TENSORS

P. VANICEK

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LECTURE NOTES
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TENSORS
(Third Printing)

Petr Vaníček

Department of Surveying Engineering
University of New Brunswick
P.O. Box 4400
Fredericton, N.B.
Canada
E3B 5A3

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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

PREFACE FOR FIRST PRINTING

This course is being offered to the post-graduate students in Surveying Engineering. Its aim is to give a basic knowledge of tensor "language" that can be applied for solving some problems in photogrammetry and geodesy. By no means, can the course claim any completeness; the emphasis is on achieving a basic understanding and, perhaps, a deeper insight into a few fundamental questions of differential geometry.

The course is divided into three parts: The first part is a very brief recapitulation of vector algebra and analysis as taught in the undergraduate courses. Particular attention is paid to the applications of vectors in differential geometry. The second part is meant to provide a link between the concepts of vectors in the ordinary Eucleidean space and generalized Riemannian space. The third, and the most extensive of all the three parts, deals with the tensor calculus in the proper sense.

The course concentrates on giving the theoretical outline rather than applications. However, a number of solved and mainly unsolved problems is provided for the students who want to apply the theory to the "real world" of photogrammetry and geodesy.

It is hoped that mistakes and errors in the lecture notes will be charged against the pressure of time under which the author has worked when writing them. Needless to say that any comment and criticism communicated to the author will be highly appreciated.

P. Vaníček
2/11/1972

PREFACE FOR SECOND PRINTING

The second printing of these lecture notes is basically the same as the first printing with the exception of Chapter 4 that has been added. This addition was requested by some of the graduate students who sat on this course.

I should like to acknowledge here comments given to me by Dr. G. Blaha and Mr. T. Wray that have helped in getting rid of some errors in the first printing as well as in clarifying a few points.

P. Vaníček
12/7/74

CONTENTS

1) VECTORS IN RECTANGULAR CARTESIAN COORDINATES

1.1) Basic Definitions

The Cartesian power E^3, where E is a set of real numbers, is called the System of Coordinates in three-dimensional space (futher only 3D-space). Any element $\vec{r}\varepsilon E^3$ is said to describe a point in the space, the elements $\vec{r}$ being obviously ordered triplets of real numbers. It is usual to denote them thus:

$$\vec{r} \equiv (x, y, z)$$

If the distance of any two points, $\vec{r}_1$ and $\vec{r}_2$ say, is given by

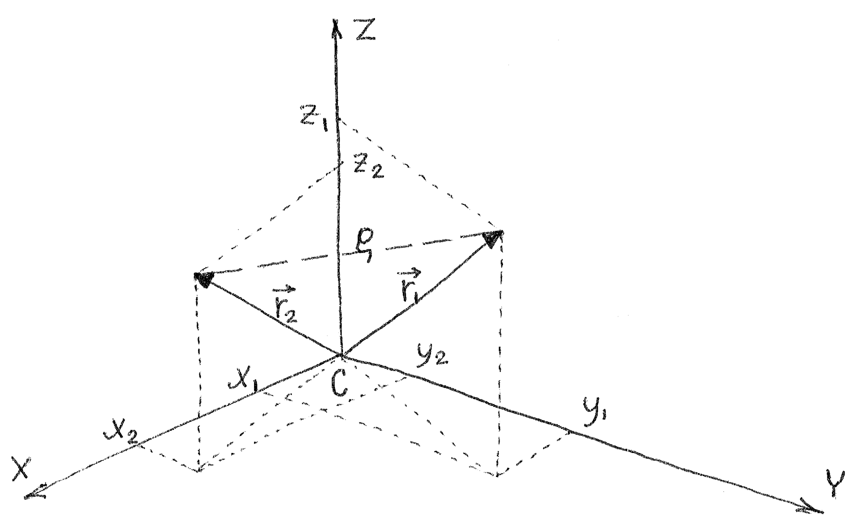
$$\rho(\vec{r}_1, \vec{r}_2) = \sqrt{[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2]}$$

where

$$\vec{r}_1 = (x_1, y_1, z_1), \qquad \vec{r}_2 = (x_2, y_2, z_2) ,$$

then the system of coordinates is known as Rectangular Cartesian.

This distance metricizes (measures) the space and this particular distance (metric) is known as the Eucleidean metric. The appropriate metric space (called usually just simply space) is called the Eucleidean space. The graphical interpretation given here is well known from the elementary geometry.



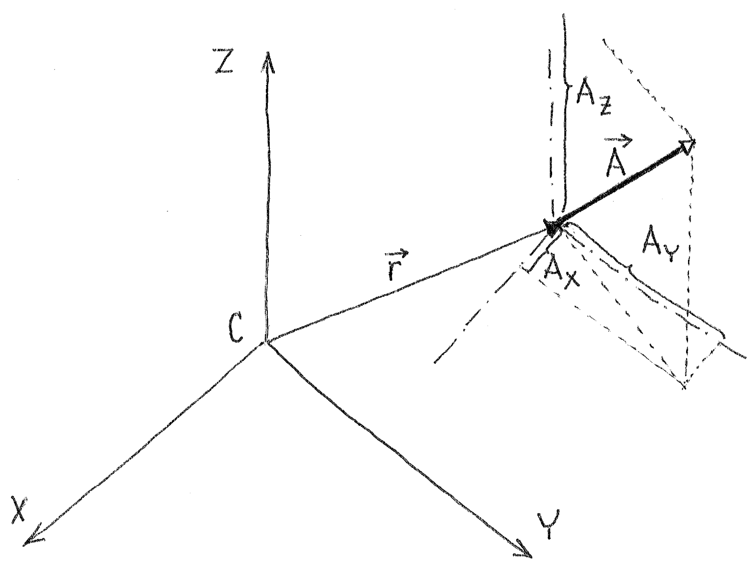
A triplet $\vec{A} \equiv (A_x, A_y, A_z)$ of real functions of three real arguments:

$$\boxed{\vec{A} \varepsilon \; \{E^3 \to E^3\}}$$

is called a <u>vector</u> in the 3D Eucleidean space or a vector in Cartesian Coordinates. It obviously can be seen as being a function of the point $\vec{r}$. It is usually interpreted as describing a segment of a straight line of a certain length and certain orientation. The length and the direction are functions of the three real functions A_x, A_y, A_z known as <u>coordinates</u> or <u>components</u> of the vector $\vec{A}$. The real function

$$\boxed{A = \surd(A_x^2 + A_y^2 + A_z^2)}$$

(sometimes denoted as $|\vec{A}|$ is called the <u>length</u> or <u>absolute value</u> of the vector $\vec{A}$. It again is evidently a function of the point $\vec{r}$.



The real functions

$$\boxed{\frac{A_x}{A}, \; \frac{A_y}{A}, \; \frac{A_z}{A}}$$

are known as the <u>direction cosines</u> of the vector $\vec{A}$ and they determine the direction of $\vec{A}$. Note that every one of the three above expressions is dependent on the other two. Squaring the equation for the absolute value and dividing it by A^2 we get

$$\boxed{\left(\frac{A_x}{A}\right)^2 + \left(\frac{A_y}{A}\right)^2 + \left(\frac{A_z}{A}\right)^2 = 1.}$$

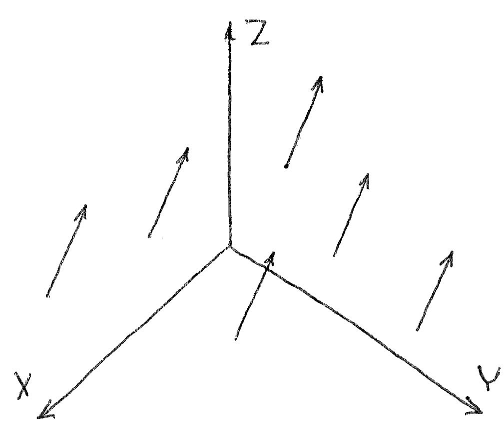
This can always be done if A is different from zero and $A \neq 0$ if and only if at least one of the components is different from zero. This leads to a statement, that a vector of zero length has got an undetermined direction.

Further, we can see that the point $\vec{r}$ can be regarded as a special case of a vector, whose argument is always the center of coordinates C:

$$\vec{r}_C \equiv (0,\ 0,\ 0) \equiv \vec{0}.$$

It is therefore also called the <u>position vector</u> or the <u>radius-vector</u> of the point. Hence we talk about the triplet of real functions $\vec{A}$ as <u>vector function of vector argument</u>.

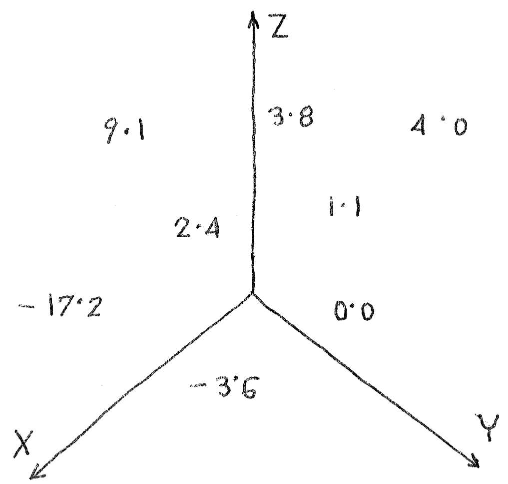
A triplet of constant functions (real numbers) is called <u>free vector</u>, meaning that its absolute value and direction (as well as its components) are independent or free from the argument (point). On the other hand, if we have a vector function of a vector argument defined for each point in a certain region $R \subset E^3$ of our space we say that there is a <u>vector field</u> defined in R. Thus obviously a free vector can be regarded as constant vector field and we shall refer to it as such.



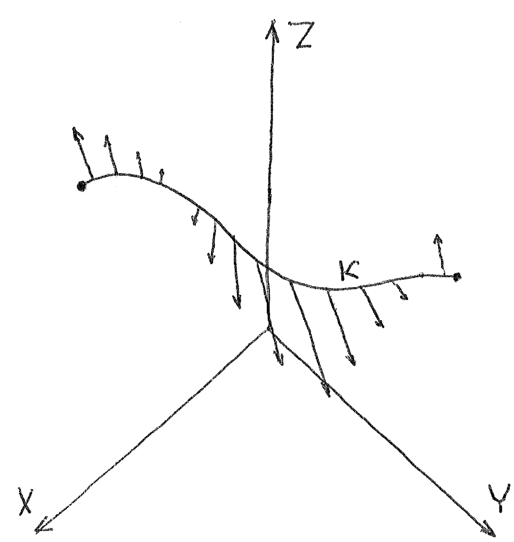
It is useful to extend the definition of a field to one-valued real functions of a vector argument as well. If we have in a certain region $R \subset E^3$ of our space a real function ϕ of the position vector defined then we say that

$$\phi \varepsilon \{E^3 \rightarrow E\}$$

is a <u>scalar field</u> in R. We thus note that vector field is a vector function of a vector variable the scalar field is a scalar function of a vector variable.



One more useful quantity can be also defined here and that is a <u>vector function of a scalar variable</u>, i.e. three-valued real functions of one real variable. This quantity is often used whenever it is necessary to consider a varying parameter (real variable) in the space. This parameter can be time, length of a curve, etc.



Hence we may have, for instance, a vector defined along a curve κ as a function of its length as shown on the diagram. The more or less trivial extension of this concept is the <u>scalar function of a scalar variable</u> or the well known real function of one real variable known from the fundamentals of mathematical analysis.

The vector function of two scalar arguments is also used particularly in the differential geometry of surfaces. The way, how this quantity is defined is quite obvious.

Note that we have confined ourselves just to 3D-space. The development for an nD-space is indeed completely analogous and often used in various branches of mathematics.

1.2) Vector Algebra

1.2.1) Zero Vector

Zero vector $\vec{0}$ is a vector whose components are all zero. The necessary and sufficient condition for this is that its absolute value equals to zero. The direction of a zero vector is undetermined.

1.2.2) Unit Vectors

Unit vector $\vec{A}$ is a vector whose absolute value equals to 1. Its direction may be arbitrary. The components of a unit vector are equal to its direction cosines as can be seen from the equation

$$A_x^2 + A_y^2 + A_z^2 = 1$$

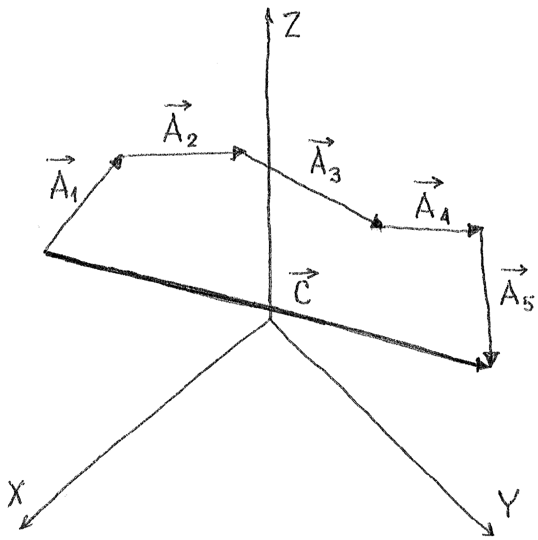
for its absolute value.

1.2.3) Summation of Vectors

Summation of n vectors $\vec{A}_i \equiv (A_{ix}, A_{iy}, A_{iz})$ is a vector $\vec{B}$ whose components are given by

$$B_x = \sum_{i=1}^{n} A_{ix}, \quad B_y = \sum_{i=1}^{n} A_{iy}, \quad B_z = \sum_{i=1}^{n} A_{iz}.$$

The geometrical interpretation of the summation is shown on the diagram. Evidently the summation is commutative and associative, i.e.



$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}).$$

The absolute value of the sum $\vec{C}$ of the two vectors $\vec{A}$ and $\vec{B}$ is given by

$$C = \sqrt{(A^2 + B^2 + 2AB\cos(\pi - \hat{AB}))},$$

where by $\hat{AB}$ we denote the angle between $\vec{A}$ and $\vec{B}$. The proof is left to the reader.

<u>Convention</u> - From now on we shall be denoting x by x_1, y by x_2 and z by x_3. The corresponding components of a vector $\vec{A}$ will accordingly be A_1, A_2, A_3.

1.2.4 <u>Multiplication of a Vector by a Constant</u>

Vector $\vec{B}$ is called the product of vector $\vec{A}$ with a constant k if and only if

$$B_x = kA_x, \quad B_y = kA_y, \quad B_z = kA_z \quad \text{or} \quad B_i = kA_i \quad i = 1, 2, 3.$$

Obviously

$$k\vec{A} = \vec{A}k$$

and

$$B = kA.$$

The direction of $\vec{B}$ is identical to the direction of $\vec{A}$.

1.2.5) Opposite Vector

Vector $\vec{B}$ is known as opposite vector to $\vec{A}$ if and only if

$$\vec{A} + \vec{B} = \vec{0} .$$

It is usual to denote the opposite vector to $\vec{A}$ by $-\vec{A}$ because

$$\vec{B} = (-1) \vec{A} .$$

1.2.6 Multiplication of Vectors

i) Scalar Product $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is the real number (scalar) k given by

$$k = \boxed{\sum_{i=1}^{3} A_i B_i = \vec{A} \cdot \vec{B} .}$$

Scalar product is obviously commutative, i.e. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$, and it is not associative, i.e. $\vec{A} (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \vec{C}$. The proof of the latter is left to the reader. The reader is also advised to show that

$$\boxed{\vec{A} \cdot \vec{B} = AB \cos \widehat{AB}} ,$$

and $$(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}.$$

Obviously, the absolute value of a vector $\vec{A}$ can be written as

$$A = \sqrt{(\vec{A} \cdot \vec{A})}.$$

Two non-zero vectors $\vec{A}$, $\vec{B}$ whose scalar product equals to zero are perpendicular because $AB \cos \widehat{AB} = 0$ implies that

$$\cos \widehat{AB} = 0 \quad \text{and} \quad \widehat{AB} = \begin{cases} \pi/2 \\ 3\pi/2 \end{cases} : \vec{A} \cdot \vec{B} = 0 \equiv \vec{A} \perp \vec{B}$$

ii) Vector Product $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is the vector $\vec{C}$ given by

$$\boxed{\vec{C} \equiv (C_1, C_2, C_3) \equiv (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1).}$$

It can be easily shown that the following formulae hold:

$$(k\vec{A}) \times \vec{B} = k(\vec{A} \times \vec{B}) = \vec{A} \times (k\vec{B})$$

and particularly

$$\vec{A} \times \vec{B} = - \vec{B} \times \vec{A},$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

The reader is urged to prove:

α) $\vec{C} = \vec{A} \times \vec{B}$ is a vector perpendicular to both $\vec{A}$ and $\vec{B}$ $(\vec{A} \cdot \vec{C} = \vec{B} \cdot \vec{C} = 0)$ whose length equals to AB sin $(\widehat{AB})$ and that points the same way with respect to $\vec{A}$ and $\vec{B}$ as the + z axis points with respect to + x and + y axes (in this order);

β) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \quad \vec{B} - (\vec{A} \cdot \vec{B}) \quad \vec{C}$;

γ) $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C}) (\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C}) (\vec{A} \cdot \vec{D})$.

Evidently, the vector product of two non-zero vectors equals to zero if and only if the two vectors are parallel. Hence the necessary and sufficient condition for two vectors to be parallel is that their vector product be zero:

$$\vec{A} \times \vec{B} = \vec{0} \equiv \vec{A} \,||\, \vec{B}.$$

iii) Dyadic Product $\vec{A} * \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a quantity C which is neither scalar nor vector. It can be interpreted as a matrix (or Cartesian tensor of second rank - see later) whose components are given by

$$\boxed{C_{ij} = A_i \, B_j} \qquad i, j = 1, 2, 3.$$

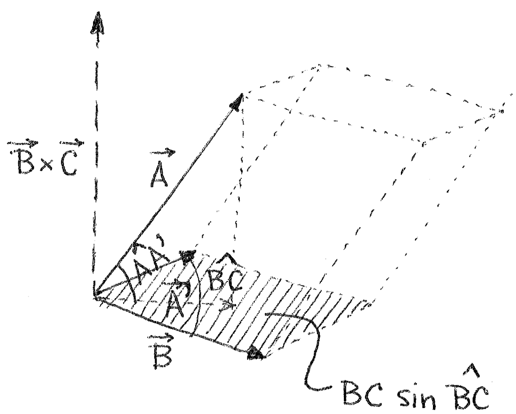
iv) Mixed Product $[\vec{A}\vec{B}\vec{C}]$ of three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ is a scalar k defined as

$$k = \boxed{\vec{A} \cdot (\vec{B} \times \vec{C}) = [\vec{A}\ \vec{B}\ \vec{C}].}$$

Its value can be seen to be

$$k = A\ B\ C \sin \widehat{BC} \sin \widehat{AA'}$$

where $\vec{A}'$ is the projection of $\vec{A}$ on the plane $\vec{B}\ \vec{C}$ (see the diagram). But this equals to the volume of the parallelepiped given by the three vectors. Evidently, the volume of this body equals to zero if and only if all the three vectors are coplanar (lay in one plane) or at least one of them is the zero-vector. Hence the necessary and sufficient condition for three non-zero vectors to be coplanar is that their mixed product equals to zero:



$$[\vec{A}\ \vec{B}\ \vec{C}] = 0 \equiv \vec{A}, \vec{B}, \vec{C} \in \kappa$$

(κ denotes a plane).

It is left to the reader to prove that

$$[\vec{A}\vec{B}\vec{C}] = [BCA] = [CAB] = - [ACB] = - [BAC] = - [CBA].$$

Problem: Show that a vector given as a linear combination of two other vectors $\vec{A}$ and $\vec{B}$, i.e. $\vec{C} = k_1\vec{A} + k_2\vec{B}$, is coplanar with the vectors $\vec{A}$ and $\vec{B}$.

1.2.7) <u>Vector Equations</u>

Equations involving vectors are known as <u>vector equations</u>. In three-dimensional geometric applications they invariably describe properties of various objects in 3D-space. For example, the vector equation

$$\vec{B} = k\vec{A}$$

tells us that vectors $\vec{A}$ and $\vec{B}$ are parallel and vector $\vec{B}$ is k-times longer than $\vec{A}$. Or the vector equation

$$\cos(\hat{AB}) = \vec{A} \cdot \vec{B} \;/\; (AB)$$

determines the angle of two non-zero vectors $\vec{A}$ and $\vec{B}$.

If we decide, for some reason, to change the coordinate system, i.e. transform the original coordinate system to another, the geometric properties of the objects do not change. Two straight lines remain parallel or perpendicular in any system of coordinates. Similarly one vector remains k-times longer than another whatever the system of coordinates may be. This fact is usually expressed by the statement that <u>vector equations are invariant in any transformation of coordinate system</u>. This is the basic reason why we prefer using vectors - and by this we mean here the described compact notation for the triplets of functions - when dealing with properties of objects in space. Another possibility would be to use coordinates instead but in that case formulae would be valid only in the one coordinate system and would not be invariant.

1.2.8) Note on Coordinate Transformations

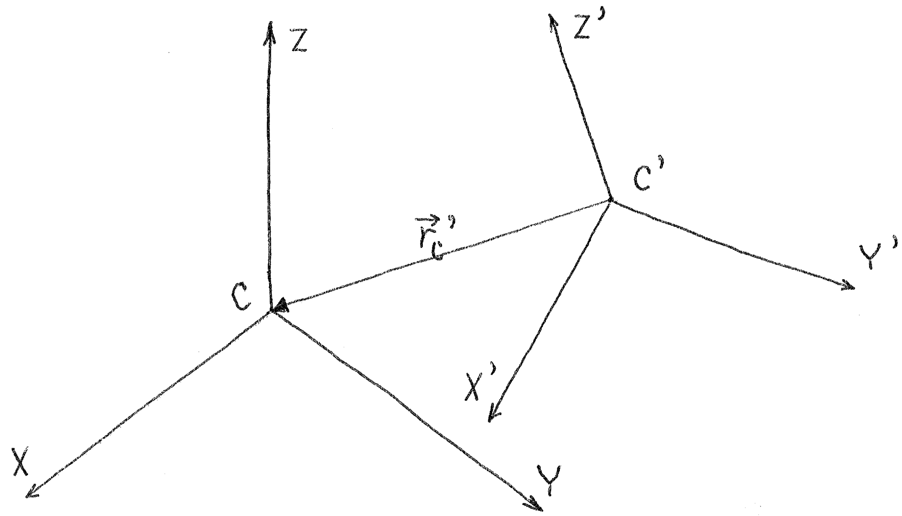
When talking about transformations of coordinate systems here we talk about relations of following kind

$$\vec{A}' = M\vec{A}$$

where $\vec{A}$, $\vec{A}'$ is a vector expressed in one and another coordinate systems and M is the transformation matrix. For the position vectors the transformation equation has a more general form, namely:

$$\vec{r}' = M\vec{r} + \vec{r}'_C$$

where $\vec{r}'_C$ is the position vector of the original center of coordinates in the new system, known also as the translation vector (see the diagram).



When transferring one Cartesian coordinate system to another, the transformation matrix can be obtained as a product of three rotation matrices representing the rotations through the three Euler's angles. Alternatively the transformation matrix can be obtained using the nine direction cosines of the new coordinate axes. In both cases, all the nine elements of M are independent of the position of the transformed vectors and can be, in addition, expressed as functions of only 3 independent variables (3 rotations of the new system of coordinates with respect to the original one). All the transformation matrices possessing these properties are known as constituting the group of Cartesian transformation matrices. Moreover, when talking about the invariance of lengths, we have to require that

$$|\det M| = 1.$$

Obviously, the Cartesian transformation matrices are something very special. Later, we shall deal with a more general group of transformations. However, it is not considered the aim of this course to deal with transformations in detail.

1.3) Vector Analysis

1.3.1 Derivative of a Vector Function of One and Two Scalar Variables

The quantity

$$\boxed{\lim_{\Delta u \to 0} \frac{\vec{A}(u + \Delta u) - \vec{A}(u)}{\Delta u} = \frac{d\vec{A}}{du}}$$

is called the derivative of vector $\vec{A}$ with respect to its scalar argument u. It is sometimes denoted by $\vec{A}'$. Geometrically, this derivative has important applications in differential geometry of curves as we shall see later.

A vector function $\vec{A}$ of two scalar arguments u and v has got two partial derivatives. These are defined completely analogously to the above case:

$$\frac{\partial\vec{A}}{\partial u} = \lim_{\Delta u \to 0} \frac{\vec{A}(u + \Delta u,\ v = \text{const.}) - A(u,\ v = \text{const.})}{\Delta u}$$

$$\frac{\partial\vec{A}}{\partial v} = \lim_{\Delta v \to 0} \frac{\vec{A}(u = \text{const.},\ v + \Delta v) - A(u = \text{const.},\ v)}{\Delta v} .$$

Geometrically, these derivatives have applications in differential geometry of surfaces and are sometimes denoted by $\vec{A}'_u$, $\vec{A}'_v$. Obviously, all the defined derivatives are again vectors.

The rules for differentiation are very much the same as those for the differentiation of real functions. Particularly we have

$$\frac{d}{du}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du} ,$$

$$\frac{d}{du}(k\vec{A}) = k\,\frac{d\vec{A}}{du} ,$$

$$\frac{d}{du}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{du} + \frac{d\vec{A}}{du} \cdot \vec{B} ,$$

$$\frac{d}{du}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{du} + \frac{d\vec{A}}{du} \times \vec{B} .$$

If A = const. then

$$\frac{d\vec{A}}{du} \cdot \vec{A} = 0 \equiv \frac{d\vec{A}}{du} \perp \vec{A} .$$

The proof of this theorem is left to the reader. The rules for partial differentiation are analogous.

1.3.2) Elements of Differential Geometry of Curves

If for all $u \varepsilon < a, b >$ a position-vector $\vec{r} = \vec{r}(u)$ is defined, we say that $\vec{r}$ describes a curve (spatial curve in 3D-space). The real variable u is called the parameter of the curve. Let us assume that $\vec{r}$ is in $< a, b >$ a continuous function and we shall hence talk about continuous curve.

If $\vec{r}$ is in $< a, b >$ continuous, we can define another scalar function of u:

$$\boxed{s(u) = \int_a^u \left|\frac{d\vec{r}}{du}\right| du}$$

that always exists for a continuous $\vec{r}$, and call it the <u>length of the curve</u> $\vec{r}$ between a and u. Since s is monotonic, there exists always its inverse function $u = u(s)$ and we conclude that for continuous curves we can always write

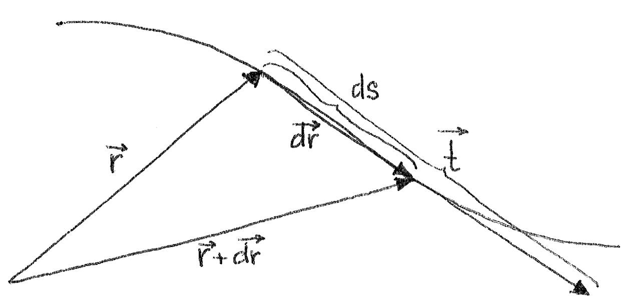
$$\vec{r}(s) = \vec{r}(u(s)).$$

This equation of a curve, using its length for parameter, is known as the <u>natural equation of the curve.</u>

The unit <u>tangent vector</u> $\vec{t}$ to the curve $\vec{r}$ is given by

$$\vec{t} = \frac{d\vec{r}}{ds}$$

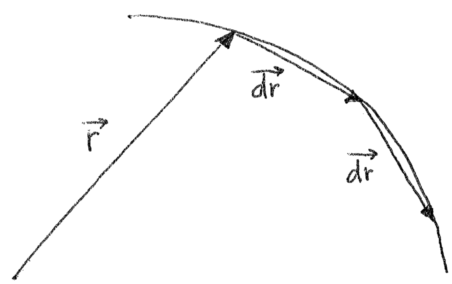
as can be easily seen from the diagram.



Since $\vec{t}$ is a vector of constant length (unit) its derivative is perpendicular to it. Denoting the length of the derivative by $1/R$ we can write

$$\boxed{\frac{d\vec{t}}{ds} = \frac{d^2\vec{r}}{ds^2} = \frac{\vec{n}}{R}}$$

where $\vec{n}$ is a unit vector perpendicular to $\vec{t}$. The interpretation of $\vec{n}$ can be based on the following reasoning. The second derivative can be regarded as a scaled difference of two infinitesimally small vectors connecting three infinitesimally close points on the curve. Hence $\vec{n}$ has to lay in the plane given by the three infinitesimally close points, i.e. in the osculating plane. Therefore $\vec{n}$ has got the direction of the principal normal (usually called just the normal) to the curve. The proof that $R = 1/\left|\frac{d^2\vec{r}}{ds^2}\right|$ is therefore the radius of curvature is left to the reader.



The vector product $\vec{b}$ of $\vec{t}$ and $\vec{n}$ (in this order)

$$\boxed{\vec{b} = \vec{t} \times \vec{n}}$$

is obviously perpendicular to both $\vec{t}$ and $\vec{n}$ and has got also unit length. It is therefore called the binormal vector. The three unit vectors create an orthonormal triplet oriented the same way as the coordinate axes.

The differential relations among these three vectors are given by Frenet's formulae

$$\boxed{\begin{aligned} \frac{d\vec{t}}{ds} &= \frac{\vec{n}}{R} \\ \frac{d\vec{n}}{ds} &= \frac{\vec{b}}{T} - \frac{\vec{t}}{R} \\ \frac{d\vec{b}}{ds} &= -\frac{\vec{n}}{T} \end{aligned}}$$

where T is the radius of torsion of $\vec{r}$, i.e. the radius of curvature of the projection of the curve onto the plane defined by $\vec{t}$ and $\vec{b}$. The proof of the second two formulae is left to the reader.

The reader is also urged to prove the following two theorems:

α) $$\boxed{T = \frac{\vec{r}''^2}{[\vec{r}' \; \vec{r}'' \; \vec{r}''']}}$$

where by primes are denoted the derivatives with respect to s;

β) if

$$\vec{\gamma} = \frac{\vec{t}}{T} + \frac{\vec{b}}{R} \qquad \text{then}$$

$$\frac{d\vec{t}}{ds} = \vec{\gamma} \times \vec{t}, \; \frac{d\vec{n}}{ds} = \vec{\gamma} \times \vec{n}, \; \frac{d\vec{b}}{ds} = \vec{\gamma} \times \vec{b} .$$

The formulae for R and T give us a handy tool for categorizing curves. If a curve $\vec{r}$ has got $1/T = 0$ in a certain interval, then it is called <u>planar</u> in this interval. If even $1/R = 0$ then $\vec{r}$ is a straight line in the same interval.

<u>Problem</u>: Show that the necessary and sufficient condition for a curve to be a straight line is

$$\vec{r}(u) = \vec{r}_o + \vec{A}u$$

where $\vec{r}_o$ is a fixed radius-vector and $\vec{A}$ is a vector

<u>Problem</u>: Determine the shortest distance of two lines

$$\vec{r}_1(u) = \vec{r}_1 + \vec{A}_1 u$$

$$\vec{r}_2(u) = \vec{r}_2 + \vec{A}_2 u$$

that do not intersect.

1.3.3) <u>Elements of Differential Geometry of Surfaces</u>

A radius-vector $\vec{r}$ given as a function of two scalar arguments u and v, defines a surface. We shall again assume that $\vec{r}$ is a continuous function of both parameters and that the surface is therefore continuous. The curves

$$\boxed{\vec{r}\ (u,\ v = \text{const}),\ \vec{r}\ (u = \text{const.},\ v)}$$

are called parametric curves on the surface, for which we can indeed use all the formulae developed in the previous paragraph.

Problem: Derive the equations of ϕ-curves and λ-curves on the surface of an ellipsoid of rotation, where ϕ and λ are geographical coordinates.

Problem: Show that the necessary and sufficient condition for a surface to be **a** plane is

$$\vec{r}\ (u,\ v) = \vec{r}_o + u\ \vec{A} + v\ \vec{B}$$

where $\vec{r}_o$, $\vec{A}$ and $\vec{B}$ are some arbitrary vectors.

Problem: Derive the shortest distance of a point $\vec{r}_p$ to the plane

$$\vec{r}\ (u,\ v) = \vec{r}_o + u\ \vec{A} + v\ \vec{B}.$$

A curve on the surface $\vec{r} = \vec{r}\ (u,\ v)$ is given by the formula

$$\boxed{\vec{r}\ (t) = \vec{r}\ (u(t),\ v(t))}$$

for which again all that has been said in the previous paragraph holds true.

The tangent plane to the surface $\vec{r}$, if it exists, is given by the two tangent vectors

$$\vec{t}_u = \frac{\partial \vec{r}}{\partial u}\ ,\quad \vec{t}_v = \frac{\partial \vec{r}}{\partial v}$$

to the two parametric curves. Hence any vector that can be expressed as a linear combination of these two vectors lays in the tangent plane. The equation

$$\boxed{\vec{r} = \vec{r}(\alpha,\ \beta) = \vec{r}_o + \alpha \left.\frac{\partial \vec{r}}{\partial u}\right|_{\vec{r} = \vec{r}_o} + \beta \left.\frac{\partial \vec{r}}{\partial v}\right|_{\vec{r} = \vec{r}_o}}$$

is therefore the equation of the <u>tangent plane</u> to the surface $\vec{r}$ (u, v) at the point $\vec{r}_o$.

<u>Problem</u>: Derive the equation of the tangent plane to the ellipsoid of rotation at the point $\vec{r}_o = \vec{r}\,(\phi_o, \lambda_o)$.

The <u>unit normal vector</u> to the surface $\vec{r}$ is given by

$$\boxed{\vec{n} = \frac{1}{D}\left(\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right)}$$

where

$$D = \sqrt{\left[\left(\frac{\partial\vec{r}}{\partial u}\right)^2 \left(\frac{\partial\vec{r}}{\partial v}\right)^2 - \left(\frac{\partial\vec{r}}{\partial u} \cdot \frac{\partial\vec{r}}{\partial v}\right)^2\right]} = \sqrt{[EG - F^2]}.$$

The normal points the same way with respect to u and v curves as the z-axis points with respect to x and y axes.

<u>Problem</u>: Derive a formula for the outward unit normal vector to the ellipsoid of rotation.

Other characteristics of a surface, as for instance various curvatures, are slightly more involved and do not serve any way as good examples for vector analysis. They are better dealt with using tensors as we shall see later. Let us just mention one more application here. The curvature of the projection of a surface curve $\vec{r}$ onto the tangent plane to the surface at the point is known as <u>geodetic curvature</u>. This quantity is usually denoted by $1/R_G$ and given by

$$\boxed{1/R_G = [\vec{r}\,'\ \vec{r}\,''\ \vec{n}]}$$

where $\vec{r}(s)$ is the surface curve.

<u>Problem</u>: What are the geodetic curvatures of ϕ and λ-curves on the surface of the ellipsoid of rotation?

As we know, one way how to define a geodesic curve on a surface is: geodesic curve is a curve whose geodetic curvature is everywhere equal to zero. Hence the equation

$$\boxed{[\vec{r}'\ \vec{r}''\ \vec{n}] = 0}$$

can be considered the equation of a geodesic curve $\vec{r}$. Here, as well as in the previous formula $\vec{r}' = \frac{d\vec{r}}{ds}$, $\vec{r}'' = \frac{d^2\vec{r}}{ds^2}$. If a curve happens to be given as a function of another parameter t then we have

$$\frac{d\vec{r}}{ds} = \left(\frac{\partial\vec{r}}{\partial u}\frac{du}{dt} + \frac{\partial\vec{r}}{\partial v}\frac{dv}{dt}\right)\frac{dt}{ds}$$

and

$$s(t) = \int_{t_o}^{t} \left|\frac{\partial\vec{r}}{\partial u}\ \frac{du}{dt} + \frac{\partial\vec{r}}{\partial v}\frac{dv}{dt}\right| dt.$$

We can see that, generally, the formulae for describing properties and relations on surfaces are complicated. It is usually simpler to deal with general surfaces using a particularly suitable system of coordinates, not necessarily Cartesian. We shall see later how to do it.

1.3.4) Differentiation of Vector and Scalar Fields

Differentiation of vector and scalar fields (vector and scalar functions of vector argument) can be defined in a multitude of ways. The three most widely used definitions can all be presented using the symbolic vector (differential operator) ∇ (nabla, del). The ∇ operator is defined as follows

$$\boxed{\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) .}$$

Applying this operator to a scalar field ϕ (multiplying the vector ∇ by the scalar ϕ) we get the derivative of the scalar field

$$\boxed{\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right)} \quad ,$$

a vector known as the <u>gradient of ϕ</u>. It is often written also as grad ϕ.

With respect to a vector field $\vec{A}$ we can obviously "apply" the operator in several ways. First we can get the scalar product of ∇ with $\vec{A}$

$$\boxed{\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \quad .}$$

This scalar product is known as the <u>divergence of $\vec{A}$</u> and often written as div $\vec{A}$.

Alternatively, we can produce a vector by taking the vector product of ∇ and $\vec{A}$ and get

$$\boxed{\nabla x\vec{A} = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right).}$$

This vector is known as <u>rotor</u> or <u>curl of $\vec{A}$</u> and often written as rot $\vec{A}$ or curl $\vec{A}$.

Another possibility would be to take the dyadic product of ∇ and $\vec{A}$ to get a matrix of derivatives. This type of the derivative of a vector is basic for tensor analysis and will be dealt with later.

All these differentiations are important in the theory of physical fields into which we are not going to go here. We shall just limit ourselves here to statements considering the rules these derivatives obey. The rules are again much the same as those for ordinary derivatives. Here are some of them:

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \ ,$$

$$\nabla(k\phi) = k\nabla\phi \ ,$$

$$\nabla(\phi\,\psi) = \phi\nabla\psi + \psi\nabla\phi ,$$

$$\nabla(f(\phi)) = \frac{df}{d\phi}\nabla\phi ,$$

$$\nabla.(\vec{A} + \vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B} ,$$

$$\nabla\cdot(\phi\vec{A}) = \phi\nabla\cdot\vec{A} + \vec{A}\cdot\nabla\phi = \phi\,\mathrm{div}\vec{A} + \vec{A}\cdot\mathrm{grad}\ \phi ,$$

$$\nabla(\vec{A}\cdot\vec{B}) = \qquad\qquad = \vec{A}\times\mathrm{rot}\vec{B} + \vec{B}\times\mathrm{rot}\ \vec{A} + (\vec{A}\cdot\nabla)\cdot\vec{B} + (\vec{B}\cdot\nabla)\cdot\vec{A} ,$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot\ \mathrm{rot}\vec{A} - \vec{A}\cdot\ \mathrm{rot}\ \vec{B} ,$$

$$\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B} ,$$

$$\nabla\times(\phi\vec{A}) = \phi\ (\nabla\times\vec{A}) - \vec{A}\times\nabla\phi = \phi\cdot\mathrm{rot}\ \vec{A} + \vec{A}\times\ \mathrm{grad}\phi \ .$$

It is left to the reader to prove the following theorems:

$$\nabla r = \frac{\vec{r}}{r} \ ,$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} \ ,$$

$$\nabla\cdot\vec{r} = 3 ,$$

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \frac{2}{r} \ ,$$

$$\nabla\times\vec{r} = \vec{0} \ ,$$

$$\nabla\cdot(\nabla\times\vec{A}) = 0, \qquad \nabla\times(\nabla\phi) = \vec{0}$$

where $\vec{r}$ is a radius vector and r its absolute value, $\vec{A}$ is a vector and ϕ is a scalar.

Very important is also the differential operator of second order Δ (delta, ∇^2). It can be obtained as the scalar product of two ∇ operators giving thus:

$$\boxed{\Delta = \nabla\cdot\nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \quad .}$$

The application of this operator to a scalar ϕ gives

$$\boxed{\Delta\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \text{div (grad } \phi) \, ,}$$

a scalar. Application to a vector $\vec{A}$ yields:

$$\boxed{\Delta\vec{A} = \frac{\partial^2\vec{A}}{\partial x_1^2} + \frac{\partial^2\vec{A}}{\partial x_2^2} + \frac{\partial^2\vec{A}}{\partial x_3^2} \quad ,}$$

a vector.

It is again left to the reader to show the following identities:

$$\Delta\,(\phi + \psi) = \Delta\phi + \Delta\psi,$$

$$\Delta\,(\phi\psi) = \phi\Delta\psi + \psi\Delta\phi + 2\nabla\phi \cdot \nabla\psi,$$

$$\Delta\,(\vec{A} + \vec{B}) = \Delta\vec{A} + \Delta\vec{B},$$

$$\Delta\nabla\phi = \nabla\Delta\phi \; ,$$

$$\Delta\,(\nabla\times\vec{A}) = \nabla\times(\Delta\vec{A}),$$

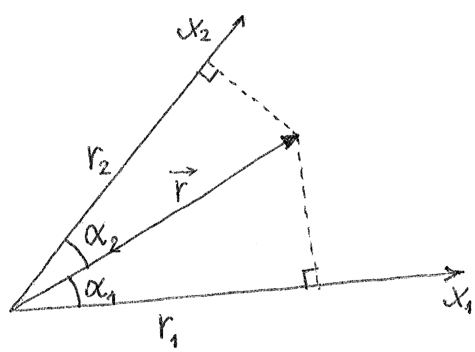
$$\nabla\,(\nabla\vec{A}) = \Delta\vec{A} + \nabla\times(\nabla\times\vec{A}).$$

Let us just mention here that the integral formulae traditionally considered a part of vector analysis are treated in another course and do not, therefore, constitute a subject of interest here.

2) VECTORS IN OTHER COORDINATE SYSTEMS

2.1) Vectors in Skew Cartesian Coordinates

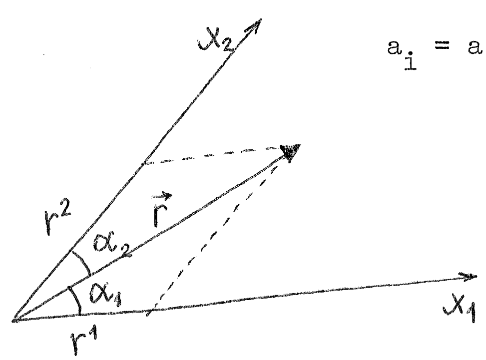
In order to see one of the difficulties encountered when dealing with curvilinear coordinates let us have a look first at the simplest departure from rectangular Cartesian system - the skew Cartesian system. For easier graphical interpretation we shall deal with 2D-space, the plane and assume that the scale along the coordinate axes is the same as the scale along the rectangular axes. The first diagram shows one way how to define the coordinates of a radius-vector in skew coordinates. These coordinates or components are known as covariant components and are obviously given by



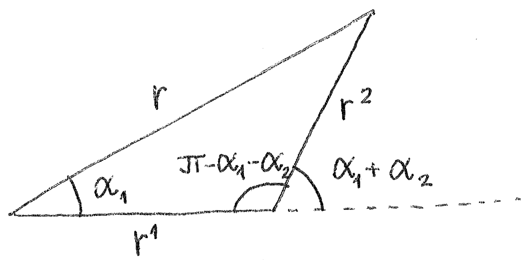
$$r_1 = r \cos \alpha_1, \quad r_2 = r \cos \alpha_2.$$

Generally, the covariant components are defined as absolute value times the directional cosine and we can generalize it immediately to any vector in 3D-space:

$$a_i = a \cos \alpha_i \quad i = 1, 2, 3.$$



The other alternative is shown on the second diagram. Applying the sine law to the lower triangle we obtain



$$\frac{r}{\sin\,(\pi-\alpha_1-\alpha_2)} = \frac{r^2}{\sin\,\alpha_1}$$

or

$$r^2 = r\,\frac{\sin\,\alpha_1}{\sin\,(\alpha_1+\alpha_2)}\ .$$

Similarly

$$r^1 = r\,\frac{\sin\,\alpha_2}{\sin\,(\alpha_1+\alpha_2)}\ .$$

Components defined in this way are called contravariant components and are denoted by a superscript to distinguish them from the covariant components. In 3D-space the expressions are more complicated and will not be dealt with here.

Note that in rectangular Cartesian coordinates, there is no difference between the covariant and contravariant components. This is the main point of departure when dealing with other than rectangular Cartesian systems.

We can now ask the obvious question: how do we determine the length of a vector $\vec{a}$ in a skew system? We either have to know at least one of the angles α_1, α_2 and some of the components or both covariant and contravariant components of $\vec{a}$. The first case is trivial, the second leads to the formula:

$$(a)^2 = a_1 a^1 + a_2 a^2 = \sum_{i=1}^{2} a_i a^i .$$

We shall prove this formula easily by just substituting for the components:

$$(a)^2 = (a)^2 \cos \alpha_1 \sin \alpha_2 / \sin(\alpha_1+\alpha_2) + (a)^2 \cos \alpha_2 \sin \alpha_1 / \sin(\alpha_1+\alpha_2) =$$

$$= (a)^2 \frac{\sin(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} .$$

We state without proof that the same formula holds even for 3D-space where we get

$$\boxed{(a)^2 = \sum_{i=1}^{3} a_i a^i .}$$

This is the generalization of the <u>scalar product</u> of $\vec{a}$ with $\vec{a}$. Note that the scalar product in rectangular Cartesian coordinates is just a special case of the above where there is no difference between contravariant and covariant components.

<u>Summation convention</u> - Since we are going to deal with summations of the type used above very extensively we shall abbreviate the equations by omitting the summation sign altogether. We shall understand, unless stated otherwise, that whenever there is an index (subscript or superscript) repeated twice in the right hand side the summation automatically takes place. Usually, we require that one index is lower and one index is upper. There will, however, be exceptions from this rule and this will be clear from the text. Such an index becomes a dummy index and does not appear on the left hand side of the equation. In order to be able to distinguish between 2D and 3D-spaces we shall use Greek indeces in 2D-space and Latin indeces in 3D-space. This means that the two scalar

products we have used in this paragraph will be written as follows:

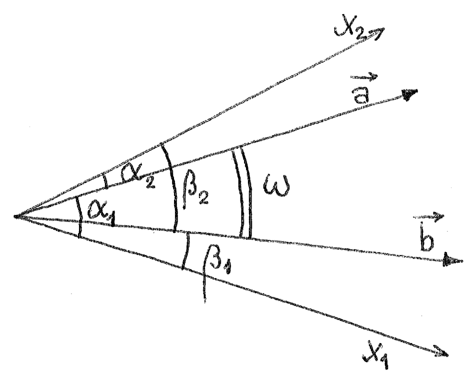
$$\boxed{(a)^2 = \sum_{i=1}^{2} a_i a^i = a_\alpha a^\alpha, \ (a)^2 = \sum_{i=1}^{3} a_i a^i = a_i a^i .}$$

This convention does not apply, of course, in case the index appears in two additive terms. Hence

$$a_i + b_i \neq \sum_{i=1}^{3} (a_i + b_i).$$

Having established the summation convention we can now ask another obvious question: how do we determine the <u>angle between two vectors $\vec{a}$ and $\vec{b}$</u> in a skew Cartesian system? Here again, if we know the direction cosines of the two vectors, the problem is trivial. More interesting is the case when only covariant components of one and contravariant components of the other vectors are known. We shall show that in this case the angle $\hat{ab} = \omega$ is given as:

$$\omega = \arccos \left(\frac{\sum_{i=1}^{2} a_i b^i}{ab}\right) = \arccos \frac{a_\alpha b^\alpha}{ab} .$$



To prove it let us first evaluate the scalar product of the two vectors. We have:

$$\sum_{i=1}^{2} a_i b^i = a_\alpha b^\alpha = ab \sin \beta_2 \cos \alpha_1/\sin(\beta_1+\beta_2) + ab \sin \beta_1 \cos \alpha_2 /\sin (\beta_1+ \beta_2).$$

Expressing α_1, α_2 in terms of β_1, β_2 and ω we get:

$$a_\alpha b^\alpha = ab \frac{\sin\beta_2 \cos(\beta_1+\omega) + \sin\beta_1 \cos(\beta_2-\omega)}{\sin(\beta_1+\beta_2)}$$

$$= ab \frac{\sin\beta_2(\cos\beta_1 \cos\omega - \sin\beta_1 \sin\omega) + \sin\beta_1(\cos\beta_2 \cos\omega + \sin\beta_2 \sin\omega)}{\sin(\beta_1+\beta_2)}$$

$$= ab \frac{\cos\omega(\sin\beta_2\cos\beta_1 + \sin\beta_1\cos\beta_2) + \sin\omega(\sin\beta_1\sin\beta_2 - \sin\beta_1\sin\beta_2)}{\sin(\beta_1+\beta_2)}$$

$$= ab\cos\omega \frac{\sin(\beta_1+\beta_2)}{\sin(\beta_1+\beta_2)}$$

The last equation is obviously equivalent to the one we set to prove.

<u>Problem</u>: Show that $a_\alpha b^\alpha = a^\alpha b_\alpha$.

The result can be generalized for three dimensional space and we can write

$$\boxed{\cos\omega = \frac{a_i b^i}{ab} = \frac{a^i b_i}{ab}}\ .$$

By answering the two questions we have shown that the two basic formulae - for the length of a vector and the angle of two vectors - remain the same in skew Cartesian coordinates as they were in the rectangular Cartesian coordinates providing we redefine the scalar product in the described manner. The same indeed holds true for all vector equations as we have stated already in 1.2.7). For this aim though the operations over vectors in non-Cartesian coordinates have to be all redefined on even more general basis which we shall do in the forthcoming paragraphs. Let us just stress here that the distinction between covariant and contravariant components is fundamental for the general development and will be strictly adhered to from now on.

2.2) Vectors in Curvilinear Coordinates

Let us assume a 3D-space with a rectangular Cartesian coordinate system X defined in it. We say that there is a curvilinear system of coordinates U defined in the same space if and only if there are three functions

$$u^i = u^i\,(x^1, x^2, x^3) \qquad i = 1, 2, 3$$

of the rectangular Cartesian coordinates defined and if these functions can be inverted to give

$$x^i = x^i\,(u^1, u^2, u^3) \qquad i = 1, 2, 3\,.$$

The necessary and sufficient condition for the two sets of functions to be reversible is that both determinants of transformation

$$\det\left(\frac{\partial u^j}{\partial x^i}\right)\;,\quad \det\left(\frac{\partial x^j}{\partial u^i}\right)$$

have to be different from zero.

The reason for writing the coordinates u as contravariant (in the case of x it makes no difference since covariant and contravariant coordinates are the same in rectangular Cartesian systems) will be explained later. At this point we shall just note that the differentials of the coordinates represent the total differentials in terms of the other coordinate system:

$$\left.\begin{aligned} du^i &= \sum_{j=1}^{3} \frac{\partial u^i}{\partial x^j}\,dx^j = \frac{\partial u^i}{\partial x^j}\,dx^j \\ dx^i &= \sum_{j=1}^{3} \frac{\partial x^i}{\partial u^j}\,du^j = \frac{\partial x^i}{\partial u^j}\,du^j \end{aligned}\right\}\; i = 1, 2, 3.$$

It can be seen that to one triplet of values x^i there corresponds only one triplet of values u^i. We say that there is a one-to-one correspondence between the two systems X and U. The examples of such curvilinear systems such as, cylindrical, spherical, geodetic etc. are well known from more elementary courses and will not be discussed here.

The basic question now arises as how to define the components (coordinates) of a vector known in the X system, or more precisely a vector field, in the new coordinate system U, and yet preserve the invariability of the vector equations even for the transformation from X to U. What we really want is that the length of a vector $\vec{a}$ as well as its orientation remain the same after expressing its components in the U system.

The way to define the components hence suggests itself as follows. Let us write the vector $\vec{a}$ in X coordinate system as

$$\vec{a} = (a \cos \alpha_1, a \cos \alpha_2) = a(\cos \alpha_1, \cos \alpha_2)$$

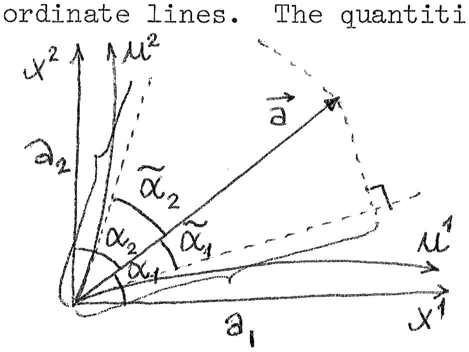
using the 2D-space for easier graphical interpretation.

In the U system we shall define

$$\vec{a} = a(\cos \tilde{\alpha}_1, \cos \tilde{\alpha}_2)$$

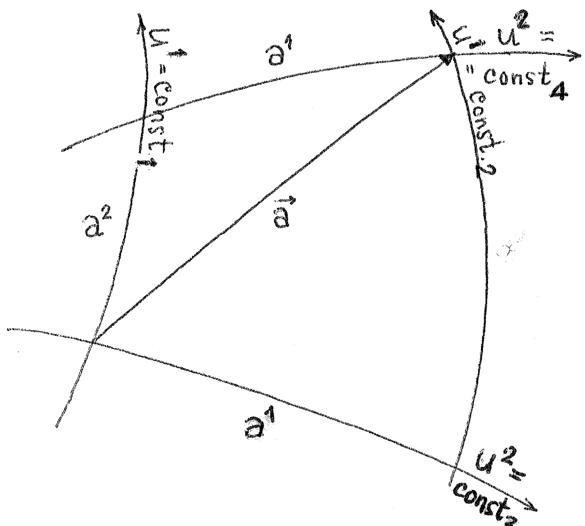
where

$\cos \tilde{\alpha}_1$, $\cos \tilde{\alpha}_2$ are the direction cosines with respect to the u^1, u^2 coordinate lines. The quantities $a \cos \tilde{\alpha}_1$, $a \cos \tilde{\alpha}_2$ are called, in agreement with the skew coordinates expressions, <u>covariant components</u> of $\vec{a}$ in U.



Here, as well as in the next section, we assume that the scale along the coordinate lines u^1, u^2 is the same as along the lines x^1, x^2. This is just for the geometric interpretation sake. If the scales are different then it becomes increasingly difficult to visualise the meaning of the components but the general results described in the next section do not change. Let us just state that in such a case one has to be careful when interpreting geometrically the components.

The <u>contravariant components</u> are then defined as segments on the coordinate lines as seen on the diagram:



$$a^2 = \text{const}_4 - \text{const}_3$$

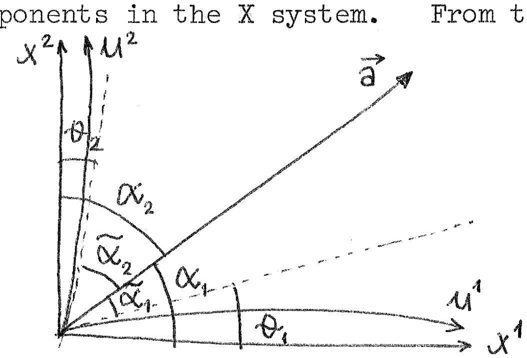
$$a^1 = \text{const}_2 - \text{const}_1.$$

We can see now the reason why we have denoted the coordinates u with a superscript (upper index rather than lower). We find that the u coordinate differences play the role of contravariant components. We shall come to this point once more later on.

It is not difficult to see that the introduced definition of covariant and contravariant components conform with the requirement of invariability of vector equations. However, the forthcoming development is going to prove it more rigorously.

2.3) <u>Transformation of Vector Components</u>

Let us now have a look at the mechanism of computing the covariant components of a vector $\vec{a}$ in the U system when we know its covariant components in the X system. From the diagram, we can write immediately



for $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ (using again for simplicity the 2D-space):

$$a \cos \tilde{\alpha}_1 = a \cos (\alpha_1 - \theta_1) =$$

$$= a(\cos \alpha_1 \cos \theta_1 + \sin \alpha_1 \sin \theta_1) ,$$

$$a \cos \tilde{\alpha}_2 = a(\cos \alpha_2 \cos \theta_2 + \sin \alpha_2 \sin \theta_2).$$

These relations can be rewritten, realizing that

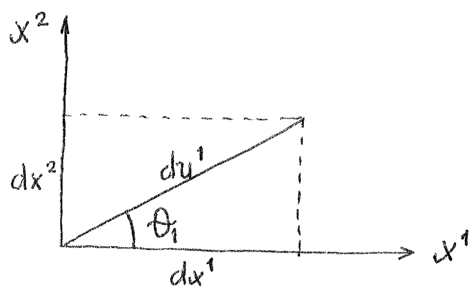
$$\alpha_1 = \frac{\pi}{2} - \alpha_2, \quad \alpha_2 = \frac{\pi}{2} - \alpha_1$$

as

$$a \cos \tilde{\alpha}_1 = a(\cos \theta_1 \cos \alpha_1 + \sin \theta_1 \cos \alpha_2)$$

$$a \cos \tilde{\alpha}_2 = a(\cos \theta_2 \cos \alpha_2 + \sin \theta_2 \cos \alpha_1).$$

On the other hand, we can write for $\cos \theta_1$, $\sin \theta_1$ from the diagram:



$$\cos \theta_1 = \frac{dx^1}{du^1} , \quad \sin \theta_1 = \frac{dx^2}{du^1} .$$

Similarly for θ_2:

$$\cos \theta_2 = \frac{dx^2}{du^2} , \quad \sin \theta_2 = \frac{dx^1}{du^2} .$$

Substituting these results back into the original equation for new covariant components we get

$$a \cos \tilde{\alpha}_i = \frac{\partial x^1}{\partial u^i} \, a \cos \alpha_1 + \frac{\partial x^2}{\partial u^i} \, a \cos \alpha_2 \qquad i = 1, 2$$

or denoting by $\tilde{a}_i$ the covariant components in the U system and using the summation convention

$$\boxed{\tilde{a}_\alpha = \frac{\partial x^\beta}{\partial u^\alpha} a_\beta} \qquad \alpha = 1, 2.$$

Here the ordinary derivatives were replaced by partial derivatives because x^i is generally an explicit function of all the u's.

This result is of a primary importance in the theory of vectors in curvilinear coordinates and bears the name the <u>transformation law for covariant components</u>. It can be derived in a more straightforward manner for the 3D-space (or space of any dimension) when we realize $(\frac{da}{dx^i} = \cos\phi_i$, thus $a_i = a\frac{da}{dx^i} = a$; $\vec{a} = a\vec{u} = a\nabla\vec{a}$ where $\vec{u} \parallel \vec{a}$, $u - 1)$ that the covariant components are given by

$$a_i = a\frac{da}{dx^i}\ , \qquad \tilde{a}_i = a\frac{da}{du^i} \qquad i = 1, 2, 3$$

in any coordinate system. Applying the rule for total differentiation we get immediately

$$\frac{\tilde{a}_j}{a} = \frac{da}{du^j} = \sum_{i=1}^{3} \frac{da}{dx^i}\frac{\partial x^i}{\partial u^j} = \frac{\partial x^i}{\partial u^j}\frac{da}{dx^i} = \frac{\partial x^i}{\partial u^j} = \frac{a_i}{a} \qquad j = 1, 2, 3$$

or $$\tilde{a}_j = \frac{\partial x^i}{\partial u^j} a_i \qquad j=1, 2, 3.$$

The reader is urged to prove to himself that the formala

$$a_j = \frac{\partial u^i}{\partial x^j}\tilde{a}_i \qquad j = 1, 2, 3$$

holds for the inverse transformation.

Since in the second derivation nowhere have we used the special properties of the X system (rectangular Cartesian) it can be seen that the transformation law holds for the transformation between any two U systems (non-Cartesian) as well. This is written as

$$\boxed{\begin{aligned} \tilde{a}_j &= \frac{\partial u^i}{\partial \tilde{u}^j} a_i \\ a_j &= \frac{\partial \tilde{u}^i}{\partial u^j}\tilde{a}_i \end{aligned}} \qquad j = 1, 2, 3.$$

The transformation law for contravariant components reads

$$\boxed{\begin{aligned} \tilde{a}^j &= \frac{\partial \tilde{u}^j}{\partial u^i} a^i \\ a^j &= \frac{\partial u^j}{\partial \tilde{u}^i} \tilde{a}^i \end{aligned}} \qquad j = 1, 2, 3$$

and its derivation is more involved. We should expect that this is the law the coordinate differentials have to obey. Comparison with the earlier developed formulae will assure the reader that it is the case. This is the basic reason why we have used the upper index for the coordinates.

As an example, let us take a pair of tangent vectors ${}^{\theta}\vec{t}$, ${}^{\lambda}\vec{t}$ to the θ and λ parametric curves on a sphere of radius R in Cartesian coordinates. Expressing the sphere as

$$\vec{r}(\theta, \lambda) \equiv \begin{cases} x = R \sin\theta \cos\lambda \\ y = R \sin\theta \sin\lambda \\ z = R \cos\theta \end{cases} \qquad (r = R)$$

we obtain:

$${}^{\theta}\vec{t} = \frac{\partial \vec{r}}{\partial \theta} = \begin{cases} x = R \cos\theta \cos\lambda \\ y = R \cos\theta \sin\lambda \\ z = - R \sin\theta \end{cases} ,$$

$${}^{\lambda}\vec{t} = \frac{\partial \vec{r}}{\partial \lambda} = \begin{cases} x = - R \sin\theta \sin\lambda \\ y = R \sin\theta \cos\lambda \\ z = 0. \end{cases}$$

The elements of the Jacobian of transformation between the Cartesian and the spherical coordinates are given by:

$$\frac{\partial x}{\partial r} = \sin\theta\cos\lambda, \quad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\lambda, \quad \frac{\partial x}{\partial \lambda} = -r\sin\theta\sin\lambda$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\lambda, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\lambda \quad \frac{\partial y}{\partial \lambda} = r\sin\theta\cos\lambda$$

$$\frac{\partial z}{\partial r} = \cos\theta\ , \quad \frac{\partial z}{\partial \theta} = -r\sin\theta\ , \quad \frac{\partial z}{\partial \lambda} = 0\ .$$

We can now evaluate the covariant components of ${}^{\theta}\vec{t}$, ${}^{\lambda}\vec{t}$ in the spherical coordinates using the transformation law for covariant components:

$$ {}^{\theta}t_1 = 0, \quad {}^{\theta}t_2 = Rr\big|_{r=R} = R^2, \quad {}^{\theta}t_3 = 0\ ,$$

$$ {}^{\lambda}t_1 = 0, \quad {}^{\lambda}t_2 = 0, \quad {}^{\lambda}t_3 = Rr\sin^2\theta\big|_{r=R} = R^2\sin^2\theta\ .$$

The reader is suggested to show that the contravariant components of the same two vectors are

$$ {}^{\theta}t^1 = 0, \quad {}^{\theta}t^2 = 1, \quad {}^{\theta}t^3 = 0, \quad {}^{\lambda}t^1 = 0, \quad {}^{\lambda}t^2 = 0, \quad {}^{\lambda}t^3 = 1.$$

Finally, we note that for a transformation between two Cartesian systems, X and $\tilde{X}$ say, all the partial derivatives

$$\frac{\partial x^i}{\partial \tilde{x}^j} = \frac{\partial \tilde{x}^j}{\partial x^i} \qquad i, j = 1, 2, 3$$

are constant and we end up with the same expression as in 1.2.8.

The transformation laws as well as the whole idea of expressing vectors in curvilinear coordinates may seem to be too complicated when we can work with rectangular Cartesian coordinates alone and use the relatively simple formulae developed in Chapter 1. This is, unfortunately, not the case generally. There are spaces, where we just cannot define Cartesian coordinates and where we have to work with curvilinear coordinates whether we like it or not. The simplest example of such spaces are surfaces that are not developable into a plane, sphere or ellipsoid being the two most commonly used ones. This matter will be dealt with more fully later.

Problem: Give the covariant and contravariant components of unit tangent vectors to ϕ and λ curves on the ellipsoid of rotation in geodetic coordinates.

3) TENSORS

3.1) Definition of a Tensor

As we have said in the last paragraph, there are spaces in which we cannot define Cartesian coordinate systems. In these spaces (but not only these) it is usually difficult to even recognise if a quantity is a vector or not. The transformation laws allow us to determine it, providing we know the quantity we deal with in two different coordinate system. We can now redefine the vector as a triplet of functions that transforms according to one of the transformation laws. Accordingly, we call the three functions either covariant or contravariant components of the vector.

This approach allows also a further generalization of vectors. We can now introduce a more general structure than the vector - the tensor. We call a structure a_{ij} of 3^2 elements a two-times covariant tensor (in 3D-space with U coordinate system) if and only if it is related to a similar structure $\tilde{a}_{ij}$ of 3^2 elements in 3D-space with $\tilde{U}$ coordinate system by following transformation equation

$$\boxed{\tilde{a}_{ij} = \frac{\partial u^k}{\partial \tilde{u}^i} \frac{\partial u^\ell}{\partial \tilde{u}^j} a_{k\ell} .}$$

Here, we, of course, use the summation convention so that the formula in the ordinary notation reads

$$\tilde{a}_{ij} = \sum_{k=1}^{3} \sum_{\ell=1}^{3} \frac{\partial u^k}{\partial \tilde{u}^i} \frac{\partial u^\ell}{\partial \tilde{u}^j} a_{k\ell} .$$

Similarly we can define, say a three-times contravariant tensor in 2D-space as obeying the following formula:

$$\boxed{\tilde{b}^{\alpha\beta\gamma} = \frac{\partial\tilde{u}^{\alpha}}{\partial u^{\delta}} \frac{\partial\tilde{u}^{\beta}}{\partial u^{\varepsilon}} \frac{\partial\tilde{u}^{\gamma}}{\partial u^{\phi}} b^{\delta\varepsilon\phi}}$$

meaning

$$\tilde{b}^{\alpha\beta\gamma} = \sum_{\delta=1}^{2} \sum_{\varepsilon=1}^{2} \sum_{\phi=1}^{2} \frac{\partial\tilde{u}^{\alpha}}{\partial u^{\delta}} \frac{\partial\tilde{u}^{\beta}}{\partial u^{\varepsilon}} \frac{\partial\tilde{u}^{\gamma}}{\partial u^{\phi}} b^{\delta\varepsilon\phi} .$$

We can also have mixed tensors with some covariant and some contravariant indeces. Hence the tensor c^{ij}_{ℓ} ought to obey the following transformation law

$$\tilde{c}^{ij}_{\ell} = \frac{\partial\tilde{u}^{i}}{\partial u^{k}} \frac{\partial\tilde{u}^{j}}{\partial u^{m}} \frac{\partial u^{n}}{\partial\tilde{u}^{\ell}} c^{km}_{n}$$

and is called once covariant and twice contravariant three-dimensional tensor. A tensor need not even have the same dimensions in all indeces. It is left to the reader to write the transformation laws for following tensors as an exercise:

$$a_{ijk},\ b^{j}_{\alpha\beta i},\ c^{\alpha i}_{j\beta}\ .$$

It seems worth mentioning here that we could have easily used the matrix notation for everything we have done so far. From now on, however, the matrix notation would not be able to express all the quantities we shall work with.

Problems: Write in full all the elements of the following tensors: $a^i a^j$ (dyade), $a_i b_i$, $a^i a_i$.

The number of indeces (excluding the dummy indeces) of a tensor is called the rank of the tensor. Thus vectors are regarded as first rank tensors, scalars as zero-rank tensors or invariants.

Note that the number of components of a tensor is given by

$$n^r$$

where n is the dimension of the space and r is the rank. If the tensor is defined in spaces of different dimension, say n_1, n_2, n_3 then the number of components is

$$n_1^{r_1} \cdot n_2^{r_2} \cdot n_3^{r_3}$$

where r_1, r_2, r_3 are the ranks in n_1, n_2, n_3 dimensional indeces. Obviously if there is a scalar ϕ defined at point $\vec{r}$, after changing the coordinate system the same scalar will remain attached to the same point (although the coordinates of the point will generally change). Hence we have the transformation equation for scalar ϕ:

$$\bar{\phi} = \phi$$

which is consistent with the transformation laws for tensor quantities.

3.2) Tensor Field, Tensor Equations

If in a region of a space a tensor is defined for each point of the region, we say that there is a tensor field defined in the region. This term evidently encompasses both special cases we have dealt with in rectangular Cartesian coordinates - vector and scalar fields - and is a direct generalization of both.

By having chosen the definition of covariant and contravariant components of a vector the way we did, we have ensured that equations involving thus defined vectors (either their covariant or contravariant components) remain insensitive (invariant) to any change of coordinate system. The same holds true for tensors of any rank. The tensors of higher ranks can be thus used to describe more complicated properties of objects in the space like measures of distortion, curvatures of surfaces etc.

3.3) Tensor Algebra

The forthcoming 9 paragraphs are common for two as well as three-dimensional tensors. For simplicity we are going to use only Latin letters for indeces and the reader can "translate" everything for himself into two dimensions by "transliteration" into Greek letters.

3.3.1) Zero Tensor

Zero tensor is a tensor whose elements are all equal to zero. It is trivial to show that the structure, say $\tilde{A}_{ij} = A_{ij} = 0$, transforms as a tensor and is therefore a tensor.

3.3.2) Kronecker δ

The structure denoted by δ_i^j and defined as

$$\delta_i^j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

in any coordinate system is also a (mixed) tensor. It is left to the reader to show that it transforms as a mixed tensor of second rank.

3.3.3) Summation of Tensors

The sum of two tensors that have the same number of covariant and contravariant indeces and the same dimension in all indeces is again a tensor of the same number of covariant and contravariant indeces:

$$C^{ij}_{k} = A^{ij}_{k} + B^{ij}_{k} \quad \text{(summation convention not applied)}$$

whose elements are all sums of the corresponding elements of the two summed tensors. The proof that the sum of two tensors is again a tensor is left to the reader

The summation is commutative and associative, i.e.

$$A^{j}_{i} + B^{j}_{i} = B^{j}_{i} + A^{j}_{i} \quad \text{(summation convention not applied)}$$

$$A_{\ell k} + (B_{\ell k} + C_{\ell k}) = (A_{\ell k} + B_{\ell k}) + C_{\ell k} \quad \text{(summation convention not applied)}$$

3.3.4) <u>Multiplication of a Tensor by a Constant</u>

The product of a constant scalar and a tensor is again a tensor of the same rank whose elements are equal to the corresponding elements (of the multiplied tensor) multiplied by the constant

$$b_{ij} = \phi A_{ij} \; .$$

The multiplication is associative and commutative. The proof that the product is a tensor is trivial.

3.3.5) <u>Opposite Tensor</u>

Opposite tensor B^{ij} to a tensor, say A^{ij}, is again a tensor of the same rank whose elements are equal to the negatively taken corresponding elements of A^{ij}. It is denoted by $-A^{ij}$ and we have

$$A^{ij} + B^{ij} = 0 \quad \text{or} \quad B^{ij} = (-1)\, A^{ij}.$$

Its tensorial properties follow immediately from 3.3.4 if we take $\phi = -1$.

3.3.6) <u>Multiplication of Tensors</u>

The product of two tensors, say A^{ij} and B^{ℓ}_{k} in this order, is a tensor of a rank that is a sum of the ranks of the two constituent tensors, in our case $C^{ij\ell}_{k}$. Its components are products of the corresponding components of the constituent tensors, for instance

$$C^{132}_{3} = A^{13} \cdot B^{2}_{3} \quad .$$

We can see that tensor product is generally not commutative, i.e.

$$C^{132}_{3} = A^{13} B^{2}_{3} \neq B^{1}_{3} A^{32}$$

$$A^{ij}B^{\ell}_{k} \neq B^{i}_{k} A^{j\ell}$$

$$(\text{but } A^{ij}B^{\ell}_{k} = B^{\ell}_{k}A^{ij}) \quad .$$

On the other hand it is always associative.

The reader can prove for himself that the product of two tensors is again a tensor by investigating its transformation from one coordinate system into an other. Note that multiplication of a tensor by a constant is just a special case of multiplication of tensors.

If one of the two tensors that are to be multiplied has got one or more covariant (contravariant) indeces identical to one or more contravariant (covariant) indeces of the second tensor then the resulting tensor will have a rank smaller by 2 or generally by a larger even integer. This is because the identical indeces become dummy indeces and we have, for example

$$A_{ij} B^{j} = C_{i}, \; A^{ik}_{j\ell} B^{\ell}_{i} = C^{k}_{j} \; .$$

Such a product bears a special name of <u>inner product</u> in one or more indeces.

Note a particular inner multiplication by δ^j_i,

$$\delta^j_i \, A^{ik}_{\ell m} = A^{jk}_{\ell m} \; ,$$

which is sometimes called the <u>change of an index</u>.

3.3.7) <u>Contraction</u>

Contraction is an operation by which we reduce the rank of the tensor by two. If we set one covariant and one contravariant indeces to be equal to each other, this automatically indicates that the indeces become dummy indeces and the summation takes place. Applying this operation to, for instance a tensor $A^{k\ell}_{ij}$ we can create four , generally different, tensors of rank two:

$$A^{i\ell}_{ij} = B^{\ell}_{j}, \quad A^{j\ell}_{ij} = C^{\ell}_{i}, \; A^{kj}_{ij} = D^{k}_{i}, \; A^{ji}_{ik} = E^{\;j}_{\;k} \; .$$

This operation is difficult to visualise unless we contract a tensor of rank two, i.e. reduce it to a scalar. In this case we have $\lambda = A^i_i$ and taking the matrix of A^j_i:

$$[A^j_i] \equiv \begin{bmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_3 & a^2_3 & a^3_3 \end{bmatrix}$$

we can see that $\lambda = \sum_{i=1}^{3} a^i_i$ = trace (A). Hence, for instance,

$$\delta^i_i = 3, \quad \delta^\alpha_\alpha = 2 \; .$$

We can show that the contracted tensor, say $A_{st}B^t$, is again a tensor by following calculations:

$$\tilde{C}_s = \tilde{A}_{st}\tilde{B}^t = \frac{\partial u^i}{\partial \tilde{u}^s} \frac{\partial u^j}{\partial \tilde{u}^t} \frac{\partial \tilde{u}^t}{\partial u^j} A_{ij}B^j$$

$$= \frac{\partial u^i}{\partial \tilde{u}^s} A_{ij}B^j$$

$$= \frac{\partial u^i}{\partial \tilde{u}^s} C_i \; .$$

Another way to view contraction is to say that it is equivalent to inner multiplication by Kronecker δ in two indeces. For example:

$$\delta^m_i \; A^{i\ell}_{mj} = A^{i\ell}_{ij} = A^{m\ell}_{mj} = B^\ell_j$$

Then the proof of the tensor character of B^ℓ_j is evident. This is the reason why some text-books do not list contraction as a special operation.

It should also be mentioned that, by the same virtue, an inner multiplication in one index can be regarded as a once contracted general tensor multiplication:

$$A^i_j \, B^\ell_{ik} = \delta^i_m \, (A^m_j \, B^\ell_{ik}) \; .$$

Therefore we shall be further speaking about a contracted product or inner product, meaning the same thing.

3.3.8) Tensor Character

We can notice that the three basic tensorial operations - summation, multiplication and contraction – produce always again a tensor. They are said to preserve the <u>tensor character</u>. This property serves as another means of distinguishing a tensor. For instance a two-index quantity A_{ij} can be tested for its tensor character by multiplying it by two <u>arbitrary</u> contravariant vectors (or a twice contravariant tensor). The result is a four index quantity, say $B_{ij}^{k\ell}$. If its double contraction results in a scalar then A_{ij} is a twice covariant tensor:

$$A_{ij}\, a^k\, b^\ell = B_{ij}^{k\ell},\; B_{ij}^{ij} = \begin{cases} \text{scalar} \Rightarrow A_{ij} \text{ is a tensor} \\ \text{something else} \Rightarrow A_{ij} \text{ is not a tensor.} \end{cases}$$

3.3.9) <u>Symmetric and Antisymmetric Tensor</u>

A tensor is called symmetric in two simultaneously either covariant or contravariant indeces if and only if its value does not change when we interchange the two indeces. For example

$$S_{ijk}^{\ell m} = S_{jik}^{\ell m}$$

is <u>symmetric</u> in the first two covariant indeces ;

$$S_i^{jk} = S_i^{kj}$$

is symmetric in its contravariant indeces.

A tensor is said to be <u>antisymmetric</u> (skew-symmetric) in two simultaneously covariant or contravariant indeces if and only if it changes sign when we interchange the two indeces. Hence $A_{ij}^{k\ell m} = -A_{ij}^{km\ell}$ is antisymmetric in the last two contravariant indeces.

Any tensor of second rank can be expressed as a sum of a symmetric and an antisymmetric tensors. To show this, let us take, for instance, a tensor B^{ij} and write

$$B^{ij} = \underbrace{1/2\ B^{ij} + 1/2\ B^{ij}}_{B^{ij}} + \underbrace{1/2\ B^{ji} - 1/2\ B^{ji}}_{0} \quad .$$

This equation can be rewritten as

$$B^{ij} = \underbrace{1/2\ (B^{ij} + B^{ji})}_{S^{ij}} + \underbrace{1/2\ (B^{ij} - B^{ji})}_{A^{ij}} \quad .$$

Here the first tensor is symmetric since it does not change when we interchange the indeces. The second changes the sign when we interchange the indeces and is therefore antisymmetric.

3.3.10) Line Element and Metric Tensor

We have seen in 1.1 that in 3D-rectangular Cartesian coordinates (or Eucleidean space), the square of a distance ΔS between any two points $\vec{r}_1$, $\vec{r}_2$ was given by

$$(\Delta S)^2 = \sum_{i=1}^{3} (\Delta X^i)^2 \ .$$

If the two points are infintesimally close, we get similarly

$$(ds)^2 = \sum_{i=1}^{3} (dx^i)^2 = dx^i\ dx_i$$

where ds is called the line element.

Let us ask now, what will be the formula for the line element in a curvilinear U system. As we have shown already (2.2), the coordinate differentials transform as contravariant vectors. Hence the differentials du^i will be given as

$$du^i = \frac{\partial u^i}{\partial x^j}\, dx^j$$

and

$$dx^i = \frac{\partial x^i}{\partial u^j} du^j \quad .$$

Substituting this result into the formula for the line element and realizing that the line element is invariant in the transformation X → U we have

$$(ds)^2 = \frac{\partial x^i}{\partial u^p} \frac{\partial x^i}{\partial u^s} \; du^p \; du^s \quad .$$

Since in this formula the summation convention applies (i,p,s are dummy indeces) the quantity

$$\boxed{\frac{\partial x^i}{\partial u^p} \frac{\partial x^i}{\partial u^s} = g_{ps}}$$

is a two-index field. Moreover, since the twice contracted product of g_{ps} with two contravariant vectors du^i and du^j is a scalar, the quantity g_{ps} has to be a twice covariant tensor. It is, perhaps, the most important tensor and is called <u>metric</u> or <u>fundamental tensor</u>. The one element, can be written using the metric tensor as

$$\boxed{(ds)^2 = g_{ij} \; du^i \; du^j}$$

in any coordinate system. Note that the metric tensor is symmetrical since $g_{ij} = g_{ji}$, or more explicitly:

$$\frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} = \frac{\partial x^k}{\partial u^j} \frac{\partial x^k}{\partial j^i} \quad .$$

Obviously, if we have the equations $x^i = x^i\,(u^1, u^2, u^3)$, i = 1,2,3, relating the curvilinear system U to the rectangular Cartesian system X, we can

derive the metric tensor of the U system from the above formulae.
It can be shown that if the U system is again rectangular Cartesian with the same scale along all the axes, we end up with

$$g_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j; \end{cases}$$

a tensor equal to the Kronecker δ and denoted therefore by δ_i^j. This leads to the equation for the line element

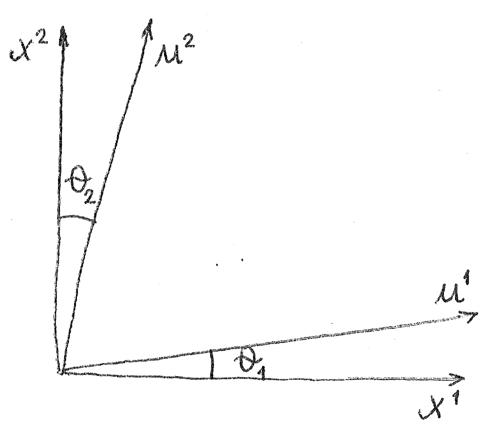
$$(ds)^2 = \delta_i^j \, du^i \, du^j = du^i \, du_i$$

the same as the one we began with; this was to be indeed expected.

On the other hand, if the U system is skew Cartesian, its metric tensor is not unit any more. The reader is advised to prove for himself that a 2D-skew Cartesian system has a metric tensor

$$[g_{\alpha\beta}] = \begin{bmatrix} 1, \sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2), 1 \end{bmatrix}$$

where θ_1, θ_2 are given according to the figure.



This holds true providing that the scale along all four axes is the same. If the scale along u^1, u^2 axes, is different from the scale along x^1, x^2 axes, the metric tensor above will be different.

As another example we may take the spherical coordinates r, θ, λ, for which we have

$$x^1 = r \sin\theta \cos\lambda$$
$$x^2 = r \sin\theta \sin\lambda$$
$$x^3 = r \cos\theta \ .$$

The reader may prove for himself that the matrix of the metric tensor in these coordinates is given by following formula

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix} .$$

Problem: Derive the metric tensor for geodetic coordinates ϕ, λ, h (geodetic latitude, longitude and height above a fixed ellipsoid of rotation given by its two axes a,b and centered upon the coordinate origin) for which

$$x^1 = (N + h) \cos\phi \cos\lambda$$
$$x^2 = (N + h) \cos\phi \sin\lambda$$
$$x^3 = (N(\frac{b}{a})^2 + h) \sin\phi \ .$$

Here

$$N = a\,(\cos^2\phi + (\frac{b}{a})^2 \sin^2\phi)^{-1/2} .$$

In 2D-space the development is completely analogous. There, the formula for the line element reads

$$\boxed{(ds)^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta}$$

and everything we have said about the 3D-metric tensor is valid for the 2D-metric tensor as well. It is left to the reader to prove that for example a sphere of radius r has got a metric tensor whose matrix equals to

$$[g_{\alpha\beta}] \equiv \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\phi \end{bmatrix} .$$

This expression can be arrived at by the following reasoning:

$$(ds)^2 = dx^i \, dx^i = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta} du^\alpha \, du^\beta = g_{\alpha\beta} \, du^\alpha \, du^\beta$$

where $u^1 = \phi$, $u^2 = \lambda$.

Problem: Derive the metric tensor for the geodetic coordinates ϕ, λ defined on the surface of a fixed ellipsoid of rotation co-centric with the Cartesian system and defined by a and b.

3.3.11) Terminological Remarks

The way we have defined a coordinate system ensures that there is always a metric tensor associated with any system of coordinates. This allows us to talk always about the metric space realized by the coordinate system. Hence we can use the two terms – coordinate system and metric space – interchangeably, which often is the case in literature on metric geometry. Note that we can talk about the metric tensor without having specified the system of coordinates, i.e., without having specified the relation between U and X systems. We can hence have just a set of 6 (or 3) independent elements of the metric tensor without knowing anything about the coordinate system it belongs to.

The metric space corresponding to the rectangular Cartesian coordinates is known as the Eucleidean space, as mentioned already in 1.1. The Eucleidean space is then characterized by the unit metric tensor.

(The tensors in Eucleidean spaces are sometimes called <u>affinors</u>). The metric space corresponding to skew Cartesian coordinates (with constant scale along each axis, not necessarily the same for all axes), is usually called the affine or pseudo-Eucleidean space. It is characterized by a metric tensor constant throughout the space. Both the Eucleidean and affine spaces are called <u>flat</u>.

If the metric tensor changes from point to point, the metric space corresponding to the used system of coordinates is called <u>Riemannian</u> (in a wider sense) or <u>curved</u>. Hence, for instance, a coordinate system with three perpendicular axes and scales varying along these axes does no longer represent a flat space. Spaces, for which the metric tensor is diagonal for every point, are called <u>locally orthogonal</u>. Most of the coordinate systems dealt with in practice (spherical, cylindrical, geodetic, geographical, etc.) are locally orthogonal. All the spaces, i.e. Eucleidean, affine, locally orthogonal, are often regarded as special cases of the Riemannian space.

Theoretically, we indeed can choose the metric in any (non-metric) space any way we want. When dealing, for instance, with a sphere, there is nothing to stop us from defining the metric as, say, Eucleidean and write

$$(ds)^2 = (d\phi)^2 + (d\lambda)^2 .$$

The only problem is that in such a case we cannot relate it properly to physical (geometric) reality. For the example mentioned above it would not be possible to immerse the sphere metricised in the described way into a three-dimensional Eucleidean space without distorting its Eucleidean metric. The distance of two points measured on the surface of the sphere (by our chosen Eucleidean metric) would not generally agree with the distance of the same two points measured by the Eucleidean distance in the 3D-space. Therefore, we have to require that the metric of a space of lower dimension immersed in a space of higher dimension be compatible with the metric of the space of higher dimension called sometimes the meta-space or super-space.

Spaces that can be metricised with the Eucleidean metric (i.e., whose any metric tensor can be transformed to the unit tensor by a transformation of coordinates) and yet remain compatible with the meta-space are called flat with respect to the meta- space or inherently flat. If this cannot be done then the space is called curved with respect to the meta-space or inherently curved. If we consider the common 3D-Eucleidean space the meta-space for the surfaces we deal with (2D-spaces) then all the developable surfaces are inherently flat, all the other surfaces are inherently curved. For instance the sphere cannot be metricised with Eucleidean metric in the 3D-Eucleidean space and is therefore curved with respect to this particular space. The terminology in literature is not completely unified in this respect.

3.3.12) Associated Metric Tensor, Lowering and Raising of Indeces.

Let us take now the metric tensor g_{ij}, multiply it by a contravariant vector A^k and contract the result (since g_{ij} is symmetrical, it does not matter whether we contract the first or second index). As we have seen already, this operation is sometimes called inner multiplication and in our case we get:

$$g_{ij} A^j = B_i .$$

Since this is a tensor equation it has to be valid in any coordinate system, therefore even in rectangular Cartesian coordinates. But we have seen in 3.3.10 that for rectangular Cartesian coordinates $g_{ij} = \delta^i_j$ and we get

$$g_{ij}\, A^j = \delta^i_j\, A^j = A^i = A_i = B_i \;.$$

(Note that we can lower and raise indeces here freely only because we work in rectangular Cartesian coordinates). Thus the metric tensor provides us with the tool for determining the covariant components of a vector if we know its contravariant components:

$$\boxed{A_i = g_{ij}\, A^j}.$$

This is the reason why we call the inner multiplication of a vector by metric tensor <u>lowering of one contravariant index</u>. This operation can be applied to any, (at least once contravariant) tensor of any rank. For example

$$g_{ij}\, A^{jk}_{\ell} = A^{\cdot k}_{i\ell}, \quad g_{ij}\, B^{k\ell j} = B^{k\ell\cdot}_{i}$$

where by the dot we indicate the initial position of the lowered index.

Let us now have a look again at the first equation of this paragraph from purely algebraic point of view. It can be evidently written as

$$\sum_{j=1}^{3} g_{ij}\, A^j = A_i$$

and regarded as a system of three algebraic equations for A^j, $j = 1,2,3$. It therefore makes sense to invert the system and write

$$\sum_{i=1}^{3} G_{ij}\, A_i = A^j \qquad (*)$$

since det (g_{ij}) for any Riemannian space is always positive.

To show that det $(g_{ij}) = g$ is always positive we can write:

$$g_{ij} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} = \sum_{k=1}^{3}\frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} \quad .$$

Hence the matrix of g_{ij} can be regarded as a matrix product of the two transformation matrices (Jacobians) $[\partial x^k/\partial u^i]$ and $[\partial x^k/\partial u^j]^T$. We know that according to Laplace's theorem, the determinant of a product of two matrices equals to the product of the determinants of the two matrices. We have hence

$$g = \det\,(g_{ij}) = \det\,(\frac{\partial x^k}{\partial u^i})\,\det\,(\frac{\partial x^k}{\partial u^j}) = \det^2\,(\frac{\partial x^k}{\partial u^i})$$

and since $\det\,(\frac{\partial x^k}{\partial u^i}) \neq 0$ (see 2.2) we get $g > 0$. The proof is even simpler for a locally orthogonal system for which we can write:

$$\det\,(g_{ij}) = \prod_{i=1}^{3} g_{ii} = \prod_{i=1}^{3}\left(\frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^i}\right)$$

$$= \prod_{i=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial x^k}{\partial u^i}\right)^2 .$$

Interpreting the equation (*) again in tensor notation we get

$$\boxed{g^{ij} A_i = A^j}$$

where g^{ij} has to be a twice contravariant tensor because when multiplied and contracted with a covariant vector it gives a contravariant vector. It is called the <u>associated metric</u> tensor. The operation described by the above formula is known as <u>raising of the covariant index</u>. It can be again applied to any (at least once covariant) tensor of any rank. For example:

$$g^{ij} A^{\ell}_{jk} = A^{i\ell}_{.k}, \quad g^{ij} B_{k\ell j} = B^{i}_{k\ell} .$$

An associated tensor to any second rank tensor whose determinant is different from zero can be defined similarly.

Problem: Derive the contravariant vectors ${}^{\theta}t^{i}$, ${}^{\lambda}t^{i}$ from the example in section 2.3 from their covariant forms by means of the associated metric tensor.

We also find that a once contracted product of the metric and associated metric tensors equals to the Kronecker δ. To show this let us write

$$g^{ij} A_{i} = A^{j}.$$

Multiplying the equation by the metric tensor and contracting in j indeces yields

$$g_{kj} g^{ij} A_{i} = g_{kj} A^{j} .$$

But the right hand side equals to A_k, therefore the left hand side must equal to A_k too. Hence

$$\boxed{g_{kj} g^{ij} = \delta^{i}_{k} .}$$

Note that even the operations "change of index" and "contraction" can be expressed in terms of the metric and associated metric tensors. We have, for example,

$$\boxed{g_{ij} g^{jk} A_{k} = A_{i}} \text{ (change of index)},$$

$$\boxed{g_{ik} g^{jk} A^{i}_{j} = A^{i}_{i}} \text{ (contraction)}.$$

Hence, we can say that all four special tensor operations (lowering, raising and change of an index as well as contraction) are particular cases of inner multiplications by the metric and/or associated metric tensors.

We also note that

$$\boxed{g_{ik}\, g^{ik} = 3, \quad g_{\alpha\beta}\, g^{\alpha\beta} = 2.}$$

3.3.13 Scalar Product of Two Vectors, Applications

As in skew Cartesian coordinates, we again call the inner product of two vectors the scalar product. The inner product of a vector with itself equals to the square of its length here as well as before:

$$(A)^2 = A_i\, A^i = A^i\, A_i \ .$$

This can be easily seen when we realize that the above is a vector equation and therefore is invariant in any coordinate transformation. Since we know that A is the length of the vector in Cartesian coordinates it has to be the same in any coordinate system.

Problem: What are the lengths of $\overset{\theta}{\vec{t}}$, $\overset{\lambda}{\vec{t}}$ used in the example in section 2.3?

Analogously, we call A the length of the vector A even in an inherently curved space. Note that it is difficult to visualize a vector in a curved space since it no longer can be interpreted as an oriented segment of a straight line. The same holds true for the components. In a curved space one cannot use the "common sense" and has to go strictly by definitions. In our development we shall be working in a flat space. The results, however, will lend themselves to straightforward generalization in a curved space with the only difference that their geometrical interpretation will be rather difficult.

Making use of the metric and associated metric tensors, we can write for the length of a vector:

$$\boxed{A = (g_{ij}\ A^i\ A^j)^{1/2} = (g^{ij}\ A_i\ A_j)^{1/2}\ .}$$

The analogy with the fundamental form is evident. Obviously, in a 2D-space the length of a vector $\vec{A}$ will be defined by

$$\boxed{A = (g^{\alpha\beta}\ A_\alpha\ A_\beta)^{1/2} = (g_{\alpha\beta}\ A^\alpha\ A^\beta)^{1/2}\ .}$$

Let us just note that the fundamental form (see 3.3.10) can also be written as

$$(ds)^2 = dx^i\ dx_i\ .$$

Dividing this equation by $(ds)^2$ we get

$$\frac{dx^i}{ds}\ \frac{dx_i}{ds} = 1\ .$$

Hence we can conclude that $\frac{dx^i}{ds}$ are contravariant components of a unit vector and $\frac{dx_i}{ds}$ are covariant components of the same.

We have seen in 2.1 that the following vector equation

$$AB \cos \omega = A_i\ B^i = A^i\ B_i$$

holds true for any two vectors $\vec{A}$, $\vec{B}$. It has to hold true even in a curvilinear coordinate system since both lengths and the angle ω are invariant. Rewriting the above equation using the metric and associated metric tensors we get:

$$\boxed{AB \cos \omega = g_{ij} A^i B^j = g^{ij} A_i B_j \ .}$$

Note that this gives us the means of distinguishing two perpendicular vectors in both flat and curved spaces. In 2D-space analogous formulae hold:

$$\boxed{AB \cos \omega = g_{\alpha\beta} A^\alpha B^\beta = g^{\alpha\beta} A_\alpha B_\beta \ .}$$

Problem: Show by direct computation that the angle between two tangent vectors to ϕ and λ curves on an ellipsoid equals to $\pi/2$.

3.3.14) Levi-Civitã Tensor

Let us define now the following tensor in 3D-space:

$$\boxed{e_{rst} = \sqrt{g}\ \delta_{rst}}$$

where

$$\delta_{rst} = \begin{cases} +1 & \text{for all even permutations of indeces} \\ -1 & \text{for odd permutations of indeces} \\ 0 & \text{for all other combinations of indeces} \end{cases}$$

and

$$g = \det\ (g_{ij}).$$

Hence

$$\delta_{123} = \delta_{231} = \delta_{312} = 1,$$

$$\delta_{321} = \delta_{132} = \delta_{213} = -1$$

$\delta_{111} = \delta_{112} = \ldots = \delta_{333} = 0$. We can show that e_{rst} is a three-times covariant tensor.

To show it, it suffices to consider e_{rst} first in rectangular Cartesian coordinates:

$$e_{rst} = \delta_{rst}$$

(in rectangular Cartesian coordinates we assume for simplicity $\det(g_{ij}) = 1$). Taking $\tilde{e}_{rst}$ in another arbitrary coordinate system U, it has to satisfy the following transformation law:

$$\tilde{e}_{rst} = \frac{\partial x^i}{\partial u^r} \frac{\partial x^j}{\partial u^s} \frac{\partial x^k}{\partial u^t} \delta_{ijk} \,. \qquad (*)$$

To prove that the above equation is satisfied we have to prove first that

$$A^r_i A^s_j A^t_k \delta_{rst} = \det(A)\, \delta_{ijk}$$

where det (A) is the determinant of the matrix of the arbitrarily chosen mixed tensor A^j_i. This can be seen directly for specific values of i,j,k. We have, for instance

$$A^r_1 A^s_2 A^t_3 \delta_{rst} = \det(A)$$

$$A^r_3 A^s_2 A^t_1 \delta_{rst} = -\det(A) \,,$$

and similarly for other permutations of i,j, k.
The proof that all the expressions

$$A^r_i A^s_j A^t_k \delta_{rst} \,,$$

where the lower indeces i,j,k are other combinations of 1,2, and 3, equal to zero is left to the reader.

Having proved this, we proceed to state that

$$\det\left(\frac{\partial x^i}{\partial u^j}\right) = \sqrt{\det(g_{ij})} = \sqrt{g} \,.$$

This has been shown in section 3.3.12 already. Applying both our findings to equation (*) we obtain

$$\tilde{e}_{rst} = \sqrt{g}\, \delta_{rst}$$

which concludes the proof.

The e_{rst} tensor is called the Levi-Civitá covariant tensor.
Analogously we can define

$$\boxed{e^{rst} = \frac{1}{\sqrt{g}}\,\delta_{rst}\ .}$$

The proof that this is a three-times contravariant tensor is left to the reader. It is known as the Levi-Civitá contravariant tensor. The Levi-Civitá tensors are sometimes called e-systems. Note that Levi-Civitá tensors can be defined only for spaces with dimensions higher than 2.

3.3.15) Vector Product of Two Vectors.

The covariant vector

$$\boxed{C_i = e_{ijk}\, A^j\, B^k}$$

is known as the covariant vector product of the two (contravariant) vectors A^j and B^k. Similarly, the contravariant vector

$$\boxed{C^i = e^{ijk}\, A_j\, B_k}$$

is called the contravariant vector product of the two (covariant) vectors.

To show that this definition of vector product is equivalent to the usual definition in rectangular Cartesian coordinates, we can spell out the components of any of the two vectors C_i or C^i (in Cartesian coordinates). We obtain, for example,

$$C_1 = A^2B^3 - A^3B^2$$

$$C_2 = A^3B^1 - A^1B^3$$

$$C_3 = A^1B^2 - A^2B^1$$

that matches the formula for the components of the vector product in 1.2.6 sub ii.

The reader is recommended to prove the following

$$C_i A^i = C^i A_i = C_i B^i = C^i B_i = 0$$

where the A,B,C vectors are those from the above equations. Note that the vector product provides us with the tool for distinguishing two parallel vectors in flat as well as curved spaces.

Problem: Write the equation for a unit normal vector to the ellipsoid of rotation in geodetic coordinates system using vector product.

3.4) Tensor Analysis

3.4.1) Constant Vector Field

Before we start talking about differential operations with tensors let us see first how we can recognize constant vector field in curvilinear coordinate systems. To do so let us take a vector field $\vec{A}$ known to be constant in Cartesian coordinates $(\tilde{A})$. Transforming the curvilinear system U to Cartesian X we get

$$\tilde{A}^i = \frac{\partial x^i}{\partial u^j} A^j .$$

Further consider a curve $C = C(t)$ in the space and ask what would be the change in $\tilde{A}^i$ when we move on the curve by an infinitesimally small step dt? The answer is given by the following equation

$$\frac{d\tilde{A}^i}{dt} = \frac{\partial^2 x^i}{\partial u^j \partial u^k} \frac{du^k}{dt} A^j + \frac{\partial x^i}{\partial u^j} \frac{dA^j}{dt} .$$

Since $\tilde{A}^i$ is considered constant (in Cartesian coordinates), its derivative $d\tilde{A}^i/dt$ is identically equal to zero. Hence we obtain the following differential equations of a constant vector field expressed in curvilinear coordinates:

$$\boxed{\frac{\partial^2 x^i}{\partial u^j \partial u^k} \frac{du^k}{dt} A^j + \frac{\partial x^i}{\partial u^j} \frac{dA^j}{dt} = 0 .}$$

These are often called in literature the equations for parallel transmission of a vector, the term borrowed from application of tensors in mechanics.

3.4.2) Christoffel Symbols

Our next goal will be to express the partial derivative of the coordinates in the equation for parallel transmission in terms of the metric tensor belonging to the curvilinear system. To achieve this, let us multiply the equation by $g^{rp}\,\partial x^i/\partial u^p$. We get

$$g^{rp}\,\frac{\partial x^i}{\partial u^p}\,\frac{\partial^2 x^i}{\partial u^j \partial u^k}\,\frac{du^k}{dt}\,A^j + g^{rp}\,\frac{\partial x^i}{\partial u^p}\,\frac{\partial x^i}{\partial u^j}\,\frac{dA^j}{dt} = 0 \;.$$

Here

$$g^{rp}\,\frac{\partial x^i}{\partial u^p}\,\frac{\partial x^i}{\partial u^j} = g^{rp}\,g_{pj} = \delta^r_j$$

(see 3.3.12) and the second term becomes dA^r/dt. The product of the two partial derivatives in the first term is denoted $[jk,\,p]$:

$$[jk,\,p] = \frac{\partial x^i}{\partial u^p}\,\frac{\partial^2 x^i}{\partial u^j \partial u^k}$$

and called the <u>Christoffel symbol of 1-st kind</u>. It is a function of the metric tensor only and it can be shown that:

$$\boxed{[ij,\,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right).}$$

To show this let us take partial derivatives of the metric tensor g_{ij} with respect to all three coordinates. We get

$$\frac{\partial g_{ij}}{\partial u^k} = \frac{\partial x^\ell}{\partial u^i}\,\frac{\partial^2 x^\ell}{\partial u^j \partial u^k} + \frac{\partial^2 x^\ell}{\partial u^i \partial u^k}\,\frac{\partial x^\ell}{\partial u^j} \;.$$

Analogously:

$$\frac{\partial g_{jk}}{\partial u^i} = \frac{\partial x^\ell}{\partial u^j}\,\frac{\partial^2 x^\ell}{\partial u^k \partial u^i} + \frac{\partial^2 x^\ell}{\partial u^j \partial u^i}\,\frac{\partial x^\ell}{\partial u^k} \;,$$

$$\frac{\partial g_{ki}}{\partial u^j} = \frac{\partial x^\ell}{\partial u^k} \frac{\partial^2 x^\ell}{\partial u^i \partial u^j} + \frac{\partial^2 x^\ell}{\partial u^k \partial u^j} \frac{\partial x^\ell}{\partial u^i} .$$

Summing up the last two equations and subtracting the first from them we get:

$$\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = 2 \frac{\partial x^\ell}{\partial u^k} \frac{\partial^2 x^\ell}{\partial u^i \partial u^j}$$

which is nothing else but 2[ij, k]. This concludes the proof.

Using the Christoffel symbol, the equation for parallel transmission can be rewritten as follows:

$$\frac{dA^r}{dt} + g^{rp} [jk, p] \frac{du^k}{dt} A^j = 0.$$

We note that in Cartesian coordinates the Christoffel symbol of the first kind is always zero and the above equation degenerates to

$$\frac{dA^r}{dt} = 0 ;$$

that had to be expected. The Christoffel symbol is <u>not</u> a tensor - hence the name. It is, however, of a fundamental importance in tensor analysis. Note that it is symmetrical within the first two indeces.

Upon introducing another quantity:

$$\boxed{\Gamma^\ell_{ij} = g^{\ell k} [ij, k] ,}$$

known as the <u>Christoffel symbol of 2-nd kind</u>, the above equation can be further simplified to

$$\frac{dA^r}{dt} + \Gamma^r_{jk} \frac{du^k}{dt} A^j = 0 .$$

It can be shown again that the Christoffel symbol of 2-nd kind is not a tensor. On the other hand, it can be used as a multiplier by an appropriate tensor, producing a non-tensorial quantity. It again equals to zero in Cartesian coordinates and is symmetrical in the two covariant indeces. It is sometimes also denoted by $\left\{ {r \atop jk} \right\}$ or otherwise.

To conclude the paragraph, let us note that the first term in the equation for parallel transmission can be rewritten as

$$\frac{dA^r}{dt} = \frac{\partial A^r}{\partial u^k} \frac{du^k}{dt} .$$

Substituting this result back in the equation we obtain

$$\left(\frac{\partial A^r}{\partial u^k} + \Gamma^r_{jk} A^j\right) \frac{du^k}{dt} = 0$$

(realizing that $\Gamma^r_{jk} a^j b^k = \Gamma^r_{jk} b^k a^j$). This can be further rewritten as

$$\boxed{\frac{\partial A^r}{\partial u^k} + \Gamma^r_{jk} A^j = 0}$$

since $du^k/dt \neq 0$ along a general curve $C(t)$. This is the final differential equation of a constant vector field in any Riemannian space (or equation of parallel transmission).

It can be shown similarly that for covariant vectors we get

$$\boxed{\frac{\partial A_r}{\partial u^k} - \Gamma^j_{rk} A_j = 0 .}$$

Analogously, in 2D-space, the Christoffel symbols are defined as follows:

$$\boxed{[\alpha\beta, \gamma] = \frac{1}{2} \left(\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right)}$$

$$\boxed{\Gamma^\gamma_{\alpha\beta} = g^{\gamma\delta} [\alpha\beta, \delta].}$$

Similarly, the equations for parallel transmission (or constant field) are

$$\boxed{\begin{aligned} &\frac{\partial A^{\alpha}}{\partial u^{\beta}} + \Gamma^{\alpha}_{\gamma\beta} A^{\gamma} = 0 \\ &\frac{\partial A_{\alpha}}{\partial u^{\beta}} - \Gamma^{\gamma}_{\alpha\beta} A_{\gamma} = 0. \end{aligned}}$$

<u>Problems</u>: Derive the expressions for $[ij, k]$ and Γ^{k}_{ij} in spherical, geodetic and geographic coordinates.

Is a tangent vector to a ϕ-curve for $\lambda = \text{const}_1$ parallel to a tangent vector to a ϕ-curve for $\lambda = \text{const}_2$ on the surface of an ellipsoid of rotation?

3.4.3) <u>Tensor Derivative with Respect to Scalar Argument (Intrinsic Derivative)</u>

In trying to define a tensor derivative of a tensor, we require that it be again a tensor so that we can remain in the realm of tensor equations when differentiating. We can see that this requirement is <u>not satisfied</u> for the ordinary derivative dA^i/dt. This quantity does not transform as a tensor and is therefore not a tensor (see 3.4.1):

$$\frac{d\tilde{A}^i}{dt} = \frac{\partial^2 x^i}{\partial u^j \partial u^k} \frac{du^k}{dt} A^j + \frac{\partial x^i}{\partial u^j} \frac{dA^j}{dt} .$$

On the other hand the ordinary derivative of a scalar ϕ can be regarded as tensor derivative since

$$\frac{d\tilde{\phi}}{dt} = \frac{d\phi}{dt}$$

conforms with the transformation law for scalars (see 3.1). Denoting the tensor derivative by $\delta/\delta t$ we can then write:

$$\boxed{\frac{\delta \phi}{\delta t} = \frac{d\phi}{dt}} .$$

To produce a satsifactory definition of a tensor derivative of a vector let us take a scalar product of two arbitrary vectors and take the derivative of thus defined scalar. We get:

$$\frac{\delta}{\delta t} (A_i B^i) = \frac{d}{dt} (A_i B^i) = \frac{dA_i}{dt} B^i + A_i \frac{dB^i}{dt} .$$

We also want the tensor derivative to obey the same rules as the ordinary derivative does. Namely, we want the following equation to be satisfied:

$$\frac{\delta}{\delta t} (A_i B^i) = \frac{\delta A_i}{\delta t} B^i + A_i \frac{\delta B^i}{\delta t} .$$

In order to determine the relationship between the tensor and the ordinary derivatives, let us consider one of the vector fields, say B^i, constant. From 3.4.2 we know that B^i has to satisfy the following differential equation

$$\frac{dB^i}{dt} = - \Gamma^i_{jk} \frac{du^k}{dt} B^j .$$

Combining the first two equations and substituting for dB^i/dt we obtain:

$$\frac{dA_i}{dt} B^i - A_i \Gamma^i_{jk} \frac{du^k}{dt} B^j = \frac{\delta A_i}{\delta t} B^i + A_i \frac{\delta B^i}{\delta t} .$$

Further, we want the tensor derivative of a constant field to be zero, to conform again with the rules for ordinary derivatives. Putting $\delta B^i/\delta t = 0$ and changing the i (dummy) indeces in the first terms on both sides to j we get

$$\left(\frac{dA_j}{dt} - A_i \Gamma^i_{jk} \frac{du^k}{dt}\right) B^j = \frac{\delta A_j}{\delta t} B^j .$$

Hence we have found finally the expression for the tensor derivative of a covariant vector:

$$\boxed{\frac{\delta A_j}{\delta t} = \frac{dA_j}{dt} - \Gamma^i_{jk} A_i \frac{du^k}{dt} .}$$

Note that the order of Γ^i_{jk} and A_i does not matter.

To show that the tensor derivative of a covariant vector, sometimes called intrinsic covariant derivative, is given as covariant vector is left to the reader.

Problem: Prove by analogous reasoning that the intrinsic contravariant derivative can be defined as

$$\boxed{\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + \Gamma^i_{jk} A^j \frac{du^k}{dt} .}$$

Show that $\delta A^i/\delta t$ is a contravariant vector.

The above definitions can be now easily generalized for a tensor of any rank. For instance:

$$\boxed{\frac{\delta A^{jk}_{\ell mn}}{\delta t} = \frac{dA^{jk}_{\ell mn}}{dt} + (-\Gamma^s_{\ell q} A^{jk}_{smn} - \Gamma^s_{mq} A^{jk}_{\ell sn} - \Gamma^s_{nq} A^{jk}_{\ell ms} + \Gamma^j_{sq} A^{sk}_{\ell mn} + \Gamma^k_{sq} A^{js}_{\ell mn})\frac{du^q}{dt} .}$$

To show that an analogous formula holds true for intrinsic derivative in 2D-space is left to the reader.

Let us conclude this paragraph by stating that the tensor derivative (with respect to its scalar argument) of the metric tensor, associated metric tensor, Kronecker's δ, and the Levi-Civitá tensor are all zero. This can be seen when we express the named tensors in Cartesian coordinates. There all the Christoffel symbols disappear and all that is left are the ordinary derivatives. But even these are all zero since all the components of the above tensors are constant. Hence the named tensors can be regarded as "constant" from the point of view of tensor differentiation. An intuitive explanation for this phenomenon is that although the tensors may vary from place to place their variation reflects just the properties of the space itself and not that of any objects in the space.

3.4.4) Tensor Derivative with Respect to Coordinates (Covariant and Contravariant Derivatives)

We know that the ordinary derivative of, say A_r, can be written as

$$\frac{dA_r}{dt} = \frac{\partial A_r}{\partial u^k} \frac{du^k}{dt} .$$

Let us require that the same rule applies on the tensor derivatives as well, i.e.

$$\frac{\delta A_r}{\delta t} = \frac{\delta A_r}{\delta u^k} \frac{du^k}{dt} . \qquad (*)$$

Substituting for the intrinsic tensor derivative on the left hand side from 3.4.3 we obtain

$$\frac{dA_r}{dt} - \Gamma^i_{rk} A_i \frac{du^k}{dt} = \frac{\delta A_r}{\delta u^k} \frac{du^k}{dt} .$$

Substituting here again for the derivative dA_r/dt from the above equation we get

$$\left(\frac{\partial A_r}{\partial u^k} - \Gamma^i_{rk} A_i\right) \frac{du^k}{dt} = \frac{\delta A_r}{\delta u^k} \frac{du^k}{dt} .$$

Hence the tensor derivative $\delta A_r/\delta u^k$, which we shall denote further by $\nabla_k A_r$ is given by

$$\boxed{\frac{\delta A_r}{\delta u^k} = \nabla_k A_r = \frac{\partial A_r}{\partial u^k} - \Gamma^i_{rk} A_i \quad .}$$

It is called usually the <u>covariant derivative of the covariant vector</u> A_r and denoted sometimes by $A_{r,k}$. We can see that the covariant derivative of a covariant vector is a twice covariant tensor from the equation (*); the covariant intrinsic derivative is once covariant tensor and du^k/dt is once contravariant tensor ($\frac{du^k}{dt} = \frac{\partial u^k}{\partial x^i} \frac{dx^i}{dt}$ and $\frac{dx^i}{dt}$ was shown in 3.3.13 to be a once contravariant tensor), hence the covariant derivative is a twice covariant tensor.

It is not difficult to see that in Cartesian coordinates the covariant derivative degenerates into the simple dyadic product of the symbolic vector ∇ and $\vec{A}$. The covariant derivative of a scalar, similarly to the intrinsic derivative, reduces to

$$\nabla_i \phi = \frac{\partial \phi}{\partial u^i} \quad .$$

<u>Problem</u>: Show that the covariant derivative of the contravariant vector A^r is a mixed tensor of second rank given by:

$$\nabla_k A^r = \frac{\partial A^r}{\partial u^k} + \Gamma^r_{jk} A^j \quad .$$

Contravariant derivatives, that have less important applications, can be obtained by raising the indeces in covariant derivates. Hence:

$$\nabla^k A^r = g^{ik} \nabla_i A^r \quad .$$

Generally, a covariant derivative of any tensor can be obtained from its intrinsic derivative using the formula

$$\boxed{\frac{\delta A^{jk}_{\ell mn}}{\delta t} = \nabla_i \, A^{jk}_{\ell mn} \frac{du^i}{dt} \quad .}$$

We may note that the covariant and contravariant derivatives of the four pseudo-constant tensors (mentioned in 3.4.3) are again zero.

This is easily seen when we consider the derivatives in Cartesian coordinates. The covariant and contravariant derivatives obey again the same rules as the ordinary derivatives. For example:

$$\nabla_i(kA_j) = k(\nabla_i A_j), \quad k = \text{const.} \;,$$

$$\nabla^i(A_j+B_j) = \nabla^i A_j + \nabla^i B_j \;,$$

$$\nabla_i(A_j B^k_\ell) = (\nabla_i A_j)B^k_\ell + A_j(\nabla_i \, B^k_\ell) \;.$$

<u>Problem</u>: Show that the Frenet's formulae for a spatial curve are given by:

$$(\nabla_s \, t_r)t^s = n_r/R \;,$$

$$(\nabla_s \, n_r)t^s = -t_r/R + b_r/T,$$

$$(\nabla_s \, b_r)t^s = -n_r/T \;.$$

If $\vec{t}$ is the unit tangent vector to a curve C on a surface then

$$(\nabla_\beta t_\alpha)t^\beta = n_\alpha/R_G$$

is a differential equation relating $\vec{t}$ to the unit normal vector n_α (on the surface) and R_G, the radius of <u>geodetic curvature of the curve</u>. Hence

$$(\nabla_\alpha t_\beta)t^\alpha = 0$$

is the equation of a <u>geodesic on a surface</u>.

3.4.5) ∇ and Δ Operators in Tensor Notation.

We have already seen in the previous paragraph that the covariant derivative of a scalar is nothing else but the gradient of the scalar.

$$\boxed{\nabla_i \phi = \frac{\partial \phi}{\partial u^i} = \text{grad } \phi .}$$

It can similarly be shown that the contraction of the covariant derivative of a contravariant vector gives us the scalar known as divergence of A. To see this, it suffices to write the expression in Cartesian coordinates:

$$\boxed{\nabla_i A^i = \frac{\partial A^1}{\partial u^1} + \frac{\partial A^2}{\partial u^2} + \frac{\partial A^3}{\partial u^3} = \text{div } \vec{A} \ .}$$

Due to the symmetrical properties of the scalar product we have:

$$\nabla_i A^i = \nabla^i A_i \ .$$

Further, by the same reasoning we can prove that

$$\boxed{e^{ijk} \nabla_j A_k = \text{rot } \vec{A} \ ,}$$

and call the resulting vector the rotor or curl of $\vec{A}$. Finally, for the Laplacean of a scalar we obtain

$$\boxed{\Delta \phi = \nabla^i \nabla_i \phi = \nabla_i \nabla^i \phi \ .}$$

The proof of this statement is again based on the equivalence in Cartesian coordinates.

In order to be able to write the differential formulae in a form ready for computations, let us first derive one very important expression, namely:

$$\boxed{\Gamma^i_{ij} = \frac{\partial (\ell n \sqrt{g})}{\partial u^j} \ .}$$

Perhaps the easiest way to derive it is to take the covariant derivative of the Levi-Civita tensor. We get

$$\nabla_\ell e_{ijk} = \frac{\partial e_{ijk}}{\partial u^\ell} - \Gamma^s_{i\ell} e_{sjk} - \Gamma^s_{j\ell} e_{isk} - \Gamma^s_{k\ell} e_{ijs} .$$

But we know that this tensor equals to zero so that all the components equal to zero. Let us then take one vector of components, e.g.

$$\nabla_\ell e_{123} = \frac{\partial e_{123}}{\partial u^\ell} - \Gamma^s_{1\ell} e_{s23} - \Gamma^s_{2\ell} e_{1s3} - \Gamma^s_{3\ell} e_{12s} = 0 .$$

Taking the products of Christoffel symbols and the Levi-Cività tensors, we can see that, due to the definition of the L-C. tensor, only one vector of components is always different from zero. We can write:

$$\frac{\partial e_{123}}{\partial u^\ell} - \Gamma^1_{1\ell} e_{123} - \Gamma^2_{2\ell} e_{123} - \Gamma^3_{3\ell} e_{123} = 0 .$$

From 3.3.4 we know that $e_{123} = \sqrt{g} = \sqrt{\det}\ (g_{ij})$. Hence we obtain

$$\frac{\partial \sqrt{g}}{\partial u^\ell} - \sqrt{g}(\Gamma^1_{1\ell} + \Gamma^2_{2\ell} + \Gamma^3_{3\ell}) = 0$$

or

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial u^\ell} = \Gamma^i_{i\ell} .$$

Recalling the formula from the analysis of real functions

$$y = \ell n\ f(x), \qquad \frac{dy}{dx} = \frac{1}{f(x)} \frac{df(x)}{dx} ,$$

we can finally rewrite the above equation in the form introduced at the outset of this proof.

Having established the formula for the contracted Christoffel symbol we can now proceed to derive the expressions for the differential operators in curvilinear coordinates. For the divergence, we may write:

$$\nabla_i A^i = \frac{\partial A^i}{\partial u^i} + \Gamma^i_{ij} A^j$$

$$= \frac{\partial A^i}{\partial u^i} + \frac{\partial(\ell n\sqrt{g})}{\partial u^j} A^j$$

$$= (\sqrt{g}\,\frac{\partial A^i}{\partial u^i} + \frac{\partial\sqrt{g}}{\partial u^i} A^i)\,\frac{1}{\sqrt{g}}$$

$$\boxed{\nabla_i A^i = \frac{1}{\sqrt{g}}\,\frac{\partial(A^i\sqrt{g})}{\partial u^i}\ .}$$

If we know only the covariant components of $\vec{A}$ we can write

$$\boxed{\nabla_i A^i = \nabla^i A_i = \frac{1}{\sqrt{g}}\,\frac{\partial(\sqrt{g}\, g^{ij} A_j)}{\partial u^i}\ .}$$

<u>Problem</u>: Derive the formulae for div $\vec{A}$ in spherical and geodetic coordinates.

For the Laplacean equation for a scalar ϕ we get

$$\Delta\phi = \nabla_i \nabla^i \phi = \nabla_i\,(g^{ij}\nabla_j\phi) = \nabla_i(g^{ij}\,\frac{\partial\phi}{\partial u^j})\ .$$

Applying to this formula the same treatment as we have applied to the divergence, we obtain

$$\boxed{\Delta\phi = \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial u^i}\,(\sqrt{g}\, g^{ij}\,\frac{\partial\phi}{\partial u^j})\ .}$$

Finally, we remark that the gradient as derived at the beginning of this paragraph may be regarded as the covariant gradient - the operation of covariant derivative of a scalar results in a covariant vector. We can also define the <u>contravariant gradient</u> as

$$\boxed{\nabla^i\phi = g^{ij}\nabla_j\,\phi = g^{ij}\,\frac{\partial\Phi}{\partial u^j}\ .}$$

As a matter of fact the contravariant gradient is the one that is more often used in applications.

Problem: Derive the formulae for contravariant gradient and Laplacean in spherical and geodetic coordinates.

3.4.6) Riemann-Christoffel Tensor

Let us see now what would be the second covariant derivative of a covariant vector. Taking an arbitrary covariant vector A_i we get

$$\nabla_k(\nabla_j A_i) = \frac{\partial}{\partial u^k}(\nabla_j A_i) - \Gamma^s_{jk}(\nabla_s A_i) - \Gamma^s_{ik}(\nabla_j A_s) .$$

Realizing that

$$\nabla_j A_i = \frac{\partial A_i}{\partial u^j} - \Gamma^\ell_{ij} A_\ell$$

we get

$$\nabla_k(\nabla_j A_i) = \frac{\partial}{\partial u^k}\left(\frac{\partial A_i}{\partial u_j} - \Gamma^\ell_{ij}A_\ell\right) - \Gamma^s_{jk}\left(\frac{\partial A_i}{\partial u^s} - \Gamma^\ell_{is}A_\ell\right) - \Gamma^s_{ik}\left(\frac{\partial A_s}{\partial u^j} - \Gamma^\ell_{sj}A_\ell\right) .$$

This can be rewritten as

$$\nabla_k(\nabla_j A_i) = \frac{\partial^2 A_i}{\partial u^k \partial u^j} - \frac{\partial}{\partial u^k}(\Gamma^\ell_{ij}A_\ell) - \Gamma^s_{jk}\frac{\partial A_i}{\partial u^s} + \Gamma^s_{jk}\Gamma^\ell_{is}A_\ell - \Gamma^s_{ik}\frac{\partial A_s}{\partial u^j} + \Gamma^s_{ik}\Gamma^\ell_{sj}A_\ell .$$

An interesting question now arises as to whether the second derivative depends on the order of differentiation. To investigate this question let us interchange the order of differentiation in the above formula. We obtain

$$\nabla_j(\nabla_k A_i) = \frac{\partial^2 A_i}{\partial u^j \partial u^k} - \frac{\partial}{\partial u^j}(\Gamma^{\ell}_{ik} A_\ell) - \Gamma^s_{kj} \frac{\partial A_i}{\partial u^s} + \Gamma^s_{kj}\Gamma^{\ell}_{is} A_\ell - \Gamma^s_{ij} \frac{\partial A_s}{\partial u^k} + \Gamma^s_{ij}\Gamma^{\ell}_{sk} A_\ell .$$

The difference of the two second derivatives then yields:

$$\nabla_k(\nabla_j A_i) - \nabla_j(\nabla_k A_i) = \frac{\partial}{\partial u^j}(\Gamma^{\ell}_{ik} A_\ell) - \frac{\partial}{\partial u^k}(\Gamma^{\ell}_{ij} A_\ell) + \Gamma^s_{ij} \frac{\partial A_s}{\partial u^k} - \Gamma^s_{ik} \frac{\partial A_s}{\partial u^j} +$$

$$+ \Gamma^s_{ik}\Gamma^{\ell}_{sj} A_\ell - \Gamma^s_{ij}\Gamma^{\ell}_{sk} A_\ell .$$

Here the first two terms on the right hand side give

$$\frac{\partial}{\partial u^j}(\Gamma^{\ell}_{ik} A_\ell) - \frac{\partial}{\partial u^k}(\Gamma^{\ell}_{ij} A_\ell) = \frac{\partial \Gamma^{\ell}_{ik}}{\partial u^j} A_\ell + \Gamma^{\ell}_{ik} \frac{\partial A_\ell}{\partial u^j} - \frac{\partial \Gamma^{\ell}_{ij}}{\partial u^k} A_\ell - \Gamma^{\ell}_{ij} \frac{\partial A_\ell}{\partial u^k} .$$

Thus the terms containing the partial derivatives of the arbitrary vector $\vec{A}$ get cancelled and we end up with the final expression:

$$\nabla_k(\nabla_j A_i) - \nabla_j(\nabla_k A_i) = \frac{\partial \Gamma^{\ell}_{ik}}{\partial u^j} A_\ell - \frac{\partial \Gamma^{\ell}_{ij}}{\partial u^k} A_\ell + \Gamma^s_{ik}\Gamma^{\ell}_{sj} A_\ell - \Gamma^s_{ij}\Gamma^{\ell}_{sk} A_\ell =$$

$$= \left(\frac{\partial \Gamma^{\ell}_{ik}}{\partial u^j} - \frac{\partial \Gamma^{\ell}_{ij}}{\partial u^k} + \Gamma^s_{ik}\Gamma^{\ell}_{sj} - \Gamma^s_{ij}\Gamma^{\ell}_{sk}\right) A_\ell .$$

Hence, the difference of the two second derivatives of a covariant vector can be expressed as a product of a quantity with four indeces with the covariant vector.

Let us now have a look at the quantity in the brackets. Since we know that the second covariant derivative of a covariant tensor is a three-times covariant tensor then obviously even the difference of two such derivatives is a three times covariant tensor. Using the theorem on tensor character (3.3.8) we conclude that the expression in the brackets is a three-times covariant and once contravariant tensor. We have

$$\boxed{\nabla_k(\nabla_j A_i) - \nabla_j(\nabla_k A_i) = R^{\ell}_{\ ijk} A_{\ell}}$$

where the tensor $R^{\ell}_{\ ijk}$ is known as the Riemann-Christoffel tensor.

It can be seen that the Riemann-Christoffel tensor is antisymmetric in j and k. Its covariant form

$$\boxed{R_{\ell ijk} = g_{\ell m} R^{m}_{\ ijk}}$$

can be shown to be antisymmetric also in the first two indeces

$$R_{\ell ijk} = - R_{i\ell jk}$$

and symmetric with respect to the two pairs of indeces

$$R_{\ell ijk} = R_{jk\ell i} \quad .$$

The Riemann-Christoffel tensor has got one more interesting property. We may note that it becomes identically equal to zero in Cartesian coordinates (Eucleidean space) because all the Christoffel symbols become zero. On the other hand, since it is a tensor, if it is zero in one coordinate system it must be zero in any other coordinate system as well, as we can easily see from its transformation law. Hence one may conclude that if there exists a possibility of transforming one system of coordinates in the space to the Cartesian system, the Riemann-Christoffel tensor equals to zero. We know already (3.3.11) that the possibility to transform any coordinate system to the Cartesian system is the necessary and sufficient condition for the space to be inherently flat. Therefore we can say that the necessary and sufficient condition for a space to be inherently flat is

$$R_{\ell ijk} = 0 \ .$$

On the other hand, if $R_{\ell ijk}$, for any system of coordinates, is different from zero, the space is inherently curved. Hence the Riemann-Christoffel tensor describes some inherent property of the space, namely its curvature and is therefore sometimes called the curvature tensor. Note that here again the second derivative is connected with curvature one way or the other.

In 2D-space, the curvature tensor is given analogously by

$$\boxed{R^{\delta}_{\alpha\beta\gamma} = \frac{\partial\Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} - \frac{\partial\Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} + \Gamma^{\sigma}_{\alpha\gamma}\,\Gamma^{\delta}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\beta}\,\Gamma^{\delta}_{\sigma\gamma}\ .} \qquad *$$

It may also be mentioned that the following formulae hold for various tensors

$$\nabla_k(\nabla_j A^i) - \nabla_j(\nabla_k A^i) = R^{i}_{\cdot\ell kj}\,A^{\ell}\ ,$$

$$\nabla_k(\nabla_j A_i) - \nabla_j(\nabla_k A_i) = R_{jk\ell i}\,A^{\ell},$$

$$\nabla_k(\nabla_j\ \phi) - \nabla_j(\nabla_k\ \phi) = 0 \quad \text{for } \phi \text{ being a scalar,}$$

$$\nabla_k(\nabla_j A^{\ell}_i) - \nabla_j(\nabla_k A^{\ell}_i) = R^{\ell}_{\cdot mkj}A^{m}_i + R^{m}_{\cdot ijk}\ A^{\ell}_m\ .$$

Problems: Derive the Riemann-Christoffel tensor for a sphere and an ellipsoid of rotation.

3.4.7) Ricci-Einstein and Lamé Tensors

The Riemann-Christoffel tensor can be shown to have only $n^2(n^2-1)/12$ independent components out of n^4. This is because of the two antisymmetries and the symmetry mentioned earlier and one more condition. In 3D-space this means that we get only $9(9-1)/12 = 6$ independent components, which leads to the idea that all the information contained in the Riemann-Christoffel tensor can be expressed completely by a symmetric tensor of second rank which has also only 6 independent components.

Ricci has come up with one such possible tensor created by contracting the Riemann-Christoffel tensor in the first and last indeces, i.e.

$$\boxed{R_{ij} = R^{k}{}_{\cdot ijk} = \delta^{k}_{\ell} R^{\ell}{}_{\cdot ijk} = g^{mk} R_{mijk} \ .}$$

Since this contracted curvature tensor is also used heavily in the Einstein's general theory of relativity it became known as the <u>Ricci-Einstein (curvature) tensor</u>. It can be obviously written as

$$R_{ij} = \frac{\partial \Gamma^{k}_{ik}}{\partial u^{j}} - \frac{\partial \Gamma^{k}_{ij}}{\partial u^{k}} + \Gamma^{s}_{ik}\Gamma^{k}_{sj} - \Gamma^{s}_{ij}\Gamma^{k}_{sk} \ .$$

Recalling the expressions for Γ^{k}_{ik} from 3.4.5 we can see immediately that R_{ij} is symmetric. A more direct proof of its symmetry is given by

$$R_{ij} = g^{mk} R_{mijk} = g^{km} R_{kjim} = R_{ji}$$

realizing that $R_{mijk} = R_{kjim}$. Obviously, the Ricci-Einstein tensor is zero if and only if Riemann-Christoffel tensor is zero, i.e. in a flat space.

A different tensor of second rank can be obtained from the Riemann-Christoffel tensor from the following formula

$$\boxed{S^{ij} = \frac{1}{4}\, e^{ik\ell}\, e^{jmn}\, R_{k\ell mn} \ .}$$

This tensor is known as the <u>Lamé (curvature) tensor</u>. It can be seen that the Lamé tensor is again symmetric due to the symmetry of the Riemann-Christoffel tensor in the two pairs of indeces:

$$S^{ij} = \frac{1}{4}\, e^{ik\ell}\, e^{jmn}\, R_{k\ell mn} = \frac{1}{4}\, e^{jmn}\, e^{ik\ell}\, R_{mnk\ell} = S^{ji} .$$

Comparing the two latter tensors, one gets

$$R_{ij} = g^{mk} R_{mijk} = g^{mk}\, e$$

since

$$e_{pmi}\, e_{qjk}\, S^{pq} = R_{mijk} \;.$$

After some development, which we are not going to show here, this equation becomes

$$\boxed{R_{ij} = S_{ij} - S^* g_{ij}}$$

where

$$S_{ij} = g_{im} g_{jn} S^{mn} \quad \text{and} \quad S^* = g_{mn} S^{mn}.$$

Problems: Derive the Ricci-Einstein tensor for a sphere and an ellipsoid of rotation.

3.4.8) Gaussian Curvature of a Surface, Classification of Spaces

In 2D-space the Riemann-Christoffel tensor has got only one independent component out of $2^4 = 16$:

$$4(4-1)/12 = 1.$$

Taking the equation (*) in 3.4.6 we can see that

$$R^{\delta}_{\cdot\alpha\beta\gamma} = 0 \quad \text{for} \quad \beta = \gamma$$

(similar equation of course holds even for the 3D case). Hence even

$$R_{\delta\alpha\beta\gamma} = 0 \quad \text{for} \quad \beta = \gamma.$$

Moreover, due to the symmetry in the two pairs of indeces, we have

$$R_{\delta\alpha\beta\gamma} = 0 \quad \text{for} \quad \delta = \alpha.$$

This leaves us with only 4 components potentially different from zero, i.e. those for which $\beta \neq \gamma$ and $\delta \neq \alpha$. These are R_{1212}, R_{2121}, R_{1221}, R_{2112}. But even these are related, due to the antisymmetry of the tensor and we have

$$R_{1212} = R_{2121} = - R_{1221} = -R_{2112} \;.$$

In terms of the Ricci-Einstein tensor

$$R_{\alpha\beta} = g^{\gamma\delta}\, R_{\gamma\alpha\beta\delta}$$

this means that its four components are given simply by the following relations

$$R_{11} = g^{22}\, R_{2112} = -\, g^{22}\, R_{1212}$$

$$R_{12} = g^{21}\, R_{2121} = g^{12}\, R_{1212}$$

$$R_{21} = g^{12}\, R_{1212}$$

$$R_{22} = g^{11}\, R_{1221} = -\, g^{11}\, R_{1212}.$$

Realizing that the components of the associated metric tensor (see 3.3.12) can be written as

$$g^{11} = g_{22}/g, \quad g^{12} = -g_{12}/g, \quad g^{22} = g_{11}/g$$

(where $g = \det\, (g_{\alpha\beta})$) we get for the matrix of $R_{\alpha\beta}$:

$$[R_{\alpha\beta}] = R_{1212} \begin{bmatrix} -g^{22} & g^{12} \\ g^{12} & -g^{11} \end{bmatrix} = -R_{1212} \begin{bmatrix} g_{11}/g, & g_{12}/g \\ g_{12}/g, & g_{22}/g \end{bmatrix} .$$

Hence

$$\boxed{R = \det\, (R_{\alpha\beta}) = R^2_{1212}\, \det\, (g^{\alpha\beta}) = R^2_{1212}/g \quad ,}$$

where the quantity R_{1212}/g is called the <u>Gaussian curvature</u> of the surface (2D-space). It is usually denoted by K so that we have

$$\boxed{R = KR_{1212}}.$$

We can obviously write also:

$$\boxed{R_{\alpha\beta} = -\,\frac{R_{1212}}{g}\, g_{\alpha\beta} = -\, Kg_{\alpha\beta}}$$

or

$$g^{\alpha\beta} R_{\alpha\beta} = R^* = -2K.$$

The Gaussian curvature can be also expressed directly in terms of the metric tensor. After some development we would obtain

$$K = -\frac{1}{2\sqrt{g}} \left[\frac{\partial}{\partial u^1} \left(\frac{1}{\sqrt{g}} \frac{\partial g_{22}}{\partial u^1} \right) + \frac{\partial}{\partial u^2} \left(\frac{1}{\sqrt{g}} \frac{\partial g_{11}}{\partial u^2} \right) \right].$$

The Gaussian curvature plays the fundamental role in the classification of spaces. As we have seen already, spaces where $K = 0$ are called Eucleidian (in wider sense) or flat; if $K \neq 0$ the space is called Riemannian (in wider sense) or curved. If $K = \text{const.} > 0$ the space is knows as elliptical or <u>Riemannian in the narrow sense</u> or a space with constant positive curvature. If $K < 0$, the space is called the <u>Lobachevski space</u>, if K is a negative constant the space is hyperbolic. Elliptical and parabolic spaces are called non-Eucleidian.

<u>Problems:</u> Derive the expression for Gaussian curvature of a sphere and ellipsoid.

4) SOME APPLICATIONS OF TENSORS IN DIFFERENTIAL GEOMETRY OF SURFACES

4.1) First and Second Fundamental Forms of a Surface (Relation of the first fundamental form to the metric tensor.)

The equation for the line element on a surface

$$(ds)^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta$$

is also known as the first fundamental (Gaussian) form of the surface and $g_{\alpha\beta}$ is sometimes called the 1-st fundemental tensor of the surface. In the non-tensor notation it is usually denoted as

$$ds^2 = E\, du^2 + 2F\, dudv + G\, dv^2 .$$

Hence, $u^1 = u$, $u^2 = v$, $g_{11} = E$, $g_{12} = F$, $g_{22} = G$. We have met the quantities E, F, G, called also the fundamental quantities of 1-st order of the surface, in 1.3.3. We have also seen that they may be defined as

$$E = \left(\frac{\partial \vec{r}}{\partial u}\right)^2 , \; F = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} , \; G = \left(\frac{\partial \vec{r}}{\partial v}\right)^2$$

if the surface is given as

$$\vec{r} = \vec{r}\,(u,v) .$$

The connection between the two definitions (tensorial and classical) of the components of the metric tensor is obvious when we realize that

$$r\ (u,v) \equiv (x\ (u,v),\ y(u,v),\ (u,v) \equiv (x^1(u^1,u^2), x^2(u^1,u^2),\ x^3(u^1,u^2)).$$

so that we get for instance

$$\frac{\partial\vec{r}}{\partial u} = \frac{\partial\vec{r}}{\partial u^1} = \left(\frac{\partial x^1}{\partial u^1} \frac{\partial x^2}{\partial u^1} . \frac{\partial x^3}{\partial u^1}\right) .$$

We shall use the notation x^i_1 for $\frac{\partial\vec{r}}{\partial u}$ and x^i_2 for $\frac{\partial\vec{r}}{\partial v}$.

At the same time we may notice that the quantity

$$D = \sqrt{(EG - F^2)}$$

introduced in 1.3.3, is nothing else but

$$\sqrt{\det(g_{\alpha\beta})} = \sqrt{g}.$$

Realizing that $\partial\vec{r}/\partial\vec{u}$, $\partial\vec{r}/\partial\vec{v}$ are tangent vectors to the u and v-curves on the surface, we have another method for deriving the metric tensor of a surface. We can write

$$[g] = \begin{bmatrix} \frac{\partial\vec{r}}{\partial u} \\ \\ \frac{\partial\vec{r}}{\partial v} \end{bmatrix} \left[\frac{\partial\vec{r}}{\partial u}, \frac{\partial r}{\partial v}\right]$$

a dyadic product in matrix form, or in tensor notation

$$g_{\alpha\beta} = x^i_\alpha\ x_{i,\beta}$$

where the x coordinate system is assumed rectangular Cartesian. Note that the surface is a Eucleidian space if and only if the two tangent vectors are unit and mutually orthogonal.

Let us now consider scalar product $\vec{r} \cdot \vec{n} = x^i n_i$. We may ask ourselves a question: What would be its total differential $d(\vec{r}.\vec{n})$? The answer to this question is given by the following development:

$$d(\vec{r} \cdot \vec{n}) = \frac{\partial\vec{r}}{\partial u}\frac{\partial\vec{n}}{\partial u} du^2 + \frac{\partial\vec{r}}{\partial u}\frac{\partial\vec{n}}{\partial v} du\, dv + \frac{\partial\vec{r}}{\partial v}\frac{\partial\vec{n}}{\partial u} du\, dv + \frac{\partial\vec{r}}{\partial v}\frac{\partial\vec{n}}{\partial v} dv^2$$

$$= (L\, du^2 + 2M du dv + N\, dv^2)$$

In tensor notation we write

$$d(x^i\ n_i) = -(x^i_1 \frac{\partial n_i}{\partial u^1} du^1\ du^1 - x^i_1 \frac{\partial n_i}{\partial u^2} du^1\ du^2 -$$

$$x^i_2 \frac{\partial n_i}{\partial u^1} du^1\ du^2 - x^i_2 \frac{\partial m_i}{\partial u^2} du^2\ du^2)$$

$$= -b_{\alpha\beta}\ du^\alpha\ du^\beta.$$

This formula is known as the <u>second fundamental (Monge's) form of the surface</u>. The tensor $b_{\alpha\beta}$ is called the second fundemantal tensor of the surface and the quantities L, M, N are known as the <u>fundamental quantities of 2-nd order</u> of a surface.

Note that

$$b_{11} = L,\ b_{22} = N$$

$$b_{12} = b_{21} = 2M.$$

Hence, if $b_{\alpha\beta}$ is symmetrical,

$$b_{12} = b_{21} = M.$$

We shall show later, that this is the case, if and only if the surface is smooth, i.e., if the curvature changes continuously. The reason for being asked the question we started with will be also seen later.

As we know already (see 1.3.3) the unit normal vector is given by

$$n = (\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}) / D$$

or, in tensor notation

$$n_i = e_{ijk} x_1^j x_2^k / \sqrt{g}.$$

We may notice that the tensor $b_{\alpha\beta}$ is given by

$$b_{\alpha\beta} = -x_\alpha^i \frac{\partial n_i}{\partial u^\beta}$$

An alternative set of formulae for the components of the 2nd fundamental tensor can be derived as follows. Let us consider the scalar products of the taneent vectors with the normal vector , i.e.

$$x_1^i n_i = x_2^i n_i = 0, \; (x_\alpha^i \perp n^i).$$

then take the deriviatives of these with respect to u^1, u^2. We get

$$\frac{\partial}{\partial u^1} (x_1^i n_i) = \frac{\partial x_1^i}{\partial u^1} n_i + x_1^i \frac{\partial n_i}{\partial u^1} = 0$$

$$\frac{\partial}{\partial u^1} (x_2^i n_i) = \frac{\partial x^i}{\partial u^1} n_i + x_2^i \frac{\partial n_i}{\partial u^1} = 0$$

$$\frac{\partial}{\partial u^2} (x_1^i n_i) = \frac{\partial x_1^i}{\partial u^2} n_i + x_1^i \frac{\partial n_i}{\partial u^2} = 0$$

$$\frac{\partial}{\partial u^2} (x_2^i n_i) = \frac{\partial x_2^i}{\partial u^2} n_i + x_2^i \frac{\partial n_i}{\partial u^2} = 0$$

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and immediately:

$$L = \frac{\partial x_1^i}{\partial u^1} \; n_i \, , \quad 2M = \frac{\partial x_2^i}{\partial u^1} n_i + \frac{\partial x_1^i}{\partial u^2} n_i = \left(\frac{\partial x_2^i}{\partial u^1} + \frac{\partial x_1^i}{\partial u^2}\right) n_i =$$

$$= \left(\frac{\partial^2 x^i}{\partial u^2 \partial u^1} + \frac{\partial^2 x^i}{\partial u^1 \, \partial u^2}\right) n_i \, , \quad N = \frac{\partial x_2^i}{\partial u^2} n_i \, .$$

Considering the surface r smooth (continuous curvature) one gets

$$\frac{\partial^2 x^i}{\partial u^2 \; \partial u^1} = \frac{\partial^2 x^i}{\partial u^1 \; \partial u^2}$$

and

$$M = \frac{\partial x^i}{\partial u^2} \; n_i = \frac{\partial x_2^i}{\partial u^1} \; n_i \; .$$

In this and only in this case $b_{\alpha\beta}$ is symmetrical. This is what we shall assume from now on. Rewriting these formulae in the classical notation, one obtains:

$$L = \left[\frac{\partial^2 \vec{r}}{\partial_u^{\;2}} \;\; \frac{\partial \vec{r}}{\partial u} \;\; \frac{\partial \vec{r}}{\partial v}\right] \Big/ D \, ,$$

$$M = \left[\frac{\partial^2 \vec{r}}{\partial u \partial v} \;\; \frac{\partial \vec{r}}{\partial u} \; \frac{\partial \vec{r}}{\partial v}\right] \Big/ D = \left[\frac{\partial^2 \vec{r}}{\partial v \partial u} \;\; \frac{\partial \vec{r}}{\partial u} \; \frac{\partial \vec{r}}{\partial v}\right] \Big/ D$$

$$N = \left[\frac{\partial^2 \vec{r}}{\partial v^2} \;\; \frac{\partial \vec{r}}{\partial u} \;\; \frac{\partial \vec{r}}{\partial v}\right] \Big/ D \, .$$

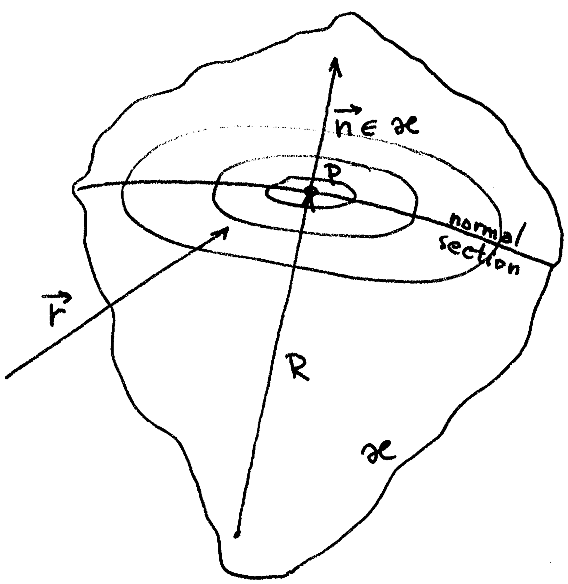
The second fundamental tensor determines the character of the surface at a point. We define three different kinds of points; according to the value of the determinant of $b_{\alpha\beta}$:

$$b = \det(b_{\alpha\beta}) \begin{cases} > 0 & \text{elliptical point} \\ = 0 & \text{parabolic point} \\ < 0 & \text{hyperbolic point.} \end{cases}$$

It can also be shown that $b = gK$, where K is the gaussian curvature.

4.2) Curvature of a Surface Curve at a Point

Let us take first a special curve on the surface, a normal section. It is given as a section of a surface by a plane κ containing the normal to the surface at the point.



The radius of curvature at P of such a normal section is given by:

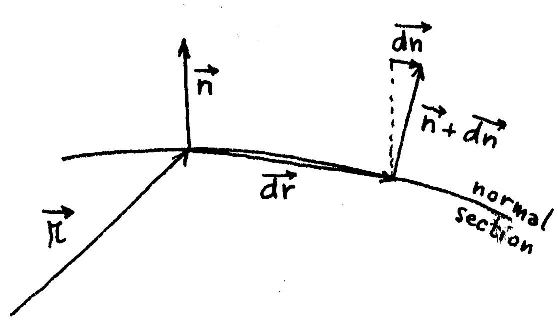
$$R(\varkappa) = \frac{g_{\alpha\beta}\, du^\alpha\, du^\beta}{b_{\alpha\beta}\, du^\alpha\, du^\beta} \qquad (*)$$

where the direction of the normal section is defined by choosing appropriate du^1/du^2, and $g_{\alpha\beta}, b_{\alpha\beta}$ are evaluated for P.

To prove this theorem we first write

$$R(\varkappa) = \frac{-ds^2}{d\vec{r} \cdot d\vec{n}},$$

the same as the above. From the diagram, one can see that if $d\vec{n} = \vec{0}$, the curvature of the normal section is equal to zero



and R $(\varkappa) \to \infty$. So the formula above works for $d\vec{n} = 0$. To show that it works under any circumstances, we write the second Frenet's formula (see 1.3.2):

$$\frac{d\vec{n}}{ds} = \frac{\vec{b}}{T} - \frac{\vec{t}}{R}$$

for the normal section. Since we deal with a plane curve, $T \to \infty$ and substituting $d\vec{r}/ds$ for $\vec{t}$ we get

$$\frac{d\vec{n}}{ds} = - \frac{d\vec{r}}{ds} \,/\, R$$

or
$$R\, d\vec{n} = - d\vec{r}.$$

Multiplying both sides by $d\vec{r}$ and realizing that

$$(d\vec{r})^2 = ds^2$$

we get

$$R = \frac{-ds^2}{d\vec{r} \cdot d\vec{n}}$$

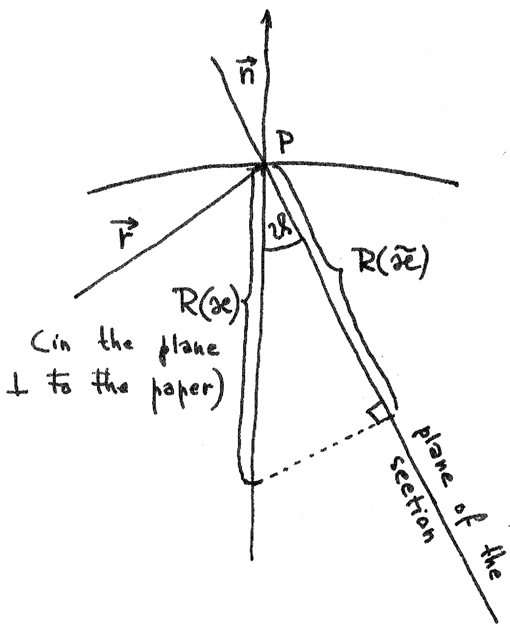
which was to be proved.

The curvature $1/R(\varkappa)$ is usually called the <u>normal curvature</u>.

Note that the radii of curvature of u and v-curves, if they are normal sections, are given by

$$R(u) = \frac{g_{11}}{b_{11}} = \frac{E}{L}\ , \quad R(v) = \frac{g_{22}}{b_{22}} = \frac{G}{N}\ .$$

Let us just state without proof, that should the second plane be inclined with respect to the normal by an angle $\theta\upsilon$ the radius of curvature R of the curve (section) would be given by

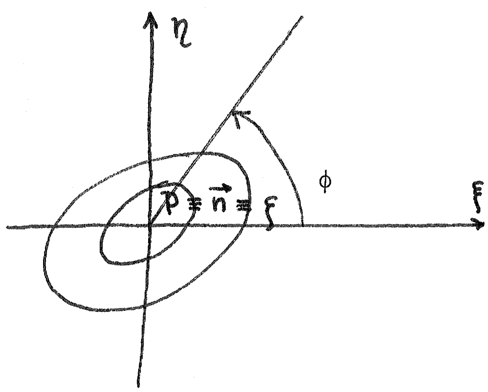


$$R(\tilde{\varkappa}) = R(\varkappa)\cos\theta \, .$$

This theorem is known as the Meusnier's theorem.

4.3 Euler's Equation

Let us now adopt the point P for the origin of a new coordinate system ξ, η, ζ, such that the ζ-axis coincides with $\vec{n}$ and ξ, η are oriented arbitrarily. We can write the equation of the surface $\vec{r}$ in this coordinate system, valid at least for a small neighbourhood of P:



$$\zeta = f(\xi,\eta) \, .$$

Then we can express the fundamental quantities E, F, G, L, M, N in equation(*) as functions of ξ, η, ζ, and we get, (after a lengthy and tedious computations)

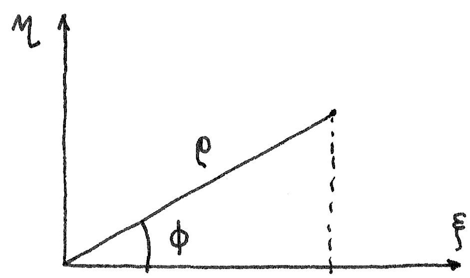
$$\pm \frac{1}{R(\phi)} = \frac{\partial^2\zeta}{\partial\xi^2}\Big|_P \cos^2\phi + 2\frac{\partial^2\zeta}{\partial\xi\partial\eta}\Big|_P \sin\phi\cos\phi + \frac{\partial^2\zeta}{\partial\eta^2}\Big|_P \sin^2\phi \quad (**)$$

where ϕ is the angle between the normal section and the ξ-axis, and $R(\phi)$ is the radius of curvature of the normal section as given by (*). The $\pm$ sign expresses the fact that within the process of transformation from (x^1, x^2, x^3) to (ξ, η, ζ) we lose (due to some square roots) track of the original sign.

We notice that for fixed P, the partial derivatives are constant so that we can write also.

$$\pm \frac{1}{R(\phi)} = A \cos^2\phi + 2C \sin\phi \cos\phi + B \sin^2\phi .$$

Expressing the angle ϕ via the coordinates ξ, η and the distance ρ



(see the diagram), we obtain

$\cos\phi = \xi/\rho$, $\sin\phi = \eta/\rho$.

Hence the above equation becomes

$$\pm \frac{1}{R(\phi)} = A \frac{\xi^2}{\rho^2} + 2C \frac{\xi\eta}{\rho^2} + B \frac{\eta^2}{\rho^2}$$

or

$$\pm \frac{\rho^2}{R(\phi)} = A\xi^2 + 2C\xi\eta + B\eta^2 .$$

Selecting here $\rho = \sqrt{R(\phi)}$, the radius-vector ρ evidently describes a curve of second order centered on P. This curve is known as <u>Dupin's indicatrix</u>. $\delta = \begin{vmatrix} A & C \\ C & B \end{vmatrix} \Rightarrow$ $\delta > 0$
($C = 0$ ellipse $A = B$ circle)
$\delta < 0$ hyperbola
$\delta = 0$ a pair of parallel lines

hence A, C, B are nothing else but L, M, N in the ξ, η local coordinate system on the surface (up to a scale factor).

The radius of curvature is extreme in two perpendicular directions. The normal sections for which the curvature is extreme are known as the <u>principal sections</u> and the radii of curvature are also called <u>principal</u> and denoted by R_1, R_2.

We can now rotate the ξ and η-axis so that ξ will coincide with the direction of the first principal section (eigenvector problem). Then the original equation (**) will become

$$\pm \frac{1}{R(\tilde{\phi})} = \tilde{A} \cos^2 \tilde{\phi} + \tilde{B} \sin^2 \tilde{\phi}.$$

The magnitudes of $\tilde{A}$ and $\tilde{B}$ can be evaluated from this equation considering first $\tilde{\phi} = 0$ or π :

$$\pm \frac{1}{R_1} = \tilde{A}$$

and then $\tilde{\phi} = \frac{\pi}{2}, 3\frac{\pi}{2}$:

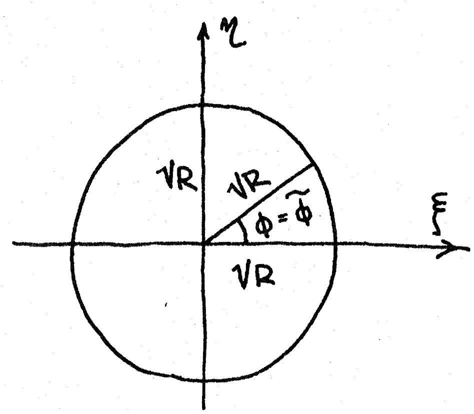
$$\pm \frac{1}{R_2} = \tilde{B}.$$

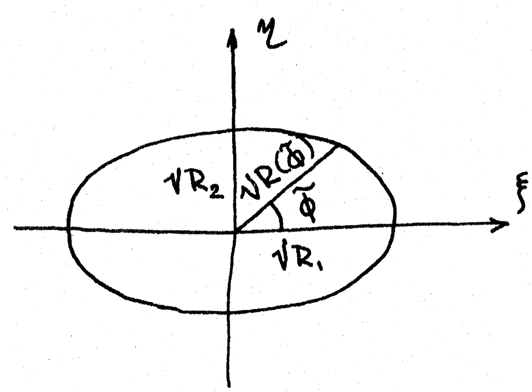
Substituting this back we obtain

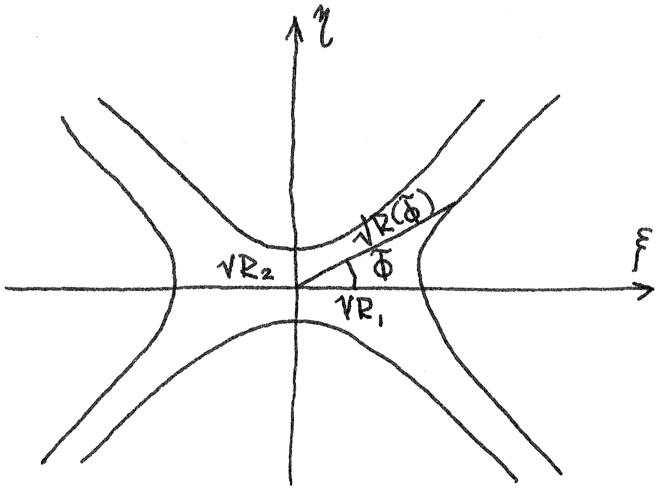
$$\frac{1}{R(\tilde{\phi})} = \pm \frac{\cos^2\tilde{\phi}}{R_1} \pm \frac{\sin^2\tilde{\phi}}{R_2}$$

the well known Euler's equation.

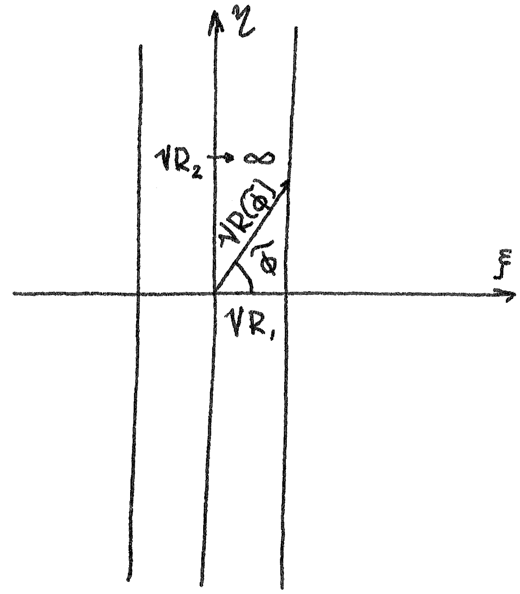
The signs $\pm$ tell us that the signs of R_1 and R_2 determine the character of the Dupin's indicatrix in its natural system of coordinates. Hence we get the four different cases as before as shown on the diagram:







η
ξ
√R₂
√R(Φ̃)
Φ̃
√R₁



η
ξ
√R₂ → ∞
√R(Φ̃)
Φ̃
√R₁

Recommended References for Further Reading

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