SYSTEMATIC ANALYSIS OF DISTORTIONS IN MAP PROJECTIONS

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A SYSTEMATIC ANALYSIS OF DISTORTIONS IN MAP PROJECTIONS

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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

A SYSTEMATIC ANALYSIS OF DISTORTIONS IN MAP PROJECTIONS

Notes for course in Map Projections, Department of Surveying Engineering.

L. Hradilek and A. C. Hamilton

Preface

These notes are laid out with the intention that this booklet will serve as a notebook for the student as well as providing him with most of the material that will be presented in the lectures.

It is assumed that the student has complete mastery of the art of manipulating expressions in differential calculus, algebra and trigonometry; numerous opportunities are provided for him to practice this skill.

Acknowledgements

These notes have evolved from the lecture notes prepared by Dr. Gottfried Konecny in 1965 supplemented by material from Mathematical Cartography by F. Fiala.

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A SYSTEMATIC ANALYSIS OF DISTORTIONS IN MAP PROJECTIONS

Notes for a course in Map Projections in
the Department of Surveying Engineering
University of New Brunswick

INTRODUCTION

If we lived on a flat earth there would be no need for a course on map projections.

As it is we live on an ellipsoidal earth and there is no way that the curved surface of the earth can be portrayed on a flat sheet of paper without introducing some distortions. There are two aspects to this problem:

(a) The mapping problem. This is the problem of representing large areas, such as continents, on a flat piece of paper. There is no perfect solution to this problem; there are only compromise solutions that have acceptable distortions.

(b) The surveying problem. Survey measurements are made on the actual, i.e. the curved, surface of the earth; it is possible using geodetic formulae to do all computations using the curved surface as reference, however this would introduce unnecessary complications to most projects and in practice most survey projects are computed assuming some plane projection. When using a plane projection the surveyor must be aware of the distortion that occurs as a result of using the projection.

Hence in both mapping and surveying we have the problem of which projection to use and of understanding the distortions that are introduced by any projection that we do use.

There are many books written for geographers and others interested in mapping at medium and small scales where a qualitative treatment is adequate. There are also many treatises on the transformations between various projection systems; in fact, this is an integral part of the course on geometrical geodesy. The purpose of these notes is to study the inherent properties of projections quantitatively and to develop a systematic approach to the evaluation and classification of map projections.

In these notes analytical expressions for distortions in the general case are developed and applied to a representative selection of projections. It is suggested that the student refer to a descriptive text, such as Map Projections by H.S. Roblin, for illustrations and a qualitative discussion of the subject and to a textbook such as Coordinate Systems and Map Projections by D.H. Maling for more detailed treatment of the subject.

CHAPTER I

GENERAL EXPRESSIONS FOR DISTORTIONS

As distortions are inevitable it is necessary to deal with them quantitatively and from this to develop their significant characteristics. We will approach this problem by developing analytical expressions for various distortions starting with distortion in length. For these developments it is assumed that the curved surface - ellipsoid, or sphere, with coordinates ϕ and λ is being mapped onto a plane with coordinates X and Y or onto a developable surface such as a cylinder or a cone. Note that a cylinder or a cone can be mapped onto a plane without any distortion.

It is assumed that there exists a functional relationship

$$X = f(\phi,\lambda) \quad : \quad Y = g(\phi,\lambda) \qquad \text{I-1a}$$

and that these are mathematically well-behaved functions having derivatives

$$dX = \frac{\partial f}{\partial \phi}\, d\phi + \frac{\partial f}{\partial \lambda}\, d\lambda \;:\; dY = \frac{\partial g}{\partial \phi}\, d\phi + \frac{\partial g}{\partial \lambda}\, d\lambda$$

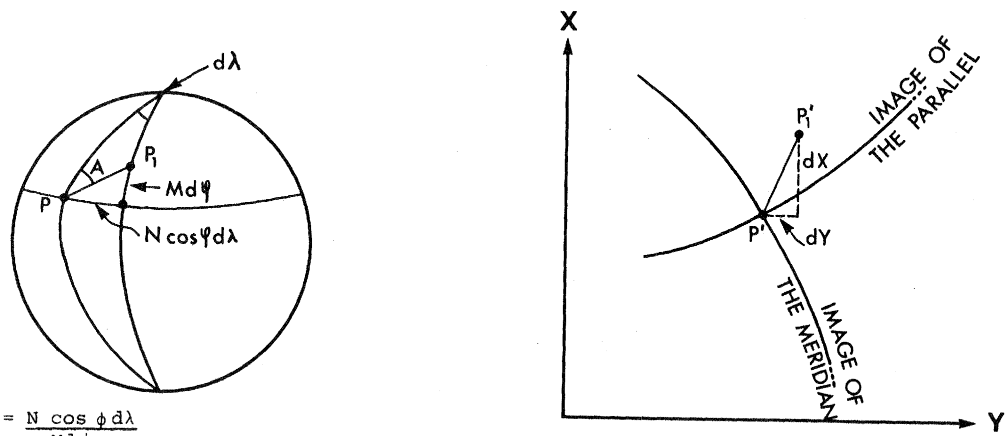
$$\text{or} \qquad dX = f_\phi\, d\phi + f_\lambda\, d\lambda \;:\; dY = g_\phi\, d\phi + g_\lambda\, d\lambda \qquad \text{I-1b}$$

The general expression for distortion will be developed using ellipsoidal parameters and then the simpler expression using spherical parameters will be found.

Distortion in length: ellipsoid

Fig. 1

(a) (b)



$\tan A = \frac{N \cos \phi d\lambda}{Md \phi}$

Earth Ellipsoid

$M = a(1-e^2)/(1-e^2\sin^2\phi)^{3/2}$ is the radius of curvature of the meridian

$N = a/(1-e^2\sin^2\phi)^{\frac{1}{2}}$ is the radius of curvature of the prime vertical in which a is the length of the semi-major axis of the earth and e is the first eccentricity of the earth.

Mapping Plane

Note that the axes have been chosen with X vertical and Y horizontal; this enables us to follow the conventional sequence i.e. ϕ and λ, X and Y.

Earth sphere: This is the sphere approximating the earth; it is assumed to have radius R where $R \approx M \approx N \approx \sqrt{MN}$.

In these notes ϕ and λ are used interchangeably for both the earth ellipsoid and the earth sphere. Transformations between the earth ellipsoid and the earth sphere are taken up in Geometrical Geodesy.

In Fig. 1 an elemental unit of length PP_1 on the ellipsoid is mapped onto the plane as $P'P_1'$.

Distortion in length is defined as

$$PP_1 - P'P_1'$$

To simplify the derivation and analyses which follow we will set

$$\frac{P'P_1'}{PP_1} = m_A \qquad \text{1-2a}$$

and call it a length distortion factor.

It follows that when $m_A = 1$ there is zero length distortion.

From Fig. 1:

$$[p'p'_1]^2 = dY^2 + dX^2 \qquad \text{1-3a}$$

$$[PP_1]^2 = (Md\phi)^2 + (N \cos \phi \; d\lambda)^2 \qquad \text{1-3b}$$

hence
$$m_A^2 = \frac{dX^2 + dY^2}{M^2 \, d\phi^2 + N^2 \cos^2\phi \, d\lambda^2} \; . \qquad \text{1-2b}$$

By substitution from 1-1b

$$m_A^2 = \frac{(f_\phi \, d\phi + f_\lambda \, d\lambda)^2 + (g_\phi \, d\phi + g_\lambda \, d\lambda)^2}{M^2 \, d\phi^2 + N^2 \cos^2\phi \, d\lambda^2}$$

and multiplying numerator and denominator by $\frac{1}{d\phi^2}$

$$m_A^2 = \frac{(f_\phi^2+g_\phi^2) + 2(f_\phi f_\lambda + g_\phi g_\lambda) \frac{d\lambda}{d\phi} + (f_\lambda^2 + g_\lambda^2) \left(\frac{d\lambda}{d\phi}\right)^2}{M^2 + N^2\cos^2\phi\left(\frac{d\lambda}{d\phi}\right)^2} \; . \qquad \text{1-2c}$$

From fig. (1a): $\tan A = \frac{N \cos \phi \; d\lambda}{M \; d\phi}$

hence
$$\frac{d\lambda}{d\phi} = \frac{M}{N \cos \phi} \tan A \; . \qquad \text{1-3c}$$

Exercise: By substitution from 1-3c into 1-2c show that:

$$m_A^2 = \frac{f_\phi^2+g_\phi^2}{M^2} \cos^2 A + \frac{2(f_\phi \, f_\lambda + g_\phi \, g_\lambda)}{MN \cos \phi} \sin A \cos A + \frac{f_\lambda^2 + g_\lambda^2}{N^2\cos^2\phi} \sin^2 A \qquad \text{1-2d}$$

$$= m_1^2 \cos^2 A + p \sin A \cos A + m_2^2 \sin^2 A \qquad \text{1-2e}$$

in which
$$m_1^2 = \frac{f_\phi^2 + g_\phi^2}{M^2} \; ; \quad p = \frac{2(f_\phi f_\lambda + g_\phi g_\lambda)}{MN \cos \phi} \; : \quad m_2^2 = \frac{f_\lambda^2 + g_\lambda^2}{N^2\cos^2\phi} \; . \qquad \text{1-2f}$$

Question: What condition is required for the distortion to be independent of azimuth, A?

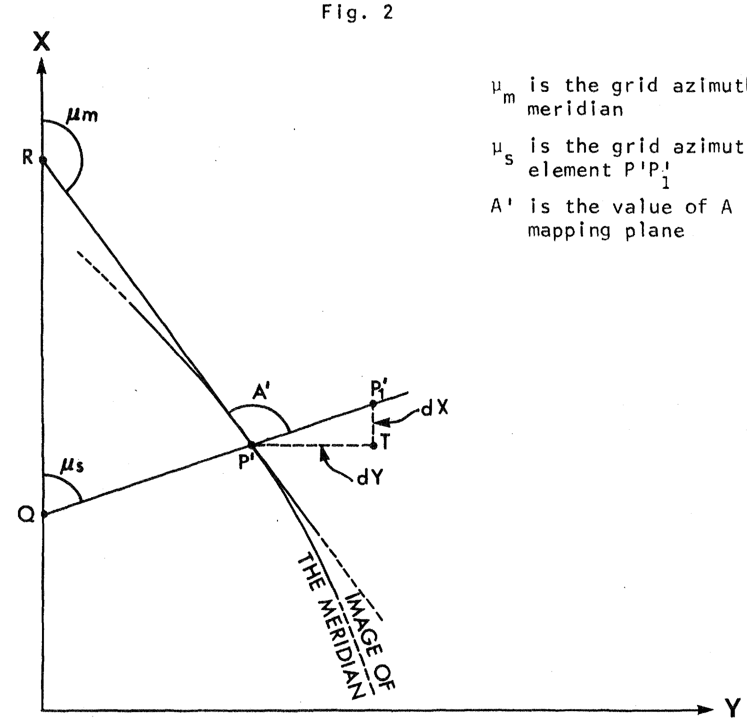
Distortion in length: sphere

When considering the earth as a sphere instead of as an ellipsoid the formulas in 1-2f become

$$m_1^2 = \frac{f_\phi^2 + g_\phi^2}{R^2} \; : \quad p = \frac{2(f_\phi f_\lambda + g_\phi g_\lambda)}{R^2 \cos \phi} \quad m_2^2 = \frac{f_\lambda^2 + g_\lambda^2}{R^2 \cos^2 \phi} \; . \qquad \text{1-2g}$$

Distortion in azimuth and angle

Fig. 2



μ_m is the grid azimuth of the meridian

μ_s is the grid azimuth of the element $P'P_1'$

A' is the value of A in the mapping plane

In $\Delta P'QR$: $(180°-A') + (180 - \mu_m) + \mu_s = 180°$

hence $180°-A' = \mu_m - \mu_s$ $\qquad$ 1-4a

and $\tan (180°-A') = \dfrac{\tan \mu_m - \tan \mu_s}{1 + \tan \mu_m \tan \mu_s}$ $\qquad$ 1-4b

from $\Delta P'P_1'T$: $\tan \mu_s = \dfrac{dY}{dX}$

from 1-1b $\left(\dfrac{dY}{dX}\right)_s = \dfrac{g_\phi d\phi + g_\lambda d\lambda}{f_\phi d\phi + f_\lambda d\lambda} = \dfrac{g_\phi + g_\lambda \frac{d\lambda}{d\phi}}{f_\phi + f_\lambda \frac{d\lambda}{d\phi}}$. $\qquad$ 1-5a

Substituting for $\dfrac{d\lambda}{d\phi}$ from 1-3c

$$\tan \mu_s = \frac{g_\phi N \cos\phi \cos A + g_\lambda M \sin A}{f_\phi N \cos\phi \cos A + f_\lambda M \sin A} \qquad \text{1-5b}$$

Taking an element of length along the meridian, i.e. with λ constant

$$\tan \mu_m = \left(\frac{dY}{dX}\right)_m = \frac{g_\phi d\phi + g_\lambda d\lambda}{f_\phi d\phi + f_\lambda d\lambda} \qquad \text{1-6a}$$

but for λ constant, $d\lambda = 0$ hence $\tan \mu_m = \dfrac{g_\phi}{f_\phi}$.

Distortion in azimuth is defined as $A'-A$ in which A' is computed from 1-4b using $\tan\mu_s$ from 1-5b and $\tan\mu_m$ from 1-6a.

To get the distortion in an angle, ω, where $\omega = A_2 - A_1$, we compute A'_1 and A'_2 from 1-4b and get the angle $\omega' = A'_2 - A'_1$ in the mapping plane.

The angular distortion is then given by:

$$\begin{aligned}\omega' - \omega &= (A'_2 - A'_1) - (A_2 - A_1) \\ &= (A'_2 - A_2) - (A'_1 - A_1) \quad .\end{aligned} \qquad \text{1-7}$$

<u>Special case: Distortion between meridian and parallel</u>

Distortion in angle between the meridian and the parallel is equal to the distortion in the azimuth, A_p, where $A_p = 90°$.

A'_p is computed by substituting

$\tan\mu_p = \left(\frac{dY}{dX}\right)_{\phi\,=\,\text{const}}$ into 1-4b in place of $\tan\mu_s$

$$\text{from 1-1b } \left(\frac{dY}{dX}\right)_{d\phi=o} = \frac{g_\lambda}{f_\lambda} \qquad \text{1-8}$$

$$\text{hence } \tan(180-A'_p) = \frac{\tan\mu_m - \tan\mu_p}{1 + \tan\mu_m \tan\mu_p} = \frac{\frac{g_\phi}{f_\phi} - \frac{g_\lambda}{f_\lambda}}{1 + \frac{g_\phi}{f_\phi} \cdot \frac{g_\lambda}{f_\lambda}}$$

$$= \frac{f_\lambda g_\phi - f_\phi g_\lambda}{f_\phi f_\lambda + g_\phi g_\lambda} \qquad \text{1-9a}$$

As $\tan(180-A'_p) = 1/\tan(A'_p - 90)$

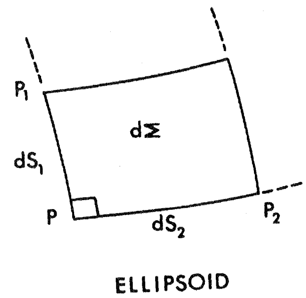
the distortion between the meridian and the parallel is given by (A'_p-90) and is evaluated from

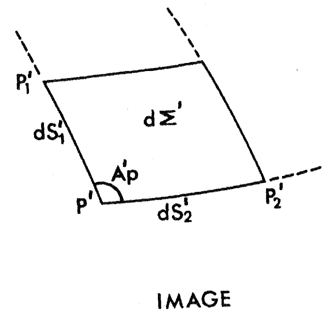
$$\tan(A'_p-90°) = \frac{f_\phi f_\lambda + g_\phi g_\lambda}{f_\lambda g_\phi - f_\phi g_\lambda} \qquad \text{1-10}$$

Question: What is the value of $A'_p-90°$ in a conformal projection?

Distortion in area

Fig. 3





Element of area:

$$d\Sigma = ds_1 \times ds_2 \qquad\qquad d\Sigma' = ds_1' \times ds_2' \times \sin A_p'$$

Distortion in area is defined as $d\Sigma - d\Sigma'$

For simplicity we will use $\frac{d\Sigma'}{d\Sigma}$ and call it an area distortion factor, noting that when $\frac{d\Sigma'}{d\Sigma} = 1$ there is zero area distortion.

$$\frac{d\Sigma'}{d\Sigma} = \frac{ds_1' \times ds_2' \sin A'_p}{ds_1 \times ds_2} = \frac{ds_1'}{ds_1} \times \frac{ds_2'}{ds_1} \times \sin A_p' \qquad \text{I-11a}$$

$$= m_1 \times m_2 \times \sin A_p'$$

where m_1 is the length distortion factor along the meridian and m_2 is the length distortion factor along the parallel.

Remembering that $\sin A_p' = \sin(180 - A_p')$ and noting that if $\tan a = \frac{c}{d}$, $\sin a = \frac{c}{\sqrt{c^2 + d^2}}$

then from I-9a

$$\sin A_p' = \frac{f_\lambda g_\phi - f_\phi g_\lambda}{\sqrt{(f_\lambda g_\phi - f_\phi g_\lambda)^2 + (f_\phi f_\lambda + g_\phi g_\lambda)^2}} \;. \qquad \text{I-9b}$$

Thus from I-2f and I-9b

$$\frac{d\Sigma'}{d\Sigma} = \sqrt{\frac{f_\phi^2 + g_\phi^2}{M^2}} \times \sqrt{\frac{f_\lambda^2 + g_\lambda^2}{N^2 \cos^2\phi}} \times \frac{f_\lambda g_\phi - f_\phi g_\lambda}{\sqrt{(f_\lambda g_\phi - f_\phi g_\lambda)^2 + (f_\phi f_\lambda + g_\phi g_\lambda)^2}} \;. \qquad \text{I-11b}$$

Exercise: Show that $\frac{d\Sigma'}{d\Sigma} = \frac{|f_\lambda g_\phi - f_\phi g_\lambda|}{MN \cos\phi}$.

I-11c

Question: What is the condition for a true-area projection?

Azimuths with minimum and maximum distortion in length

We apply the condition for extrema to 1-2e, namely: equate the first derivative to zero and solve

From 1-2e

$$m_A^2 = m_1^2 \cos^2 A + p \cos A \sin A + m_2^2 \sin^2 A$$

$$\frac{d(m_A^2)}{dA} = 2m_A \left(\frac{dm_A}{dA}\right), \qquad \frac{dm_A}{dA} = \frac{1}{2m_A}\left(\frac{dm_A^2}{dA}\right)$$

$$\text{if } \frac{d(m_A^2)}{dA} = 0 \quad \text{then } \frac{dm_A}{dA} = 0 \quad \text{provided that } m_A \neq 0.$$

$$\frac{d(m_A^2)}{dA} = -2m_1^2 \cos A \sin A + p(-\sin^2 A + \cos^2 A) + 2m_2^2 \cos A \sin A \qquad \text{1-12a}$$

$$= -m_1^2 \sin 2A + p \cos 2A + m_2^2 \sin 2A \quad . \qquad \text{1-12b}$$

Condition for extrema:

$$(m_2^2 - m_1^2) \sin 2A_E + p \cos 2A_E = 0 \qquad \text{1-13a}$$

therefore

$$\tan 2A_E = \frac{p}{m_1^2 - m_2^2} \quad . \qquad \text{1-13b}$$

This equation can be satisfied by a value, A_{E1} and by a value, A_{E2}, equal to A_{E1} + 90. These are the azimuths of the elements of extreme distortion at point P.

To establish which of A_{E1} and A_{E2} has maximum distortion and which has minumum distortion, it is, of course, necessary to take the second derivative, substitute at the extrema; if the value is positive then it is minimum, if negative it is maximum.

CHAPTER II

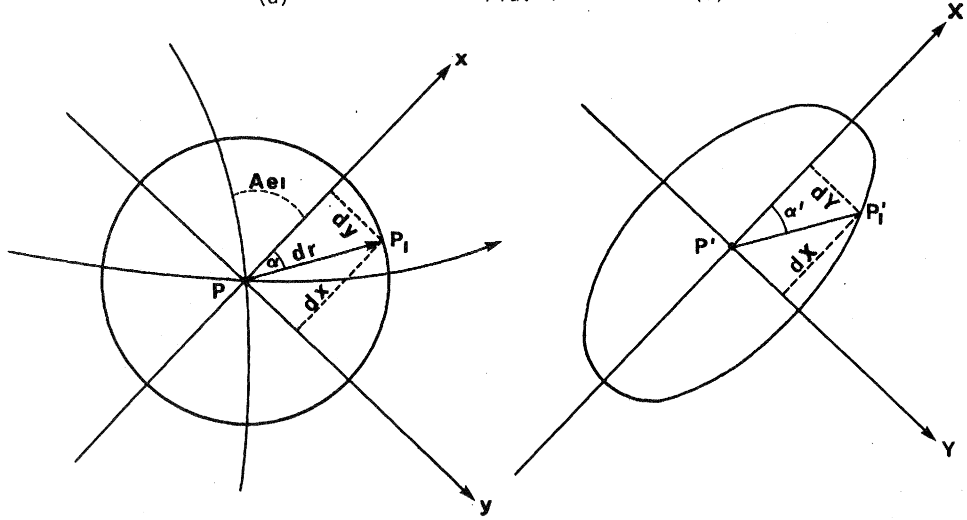
SIMPLIFIED EXPRESSIONS FOR DISTORTIONS

Appreciable simplifications of formulas expressing distortions may be achieved by using the directions of the extreme distortions as reference axes.

Tissot's Indicatrix

A differentially small circle on the ellipsoid is mapped as a differentially small ellipse on the plane. [This statement will not be proved here; a proof can be found in U.S.C. & G.S. Spec. Pub. No.57 by Oscar S. Adams].

(a) FIG. 4 (b)



Ellipsoid (sphere)

Image

Coordinate axes chosen in the directions of extreme distortion in length.

From the definition of distortion in 1-2a the magnitude of the length distortion factors in the directions of the extremes can be designated:

$$a = \frac{dX}{dx} \qquad \text{and} \qquad b = \frac{dY}{dy} \quad . \qquad \text{II-1a}$$

Note that this a is not to be confused with the a defined on page 2.

$$dX = a\ dx \qquad dY = b\ dy \qquad \text{II-1b}$$

The equation of the small circle in Fig. 4a is

$$dx^2 + dy^2 = dr^2 \qquad \text{II-1c}$$

and after substitution from II-1b the equation of the ellipse in Fig. 4b is

$$\frac{dX^2}{a^2} + \frac{dY^2}{b^2} = dr^2 \qquad \text{II-1d}$$

hence
$$\frac{dX^2}{a^2 dr^2} + \frac{dY^2}{b^2 dr^2} = 1 \quad . \qquad \text{II-1e}$$

This is the equation of Tissot's Indicatrix.

By using the axes of Tissot's Indicatrix as reference axes alternative expressions for the length distortion factor (Equation 1-2d) and the distortion in angle (Equation 1-7) can be derived.

Length distortion factor

From Fig. 4:

$$(P\,P_1)^2 = dx^2 + dy^2 = dr^2$$

$$(P'P'_1)^2 = dX^2 + dY^2 = a^2 \cos^2\alpha\; dr^2 + b^2 \sin^2\alpha\; dr^2$$

as $dX = a\,dx$ (from 11-1b) and $dx = dr \cos\alpha$ (from Fig.4a)

and similarly $dY = b\,dy$, $dy = dr \sin\alpha$

thus

$$m_\alpha^2 = \left(\frac{P'P'_1}{P\,P_1}\right)^2 = \frac{a^2\cos^2\alpha\; dr^2 + b^2\sin^2\alpha\; dr^2}{dr^2}$$

$$= a^2\cos^2\alpha + b^2\sin^2\alpha \qquad 11\text{-}2$$

where α is the bearing measured from the x-axis, i.e. from the direction of extreme distortion, a, the azimuth of which is A_{E1}. The azimuth of the PP_1 direction is $A_{E1} + \alpha$. This compares with

$$m_A^2 = m_1^2 \cos^2 A + p \sin A \cos A + m_2^2 \sin^2 A\,.$$

Area distortion factor

In Fig. 4, the area of the differentially small circle is

$$d\Sigma = \pi dr^2$$

and the area of the differentially small ellipse is

$$d\Sigma' = \pi\; adr \times bdr$$

hence

$$\frac{d\Sigma'}{d\Sigma} = \frac{\pi\; ab\; dr^2}{\pi\; dr^2} = ab \qquad 11\text{-}3$$

as compared with $\dfrac{d\Sigma'}{d\Sigma} = \dfrac{f_\lambda g_\phi - f_\phi g_\lambda}{MN \cos\phi}$ (in 1-11c) .

Distortion in bearing

From Fig. 4

$$\tan\alpha = \frac{dy}{dx}\ ; \quad \tan\alpha' = \frac{dY}{dX} \qquad \text{II-4a}$$

hence from II-1b

$$\tan\alpha' = \frac{b}{a}\frac{dy}{dx} = \frac{b}{a}\tan\alpha\ . \qquad \text{II-4b}$$

Distortion in bearing is defined as $\alpha'-\alpha$.

Extreme distortion in bearing and azimuth

We take the derivative of $\alpha' - \alpha$ and equate to zero.

Thus

$$\frac{d(\alpha' - \alpha)}{d\alpha} = 0, \quad d\alpha'_E = d\alpha_E\ . \qquad \text{II-4c}$$

Applying this condition to the derivative of II-4b

$$\frac{d(\tan\alpha')}{d\alpha'_E} = \frac{b}{a}\ \frac{d\ (\tan\alpha)}{d\alpha} \qquad \text{II-4d}$$

$$\frac{1}{\cos^2\alpha'_E} = \frac{b}{a}\cdot\frac{1}{\cos^2\alpha_E}$$

or

$$1 + \tan^2\alpha'_E = \frac{b}{a}\ (1 + \tan^2\alpha_E)\ . \qquad \text{II-4e}$$

Substituting from II-4b

$$1 + \frac{b^2}{a^2}\tan^2\alpha_E = \frac{b}{a} + \frac{b}{a}\tan^2\alpha_E$$

$$\tan\alpha_E = \sqrt{\frac{a}{b}}\ . \qquad \text{II-4f}$$

But from II-4b

$$\tan\alpha'_E = \frac{b}{a}\tan\alpha_E = \frac{b}{a}\sqrt{\frac{a}{b}} = \sqrt{\frac{b}{a}}\ . \qquad \text{II-4g}$$

For the computation of $\alpha'_E - \alpha_E$

$$\sin(\alpha'_E - \alpha_E) = \sin\alpha'_E\cos\alpha_E - \cos\alpha'_E\sin\alpha_E\ . \qquad \text{II-5a}$$

From II-4f $\quad \sin\alpha_E = \sqrt{\frac{a}{a+b}}\ , \qquad \cos\alpha_E = \sqrt{\frac{b}{a+b}}\ .$

From II-4g $\quad \sin\alpha'_E = \sqrt{\frac{b}{a+b}}\ , \qquad \cos\alpha'_E = \sqrt{\frac{a}{a+b}}\ .$

Hence $\quad \sin(\alpha'_E - \alpha_E) = \frac{b}{a+b} - \frac{a}{a+b} = \frac{b-a}{a+b}\ . \qquad \text{II-5b}$

Maximum distortion in angles

The distortion $\Delta\omega_E$ in the angle $\omega = \alpha_2 - \alpha_1$ is estimated by its maximum value which may be two times larger than the distortion in bearing.

Thus

$$\Delta\omega_E = 2(\alpha'_E - \alpha_E) \qquad \text{II-6a}$$

and

$$\sin\left(\frac{\Delta\omega_E}{2}\right) = \sin(\alpha'_E - \alpha_E) = \frac{b-a}{b+a} \quad . \qquad \text{II-6b}$$

Note we did not compute an extremum for angle previously. We merely found $\omega'-\omega = (A'_2-A_2) - (A'_1 - A_1)$ where $A'_2 + A'_1$ were expressed in terms of $\mu_m + \mu_s$.

Note: The simplified formulas for distortions are well suited to Genuine Projections; i.e. to those in which the images of meridians and parallels remain perpendicular. In other words, those projections for which p = 0, i.e. those for which $A_{E1} = 0$ and $A_{E2} = 90$. The meridians and parallels are then the directions with extreme scale factors.

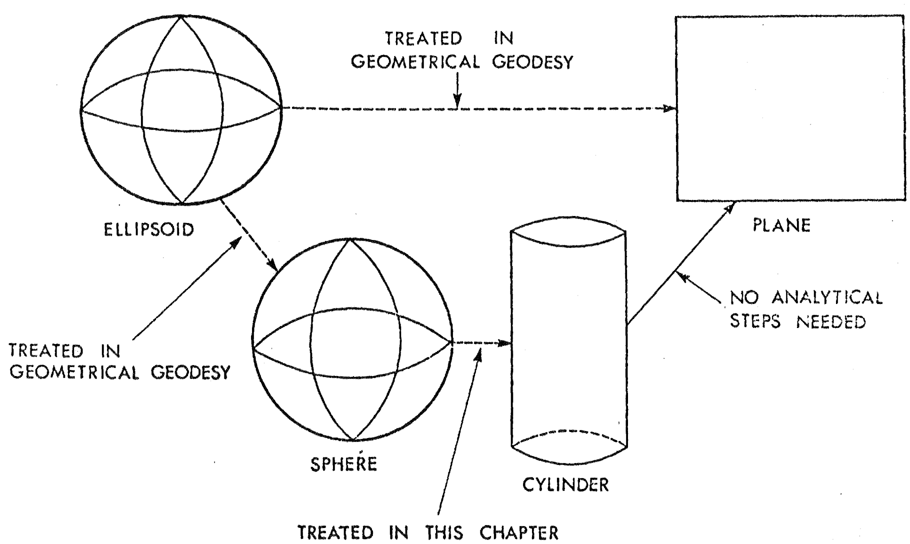
CHAPTER III

ANALYSIS OF DISTORTION IN CYLINDRICAL PROJECTIONS

The cylindrical family of projections are illustrated and their characteristics are discussed in Roblin (pp 18-25) and in many other texts. In these notes, it will be assumed that the reader has referred to Roblin or to some similar reference for a description of the projection hence only the analytical expressions for distortion will be developed here; there will be applications of the expressions developed in Chapters I and II.

The cylinder is a useful intermediate surface between the sphere and the plane. It is called a developable surface as it may be mapped onto a plane without distortion; it is of special value in visualizing the transformation. The steps in transforming, i.e. projecting, from the ellipsoid to the plane are illustrated schematically:

FIG. 5



All transformations from the ellipsoid, whether directly to the plane or to an intermediate surface are somewhat involved mathematically. As our immediate objective is to gain an understanding of map projections in general, we will in most cases start with a spherical surface.

Simple Cylindrical

For this projection the explicit functional relationships and the derivatives corresponding to the general form in 1-1a and 1-1b are:

$$X = R\phi \qquad Y = R\lambda \qquad \text{III-1a}$$

$$f_\phi = R \qquad g_\phi = 0$$

$$f_\lambda = 0 \qquad g_\lambda = R \quad . \qquad \text{III-1b}$$

From (1-2g):

$$m_1^2 = \frac{f_\phi^2 + g_\phi^2}{R^2} \quad ; \quad p = \frac{2(f_\phi f_\lambda + g_\phi g_\lambda)}{R^2 \cos\phi} \quad ; \quad m_2^2 = \frac{f_\lambda^2 + g_\lambda^2}{R^2 \cos^2\phi}$$

$$m_1 = 1 \; ; \qquad p = 0 \; ; \qquad m_2 = \frac{1}{\cos\phi} \quad . \qquad \text{III-1c}$$

What are the directions of the extrema?

$$\tan 2A_E = \frac{p}{m_1^2 - m_2^2} = 0 \qquad A_{E1} = 0 \; , \quad A_{E2} = 90°$$

therefore $a = m_2 \qquad b = m_1$.

Is the projection equal-area? No

What is the area distortion factor? $a \times b = m_1 \times m_2 = \frac{1}{\cos\phi}$. III-1d

Is the projection conformal? No. $p = 0$

but $m_1 \neq m_2$.

Maximum distortion in angle?

$$\sin\left(\frac{\Delta\omega_E}{2}\right) = \frac{b-a}{b+a} = \frac{\sec\phi - 1}{\sec\phi + 1} \quad . \qquad \text{III-1e}$$

<u>Cylindrical equal-area</u>

The functional relationships are:

$$X = R \sin \phi \qquad Y = R\lambda \qquad \text{III-2}$$

Exercises:

(i) Show that $m_1 = \cos \phi$, $p = 0$, $m_2 = 1/\cos \phi$ and hence that the projection is equal-area.

(ii) Show that the maximum distortion in angle is given by

$$\sin \left(\frac{\Delta\omega_\varepsilon}{2}\right) = \frac{\sec \phi - \cos \phi}{\sec \phi + \cos \phi}$$

Gall's projection

The functional relationshipsand first derivatives are:

$$X = \frac{1.7R \sin \phi}{1 + \cos \phi} \qquad Y = 0.7R\,\lambda \qquad \text{III-3a}$$

$$f_\phi = 1.7R \left[\frac{1}{1 + \cos \phi}\right] \qquad g_\phi = 0 \qquad \text{III-3b}$$

$$f_\lambda = 0 \qquad g_\lambda = 0.7R$$

From 1-2g

$$m_1^2 = \frac{[1.7R\,(\frac{1}{1 + \cos \phi})]^2 + 0}{R^2} ; p = 0 \;;\; m_2^2 = \frac{0 + (0.7R)^2}{R^2 \cos^2 \phi}$$

$$m_1 = 1.7\,(1 + \cos\phi)^{-1} \qquad m_2 = \frac{0.7}{\cos \phi} \qquad \text{III-3c}$$

What are the directions of the extrema?

$$\tan 2A_E = \frac{p}{m_1^2 - m_2^2} = 0 \quad ; \quad A_{E1} = 0,\ A_{E2} = 90°$$

therefore $a = m_2$, $b = m_1$

Is the projection equal-area? No

Area distortion factor?

$$a \times b = \frac{1.7}{1 + \cos \phi} \times \frac{0.7}{\cos \phi} \qquad \text{III-3d}$$

Is the projection conformal? No,

$p = 0$, but $m_1 \neq m_2$

Maximum distortion in angle?

$$\sin\left(\frac{\Delta\omega_E}{2}\right) = \frac{b-a}{b+a} = \frac{\frac{0.7}{\cos\phi} - 1.7(\frac{1}{1+ \cos\phi})}{\frac{0.7}{\cos\phi} + 1.7(\frac{1}{1+ \cos\phi})} \qquad \text{III-3e}$$

Questions:

(i) Under what condition is there no distortion in angle?

(ii) How can this be reconciled with the fact that the projection is not conformal?

Mercator's projection

The functional relationships and first derivatives are:

$$X = R \ln \tan (45° + \frac{\phi}{2}) \qquad Y = R\lambda \qquad \text{III-4a}$$

$$f_\phi = \frac{R}{\cos\phi} \qquad g_\phi = 0 \qquad \text{III-4b}$$

$$f_\lambda = 0 \qquad g_\lambda = R \;.$$

From I-2g

$$m_1^2 = \frac{f_\phi^2 + g_\phi^2}{R^2}; \; p = \frac{2(f_\phi f_\lambda + g_\phi g_\lambda)}{R^2 \cos\phi} \; ; \; m_2^2 = \frac{f_\lambda^2 + g_\lambda^2}{R^2 \cos^2\phi} \;.$$

Hence

$$m_1 = \frac{1}{\cos\phi} = \sec\phi \, ; \; p = 0 \; ; \; m_2 = \frac{1}{\cos\phi} = \sec\phi \;. \qquad \text{III-4c}$$

Do the directions of extreme distortion exist?

No, because distortion is equal in all directions.

Is the projection equal-area? No.

Area distortion factor?

$$a \times b = m_1 \times m_2 = \frac{1}{\cos\phi} \times \frac{1}{\cos\phi} = \frac{1}{\cos^2\phi} \;. \qquad \text{III-4d}$$

Is the projection conformal? Yes

$$p = 0, \text{ and } m_1 = m_2 \;.$$

Maximum distortion in angle? 0

$$\sin \left(\frac{\Delta\omega_E}{2}\right) = \frac{b-a}{b+a} = \frac{\sec\phi - \sec\phi}{\sec\phi + \sec\phi} = 0 \;.$$

Definition: The Loxodrome or rhumb line is defined as a curve on the sphere intersecting all meridians in the same azimuth. The loxodrome is mapped as a straight line on Mercator's projection. (see Roblin p.22)

The Transverse Mercator (T.M.) projection is used mainly for bands of only a few degrees in width, the band that is 6 degrees in width is known as the Universal Transverse Mercator (U.T.M.) and is used for the National Topographic Series of maps in Canada and in many other countries. To reduce the distortion in scale a secant cylinder is generally used instead of a tangent cylinder. For a 6° zone, the cylinder is made to intersect the sphere 2° each side of the central meridian.

The Transverse Mercator projection and procedures for converting from the ellipsoid are discussed in Coordinate Systems and Map Projections by D. H. Maling (pp. 217-233).

CHAPTER IV

ANALYSIS OF DISTORTION IN AZIMUTHAL (ZENITHAL) PROJECTIONS

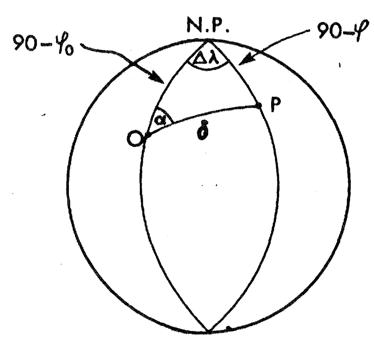
The azimuthal or zenithal family of projections are illustrated and their characteristics are discussed in Roblin (pp. 39-55) and in many other texts. As in Chapter III it will be assumed that the reader has referred to a descriptive text and only the analytical expressions for distortion will be discussed here. Also, as in Chapter III, discussion will be restricted to the transformation from sphere to plane. Some generality will be achieved, however, by the introduction of oblique spherical coordinates (cartographic coordinates) in place of normal spherical coordinates; by this artifice the analyses that follow will be valid regardless of the location of the tangent point.

Transformation to oblique spherical coordinates from normal spherical coordinates

Oblique spherical coordinates, α and δ are the coordinates with respect to some pole other than the north or south pole.

This is illustrated in Fig. 6 in which:

Fig. 6



N.P. is North Pole;

0 (ϕ_o, λ_o) is the tangent point of an oblique projection;

ϕ, λ are the normal spherical coordinates of point P;

α, δ are the oblique spherical (cartographic) coordinates of point P with respect to a pole at 0.

Exercise: Prove that the two systems are related by:

(1) $\sin\delta \sin\alpha = \cos\phi \sin\Delta\lambda$

(2) $\sin\delta \cos\alpha = \sin\phi \cos\phi_o - \cos\phi \sin\phi_o \cos\Delta\lambda$ IV-1

(3) $\cos\delta = \sin\phi \sin\phi_o + \cos\phi \cos\phi_o \cos\Delta\lambda$

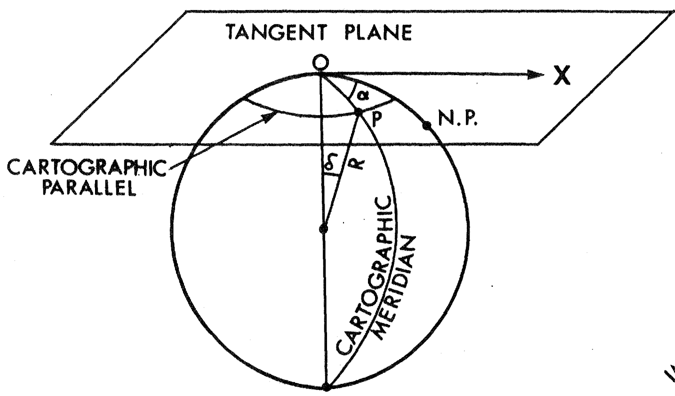
If necessary refer to a text on spherical trigonometry such as Schaum's notes (pp. 168-172) to verify your proof.

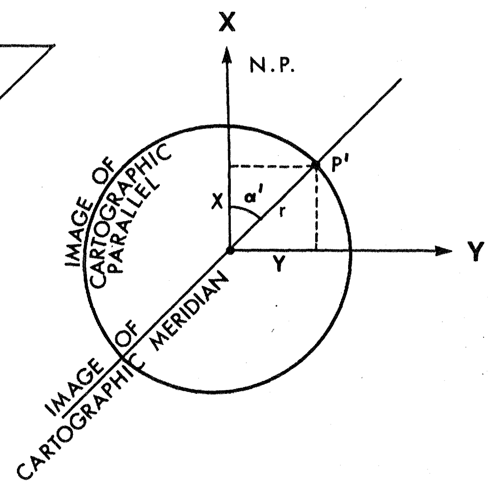
General properties of azimuthal projections

As azimuthal projections are symmetric about the tangent point the mathematical relationships can be conveniently expressed in polar coordinates.

Fig. 7

(a) (b)





In Fig. 7(b)

$X = r \cos \alpha'$ $\qquad$ $Y = r \sin \alpha'$ $\qquad$ IV-2

$\alpha' = \alpha$ $\qquad$ $r = f(\delta)$ $\qquad$ IV-3

in which $f(\delta)$ varies for different projections.

Note:

(i) The images of cartographic meridians are straight lines, i.e. α is constant.

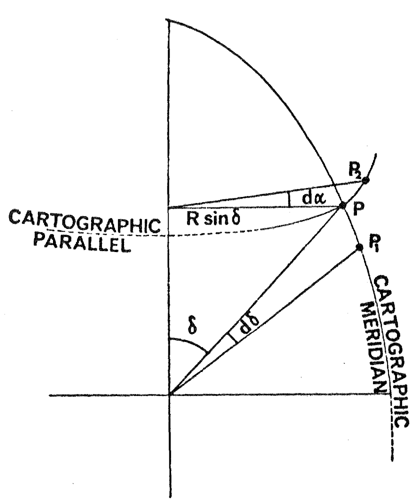
(ii) The images of cartographic parallels are circles, i.e. δ is constant.

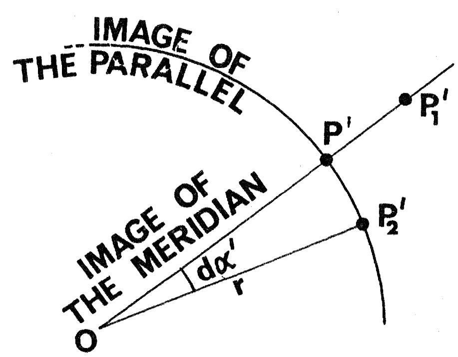
(iii) The images of the cartographic meridians and parallels intersect perpendicularly.

Expressions for a and b (Tissot's Indices):

Fig. 8

(a) (b)





Applying the definition of length distortion factor in I-2a to Fig. 8

$$a = \frac{P'P'_1}{P\ P_1} = \frac{dr}{Rd\delta} \qquad \text{IV-4a}$$

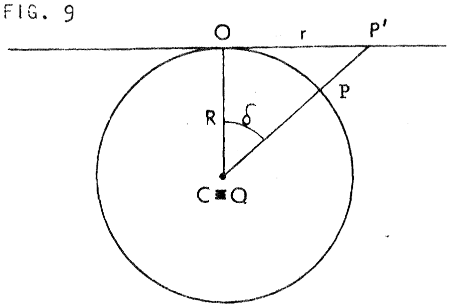
$$b = \frac{P'P'_2}{P\ P_2} = \frac{rd\alpha'}{R\sin\delta d\alpha}\ . \qquad \text{IV-4b}$$

Exercise: Show that for all azimuthal projections P=O.

Gnomonic Projection

The perspective centre is the centre of the sphere.

FIG. 9



$$\alpha' = \alpha$$

$$r = R\tan\delta \qquad \text{IV-5a}$$

$$a = \frac{1}{R} \cdot \frac{dr}{d\delta} = \frac{1}{R}\frac{R}{\cos^2\delta} = \frac{1}{\cos^2\delta} \qquad \text{IV-5b}$$

$$b = \frac{r\ d\alpha'}{R\sin\delta d\alpha} = \frac{R\tan\delta}{R\sin\delta} = \frac{1}{\cos\delta}\ . \qquad \text{IV-5c}$$

Is the projection equal-area? No.

$$ab \neq 1\ .$$

Area distortion factor?

$$a \times b = \frac{1}{\cos^2\delta} \cdot \frac{1}{\cos\delta} = \frac{1}{\cos^3\delta}\ . \qquad \text{IV-5d}$$

Is the projection conformal? $a \neq b$

Maximum distortion in angle?

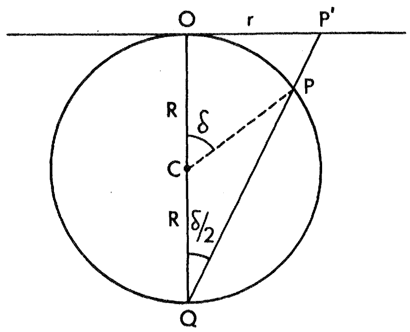
$$\sin\left(\frac{\Delta\omega_\varepsilon}{2}\right) = \frac{b-a}{b+a} = \frac{\frac{1}{\cos\delta} - \frac{1}{\cos^2\delta}}{\frac{1}{\cos\delta} \quad \frac{1}{\cos^2\delta}} = \frac{\cos\delta - 1}{\cos\delta + 1} = -\tan^2\frac{\delta}{2}\ . \qquad \text{IV-5e}$$

Note: The most important property of the gnomonic projection is that a great circle on the sphere maps as a straight line on the projection. Such an image is called an <u>orthodrome</u>.

Stereographic Projection

The perspective centre is diametrically opposite the tangent point.

Fig. 10



$$\alpha' = \alpha \qquad \text{IV-6a}$$

$$r = 2R \tan \delta/2$$

$$a = \frac{1}{R} \cdot \frac{dr}{d\delta} = \frac{1}{R} \; \frac{R}{\cos^2\delta/2} = \frac{1}{\cos^2\delta/2} \qquad \text{IV-6b}$$

$$b = \frac{r}{R \sin \delta} = \frac{2R \tan \delta/2}{R \sin \delta} = \frac{1}{\cos^2\delta/2} \quad . \qquad \text{IV-6c}$$

Similarly, as for the gnonomic projection

$$p = 0 \quad \text{and} \quad A_{E1} = 0 \quad .$$

Is the projection equal-area? No. $ab \neq 1$

Area distortion factor?

$$a \times b = \frac{1}{\cos^2\delta/2} \times \frac{1}{\cos^2\delta/2} = \frac{1}{\cos^4\delta/2} \quad . \qquad \text{IV-6d}$$

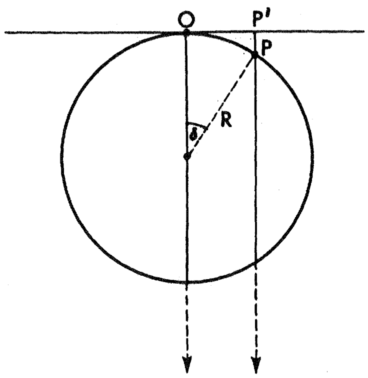
Is the projection conformal? Yes. $a = b$

Exercise: Starting with $X = r \cos \alpha'$ and $Y = r \sin \alpha'$ show that

$$m_1 = \frac{1}{\cos^2\delta/2} \text{ and } m_2 = \frac{1}{\cos^2\delta/2}$$

Orthographic Projection

Fig. 11



$$\alpha' = \alpha$$

$$r = R \sin \delta \qquad \text{IV-7a}$$

$$X = r \cos \alpha' \; ; \quad Y = r \sin \alpha'$$

$$\alpha' = \alpha$$

$$X = R \sin \delta \cos \alpha \; ; \; Y = R \sin\delta \sin\alpha$$

$$f_\delta = R \cos\delta \cos\alpha \qquad g_\delta = R \cos\delta \sin\alpha$$

$$f_\alpha = -R \sin\delta \sin\alpha \qquad g_\alpha = R \sin\delta \cos\alpha$$

$$m_1^2 = \frac{f_\delta^2 + g_\delta^2}{R^2} \qquad m_2^2 = \frac{f_\alpha^2 + g_\alpha^2}{R^2 \sin^2\delta}$$

$$= \frac{R^2 \cos^2\delta \cos^2\alpha + R^2 \cos^2\delta \sin^2\alpha}{R^2} \qquad = \frac{R^2 \sin^2\delta \sin^2\alpha + R^2 \sin^2\delta \cos^2\alpha}{R^2 \sin^2 \delta}$$

$$m_1 = \cos \delta \qquad = \frac{R^2 \sin^2\delta(\sin^2\alpha + \cos^2\alpha)}{R^2 \sin^2 \delta} \qquad \text{IV-7b}$$

$$m_2 = 1$$

By substitution it can be shown that $p = 0$, hence $m_1 = a$, $m_2 = b$.

Is the projection equal - area? No.

What is the area distortion factor? $\cos \delta$

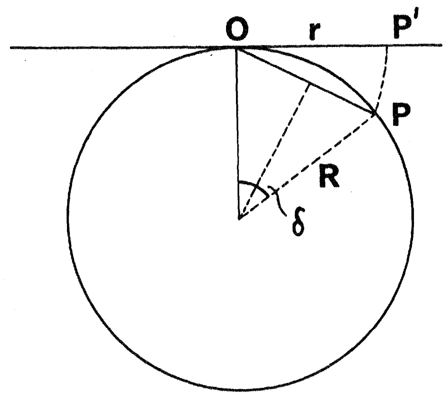
What is the maximum angular distortion?

It is given by:

$$\sin \left(\frac{\Delta\omega_\varepsilon}{2}\right) = \frac{b - a}{b + a} = \frac{1 - \cos \delta}{1 + \cos \delta} \qquad \text{IV-7c}$$

Lambert's True Area Projection

Fig. 12



$$OP = OP'$$
$$OP = 2R\sin \delta/2$$

What is the condition for a true area projection? $a \times b = 1$

Thus if $a = \frac{dr}{Rd\delta}$ and $b = \frac{r}{R^2 \sin^2\delta}$

$$a \cdot b = \frac{dr}{Rd\delta} \cdot \frac{r}{R\sin\delta} = 1$$

$$\text{i.e.} \quad r\,dr = R^2 \sin\delta\, d\delta$$

$$\int r\,dr = R^2 \int \sin\delta\, d\delta$$

$$\frac{1}{2} r^2 = -R^2 \cos\delta + C$$

For $\delta = 0$, $r = 0$, thus $C = R^2$

and $\frac{1}{2} r^2 = -R^2 \cos\delta + R^2 = R^2(1-\cos\delta)$

$$r^2 = 2R^2 (1-\cos\delta)$$

$$r = 2R\sqrt{\frac{1-\cos\delta}{2}} = 2R\sin\delta/2 \qquad \text{IV-8a}$$

$$OP' = OP = 2 \times (R\sin\delta/2)$$

$x = r\cos\alpha$ $\qquad$ $y = r\sin\alpha$

$X = 2R\sin\delta/2 \cos\alpha$ $\qquad$ $Y = 2R\sin\delta/2 \sin\alpha$

$f_\delta = \frac{1}{2} 2R\cos\delta/2 \cos\alpha$ $\qquad$ $g_\delta = \frac{1}{2} 2R\cos\delta/2 \sin\alpha$

$f_\alpha = -2R\sin\delta/2 \sin\alpha$ $\qquad$ $f_\alpha = 2R\sin\delta/2 \cos\alpha$

$$m_1^2 = \frac{R^2\cos^2\delta/2\,\cos^2\alpha + R^2\cos^2\delta/2\,\sin^2\alpha}{R^2} \qquad m_2^2 = \frac{4R^2\sin^2\delta/2\,\sin^2\alpha + 4R^2\sin^2\delta/2\,\cos^2\alpha}{R^2\,\sin^2\delta}$$

$$m_1 = \cos\delta/2 \qquad = \frac{4R^2\,\sin^2\delta/2}{4R^2\,\sin^2\delta/2\,\cos^2\delta/2}$$

$$m_2 = \frac{1}{\cos\delta/2} \qquad \text{IV-8b}$$

By substitution it can be shown that $p = 0$

Is the projection equal area? Yes

Is the projection conformal? No

What is the maximum angular distortion? $\frac{b-a}{b+a} = \frac{\sec\delta/2 - \cos\delta/2}{\sec\delta/2 + \cos\delta/2}$

Is it a genuine projection?

True Meridian Projection

For the projection to have zero length distortion on the meridian a must be equal to unity. Given this condition show that:

1. $r = R\delta$ $\qquad$ IV-9a
2. Area distortion factor is $\delta/\sin\delta$
3. Maximum angular distortion is $\frac{\delta - \sin\delta}{\delta + \sin\delta}$ $\qquad$ IV-9b

CHAPTER V

ANALYSIS OF DISTORTION IN CONICAL PROJECTIONS

The conical family of projections are illustrated and the characteristics of some representative examples are presented in Roblin (pp. 26-33) and in many other texts. As for the previous chapters, it will be assumed that the reader has referred to a descriptive text and only the analytical expressions for distortions will be discussed here.

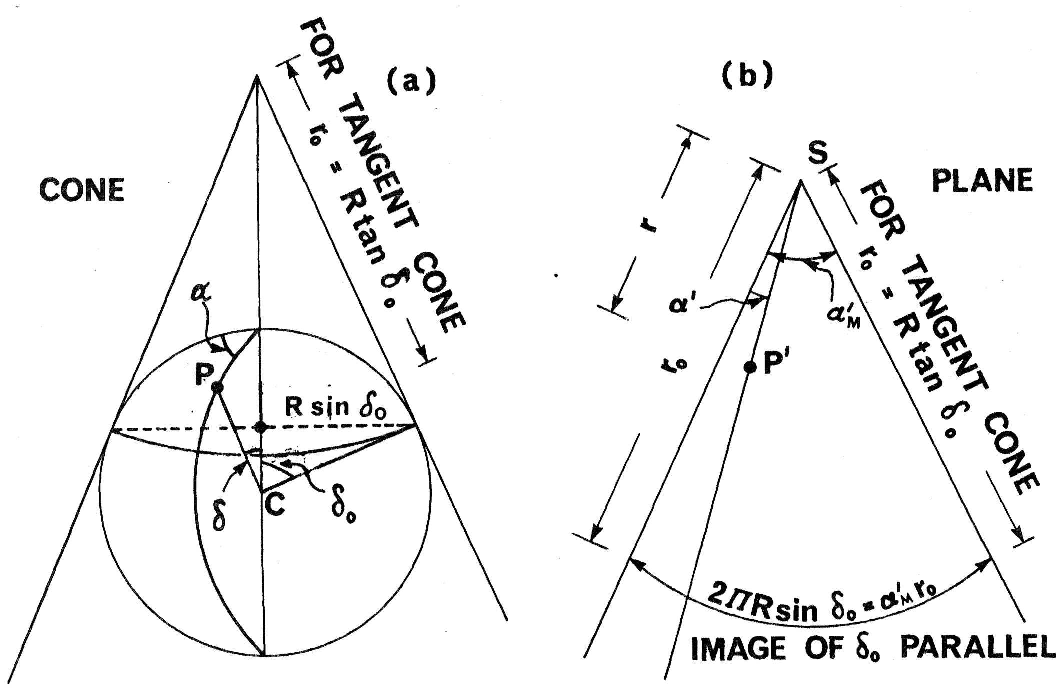
Also, as for zenithal projections the cartographic coordinates (δ,α) will be used.

For "Genuine" conical projections the images of cartographic meridians are straight lines which are convergent to one point and the images of cartographic parallels (δ = const) are concentric circles. By virtue of this similarity the Tissot's indices, a and b, are the same as for zenithal projections (see Fig. 8 and equations IV-4a and b) in which

$$a = \frac{dr}{Rd\delta} \quad ; \quad b = \frac{rd\alpha'}{R \sin \delta \, d\alpha}$$

$$\text{if } \frac{d\alpha'}{d\alpha} = n \quad ; \quad b = \frac{rn}{R \sin \delta}$$

Fig. 14



General equations of "genuine" conical projections

$$\alpha' = n\,\alpha \qquad o < n \leq 1$$
$$r = f(\delta) \qquad \text{V-1}$$

α'_M is defined as the value of α' for $\alpha = 2\pi$

Thus $\alpha' : \alpha = \alpha'_M : 2\pi$

$$\alpha' = \frac{\alpha'_M}{2\pi} \cdot \alpha \text{ , hence } n = \frac{\alpha'_M}{2\pi} \qquad \text{V-2}$$

For one standard parallel:

from fig. 14(b) $\alpha'_M = \dfrac{2\pi R \sin\delta_o}{r_o}$

$$\therefore \alpha' = \frac{2\pi R \sin\delta_o}{2\pi r_o} \cdot \alpha \text{, hence } n = \frac{R\sin\delta_o}{r_o} \qquad \text{V-3}$$

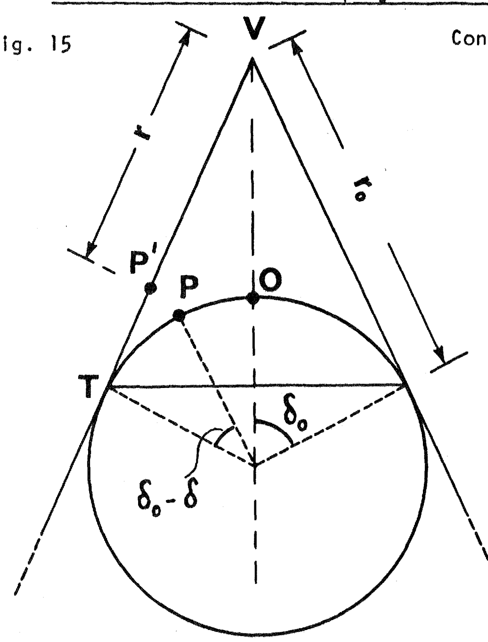
For tangent cone:

$$r_o = R\tan\delta_o$$

$$\alpha' = \frac{R\sin\delta_o}{R\tan\delta_o} \cdot \alpha = \cos\delta_o \cdot \alpha \text{ , hence } n = \cos\delta_o \qquad \text{V-4}$$

Distortionless meridian projection on tangent cone

Fig. 15



Condition for distortionless meridians:

$TP = TP'$: $TP = R\,(\delta_o - \delta)$
$r = r_o - TP' = r_o - TP$
$r = r_o - R(\delta_o - \delta)$

Equations of the projection:

$$\alpha' = \cos\delta_o\,\alpha$$
$$r = r_o - R(\delta_o - \delta) \qquad \text{V-5}$$

where $r_o = R\tan\delta_o$

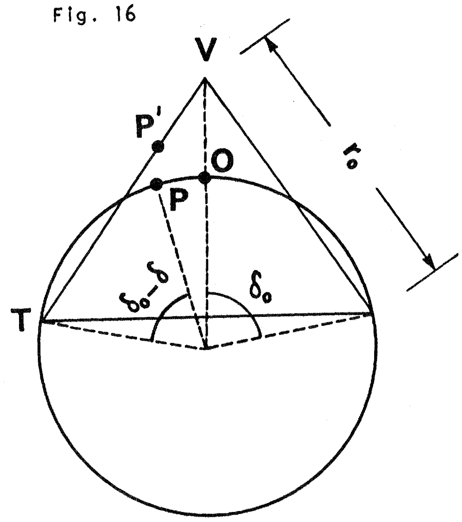
For the cartographic pole, $\delta = 0$, and $r = r_o - R\delta_o$
(i.e. The pole maps as a circle)

Exercises: (1) Show that $a = 1$, i.e. that the meridian is distortionless.

(2) Show that $b = \dfrac{\sin\delta_o + (\delta - \delta_o)\cos\delta_o}{\sin\delta}$

Distortionless meridian projection when the pole is mapped as a point

Fig. 16



Condition: $r_o = TV = TO$

where $TO = R\delta_o$

$$r_o = R\delta_o$$

$$\alpha' : \alpha = \alpha'_M : 2\pi$$

$$\alpha' = \frac{\alpha'_m}{2\pi} \cdot \alpha$$

But $\alpha'_M = \dfrac{2\pi\ R \sin \delta_o}{r_o}$

$$\alpha' = \frac{2\pi\ R \sin \delta_o}{R\ \delta_o\ .\ 2\pi} \ .$$

Hence $n = \dfrac{\sin \delta_o}{\delta_o}$ V-6

Equations of the projection:

$$\alpha' = \frac{\sin \delta_o}{\delta_o}\alpha$$

$$r = r_o - R(\delta_o - \delta) = r_o - R\delta_o + R\delta = R\delta \qquad \text{V-7}$$

To find Tissot's Indicatrices:

$$a = \frac{dr}{Rd\delta} = \frac{Rd\delta}{Rd\delta} = 1$$

$$b = \frac{r}{R \sin \delta} \cdot \frac{d\alpha'}{d\alpha} = \frac{R\delta}{R \sin \delta} \cdot \frac{\sin \delta_o}{\delta_o} = \frac{\delta \sin \delta_o}{\delta_o \sin \delta}$$

Is this projection equal-area? No

What is the area distortion factor?

What is the maximum angular distortion?

Equal-area projection

Condition: $a.b = 1$

i.e. $\frac{dr}{Rd\delta} \cdot \frac{r}{R \sin \delta} \cdot \frac{d\alpha'}{d\alpha} = 1$

But we define $\alpha' = n\alpha$

hence $d\alpha' = nd\alpha$

Thus $\frac{n\ r\ dr}{R^2 \sin \delta\ d\delta} = 1$

Integration gives

$$n \int r\ dr = R^2 \int \sin \delta\ d\delta$$

and $\frac{1}{2} n\ r^2 = -R^2 \cos \delta + C\ R^2$

in which CR^2 is an arbitrary constant,

which gives $r = R \sqrt{\frac{2}{n}} \sqrt{C - \cos \delta}$ V-8

and along with $\alpha' = n\ \alpha$ defines any equal-area conical projection

n, and c are constants which may be derived to satisfy any two conditions of our choice

e.g. (1) Pole to be mapped as a point

(2) Parallel δ_o mapped without distortion (one std. parallel)

From (1) when $\delta = 0$, $r = 0$ $\qquad 0 = R \sqrt{\frac{2}{n}} \sqrt{C - 1}$

hence $C = 1$

From (2) and V-3 for one std. parallel $\qquad n = \frac{R \sin \delta_o}{r_o}$

Thus the equations for the equal-area projection with one standard parallel and in which the pole is mapped as a point are:

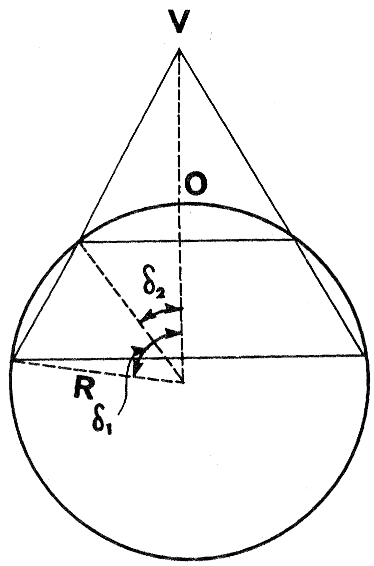
$$r = R \sqrt{\frac{2}{n}} \sqrt{1 - \cos \delta}$$

$$\alpha' = n\alpha = \frac{R \sin \delta_o}{r_o} \alpha \qquad \text{V-9}$$

Exercise: Show that $a.b = 1$

Alber's equal-area projection with two standard parallels

Fig. 17



Given R, δ_1 and δ_2

$$\alpha'_M = \frac{2\pi\ R \sin \delta_1}{r_1}$$

From V-2 $\quad \frac{\alpha'_M}{2\ \pi} = n$

For one standard parallel

$$R \sin \delta_1 = n\ r_1$$

Similarly, for the second standard parallel

$$R \sin \delta_2 = n\ r_2$$

Thus, we have two conditions for finding the unknown constants n and c:

$$R \sin \delta_1 = n\ R\sqrt{\frac{2}{n}\ \ C-\cos \delta_1}$$

$$R \sin \delta_2 = n\ R\sqrt{\frac{2}{n}\ \ C-\cos \delta_2}$$

Squaring:

$$\sin^2\delta_1 = 2n\ (C-\cos \delta_1) \qquad \text{V-10a}$$

$$\sin^2\delta_2 = 2n\ (C-\cos \delta_2) \qquad \text{V-10b}$$

$$\therefore \quad \frac{\sin^2\delta_1}{\sin^2\delta_2} = \frac{C-\cos \delta_1}{C-\cos \delta_2} \quad \text{i.e.} \quad \frac{1-\cos^2\delta_1}{1-\cos^2\delta_2} = \frac{C-\cos \delta_1}{C-\cos \delta_2}$$

By cross multiplication:

$$C-C\ \cos^2\delta_1 - \cos \delta_2 + \cos^2\delta_1\ \cos \delta_2 =$$
$$C-C\ \cos^2\delta_2 - \cos \delta_1 + \cos \delta_1\ \cos^2\ \delta_2$$

$$C(\cos^2\delta_2-\cos^2\delta_1) = \cos \delta_2 - \cos \delta_1 + \cos \delta_1\ \cos^2\delta_2 - \cos^2\delta_1\ \cos \delta_2$$

$$\text{''} \quad = \quad \text{''} \quad + \cos \delta_1\ \cos \delta_2\ (\cos \delta_2 - \cos \delta_1)$$

$$C = \frac{(\cos \delta_2-\cos \delta_1)\ \ (1+\cos \delta_1\ \cos \delta_2)}{(\cos \delta_2-\cos \delta_1)\ \ (\cos \delta_2+\cos \delta_1)} \qquad \text{V-11}$$

To get n, subtract V-10b from V-10a:

$$1-\cos^2\delta_1 = 2\ n\ C - 2\ n\ \cos \delta_1$$
$$1-\cos^2\delta_2 = 2\ n\ C - 2\ n\ \cos \delta_2$$
$$\cos^2\delta_2 - \cos^2\ \delta_1 = 2n\ (\cos \delta_2 - \cos \delta_1)$$

$$n = \frac{\cos \delta_2 + \cos \delta_1}{2} \qquad \text{V-12}$$

Hence:

$$\alpha' = \frac{\cos \delta_2 + \cos \delta_1}{2} \cdot \alpha \qquad \text{V-13}$$

$$r = \frac{2R}{\sqrt{\cos \delta_1 + \cos \delta_2}} \sqrt{\frac{1+\cos \delta_1 \cos \delta_2}{\cos \delta_1 + \cos \delta_2} - \cos \delta}$$

To find a and b: $a = \frac{dr}{Rd\delta}$; $b = \frac{r}{R \sin \delta} \cdot \frac{d\alpha'}{d\alpha}$

$$r = R \sqrt{\frac{2}{n}} \sqrt{C - \cos \delta}$$

$$dr = R \frac{1}{2} \sqrt{\frac{2}{n}} \cdot \frac{\sin \delta \, d\delta}{\sqrt{C - \cos \delta}}$$

$$a = \frac{R}{\sqrt{2n}} \cdot \frac{\sin \delta \, d\delta}{\sqrt{C - \cos \delta}} \cdot \frac{1}{R \, d\delta} = \frac{1}{\sqrt{2n}} \cdot \frac{\sin \delta}{\sqrt{C - \cos \delta}}$$

$$b = \frac{R\sqrt{\frac{2}{n}} \sqrt{C - \cos \delta}}{R \sin \delta} \cdot \frac{\cos \delta_2 + \cos \delta_1}{2}$$

but $\underline{\cos \delta_2 + \cos \delta_1} = n$

$$b = \frac{\sqrt{2n} \sqrt{C - \cos \delta}}{\sin \delta}$$

$$a.b = 1$$

To show that the parallel defined by δ_1 is standard (i.e. b = 1)

$$b = \frac{\sqrt{2n} \sqrt{C - \cos \delta}}{\sin \delta} \qquad \frac{\sqrt{2n} \sqrt{C - \cos \delta_1}}{\sin \delta_1}$$

$$= \frac{\sqrt{2} \cdot \sqrt{\frac{\cos \delta_2 + \cos \delta_1}{2}} \cdot \sqrt{\frac{1 + \cos \delta_1 \cos \delta_2}{\cos \delta_1 + \cos \delta_2} - \cos \delta_1}}{\sin \delta_1}$$

$$= \frac{\sqrt{\cos \delta_2 + \cos \delta_1} \cdot \sqrt{\frac{1 + \cos \delta_1 \cos \delta_2 - \cos \delta_1 (\cos \delta_1 + \cos \delta_2)}{\cos \delta_1 + \cos \delta_2}}}{\sin \delta_1}$$

$$= \frac{\sqrt{1 + \cos \delta_1 (\cos \delta_2 - \cos \delta_1 - \cos \delta_2)}}{\sin \delta_1} = \frac{\sqrt{1 - \cos^2 \delta_1}}{\sin \delta_1} = 1$$

<u>Note</u>: In this discussion of Alber's projection it was assumed that δ_1 and δ_2 were given. It may be that we want to choose δ_1 and δ_2 to satisfy some condition, e.g. the condition that the angular distortion at the upper and lower ends of the map should be equal to each other and also equal to the angular distortion in the central part of the map.

If δ' is the cartographic latitude of the top and δ'' the cartographic latitude of the bottom of the map then the condition specified above will be satisfied when

$$\cos \frac{\delta_2 - \delta_1}{2} = \sqrt{\cos \frac{\delta'' - \delta'}{2}}$$

$$\cos \frac{\delta_2 + \delta_1}{2} = \frac{\cos \frac{\delta'' + \delta'}{2}}{\sqrt{\cos \frac{\delta'' - \delta'}{2}}}$$

The proof for this can be found in T.R. No. 5, D.S.E., Map Projections by G. Konecny, pp. 29-34.

Conformal conical projections

For conformality $a = b$

$$a = \frac{dr}{Rd\delta} \quad , \qquad b = \frac{n\,r}{R \sin\delta}$$

Hence $\frac{dr}{r} = n \frac{d\delta}{\sin\delta}$

$$\int \frac{dr}{r} = \ell n\ r + c$$

$$\int \frac{d\delta}{\sin\delta} = \int \frac{\sin^2\delta/2 + \cos^2\delta/2}{\sin\delta/2 \cos\delta/2}\ d\ \delta/2$$

$$= \int \frac{\sin\delta/2}{\cos\delta/2}\ d\ \delta/2 + \int \frac{\cos\delta/2}{\sin\delta/2}\ d\ \delta/2$$

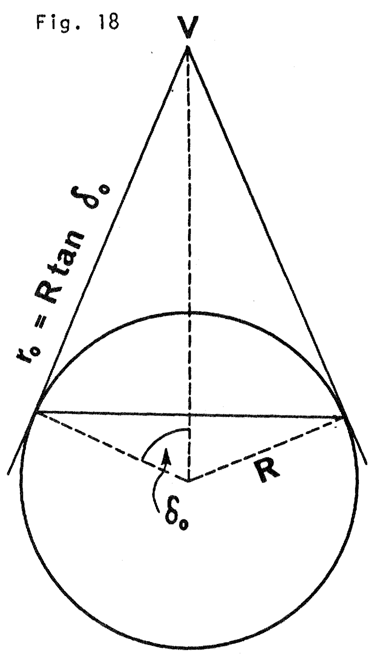
$$= -\ell n \cos\delta/2 + \ell n \sin\delta/2 + c = \ell n \tan\delta/2 + c$$

$$\ell n\ r = n \cdot \ell n \tan\delta/2 + \ell n\ c$$

$$\left.\begin{array}{l} r = c\ (\tan\delta/2)^n \\ \text{and } \alpha' = n\ \alpha \end{array}\right\} \text{General equations of conformal conical projections.} \qquad \text{V-14}$$

Conformal projection on tangent cone with one standard parallel

Fig. 18



From V-4 for tangent cone:

$n = \cos\delta_o$ and $r_o = R \tan\delta_o$.

For $r = r_o$

$$R \tan\delta_o = C(\tan\delta_{o/2})^{\cos\delta_o}$$

i.e. $c = \dfrac{R \tan\delta_o}{(\tan\delta_{o/2})^{\cos\delta_o}}$, and

$$r = \frac{R \tan\delta_o}{(\tan\delta_{o/2})^{\cos\delta_o}} (\tan\delta/2)^{\cos\delta_o}$$

$$\alpha' = \cos\delta_o\ \alpha \quad . \qquad \text{V-15}$$

For the pole:

$\lim\limits_{\delta \to o} r = 0$ $\therefore$ the pole maps as a point.

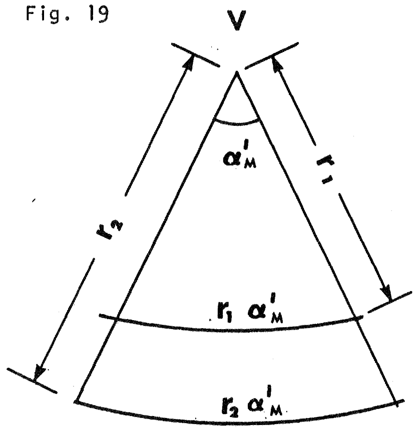
Exercise:

Show that $a = \dfrac{dr}{Rd\delta} = \dfrac{\sin\delta_o}{\sin\delta} \cdot \dfrac{(\tan\delta/2)^{\cos\delta_o}}{(\tan\delta_{o/2})^{\cos\delta_o}}$

$$b = \frac{nr}{R \sin\delta} = \frac{\sin\delta_o}{\sin\delta} \cdot \frac{(\tan\delta/2)^{\cos\delta_o}}{(\tan\delta_{o/2})^{\cos\delta_o}} \quad .$$

Conformal conical projection with two standard parallels (Lambert's conformal)

Fig. 19



Condition for standard parallels:

Sphere		Plane
$2\pi\ R \sin \delta_1$	=	$r_1\ \alpha'_M = r_1\ 2\pi\ n$
$2\pi\ R \sin \delta_2$	=	$r_2\ \alpha'_M = r_2\ 2\pi\ n$.

But for conformal projection

$$r = c\ (\tan \delta/2)^n \quad .$$

For $r = r_1$: $2\pi\ R \sin \delta_1 = c(\tan \delta_{1/2})^n \cdot 2\pi\ n$.

V-16

For $r = r_2$: $2\pi\ R \sin \delta_2 = c(\tan \delta_{2/2})^n \cdot 2\pi\ n$.

Dividing: $\dfrac{\sin \delta_1}{\sin \delta_2} = \dfrac{(\tan \delta_{1/2})^n}{(\tan \delta_{2/2})^n}$

$$n = \frac{\ln \sin \delta_1 - \ln \sin \delta_2}{\ln \tan \delta_{1/2} - \ln \tan \delta_{2/2}} \qquad \text{V-17}$$

and from V-16 $\quad c = \dfrac{R \sin \delta_1}{n(\tan \delta_{1/2})^n} = \dfrac{R \sin \delta_2}{n(\tan \delta_{2/2})^n}$. V-18

By using a series of these the world, except for polar regions, can be mapped conformally.

CHAPTER VI

NON-GENUINE PROJECTIONS

In non-genuine projections:

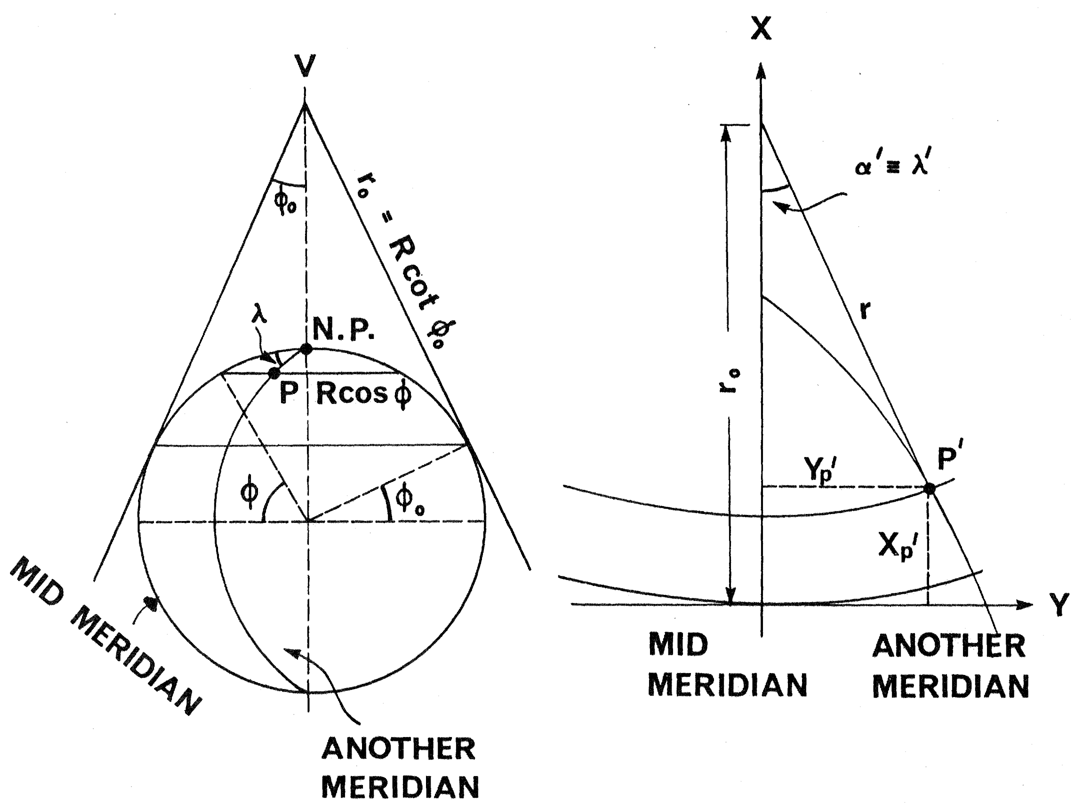
(1) The images of meridians and parallels do <u>not</u> intersect perpendicularly.

(2) The meridians and parallels are <u>not</u> the directions of extreme distortion. Thus the simplified expressions (Ch. II) for computing distortions cannot be used; it is necessary to use the general expressions developed in Ch. I.

There is no limit to the number of non-genuine projections that one can devise. Bearing in mind that these projections are not conformal their usefulness to surveyors is limited. Nevertheless countless non-genuine projections have been developed and a few have been found to be useful; Bonne's projection is one of these. It is analyzed here to illustrate the flexibility and the problems that arise in the use of a non-genuine projection. It can be classified as a pseudo-conical projection.

Fig. 20

(a) (b)



Bonne's Projection

As illustrated and described in Roblin, p. 31 in this projection the sphere is mapped onto the tangent cone and

(1) The images of the parallels are concentric circles without distortion in length.

(2) The mid-meridian is standard.

For simplicity we will use geographic rather than cartographic coordinates.

For the tangent cone $r_o = R \cot \phi_o$ VI-1

For standard parallels (i.e. all of them) $R\lambda \cos\phi = r\alpha'$ VI-2

For the standard mid-meridian $r = r_o + R(\phi_o - \phi)$ VI-3

Thus the equations of Bonne's projection are:

$$r = r_o + R(\phi_o - \phi) \ ; \quad \alpha' = \frac{R \cos \phi}{r} \lambda$$

i.e. $$X = r_o - r \cos \alpha' \ ; \quad Y = r \sin \alpha' \qquad \text{VI-4}$$

$$\frac{\partial X}{\partial r} = -\cos \alpha' \qquad \frac{\partial Y}{\partial r} = \sin \alpha'$$

$$\frac{\partial X}{\partial \alpha'} = r \sin \alpha' \qquad \frac{\partial Y}{\partial \alpha'} = r \cos \alpha' \qquad \text{VI-5}$$

$$\frac{\partial r}{\partial \phi} = -R \qquad \frac{\partial \alpha'}{\partial \phi} = \frac{(r_o+R\phi_o-R\phi)(-\sin \phi)-\cos \phi(-R)}{(r_o+R\phi_o-R\phi)^2} . R\lambda$$

$$\frac{\partial r}{\partial \lambda} = 0 \qquad = [-r \sin \phi + R \cos \phi] \frac{R\lambda}{r^2}$$

$$\frac{d\alpha'}{\partial \lambda} = \frac{R \cos \phi}{r}$$

$$f_\phi = \frac{\partial X}{\partial r} \cdot \frac{\partial r}{\partial \phi} + \frac{\partial X}{\partial \alpha'} \cdot \frac{\partial \alpha'}{\partial \phi} = -\cos\alpha'(-R) + r \sin \alpha'[R \cos\phi - r \sin \phi] \frac{R\lambda}{r^2}$$

$$= R[\cos \alpha' - \lambda \sin \alpha'(\sin \phi - \frac{R \cos \phi}{r})]$$

$$f_\lambda = \frac{\partial X}{\partial r} \cdot \frac{\partial r}{\partial \lambda} + \frac{\partial X}{\partial \alpha'} \cdot \frac{\partial \alpha'}{\partial \lambda} = 0 + r \sin \alpha' x \frac{R \cos \phi}{r} = R \sin \alpha' \cos \phi$$

VI-6

$$g_\phi = \frac{\partial Y}{\partial r} \cdot \frac{\partial r}{\partial \phi} + \frac{\partial Y}{\partial \alpha'} \cdot \frac{\partial \alpha'}{\partial \phi} = \sin \alpha'(-R) + r \cos \alpha'[-r \sin \phi + R \cos\phi] \frac{R\lambda}{r^2}$$

$$= -R[\sin \alpha' + \lambda \cos \alpha'(\sin \phi - \frac{R \cos \phi}{r})]$$

$$g_\lambda = \frac{\partial Y}{\partial r} \cdot \frac{\partial r}{\partial \lambda} + \frac{\partial Y}{\partial \alpha'} \cdot \frac{\partial \alpha'}{\partial \lambda} = 0 + r \cos \alpha' \ x \frac{R \cos \phi}{r} = R \cos \alpha' \cos \phi$$

$$m_1 = \sqrt{\frac{f_\phi^2 + g_\phi^2}{R^2}} = \sqrt{1 + \lambda^2 (\sin\phi - \frac{R\cos\phi}{r})^2}$$

$$p = 2\,\frac{f_\phi\, f_\lambda + g_\phi\, g_\lambda}{R^2 \sin\phi} = 2\lambda\,(\frac{R\cos\phi}{r} - \sin\phi) \qquad \text{VI-7}$$

$$m_2 = \sqrt{\frac{f_\lambda^2 + g_\lambda^2}{R^2\cos^2\phi}} = \sqrt{\frac{R^2\cos^2\phi(\sin^2\alpha' + \cos^2\alpha')}{R^2\cos^2\phi}} = 1$$

Hence we get

$$m_A^2 = [1+\lambda^2(\sin\phi - \frac{R\cos\phi}{r})^2]\cos^2A + \sin^2A - 2\lambda(\sin\phi - \frac{R\cos\phi}{r})\sin A\cos A$$

Let $q = \lambda(\sin\phi - \frac{R\cos\phi}{r})$

$$m_A^2 = (\cos^2A + \sin^2A) + q^2\cos^2A - 2q\sin A\cos A$$

$$\frac{dm_A^2}{dA} = -2q^2\cos A\sin A - 2q(\cos^2A - \sin^2A)$$

$$= -q^2\sin 2A_E - 2q\cos 2A_E = 0$$

$$\tan 2A_E = -2/q = \frac{-2}{\lambda(\sin\phi - \frac{R\cos\phi}{r})}$$

which yields A_{E1} and $A_{E1} + 90°$

Exercise: Show that Bonne's projection is an equal-area projection, i.e. show that:

$$|f_\lambda\, g_\phi - f_\phi\, g_\lambda| = R^2\cos\phi \quad .$$