Let us assume, we know the deformation:
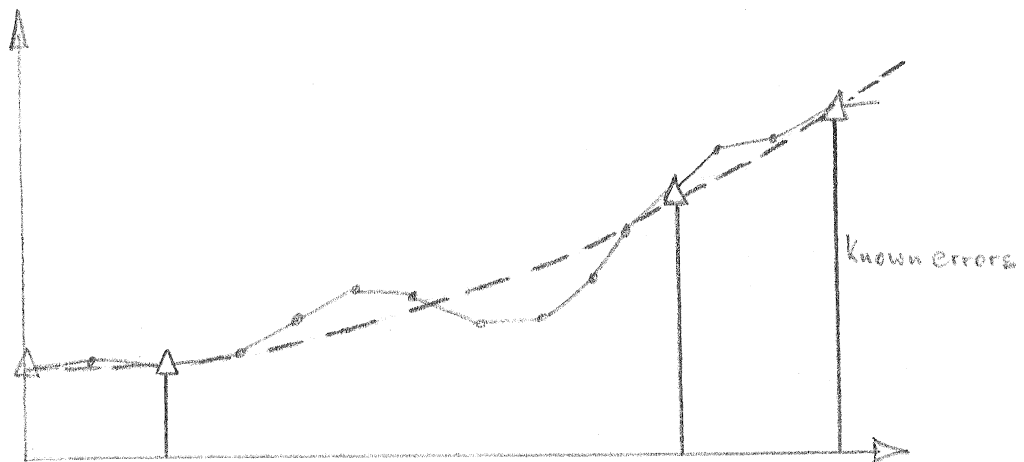
One tries to determine the plausible deformation, using these known points. The deformation is considered to be steady. Such a curve can for example be obtained using a plastic ruler which gives you a better curve than any numerically determined one. However, there is the additional problem of measuring errors in the known points.
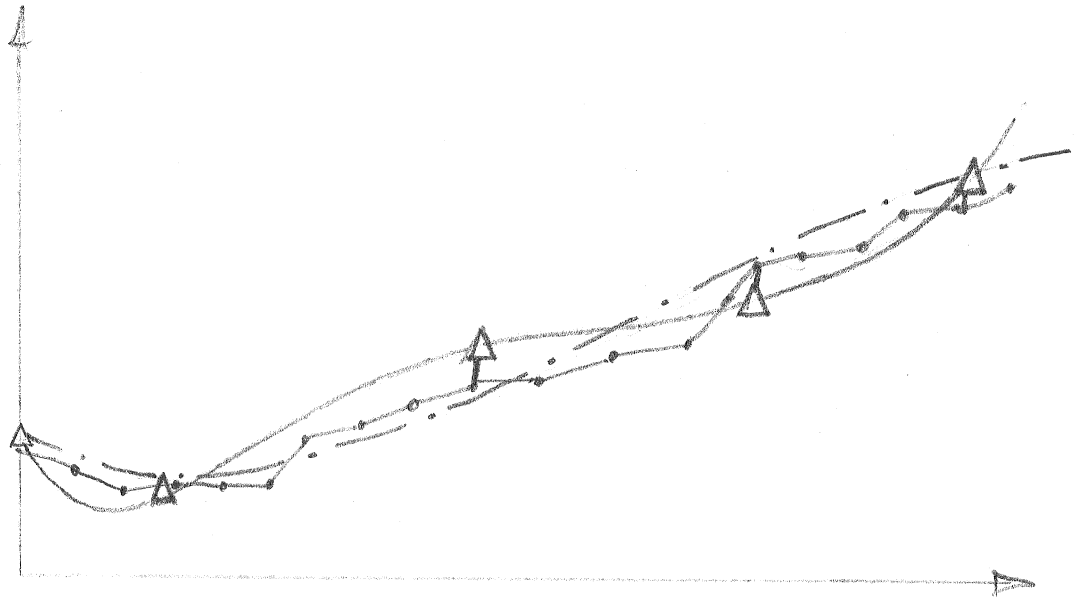
What is better?

One thing is quite obvious that points close together might cause large errors.
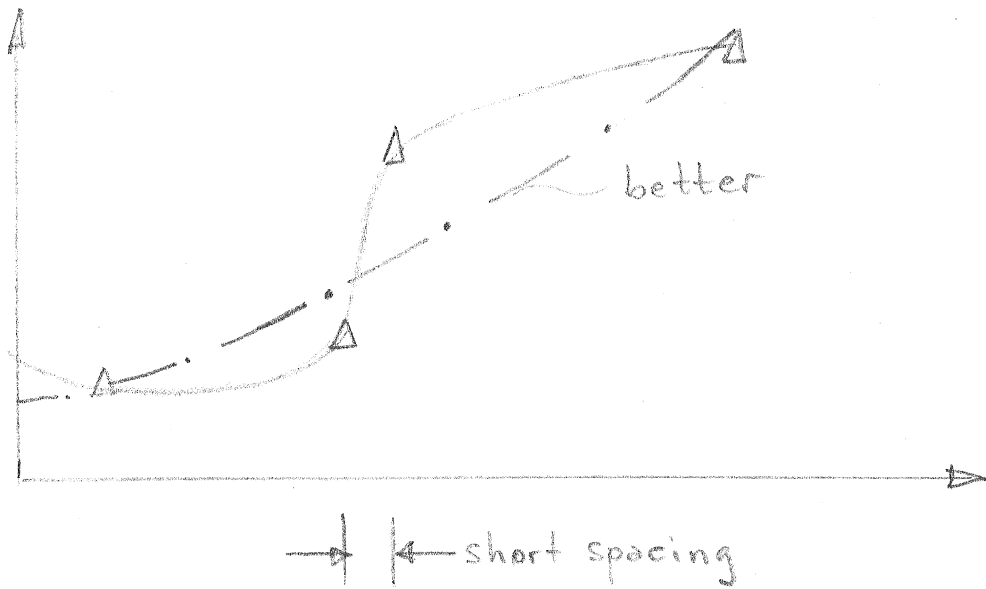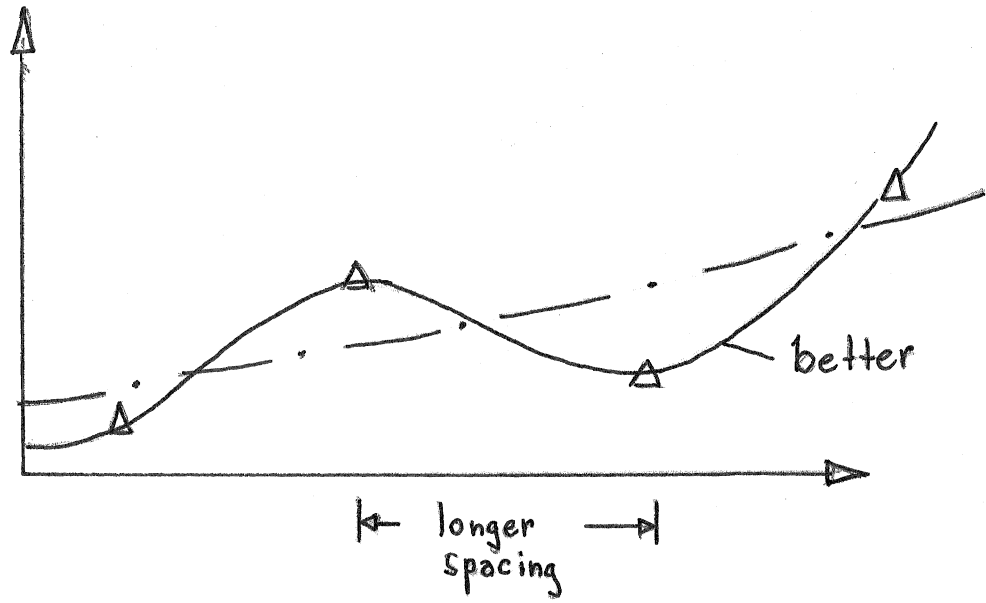
Extreme case: one model!
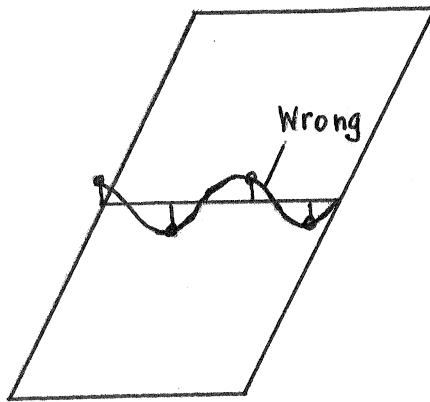
There has to be a certain restriction to the degree of the polynomial.

5.2 The U.S. Coast & Geodetic Survey Method, as Example

Extract from: Aerotriangulation Strip Adjustment

by M. Keller and G.C. Tewinkel
U.S. Coast and Geodetic Survey

AEROTRIANGULATION is a photogrammetric technique for deriving ground coordinates of objects from a set of overlapping aerial photographs that show images both of those objects and also of a relatively sparse distribution of other objects whose coordinates are known from previous classical measurements on the ground. Provisional photogrammetric coordinates of objects can be determined by at least two general methods:

(1) through the use of a high-order photogrammetric plotting instrument, or

(2) through analytic computations based on observed coordinates of

   images on the photographs.

The discussion pertains specifically to both cases. In each instance, the photogrammetric strip coordinates of points comprise a thick, three-dimensional ribbon in space generally not referred specifically to any ground surveyed system of points.

Finally, the strip coordinates, in order for them to be useful, must be related to the ground system through the application of polynomial transformations in a curve-fitting procedure to adjust the photogrammetric strip coordinates to agree with known ground surveyed coordinates. This fitting technique is called Strip Adjustment. The polynomials are nonlinear because of the systematic accumulation of errors throughout the strip. The application of least squares provides a logical analysis of redundant data.

This paper comprises a documented computer program concerned only with the transformation and adjustment of the strip coordinates of points to fit ground control data.

Technical Bulletins No. 1[1] and No. 10[2] presented the principal formulation still being applied in the Coast and Geodetic Survey, and No. 21[3] included an application of those ideas. The present bulletin includes helpful modifications which have been added since the dates of the original releases, combines the ideas of the three former bulletins into a single operation, adapts the program for either instrumental or analytica aerotriangulation, and embodies a systematic technique for correcting horizontal coordinates for the local inclinations of the strip.

This program is considered to be the first of a series of two or three new programs for analytic aerotriangulation. Chronologically, this program will be used as the third step in the provisional adjustment of strips. The other two programs will consist of

(1) the reduction of observed image coordaintes and

(2) relative orientation, including the assembly of the oriented data. The three programs will comprise a complete practical set for analytic strip aerotriangulation for use on a medium size computer.

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[1] Aerotriangulation adjustment of instrument data by computational methods by W.D. Harris, Technical Bulletin No. 1, Coast and Geodetic Survey, January 1958.

[2] Vertical adjustment of instrument aerotriangulation by computational methods by W.D. Harris, Technical Bulletin No. 10, Coast and Geodetic Survey, September 1959.

[3] Analytic aerotriangulation by W.D. Harris, G.C. Tewinkel and C.A. Whitten, Technical Bulletin No. 21, Coast and Geodetic Survey, Corrected July 1963.

INTRODUCTION

The adjustment of aerotriangulated strips has been the subject of numerous articles and publications for a few decades, as indicated presently by the references. The articles agree in general that the equations for transforming photogrammetric coordinates into ground coordinates can be expressed by polynomials which need to be at least second degree. Schut[4] and Mikhail[5] show that a conformal transformation in three dimensions is not possible if the degree is greater than one, although a conformal transformation in a plane is not limited as to degree. Inasmuch as an ideal solution of the three-dimensional problem does not seem to exist, photogrammetrists have been free to devise approximate, quasi-ideal, and impirical solutions that seem to give practical and usable solutions to their problems. Several examples are cited.

The Coast Survey formulas applied herein are:

$$x' = x - \Delta z(3hx^2 + 2ix + j) + ax^3 + bx^2 + cx - 2dxy - ey + f$$

$$y' = y - \Delta z(kx^2 + \ell x + m) + 3ax^2y + 2bxy + cy + dx^2 + ex + g$$

$$z' = z[1 + (3hx^2 + 2ix + j)^2 + (kx^2 + \ell m + m^2]^{1/2} + hx^3 + ix^2 + jx + kx^2y + my + n .$$

(The x, y coordinates refer to the axis-of-flight system after the application of Equations 22 and 23). Schut[4] states the following conformal relations for horizontal coordinates to third degree:

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[4] Development of programs for strip and block adjustment by the National Research Council of Canada by G.H. Schut, _Photogrammetric Engineering_, Vol. 30, No. 2, page 284, 1964.

[5] Simultaneous three-dimensional transformation of higher degrees by E.M. Mikhail, _Photogrammetric Engineering_, Vol. 30, No. 4, page 588, 1964.

$$x' = x + a_1 + a_3x - a_4y + a_5(x^2-y^2) + 2a_6xy + a_7(x^3-3xy^2)-a_8(3x^2y-y^3) + ..$$

$$y' = y + a_2 + a_4x + a_3y + a_6(x^2-y^2)+2a_5xy + a_7(3x^2y-y^3) + a_8(x^3-3xy^2) + .. \qquad (2)$$

Webb and Perry[6] used

$$x' = x + a_3x^2 + b_3x + c_3xy + d_3xz + e_3$$

$$y' = y + a_4x^2 + b_4xy + c_4y + d_4 \qquad (3)$$

$$z' = z + a_5x^2 + b_5xy + c_5xy + d_5y + e_5x + f_5 .$$

The following equations in a plane appear in Schwidefsky's textbook:

$$x' = x + a_1x^3 + b_1x^2 - 3a_2x^2y - 2b_2xy$$

$$\qquad (4)$$

$$y' = y + a_2x^3 + b_2x^2 + 3a_1x^2y + 2b_1xy .$$

Arthur[8] of the Ordnance Survey of Britain published:

$$x' = x + a_1 + a_4x + a_6z - a_7y + 1/2a_8x^2 + a_{10}xz - a_{11}xy$$

$$y' = y + a_2 + a_4y + a_5z + a_7x + a_8xy + a_9xz + 1/2a_{11}x^2 \qquad (5)$$

$$z' = z + a_3 + a_4z - a_5y - a_6x + a_8xy - 1/2y_{10}x^2 .$$

Norwicki and Born[9] suggest variable degree polynomials depending on the

special conditions relative to the number and distribution of control

points. The study included sixth degree.

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[6] Forest Service procedure for stereotriangulation adjustment by elevtronic computer by S.E. Webb and O.R. Perry, Photogrammetric Engineering, Vol. 25, No. 3, page 404, 1959.

[7] An outline of photogrammetry by K. Schwidefsky, Pitman Publishing Corp., p. 272, 1959.

[8] Recent developments in analytic aerial triangulation at the Ordnance Survey by D.W.G. Arthur, Photogrammetric Record, Vol. 3, No. 14, page 120, 1959.

[9] Improved stereotriangulation adjustments with electronic computers by A.L. Norwicki and C.J. Born, Photogrammetric Engineering, Vol. 26, No. 4, page 599, 1960.

The unusual terms in the Coast Survey formulas are designed to compensate for the local tilts of the strip: otherwise these formulas are not greatly different from the others. Justifications for their existence is given in the next section.

It may be appropriate at this stage to state the precision toward which this study is directed. Accuracies of a few feet have been experienced where the flight altitude is 20,000 feet; fractions of a foot are significant. Consequently, this program is prepared so as to preserve thousandths of a foot for round-off reasons even though the small distances are not ordinarily significant in themselves. Thus the fine precision being sought causes one to consider carefully the type of transformation being applied lest the transformation itself add systematic errors due to excessive constraint or relaxation.

The best root-mean-square accuracy that can be expected using the piecemeal, provisional Coast Survey analytic solution is probably in the neighbourhood of 1/10,000 to 1/20,000 of the flight altitude where film is used in the aerial camera; that is, 1 foot if the altitude is 10,000 feet. If glass plates are used in the camera, and if the results are refined by a subsequent block adjustment technique, present results suggest that 1/50,000 can be approached; that is, about 2 1/2 inches if the altitude is 10,000 feet.

BASIS FOR THE FORMULATION

Analysis of the Curve Forms

The basis for the Coast Survey formulation is essentially that of Brandt[10] and Price[11] and is restated here for the sake of completeness.

Considering the abscissa direction first, the "new" or correct value x' (referred to the axis-of-flight coordinate system) for a point on the centerline (axis of flight) of a strip of aerotriangulation is considered to be composed of the "old" value plus a correction c_x:

$$x' = x + c_x$$

(which also serves to define $c_x = x' - x$). The correction is expressed by means of a polynomial of third degree in terms of the "old" coordinate:

$$x' = x + ax^3 + bx^2 + cx + f \, . \tag{6}$$

Similarly, the new y-coordinate of a point on the centerline is expressed using a quadratic polynomial:

$$y' = y + dx^2 + ex + g \tag{7}$$

also in terms of the abscissa x inasmuch as the magnitude of the correction is obviously related to the distance from the beginning of the strip.

Adequate theoretical and operational justification exists for assuming that the x and z equations need to be cubic whereas the y-

[10] Resume of aerial triangulation adjustment at the Army Map Service by R.S. Brandt, Photogrammetric Engineering Vol. 17, No. 4, page 806, 1951.

[11] Some analysis and adjustment methods in planimetric aerial triangulation by C.W. Price, Photogrammetric Engineering, Vol. 19, No. 4, page 627, 1953.

equation does not need to be more than quadratic, but it was the experience of Coast Survey photogrammetrists based on having performed perhaps a hundred or more graphic solutions prior to the use of the computer that led to the final conclusions. The x-curve was almost never symmetrical (a quadratic curve could by symmetrical) inasmuch as the second half invariably had a greater degree of curvature than the first half. A third degree correction polynomial adequately removed the discrepancy whereas the quadratic form left a residual error too large to be acceptable, although the situation could not be completely explained through a theoretical analysis. However, the y-curve was both smaller in magnitude and more nearly symmetrical. The graphic z-curves also were perceptably greater than second degree and the theoretical reasons seemed to be even more convincing.

Considering the y-curve for a moment, it is emphasized that Equation 7 applies to points on the centerline of the strip (fig. 1). The point m on the centerline needs to be corrected by moving it to n, and the magnitude of the correction is given by the equation. However, the point p on the edge of the strip needs to be corrected in two directions:

(1) one component pq is equal to the correction mn at m, and

(2) the second component is rq in the x-direction.

In the right triangle mps, the angle at m is given by the first derivative of the equation of the y-curve, which is the centerline (Equation 7):

$$\tan \theta = (2d)\ x + e.$$

Then

$$\overline{ps} = \overline{mp} \tan \theta.$$

But mp is the ordinate y of the point:

$$\overline{qr} = \overline{ps} = y[(2d)x + e] \ . \qquad (8)$$

The complete x-equation is therefore composed of both equations 6 and 8:

$$x' = x + ax^3 + bx^2 + ex - (2d) xy - ey + f \qquad (9)$$

where the minus signs derive from the analytic definition of the direction of the slope; i.e., the relative direction of rq.

The complete y-equation is formed through a comparable analysis:

$$y' = y + 3ax^2y + 2bxy + cy + dx^2 + ex + g. \qquad (10)$$

Equation 6 depicts a lengthwise stretching or compressing of the strip. The term $(3ax^2y + 2bxy + cy)$ in Equation 10 is the effect of the local stretch or compression in the y-direction both as a function of the abscissa x of point in the strip and also as a function of the distance y that the point is off the centerline.

The vertical dimension is explained by a similar analysis. The basic equation is the cubic form
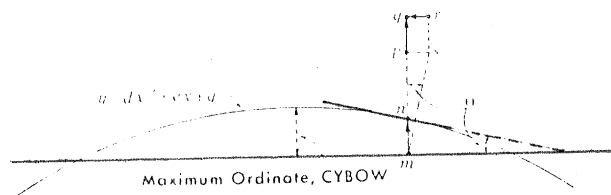
Fig. 1. - Sketch of the center line of the y or azimuth curve illustrating the derivation of a component x-correction for a point not on the center line. The definition of the term maximum ordinate or CYBOW is also indicated.

$$z' = z + hx^3 + ix^2 + jx + kx^2y + \ell xy + my + n \qquad (11)$$

which can be considered as composed of the sum of the two principal geometric parts

$$hx^3 + ix^2 + jx \ . \qquad (12)$$

$$kx^2y + \ell xy + my \ . \qquad (13)$$

The first part (equation 12) is sometimes called the "BZ-curve". This curve is the projection of the center line of the strip onto the xz-plane. The second part (Equation 13) has been described as "twist" and as "cross tilt." The latter is considered to be quadratic, again based largely on experience gained from the graphic analyses of many strips, inasmuch as the graphic curves invariably were not straight lines. If they were linear, the effect would resemble a helix of constant pitch, like a screw thread, but the quadratic form fits most situations more closely. The terms in Equation 13 have as their common factor the ordinate y of the point so that the farther the point is off the axis of the strip, the greater is the correction.

The Slope Corrections

If y is factored out of Equation 13,

$$(kx^2 + \ell x + m)y \ . \tag{14}$$

it is obvious that the parenthetical expression represents the slope of the strip perpendicular to the centerline:

$$\tan \omega = kx^2 + \ell x + m \tag{15}$$

Moreover, the first derivative of Equation 12 is the instantaneous slope of the BZ-curve in the direction of the strip:

$$\tan \phi = d/dx\ (hx^3 + ix^2 + jx) = 3hx^2 + 2fx + j \ . \tag{16}$$

Thus equations 15 and 16 depict the slopes of the strip "ribbon" in the x and y directions at any given abscissa x. Then the resultant tilt τ (deviation from the vertical) of the normal to the strip is given by
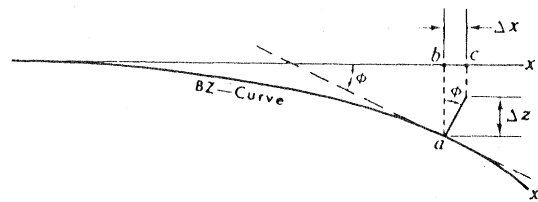
Fig. 2 - Sketch of the center line vertical BZ-curve illustrating how the local inclination of the curve causes a component horizontal correction in the x-direction.

$$\sec \tau = (1 + \tan^2\phi + \tan^2\omega + \tan^2\phi \tan^2\omega)^{1/2} \qquad (17)$$

which, inasmuch as ϕ and ω are both small angles, is approximated with sufficient accuracy for practical operations by

$$\sec \tau = (1 + \tan^2\phi + \tan \omega)^{1/2} . \qquad (18)$$

In aerotriangulation, both in instrumental and in the Coast Survey analytic systems, the observed or computed coordinates of points are related to the initial rectangular axes of the strip at the first model or first photograph rather than the curved centerline of the strip (fig. 2). Consequently, the base and top of an elevated object have different horizontal coordinates. In the figure, the base a has the abscissa of point b whereas the top has the abscissa of point c. The previous photogrammetric solution yields the abscissa of c and the elevation Δz, which introduce a discrepancy Δx equal to the distance from b to c:

$$\Delta x = \Delta z \tan \phi = \Delta z (3hx^2 + 2ix + j). \qquad (19)$$

The value is subtracted from the abscissa of c as a correction indicated in the first formula of Equation 1.

In a similar manner, the correction of the ordinate is:

$$\Delta y = \Delta z \tan \omega = \Delta z(kx^2 + \ell x + m) \qquad (20)$$

as indicated in the second formula of Equation 1.

The z-correction is probably of minor consequence; nevertheless, it is also applied. The photogrammetric elevation is too small and needs to be increased by multiplying it by the secant of the resultant inclination τ of the line perpendicular to the surface of the strip. The term is applied in the third formula of Equation 1.

Two Preliminary Affine Transformations

In Equations 1 it was tactily assumed both that

(1) the directions of the x, y, z axes were essentially parallel to the x', y', z' axes, and also that

(2) the x-axis represented the centerline or axis of flight of the photogrammetric strip.

Both of these conditions are violated in practice; consequently, two preliminary affine transformations are utilized for rotation, translation, and scale change. Inasmuch as these conditions need not be exactly adhered to, unique transformations are utilized so as to impose as few fixed conditions as possible and to simplify the computations for determining the constants of the transformation equations.

Perhaps an explanation is in order as to the reasons for assuming that these conditions are necessary. The systematic errors depicted by the polynomials of Equation 1 are direction-sensitive inasmuch as they are propagated as functions of the length of the strip, or the number of photographs in the strip, or simply the abscissa of a point in the strip. However, the photogrammetric direction of the strip (axis of flight) is not known with sufficient accuracy until after the strip has been aerotriangulated, at which time the easiest way to obtain the desired coordinates is to transform them by means of a computer rather than to reobserve them.

Secondly, whereas the photogrammetric strip (model) coordinates may progress in any direction of the compass, the ground coordinates are oriented with +X eastward. But the X-coordinates must also be reoriented into the axis-of-flight direction, which possibly would be unnecessary if all strips were flown north-south or east-west. Again, it is fairly easy to rotate, scale and translate the ground coordinate system into the axis-of-flight system with an electronic computer.

Finally, after the correction Equations 1 have all been applied to the coordinates of a point, it is necessary to convert the coordinates back into the ground system by applying the inverse of the second affine transformation above so that the resulting coordinates are meaningful and useful in surveying and mapping work.

Model Coordinates to Axis-of-Flight System

By "model" coordinates is meant the form of the data from a stereoplanigraph bridge. A comparable form results from the preliminary computer solution of the Coast Survey analytic aerotriangulation.

Let the model coordinates of a point near the center of the initial model be x_1, y_1, and near the center of the terminal model be x_2, y_2. The axis of flight is arbitrarily defined as passing through these two points. This axis of flight is to be the new x-axis: the new ordinates of both the above points are therefore zeros. The origin of the axis of flight is defined as midway between these initial and terminal points in order to reduce the numerical magnitude of the x-coordinates, which has added importance inasmuch as the x-coordinates are squared and cubed as indicated in Equation 1. The distance D

between the initial and terminal points is given by analytic geometry to be

$$D = (\Delta x^2 + \Delta y^2)^{1/2} \ ,$$

$$\Delta x = x_1 - x_2, \quad \Delta y = y_1 - y_2 \ , \tag{21}$$

Consequently, the coordinates of the initial and terminal points in the axis-of-flight system are

$$x_1' = - 1/2 \ D, \ y_1' = 0$$

$$x_2' = + 1/2 \ D, \ y_2' = 0$$

The axis-of-flight coordinates of other points can be computed using the following set of affine transformation formulas (which comprise a special form of the more general Equation 25 discussed later):

$$x' = a_1 x - b_1 y + c_1$$

$$y' = b_1 x + a_1 y + d_1 \ . \tag{22}$$

$$a_1 = -\Delta x/D$$

$$b_1 = \Delta y/D$$

$$c_1 = -a_1 x_1 + b_1 y_1 - 1/2D \tag{23}$$

$$d_1 = -a_1 x_1 - a_1 y_1 \ .$$

Equations 22 constitute simply a rotation and translation of the "model" coordinates into the "axis-of-flight" coordinates maintaining the same original model scale. The coefficients a_1 ..... d_1 are the constants for the transformation: their values are determined once only.

Ground Horizontal Coordinates Into the Axis-of-Flight System, and the Inverse

The coordinates of horizontal ground control stations also need to be transformed into the same axis-of-flight system. The transformation is based on only two of the stations, one near the initial end of the strip and one near the terminal end. In the program, the coordinates of the two stations are listed as the first and the last ones used in the adjustment. Two sets of coordinates are given for each point: one set is in the form of model coordinates $x_1 \ldots y_2$ and the other set in the ground survey system as $X_1 \ldots Y_2$.

The first step is to transform the model values x_1, etc., into the axis-of-flight system by applying Equations 22 and then applying the slope corrections as indicated by Equations 19 and 20. If $x_1' \ldots y_2'$ are the axis-of-flight coordinates of the initial and terminal control stations, the following differences can be expressed:

$$\Delta x' = x_1' - x_2', \quad \Delta y' = y_1' - y_2'$$

$$\Delta X = X_1 - X_2, \quad \Delta Y = Y_1 - Y_2 . \tag{24}$$

(ΔX is called DGX in the Fortran program Statements 12+4 and 80+3). The square of the axis-of-flight distance D^2 between the terminal model control points is

$$D^2 = \Delta x'^2 + \Delta y'^2 .$$

The same type of affine transformations as Equation 22 is applied to the axis-of-flight coordinates to convert them into the ground system of coordinates (Statements 70+1 and 70+2 in the Fortran program) which is applied to all new (bridge) points as the last stage of the computation after implementing the corrections of Equation 1:

$$X = a_{20}x' - b_{20}y' + c_{20}$$

$$Y = b_{20}x' + a_{20}y' + d_{20} .$$

(25)

If the values of x_1', etc., and X_1, etc. are substituted into Equations 25, one obtains four simultaneous linear equations in which four constant coefficients a_{20} ..... d_{20} are the only unknowns. The solution of the equations gives

$$a_{20} = (\Delta X \cdot \Delta x' + \Delta Y \cdot \Delta y')/D^2$$

$$b_{20} = (\Delta Y \cdot \Delta x' - \Delta X \cdot \Delta y')/D^2$$

(26)

$$c_{20} = X_1 - a_{20} x_1' + b_{20}y_1'$$

$$d_{20} = Y_1 - b_{20}x_1' - a_{20}y_1' .$$

Equations 25 embody a change in scale in addition to rotation and translation.

The inverse form of Equations 25 shows the corresponding transformation of ground coordinates into axis-of-flight values:

$$x' = a_{21}X + b_{21}Y - c_{21}$$

$$y' = -b_{21}X + a_{21}Y - d_{21} .$$

(27)

The values of the four new constants can be computed from those of Equations 25 by applying the following relations:

$$d* = 1/(a_{20}^2 + b_{20}^2)$$

$$a_{21} = a_{20} \cdot d*$$

$$b_{21} = b_{20} \cdot d*$$

(28)

$$c_{21} = (a_{20} c_{20} + b_{20} d_{20})d*$$

$$d_{21} = (a_{20} d_{20} - b_{20} c_{20})d* .$$

The evaluation of all eight of the constants of Equations 25 and 27 are indicated after Statement 80 of the Fortran program.

It is noted that $(a_{20}^2 + b_{20}^2)^{1/2}$ expresses the scale change included in the transformation.

It should be noted that the application of transformation Equations 27 is such that no discrepancy c_x or c_y exists or remains at the two end horizontal control points on which the transformation is based. Consequently, if the two polynomial curves are plotted, they will both cross the zero line, or x-axis, at the location of these two control stations. (Fig. 1)

Normally each curve describes a sweeping arc joining these two end points. Midway between the end points the y-curve will be at its greatest distance from the x-axis, indicating that the correction is maximum. One is assured of this feature because the equation is quadratic and is symmetrical with respect to the y-axis. Moreover Equations 22 have already been applied so that the origin of the observed coordinate system is at the center of the strip. Consequently the maximum height of the curve is where it crosses the y-axis which is where $x = 0$. If in Equation 10 x is set equal to zero, then the height of the curve is given simply by $c_y = g$. The value g may be called the "maximum ordinate" which is symbolized in the program as CYBOW. This quantity is a diagnostic term whose magnitude as determined through experience normally does not exceed certain practical limits unless a blunder occurs in some of the data.

Similarly, from Equation 9 the CXBOW is $c_x = f$. Inasmuch as the x-curve is cubic, it is not symmetrical and the value f is not the maximum. Nevertheless, f is sufficiently near the maximum that it still is useful for detecting a gross blunder.

Preliminary Vertical Affine Transform

The value of z is derived from the photogrammetric analysis and its exact relation to the ground system of measurement is not known until later. The following transformation and its inverse relating the two systems are stated and explained:

$$z' = z_0 + Z/s$$
$$Z = s(z' - z_0) \ . \tag{29}$$

(In Fortran notation, z_0 is designated as EINDEX. These equations occur in the Fortran program as Statements 31+1 and 70+3). The instrument or photogrammetric model elevation of a point in the axis-of-flight system is symbolized as z' whereas Z refers to the ground elevation, s is the horizontal (as well as vertical) scale factor that relates the magnitudes of z' and Z, and z_0 is an index elevation, or translation term. Thus Equation 29 implies simply dilation (scale) and translation. The scale factor is determined after Equation 24 above:

$$s = [(\Delta X^2 + \Delta Y^2)/(\Delta x'^2 + \Delta y'^2)]^{1/2} \ . \tag{30}$$

(Statement 12+5 in the Fortran program). Obviously the scale factor is normally a number greater than 1, such as 10,000, and is the number for multiplying instrument or model dimensions to convert them into equivalent ground dimensions.

The index elevation z_0 is defined as an average value for a list of control points:

$$z_0 = z'_a - Z_a/s \ .$$

(Fortran statement 14+2). Thus the index is the difference between the average of the instrument elevations and the average of the corresponding ground elevations that have been scaled into the instrument units. The index serves not only as a translational element between the two systems, but also confines the adjustment to the actual zone of elevations rather than simply applying the adjustment to a set of abstract numbers which in practice would each ordinarily be of considerably greater magnitude than its corresponding elevation.

The Solution of Simultaneous Equations

The first two formulas of Equation 1 give rise to two simultaneous observation equations for each horizontal control point in which the unknowns are the seven constants a ..... g. (The third formula gives rise to a single observation equation of a different set for each vertical control point where the unknowns consist of seven additional constants h ... n). Four or more horizontal stations result in eight or more observation equations in seven unknowns. A least-squares solution applies in which seven normal equations are first formed from the set of observation equations by classical methods and then solved uniquely using a routine commonly applied in Coast Survey geodetic work. The routine consists principally of Gaussian elimination incorporating the square-root technique. The system is described in detail in Technical Bulletin 21.[12] (See also Hildebrand.[13])

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[13] Introduction to Numerical Analysis by F.B. Hildebrand, McGraw-Hill Book Company, Inc., page 431, 1956.

[12] See note 3 above.

If a set of m observation equations have the form

$$a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n = a_{n+1} \, ,$$

(31)

each coefficient of the set of corresponding normal equations can be stated as

$$b_{ij} = \sum_{k=1}^{m} a_i \, a_j, \quad \begin{array}{l} i = 1 \ldots\ldots n \\ j = i \ldots\ldots (n+1) \end{array} \, .$$

(32)

in which i, j are the row and column numbers of the coefficient, m is the number of observation equations and n is the number of unknowns (as well as the number of normal equations). (Statement 23 of the Fortran program shown in Table 8). Inasmuch as the normal equation system is symmetric with respect to its principal diagonal, the terms below the diagonal are not computed nor are the spaces in the matrix used.

An auxiliary c-matrix is formed from the normal equation b-matrix in the _forward_ concept of the solution. For the terms in the first row

$$c_{ij} = b_{ij}/(b_{ii})^{1/2} \cdot \quad j = 1 \ldots (n+1)$$

(33)

The formula for the diagonal terms is stated explicitly although it is essentially identical to that for the general terms:

$$c_{ii} = [b_{ii} - \sum_{k=1}^{(i-1)} (c_{ki})^2]^{1/2}, \quad i = 2 \ldots\ldots n \, .$$

(34)

The general term is

$$c_{ij} = b_{ij} - \sum_{k=1}^{(i=1)} c_{ik}c_{ij} \, (c_{ii})^{1/2} \, ,$$

$$i = 2 \ldots n$$

$$j = (i + 1) \ldots (n + 1)$$

(35)

This completes the forward solution. It is accomplished by the Fortran Statements following number 23 through 99, which apply the formulas in a different order than stated here in order to exploit the facilities of the DO and IF instructions of the program language. Again, terms below the diagonal are not computed. The terms are stored in the same spaces as the normal equations, that is, c_{ij} is stored in the space formerly occupied by b_{ij} inasmuch as the latter is no longer needed.

The final <u>back</u> concept of the solution d is given by:

$$d_{n,n+1} = c_{n,n+1}/c_{nn} \tag{36}$$

$$d_{i,n+1} = (c_{i,n+1} - \sum_{k=1}^{(n-1)} c_{i-1,n+1}\, c_{in})/c_{ii}, \quad i = n-1,\, n-2,\, n-3\, \ldots\, 1. \tag{37}$$

(Statements 96 through 91 of Table 8). These are the new values in the last column that comprise the "answers" to the solution. Again they are stored in the same matrix space as the corresponding b and c terms. The coefficients of the observation equations are symbolized as A in the program whereas the b, c and d terms have the common designation EN.

6. Aerotriangulation with Independent Models

6.1 General Remarks

Numerous photogrammetrists and agencies have developed semi-analytical aerotriangulation methods and/or procedures. Most of them were not developed to full operational capacity, while others are incorporated into everyday production. Besides differences in mathematical approaches, especially in setting up rotational matrices and solving large systems of equations, there are either approximate solutions using polynomial adjustments or similar approaches or rigorous adjustments. Viewing the situation from the Canadian angle, I would like to discuss the NRC, EMR and Stuttgart methods - better known by the names of the people directly connected with them, namely Schut, Blais and Ackermann.

The basic idea is common to all methods. As the name indicates, each model is measured independently, which means that no absolute orientation is necessary. The model coordinates x, y, z of all the points are measured in the machine - or model coordinate system. A base change is not necessary, which makes the modern precision plotters (e.g. Kern PG2, PG3, Wild A8, A10, Zeiss Planimat etc.) suitable for this method. This means that practically the same accuracy can be obtained with less expensive instruments. One important fact that has to be kept in mind, is that the two projection centres are needed for the model connection and therefore to be measured and included in the list of model coordinates.

The connection of the models to strip- or block formations is then performed analytically within a least squares adjustment.

If only planimetric information is desired tie- or pass
points in the model corners are theoretically sufficient for model
connection. Additional tie points will strengthen the solution and
increase the accuracy. For three dimensional blocks it is essential
to also use the projection centres as pass points because of an
otherwise undetermined ϕ-tilt between adjacent models.
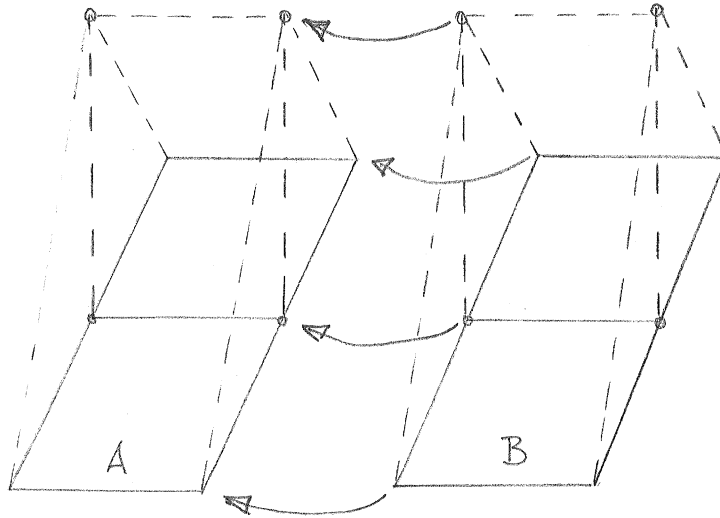
Before going into the details of the already mentioned
methods, I would like to name a few other approaches, just to give
you an idea of their varieties.
Inghilleri and Galetto (Italy) - rigorous solution with the following
three steps:

1) Determination of projection centres (LS adj. of space resection)

2) Relative orientation refinement (measurement of residual y-parallaxes - projective equations to refine models (angular elements only!) directional tangents in space).

3) Successive model linking (7 parameter solution). This method reduces the operator time by nearly a factor two, while increasing the computer time to perform step 2).

4) Strip Adjustment with any available program.

Weissman (Israel) - This is a development based on Inghilleri/Galetto's approach with using a 3D-coordinate transformation reduced to the centre of gravity for model linkage. Again the method ends at the strip formation stage and requires a strip or block adjustment afterwards.

Thompson (London, England) - has a different approach to the model linkage. He compares the vertices of triangles formed in two different models. These triangles are defined by the projection centre and two points located symmetrically with respect to the principal point so that their x-coordinates are zero. The unique feature of Thompson's method is the direct solution for 6 elements of the rotational matrix using linear equations. The remaining three may then be determined using orthogonality relationships. Thompson's approach was somewhat modified by Roelof (Delft, Holland).

6.2 Determination of Projection Centre Coordinates

There are several methods of determining projection centres. Several plotters, especially the ones where the projection centre coordinates change due to relative orientation, have special arrangements for simple determination.

Avoiding unnecessary duplication with your laboratory sheets, I would like to mention that most of the methods utilized fall into one of the following categories:

- space resection

- grid measurements in two planes

- direct observations with vertical space rods

- similarity transformation.

Since the first three are either self explanatory or explained in the lab sheet, I would like to present the last one as proposed by Dr. Ebner for use with PAT-M.

In this case, the projection centre coordinates are determined by stereo grid measurements. For this purpose, grid plates are centered very carefully in the left and right plate carriers. Then a relative orienation is performed. Afterwards the grid points 1-6 are measured and recorded. Using a spacial similarity transformation a true grid model is formed by the grid width "a" and the mean calibrated principal distance "c" is transformed onto the recorded grid model. The desired projection centres are then the transformed points 7 and 8 of the true grid model.
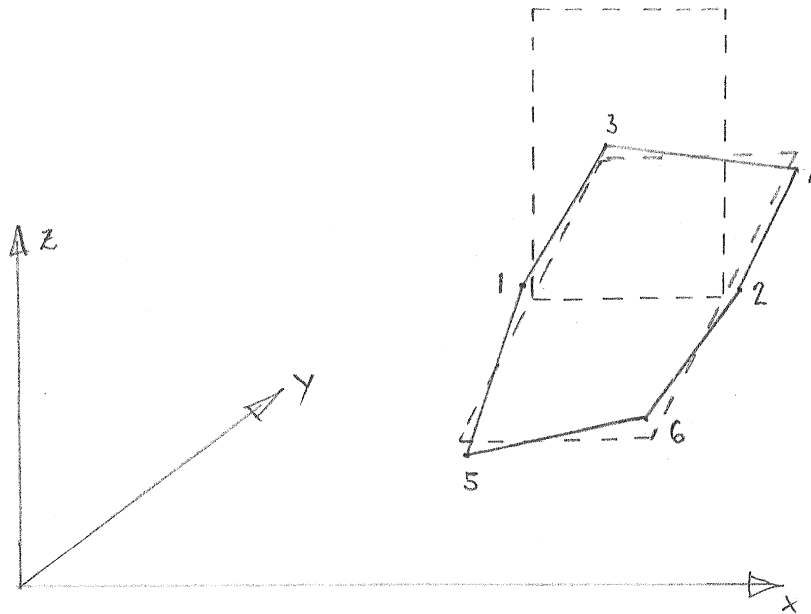
An advantage of this method is that the accuracy of the instrument can easily be derived from the residuals of the x, y, z-coordinates of the 6 grid points.

6.3 The NRC-Approach (Schut)

6.3.1 Formation of Strips from Independent Models

The model coordinate system of the first model is retained as the strip system. Subsequently, each following model in succession can be transformed to that system by connecting it to the preceeding model, using the following steps:

1) translation to make the coordinates of the common projection centre the same as in preceeding model

2) rotation which makes orientation of vectors from common projection centre to common points the same

3) scaling to give those vectors equal length.

Prior to Thompson's work, the coefficients of transformation for space rotation were non-linear functions of the rotation parameters and usually solved via an iteration process.

Based on Thompson's knowledge, that by proper selection of the parameters, linear equations are possible, Schut derived his own rotational equations. He was able to derive a set of four linear equations which are homogeneous with respect to four parameters. However, the derivation by means of matrix algebra leads directly only to three of them, and quaternion algebra is little known.

The best way of writing these formulae is as a transformation in a four dimensional space by means of 4x4 matrices. Then they can be obtained by matrix algebra.

The rectangular coordinate systems x, y, z have their origins in the common projection centre and a fourth variable t is associated with x, y and z. These four parameters are arranged in a special way as the elements of a matrix T:

$$T = \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

Similarly the elements a, b, c, d are arranged as the elements of a matrix D:

$$D = \begin{pmatrix} d & -a & -b & -c \\ a & d & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{pmatrix}$$

Any matrix arranged this way has the following characteristics:

Both column and row vectors are mutually orthogonal and have the same length. The determinant of the matrix is equal to the 4th power of the length of these vectors, therefore, except if all parameters are zero, the matrix is non singular. The product of two matrices of this type will be another matrix of this type, so is the inverse. Now the D matrices are restricted such, that their vectors have unit length:

$$|D| = (d^2 + a^2 + b^2 + c^2)^2 = +1$$

and

$$D^{-1} = D^T$$

Considering the four parameters of T as the coordinates of a point in 4D-space and the matrix D as a transformation matrix, the elements of the first column of their product matrix are the coordinates after rotation:

$$T' = DT$$

or in longhand:

$$t' = dt - ax - by - cz$$
$$x' = at + dx - cy + bz$$
$$y' = bt + cx + dy - az$$
$$z' = ct - bx + ay + dz$$

since

$$|T'| = |T| \text{ (due to } |D| = +1)$$

the vectors keep their length and therefore the scale is preserved and we have a pure rotation in 4D-space. However these equations have the undesired property that the transformed coordinates are not only functions of x, y and z but also of t which is undefined.

Another transformation (also a rotation) would be

$$T' = D_1 T D_2$$

by premultiplication with DT we get

$$D_1^T \, T^1 = T D_2$$

and in terms of the first column elements:

$$d_1 t' + a_1 x' + b_1 y' + c_1 z' = t d_2 - x a_2 - y b_2 - z c_2$$

$$-a_1 t' + d_1 x' + c_1 y' - b_1 z' = x d_2 + t a_2 - z b_2 + y c_2$$

$$-b_1 t' - c_1 x' + d_1 y' + a_1 z' = y d_2 + z a_2 + t b_2 - x c_2$$

$$-c_1 t' + b_1 x' - a_1 y' + d_1 z' = z d_2 - y a_2 + x b_2 + t c_2$$

We wish to select only those for which x', y', z' are independent of t.

If in 3D-space we require that x' and y' are independent of z, it implies a rotation around the z-axis, therefore, this selection implies a rotation around the t-axis, which means

$$\underline{t' = t.}$$

By inspection it was found out that this independence is obtained if $D_2 = D_1^T$. The the original transformation becomes

$$T' = D T D^T$$

giving the following linear equations

$$(x'-x)a + (y'-y)b + (z'-z)c \qquad\qquad = 0$$

$$- (z'+z)b + (y'+y)c + (x'-x)d = 0$$

$$(z'+z)a \qquad\qquad - (x'+x)c + (y'-y)d = 0$$

$$-(y'+y)a + (x'+x)b \qquad\qquad + (z'+z)d = 0$$

Since t remains unchanged, the rotation is restricted to the x, y, τ
subspace of the 4-D space. The latter equations are therefore used
for the computation of the parameters of the rotation

$$T' = DTD^T .$$

The connection of a model of a strip to the previous model
requires the coordinates of the common projection centre and of at
least 2 other points. From the reduced coordinates of each point
(to common proj. centre) four linear equations can be obtained. A
least squares adjustment is used to solve for the rotational parameters
a, b, c and d. In this case, the normal equations become very simple
functions of those linear equations. The elements s_{ij} of the normal
equations become:

$$s_{11} = \Sigma \ [(x'-x)^2 + (y'+y)^2 + (z'+z)^2]$$

$$s_{22} = \Sigma \ [(x'+x)^2 + (y'-y)^2 + (z'+z)^2]$$

$$s_{33} = \Sigma \ [(x'+x)^2 + (y'+y)^2 + (z'-z)^2]$$

$$s_{44} = \Sigma \ [(x'-x)^2 + (y'-y)^2 + (z'-z)^2]$$

$$s_{12} = s_{21} = -2\Sigma yx' + y'x$$

$$s_{13} = s_{31} = -2\Sigma xz' + x'z$$

$$s_{23} = s_{32} = -2\Sigma zy' + z'y$$

$$s_{14} = s_{41} = 2\Sigma zy' - z'y$$

$$s_{24} = s_{42} = 2\Sigma xz' - x'z$$

$$s_{34} = s_{43} = 2\Sigma yx' - y'x$$

From the normal equations only the ratio of the four parameters can be determined. Therefore, one is set equal to 1 and the others are obtained by solving the equations.

If d = 1 then a and b become finite, since they are related to longitudinal and transversal tilts. The parameter c is related to a rotation in the horizontal plane and becomes infinitely large for a rotation of 180°. In this case, c = 1 would have to be chosen.

With this the following simple algorithm can be used:

$$s'_{22} = s_{22} - s_{12}s_{12}/s_{11}$$

$$s'_{23} = s_{23} - s_{13}s_{12}/s_{11}$$

$$s'_{24} = s_{24} - s_{14}s_{12}/s_{11}$$

$$s'_{33} = s_{33} - s_{13}s_{13}/s_{11} - s'_{23} \cdot s'_{23}/s'_{22}$$

$$s'_{34} = s_{34} - s_{14}s_{13}/s_{11} - s'_{24}s'_{23}/s'_{22}$$

$$s'_{44} = s_{44} - s_{14}s_{14}/s_{11} - s'_{24}s'_{24}/s'_{22}$$

Then

$$d = 1 \quad \text{and} \quad c = -s'_{34}/s'_{33}$$

or

$$c = 1 \quad \text{and} \quad d = -s'_{34}/s'_{44}$$

and

$$b = -(s'_{24}d + s'_{23}c)/s'_{22}$$

$$a = -(s_{14}d + s_{13}c + s_{12}b)/s_{11}$$

Now all the elements of D are known and the <u>rotation</u> can be performed <u>for each point</u> according to

$$T' = DTD^T .$$

Since this involves a new T-matrix for each point and therefore many computational operations, it is better to transform all coordinates as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \lambda R \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x'_p \\ y'_p \\ z'_p \end{pmatrix}$$

where λ = scale factor and R, as expressed in terms of D:

$$R = \begin{pmatrix} d^2+a^2-b^2-c^2 & 2ab - 2cd & 2ac + 2bd \\ 2ab + 2cd & d^2-a^2+b^2-c^2 & 2bc + 2ad \\ 2ac - 2bd & 2bc + 2ad & d^2-a^2-b^2+c^2 \end{pmatrix} \frac{1}{d^2+a^2+b^2+c^2}$$

and x'_p, y'_p and z'_p are the coordinates of the common projection centre in the model to which the correction is made.

6.3.2 Strip and Block Adjustment (Schut)

The adjustment of a strip is performed in three steps:

1) Similarity transformation (scaling, rotating and translating strip with the aid of specified ground control points)

2) Height adjustment

3) Planimetric adjustment.

The block adjustment is an iterative solution in which each strip in turn is transformed. For computational purposes, both systems (ground-coordinates and strip coordinates) are shifted such that the origin lies inside the strip, e.g. first control point.

1) Similarity transformation

Since translations are considered at a later stage, the similarity transformation can be written as:

$$\begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (1)$$

The subscript "4" denotes the fact that this rot. matrix is built up in four steps, while A is this rotational matrix multiplied by a scale factor.

These step by step procedures to determine the rotation starts with a similarity transformation of the x and y coordinates.

Using complex numbers, this transformation can be expressed as

$$(x_1 + iy_1) = (x + iy) + (e_3 + ie_4)(x + iy) \qquad (2)$$

and separating real and imaginary terms, while multiplying z with the obtained scale factor, we get

$$x_1 = (1 + e_3) + \qquad - e_4 y$$

$$y_1 = \qquad e_4 x + (1 + e_3)y \qquad (2')$$

$$z_1 = \qquad \lambda z$$

where

$$\lambda = \sqrt{(1 + e_3)^2 + e_4^2}$$

or in matrix notation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = R_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{with} \quad R_1 = \begin{pmatrix} 1+e_3 & -e_4 & 0 \\ e_4 & 1+e_3 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad (2'')$$

The second step is a levelling of the strip

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = R_2 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

where

$$R_2 = \begin{pmatrix} 1 - \frac{1}{4} b_1^2 + \frac{1}{4} c_1^2 & -\frac{1}{2} b_1 c_1 & -b_1 \\ -\frac{1}{2} b_1 c_1 & 1 + \frac{1}{4} b_1^2 - \frac{1}{4} c_1^2 & -c_1 \\ b_1 & c_1 & 1 - \frac{1}{4} b_1^2 - \frac{1}{4} c_1^2 \end{pmatrix} \quad (3)$$

The matrix R_2 were orthogonal except for the omittance of a factor $1 + \frac{1}{4} b_1^2 + \frac{1}{4} c_1^2$, which means that with this rotation a small scale change takes place.

b_1 and c_1 correct for longitudinal and transversal tilts. They are computed from linear formulae extracted from the previous matrix by neglecting squared and mixed terms. Therefore, for large tilts, the levelling is only approximate, a fact which is corrected in the final height adjustment.

Now again a planimetric similarity transformation takes place

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = R_3 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad (4)$$

where R_3 is constructed in the same manner than R_1. This is to give the strip the correct scale for the final height adjustment and is needed for large tilts and to correct for the scale change introduced with R_2.

Finally, if a correction for longitudinal curvature or torsion is to be applied, the x,y,z origin has to be placed mid way between two points which define the strip axis. This has to be done at the initial stage when shifting coordinate systems. After all the mentioned transformations, the origin remains the same, but the strip is rotated around the z-axis to get these two points into the x/z plane, which means that the new coordinates are related ot the axis of flight.

$$\begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = R_4 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

with

$$R_4 = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (5)$$

$$\cos \alpha = \frac{x_3}{\sqrt{x_3^2+y_3^2+z_3^2}} \qquad \text{and} \qquad \sin \alpha = \frac{y_3}{\sqrt{x_3^2+y_3^2+z_3^2}}$$

With this the similarity transformation is complete:

$$A = R_4 R_3 R_2 R_1 \qquad (6)$$

2) Height Adjustment

$$x_5 = x_4 - z_4 b_1 - 2x_4 z_4 b_2 - 3x_4^2 z_4 b_3 - \ldots \ldots -2x_4 z_4 d_1$$

$$y_5 = y_4 - z_4 c_1 - x_4 z_4 c_2 - x_4^2 z_4 c_3 - \ldots \ldots -2y z_4 d_1 \tag{7}$$

$$z_5 = z_4 + x_4 b_1 + x_4^2 b_2 + x_4^3 b_3 + \ldots$$

$$+ y_4 c_1 + x_4 y_4 c_2 + x_4^2 y_4 c_3 + \ldots + (x_4^2 + y_4^2) d_1$$

earth curvature
correction

b_1 and c_1 serve for final tilt corrections. The other b and c

terms serve for curvature and torsion corrections.

3) Planimetric Adjustment

At this stage the system has to be returned to the ground

coordinate system from the flight-axis system.

$$x_6 = \cos \alpha \, x_5 - \sin \alpha \, y_5$$

$$y_6 = \sin \alpha \, x_5 + \cos \alpha \, y_5 \tag{8}$$

Then the planimetric adjustment, again disregarding the translations,

is performed

$$(x_7 + iy_7) = (x_6 + iy_6) + (e_3 + ie_4)(x_6 + iy_6)$$

$$+ (e_5 + ie_6)(x_6 + iy_6)^2 \tag{9}$$

$$+ (e_7 + ie_8)(x_6 + iy_6)^3 + \ldots$$

So far the translations e_1 and e_2 in the planimetric trans-

formations and a_1 in the height computations have been omitted. They

are computed however together with the coefficients of these formulae.

Then, instead of employing them to the transformation formulae, they

are added to the ground control system directly after computation.

Therefore the ground control origin shifts during the procedure.

As a final step, they are all added:

$$x_{tr} = x_7 + c_E + e_1^{(2")} + e_1^{(4)} + e_1^{(9)}$$

$$y_{tr} = y_7 + c_N + e_2^{(2")} + e_2^{(4)} + e_2^{(9)}$$

$$z_{tr} = z_5 + c_H + a_1^{(3)} + a_1^{(7)}$$

transformed   initial shifts in easting,
              northing and height.

For the parameter computation, each control pt. supplies one or more condition equations, stating that the transformed strip coordinates should be equal to the known ground coordinates. If the same program is used for block adjustment, each tie point provides three additional conditions, stating that the transformed coordinates should be equal to those of an earlier transformed overlapping strip. The following correction equations are: <u>for planimetry</u>:

$$(e_1+ie_2) + (x+iy)(e_3+ie_4) + (x+iy)^2(e_5+ie_6) + \ldots$$

$$= \underbrace{(E - iN) - (x + iy)}_{\text{orig. ground coords.}}$$

<u>for heights:</u>

$$a_1 + xb_1 + x^2b_2 + x^3b_3 + \ldots$$
$$+ yc_1 + xyc_2 + x^2yc_3 + \ldots = H - Z - (x^2 + y^2)d_1$$

6.4 <u>The Stuttgart Program PAT-M</u>

Although this program package is generally associated with the name Ackermann, I would like to attach two other names as well, namely Ebner and Klein.

Ackermann, as director of the Stuttgart University Institute for Photogrammetry was the initiator of this program and supplied the theoretical background. Ebner performed numerous detail investigations into photogrammetric problems and supplied the final formualtion, while Klein wrote and optimized the computer program. The name stands for Program Aerial Triangulation, independent Models.

Mathematical Formulation

The mathematical system is based on the relationship between the terrain point i and its measured model coordinates in the model j. Each model is subjected to a similarity transformation during the block adjustment process which performs it simultaneously for all models, taking tie- and control points into account appropriately.

The following non-linear observation equations are used:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{ij} = -\lambda_j R_j \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{ij} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}_j + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i \tag{1a}$$

where

$(x\ y\ z)_{ij}^T$ = vector of model coordinates of point i in model j.

$(X\ Y\ Z)_i^T$ = vector of unknown ground coordinates of point i.

$(v_x\ v_y\ v_z)_{ij}^T$ = vector of corrections associated with the transformed point i in model j.

these include the 7 unknown orientation parameters for model j.
$(X_0\ Y_0\ Z_0)_j^T$ = origin of model coordinate system j.

λ_j = scale factor for model j.

R_j = orthogonal rotation matrix for model j.

A modified version of the Rodrigues - Caylay matrix is used as the rotational matrix R_j:

$$R_j = \frac{1}{k} \begin{pmatrix} 1 + 1/4(a^2-b^2-c^2) & -c + 1/2\ ab & b + 1/2\ ac \\ c + 1/2\ ab & 1 + 1/4(-a^2+b^2-c^2) & -a + 1/2\ bc \\ -b + 1/2\ ac & a + 1/2\ bc & 1+1/4(-a^2-b^2+c^2) \end{pmatrix} \quad (1b)$$

where

$$k = 1 + \frac{1}{4}(a^2+b^2+c^2)$$

and a, b, c = three independent rotation parameters.

As already mentioned, there are 7 unknown parameters per model (a, b, c, λ, X_o, Y_o, Z_o) and 3 unknown coordinates per terrain point.

The observed model coordinates $(x, y, z)_{ij}$ can be weighted. The following weight coefficient matrix, scaled to terrain units can be introduced:

$$Q_{(ij)(ij)} = \begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix} \quad (1c)$$

Since the model coordinates are treated as uncorrelated,

$$Q_{(ij)(k\ell)} = 0 \ .$$

For computational reasons it is not practical to weight each individual point. Therefore one set of weight coefficients is used for all model points, another one for the projection centers.

The coordinates of the ground control points are not ncessarily kept fixed during the adjustment. This means that these coordinates are also treated as observations which leads to an additional set of observation equations, namely

$$
\begin{pmatrix} v_x^c \\ v_y^c \\ v_z^c \end{pmatrix}_i = - \begin{pmatrix} x^c \\ y^c \\ z^c \end{pmatrix}_i + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i \qquad (2)
$$

where

$(X_c \ Y_c \ Z_c)_i^T$ = vector of terrestrial coordinates of control point i

$(V_x^c \ V_y^c \ V_z^c)_i^T$ = vector of corrections associated with the terrestrial

coordinates of control point i

$(X \ Y \ Z)_i^T$ = vector of unknown terrain coordinates of (control)

point i.

This last vector ties equations (2) with equations (1a).

The terrestrial coordinates of the ground control points are again weighted according to equation (1c). It is, however, possible to assign different weights to each individual control point, which is desirable, since some are horizontal control points or vertical ones or both.

The observation equations (1a) are linearized with the aid of approximate values. For the rotation parameters a, b, c an initial zero value is used as approximation. The linearized error equations are then:

$$
\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{ij} = \begin{pmatrix} 0 & -z & y & -x \\ z & 0 & -x & -y \\ -y & x & 0 & z \end{pmatrix}_{ij} \begin{pmatrix} da \\ db \\ dc \\ d\lambda \end{pmatrix}_j - \begin{pmatrix} dX_o \\ dY_o \\ dZ_o \end{pmatrix}_j + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i - \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{ij} \qquad (3)
$$

The symbols have the same meaning as in (1a). da, db, dc and dλ represent increments to the parameters a, b, c and λ. x, y, z are the model coordinates as measured.

Since the linearization is based on approximate values, an iterative process is used for solving the problem. Therefore, x, y, z will become the corrected model coordinates of the previous iteration.

The block adjustment program which is directly based on equations (2) and (3) is called PAT-M7 (_P_rogram _A_erial _T_riangulation, independent _M_odels _7_ parameter transformations). The programming of this system is presently not yet optimized. Instead the version PAT-M43 was first completed, which iterates sequential horizontal and vertical adjustments, applying 4-parameter - and 3-parameter transformations respectively. By doing this, the computer time is reduced by a factor of 3. In order to accomplish that, equations (2) and (3) have to be replaced.

For the horizontal block adjustment, the following observation equations are used:

- for photogrammetrically measured model coordinates:

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}_{ij} = - \begin{pmatrix} x & -y \\ y & x \end{pmatrix}_{ij} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}_j - \begin{pmatrix} x_o \\ y_o \end{pmatrix}_j + \begin{pmatrix} X \\ Y \end{pmatrix}_i \qquad (4a)$$

and for ground control points:

$$\begin{pmatrix} v_x^c \\ v_y^c \end{pmatrix}_i = - \begin{pmatrix} X^c \\ Y^c \end{pmatrix}_i + \begin{pmatrix} X \\ Y \end{pmatrix}_i \qquad (4b)$$

In order to avoid confusion with equation (1b) the rotational parameters in equation (4a) are noted as $\bar{a}$ and $\bar{b}$.

The weight coefficients are the same as in the planimetric part of equation (1c).

The equations correspond with the "Anblock" approach, which in this case has the special advantage of being linear functions of the unknowns $(\bar{a}, \bar{b}, X_0, Y_0)_j$ and $(X, Y)_i$. Therefore no approximations and no iterations if the models have been levelled beforehand are necessary.

The projection centres are <u>not</u> included in the determination of the horizontal transformation parameters, because of disturbing effects on the convergence of the plan-height iteration cycle.

The observation equations for the vertical adjustment are non-linear. Their linearized form is for model tie points:

$$(v_z)_{ij} = (-y \quad x)_{ij} \begin{pmatrix} da \\ \\ db \end{pmatrix}_j - (dZ_0)_j + (Z)_i - (z)_{ij} \qquad (5a)$$

for height control points

$$(v_z^c)_i = -(Z^c)_i + (Z)_i \qquad (5b)$$

and for the projection centres:

$$\begin{pmatrix} v_x^{PC} \\ v_y^{PC} \\ v_z^{PC} \end{pmatrix}_{ij} = \begin{pmatrix} 0 & -z \\ z & 0 \\ -y & x \end{pmatrix}_{ij} \begin{pmatrix} d_a \\ d_b \end{pmatrix}_j - \begin{pmatrix} 0 \\ 0 \\ dZ_0 \end{pmatrix}_j + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_i - \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{ij} \qquad (5c)$$

The increments da and db refer to the same parameters a, b as in equations (1b) and (3). The weighting again corresponds to equation (1c).

The program PAT-M43 starts with a horizontal adjustment which does not require any approximations. With the measured model coordinates a rigorous least squares solution using equations (4a) and (4b) leads to the transformation parameters $(\bar{a}, \bar{b}, X_0, Y_0)$. Subsequently all the horizontal coordinates of each model are transformed with these parameters while the heights are corrected for scale.

These "new" model coordinates are entered into the vertical adjustment utilizing equations (5a-c). A straightforward least square solution results in the increments $(da, db, dz_o)_j$ for the respective transformation parameters, which are then used to transform the model coordinates rigorously with the complete spacial similarity transformation formula.

The model coordinates $(x, y, z)_{ij}$ refer therefore always to the latest stage of transformation, which means that an iteration process can be restarted after any iteration.

Each iteration repeats the plan-height sequence of adjustment until sufficient convergence is reached. This occurs usually after three steps. Then the final terrain coordinates for the unknown points are computed together with residual errors at tie and control points.

6.5 The E.M.R. Program (Blais)

This approach is a 7 parameter simultaneous solution and in concept quite closely related to PAT-M-7. It has the big advantage, that it is free of charge for Canadian users. There are several restrictions and original assumptions, namely: The photogrammetric data for each stereo model is assumed to have been corrected for all known systematic errors, e.g. earth curvature, lens distortion, refraction. This means the photogrammetric model data are assumed to be affected only by uncorrelated random errors.

The ground control coordinates are assumed error free for adjustment purposes. (Manual removal of questionable ground control is necessary!)

Since the transformation parameters are non-linear, initial
approximations are necessary. Unlike PAT-M, this is not an iteration
procedure, therefore better approximations of the rotational angles are
required. Blais specifies 2-3 degrees, which is not always the case,
especially in κ! He therefore suggests an initial correction using
an azimuth reading of the air base on the flight index map. (2-3
degrees is even then quite difficult to achieve). These restrictions
are somewhat overcome by the fact that the output file has the same
format as the input file. Another run with the output of the previous
one would then be needed.

The basic equations for the transformation are:

$$X = ax + by - cz + e$$

$$Y = ay - bx - dz + f$$

$$Z = az + cx + dy + g$$

where X, Y, Z are the terrestrial coordinates, x, y, z the model
coordinates and a-g, 7 unknown parameters. With some mathematical
manipulations this leads to

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \end{pmatrix}$$

where R is a rotational matrix, e.g. one like the one used in PAT-M-7.

Since it is rather sensitive to the magnitude of the rotations -
which is not critical in case of an iteration approach - Blais used a
somewhat different formulation for his program SPACE-M. Again his
initial equations are:

$$X = ax + by - cz + e$$

$$Y = ay - bx - dz + f \qquad\qquad (1)$$

$$Z = az + cx + dy + g$$

ground
system   (x, y, z) = model coordinates

First a planimetric similarity is considered:

$$X = \alpha x + \beta y + \xi$$

$$Y = \alpha y - \beta y + \eta$$

The scale in this case is

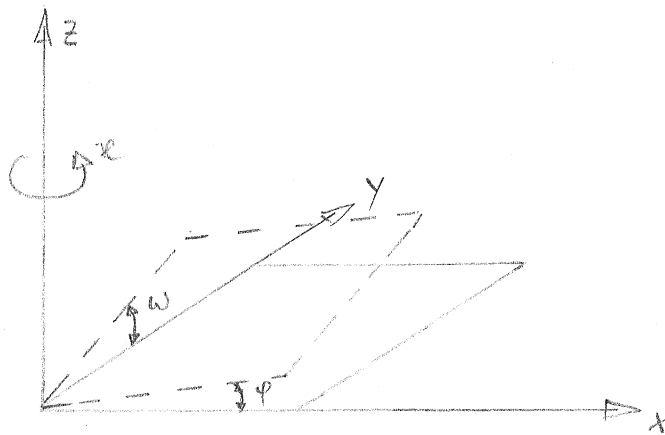$$\lambda = \sqrt{\alpha^2 + \beta^2}$$

while the rotation angle

$$\kappa = arc \tan \frac{-\beta}{\alpha}$$

(counter clockwise about z-axis).

Next he considers a plane, approximately parallel to the x, y-plane:

$$Z = \gamma x + \delta y + \zeta$$

The two levelling angles are (as measured from x, y- plane)

$$\phi = \text{arc tan } \gamma \quad \text{(around y-axis)}$$

$$\omega = \text{arc tan } \delta \quad \text{(around x-axis)}$$

In the 3D x, y, z space these inclinations of the x/y plane lead to:

$$z_{\text{rotated}} = z + \gamma x + \delta y$$

which has the following (linearized) effects on x and y:

$$x_{\text{rot.}} = x - \gamma z$$

$$y_{\text{rot.}} = y - \delta z$$

as obtained using the orthogonality of the x, y, z-system. Using these facts and the linear approximation of the scale:

$$\sqrt{\alpha^2 + \beta^2} \approx \alpha$$

he sets:

$$\lambda = \sqrt{a^2 + b^2}$$

$$\kappa = \text{arc tan } -b/a$$

$$\phi = \text{arc tan } c/\lambda$$

$$\omega = \text{arc tan } d/\lambda$$

Normally, the scale would be estimated from a 3D-volume change, however Blais indicated that the planimetric scale factor is advantageous. He also applies the rotations in the order ϕ, ω, κ and obtains the following transformation:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = (a^2+b^2)^{1/2} \begin{pmatrix} \kappa_1 & -\kappa_2 & 0 \\ \kappa_2 & \kappa_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_1 & -\omega_2 \\ 0 & \omega_2 & \omega_1 \end{pmatrix} \begin{pmatrix} \phi_1 & 0 & -\phi_2 \\ 0 & 1 & 0 \\ \phi_2 & 0 & \phi_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e' \\ f' \\ g' \end{pmatrix}$$

where the subscripts 1 and 2 correspond to the cosine and sine of the angle respectively.

These functions have not to be used explicitly, since

$$\kappa_1 = \cos \kappa = a/\lambda$$

$$\kappa_2 = \sin \kappa = -b/\lambda$$

$$\phi_1 = \cos \phi = a/\kappa_1 \ (a^2+b^2+c^2)^{1/2}$$

$$\phi_2 = \sin \phi = c\phi_1$$

$$\omega_1 = \cos \omega = a/\kappa_1 \ (a^2+b^2+d^2)^{1/2}$$

$$\omega_2 = \sin \omega = d\omega_1$$

The observation equations for a control point (X, Y, Z) with the photogrammetric model coordinates (x_i, y_i, z_i) in the ith model are

$$X = a_i x_i + b_i y_i - c_i z_i + e_i$$

$$Y = a_i y_i - b_i x_i - d_i z_i + f_i \qquad (2)$$

$$Z = a_i z_i + c_i x_i + d_i y_i + g_i$$

If the control is horizontal only, Z obtains zero weight while for a vertical control point X and Y obtain zero weight. Unknown tie or pass points, appearing in models j and k have the equations

$$X' = a_j x_j + b_j y_j - c_j z_j + e_j$$

$$Y' = a_j y_j - b_j x_j - d_j z_j + f_j \qquad (3)$$

$$Z' = a_j z_j + c_j x_j + d_j y_j + g_j$$

and also

$$X' = \cdots$$
$$Y' = \cdots \qquad \text{similar with subscript k at all parameters.}$$
$$Z' = \cdots$$

6.6 Comparison of the Three Approaches

a) Availability:

Schut's program is generally available and suitable for medium size (even smaller) computers. It is free of charge.

PAT-M-43 costs approximately $15,000 to $20,000 and is available and tested for both IBM and CDC computer systems. EMR and LRIS possess copies, the latter being installed at U.N.B. with the Department of Surveying Engineering having user rights. Any federal or provincial projects can be executed while the copyright does not permit the use of the program for private industry.

SPACE-M is principally available for all Canadian users free of charge. Presently only a CDC version is available, however, a conversion to the IBM system is planned.

Both PAT-M and SPACE-M require large computers!

b) Control Requirements and Accuracies

If there is a dense ground control, there is virtually no difference in accuracy. This, however is not really the purpose of aerotriangulation. For proper use, Schut's approach requires control in every strip, both vertical and horizontal.

To give you an indication on what can happen, I shall present you some results obtained by EMR, using the Calgary Test block and also some of my own experiences. The accuracy and results of Schut's approach vary with the degree of polynomial, while with PAT-M and SPACE-M one set of results is obtained. Schut's results are also very dependent on the control of the first strip!

EMR published some results on the Calgary Test Block:

This is a block of 142 models of wide angle photography at 1:27,500 scale flown over a mixed rural and urban area. Measurements were made on a Wild STK-1 Stereocomparator and independent models formed analytically. There are 19 horizontal control points and 376 vertical control points distributed fairly evenly in the 9 lines of 15-16 models.

The different control configurations were done with the three adjustment programs:

APAAT - G. Schut's polynomial adjustment of strips;

PAT-M - Stuttgart planimetric-levelling adjustment of independent models;

SPACE-M - the author's spatial adjustment of independent models.

The independent models were corrected for systematic errors (earth curvature, atmospheric refraction and lens distortion) and in the case of APAAT, the strips formed analytically before the adjustment. The control configurations are shown graphically using the model index with the symbols:

▲ - horizontal control point

● - vertical control point

and any unused control point is a check point.

First, the results of adjustments with all the horizontal and vertical control points included are given. Then, for all other control configurations, only four horizontal control points were used, one in each corner of the block. The results of 3 adjustments using 4, 3 and 2 cross-chains of vertical control are given on the following

pages. It should be noted in the cases of 3 and 2 cross-chains, that in two cases there are two adjacent strips that do not contain any vertical control points. In practice, this situation is unacceptable unless the tie points between the two strips are laterally staggered since there is no condition to control the lateral tilt in each strip. The fact that the results are good in our case is due to the presence of this condition.

CALGARY TEST BLOCK

ALL CONTROL INCLUDED
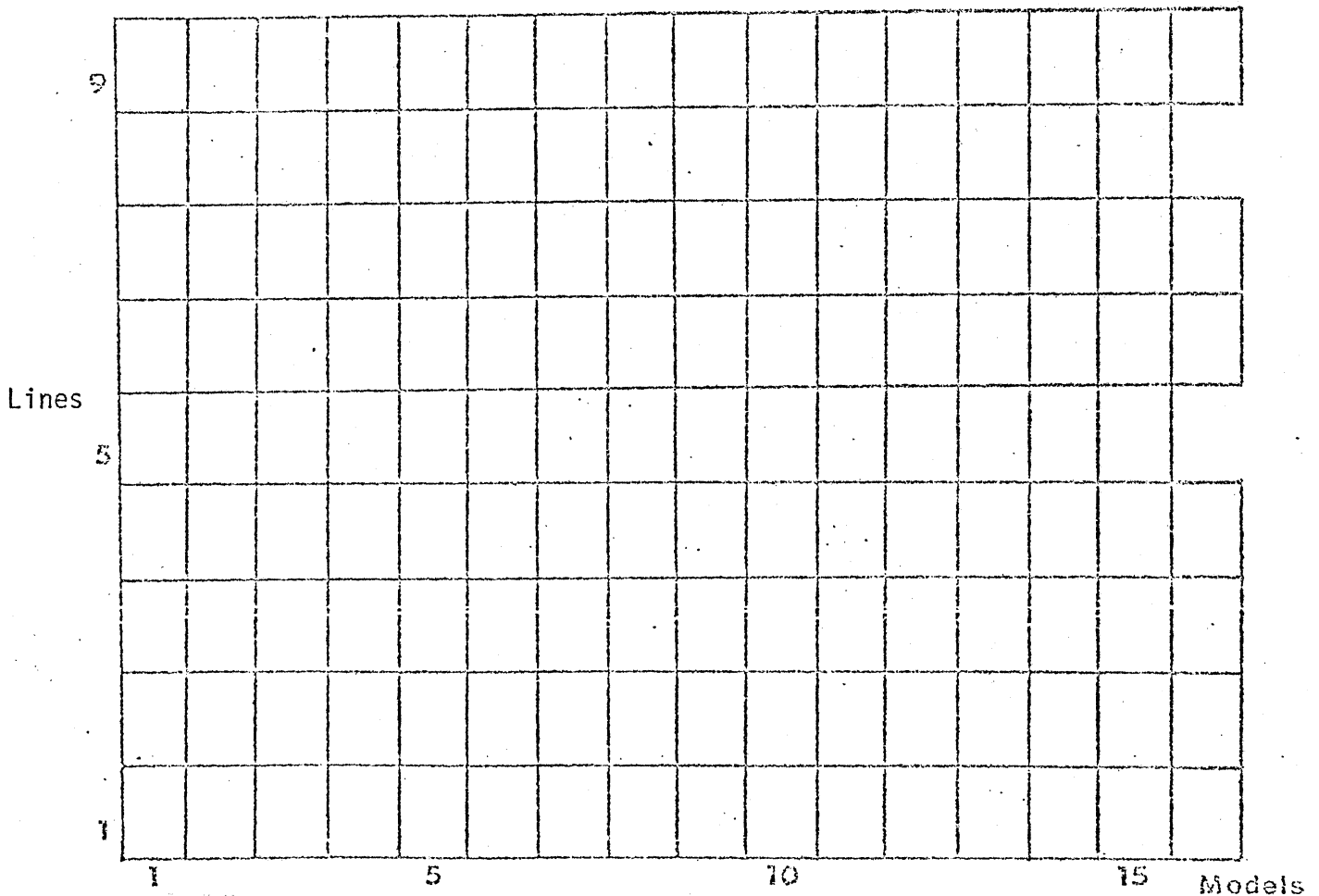
Control Points In X, Y, Z:

PAT- M	1.40 m	1.76 m	0.33 m
SPACE-M	1.18 m	1.38 m	0.53 m
APAAT (Schut)	1.57 m	1.91 m	0.76 m

CALGARY TEST BLOCK

4 BANDS OF VERTICAL CONTROL

CHECK POINTS IN X, Y, Z:

PAT-M	4.30 m	4.71 m	0.85 m
SPACE-M	4.72 m	4.99 m	0.85 m
APAAT	10.76 m	7.02 m	0.99 m

CALGARY TEST BLOCK

3 BANDS OF VERTICAL CONTROL
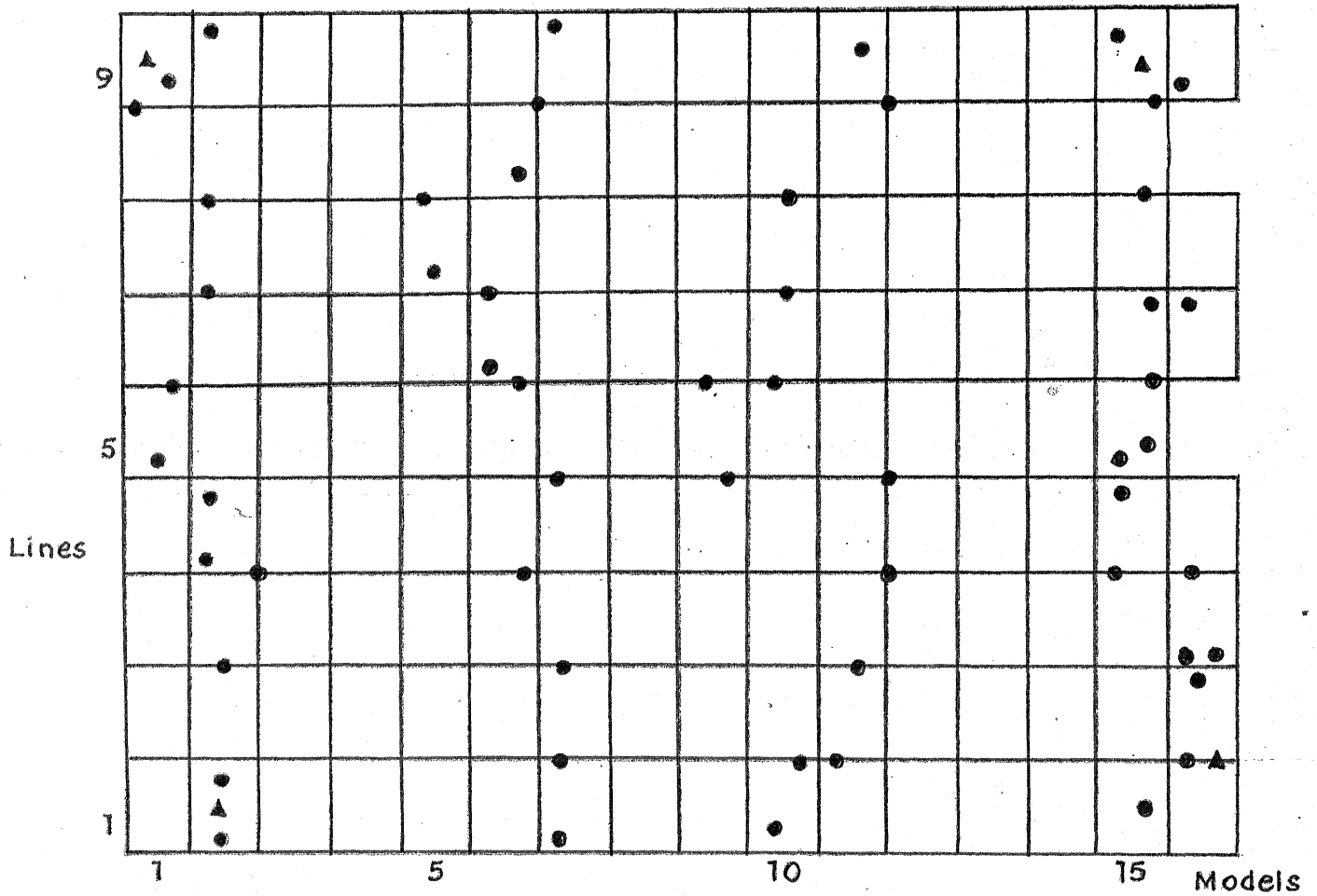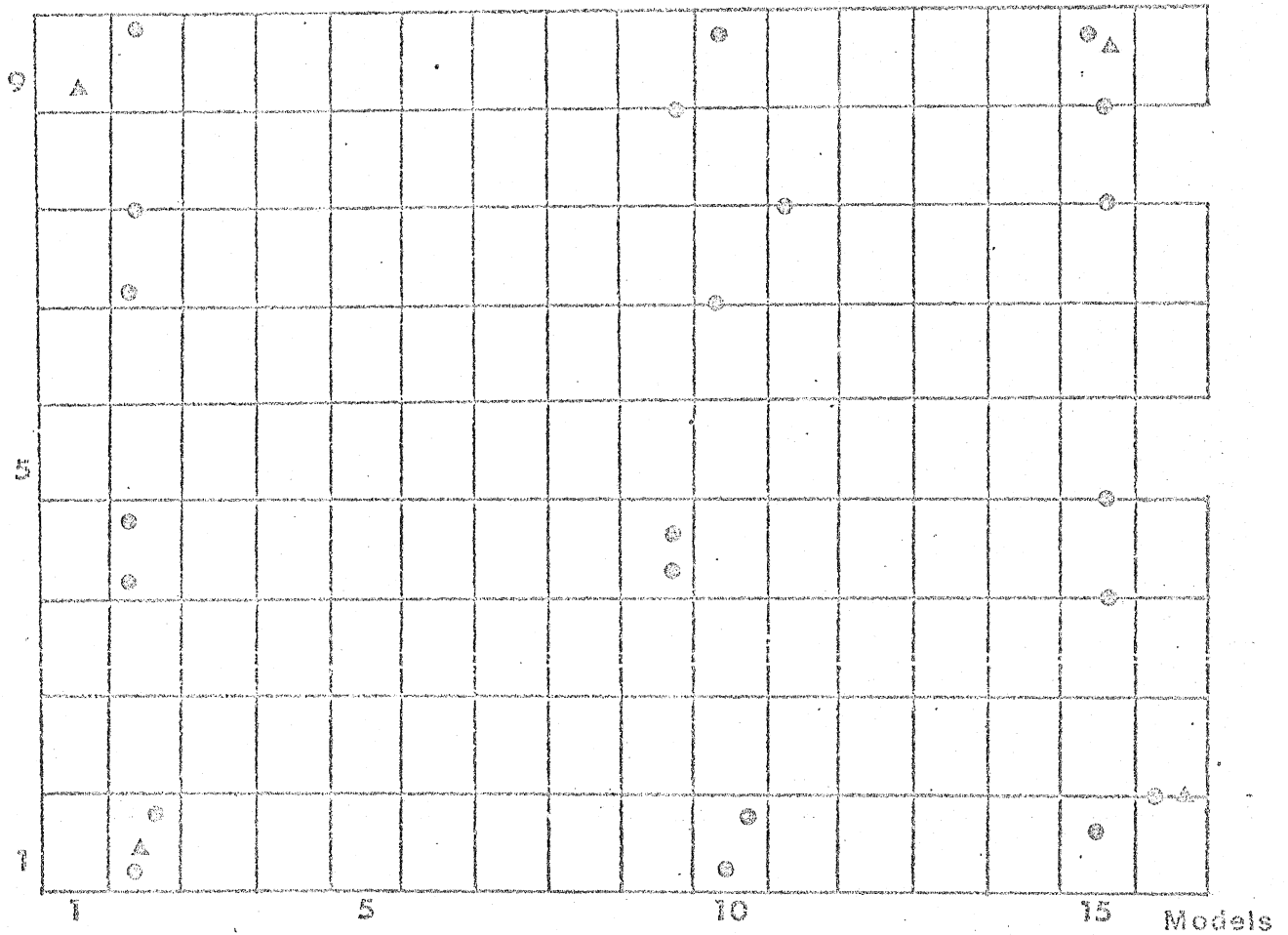
CHECK POINTS IN X, Y, Z:

PAT-M	4.80 m	4.71 m	1.13 m
SPACE-M	4.72 m	5.00 m	1.08 m
APAAT	10.76 m	7.01 m	1.04 m

CALGARY TEST BLOCK

VERTICAL CONTROL AT ENDS OF LINES
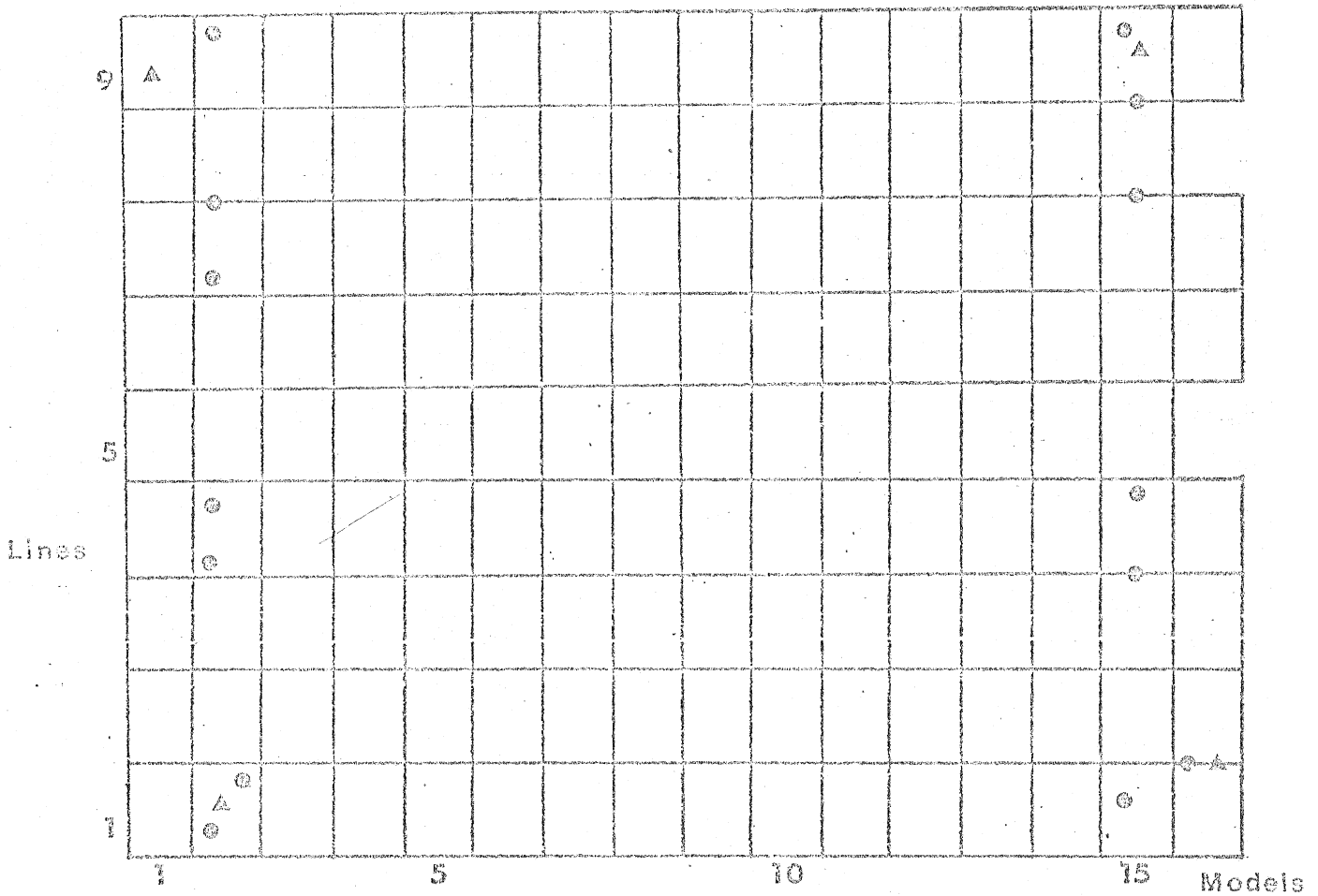
CHECK POINTS IN X, Y, Z:

PAT M	4.79 m	4.72 m	7.04 m
SPACE M	4.72 m	5.00 m	1.75 m
APAAT	10.75 m	7.02 m	2.32 m *

---

* Depends largely on control situation in 1st line (here 2 + 4) sequential procedure.

Standard deviations obtained with PAT-M-43 of

Block	Photo Scale	number of control points hor.	vert.	tie points σx	σy	σz	proj. centers σx	σy	σz	control pts. σx	σy	σz	σ hor.	σ vert.	With Schut's program A-8 results tie points σx	σy	σz	control pts. σz hor.	vert.
	7,200	15	80	0.43	0.98	0.43	0.61	2.16	1.10	0.17	0.18	0.35	1.08	0.82	3.41	1.99	1.33	0.76	0.82
	7,200	10	21	0.43	0.98	0.39	0.61	2.38	1.01	0.16	0.19	0.84	1.08	0.88	---	---	---	---	---
	7,200	5	14	0.43	0.98	0.38	0.56	2.39	1.01	0.16	0.38	0.91	1.09	0.85	---	---	---	---	---
	40,000	27	62	1.14	1.05	1.94	6.42	5.66	7.41	1.26	1.29	1.57	1.67	3.95	14.67	7.84	5.26	?	2.69
	40,000	10	21	1.12	1.03	2.17	6.24	5.58	6.67	0.86	0.91	1.99	1.65	4.96	---	---	---	---	---
	40,000	4	12	1.10	1.03	2.17	6.11	5.51	6.65	0.55	0.33	2.17	1.64	5.07	---	---	---	---	---

Results obtained at UNB

For rigorous solutions the following basic rules can be given:

Horizontal Control: Use control at the perimeter only! Control points inside the area do not improve the solution. Points in the four corners are usually sufficient; it is better however to have a control point at about every 8 base lengths.

Vertical Control: Although this question is still under investigation, I would like to mention that there is relatively little differences between Schut and PAT-M while SPACE-M appears to show some advantages in relatively ill-conditioned blocks.

Generally, rather dense cross-chains every 4-6 base-lengths are needed. An additional perimeter point provides significant improvement, while more than one provides no further improvement.

c) Generality:

With the above mentioned additions, PAT-M provides the most general approach. Even in its present form, it is much more general than both Schut and SPACE-M. It is independent of the strip, can accommodate cross-strips and flights of different dates, scales and directions which both is impossible with the other approaches. There are other restrictions as well, which are quite important for production work.

d) Auxiliary Vertical Control

Both SPACE-M and PAT-M (APR-version) incorporate lake surfaces as additional vertical control. In addition, PAT-M (APR-version) can handle Statoscope and APR data as well.

The following publication from The Canadian Surveyor 1972, provides the necessary details:

"Combined Block Adjustment of APR - Data and Independent Photogrammetric Models" by F. Ackermann, H. Ebner and H. Klein, Stuttgart University.

7. Bundle Adjustment

7.1 General Remarks

This is the most general method and is based on the measurement of image coordinates utilizing stereo- or mono comparators (+ point transfer devide). The image points together with the projection centres (given in the photo system by principal point and camera constant) define the bundles for each photograph. The orientation of all bundles relatively to each other and in relation to terrestrial control points is performed simultaneously by least squares. Using observation equations, the following non-linear error equations appear:

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}_{ij} = c_j \begin{pmatrix} \bar{x}/\bar{z} \\ \bar{y}/\bar{z} \end{pmatrix}_{ij} - \begin{pmatrix} x \\ y \end{pmatrix}_i \qquad (1)$$

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = R_j \begin{pmatrix} x_i - x_{o_j} \\ y_i - y_{o_j} \\ z_i - z_{o_j} \end{pmatrix} \qquad (2)$$

where

$[x, y]_{ij}^T$ = vector of image coordinates of point i in photo j reduced to principal point

$[v_x, v_y]_{ij}^T$ = correction vector

c_j = camera constant for bundle j

$[x_i \; y_i \; z_i]^T$ = vector of terrain coordinates of point i

$[x_{oj} \; y_{oj} \; z_{oj}]^T$ = vector of terrain coordinates of projection centre j

R_j = orthogonal rotational matrix for bundle j.

When using the Rodriguez matrix for R_j, it has the advantage that there are no trig. functions (see PAT-M), instead there are the three parameters

$$a, b, c.$$

These error equations are formed for all measured points. The unknowns are the six orientation parameters a_j, b_j, c_j, x_{oj}, y_{oj}, z_{oj} for each bundle j and the terrain coordinates of all points i.

Since these error equations are non-linear, linearization and iteration is required. There are again several bundle adjustment programs available. Ackermann's Institute in Stuttgart has PAT-B; D.C. Brown uses his own approach, which I believe, was one of the pioneer programs, the major breakthrough coming with skillful partitioning and forming of the normal equation matrix and its inversion. As an example, I would like to present to you the NOAA (formerly USCGS) approach, since it is readily available. It is published by Keller and Tewinkel as "Block Analytic Aerotriangulation".

7.2 <u>USCGS - Block Analytic Aerotriangulation</u>

This program uses as input <u>refined</u> image coordinates and provisional object coordinates. These provisional object coordinates are obtained after three initial steps:

1) Three photo orientations

2) Strip adjustment with polynomials

3) Coordinate transformation, taking into account the earth
   curvature.

The actual adjustment however is <u>not</u> influenced by the polynomial formulae used in step 2).

In step 1) the sequence of the input cards is tested. Multiple readings of image coordinates are averaged and tested for blunders. Then the image coordinates are corrected for symmetric and asymetric radial distortion, dimensional instability of the film and atmospheric refraction. Comparator calibration results can also be incorporated. Then each photo is oriented relative to the previous two in a strip and arbitrary model coordinates are formed. Residuals are analysed with automatic blunder rejection. The colinearity condition is imposed such that discrepancies of the observed image coordinates are minimized.

The three photo units are called triplets. Their use has been advocated but with the exception of the USCGS did not gain much recognition.

In step 2) the triplet coordinates of objects are transformed into the ground coordinate system by application of polynomial formulae, as discussed in chapter 5.2. The purpose of this is to obtain approximations for the subsequent adjustment.

In step 3) the approximate coordinates, which are usually in a state plane coordinate system or an equivalent are transformed into geocentric system. This is done in two steps, namely from X, Y, H, to ϕ, λ, H and then $\bar{X}$, $\bar{Y}$, $\bar{Z}$. After the block adjustment, these geocentric coordinates are transformed back into state plane coordinates, again via geographic coordinates. The block adjustment is basically a

"simultaneous resection", which is used to determine the three linear elements of position and the three angular elements of orientation of the exposure camera. Again, as in the triplet approach, the colinearity equations are applied, in this case simultaneously for all photos.

The following colinearity equations are used:

$$\frac{x}{z} = \frac{(X-X_0)a_{11} + (Y-Y_0)a_{12} + (Z-Z_0)a_{13}}{(X-X_0)a_{31} + (Y-Y_0)a_{32} + (Z-Z_0)a_{33}}$$

$$\frac{y}{z} = \frac{(X-X_0)a_{21} + (Y-Y_0)a_{22} + (Z-Z_0)a_{23}}{(X-X_0)a_{31} + (Y-Y_0)a_{32} + (Z-Z_0)a_{33}}$$

(1)

x and y are measured image coordinates; z the calibrated focal length (camera constant); X, Y, Z ground coordinates; X_0, Y_0, Z_0 unknown coordinates of exposure station and the a-values represent direction cosines, indicating the relative angular orientation of image and ground coordinate axes.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \text{fct. of } \phi, \omega, \kappa \qquad (2)$$

In equation (1) x, y, z are considered known, with x and y being subject to random errors of observation, while z is considered to be a fixed constant. All the other terms (ω, ϕ, κ, X_0, Y_0, Z_0, X, Y, Z) are unknowns to be determined. Only X, Y, Z ground coordinates of an object are of prime interest, while the other terms are necessary intermediate values. Step 2) provides approximate values for the desired ground coordinates while step 1) provides approximations for the six exterior orientation values of the bundle. Equations (1) are linearized to:

$$(p_{12}\Delta\omega + p_{13}\Delta\phi + p_{14}\Delta\kappa - p_{15}\Delta X_0 - p_{16}\Delta Y_0 - p_{17}\Delta Z_0 + p_{15}\Delta X + p_{16}\Delta Y + p_{17}\Delta Z)q = -p_{11}/q$$

$$(p_{22}\Delta\omega + p_{23}\Delta\phi + p_{24}\Delta\kappa - p_{25}\Delta X_0 - p_{26}\Delta Y_0 - p_{27}\Delta Z_0 + p_{25}\Delta X + p_{26}\Delta Y + p_{27}\Delta Z)/q = -p_{21}/q$$

(3)

where the coefficients $\underline{p_{ij}}$ are defined in determinant notation in Table 1:

Table 1

$$p_{11} = \begin{vmatrix} x & z \\ A_1 B & A_3 B \end{vmatrix} \qquad p_{21} = \begin{vmatrix} y & z \\ A_2 B & A_3 B \end{vmatrix}$$

$$p_{12} = \begin{vmatrix} x & z \\ \frac{\partial A_1}{\partial \omega} B & \frac{\partial A_3}{\partial \omega} B \end{vmatrix} \qquad p_{22} = \begin{vmatrix} y & z \\ \frac{\partial A_2}{\partial \omega} B & \frac{\partial A_3}{\partial \omega} B \end{vmatrix}$$

$$p_{13} = \begin{vmatrix} x & z \\ \frac{\partial A_1}{\partial \phi} B & \frac{\partial A_3}{\partial \phi} B \end{vmatrix} \qquad p_{23} = \begin{vmatrix} y & z \\ \frac{\partial A_2}{\partial \phi} B & \frac{\partial A_3}{\partial \phi} B \end{vmatrix}$$

$$p_{14} = \begin{vmatrix} x & z \\ \frac{\partial A_1}{\partial \kappa} B & \frac{\partial A_3}{\partial \kappa} B \end{vmatrix} \qquad p_{24} = \begin{vmatrix} y & z \\ \frac{\partial A_2}{\partial \kappa} B & \frac{\partial A_3}{\partial \kappa} B \end{vmatrix}$$

$$p_{15} = \begin{vmatrix} x & z \\ a_{11} & a_{31} \end{vmatrix} \qquad p_{25} = \begin{vmatrix} y & z \\ a_{21} & a_{31} \end{vmatrix}$$

$$p_{16} = \begin{vmatrix} x & z \\ a_{12} & a_{32} \end{vmatrix} \qquad p_{26} = \begin{vmatrix} y & z \\ a_{22} & a_{32} \end{vmatrix}$$

$$p_{17} = \begin{vmatrix} x & z \\ a_{13} & a_{33} \end{vmatrix} \qquad p_{27} = \begin{vmatrix} y & z \\ a_{23} & a_{33} \end{vmatrix}$$

The term $\underline{q}$ is defined as:

$$q = A_3B = [a_{31} a_{32} a_{33}] \begin{pmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{pmatrix}.$$

The terms on the right sides of Equation 3 are the residuals $\underline{v_x}$ and $\underline{v_y}$.

The partial derivative terms shown in Table 1 are defined with reference to Equation 2 as follows:

$$\frac{\partial A}{\partial \omega} = \begin{pmatrix} 0 & -\sin \omega \sin \kappa + \cos \omega \sin \phi \cos \kappa & \cos \omega \sin \kappa + \sin \omega \sin \phi \cos \kappa \\ 0 & -\sin \omega \cos \kappa - \cos \omega \sin \phi \sin \kappa & \cos \omega \cos \kappa - \sin \omega \sin \phi \sin \kappa \\ 0 & -\cos \omega \cos \phi & -\sin \omega \cos \phi \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -a_{13} & a_{12} \\ 0 & -a_{23} & a_{22} \\ 0 & -a_{33} & a_{32} \end{pmatrix}$$

$$\frac{\partial A}{\partial \phi} = \begin{pmatrix} -\sin \phi \cos \kappa & \sin \omega \cos \phi \cos \kappa & -\cos \omega \cos \phi \cos \kappa \\ \sin \phi \sin \kappa & -\sin \omega \cos \phi \sin \kappa & \cos \omega \cos \phi \sin \kappa \\ \cos \phi & \sin \omega \sin \phi & -\cos \omega \sin \phi \end{pmatrix}$$

$$\frac{\partial A}{\partial \kappa} = \begin{pmatrix} -\cos \phi \sin \kappa & \cos \omega \cos \kappa - \sin \omega \sin \phi \sin \kappa & \sin \kappa \cos \kappa + \cos \omega \sin \phi \sin \kappa \\ -\cos \phi \cos \kappa & -\cos \omega \sin \kappa - \sin \omega \sin \phi \cos \kappa & -\sin \omega \sin \kappa + \cos \omega \sin \phi \cos \kappa \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} \\ 0 & 0 & 0 \end{pmatrix}. \qquad (4a)$$

The subscript, as in $(\partial A_1/\partial\omega)B$ and $(\partial A_1/\partial\phi)B$ represents the matrix notation for the scalar product

$$[\partial a_{11}/\partial\omega, \ \partial a_{12}/\partial\omega, \ \partial a_{13}/\partial\omega] \begin{pmatrix} X-X_0 \\ Y-Y_0 \\ Z-Z_0 \end{pmatrix} \qquad (4b)$$

Since the linearization is based on approximation, an iterative process is applied which terminates when $\Delta\omega$, $\Delta\phi$, $\Delta\kappa$ are all less than 10^{-5} radian ($\approx 2''$).

First, however, resections are performed for each photograph to determine initial approximations for ω, ϕ, κ, X_0, Y_0, Z_0 for each photograph based on approx. coordinates of 18 object points. The error equation is a shorter form of equation (3):

$$(p_{12}\Delta\omega + p_{13}\Delta\phi + p_{14}\Delta\omega - p_{15}\Delta X_0 - p_{16}\Delta Y_0 - p_{17}\Delta Z_0)/q = -p_{11}/q$$

$$(5)$$

$$(p_{22}\Delta\omega + p_{23}\Delta\phi + p_{24}\Delta\omega - p_{25}\Delta X_0 - p_{26}\Delta Y_0 - p_{27}\Delta Z_0)/q = -p_{21}/q$$

The p_{ij} and q are the same as in equation (3).

After completion of the resection routine, the block adjustment observation equations are formed utilizing the resection data which are already in the computer memory.

The equations are solved by a modified Gauss-Cholesky elimination approach.

There are two types of weights:

1st for image coordinates as a fct. of the distance from the centre
    of the photograph due to the fact that resolution degrades with
    radial distance.

2nd control points.

The X, Y, Z ground coordinates of object points based on measured image coordinates, camera stations and orientation for two (or more) photographs and computed by intersection.

The colinearity equation (1) can also be expressed as

$$\begin{vmatrix} x & z \\ (A_1 \cdot B) & (A_3 \cdot B) \end{vmatrix} = 0 \quad (= p_{11})$$

or

$$(A_3 \cdot B)x - (A_1 \cdot B)z = 0$$
$$(A_4 \cdot B)y - (A_2 \cdot B)Z = 0 \qquad (6)$$

A photo residual v is defined as

$$v_x = x_{obs.} - x_{comp.} \qquad (7)$$

The computed coordinates are obtained according to (6) as:

$$x = z(A_1 \cdot B)/(A_3 \cdot B)$$
$$y = z(A_2 \cdot B)/(A_3 \cdot B) \qquad (8)$$

A combination of (7) and (8) gives:

$$v_x = x - z\,(A_1 \cdot B)/(A_3 \cdot B)$$
$$v_y = y - z\,(A_2 \cdot B)/(A_3 \cdot B) \qquad (9)$$

or

$$(A_3 \cdot B)v_x = (A_3 \cdot B)x - z(A_1 \cdot B)$$
$$(A_3 \cdot B)v_y = (A_3 \cdot B)y - z(A_2 \cdot B) \qquad (10)$$

Using the values of table 1, this leads to

$$v_x = (p_{15}X + p_{16}Y + p_{17}Z)/q - (p_{15}X_0 + p_{16}Y_0 + p_{17}Z_0)/q$$
$$v_y = (p_{25}X + p_{26}Y + p_{27}Z)/q - (p_{25}X_0 + p_{26}Y_0 + p_{27}Z_0)/q \qquad (11)$$

This is done after the actual block adjustment and since it is done in sequence, an unlimited number of points can be computed.

8. Conclusions and Outlook

The purpose of aerotriangulation, namely control densification, has been fulfilled. Modern rigorous block adjustments have reduced the horizontal control requirements to the perimeter of the block with rather large spacings. However, rather dense chains of vertical control points across the strips - except for 60% sidelap - are required at certain intervals unless auxiliary vertical control is utilized.

Several program systems have incorporated the latter and with proper statoscope use it is practically not necessary for mapping purposes to have vertical control points within the block, especially if lakes are also available. APR is also quite useful, especially when tied into known water elevation as it permits then to even reduce the vertical control at the both ends.

Finally, inertial navigation systems are now providing accuracies well within the mapping requirements and will in future play a bigger part, especially in difficult areas, e.g. for coastal mapping.

Many practical tests have shown that bundle adjustments hardly give better accuracies than independent models, a fact which contradicts theoretical expectations. By now it is apparent that this is caused by the fact that all the rigorous adjustments basically just model random errors. Systematic components, so adequately covered in interpolation type approaches (e.g. polynomials) are still present.

A new generation of block adjustment programmes uses additional parameters to model systematic errors which cause deformation of the photogrammetric models or images. It is thereby immaterial whether the causes or the effects are modelled, both procedures involve additional parameters and are either labeled as such or as "self calibration". The results for both bundles and independent models have been excellent and have reached the photogrammetric noise level of 3 to 4 μm in the photograph.

With this the present rapid development of aerotriangulation may well subside, although first attempts are being made to move from large computers to minicomputers at the expense of generality, at least presently.

Unfortunately there is no textbook covering modern aerotriangulation. The book "Phototriangulation" by S.K. Ghosh covers the analogue methods very well and gives many references. Otherwise the student is encouraged to read articles in Photogrammetric Engineering and Remote Sensing and other leading journals as well as the publications from such important centres of aerotriangulation as DBA-Systems (D.C. Brown), Stuttgart University (Ackermann), NRC (Schut), University of Illinois (Wong's SAPGO approach), and the governmental agencies in the United States (NOAA, U.S. Geological Survey) and Canada (EMR), and others.