GRAVIMETRIC SATELITE GEODESY

P. VANICEK

March 1973

LECTURE NOTES
NO. 32

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By

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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

FORWORD

When setting up this course I tried to be faithful to my principles:

- to begin with assuming as little initial knowledge from the students as possible;

- to define all the used terms properly;

- to present all the logical arguments behind the structure of the subject avoiding "logical gaps";

- to concentrate on the concepts and go into applications and technicalities only if time permits. Due to the breadth of the presented subject, I found it rather difficult to do so within one term course. Hence, the student will find it necessary to bridge the inevitable gaps from outside sources referenced in the lecture notes. Also, somebody may grumble that the course is on the heavier side as far as the use of mathematics is concerned. This is so, because, as A. Einstein put it once, "the approach to a more profound knowledge of the basic principles of physics is tied up with the most intricate mathematical methods."

I should like to acknowledge the kind help of Dr. E.J. Krakiwsky and Mr. D.E. Wells who made me aware of some of my "overly original ideas." In addition, Mr. Wells suggested a reorganization of the first section that greatly improved the logical structure. I also owe many thanks to Mrs. Debbie Smith who expertly typed these notes from my atrocious hand-written manuscript. Any comments communicated to the author will be greatly appreciated.

P. Vanicek
Fredericton, N.B.
25/3/1973

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1) Review of Classical Mechanics

1.1 Introduction

Let us begin with the convention that throughout the course we shall be using subscript notation for vectors and matrices. The Latin letters will denote indices running from 1 to 3; Greek letters will be used for indices acquiring other values. Whenever the same subscript will be used twice in a product of two quantities, it will automatically imply that summation over the subscript takes place; the subscript becomes a dummy subscript.

The space we shall be working in will be the classical (Newtonian) physical space defined as follows. The physical space is a metric space, metricized with Euclidean metric. Positions of points are given by position vectors $x_i \varepsilon E_1$ $(i = 1,2,3)$, where by E_1 we denote a set of all real numbers; x_i are called the (rectangular) Cartesian coordinates. The distance between two points $x_i^{(1)}$, $x_i^{(2)}$ is hence given by:

$$s_{1,2} = \sqrt{\sum_{i=1}^{3} (x_i^{(1)} - x_i^{(2)})^2} \ \varepsilon \ \tilde{E}_1 \ .$$

(By $\tilde{E}_1$ we denote a set of all non-negative real numbers.) The points are allowed to "move in the space", which means that their positions may vary with a parameter t, called time. This is usually denoted as

$$x_i = x_i(t), \ t\varepsilon E_1 \ ,$$

meaning that the position vector x_i is a function of the scalar argument-time.

In the physical space there are physical objects possessing certain physical properties and acting upon each other according to certain physical laws. In classical mechanics, we deal with two kinds of physical objects: particles and physical bodies. The properties we shall be interested in are motion, velocity, acceleration, mass, gravitational force, momentum, kinetic energy.

In classical mechanics, no other interactions are considered but gravitation. Thus neither electromagnetic nor the nuclear interactions are regarded as being present. Also, velocities are assumed to be very low compared with the speed of light. The action of the gravitational force is considered instantaneous, i.e., the velocity of propagation of gravitation is considered infinite which distinguishes the classical mechanics from the relativistic mechanics.

Our use of classical mechanics will be limited to the study of the motion of particles in the physical space. The motion is described by equations known as equations of motion. Hence, derivation of various kinds of equations of motion will be considered our primary aim in this section.

1.2 Fundamental Definitions

By a particle (mass point) in classical mechanics, we understand a pair of elements

$$(x_i, m) \in E_3 \times E_1^+ ,$$

(by E_3 we denote the Cartesian product $E_1 \times E_1 \times E_1$, and E_1^+ is a set of all positive real numbers), where x_i is the vector of the point, taken as varying in time (x is a function of time), and m is a real number considered independent of time ($m \neq m(t)$) and called the <u>mass</u> of the particle. The position vector x_i is also known as the <u>motion</u> of the particle.

Taking a particle (x_i, m), it is useful to define its <u>velocity</u> $\dot{x}_i$ as

$$\frac{dx_i}{dt} = \dot{x}_i(t) \in E_3$$

and <u>acceleration</u> $\ddot{x}_i$ by

$$\frac{d\dot{x}_i}{dt} = \frac{d^2x_i}{dt^2} = \ddot{x}_i(t) \in E_3 \ .$$

We shall assume that these two functions, describing the same motion x_i of the particle,always exist. More will be said about it later.

If the velocity equals to zero, we say that the particle does not move. If the acceleration equals to zero, the particle is said to move inertially. A particular value of the motion x_i, i.e., $x_i(t)$, is called the <u>instantaneous position</u> of the particle. Similarly, $\dot{x}_i(t)$ is its <u>instantaneous velocity</u>, and $\ddot{x}_i(t)$ its <u>instantaneous acceleration</u>.

It has been determined from physical experiments with two particles that a presence of one particle influences the motion of the other. It has been observed that the two particles attract each other so that each particle acquires an acceleration directed towards the other. The two accelerations, observed with respect to the coordinate system (more precisely a stationary coordinate system--see section 1.5), are generally different. The property of the particle that determines its acceleration is the mass. Calling the mass of the first particle m, its motion x_i, the mass of the second particle μ, and its motion x_i, we define the mass as being inversely proportional to the acceleration, i.e.,

$$m\ddot{x} = \mu\ddot{X} \ . \qquad (*)$$

Using the vector notation and realizing that the two accelerations have opposite signs, we have

$$m\ddot{x}_i = -\mu\ddot{X}_i \ .$$

This vector quantity is called the gravitational force (acting on the particles). Denoting the forces acting on the two particles by f_i and F_i, we have

$$f_i = m\ddot{x}_i \ ,$$

$$F_i = \mu\ddot{X}_i$$

and

$$f_i = -F_i \ .$$

These formulae are known as Newton's law, and they link the mass, force, and acceleration related to a particle. Since $\ddot{x}_i$ $(\ddot{X}_i)$ is considered a function of time, then even the force $f_i(F_i)$ is a function of time. On the other hand, the mass in classical mechanics is always regarded as constant with respect to time, or conservative.

It has already been observed that the two accelerations remain inversely proportional to the square of the distance of the two particles. Denoting

$$x_i - X_i = - a_i \ , \qquad |a_i| = a \ ,$$

we have to require that

$$\ddot{x} = c/a^2 \ , \quad \ddot{X} = C/a^2 \ , \qquad (**)$$

where c and C are some constants characterising the two particles. Combining the two sets of equations, (*) and (**), we obtain

$$UC = mc \quad .$$

In other words, the ratio C/m, or c/U, is constant. Denoting this constant by κ, we can express the constants c and C in terms of the masses as follows:

$$c = \kappa\mu \ , \quad C = \kappa m \ .$$

Simple substitution into eqn. (**) yields

$$\ddot{x} = \kappa\mu/a^2 \ , \quad \ddot{X} = \kappa m/a^2 \ ,$$

and we can see that each particle renders the other an <u>acceleration proportional to its own mass</u>.

These equations can now be rewritten using the forces f_i and F_i yielding

$$\boxed{f_i = - F_i = \frac{\kappa\mu m}{a^3} \quad a_i \ .}$$

This formula is known as Newton's law of universal attraction or the <u>law of universal gravitation</u>. The constant of proportionality κ is called Newton's or the gravitation constant. It can be regarded as the ratio between the behaviour of the mass of a particle as a "source" of gravitation and the behaviour of the mass of the same particle as a "source" of gravitation. Its value, determined from experiments, is

$$\kappa \approx 6.67\ 10^{-11}\ m^3\ kg^{-1}\ sec^{-2}\ .$$

The vector function $\tilde{p}_i$ of position and time given as

$$\boxed{\tilde{p}_i(t) = m\dot{x}_i(t) \in E_3}$$

is called the <u>momentum</u> of the particle (x_i, m). It is not difficult to see that gravitational force f_i is linked with the momentum $\tilde{p}_i$ through the following equation:

$$\boxed{f_i(t) = \frac{d}{dt}(m\dot{x}_i(t)) = \dot{\tilde{p}}_i(t)\ .}$$

It is an empirically established fact that the acceleration is always a continuous and bounded function of time. Therefore, even the force is a continuous and bounded function of time. The velocity, hence, must be not only bounded and continuous but also a smooth function of time and so must be the motion of the particle.

The scalar function $\tilde{T}$ of position and time related to the particle (x_i, m) by the formula

$$\boxed{\tilde{T}(t) = \frac{1}{2} m\ \dot{x}_i(t)\ \dot{x}_i(t) \in E_1\ ,}$$

is called the kinetic energy of the particle. Note that $x_i\ x_i = x^2$ and that the kinetic energy can also be expressed by the momentum as

$$\tilde{T}(t) = \frac{\tilde{p}_i(t)\tilde{p}_i(t)}{2m} \quad .$$

Writing the first formula for kinetic energy in the classical notation, i.e.,

$$\tilde{T} = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 \quad ,$$

it is not difficult to see that

$$\frac{\partial \tilde{T}}{\partial \dot{x}_j} = \frac{1}{2} m \, 2\dot{x}_j = m\dot{x}_j = \tilde{p}_j \quad .$$

Since $\dot{\tilde{p}}_i = f_j$, we can also write

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{x}_j}\right) = \dot{\tilde{p}}_j = f_j \quad ,$$

which is the <u>relationship</u> between <u>kinetic energy and force</u>.

1.3 <u>Gravitational Field, Potential</u>

Let us now return to the law of universal gravitation. It is obviously valid for both particles involved, and its meaning depends on which particle one associates himself with. This "ambiguity" may prove difficult to keep track of. It is, therefore, convenient to regard one of the particles as a "source" of gravitation or attracting particle and the other as "sensing" the gravitation or attracted particle. This is known as the concept of <u>gravitational field</u>. It can be mathematically formulated as

$$\ddot{x}_i = \frac{\kappa\mu}{a^3} a_i .$$

Note that this is nothing else but the law of universal gravitation divided by the mass of the sensing (or attracted) particle, where a_i is the vector joining the attracted with the attracting particles.

In the above equation, $\ddot{x}_i$ is no longer an acceleration of a specific particle but an acceleration field; from the mathematical point of view, $\ddot{x}_i$ is a central vector field. If a particle with mass m happens to occur at a particular place x_i in the field, then obviously the field would start attracting it with a force

$$f_i = m\ddot{x}_i = \frac{\kappa\mu m}{a^3} a_i .$$

The field can exert a force at a point x_i if and only if m at x_i is different from zero, i.e., only if there is a particle present at x_i.

Since a vector field is more awkward to deal with than a scalar field, we shall try to simplify the concept of the gravitational field further. We define a scalar field U such that it satisfies the following equation:

$$\ddot{x}_i = - \frac{\partial U}{\partial x_i} = - \text{grad } U .$$

This scalar field is called gravitational potential or attracting potential. It can be thought of as again generating gravitation given by

$$f_i = - m \frac{\partial U}{\partial x_i} = m\ddot{x}_i .$$

It can be shown that the potential U of the particle(X_i, μ) is given by

$$U = - \frac{\kappa\mu}{a} \quad .$$

The proof of this formula reads as follows:

$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial a} \frac{\partial a}{\partial x_i}$$

$$\frac{\partial U}{\partial a} = \frac{\kappa\mu}{a^2}$$

$$\frac{\partial a}{\partial x_i} = \frac{\partial}{\partial x_i} \sqrt{\sum_{j=1}^{3} (x_j - X_j)^2} = \frac{1}{2a} \frac{\partial}{\partial x_i} \sum_{j=1}^{3} (x_j - X_j)^2 = \frac{(x_i - X_i)}{a} = - \frac{a_i}{a} \, .$$

Hence,

$$\frac{\partial U}{\partial x_i} = - \kappa\mu \frac{a_i}{a^3} = - \ddot{x}_i \; ,$$

which was to be proved.

Considering a cluster of rigidly connected particles and an attracted particle, the situation will be very much the same. Disregarding the effect of the particle on the cluster, or connecting the cluster to the coordinate system, which is the same thing, the attracted particle will move according to the sum of all the forces generated by the particles in the cluster. The sum F of the forces will no longer be described by a central field unless the cluster is infinite and arranged with a spherical symmetry around its centre of gravity.
This gravitational field can now again be represented by its potential U, defined in exactly the same way as in the case of one particle field, i.e.,

$$m\ddot{x}_i = - m \frac{\partial U}{\partial x_i} \quad .$$

The just described situation can be immediately applied to a rigid body that is nothing else but a cluster of rigidly connected particles. Hence we may also talk about the gravitational potential of a rigid body or, for that matter, the gravitational potential of any physical body.

The gravitational potential U can either be an explicit function of time, i.e., vary not only with place but also with time, or may not. When it is, it is called non-conservative potential and the force it delivers (is also a function of time) is known as non-conservative force. If the potential is not varying with time, it is called stationary or conservative and the corresponding force -- m $\partial U/\partial x_i$ -- is also known as conservative.

At this point, it comes in handy to realize that we have been dealing with two parallel sets of quantities. One set can be obtained from the other just by considering the mass m or omitting it. Thus we can distinguish the following corresponding pairs:

(x_i, m)	. . . x_i	(particle	. . . motion)
$\tilde{p}_i$	. . . $\dot{x}_i$	(momentum	. . . velocity)
f_i	. . . $\ddot{x}_i$	(force	. . . acceleration).

Defining two more quantities, namely, gravitational (attracting) energy $\tilde{U}$ (usually called the potential energy) as

$$\tilde{U} = mU$$

and <u>kinetic potential</u> T as

$$T = \tilde{T}/m,$$

we can complete the list as follows

$\tilde{T}$ . . . . . . T (kinetic energy ..... kinetic potential)

$\tilde{U}$ . . . . . . U (gravitational energy .... gravitational potential).

Based on these two sets of quantities are two branches of mechanics: <u>kinetics</u> and <u>kinematics</u>. While kinetics deal with particles, and therefore masses, the kinematics deal only with motions and its relatives. All the equations we have derived so far, can be formulated in both ways. For example

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{x}_i}\right) = f_i \quad . \; . \; . \; . \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = \ddot{x}_i \ ,$$

$$f_i = \frac{\partial \tilde{U}}{\partial x_i} \quad . \; . \; . \; . \quad \ddot{x}_i = - \frac{\partial U}{\partial x_i} \ .$$

We shall tend to use more the kinematic approach.

To conclude this paragraph, let us state without proof one useful physical law. It has been established that if the gravitational potential U (and therefore even $\tilde{U}$) is conservative, the total energy $\tilde{T} + \tilde{U}$ of a particle moving in the field is conserved. This is known as the <u>law of conservation of energy</u> and it means that

$$\tilde{E} = \tilde{T} + \tilde{U}$$

is not an explicit function of time. The consequence is that the kinetic energy is not an explicit function of time either. We have for a conservative field:

$$\frac{\partial\tilde{E}}{\partial t} = \frac{\partial}{\partial t}(\tilde{T} + \tilde{U}) = \frac{\partial\tilde{T}}{\partial t} + \frac{\partial\tilde{U}}{\partial t} = 0 \ .$$

But

$$\frac{\partial\tilde{U}}{\partial t} = 0$$

since $\tilde{U}$ is not an explicit function of time, hence

$$\frac{\partial\tilde{T}}{\partial t} = 0$$

and $\tilde{T}$ is not an explicit function of time either.

1.4) Equations of Motion

In section 1.2 we have defined the gravitational force by

$$f_i(t) = m\,\ddot{x}_i(t) \quad .$$

In classical mechanics, we think about the gravitational energy as the only cource of force f_i. This leads immediately to the conclusion that it is only the gravitational energy that generates the motion of particles in classical mechanics. Mathematically, this is expressed by our known equation

$$m\,\ddot{x}_i(t) = -\,m\,\frac{\partial U(t)}{\partial x_i} \quad ,$$

or, equivalently,

$$\ddot{x}_i(t) = -\,\frac{\partial U(t)}{\partial x_i} \quad .$$

These differential equations of second order are hence known as the <u>equations of motions</u> (of the particle) in the potential field U.

Since the potential U is a function of position (as well as time) they have got the following form:

$$\ddot{x}_i(t) + \phi_i(t, x_j(t)) = 0 ,$$

where, by the symbol ϕ_i, we denote the partial derivative $\partial U/\partial x_i$. Hence, the equations are not independent--they represent a system of three differential equations of second order that can be solved only if ϕ_i is a very simple function.

The solution $x_i(t)$ is a function of time containing 6 integration constants determined usually from the initial conditions, i.e., from the state of the motion x_i at a particular initial time t_o. We may note that since 6 constants have to be determined, it is not enough to know just the values of the 3 components $x_1(t_o)$, $x_2(t_o)$, $x_3(t_o)$ at t_o. We usually have to know also the velocities $\dot{x}_1(t_o)$, $\dot{x}_2(t_o)$, $\dot{x}_3(t_o)$ at the same time.

One such solution is possible if U is the central field potential, i.e.,

$$U = - \frac{\kappa\mu}{r} ,$$

where $r = \sqrt{\sum_{i=1}^{3} x_i^2}$ is the distance of the attracted point from the centre of the central field taken as the origin of the coordinate system at the same time. Then the gradient is given, as we have seen in section 1.3, by

$$\partial U/\partial x_i = - \kappa\mu \frac{x_i}{r^3} .$$

The solution was arrived at by J. Kepler at the beginning of the seventeenth century on the basis of Tycho's observations and is summarized in his three famous laws [Kovalevsky, 1967]:

(1) The particle moves on a plane curve of second order and its radius-vector sweeps out equal areas in equal time intervals.

(2) The plane of the curve contains the centre of coordinates that coincides with one of the foci of the curve.

(3) The squares of the periods of orbit on elliptical (circular) curves are proportional to the cubes of the semi-major axes of the ellipses (circles).

Let us state here that for a more complicated form of U we are not able to use this simple approach and have to go for a more sophisticated mathematical treatment. This treatment involves the so-called canonic equations of motion and will constitute the rest of the first section. The first step towards such a formulation is to rewrite the equations of motion in a slightly different manner.

Since the attracting energy is the only source of force in classical mechanics, what is the kinetic energy? It is useful to think about kinetic energy as a "measure of force." Adopting this approach, we can say that the force "exerted on a particle by the attracting energy" must equal the force as "indicated by its kinetic energy." In mathematical terms, this reads

$$-\frac{\partial \tilde{U}}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{x}_i}\right) .$$

Evidently, this is another formulation of the equations of motion. In kinematics, they read

$$\boxed{-\frac{\partial U}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)}$$

and we may recall having seen them in section 1.3 already. We shall see later that these euqations of motion can be further generalized and then converted into the canonic equations we are looking for.

1.5) Generalized Coordinates

When describing problems in classical mechanics, we do not have to use the Cartesian coordinates as we have done thus far. As a matter of fact, it is normally more convenient to use different systems for different problems. Some such systems are almost dictated by the character of the problem we are to deal with.

Generally, any triplet of functions

$$q_i = q_i(x_j)$$

of x_j can be used for the coordinate system. However, we usually require that the two coordinate systems, x_i, q_j, are in one-to-one relation, i.e., that to each triplet x_i there corresponds one and only one triplet q_j, and vice-versa. Hence, there also exists a triplet of functions, inverse to the above:

$$x_i = x_i(q_j) \quad .$$

This means that the Jacobian matrix of transformation

$$\mathcal{J}_{ij} = \frac{(q_i)}{(x_j)}$$

is a non-singular matrix and so is its inverse

$$\tilde{\mathcal{J}}_{ij} = \frac{(x_i)}{(q_j)} \quad .$$

It is usual in mechanics to extend the definition of such arbitrary new coordinate systems to a set of ν functions q_α such that the original (rectangular) Cartesian coordinates are expressed as

$$x_i = x_i(q_\alpha) \quad .$$

The number ν of such new functions must equal the number of degrees of freedom of the mechanical system that we want to study. Such functions q_α are usually called generalized coordinates, since they cannot be considered natural coordinates in the three-dimensional space in which we work.

In our case, we shall be dealing with only one particle (point, motion) moving in an attracting field considered stationary in the coordinate system. Hence, the number of degrees of freedom of our mechanical system (consisting of the attracting field and the moving point) will be equal to 3. These degrees of freedom can be visualized as representing the three coordinates of the point; the remaining 6 degrees of freedom belonging to the body emanating the attracting field are removed by fixing the coordinate system to the body. We shall be then dealing with just three coordinates q_j and yet call them also generalized coordinates, conforming to the custom in mechanics.

In general, the generalized, as well as the Cartesian coordinates, can be defined as varying with time:

$$x_i = x_i(q_j,\ t) \quad , \qquad q_i = q_i(x_j,\ t) \ .$$

Let us have a look at what happens to the velocities and accelerations if this is the case. Defining velocity $\dot{x}_i$ as total derivative of x_i with respect to time, we have the following relationship:

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial q_j}\frac{dq_j}{dt} = \frac{\partial x_i}{\partial t} + \tilde{J}_{ij}\,\dot{q}_j$$

where $\dot{q}_j$ is the generalized velocity. Similarly

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial x_j}\frac{dq_i}{dt} = \frac{\partial x_i}{\partial t} + J_{ij}\,\dot{x}_j \quad .$$

Thus the fact that the coordinate systems depend on time gives rise to the terms $\partial x_i/\partial t$ and $\partial q_i/\partial t$ (partial derivatives) that can be regarded as <u>virtual velocities</u>.

For the accelerations, we obtain

$$\ddot{x}_i = \frac{\partial^2 x_i}{\partial t^2} + \frac{\partial x_i}{\partial q_j}\frac{d^2 q_j}{dt^2} + \frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\frac{dq_j}{dt}\frac{dq_\ell}{dt}$$

$$= \frac{\partial^2 x_i}{\partial t^2} + \tilde{J}_{ij}\,\ddot{q}_j + \frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\dot{q}_j\,\dot{q}_\ell$$

and, similarly,

$$\ddot{q}_i = \frac{\partial^2 q_i}{\partial t^2} + J_{ij}\,\ddot{x}_j + \frac{\partial^2 q_i}{\partial x_j \partial x_\ell}\dot{x}_j\,\dot{x}_\ell$$

Here again the second partial derivatives with respect to time disappear if the coordinate systems are not functions of time or, more precisely, if they are only linear functions of time ($\partial x_i/\partial t$ = const., $\partial q_i/\partial t$ = const.).

Thus these terms can be again regarded as virtual (accelerations) only, i.e. depending on the mutual motion of the coordinate systems, and having nothing to do with the mechanical system we describe.

Two coordinate systems that move with respect to each other with constant relative velocity ($\partial x_i/\partial t$ = const., $\partial q_i/\partial t$ = const.) are called mutually inertial. Conversely, if their mutual velocities are not constant, they are known as mutually non-inertial. We can see that when dealing with a mechanical system in two inertial systems of coordinates we do not observe any virtual accelerations or forces; this considerably simplifies the investigations.

To make things even more simple, we usually choose the rectangular Cartesian coordinate system so that

$$\frac{\partial x_i}{\partial t} = 0 \ .$$

Such coordinate system is called stationary and can be realized by letting the individual coordinate axes point to fixed directions among the stars. In this system, the fixed stars do not appear to move and we say that the coordinate system does not vary with time. It is called the inertial frame.

From now on, we shall assume that neither x_i nor q_j systems depend on time. Then the relations between the velocities and accelerations in the two systems become

$$\dot{x}_i = \tilde{\mathcal{Y}}_{ij}\, \dot{q}_j, \quad \dot{q}_i = \mathcal{Y}_{ij}\, \dot{x}_j$$

$$\boxed{\ddot{x}_i = \tilde{\tilde{\mathcal{Y}}}_{ij}\, \ddot{q}_j + \frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\, \dot{q}_j \dot{q}_\ell}\ , \qquad \ddot{q}_i = \mathcal{Y}_{ij}\, \ddot{x}_j + \frac{\partial^2 q_i}{\partial x_j \partial x_\ell}\, \dot{x}_j \dot{x}_\ell \ .$$

1.6) Lagrangian Equations of Motion

At the end of 1.4 we have developed the generalized equations of motion. Let us have a look now at what form will they acquire in a system of generalized coordinates q_j. Considering, to begin with, the attractive potential $U(x_i)$ to be conservative, we have

$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial q_k}\frac{\partial q_k}{\partial x_i} = \mathcal{J}_{ki}\frac{\partial U}{\partial q_k} .$$

These equations can be rewritten as

$$-\frac{\partial U}{\partial X_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = \ddot{x}_i = \tilde{\mathcal{J}}_{ij}\ddot{q}_j + \frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\dot{q}_j \dot{q}_\ell .$$

We shall show that this term, multiplied by $\tilde{\mathcal{J}}_{ki}$ equals to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} .$$

To show it, let us first express the kinetic potential in generalized coordinates. We get

$$T = \frac{1}{2}\dot{x}_i \dot{x}_i = \frac{1}{2}\tilde{\mathcal{J}}_{ij}\dot{q}_j \tilde{\mathcal{J}}_{i\ell}\dot{q}_\ell$$

or

$$T = \frac{1}{2}\dot{q}_j \dot{q}_\ell \frac{\partial x_i}{\partial q_j}\frac{\partial x_i}{\partial q_\ell} .$$

Differentiation with respect to q_k yields

$$\frac{\partial T}{\partial q_k} = \frac{1}{2}\left(\dot{q}_j \dot{q}_\ell \frac{\partial^2 x_i}{\partial q_j \partial q_k}\frac{\partial x_i}{\partial q_\ell} + \dot{q}_j \dot{q}_\ell \frac{\partial x_i}{\partial q_j}\frac{\partial^2 x_i}{\partial q_\ell \partial q_k}\right.$$

realizing that

$$\frac{\partial \dot{q}_i}{\partial q_k} = \frac{\partial^2 q_i}{\partial x_j \partial q_k}\dot{x}_j = \frac{\partial}{\partial x_j}\left(\frac{\partial q_i}{\partial q_k}\right)\dot{x}_j = \frac{\partial}{\partial x_j}(\delta_i^k)\dot{x}_j = 0\,\dot{x}_j = 0 .$$

where δ_i^k is the Kronecker δ defined as

$$\delta_i^k = \begin{cases} 1 & k = i \\ 0 & k \neq i. \end{cases}$$

Interchanging the subscripts j and ℓ in the first term of the equation for $\partial T/\partial q_k$ we get

$$\boxed{\frac{\partial T}{\partial q_k} = \dot{q}_j \, \dot{q}_\ell \, \frac{\partial^2 x_i}{\partial q_\ell \partial q_k} \frac{\partial x_i}{\partial q_j} \quad .}$$

This interchange is permissible since we deal just with the summation indices.

Let us now evaluate the second expression, $\frac{d}{dt}(\partial T/\partial \dot{q}_k)$. We obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{1}{2} \frac{\partial \dot{q}_j}{\partial \dot{q}_k} \dot{q}_\ell \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_\ell} + \frac{1}{2} \dot{q}_j \frac{\partial \dot{q}_\ell}{\partial \dot{q}_k} \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_\ell}\right)$$

since obviously $\partial x_i/\partial q_j$ is not a function of $\dot{q}_k$ and

$$\frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial x_i}{\partial q_j}\right) = 0 \ .$$

Realizing that again

$$\frac{\partial \dot{q}_j}{\partial \dot{q}_k} = \delta_j^k, \qquad \frac{\partial \dot{q}_\ell}{\partial \dot{q}_k} = \delta_\ell^k$$

we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{1}{2} \dot{q}_\ell \frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_\ell} + \frac{1}{2} \dot{q}_j \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k}\right) \ .$$

Interchanging the dummy index ℓ in the first term on the right-hand side for j we end up with

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\dot{q}_j \frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_j}\right) .$$

Carrying out the differentiation with respect to time yields

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \ddot{q}_j \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_j} + \dot{q}_j \frac{\partial^2 x_i}{\partial q_k \partial q_\ell}\dot{q}_\ell \frac{\partial x_i}{\partial q_j} + \dot{q}_j \frac{\partial x_i}{\partial q_k}\frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\dot{q}_\ell .}$$

Now we can subtract from this equation the equation for $\partial T/\partial q_k$ with the result

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \ddot{q}_j \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_j} + \dot{q}_j\,\dot{q}_\ell \frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\frac{\partial x_i}{\partial q_k}$$

$$= \ddot{q}_j \tilde{\mathcal{J}}_{ij}\tilde{\mathcal{J}}_{ik} + \dot{q}_j\,\dot{q}_\ell \frac{\partial^2 x_i}{\partial q_j \partial q_\ell}\tilde{\mathcal{J}}_{ik}$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\tilde{\mathcal{J}}_{ik},$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = \mathcal{J}_{ki}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right)$$

that was to be proved.

Substituting now back into the generalized equation of motion we have

$$-\mathcal{J}_{ki}\frac{\partial U}{\partial q_k} = \mathcal{J}_{ki}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right).$$

Multiplying the equation by $\tilde{\mathcal{J}}_{ik}$ we get finally

$$\frac{\partial U}{\partial q_k} - \frac{\partial T}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = 0 .$$

Recalling that conservative U we work with is a function of q_k only, we have

$$\frac{\partial U}{\partial \dot{q}_k} = 0$$

and also

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) = 0.$$

Hence, defining a new potential

$$L = T - U$$

called Lagrangian potential, or just Lagrangian, we can rewrite the equations above in a simpler form (for U conservative)

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0.}$$

This is the new form of our generalized equations of motion of which the original equations were only a special case. These new equations are known as Lagrangian equations of motion. An altogether different derivation of these equations is given in Appendix 1.

Note that the Lagrangian equations are derived under the assumption that U was conservative. In various physical problems, U may be given as a function of velocities as well as coordinates. This is the case with friction, electromagnetic forces, etc. Such potential is also considered non-conservative. Its partial derivatives $\partial U/\partial \dot{q}_k$ are then different from zero and the Lagrangian equations read

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = -\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) .$$

where the left hand side is evaluated under the assumption of U being conservative.

1.7) Canonic Equations of Motion

Let us first call the quantity $\partial L/\partial \dot{q}_i$ the generalized momentum p_i. For comparison see 1.2 where the formula $\partial \tilde{T}/\partial \dot{x}_i = \tilde{p}_i$ was derived. But using the Lagrangian equations we get:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i \ .$$

For comparison see 1.3, namely $-\partial\tilde{U}/\partial x_i = f_i$ and 1.2, namely $f_i = \dot{\tilde{p}}_i$.

As we have done so far, we shall require that, neither T nor U be an explicit functions of time. Hence

$$L = L\ (q_i,\ \dot{q}_i).$$

Then the total differential of the Lagrangian potential is given by

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \ .$$

Upon substitution for the partial derivatives from the above formulae we obtain

$$dL = \dot{p}_i\ dq_i + p_i\ d\dot{q}_i \ .$$

But the second term on the right hand side can also be expressed from the following formula for the total differential of the product $p_i\dot{q}_i$.

$$d(p_i\dot{q}_i) = p_i d\dot{q}_i + \dot{q}_i dp_i \ .$$

Thus we get

$$dL = \dot{p}_i\ dq_i + d(p_i\dot{q}_i) - \dot{q}_i\ dp_i \ ,$$

or

$$d(p_i\dot{q}_i - L) = \dot{q}_i dp_i - \dot{p}_i dq_i \ .$$

Let us have a look now at the scalar product $p_i\dot{q}_i$. For $q_i = x_i$ we get

$$\dot{q}_i = \dot{x}_i \ ,$$

Hence

$$p_i\dot{q}_i = \dot{x}_i\dot{x}_i = 2T \ .$$

Then even in the generalized coordinates the product $p_i\dot{q}_i$ has to equal to 2T because T is a scalar invariant in any coordinate transformation. Realizing that L = T - U, we get

$$p_i\dot{q}_i - L = 2T - T + U = T + U = E.$$

We can therefore write

$$\boxed{dE = \dot{q}_i dp_i - \dot{p}_i dq_i .}$$

The total potential $E(q_i, \dot{q}_i)$ used in this context is sometimes called the Hamiltonian function. As we have seen in 1.3, E in a conservative field is not an explicit function of time and thus its total differential does not contain the time differential.

We can now solve the differential equation by writing first

$$\frac{\partial E}{\partial q_j} = \dot{q}_i \frac{\partial p_i}{\partial q_j} - \dot{p}_j \ .$$

This equation is obtained from the above through a formal division by dq_j. Here p_i is not a function of q_j. Both vectors (velocity and position) must be considered normally independent since we are entitled to choose them both arbitrarily at the beginning of the trajectory (see 1.4). The only relationship between them is given by the equations of motion into which they figure as two sets of independent parameters. Therefore we must consider

$$\partial p_i / \partial q_j = 0.$$

Hence

$$\boxed{\frac{\partial E}{\partial q_i} = - \dot{p}_i .} \qquad (*)$$

Then we can write similarly, dividing the original equation by dp_j:

$$\frac{\partial E}{\partial p_j} = \dot{q}_j - \dot{p}_i \frac{\partial q_i}{\partial p_j}$$

and again q_i being not an explicit function of p_j the second term disappears. Thus

$$\boxed{\frac{\partial E}{\partial p_i} = \dot{q}_i \ .} \qquad (**)$$

The equations (*) and (**) are the canonic equations of motion we have been looking for.

Note that this system of six differential equations of 1-st order replaces the system of three differential equations of 2-nd order. They are usually much easier to solve than the Lagrangian equations.

We shall conclude this section by remarking that for the rectangular Cartesian coordinates, the canonic equations can be developed almost immediately by writing two sets of equations: the equations of motion

$$\ddot{x}_i = - \frac{\partial U}{\partial x_i}$$

and the equation for the kinetic potential

$$T = \frac{1}{2} \dot{x}_i \ \dot{x}_i \ .$$

We have obviously

$$\frac{\partial T}{\partial \dot{x}_i} = \dot{x}_i$$

and substituting p_i for some of the $\dot{x}_i$ we get

$$\dot{p}_i = -\frac{\partial U}{\partial x_i} , \quad \dot{x}_i = \frac{\partial T}{\partial p_i} .$$

Denoting by $E(x_i, p_i)$ the sum $T(p_i) + U(x_j)$, we finally end up with the canonic equations of motion in Cartesian coordinates.

The question then arises as why to bring the Langragian potential and all the subsequent quantities into the discussion at all. The answer is that we have to in order to show that the canonic equations are valid not only for Cartesian coordinates but for any system of generalized coordinates, i.e. to show that the canonic equations in Cartesian coordinates above are just a special case of a more general formulation. We have thus established that the canonic equations of motion are invariant in any admissible coordinate transformation.

2) Close Satellite Orbits

2.1) Basics of Celestial Mechanics

When a satellite orbits around the earth there are various forces - some of them gravitational some of them not - acting on it. It is advantageous not to talk about these forces directly but deal with the potentials corresponding to these forces. By far the most predominant among all these potentials is the attracting potential of the earth.

The attracting potential of the earth is not too different from an attracting potential of a central field. The deviations of the actual potential from that of a central field are at most of the order of 10^{-3} (measured by the potential of the central field) as we shall see later. It is thus customary to write the formula for the actual potential in the following form

$$V = -U = \frac{\kappa \mathcal{M}}{r} + R$$

where the first term obviously describes the potential of a central field with $\mathcal{M}$ denoting the mass of the earth and r the distance of the satellite from the center of gravity of the earth. R represents the deviation of the actual potential from the potential of the central field. R, as a whole, is as stated above of the order of 10^{-3} of the first term and is usually called the disturbing potential or perturbing potential.

Since the disturbing potential is very small with respect to the main central field potential, it is very convenient to regard in the first approximation the problem of motion of an earth satellite as motion in a central field. For this type of motion the theory developed by J. Kepler holds and the motion presents no serious theoretical problem.

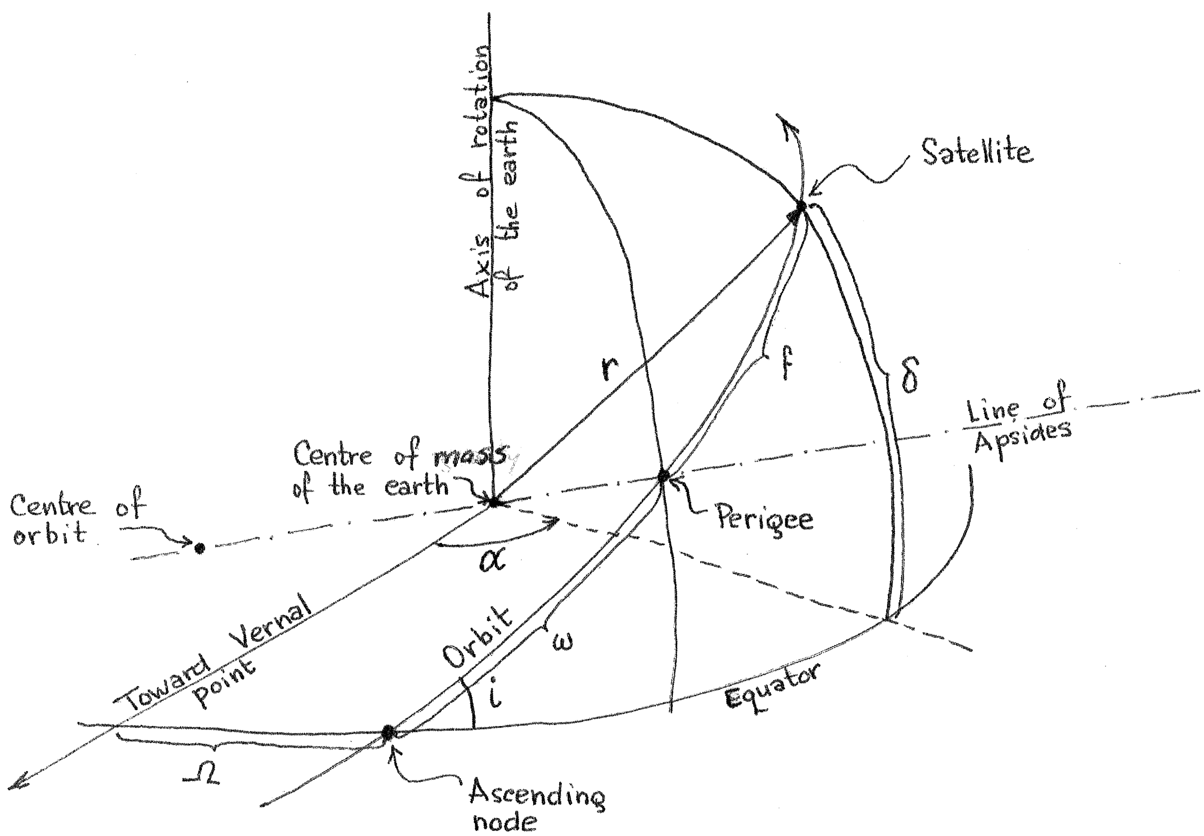
To describe the actual motion, any system of coordinates can be used. However, some coordinate systems are better suited for the investigations than other. The best one, from the theoretical point of view, would be a stationary system, related to the sun. From the computational point of view though, this would present us with great difficulties because of the complexity of motion of the earth that would have to be described. Also, we would loose the opportunity to view the motion in first approximation as a motion in central field centered on the coordinate origin. Hence, for computational convenience, we generally sacrifice the inertiality (with respect to fixed stars) of the system and content ourselves with a non-inertial system concentric with the centre of mass of the earth. To make the system close to inertial, however, we take the directions of the coordinate axes fixed in the star space. This is done by having one axis pointing towards the mean vernal point. Second axis is let to coincide with the mean axis of rotation of the earth and the third completes the rectangular Cartesian triade [Krakiwsky and Wells, 1971]. The mean positions of vernal point and the axis of rotation are referred to a convenient epoch.

Since the described coordinate system is not inertial with respect to fixed stars, its acceleration (with respect to an inertial system) can be observed as a change of geometry of the potential. This appears as non-conservative part of the disturbing potential with annual period and is treated usually together with the rest of the so called "tidal part" of the disturbing potential as we shall see later.

A more serious problem arises from the fact that in the described coordinate system, the earth, together with the attracting potential it radiates, is moving. It rotates around its immediate axis of rotation and it also precesses and nutates. Hence, its attracting potential becomes, in this coordinate system, non-conservative. However, there are parts of the earth attracting (gravitational) potential that possess a rotational symmetry with respect to the instantaneous axis of rotation. These can be considered as approximately conservative, if we disregard the precession and nutation that, due to their long periods, introduce only very minute virtual accelerations.

Having established this we can now start thinking about a more convenient generalized coordinate system linked with the above rectangular Cartesian framework. The most widely used such generalized system is the system of 6 orbital elements known also as Keplerian elements. Out of these, only 3 play the role of proper generalized coordinates, the rest being linked with generalized velocities or generalized momenta as we shall see later.

Devised by Kepler they were meant to describe an elliptical motion of a particle (originally a planet) in a central field (originally that of the Solar attraction). Although the orbital elements are well known to any student of astronomy we shall recapitulate them here for the sake of completeness (see also [Krakiwsky and Wells, 1971]):



ı . . . semi-major axis of the orbital ellipse

. . . eccentricity of the orbital ellipse

. . . inclination of the orbital plane with respect to the equator

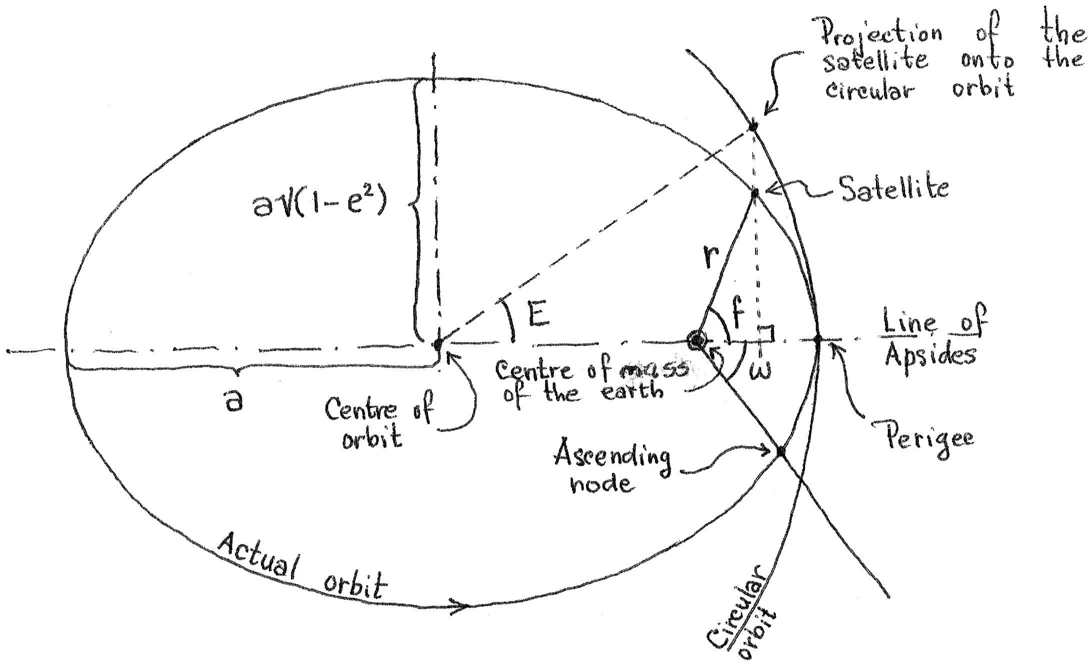
. . right ascention of the ascending node

. . argument of the perigee

. . mean anomaly.

The geometric meaning of i, Ω, ω is clear from the diagr

To show the geometric meaning of a and e we have to draw a figure of the orbital ellipse:



Finally we have to explain what the mean anomaly M is. To do so we first define the true (real) anomaly f as an angle between the satellite and its perigee measured from the centre of gravity of the earth. Then we can say that the mean anomaly is an angle between a hypothetical satellite moving with constant angular velocity (observed at the centre of mass of the earth) on the actual orbit and the perigee. Hence M is a linear function of time

$$M = M_o + M_1 t$$

while f is a more complicated function of time governed by first Kepler law.

The meaning of the eccentric anomaly E is obvious from the figure. Its importance is in linking the two aforementioned anomalies via two known formulae:

$$E = 2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \, \mathrm{tg} \, \frac{f}{2}\right)$$

$$M = E - e \sin E$$

see [Krakiwsky and Wells, 1971]. While the first formula has got a closed form inverse

$$f = 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \, \mathrm{tg} \, \frac{E}{2}\right),$$

the inversion of the second formula leads to an infinite series.

We remark that for a motion in a central field the first 5 Keplerian elements are constant. They describe the size, shape and orientation of the orbit. The only element that depends on time is the sixth, the mean anomaly in our case. It describes the instantaneous position of the satellite on the otherwise stationary orbit. In some developments, other anomalies are preferred to the mean anomaly. In our case we shall try to work with M wherever possible.

Using the Kepler laws we can derive the expression for the kinetic potential of a satellite. The formula for twice the kinetic potential is called the <u>Vis-Viva Integral</u> in celestial mechanics, and reads

$$\boxed{\dot{X}_i \cdot \dot{X}_i = |\dot{F}_i|^2 = 2T = \kappa \mathcal{M} \left(\frac{2}{r} - \frac{1}{a}\right) .}$$

Here r is the length of the radius-vector of the satellite given by the known formulae [Krakiwsky and Wells, 1971]:

$$r = a \sqrt{[(1-e^2) \sin^2 E + (\cos E - e)^2]} = a \frac{1-e^2}{1+e \cos f} .$$

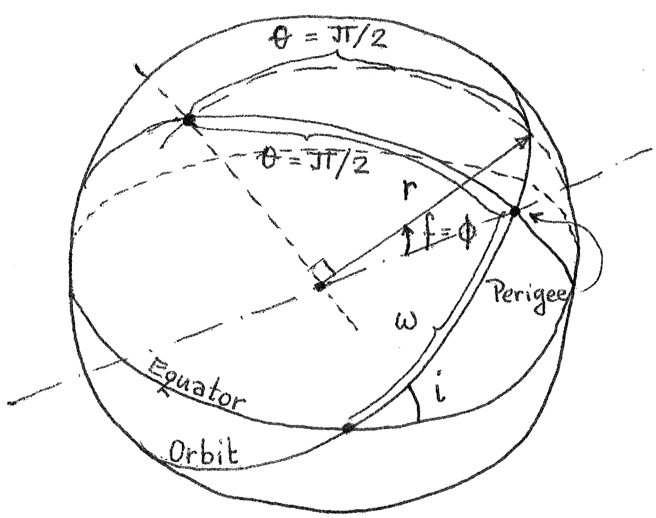
To derive the Vis-Viva integral, let us begin again with the motion in central field. The velocity of the satellite $\dot{x}_i$ in rectangular Cartesian coordinates can be expressed in spherical coordinates $r_i = (r, \vartheta, \phi)$ as follows

$$\dot{x}_i = \mathcal{M}_{ij}\, \dot{r}_j$$

where the matrix $\mathcal{M}$ is nothing else but again the Jacobian matrix of transformation:

$$\mathcal{M}_{ij} = \frac{\partial(x_i)}{\partial(r_j)} .$$

We can choose the spherical coordinates r_i in such a way as to make ϑ equal to $\pi/2$ and ϕ equal to the true anomaly f (see the Figure). This means that ϑ is measured in the reference plane perpendicular to the orbit and containing the perigee, f is measured in the plane of orbit from the perigee. The radius-vector r is measured from the origin of the coordinate system which is the focus of the orbital ellipse(coincides with the centre of the sphere on our Figure).



The reader can prove that for such spherical coordinates the square of the velocity of the satellite is given by

$$\dot{x}_i \dot{x}_i = \dot{r}^2 + r^2 \dot{f}^2$$

(Note that $\dot{\vartheta} = 0$!).

This equation can be rewritten by means of the chain rule for derivatives

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial f}\frac{df}{dt} = \frac{\partial r}{\partial f}\dot{f}$$

(Note that $\dot{a} = \dot{e} = 0$ for the motion in central field!). We get

$$\dot{x}_i\dot{x}_i = ((\frac{\partial r}{\partial f})^2 + r^2)\ \dot{f}^2$$

where $\partial r/\partial f$ can be evaluated from the known formula for r as follows:

$$\frac{\partial r}{\partial f} = r\ \frac{e \sin f}{1+e \cos f} .$$

Substituting this back into the equation for velocity we get

$$\dot{x}_i\dot{x}_i = (1 + \frac{e^2 \sin^2 f}{(1+e \cos f)^2})\ r^2\ \dot{f}^2 = \frac{1+e^2 + 2e \cos f}{(1 + e \cos f)^2}\ r^2\ \dot{f}^2$$

$$= (\frac{2(1+e \cos f)}{(1+e \cos f)^2} + \frac{e^2 - 1}{(1+e \cos f)^2})\ r^2\ \dot{f}^2$$

$$= (\frac{2}{1+e \cos f} - \frac{1 - e^2}{(1+e \cos f)^2})\ r^2\ \dot{f}^2 .$$

Using again the formula for r we obtain

$$\frac{2}{1+e \cos f} = \frac{2r}{a(1-e^2)}\ ,\quad \frac{1-e^2}{(1+e \cos f)^2} = \frac{r^2}{a^2(1-e^2)}$$

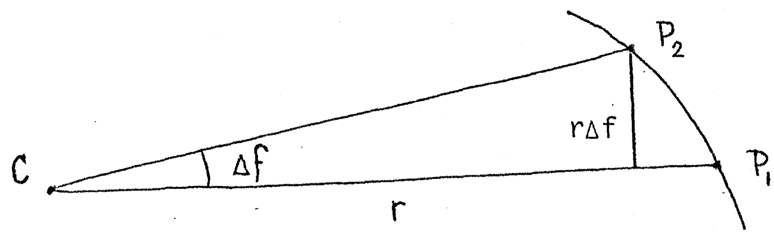
so that we can write

$$\dot{x}_i\dot{x}_i = (\frac{2}{r} - \frac{1}{a})\ \frac{r^4}{a(1-e^2)}\ \dot{f}^2 = (\frac{2}{r} - \frac{1}{a})\ \frac{r^4}{a\nu^2}\ \dot{f}^2$$

where $\nu = \sqrt{(1 - e^2)}$.

To evaluate $\dot{f}$ let us recall the 1-st Kepler law. It requires that the area A of the quasi-triangle P_1P_2C swept by the radius-vector r in a time interval Δt be constant, i.e.

$$A/\Delta t = k = \text{const.}$$



Making the time interval Δt infinitely small, the area A can be computed from the following formula:

$$A = \lim_{\Delta t \to 0} r \frac{r}{2} \Delta f.$$

Substituting this back we obtain

$$k = \lim_{\Delta t \to 0} \frac{r^2}{2} \frac{\Delta f}{\Delta t} = \frac{r^2}{2} \dot{f}.$$

Hence $\dot{f}$ is inversely proportional to r^2.

We can now rewrite the formula for the velocity as

$$\dot{x}_i \dot{x}_i = \left(\frac{2}{r} - \frac{1}{a}\right) \frac{k^*}{a\nu^2},$$

where $k^* = r^4 \dot{f}^2 = 4k^2$.

Let us now multiply the equation by half of the mass of the satellite. We get

$$\frac{m}{2} \dot{x}_i \dot{x}_i = \tilde{T} = \left(\frac{1}{r} - \frac{1}{2a}\right) \frac{mk^*}{a\nu^2},$$

the kinetic energy. The first term on the right hand side obviously varies with time while the second is constant. The only explanation for it is that the first term represents the negative attracting energy:

$$\frac{mk^*}{ra\nu^2} = -\tilde{U}$$

while the second term is nothing but the total energy $\tilde{E}$, which we know is constant for a motion in conservative field

$$\frac{mk^*}{2a^2\nu^2} = -\tilde{E}.$$

But, from our earlier explanations, we know that the potential energy of the central field is given by

$$\tilde{U} = -m\frac{\kappa\mathcal{M}}{r}$$

which must result in the following equation

$$\frac{k^*}{a\nu^2} = \kappa\mathcal{M},$$

where $\mathcal{M}$ is the mass of the earth. Thus we finally end up with the vis-viva integral which we set out to prove.

Let us now go back to the real earth and the actual potential U governing the motion of the satellite. This motion will no longer be a plane Keplerian motion. In the real case, all the orbital elements vary with time (not only the anomalies) and we are faced with a much more difficult problem.

Probably the easiest way to solve the problem, i.e. to derive the expressions for motion or as we say in celestial mechanics, to predict the orbit, is to first establish a system of canonic equations of motion using a convenient system of generalized coordinates. These equations can be then solved or transformed to something else.

2.2) Delauney Coordinates, Lagrangian and Hamiltonian in Delauney Coordinates.

The most natural choice of the generalized coordinates is the choice of the last three orbital elements:

$$q_1 = M, \quad q_2 = \omega, \quad q_3 = \Omega$$

in this order [Kovalevsky, 1967, and other]. These particular coordinates were first suggested by a French astronomer Delauney, whose name they usually bear. Delauney has also shown that choosing these coordinates q_i we get the generalized momenta $p_i = \partial L/\partial \dot{q}_i$ (see 1.7) given by the following equations:

$$p_1 = \sqrt{(\kappa \mathcal{M} a)} \quad , \quad (= p_1(a))$$

$$p_2 = \sqrt{(\kappa \mathcal{M} a \, \nu^2)} = p_1 \nu \quad , \quad (= p_2(a, e))$$

$$p_3 = \cos i \, \sqrt{(\kappa \mathcal{M} a \, \nu^2)} = p_2 \cos i \quad (= p_3(a, e, i)).$$

Since the derivation of the generalized momenta is quite involved, we are not going to prove the above formulae here [Kovalevsky, 1967].

We can now derive the Lagrangian and Hamiltonian potentials for this particular system of coordinates. Recalling the formula for the Lagrangian potential (1.6)

$$L = T - U$$

and making use of the Vis-Viva integral as well as the formula for U from the beginning of 2.1, we get

$$L = \frac{1}{2} \kappa \mathcal{M} \left(\frac{2}{r} - \frac{1}{a}\right) + \left(\frac{\kappa \mathcal{M}}{r} + R\right).$$

Note that we can use the Vis-Viva integral even for motion in non-central field because, although it was developed for Keplerian motion originally, it was shown to hold true for any motion, that is even if a and r changes with time. The above equation yields

$$\boxed{\begin{aligned} L &= \frac{2\kappa \mathcal{M}}{r} - \frac{\kappa \mathcal{M}}{2a} + R \\ &\kappa \mathcal{M}\left(\frac{2}{r} - \frac{1}{2a}\right) + R \end{aligned}}$$

and L can be subsequently expressed in terms of q_i and p_i. We are not going to do it because we do not need this result directly.

The Hamiltonian function (see 1.7) needed for the canomic equation

$$E = T + U$$

is similarly given by

$$E = \frac{\kappa \mathcal{M}}{2}\left(\frac{2}{r} - \frac{1}{a}\right) - \left(\frac{\kappa \mathcal{M}}{r} + R\right)$$

$$= - \frac{\kappa \mathcal{M}}{2a} - R .$$

Expressing a in terms of p_1 we obtain

$$E = - \frac{1}{2}\left(\frac{\kappa \mathcal{M}}{p_1}\right)^2 - R.$$

It should be noted that the validity of the canonic equations based on this Hamiltonian is guaranteed only if U is conservative and hence if R is not an explicit function of time. In the forthcoming discussion, we shall assume that R does not depend on time while keeping in mind that there are components in R which are definitely time dependent. The consequences of this assumption will be pointed out wherever appropriate.

Let us conclude this section by stating that this choice of generalized coordinates is not suitable for circular or equatorial orbits. In the case of a circular orbit ($e = 0$) we obviously get $p_2 = p_1$; for equatorial orbit ($i = 0$) we have $p_3 = p_2$. This reduces the number of independent canonic equations and prevents us from solving them. For this reason, Delauney came up with another set of coordinates suitable for circular and equatorial orbits. These are

$$\tilde{q}_1 = q_1 + q_2 + q_3, \quad \tilde{q}_2 = q_2 + q_3, \quad \tilde{q}_3 = q_3 .$$

These yield

$$\tilde{p}_1 = p_1, \quad \tilde{p}_2 = p_2 - p_1, \quad \tilde{p}_3 = p_3 - p_2 .$$

[Kaula, 1962]. In our development we are going to deal with the first set of Delauney's coordinates only.

2.3) Canonic Equations and their Transformation to Velocities in Orbital Elements

The canonic equations of motion can now be written (see 1.7) as

$$\dot{p}_i = - \frac{\partial E}{\partial q_i} \quad , \quad \dot{q}_i = \frac{\partial E}{\partial p_i}$$

where q_i are the Delauney's coordinates and p_i the Delauney's generalized momenta. In order to be able to use a compact notation in the forthcoming argument, let us perform one change in the Delauney's variables, namely the first, i.e. M.

The 4-th canonic equation reads

$$\dot{q}_1 = \frac{\partial E}{\partial p_1}$$

or

$$\dot{q}_1 = \dot{M} = \frac{(\kappa \mathcal{M})^2}{p_1^3} - \frac{\partial R(q_i, p_i)}{\partial p_1}$$

$$= \sqrt{\frac{\kappa \mathcal{M}}{a^3}} - \frac{\partial R}{\partial p_1} .$$

Obviously, if the disturbing potential R equals to 0, the above equation describes the time change of the mean anomaly of the motion in central field, i.e. the Keplerian motion. Denoting the mean anomaly of this Keplerian motion by M* we get

$$\dot{M}^* = \sqrt{\frac{\kappa \mathcal{M}}{a^3}} .$$

This result provides us with an opportunity to see how the 3rd Kepler law can be used when working with the Keplerian motion. The above formula can be indeed obtained directly from the 3rd Kepler law and the derivation is given in Appendix 2 .

Substituting this result back into the canonic equation of motion we get

$$\frac{d}{dt}(M-M^*) = \Delta\dot{M} = -\frac{\partial R}{\partial p_1} .$$

where ΔM is the deviation from Keplerian motion.

On the other hand the remaining equations can be all written in a straightforward manner:

$$\dot{p}_1 = -\frac{\partial E}{\partial q_1} = \frac{\partial R}{\partial q_1} = \frac{\partial R}{\partial M} ,$$

$$\dot{p}_2 = -\frac{\partial E}{\partial q_2} = \frac{\partial R}{\partial q_2} = \frac{\partial R}{\partial \omega} ,$$

$$\dot{p}_3 = -\frac{\partial E}{\partial q_3} = \frac{\partial R}{\partial q_3} = \frac{\partial R}{\partial \Omega} ,$$

and

$$\dot{q}_2 = \dot{\omega} = \frac{\partial E}{\partial p_2} = -\frac{\partial R}{\partial p_2} ,$$

$$\dot{q}_3 = \dot{\Omega} = \frac{\partial E}{\partial p_3} = -\frac{\partial R}{\partial p_3} .$$

Hence, taking the difference ΔM of the two anomalies M and M* instead of M as the first generalized coordinate, we end up with a new system of equations

$$\dot{p}_i = -\frac{\partial E}{\partial q_i} = \frac{\partial R}{\partial q_i}\ , \quad \dot{\tilde{q}}_i = \frac{\partial E}{\partial p_i} = -\frac{\partial R}{\partial p_i}\ ,$$

with $\tilde{q}_i$ denoting the vector $(\Delta M, \omega, \Omega)$ and $q_i \equiv (M, \omega, \Omega)$. Inspecting the new system of equations we realize that the right hand sides are nothing else but gradients of R expressed in the two coordinate systems q_i and p_i.

Even this new system of equations is still difficult to handle because of the presence of generalized momenta. It would be preferable to have the equations formulated in such a way as to contain only the orbital elements so that we could use the constant values characterizing the plane motion as direct first approximation. This can be done without too much of a problem in the following manner.

Let us denote the vector (a, e, i) by k_j. Then we can write

$$\frac{\partial R}{\partial k_j} = \mathcal{A}_{ij} \frac{\partial R}{\partial p_i}$$

where $\mathcal{A}_{ij}$ is the Jacobian matrix of transformation $p_i \rightarrow k_j$

$$\mathcal{A}_{ij} = \frac{\partial(p_i)}{\partial(k_j)}\ .$$

Then

$$\frac{\partial R}{\partial p_i} = \tilde{\mathcal{A}}_{ji} \frac{\partial R}{\partial k_j}$$

where $\tilde{\mathcal{A}}_{ji}$ is the inverse of the Jacobian matrix $\mathcal{A}_{ji}$, if it exists.

On the other hand, we can write for the generalized momenta:

$$\dot{p}_j = \frac{dp_j}{dt} = \frac{\partial p_j}{\partial k_i}\frac{dk_i}{dt} = \frac{\partial p_j}{\partial k_i}\dot{k}_i\ .$$

Here $\partial p_j/\partial k_i$ is nothing else but the transposed Jacobian matrix $\mathcal{A}_{ji}$ so that we have:

$$\dot{p}_j = \mathcal{A}_{ji}\,\dot{k}_i\ .$$

Substituting into our modified canonic equations for $\partial R/\partial p_i$ and for $\dot{p}_j$ we obtain

$$\mathcal{A}_{ji}\,\dot{k}_i = \frac{\partial R}{\partial q_j}\ ,$$

$$\boxed{\dot{\tilde{q}}_i = -\tilde{\mathcal{A}}_{ji}\,\frac{\partial R}{\partial k_j}\ .}$$

The first equation gives

$$\boxed{\dot{k}_i = \tilde{\mathcal{A}}_{ij}\,\frac{\partial R}{\partial q_j}\ .}$$

These two systems of equations relate the velocities of orbital elements ($\dot{k}_i$, $\dot{\tilde{q}}_i$) with the gradient of R expressed in k and q coordinates respectively, i.e. the acceleration of the disturbing force expressed in k and q coordinates.

Problem:

Show that (the "transposed" inverse of t_{ij}).

$$\tilde{\mathcal{A}}_{ji} = \frac{\partial(k_j)}{\partial(p_i)} = \frac{1}{\sqrt{\kappa \mathcal{M}\, a)}}\begin{bmatrix} 2a, & \nu^2/e, & 0 \\ 0, & -\nu/e, & \dfrac{\cot\ i}{\nu} \\ 0, & 0\ , & \dfrac{-1}{\nu\ \sin\ i} \end{bmatrix}$$

where $\nu^2 = 1 - e^2$.

Since the derived equations are of fundamental importance to us we shall spell them out in full.

$$\dot{a} = \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} 2a \frac{\partial R}{\partial M} ,$$

$$\dot{e} = \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} \left(\frac{\nu^2}{e} \frac{\partial R}{\partial M} - \frac{\nu}{e} \frac{\partial R}{\partial \omega}\right) ,$$

$$\dot{i} = \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} \left(\frac{\cot i}{\nu} \frac{\partial R}{\partial \omega} - \frac{1}{\nu \sin i} \frac{\partial R}{\partial \Omega}\right) ,$$

$$\Delta\dot{M} = \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} \left(- 2a \frac{\partial R}{\partial a} - \frac{\nu^2}{e} \frac{\partial R}{\partial e}\right) ,$$

$$\dot{\omega} = \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} \left(\frac{\nu}{e} \frac{\partial R}{\partial e} - \frac{\cot i}{\nu} \frac{\partial R}{\partial i}\right) ,$$

$$\dot{\Omega} = \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} \frac{1}{\nu \sin i} \frac{\partial R}{\partial i} ,$$

where we have substituted the orbital parameters for the generalized coordinates k_i, $\tilde{q}_i$, q_i. Denoting by K_α the vector (k_i, q_i) and by $\tilde{K}_\alpha$ the vector $(k_i, \tilde{q}_i)$ $(\alpha = 1, 2, \ldots, 6)$ and introducing a 6 by 6 matrix $\mathcal{B}_{\beta\alpha}$:

$$[\mathcal{B}] = \begin{bmatrix} 0 & \tilde{\mathcal{A}}_{ij} \\ -\tilde{\mathcal{A}}_{ji} & 0 \end{bmatrix}$$

we can write the above equations of motion in orbital parameters as

$$\boxed{\dot{\tilde{K}}_\alpha = \mathcal{B}_{\beta\alpha} \frac{\partial R}{\partial K_\beta} .}$$

These equations can be regarded as transformed equations of motion again. We can notice that in absence of any disturbing potential $(R = 0)$ we get $\partial R/\partial K_\beta = 0$ and therefore $\dot{\tilde{K}}_\alpha = 0$. Then none of the orbital elements a, e, i, ω, Ω varies with time and we obtain again a stationary planar (Keplerian) orbit. As far as ΔM is concerned, we get:

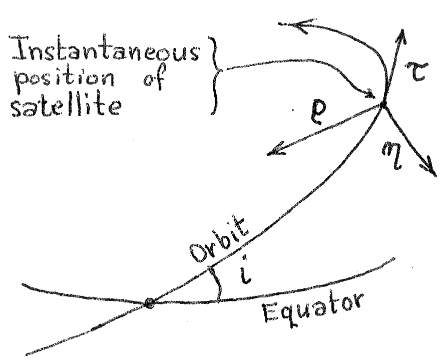
$$\dot{M} - \dot{M}^* = 0$$

or, in other words, the mean anomaly of the motion equals the mean anomaly of the Keplerian motion, up to a constant term.

2.4) Velocities in Orbital Elements in Terms of Orbital Forces

It is sometimes handy to know the "response" of a satellite orbiting around the earth to forces expressed in a more readily understandable form than gradients in k and q systems. Thus, for instance, we want to know, what would be the orbital element velocities $\dot{\tilde{K}}_\alpha$ in response to radial, tangential and normal orbital forces. To answer this question we define a new system of coordinates ξ_i, moving with the hypothetical Keplarian satellite, with one axis pointing in the direction opposite to the direction of the radius vector, one pointing in the direction perpendicular to the radius-vector and laying in the osculating half-plane containing the positive branch of the orbit to which it is amost tangential and the third normal to the first two making a positively oriented orthogonal system of axes. Denoting the coordinates along these three axes by ρ, τ, η we can write

$$(\rho,\ \tau,\ \eta) = \xi_i .$$



The forces acting on the satellite can be split into 2 parts: 1st due to the central field, 2nd, due to the disturbing field. As we have seen already, the first part does not cause any changes in the orbital elements $\tilde{K}_\alpha \equiv (a,\ e,\ i,\ \Delta M,\ \omega,\ \Omega)$. Only the disturbing field has any effect on these. Hence in this development we shall assume the disturbing force only, for which we can write

$$F_i = -\frac{\partial R}{\partial \xi_i}$$

where F_i are the components of the acceleration belonging to the disturbing force in our new system ξ_i.

Between the accelerations expressed in ξ, k and q coordinates, the following transformations hold

$$\frac{\partial R}{\partial k_i} = \mathcal{C}_{ji} \frac{\partial R}{\partial \xi_j} \quad ,$$

$$\frac{\partial R}{\partial q_i} = \mathcal{D}_{ji} \frac{\partial R}{\partial \xi_j}$$

where

$$\mathcal{C}_{ji} = \frac{\partial(\xi_j)}{\partial(k_i)} \quad ,$$

$$\mathcal{D}_{ji} = \frac{\partial(\xi_j)}{\partial(q_i)}$$

are the Jacobian matrices of transformation.

Substituting these equations back into the equations for orbital element velocities we obtain

$$\dot{k}_i = \mathcal{A}_{ij} \mathcal{D}_{\ell j} \frac{\partial R}{\partial \xi_\ell} = \mathcal{E}_{\ell i} \frac{\partial R}{\partial \xi_\ell} = -\mathcal{E}_{\ell i} F_\ell \quad ,$$

$$\dot{\tilde{q}}_i = -\tilde{\mathcal{A}}_{ji} \mathcal{C}_{\ell j} \frac{\partial R}{\partial \xi_\ell} = \mathcal{F}_{\ell i} \frac{\partial R}{\partial \xi_\ell} = -\mathcal{F}_{\ell i} F_\ell \; .$$

Denoting again $(\dot{k}_i, \dot{\tilde{q}}_i)$ by $\dot{\tilde{K}}_\alpha$ we can combine the two equations into one and write

$$\dot{\tilde{K}}_\alpha = -\mathcal{G}_{\ell\alpha} F_\ell$$

where

$$[\mathcal{G}] = \begin{bmatrix} \mathcal{E} \\ \mathcal{F} \end{bmatrix}$$

is a 6 by 3 matrix of functions of orbital elements. The reader may prove that the explicit form of the above equation reads [Tucker et al., 1970]:

$$\dot{a} = Q\left(\frac{2a^2 e}{r} \sin f \; F_1 + \frac{2a^3 \nu^2}{r^2} F_2\right) ,$$

$$\dot{e} = Q\left(\frac{a\nu^2}{r} \sin f \; F_1 + \left(e + \left(1 + \frac{a\nu^2}{r}\right) \cos f\right) F_2\right) ,$$

$$\dot{i} = Q \cos (\omega + f) \; F_3 ,$$

$$\Delta\dot{M} = Q\nu\left(\left(2 - \frac{a\nu^2}{re} \cos f\right) F_1 - \left(1 + \frac{a\nu^2}{r}\right) \frac{\sin f}{e} F_2\right) ,$$

$$\dot{\omega} = Q\left(- \frac{a\nu^2}{re} \cos f \; F_1 + \left(1 + \frac{a\nu^2}{r}\right) \frac{\sin f}{e} F_2 - \frac{\sin(\omega+f)}{\tan i} F_3\right) ,$$

$$\dot{\Omega} = Q \, \frac{\sin (\omega + f)}{\sin i} F_3 ,$$

where by Q we denote

$$Q = \frac{r}{\nu} \sqrt{\frac{1}{\kappa \mathcal{M} a}} .$$

These equations are helpful to study the influence of the individual disturbing forces acting on the satellite, even though the forces may not be stationary and hence may not satisfy the equations. Note that for satellites with small excentricity e, F_2 practically coincides with the tangent to the positive branch of the orbit.

2.5) Disturbing Potential

In 2.1, we have introduced the term disturbing (perturbing) potential R for the deviation of the actual potential of all the forces acting on the satellite from the potential of the central field. We have also seen that in order that the developed theory hold, R has to be conservative, i.e. not explicitly dependent on time. We shall now try to track down the individual components of the disturbing potential.

2.5.1) The far most important component of the disturbing potential is the one arising from the flattening (ellipticity) of the earth. The fact, that the earth is more or less ellipsoidal, with flattening of the order of 1/300 [IAG Special Publication, 1981], causes the close external gravitational field to deviate from spherical symmetry. The equipotential surfaces, that can be used to depict or map the attracting field, have approximately the same flattening. This flattening corresponds to the difference of equatorial and polar radii, of the same equipotential surface, which is of the order of 21 km.

The component of R due to this ellipticity is known as the ellipticity term and we shall denote it by R_E. The elliptic term is rotationally symmetric, which means that the fact that the earth rotates underneath the orbit does not introduce any variations in R_E. Hence, in our coordinate system, R_E is stationary (up to the influence of the precession and nutation).

2.5.2) Remaining irregularities of the earth external gravitation field can be lumped into another part of the disturbing potential, R_I. Since these two terms, R_E and R_I are of the utmost interest to the geodesists, we shall deal with them in a separate section. Let us just mention here, that the irregularities of the gravitational field can be split into two sub-parts, R_{IS} and R_{IN}, where the first sub-part is again rotationally symmetric. Obviously, neither R_E nor R_I depends on anything not even the mass of the satellite. It influences "massive" as well as "light" satellites in the same way.

2.5.3) As the satellite moves through the earth atmosphere it collides with the air molecules. These collusions result in "friction" between the satellite and the air which is usually termes air drag. The force due to the friction is known to be acting opposite to the direction of motion and is in magnitude proportional to the velocity of the motion. We shall denote the potential of this force by R_D.

Since the atmosphere is denser at lower and thinner at higher altitudes, the air drag causes the satellite to slow down more at the vicinity of the perigee. The consequence is that the altitude of the apogee gets reduced and the orbit becomes more and more circular ($e \rightarrow 0$). From the formulae for orbital parameter velocities (2.4), we can see that i and Ω are not influenced at all, at least in the first approximation.

The magnitude of the air drag depends on the shape and size of the satellite as well as on the density of the atmosphere. Evaluation of air drag is a difficult proglem, not only because of our incomplete knowledge of the air density distribution but also because the potential R_D is not strictly stationary. It would be stationary only for a perfectly symmetrical atmosphere.

2.5.4) Solar radiation in a whole spectrum of frequencies introduces another force - solar radiation pressure. The direction of this force is always given by the direction Sun - satellite and its magnitude depends largely on the specific mass of the satellite. Lighter and larger satellites are subjected to more pressure than heavier and smaller ones. It has been established from experiments that the solar radiation pressure becomes particularly significant at altitudes above 1000 km, where it becomes more important than the air drag even for small and heavy satellites.

The potential of the solar radiation pressure, R_P, is, thus non-stationary. It influences all the orbital elements.

2.5.5) The Sun, and also the Moon, influence the satellite also by their Newtonian attraction. These two celestial bodies (strictly speaking all the celestial bodies) radiate their own attracting potentials that interfere with that of the earth. This part of the disturbing potential, R_T, is known as tidal and obviously varies with the positions of the Sun and Moon, and hence with time. It therefore is also non-stationary and influences all the orbital elements.

2.5.6) Relativistic effects are of several different kinds. The largest, (almost 100-times larger than the rest), and therefore most important, is the secular influence on the motion of perigee ω. It is due to the fact that the equation of motion in a central field formulated by the general theory of relativity is not linear (as it is in the classical mechanics). The influence of the non-linear terms results in the orbital period to be slightly longer than 2π making thus the perigee to advance by a small amount every turn. Although the potential R_R of this virtual force is non-stationary, the relativistic effect can be built in the mathematical model and accounted for quite properly without too much of a problem.

2.5.7) As the satellite moves through the ionosphere, it acquires an electrical charge. Then its electrostatic field starts interacting with the magnetic field of the earth influencing the motion of the satellite to a certain degree. The potential of this disturbing force is very similar to that of the air drag but even more difficult to deal with mathematically. This <u>electromagnetic disturbing potential</u> R_M is definitely non-conservative and depends on the electrical properties of the satellite, its direction with respect to the earth magnetic field and other parameters.

In recapitulation we can just state that the disturbing potential R can be written in first approximation as a sum of 8 terms:

$$R = R_E + R_{IS} + R_{IN} + R_D + R_P + R_T + R_R + R_M$$

where only the first two can be regarded as strictly conservative. Hence, the theory we have developed thus far, can be applied only to the first two terms. The rest has to be dealt with in a more sophisticated way which shall not be treated in this outline. More detail can be found in [Kaula, 1962; Kaula, 1966].

Finally, let us mention here that the centrifugal force arising from the earth's rotation that plays an important role in physical geodesy does not appear here at all. The reason for this fact is that the satellite is not rigidly connected to the earth; the earth rotates underneath its orbit freely and its rotation is felt by the satellite only in the time changes of R_{IN} (and R_D, R_M in a lesser degree). Thus the necessary link giving rise to the centrifugal force is missing altogether.

2.6) Orbit Prediction

If we knew the disturbing potential R then the equations of motion in orbital elements derived in 2.3 could be used immediately for orbit prediction. The orbital elements at any given instant t_1 could be computed by integrating the mentioned differential equations as follows

$$\tilde{K}_\alpha(t_1) = \tilde{K}_\alpha(t_o) + \int_{t_o}^{t_1} \dot{\tilde{K}}_\alpha(t)\, dt$$

$$= \tilde{K}_\alpha(t_o) + \int_{t_o}^{t_1} \mathcal{B}_{\alpha\beta}(t) \frac{\partial R}{\partial K_\beta}\, dt$$

where $\tilde{K}_\alpha(t_o)$ is the known position of the satellite at the time t_o (initial position).

Unfortunately, the disturbing potential is never known precisely and in addition R is a function of time too. Hence the integration above can yield only an approximate orbit. However, by comparing the approximate orbit with the actual (observed), we get a discrepancy that can be further analysed, R improved and the prediction bettered on the basis of this improvement.

An alternative approach that does not require the knowledge of R is used by the Smithsonian Astrophysical Observatory [Veis and Moore, 1960] by which the orbital elements at an instant t_1 are computed from the following formula

$$\tilde{K}_\alpha(t_1) = \tilde{K}_\alpha(t_o) + P_{N,\alpha}(t_1 - t_o)$$

where $P_{N,\alpha}(t_1-t_o)$ is a generalized polynomial of N-th order. It is composed of algebraic, trigonometric, hyperbolic and exponential functions of the time interval t_1-t_o. The coefficients of this polynomial are determined by the least-squares approximation of a known piece of orbit (called sometimes arc). These coefficients can be, of course, updated in much the same way as the solution of the equations of motion in the first approach, using again the observed discrepancies between the predicted and actual orbits.

In the orbit prediction we are not restricted to use the orbital elements only. It is quite common to use other coordinate systems too. The geocentric rectangular Cartesian coordinates are quite popular for this task.

3) Gravitational Potential of the Earth

3.1) Gravitational Potential in Spherical Harmonics

In Physical Geodesy the earth gravity potential (in spherical coordinates r, θ, λ)

$$W = -U + \frac{1}{2} r^2 \tilde{\omega}^2 \cos^2\theta$$

is used [Vaníček, 1971]. Here by U we again denote the attracting or gravitational potential and the second term is the potential of the centrifugal force where $\tilde{\omega}$ denotes the angular velocity of the rotation of the earth. For the gravitational potential U outside a sphere of radius r_o, concentric with the coordinate system and containing the whole earth, it was found [Vaníček, 1971]

$$U = -\sum_{n=0}^{\infty} \left(\frac{r_o}{r}\right)^{n+1} \sum_{m=0}^{n} (A_{nm}\cos m\lambda + B_{nm} \sin m\lambda)\, P_{nm}(\cos\theta),$$

a series of spherical harmonics.

In this formula $r > r_o$, $P_{nm}(\cos\theta)$ is the associated Legendre function of n-th order and m-th degree [Heiskanen and Moritz, 1967], and A_{nm}, B_{nm} are some coefficients depending on the distrubition of masses within the earth as well as on the position of the earth with respect to the coordinate system. They are usually called harmonic or potential coefficients. As we have already said earlier, when dealing with extraterrestrial objects like satellites, we do not have to worry about the centrifugal force. Hence our dealings will be solely with the gravitational potential U.

A considerable simplification can be achieved by choosing the coordinate system so that its origin coincides with the centre of mass of the earth. In this case all the terms of 1-st degree (containing $n = 1$) disappear. It also is customary to orient the fundamental axis of the coordinate system ($\theta = 0$) to coincide with the mean rotational axis of the earth. The plane $\lambda = 0$ is required to pass through the Mean Greenwich observatory. In addition, the reference sphere of radius r_o is usually chosen so that its radius equals the semi major axis a_e of the mean-earth ellipsoid as defined in Physical Geodesy. Then the expression for U becomes

$$U = -\frac{a_e}{r}\left[A_{oo} + \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n}(A_{nm}\cos m\lambda + B_{nm}\sin m\lambda)P_{nm}(\cos\theta)\right].$$

We can see now that the only term that does not depend on θ or λ is $-(a_e/r)A_{oo}$. This is, hence, the term describing the part of the earth gravitational potential corresponding to the central field (we recall that the central field is defined so that it is not a function of either θ or λ). But the central field potential is given also by $-(\kappa\mathcal{M})/r$, where $\mathcal{M}$ is the mass of the earth including its atmosphere. Thus we get

$$A_{oo} = \frac{\kappa\mathcal{M}}{a_e}.$$

The gravitational potential can then be written as

$$U(r,\phi,\lambda) = -\frac{\kappa\mathcal{M}}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n}(J_{nm}\cos m\lambda + K_{nm}\sin m\lambda)P_{nm}(\cos\theta)\right]$$

where

$$J_{nm} = -A_{nm}\frac{a_e}{\kappa\mathcal{M}}, \quad K_{nm} = -B_{nm}\frac{a_e}{\kappa\mathcal{M}}.$$

It should be noted that some authors use the normalized Legendre associated functions $\tilde{P}_{nm}$ instead of P_{nm}:

$$\tilde{P}_{nm} = \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}}\, P_{nm}.$$

Use of $\tilde{P}_{nm}$ instead of P_{nm} results in a different set of harmonic coefficients, $\tilde{C}_{nm}$, $\tilde{S}_{nm}$ that are, in addition, taken usually also with the opposite signs to J_{nm}, K_{nm} giving thus

$$\tilde{C}_{nm} = -\sqrt{\frac{(n+m)!}{2(2n+1)(n-m)!}}\, J_{nm}$$

$$\tilde{S}_{nm} = -\sqrt{\frac{(n+m)!}{2(2n+1)(n-m)!}}\, K_{nm}\ .$$

These coefficients are known as normalized harmonic coefficients.

Referring to 2.5 it is not difficult to see that

$$R_G = R_E + R_{IS} + R_{IN} = -\frac{\kappa \mathcal{M}}{r} \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} (J_{nm} \cos m\lambda + K_{nm} \sin m\lambda) P_{nm}(\cos\theta).$$

The question arises now as to which terms of the series of spherical harmonics can be listed under the individual parts R_E, R_{IS}, R_{IN}. Evidently, only the terms that do not depend on λ will represent the stationary part of the disturbing potential, i.e. $R_E + R_{IS}$. This is because they represent a field possessing rotational symmetry so that when the potential rotates with the earth there are no time variations in this part of the disturbing potential. Further, by R_E we understand that part of U containing J_{20} only. Thus we get

$$\boxed{R_E = -\frac{\kappa \mathcal{M}}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2\ (\cos\theta),}$$

$$R_{IS} = -\frac{\kappa\mathcal{M}}{r} \sum_{n=3}^{\infty} \left(\frac{a_e}{r}\right)^n J_n P_n \ (\cos\theta) \ ,$$

$$R_{IN} = -\frac{\kappa\mathcal{M}}{r} \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=1}^{n} (J_{nm}\cos m\lambda + K_{nm}\sin m\lambda) P_{nm} \ (\cos\theta) \ ,$$

where we have denoted J_{no} by J_n and P_{no} by P_n. It can be shown that $P_n(\cos\theta)$ are just the ordinary Legendre polynomials of m-th degree.

We conclude this section by stating that the terms in the series expressing R_{IS} are called zonal harmonics. The terms in the double-series for R_{IN} are known as tesseral harmonics. It is particularly popular to use the normalized harmonic coefficients for the tesseral harmonics yielding:

$$R_{IN} = \frac{\kappa\mathcal{M}}{r} \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=1}^{n} (\tilde{C}_{nm}\cos m\lambda + \tilde{S}_{nm} \sin m\lambda) \tilde{P}_{nm}(\cos\theta) .$$

In this outline we shall be using only the common harmonic coefficients J_{nm}, K_{nm} wherever possible.

3.2) Connection Between the Disturbing Potentials R_G and T

The disturbing potential R_G as developed in 3.1 is obviously a scalar function of the 3 spherical coordinates r, θ, λ. What is probably less obvious is the fact that it also is related to the sphere of radius a_e or, as we would say, to the reference sphere $r = a_e$. If we regard the sphere as being massive with spherically distributed mass then the reference sphere can be visualized as generating the central field with potential $-(\kappa\mathcal{M})/r$ for $r > a_e$.

Similar approach is used in Physical Geodesy where instead of reference sphere a reference ellipsoid is introduced. This mean-earth ellipsoid is defined by its position and orientation (concentric with the earth and its semi-minor axis coinciding with the mean axis of rotation of the earth), size and shape (usually a_e and flattening f - do not mix up with the true anomaly of the Keplerian motion) and some physical properties. The physical properties are its mass (equal to that of the earth), the distribution of masses within the ellipsoid (chosen so that its equipotential surface $V^* = W_o$, of its attracting potential, coincides with the surface of the ellipsoid - where W_o is the potential on the geoid) and its rotation. It rotates around its semi-minor axis with the same velocity as the earth does, i.e. with velocity $\tilde{\omega}$.

The mean-earth ellipsoid is then said to generate the so-called normal gravity potential which can be written as follows

$$V^* = -U^* + \frac{1}{2} r^2 \tilde{\omega}^2 \cos^2 \theta$$

where U^* is the gravitational (attraction) part and the second term on the right-hand side is the potential of the centrifugal force, identical to that of the real earth. Since our "massive" ellipsoid is not only rotationally symmetric but also symmetric with respect to its equator, the gravitational potential U^* must have the same properties too. Hence developing U^* into spherical harmonics they must i) be functions of θ only; ii) contain only even degree harmonics (the odd degree harmonics acquire opposite values on Northern and Southern "hemispheres"). We can thus write

$$U^* = -\frac{\kappa\mathcal{M}}{r}\left(1 - \sum_{n=2,4,..}\left(\frac{a_e}{r}\right)^n J_n^* P_n(\cos\theta)\right)$$

where J_n^* are some harmonic coefficients, generally different from J_n in 3.1.

The difference between the actual potential of the earth W and the normal potential is called the <u>disturbing potential T</u> (do not mix up with kinetic potential):

$$T = W - V^* = W - U^* - \frac{1}{2} r^2 \tilde{\omega}^2 \cos^2\theta$$

The determination of the disturbing potential T (and some quantities closely related to it) is one of the main tasks of geodesy. It will also be our main task in this outline. Thus the question arises as what is the relationship between R_G and T, i.e. can we use the "terrestrial" disturbing potential T for orbit prediction and inversely, can we use the R_G sensed by the satellite to help us solving the terrestrial problems?

The question can be easily answered by equating the actual potential of the earth using the two gravitational potentials U and U*. We get

$$U + \frac{1}{2} r^2 \tilde{\omega}^2 \cos^2\theta = U^* + \frac{1}{2} r^2 \tilde{\omega}^2 \cos^2\theta + T.$$

According to 3.1

$$U = -\frac{\kappa\mathcal{M}}{r} - R_G .$$

On the other hand

$$U^* = -\frac{\kappa\mathcal{M}}{r} + \frac{\kappa\mathcal{M}}{r} \sum_{n=2,4,..}\left(\frac{a_e}{r}\right)^n J_n^* P_n(\cos\theta) = -\frac{\kappa\mathcal{M}}{r} + Z_t .$$

Hence we get

$$R_G = - Z_t + T$$

which is the relation we have been seeking.

Let us now have a closer look at the series Z_t of even-degree zonal terrestrial harmonics. Since the potential V^* is defined so that the equipotential surface

$$V^* = + \frac{\kappa\mathcal{M}}{r} - Z_t + \frac{1}{2} r^2 \tilde{\omega}^2 \cos^2 \theta = W_o$$

is a surface of an ellipsoid of rotation with semi-major axis a_e and flattening f, there must be a relationship between the harmonic coefficients J_n^* and the geometric parameters of the ellipsoid. Such relationships were found by various authors and we shall just state without proof the formula for J_2^* that can be found in most geodetic literature, e.g. [Heiskanen and Moritz, 1967]:

$$J_2^* = \frac{2}{3} f - \frac{1}{3} m - \frac{1}{3} f^2 + \frac{2}{21} fm$$

where

$$m = \frac{\tilde{\omega}^2 a_e}{\gamma_e} ,$$

γ_e stands for the magnitude of normal gravity ($|\nabla V|^*$) on the equator of the mean-earth ellipsoid. Similar formulae exist also for higher degree harmonic coefficients J_n^*.

J_2^* can be also expressed in terms of other parameters of the mean-earth ellipsoid. The following equation

$$J_2^* = \frac{C - A}{\mathcal{M} a_e^2}$$

is well known [Heiskanen and Moritz, 1967]. Here C and A are the moments of inertia of the ellipsoid with respect to its axis of rotation and with respect to any line in the equatorial plane.

The perphaps most disputed technique of determining the J_2^* and other J_n^* (n even) terms is based on the theory of equilibrium of a rotating liquid body. This theory teaches us that if the earth were completely liquid and spinning with velocity $\tilde{\omega}$, its shape would be very close to an ellipsoid of rotation. The flattening of this hydrostatic equilibrium shape would be given by approximately 1/300.0 as opposed to 1/298.25 as derived from the actual observations, terrestrial or satellite [Caputo, 1967] hence giving two different values of J_2^*. This discrepancy cannot be explained by observational errors and has to be considered real. It is probably caused by the departure of the real earth from fluid (or plastic) state. The existing hypotheses are still matter of a controversy.

Finally, let us say something about the magnitude of the harmonic coefficients generally. The far predominant among all of them is the coefficient J_2^* or J_2 which is of the order 10^{-3}. It is in absolute value, about a 1000 times larger than any other harmonic coefficient. This is the main reason why its contribution to the distrubing potential (R_E) is singled out and dealt with separately. The rest of the harmonic coefficients, as experience has shown, decrease with increasing degree. The best known experimental rule for this decrease is due to Kaula and reads

$$\frac{1}{2n+1} \sqrt{\sum_{m=0}^{n} (J_{nm}^2 + K_{nm}^2)} \doteq \frac{1}{n^2 10^5} \quad (n > 2)$$

[Gaposhkin and Lambeck, 1970]. It became known as the Kaula's rule of thumb. More recent investigations have produced more complicated experimental formulae but the improvement gained does not seem to be too significant.

3.3) Gravitational Disturbing Potential in Orbital Elements

In 2.3 we have derived the equations of motion in orbital elements containing the gradient of R, i.e. $\partial R/\partial q_i$ and $\partial R/\partial k_i$. Considering the influence of all other parts of R but R_G removed beforehand through corrections to the observed orbits, we can express the orbital element velocities $\dot{\tilde{K}}_\alpha$ as functions of the gradient of the gravitational disturbing potential R_G:

$$\dot{\tilde{K}}_\alpha = \mathcal{B}_{\beta\alpha} \frac{\partial R_G}{\partial K_\beta} .$$

In 3.1 we have come up with the expression for R_G as a function of spherical coordinates $r_i = (r, \theta, \lambda)$. To be able to use this expression in the above formula we would have to compute the Jacobian of transformation

$$\mathcal{H}_{i\alpha} = \frac{\partial(r_i)}{\partial(K_\alpha)}$$

and write

$$\dot{\tilde{K}}_\alpha = \mathcal{B}_{\beta\alpha} \mathcal{H}_{i\beta} \frac{\partial R_G}{\partial r_i} .$$

This approach would be equivalent to the classical one which we are going to outline.

The classical approach of celestial mechanics is based on the transformation of $R_G(r_i)$ to $R_G(K_\alpha)$. It is preferable to the above technique because of the possibility to treat K_α in the 1-st approximation as independent of time (because of the closeness of the actual orbit to the plane Keplerian orbit). On the other hand, the derivation of the formula for $R_G(K_\alpha)$ is very laborious. For the reason of avoiding the voluminous manipulations we shall just introduce the final result leaving it on the interested reader to fill the gap from any textbook on celestial mechanics [Kovalevsky, 1967; Caputo, 1967]. The final formula reads

$$R_G = \frac{\kappa \mathcal{M}}{a} \sum_{n=2}^{\infty} \left(\frac{a_e}{a}\right)^n \sum_{m,p=0}^{n} F_{nmp}(i) \sum_{q=-\infty}^{\infty} G_{npq}(e) S_{nmpq} (M, \omega, \Omega, \theta)$$

where $F_{nmp}(i)$ and $G_{npq}(e)$ are some complicated functions of i (inclination) and e (excentricity). They again can be found in the textbooks and for the sake of completeness we include the list of some of their components in Appendix 3. The 3-rd function S is of a less complicated nature and reads

$$S_{nmpq} = \begin{cases} - J_{nm} \cos \psi - K_{nm} \sin \psi & \text{for n-m even} \\ \div J_{nm} \sin \psi + K_{nm} \cos \psi & \text{for n-m odd} \end{cases}$$

where $\psi = \psi$ (n, m, p, q; M, ω, Ω, θ) is a linear function ω,M,Ω given by

$$\psi = (n-2p)\omega + (n-2p+q)M + m(\Omega-\theta)$$

and θ denotes the true Greenwich Sidereal Time (do not mix up with the second spherical coordinate), describing the rotation of the earth underneath the orbit.

Using the formula for R_G, its three constituents R_E, R_{IS} and R_{IN} can be written as follows:

$$R_E = \frac{\kappa \mathcal{M}}{a} \left(\frac{a_e}{a}\right)^2 \sum_{p=0}^{2} F_{2op}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e)\, S_{2opq}\,(M, \omega, \Omega, \theta)\ .$$

Here

$$S_{2opq} = - J_2 \cos\,[(2-2p)\omega + (2-2p+q)M]$$

so that we obtain

$$R_E = -\frac{\kappa \mathcal{M} a_e^2}{a^3} J_2 \sum_{p=0}^{2} \sum_{q=-\infty}^{\infty} F_{2op}(i)\, G_{2pq}(e)\, \cos\,[(2-2p)\omega + (2-2p+q)\,M]\ ,$$

a linear function of J_2. The zonal harmonics contribution yields

$$R_{IS} = \frac{\kappa \mathcal{M}}{a} \sum_{n=3}^{\infty} \left(\frac{a_e}{a}\right)^n \sum_{p=0}^{n} F_{nop}(i) \sum_{q=-\infty}^{\infty} G_{npq}(e) \begin{Bmatrix} J_n \cos\,[(2-2p)\omega+(n-2p+q)\,M] \\ J_n \sin\,[(2-2p)\omega+(n-2p+q)\,M] \end{Bmatrix}$$

where the upper expression in the braces { } is valid for n even and the lower for n odd. This can evidently be simplified to

$$R_{IS} = -\frac{\kappa \mathcal{M}}{a} \sum_{n=3}^{\infty} \left(\frac{a_e}{a}\right)^n J_n \sum_{p=0}^{n} \sum_{q=-\infty}^{\infty} F_{nop}(i)\, G_{npq}(e) \begin{Bmatrix} \cos[(2-2p)\omega+(n-2p+q)M] \\ \sin[(2-2p)\omega+(n-2p+q)M] \end{Bmatrix},$$

obviously again a linear function of the harmonic coefficients J_n. The tesseral harmonics contribution will have the same form as the original formula for R_G with two minor changes: the summation over n begins with n = 3 and the summation over m begins with m = 1.

Having expressed R_G in orbital elements K_α we could take the required derivatives $\partial R_G/\partial K_\alpha$, substitute them into the formula for $\dot{\tilde{K}}_\alpha$ and obtain the velocities in orbital elements. Then these equations would have to be integrated to give the actual orbit as a function of the harmonic coefficients. Easier appears to be the alternative approach based on the idea of perturbations which will be outlined in the next chapter.

4) PERTURBATIONS

4.1) Perturbations in Orbital Elements

As we have said several times already the actual orbit does not deviate much from the Keplerian plane motion because the disturbing potential R is much smaller than the potential of the central field. This allows us to treat the deviations of the actual orbit from the planar orbit as quantities of second order of importance or in other words, as perturbations of the planar orbit. These perturbations can be of course described in any coordinate system.

The most common way is to express the perturbations in orbital elements. We know that for the Keplerian motion, the orbital elements, $\tilde{K}_\alpha$ are constant, i.e. they do not depend on time. Hence, any time variations of $\tilde{K}_\alpha$ we observe are the perturbations in orbital elements. We have already met the perturbations in 2.6 when dealing with orbit prediction. There, the difference between $\tilde{K}_\alpha(t_1)$ and $\tilde{K}_\alpha(t_o)$ was experienced due to the perturbations in the time interval $< t_o, t_1 >$. In this chapter we are going to express the perturbations as functions of time in a systematic fashion and denote them by $\delta\tilde{K}_\alpha(t)$. It is not difficult to see that between the perturbations and the velocities the following relation holds

$$\delta\tilde{K}_\alpha\ (t) = \int \dot{\tilde{K}}_\alpha(t)\ dt.$$

Using this symbolism the orbital elements for an instant t_1 can be derived from the orbital elements at t_o from

$$\tilde{K}_\alpha(t_1) = \tilde{K}_\alpha(t_o) + \delta\tilde{K}_\alpha(t_o, t_1 - t_o)$$

where $\delta\tilde{K}_\alpha$ is a function of the disturbing potential.

$$\delta\tilde{K}_\alpha(t) = \int \mathcal{B}_{\beta\alpha} \frac{\partial R}{\partial K_\beta} \, dt .$$

We indeed see that if R = 0 we get $\delta\tilde{K}_\alpha(t) = 0$ as required.

The integration of the equations of motion in orbital elements is a formidable task. All the elements of the matrix $\mathcal{B}$ are functions of time and so are the derivatives of R. Hence, the integration is usually done approximately only. We can regard the Keplerian orbit as a "zero approximation", yielding zero perturbations. This corresponds to R = 0. Taking $R = R_E$ we get the "1-st approximation", involving only the most predominant term in the disturbing potential. Based on this approximation are the so-called <u>linear perturbations</u> to which we shall devote most of the forthcoming chapter.

4.2) First Approximation of the Equations of Motion

To be able to get the 1-st approximation of $\mathcal{B}$ we have to see first how the R_E term influences the orbital elements. To see this we have to solve the equations of motion for $R = R_E$. Therefore, we have to derive the expressions for $\partial R_E/\partial K_\alpha$.

The formula for R_E as a function of K_α was derived in 3.3. It has been shown by various authors that the dependence of R_E on M is much weaker than the dependence on ω [Kaula, 1966]. Neglecting M in the expression for R_E we make an error of the order of J_2^2, i.e. of the same order as the so far neglected harmonic coefficients J_n. This neglect is hence perfectly justifiable and we shall make a good use of it.

When we say that we neglect M in the formula for R_E, this is equivalent to saying that we take only such combinations of the indices p and q that satisfy the equation

$$(2 - 2p + q)M = 0.$$

This yields the following 3 combinations for the allowable p and q, i.e. $0 \leq p \leq n = 2$, and $-\infty < q < \infty$:

$$(p, q) \equiv (0, -2), (1, 0), (2, 2).$$

Inspecting the tables for the function G_{2pq}, one discovers that

$$G^{(e)}_{20-2} = G^{(e)}_{222} = 0, \; G^{(e)}_{210} = (1-e^2)^{-3/2} = \nu^{-3}.$$

Hence the only combination of p and q that has to be taken into account is (1, 0) for which we find

$$F_{201}(i) = \frac{3}{4}\sin^2 i - \frac{1}{2}.$$

Thus the first order approximation of R_E gives

$$R_E \approx -\frac{\kappa \mathcal{M} a_e^2}{a^3} \; J_2 F_{201}(i) \; G_{210}(e) \; \cos\,[(2 - 2)\omega]$$

$$= -\frac{\kappa \mathcal{M} a_e^2}{\nu^3 a^3}\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right) J_2.$$

Using this equation we can now derive the expressions for the gradient $\partial R_E/\partial q_i$, where according to 2.3, we understand $q_i = (M, \omega, \Omega)$. We get

$$\frac{\partial R_E}{\partial q_i} \approx 0$$

and immediately

$$\dot{k}_i \approx 0.$$

where $k_i = (a, e, i)$.

Thus, in the first approximation, the first 3 orbital elements a, e, i are not perturbed by R_E at all. They remain constant and so does the matrix $\mathcal{B}$ of the system of equations of motion.

4.3) Linear Perturbations Due to the Elliptical Term

The approximate equations of motion can now be used to develop the perturbations due to the elliptical term R_E. Evaluating the gradient $\partial R_E/\partial k_i$ we obtain:

$$\frac{\partial R_E}{\partial k_1} = \frac{\partial R_E}{\partial a} \approx 3\, \frac{\kappa \mathcal{M} a_e^2}{\nu^3 a^4} \left(\frac{3}{4} \sin^2 i - \frac{1}{2}\right) J_2,$$

$$\frac{\partial R_E}{\partial k_2} = \frac{\partial R_E}{\partial e} \approx -3\, \frac{\kappa \mathcal{M} a_e^2}{\nu^5 a^3}\, e \left(\frac{3}{4} \sin^2 i - \frac{1}{2}\right) J_2,$$

$$\frac{\partial R_E}{\partial k_3} = \frac{\partial R_E}{\partial i} \approx -\frac{3}{4}\, \frac{\kappa \mathcal{M} a_e^2}{\nu^3 a^3} \sin 2i \; J_2 \; .$$

Substituting the gradient $\partial R_E/\partial k_i$ back into the equations of motion we obtain

$$\tilde{\dot{q}}_1 = \Delta \dot{M} \approx \frac{-1}{\sqrt{(\kappa \mathcal{M} a)}}\; 3\, \frac{\kappa \mathcal{M} a_e^2}{\nu^3 a^3} \left(\frac{3}{4} \sin^2 i - \frac{1}{2}\right) (2-1) J_2$$

$$= -\frac{3}{\nu^3} \sqrt{\frac{\kappa \mathcal{M}}{a^3}} \left(\frac{a_e}{a}\right)^2 \left(\frac{3}{4} \sin^2 i - \frac{1}{2}\right) J_2 \; ,$$

$$\dot{q}_2 = \dot{\omega} \approx \frac{-1}{\sqrt{(\kappa \mathcal{M} a)}} 3 \frac{\kappa \mathcal{M} a_e^2}{\nu^3 a^3} \left(\frac{1}{\nu} \left(\frac{3}{4} \sin^2 i - \frac{1}{2}\right) - \frac{\cot g \frac{i}{2}}{4\nu} \sin 2i\right) J_2$$

$$= - \frac{3}{4\nu^4} \sqrt{\frac{\kappa \mathcal{M}}{a^3}} \left(\frac{a_e}{a}\right)^2 (1 - 5 \cos^2 i) J_2 ,$$

$$\dot{q}_3 = \dot{\Omega} \approx \frac{-1}{\sqrt{(\kappa \mathcal{M} a)}} 3 \frac{\kappa \mathcal{M} a_e^2}{\nu^3 a^3} \frac{1}{4\nu} \frac{\sin 2i}{\sin i} J_2$$

$$= - \frac{3}{2\nu 4} \sqrt{\frac{\kappa \mathcal{M}}{a^3}} \left(\frac{a_e}{a}\right)^2 \cos i \; J_2 .$$

Since in the 1-st approximation none of the elements a, e, i is a function of time, these three differential equations yield, upon integration, pertrubations that are linear functions of time. These are

$$\delta \Delta M(t) \approx P\left(\frac{3}{4} \sin^2 i - \frac{1}{2}\right) J_2 t$$

$$\delta \dot{\omega}(t) \approx \frac{P}{4\nu} (1 - 5 \cos^2 i) J_2 t$$

$$\delta \Omega(t) \approx \frac{P}{2\nu} \cos i \; J_2 t$$

where by P we understand

$$P = - \frac{3}{\nu^3} \sqrt{\frac{\kappa \mathcal{M}}{a^3}} \left(\frac{a_e}{a}\right)^2 .$$

These linear perturbations are linear in time and could be called linear linear perturbations. This would be awkward so we call them <u>secular</u> linear perturbations. They can be used for the approximate evaluation of J_2.

<u>Problem:</u> Determine the approximate rates of change of individual orbital elements for a typical geodetic satellite of $e \approx 10^{-2}$, perigee height of approximately 10^3 km and various acceptable values of i. Consider $\kappa \mathcal{M} = 398\ 603 \times 10^9\ m^3\ sec^{-2}$, $a_e = 6.378{,}137$ m and $J_2 \approx 1.082\ 10^{-3}$.

4.4) Formal Integration of Equations of Motion for R_G

Having determined the first approximation to the perturbations (linear perturbations) due to the elliptical term (secular linear perturbations) we now can proceed to establish the linear perturbation due to the rest of the gravitational disturbing potential. As we have seen earlier, the equations of motion we use in this development (see 2.3) are valid only for R stationary. But we have also discovered that the tesseral harmonics part of R_G, i.e. R_{IN} is not stationary and therefore, strictly speaking, cannot be used in the equations of motion. However, it is useful to treat all components R_E, R_{IS} and R_{IN}, formally as being stationary and then correct for the non-stationarity of R_{IN} later. This is what we are going to do here.

When integrating the equations of motion formally for $R = R_G$, we shall use the first approximation of the orbital elements, i.e. we shall assume that in the matrix $\mathcal{B}$ a, e, i are constant. This amounts to neglecting the time variations in these elements due to the terms of the order of J_2^2, in other words, terms of the same order as the largest of the rest of harmonic coefficients. This is the second flaw of the linear perturbations. The effect of this flaw has to be also corrected for later.

Let us take now the general equation of R_G as developed in 3.3 and write it briefly as

$$R_G = \frac{\kappa \mathcal{M}}{a} \sum_n \sum_m \sum_p \sum_q \left(\frac{a_e}{a}\right)^n F_{nmp} G_{npq} S_{nmpq} = \sum_{n,m,p,q} R_{G;nmpq} .$$

Each of the components $R_{G;nmpq}$ can be obviously treated separately in the equations of motion. Substituting the above expression into the equations of motion we get

$$\dot{\tilde{K}}_\alpha = \Sigma_{nmpq} \mathcal{B}_{\alpha\beta} \frac{\partial R_{G;nmpq}}{\partial K_\beta} = \Sigma_{nmpq} \dot{\tilde{K}}_{\alpha;nmpq}$$

and each component $R_{G;nmpq}$ can be regarded as contributing $\dot{\tilde{K}}_{\alpha,nmpq}$ toward the velocity $\dot{\tilde{K}}_\alpha$. Similarly, it can be regarded as contributing $\delta\tilde{K}_{\alpha,nmpq}$ toward the overall perturbation $\delta\tilde{K}_\alpha$. We shall now proceed to evaluate these individual contributions, leaving out the subscripts for simplicity.

The first individual perturbation $\delta\tilde{K}_1 = \delta a$ is given by

$$\delta a(t) = \int \dot{a} dt \ .$$

Substitution for $\dot{a}$ from the equations of motion yields

$$\delta a(t) = \int \frac{1}{\sqrt{(\kappa \mathcal{M} a)}} 2a \frac{\partial R}{\partial M} dt \approx 2 \sqrt{\frac{a}{\kappa \mathcal{M}}} \int \frac{\partial R}{\partial M} \ dt \quad ,$$

taking a as constant, i.e. independent of time.

The integral here can be written as

$$\int \frac{\partial R}{\partial M} dt = \int \frac{\partial R}{\partial \psi} \frac{\partial \psi}{\partial M} \ dt.$$

Recalling the formula for ψ (see 3.3) we get

$$\frac{\partial \psi}{\partial M} = n - 2p + q \ .$$

This is obviously not a function of time either and can therefore be taken out of the integration, giving

$$\int \frac{\partial R}{\partial M} dt = (n-2p+q) \int \frac{\partial R}{\partial \psi} dt \ .$$

The integral $\int \frac{\partial R}{\partial \psi} dt$ can now be solved by changing the variables. We can write

$$\int \frac{\partial R}{\partial \psi} dt = Q(\psi) \ .$$

Then

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial \psi}\frac{d\psi}{dt} = \frac{\partial R}{\partial \psi}$$

or

$$\frac{\partial Q}{\partial \psi} = \frac{\partial R}{\partial \psi}/\dot{\psi} \ .$$

Hence

$$dQ = \dot{\psi}^{-1} \ dR$$

and finally

$$Q = \int dQ = \int \dot{\Psi}^{-1} dR.$$

Here $\dot{\psi}$ is given by

$$\dot{\psi} = (n-2p)\dot{\omega} + (n-2p+q)\ \dot{M} + m\ (\dot{\Omega} - \dot{\theta}).$$

Here, $\dot{\omega}$, $\dot{M}$, $\dot{\Omega}$, can be considered, in the first approximation, as depending only on a, e, i and J_2. $\dot{\theta}$ is the frequency of the rotation of the earth. We thus get

$$\delta a(t) \approx 2\sqrt{\frac{a}{\kappa \mathcal{M}}}\ \frac{(n-2p+q)}{\dot{\psi}} \int dR = \sqrt{\frac{a}{\kappa \mathcal{M}}}\ \frac{R}{\dot{\psi}}\ 2(n-2p+q).$$

Substituting here for R and $\dot{\psi}$ we can write the complete formula as

$$\delta a(t) \approx 2\sqrt{\frac{k\mathcal{M}}{a}} \quad \Sigma_{nmpq}\left(\frac{a_e}{a}\right)^n \frac{F_{nmp}(i)\ G_{npq}(e)\,(n-2p+q)}{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})}\ S_{nmpq}(\psi).$$

The reader can show analogically that the following formulae hold for e and i [Kaula, 1966]:

$$\delta e_{nmpq}(t) \approx \sqrt{\frac{k\mathcal{M}}{a}}\ \frac{\nu}{ae}\ \frac{R_{G;nmpq}}{\dot{\psi}} \quad \left(\nu\ \frac{\partial \psi}{\partial M} - \frac{\partial \psi}{\partial \omega}\right) \ ,$$

$$\delta e(t) \approx \sqrt{\frac{\kappa\mathcal{M}}{a}}\ \frac{\nu}{ae} \qquad \Sigma_{nmpq} \left(\frac{a_e}{a}\right)^n \frac{F_{nmp}(i)\, G_{npq}(e)\, [\nu(n-2p+q)-(n-2p)]}{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})}\ S_{nmpq}(\psi),$$

$$\delta i_{nmpq}(t) \approx \sqrt{\frac{\kappa\mathcal{M}}{a}}\ \frac{1}{a\nu \sin i}\ \frac{R_{G;\,nmpq}}{\dot{\psi}}\ (\cos i \frac{\partial\psi}{\partial\omega} - \frac{\partial\psi}{\partial\Omega})\ ,$$

$$\delta i(t) \approx \sqrt{\frac{\kappa\mathcal{M}}{a}}\ \frac{1}{a\nu \sin i} \qquad \Sigma_{nmpq} \left(\frac{a_e}{a}\right)^n \frac{F_{nmp}(i)\, G_{npq}(e)\, [\cos i(n-2p)-m]}{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})}\ S_{nmpq}(\psi)\ .$$

The integration of the other three equations of motion is done similarly. We shall show only the integration of the first equation. We can write, to begin with:

$$\delta\Delta M = \delta(M-M^*) = \int\Delta\dot{M}dt = \int\dot{M}dt - \int\dot{M}^*dt = \delta M - \int\sqrt{\frac{\kappa\mathcal{M}}{a^3}}dt = \delta M - \dot{M}^*t\ .$$

Considering $\Delta\dot{M}$ as given by the equations of motion we get

$$\delta\Delta M(t) \approx - \frac{1}{\sqrt{(\kappa\mathcal{M}a)}} \int (2a \frac{\partial R}{\partial a} + \frac{\nu^2}{e} \frac{\partial R}{\partial e})\ dt$$

$$\approx \frac{-1}{\sqrt{(\kappa\mathcal{M}a)}}\ [2a \int \frac{\partial R}{\partial a}\ dt + \frac{\nu^2}{e} \int \frac{\partial R}{\partial e}\ dt\ ],$$

Concentrating again on the individual contributions due to $R_{G;nmpq}$ and leaving the subscripts out we obtain

$$\frac{\partial R}{\partial a} \approx - \frac{(n+1)}{a}\ R\ ,$$

$$\frac{\partial R}{\partial e} \approx \frac{\kappa\mathcal{M}}{a} \left(\frac{a_e}{a}\right)^n F(i)\ \frac{\partial G(e)}{\partial e}\ S(\psi)\ .$$

Then the two integrals in the above equation can be written as

$$\int \frac{\partial R}{\partial a}\ dt \approx - \frac{(n+1)}{a} \int R dt \approx - \frac{(n+1)}{a}\ \frac{\kappa\mathcal{M}}{a} \left(\frac{a_e}{a}\right)^n F(i)\ G(e) \int S(\psi)\ dt,$$

$$\int \frac{\partial R}{\partial e}\ dt \approx \frac{\kappa\mathcal{M}}{a} \left(\frac{a_e}{a}\right)^n F(i)\ \frac{\partial G(e)}{\partial e} \int S(\psi)\ dt$$

because, in the first approximation, only S (or more precisely ψ) is a function of time. Here, the integral over S can be evaluated using the same idea of changing the variables as we have used above. We have:

$$\int S(\psi)dt = Q(\psi).$$

Hence

$$\frac{dQ(\psi)}{dt} = \frac{dQ(\psi)}{d\psi}\frac{d\psi}{dt} = S(\psi)$$

or

$$\frac{dQ(\psi)}{d\psi} = 1/\frac{d\psi}{dt} S(\psi) = S(\psi)/\dot{\psi} .$$

Finally

$$Q(\psi) = \int \frac{dQ(\psi)}{d\psi} d\psi = \int \frac{S(\psi)}{\dot{\psi}} d\psi$$

where $\dot{\psi}$ can be again, in the first approximation, regarded as constant with respect to ψ (ψ contains M, Ω, ω, θ while $\dot{M}$, $\dot{\Omega}$, $\dot{\omega}$, are functions of a, e, i, J_2 and $\dot{\theta} \approx$ const. $\approx$ 1 revolution/day). Hence we can write

$$\int S(\psi)dt \approx \frac{1}{\dot{\psi}} \int S(\psi)d\psi .$$

This integral from S can now be evaluated directly. Using the formula for S introduced in 3.3 and denoting the integral by $\bar{S}$, to conform with the custom in this discipline, we get

$$\bar{S}_{nmpq}(\psi) = \begin{cases} -J_{nm} \sin\psi + K_{nm} \cos\psi & n-m \text{ even} \\ +J_{nm} \cos\psi + K_{nm} \sin\psi & n-m \text{ odd} . \end{cases}$$

Substituting this back into the formula for the perturbation in ΔM and denoting, in addition, $\partial G/\partial e$ by G' we obtain

$$\delta\Delta M_{nmpq}(t) \approx -\sqrt{\frac{\kappa \mathcal{M}}{a^3}} \left(\frac{a_e}{a}\right)^n [-2(n+1) G_{npq}(e) + \frac{\nu^2}{e} G'_{npq}(e)] F_{nmp}(i) \bar{S}_{nmpq}(\psi)/\dot{\psi} .$$

Then the overall perturbation in M is given by

$$\delta M(t) \approx \sqrt{\frac{\kappa\mathcal{M}}{a^3}}\{t - \sum_{nmpq} \left(\frac{a_e}{a}\right)^n \frac{F_{nmp}(i)[2(n+1)G_{npq}(e) - \frac{\nu^2}{e} G'_{npq}(e)]}{(n-2p)\dot{\omega}+(n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})} \bar{S}_{nmpq}(\psi)\}.$$

The reader can show that similar equations hold for the other 2 elements, Ω and ω [Kaula, 1966]:

$$\delta\Omega(t) \approx \sqrt{\frac{\kappa\mathcal{M}}{a^3}} \frac{1}{\nu \sin i} \sum_{nmpq} \left(\frac{a_e}{a}\right)^n \frac{F'_{nmp}(i)\ G_{npq}(e)}{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})} \bar{S}_{nmpq}(\psi),$$

$$\delta\omega(t) \approx \sqrt{\frac{\kappa\mathcal{M}}{a^3}} \frac{1}{\nu} \sum_{nmpq} \left(\frac{a_e}{a}\right)^n \frac{\nu/e\ F_{nmp}(i)G'_{npq}(e) - \mathrm{cotg}\ i/\nu\ F'_{nmp}(i)G_{npq}(e)}{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})} \bar{S}_{nmpq}(\psi),$$

where by F' we denote $\partial F/\partial i$.

Realizing again that $\dot{\theta} \approx$ const. and $\dot{M}$, $\dot{\Omega}$, $\dot{\omega}$, are in the 1-st approximation functions of a, e, i only (realizing that J_2 is not a function of time), all the partial perturbations can be written as

$$\delta\tilde{K}_{\alpha,\ nmpq}(t) \approx A_\alpha(n,m,p,q;\ a,e,i)\ (\sigma(\psi)J_{nm} + \tilde{\sigma}(\psi)K_{nm})$$

where $\sigma(\tilde{\sigma})$ is either cos (sin) or sin (cos) taken with the appropriate sign depending on i) the difference n-m,

ii) the presence of S or $\bar{S}$ in the formula. A_α are the functions already specified, i.e.

$$A_1 = 2\sqrt{\frac{\kappa\mathcal{M}}{a}}\left(\frac{a_e}{a}\right)^n \frac{F_{nmp}(i)\ G_{npq}(e)\ (n-2p+q)}{(n-2p)\dot{\omega}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})}$$

for a, etc. Then, applying the summation with respect to p and q, we get

$$\delta\tilde{K}_{\alpha,\ nm}(t) \approx \sum_{p=0}^{n} \sum_{q=-\infty}^{\infty} A_\alpha(\sigma J_{nm} + \tilde{\sigma}K_{nm})$$

$$= J_{nm} \sum_{p\ q} A_\alpha \sigma + K_{nm} \sum_{p\ q} A_\alpha \tilde{\sigma}\ .$$

The complete linear perturbations are then given by the following formulae

$$\delta\tilde{K}_{\alpha}(t) \approx \sum_{n=2}^{\infty} \sum_{m=0}^{n} [J_{nm} \sum_{p} \sum_{q} A_{\alpha}\sigma + K_{nm} \sum_{p} \sum_{q} A_{\alpha}\tilde{\sigma}] ,$$

i.e. as linear combinations of the harmonic coefficients.

4.5) Non-Linear Perturbations

As we have already stated (4.4), the linear perturbations cannot be used for two reasons. Firstly, they include the influence of the non-stationary part of the gravitational disturbing potential which contradicts the basic assumption of the Hamiltonian approach. Secondly, they do not account for the quadratic term of J_2, i.e. J_2^2, which is of the order of J_n, $n = 3, 4, \ldots$.

Both these hindrances can be corrected for by means of developing corrections to the linear formulae and produce a set of formulae for what became known as the non-linear perturbations. The methods for developing the formulae are tedious and very involved and hence considered beyond the scope of this course. The interested reader can find them for instance in [Kaula, 1962; Kaula, 1966].

In order to discuss at least the generalities of the earth gravity field determination let us just state here that the formulae for the non-linear perturbations can be again brought to the same form as those for the linear perturbations. This means that even the non-linear perturbations can be expressed as linear combinations of the harmonic coefficients, involving the same trigonometric functions σ, $\tilde{\sigma}$. Denoting the non-linear perturbations by $\delta^*\tilde{K}_{\alpha}(t)$ we have

$$\delta^*\tilde{K}_\alpha(t) = \sum_{n=2}^{\infty}\sum_{m=0}^{n}[J_{nm}\sum_{p}\sum_{q} A^*_\alpha\,\sigma(\psi) + K_{nm}\sum_{p}\sum_{q} A^*_\alpha\,\tilde{\sigma}(\psi)] ,$$

where only the functions A^*_α (n,m,p,q; a, e, i) are different from A_α.

It has been established [Gaposhkin and Lambeck, 1970] that the functions A^*_α, although quite complicated in nature, are proportionate to $e^{|q|}$. Since $e < 1$ (usually $e \ll 1$), the magnitude of A^*_α decreases with increasing absolute value of q. Hence the smaller e is, the sooner the series can be truncated in q. For typical geodetic satellites ($e \approx 10^{-2}$) the summation over q does not have to go beyond $|q| = 10$.

We may also note that the functions A^*_α, for a specific satellite, are functions of time only through a, e, i. Since these elements vary with time very slowly, A^*_α can be considered as functions of only n,m,p,q even for very long arcs, if mean values of a, e, i are taken.

To conclude with, let us repeat again that the perturbations can be expressed in other coordinate systems as well. It is quite a common practice to express them in geocentric rectangular Cartesian coordinates. The resulting equations give approximately the same results - approximately because within the developments, different approximations are used.

4.6) Frequencies of Perturbations, Resonance

When we take a close look at the formulae for perturbations we discover that each perturbation is expressed as a function of time through the trigonometric terms $\sigma(\psi)$ and $\tilde{\sigma}(\psi)$. As we have seen, M, ω, Ω are in the 1-st approximation linear functions of time and so is θ. Hence denoting

$$M = M_o + \dot{M}\, t, \quad \omega = \omega_o + \dot{\omega}\, t, \quad \Omega = \Omega_o + \dot{\Omega}\, t, \quad \theta = \theta_o + \dot{\theta}\, t$$

we can write

$$\psi = [(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})]t$$
$$+ (n-2p)\omega_o + (n-2p+q)M_o + m(\Omega_o - \theta_o)$$
$$= \dot{\psi}_t + \psi_o .$$

The term $\dot{\psi}$ represents the frequency and ψ_o is the phase of σ or $\tilde{\sigma}$. From the point of view of frequency analysis the phase ψ_o does not interest us and we shall concentrate on $\dot{\psi}$.

In the expression for $\dot{\psi}$; $\dot{M} > \dot{\Omega} - \dot{\theta} > \dot{\omega}$; $\dot{M}$ is the frequency of the orbiting satellite, i.e. several times per day; $\dot{\Omega} - \dot{\theta} \sim - \dot{\theta}$ $(\dot{\theta} >> \dot{\Omega})$ is the frequency of the rotation of the earth, i.e. approximately once per day; finally $\dot{\omega}$ is the frequency of the motion of perigee within the orbital plane that is much slower than the other two motions. This ordering gave raise to the terminology used. The frequencies of the order of $\dot{\theta}$ and higher are known as corresponding to <u>short periods</u>. Lower frequencies of the order of $\dot{\omega}$ and around this value, are said to correspond to <u>long periods</u>. Lower frequencies still, due to periodic changes of A^*_α are usually thought of as depicting linear variations called <u>secular</u> as in 4.3.

If we have a look now at $\dot{\psi}$ we can see that in case n-2p+q = 0 and m = 0, the corresponding partial perturbation will have only long periodic and secular terms. If, in addition, even n-2p=0 then the corresponding partial perturbation will vary only secularly, i.e. with A^*_α. Such will be the case for the combination of subscripts (n, m, p, q) = (2k, 0, k, 0) for any value of k. Since n has to be an even number and m has to equal to zero to get a purely secular partial perturbation, we conclude that only zonal harmonics of even order, i.e., harmonics related to the ellipticity, cause purely secular variations or orbital elements. Note that for $J_{2k,0}$ the perturbation equations are undefined. This is the practical reason why the perturbation equations cannot be used and why the secular linear perturbation equation (section 4.3) are used. Hence, in practice, the zonal harmonic coefficients of even order are generally determined from observations of secular perturbations because they are not influenced by other harmonics.

On the other hand, all zonal harmonics ($m = 0$) give raise to long periodic terms because they always contain combinations $n-2p+q = 0$. They are therefore determined mainly from long periodic variations of the perturbations. The tesseral harmonics have always short periodic terms ($m \neq 0$) and their coefficients have to be determined from short periodic variations.

As in the case of linear perturbations, the functions A^*_α in the non-linear formulae are also inversely proportional to $\dot{\psi}$. What then may happen, and in fact often happens in reality, is that $\dot{\psi}$ becomes very small for certain combinations (n, m, p, q). When this occurs, the value of A^*_α increases and magnifies certain low frequencies of the perturbations. These frequencies, known as <u>resonant frequencies</u>, are then particularly useful for evaluation of the corresponding harmonic coefficients. We say, that the satellite is specially sensitive to the resonant frequencies.

5. DETERMINATION OF THE TERRESTRIAL GRAVITY FIELD CHARACTERISTICS

5.1) Evaluation of Harmonic Coefficients from Perturbations

It is not difficult to see that the equations for perturbations give us a handy tool for evaluating the harmonic coefficients of the earth gravity field. They can be immediately taken as observation equations for adjustment (or more specifically for harmonic analysis) since they provide us with the linear relationship between the unknowns J_{nm}, K_{nm} and the perturbations. For this purpose, the perturbations in individual elements are evaluated as differences between the "observed" elements, i.e. orbital elements as derived from the observed positions of the satellite, and some adopted constant values. These constant orbital elements are usually chosen close to the mean values for the period of observations (generally several days) and are said to describe an intermediate orbit. Evidently, the intermediate orbit is a Keplerian orbit chosen to represent the actual orbit quite closely. The actual orbit for the period of observations is called orbital arc.

The coefficients for the unknowns (harmonic coefficients) can be written as

$$X_{\alpha,nm}(t) = \sum_{p\ q} A^*_\alpha \sigma(\psi), \quad Y_{\alpha,nm}(t) = \sum_{p\ q} A^*_\alpha \tilde{\sigma}(\psi) .$$

As we have seen in 4.5, they are functions of the orbital elements and also of n, m and θ ($\dot{\omega}$, $\dot{\Omega}$, $\dot{M}$ are treated as functions of a, e, i, J_2).

To evaluate the coefficients $X_{\alpha,nm}(t)$, $Y_{\alpha,nm}(t)$ we use the orbital elements describing the intermediate orbit and the value of J_2 determined from the 1-st approximation of the equations of motion (see 4.3). Hence, in practice, the evaluation of the harmonic coefficients is usually carried out in two steps, consisting of the determination of the first approximation of J_2 and the determination of the rest of the coefficients including the correction to the first approximation to J_2.

Having determined the first approximation of J_2 and the values of A^*_α we then proceed to evaluate the zonal harmonic coefficients from the secular and long-periodic variations. These variations are influenced by tesseral harmonics in resonant frequencies so that the determination generally includes these also. Finally, the parameters of non-gravitational variations (see 2.5) have to be usually solved for together with the zonal harmonic coefficients as well. The predominant effects are due to the air-drag, solar radiation pressure and luni-solar tides [Kaula, 1966].

Then, in what is effectively the third step, we can determine the tesseral coefficients from the short-periodic variations. Since there are also short-periodic variations due to the zonal harmonics (see 4.6) these have to be first removed. The removal is done using the zonal harmonic coefficients determined previously.

There is still one complication involved in the described harmonic analysis that should be mentioned here. Always a whole set of harmonic coefficients is connected to a certain period (and inversely, of course, there is a whole series of periods related to each coefficient). Hence what we can determine from the harmonic analysis, i.e. amplitudes of specific trigonometric terms, are just various linear combinations of harmonic coefficients. If the combinations varied from period to period they could be then disentangled, i.e. the individual coefficients could be obtained from a system of linear algebraic equations. This may or may not be the case.

To eliminate any possible linear dependence of the individual harmonic coefficients it is necessary to use several different orbital arcs in the solution. These usually are not only arcs belonging to one satellite at different epochs but also arcs for different satellites. It is not rare to use hundreds of arcs in one solution. One such solution is described in an illustrative way in [Gaposchkin & Lambeck, 1970].

Two more things are of importance for the harmonic analysis. The first is the choice of the maximum order n to which the harmonic analysis should be carried out. This choice is governed by the precision with which the perturbations, or more specifically the "observed" orbital elements, can be determined. This precision is, at present, a few metres at best. On the other hand, we can determine the magnitude of any such partial contribution, due to a certain order, from the Kaula's rule of thumb (see 3.2). These contributions decrease rapidly not only because of the natural decrement in magnitude of coefficients but also due to the decrease of the damping factor $(\frac{a_e}{a})^n$ (see 3.3) with the order of the harmonic coefficient. The highest order of harmonic coefficients that can be estimated with the present best precision of orbit is about 16 [Sandson and Strange, 1972].

The second remark concerns the weighting of perturbations for the harmonic analysis. This has to be done on the basis of observation precision and we shall not dwell on this point here. It is more appropriately dealt with in the geometric aspects of satellite geodesy where it is known as weighting of the orbit.

To conclude with, let us state that there is no universal rule as to which orbital element should be used in the perturbation analysis as described above. They may be selected differently for different satellites as these are again influenced differently by the various non-gravitational forces and thus differently suitable for the analysis.

5.2) Determination of Gravity Anomalies

Once the harmonic coefficients J_{nm} and K_{nm} are known, they can be easily used to describe other characteristics of the earth gravity field. Although J_{nm} and K_{nm} describe the gravitational potential, they can also express gravity anomalies, geoidal heights, etc.

In order to derive the relation between the gravitational potential and the gravity anomalies we first recall the formula linking the terrestrial disturbing potential T with the gravity anomaly developed in spherical harmonics [Vaníček, 1971, 3.15]:

$$T \approx \sum_{n=2}^{\infty} \frac{R}{n-1} \Delta g_n \quad ,$$

where R is the mean radius of the earth and should not be confused with the disturbing potential. Δg_n are the spherical harmonics of the gravity anomaly Δg. Here T is assumed to have been determined with respect to an ellipsoid (see 3.2) concentric with the earth and having the same mass $\mathcal{M}$ as the earth. The consequence of this assumption is that developing T in spherical harmonics the first two degree terms are identically equal to zero, i.e.,

$$T = \sum_{n=2}^{\infty} T_n$$

with T_n denoting the spherical harmonics of T.

We have also seen that T can be written as (3.2)

$$T = R_G + Z_t .$$

Substitution for R_G and Z_t yields

$$T = -\frac{\kappa \mathcal{M}}{r} \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} (J_{nm} \cos m\lambda + K_{nm} \sin m\lambda) P_{nm} (\cos \theta) +$$

$$+ \frac{\kappa \mathcal{M}}{r} \sum_{n=2,4,..} \left(\frac{a_e}{r}\right)^n J_n^* P_n (\cos \theta).$$

This can be rewritten again

$$T = -\frac{\kappa \mathcal{M}}{r} \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} [(J_{nm} - J_{nm}^*) \cos m\lambda + K_{nm} \sin m\lambda] P_{nm} (\cos \theta) = \sum_{n=2}^{\infty} T_n$$

where

$$J_{nm}^* = \begin{cases} J_n^* & n = 2,4,\ldots; \quad m = 0 \\ 0 & \text{all other combinations of n and m.} \end{cases}$$

Hence we can equate

$$T_n \approx \frac{R}{n-1} \Delta g_n$$

and since

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n$$

(the summation does not include the terms of 0-th and 1-st order since under the accepted assumptions they are again both equal to zero [Vaníček, 1971]), we obtain finally

$$\Delta g \approx \sum_{n=2}^{\infty} \frac{n-1}{R} T_n$$

$$= -\frac{\kappa \mathcal{M}}{rR} \sum_{n=2}^{\infty} (n-1) \left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} [(J_{nm} - J^*_{nm}) \cos m\lambda + K_{nm} \sin m\lambda] P_{nm}(\cos \theta) .$$

It is usual in practice to put

$$r \approx a_e \approx R .$$

This spherical approximation changes the above formula to

$$\Delta g \approx -\frac{\kappa \mathcal{M}}{R^2} \sum_{n=2}^{\infty} (n-1) \sum_{m=0}^{n} [(J_{nm} - J^*_{nm}) \cos m\lambda + K_{nm} \sin m\lambda] P_{nm} (\cos \theta) .$$

The average difference (in absolute value) between the last two formulae is approximately 0.1 mgal with a maximum of about 1 mgal [Rapp, 1972].

We should note that the values of J^*_n depend on the flattening of the selected geocentric reference ellipsoid (see 3.3) which we are free to choose more or less arbitrarily. Practically only the first two terms, J^*_2 and J^*_4, can be taken as different from zero since they describe any chosen ellipsoid adequately. It is interesting to see that the selection of a particular flattening influences the values of gravity anomalies quite strongly [Gaposchkin and Lambeck, 1970].

Let us conclude by stating that should the reference ellipsoid have a mass different from that of the earth an additional absolute term would have to be taken into account in the formula above. Also, we should be aware of the fact that other formulae have been derived for Δg by various scholars.

5.3) Determination of Geoidal Heights

To develop the equation linking the geoidal height with the harmonic coefficients, let us again assume that the reference ellipsoid is geocentric and its mass equals to the mass of the earth. The 2nd Bruns formula [Vaníček, 1971, section 3.11] can be used in its simpler form

$$N = T/\gamma$$

where by N we denote the geoidal height and γ is the normal gravity on the chosen best fitting reference ellipsoid. Taking the expression for T (from 5.2) we obtain

$$N = -\frac{\kappa\mathcal{M}}{r\gamma} \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} [(J_{nm} - J^*_{nm}) \cos m\lambda + K_{nm} \sin m\lambda]\, P_{nm} (\cos \theta) .$$

Here a_e and r can be again replaced by the mean radius of earth. Moreover, the normal gravity, in the first approximation, can be written as

$$\gamma \approx \frac{\partial}{\partial r} \left(- \frac{\kappa\mathcal{M}}{r}\right) = \frac{\kappa\mathcal{M}}{r^2} \approx \frac{\kappa\mathcal{M}}{R^2} .$$

Substituting these approximations back into the above equation we get

$$N \approx -R \sum_{n=2}^{\infty} \sum_{m=0}^{n} [(J_{nm} - J^*_{nm}) \cos m\lambda + K_{nm} \sin m\lambda] P_{nm} (\cos \theta) .$$

The difference between the exact and the approximate formulae is in average (in absolute value) 0.2 m. The maximum is of the order of 1 m [Rapp, 1972].

We again note that the geoid can be computed with respect to an ellipsoid of arbitrary flattening, the selection being made possible through the J^*_n coefficients. The various versions of the geoid derived from satellite observations by different authors are always very smooth since the perturbation analysis cannot give the higher order harmonic coefficients. The smallest details one may expect of any such "satellite geoid" are of the order of 2000 km in length (for degree 20). This is the reason why many geodesists combine the satellite data (lower degree harmonics) with terrestrial gravity data (higher degree harmonics) to obtain the so-called "combined solution". To venture into this domain is considered beyond the scope of this course.

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APPENDIX 1

Alternative Derivation of Lagrangian Equations of Motion

We first define the Lagrangian potential L as

$$L\ (q, \dot{q}) = T - U.$$

Then we define the action function S, known sometimes as Hamilton's principal function, as

$$S = \int_{t_o}^{t_1} L\ dt.$$

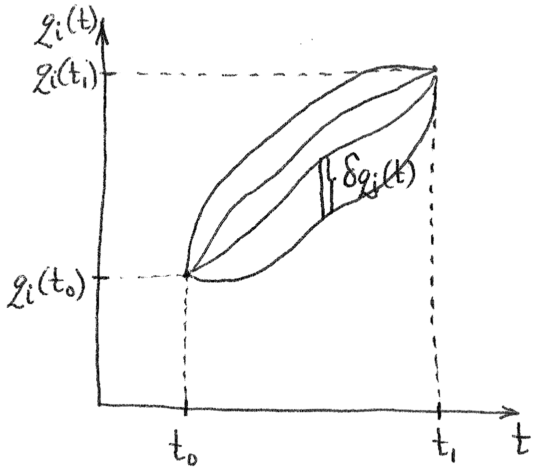
It describes the action of the potential change L in the period of time t_o, t_1.

The Hamilton's principle, one of the fundamental laws of classical physics says that the particle moves in a conservative field U in such a way that the action of the potential change L within any period of time is minimum, see for instance [Landau and Lifschitz, 1965]. In mathematical language it can be stated as

$$\min_{q_i} S.$$

It obviously is an extremum condition and can be formulated in an equation form using variational calculus, leading then to our system of differential equations of motion.

To be able to apply the variational calculus, let us first formulate the extremum condition in terms of variations. We assume that we know the position $q_i(t_o)$ of the point (or generally a whole system of points) at the time t_o, as well as the position $q_i(t_1)$ at the time t_1. We shall be trying to determine the path of the motion (the trajectory) of our point between these two positions such as to minimize the action function S. Obviously, for different trajectories we get different values of S. This means that varying the trajectories, we can vary the value of S belonging to them. Let us take then one trajectory $q_i(t)$ and another trajectory $q_i(t) + \delta q_i(t)$ infinitesimally close to the first one. The difference $\delta q_i(t)$ is called the variation of trajectory and we note that



$$\delta q_i(t_o) = \delta q_i(t_1) = 0$$

since all the trajectories go through the two end points. Variation of the trajectory $\delta q_i(t)$ will produce the following variation of the action function:

$$\delta S = \int_{t_o}^{t_1} L(q_i+\delta q_i, \dot{q}_i+\delta\dot{q}_i)dt - \int_{t_o}^{t_1} L(q_i, \dot{q}_i)\,dt\ ,$$

where the variation $\delta\dot{q}_i$ of the generalized velocity can be expressed as

$$\delta\dot{q}_i = \frac{d}{dt}\,\delta q_i .$$

Because the variation of the trajectory is assessed infinitesimally small, we can develop the Lagrangian $L(q_i+\delta q_i, \dot{q}_i+\delta\dot{q}_i)$ into the Taylor series and retain just the first two terms. We obtain

$$L(q_i+\delta q_i, \dot{q}_i+\delta\dot{q}_i) \doteq L(q_i, \dot{q}_i) + \frac{\partial L}{\partial q_i}\,\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\,\delta\dot{q}_i$$

and the variation of the action function becomes

$$\delta S = \int_{t_o}^{t_1} \left(\frac{\partial L}{\partial q_i}\,\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\,\delta \dot{q}_i\right) dt \,.$$

The second term can now be integrated by parts:

$$\int_{t_o}^{t_1} \left(\frac{\partial L}{\partial \dot{q}_i}\,\delta \dot{q}_i\right) dt = \left[\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right]_{t_o}^{t_1} - \int_{t_o}^{t_1} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right) dt \,.$$

The first term on the right hand side can be written as

$$\left[\frac{\partial L}{\partial \dot{q}_i}\quad \delta q_i\right]_{t_o}^{t_1} = \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{t_1} \delta q_i(t_1) - \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{t_o} \delta q_i(t_o)$$

and equals thus to zero since both variations equal to zero. Hence we get finally:

$$\delta S = \int_{t_o}^{t_1} \left(\frac{\partial L}{\partial q_i}\,\delta q_i - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right) dt$$

$$= \int_{t_o}^{t_1} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i \; dt \,.$$

Everthing else being fixed, the action function is just a function of the trajectory. The trajectory that minimizes the action function is the one that renders the variation of the function zero. Hence the equations

$$\int_{t_o}^{t_1} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i \; dt = 0$$

can be regarded as the equations of the motion satisfying the Hamiltonian principle of least action. But the only way how to satisfy the equations above is to make the expression in the brackets equal to zero, since the variation δq_i is an arbitrary (though small) function of time. Thus we obtain the Lagrange's equations of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 .$$

APPENDIX 2

Derivation of Keplerian Orbital Frequency from Third Kepler Law.

The 3-rd Kepler law states

$$P^2/a^3 = \text{const.} = \tilde{k},$$

where P is the period of the orbit of the satellite. P can be obtained from M* by realizing that i) M* is a linear function of time (see 2.1):

$$M^*(t) = M_o^* + M_1^* \, t \;;$$

ii) P is the time interval $t_2 - t_1$ necessary to make

$$M^*(t_2) = M^*(t_1) + 2\pi \; .$$

Hence

$$M^*(t_2) - M^*(t_1) = M_1^*(t_2 - t_1) = M_1^* P = 2\pi$$

where M_1^* is nothing else but $\dot{M}^*$. Thus

$$P = 2\pi/\dot{M}^*$$

and we see that $\dot{M}^*$ is the angular frequency of the orbital motion as we might have expected.

Substituting for P in the 3-rd Kepler law, we obtain

$$\dot{M}^{*2} a^3 = 2\pi/\tilde{k} \; .$$

On the other hand, taking the Vis-Viva integral for a circular orbit (e = 0, r = a), we get

$$\dot{x}_i \dot{x}_i = \left(\frac{2}{a} - \frac{1}{a}\right) \kappa \mathcal{M} = \frac{\kappa \mathcal{M}}{a} \; .$$

Here the linear instantaneous velocity can be also computed from the obvious formula

$$|\dot{x}_i| = \frac{2\pi a}{P} .$$

Substituting for P here and evaluating the Vis-Viva integral for the circular orbit we obtain

$$\dot{M}*^2 a^2 = \frac{\kappa \mathcal{M}}{a}$$

and $\dot{M}*^2 a^3$ (for the circular orbit) equals to $\kappa\mathcal{M}$. But if it equals to $\kappa\mathcal{M}$ for a circular orbit, it has to equal to the same constant for any orbit and we end up with the equation

$$\dot{M}* = \sqrt{\frac{\kappa \mathcal{M}}{a^3}} .$$

APPENDIX 3

List of the principal components of F(i) and G(e) (see 3.3) as they appear in [Kaula, 1966].

n	m	p	$F_{nmp}(i)$
2	0	0	$-3(\sin^2 i)/8$
2	0	1	$3(\sin^2 i)/4 - 1/2$
2	0	2	$-3(\sin^2 i)/8$
2	1	0	$3 \sin i(1 + \cos i)/4$
2	1	1	$-3(\sin i \cos i)/2$
2	1	2	$-3 \sin i(1 - \cos i)/4$
2	2	0	$3(1 + \cos i)^2/4$
2	2	1	$3(\sin^2 i)/2$
2	2	2	$3(1 - \cos i)^2/4$
3	0	0	$-5(\sin^3 i)/16$
3	0	1	$15(\sin^3 i)/16 - 3(\sin i)/4$
3	0	2	$-15(\sin^3 i)/16 + 3(\sin i)/4$
3	0	3	$5(\sin^3 i)/16$
3	1	0	$-15 \sin^2 i(1 + \cos i)/16$
3	1	1	$15 \sin^2 i(1 + 3 \cos i)/16 - 3(1 + \cos i)/4$
3	1	2	$15 \sin^2 i(1 - 3 \cos i)/16 - 3(1 - \cos i)/4$
3	1	3	$-15 \sin^2 i(1 - \cos i)/16$
3	2	0	$15 \sin i(1 + \cos i)^2/8$
3	2	1	$15 \sin i(1 - 2 \cos i - 3 \cos^2 i)/8$
3	2	2	$-15 \sin i(1 + 2 \cos i - 3 \cos^2 i)/8$
3	2	3	$-15 \sin i(1 - \cos i)^2/8$
3	3	0	$15(1 + \cos i)^3/8$
3	3	1	$45 \sin^2 i(1 + \cos i)/8$
3	3	2	$45 \sin^2 i(1 - \cos i)/8$
3	3	3	$15(1 - \cos i)^3/8$
4	0	0	$35(\sin^4 i)/128$
4	0	1	$-35(\sin^4 i)/32 + 15(\sin^2 i)/16$

n	m	p	$F_{nmp}(i)$
4	0	2	$(105/64)\ \sin^4 i - (15/8)\ \sin^2 i + 3/8$
4	0	3	$-\ (35/32)\sin^4 i + (15/16)\sin^2 i$
4	0	4	$(35/128)\ \sin^4 i$
4	1	0	$-\ (35/32)\sin^3 i(1 + \cos i)$
4	1	1	$(35/16)\sin^3 i(1 + 2\cos i) - (15/8)(1 + \cos i)\ \sin i$
4	1	2	$\cos i(15(\sin i)/4 - 105(\sin^3 i)/16)$
4	1	3	$-\ (35/16)\sin^3 i(1-2\cos i)+(15/8)\sin i(1 - \cos i)$
4	1	4	$(35/32)\sin^3 i(1 - \cos i)$
4	2	0	$-\ (105/32)\ \sin i(1 + \cos i)^2$
4	2	1	$(105/8)\sin^2 i\ \cos i(1+\cos i) - (15/8)(1 + \cos i)^2$
4	2	2	$(105/16)\sin^2 i(1 - 3\cos^2 i) + (15/4)\ \sin^2 i$
4	2	3	$-\ (105/8)\sin^2 i\ \cos i(1 - \cos i)-(15/8)(1-\cos i)^2$
4	2	4	$-\ (105/32)\sin^2 i(1 - \cos i)^2$
4	3	0	$(105/16)\ \sin i(1 + \cos i)^3$
4	3	1	$(105/8)\ \sin i(1 - 3\cos^2 i - 2\cos^3 i)$
4	3	2	$-\ (315/8)\sin^3 i\ \cos i$
4	3	3	$-\ (105/8)\ \sin i(1 - 3\cos^2 i + 2\cos^3 i)$
4	3	4	$-\ (105/16)\ \sin i(1 - \cos i)^3$
4	4	0	$(105/16)(1 + \cos i)^4$
4	4	1	$(105/4)\sin^2 i(1 + \cos i)^2$
4	4	2	$(315/8)\ \sin^4 i$
4	4	3	$(105/4)\ \sin^2 i(1 - \cos i)^2$
4	4	4	$(105/16)(1 - \cos i)^4$

n	p	q	n	p	q	$G_{npq}(e)$
2	0	-2	2	2	2	0
2	0	-1	2	2	1	$-e/2 + e^3/16 + \ldots$
2	0	0	2	2	0	$1 - 5e^2/2 + 13e^4/16 + \ldots$
2	0	1	2	2	-1	$7e/2 - 123e^3/16 + \ldots$
2	0	2	2	2	-2	$17e^2/2 - 115e^4/6 + \ldots$
2	1	-2	2	1	2	$9e^2/4 + 7e^4/4 + \ldots$
2	1	-1	2	1	1	$3e/2 + 27e^3/16 + \ldots$
			2	1	0	$(1-e^2)^{-3/2}$
3	0	-2	3	3	2	$e^2/8 + e^4/48 + \ldots$
3	0	-1	3	3	1	$-e + 5e^3/4 + \ldots$
3	0	0	3	3	0	$1 - 6e^2 + 423e^4/64 + \ldots$
3	0	1	3	3	-1	$5e - 22e^3 + \ldots$
3	0	2	3	3	-2	$127e^2/8 - 3065e^4/48 + \ldots$
3	1	-2	3	2	2	$11e^2/8 + 49e^4/16 + \ldots$
3	1	-1	3	2	1	$e(1 - e^2)^{-5/2}$
3	1	0	3	2	0	$1 + 2e^2 + 239e^4/64 + \ldots$
3	1	1	3	2	-1	$3e + 11e^3/4 + \ldots$
3	1	2	3	2	-2	$53e^2/8 + 39e^4/16 + \ldots$
4	0	-2	4	4	2	$e^2/2 - e^4/3 + \ldots$
4	0	-1	4	4	1	$-3e/2 + 75e^3/16 + \ldots$
4	0	0	4	4	0	$1 - 11e^2 + 199e^4/8 + \ldots$
4	0	1	4	4	-1	$13e/2 - 765e^3/16 + \ldots$
4	0	2	4	4	-2	$51e^2/2 - 321e^4/2 + \ldots$
4	1	-2	4	3	2	$(3e^2/4)\ (1 - e^2)^{-7/2}$
4	1	-1	4	3	1	$e/2 + 33e^2/16 + \ldots$
4	1	0	4	3	0	$1 + e^2 + 65e^4/16 + \ldots$
4	1	1	4	3	-1	$9e/2 - 3e^3/16 + \ldots$
4	1	2	4	3	-2	$53e^2/4 - 179e^4/24 + \ldots$
4	2	-2	4	2	2	$5e^2 + 155e^4/12 + \ldots$
4	2	-1	4	2	1	$5e/2 + 135e^3/16 + \ldots$
			4	2	0	$(1 + 3e^2/2)\ (1 - e^2)^{-7/2}$