BRIEF OUTLINE OF THE MOLODENSKIJ THEORY

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BRIEF OUTLINE OF THE MOLODENSKIJ THEORY

(the determination of the figure of the earth)

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PREFACE

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1) Introduction

The classical approach in geodesy is based on the use of two different reference surfaces -- the ellipsoid of rotation and the geoid. To establish the relation between the two surfaces is one of the main tasks of geodesy. The, ages'-old problem of "determination of the figure of the earth", actually the mean earth ellipsoid, fitting in the best possible way to the geoid is another of the traditional tasks of geodesy.

There are two major hinderances in the classical concept of geodesy:

i) the definition of geoid is not completely rigorous -- i.e., the value of the potential on geoid is not known;

ii) the process of the removal of the effect of masses outside the geoid is based on various assumptions and hypotheses concerning the distribution of density within the earth.

The basic idea of Molodenskij is that he does not use (and therefore seek) the geoid at all. Instead, he uses the topographic surface of the earth as a reference surface. Hence he does not have to assume anything about the internal structure of the earth. For the purpose of heights, he defines a purely superficial surface, with no physical interpretation, that does not depart too far from the geoid though, and calls it quasigeoid.

The reason that Molodenskij does not use geoid, sets some scholars against his theory. In a way, the geoid is the most real and most concrete surface one can use.

2) Some integral formulae

The Gauss formula

$$\boxed{\oint\!\!\oint_s f_n dS = \iiint_V \operatorname{div} \vec{f}\, dV}$$

is well known from integral calculus. The meaning of the symbols is -s the closed surface enclosing the volume V, $f_n = \vec{f}\cdot\vec{n}$ where $\vec{n}$ is the normal unit vector towards the surface s and $\vec{f}$ is a vector field. The Gauss formula can be regarded as an "inversion" of the expression for divergance:

$$\operatorname{div} \vec{f} = \lim_{V\to 0} \frac{\oint\!\!\oint_s f_n \, dS}{V} .$$

Denoting by U the potential of $\vec{f}$, if it exists, we can write

$$f_n = \vec{f}\cdot\vec{n} = \nabla U\cdot\vec{n} = \frac{\partial U}{\partial n}$$

and

$$\operatorname{div} \vec{f} = \nabla\vec{f} = \Delta U.$$

Substituting these into the Gauss formula we get

$$\boxed{\oint\!\!\oint_s \frac{\partial U}{\partial n}\, dS = \iiint_V \Delta U dV}$$

which is known as the Gauss formula for potential.

Let us now have two vector fields $\vec{f},\vec{g}$. ($\vec{g}$ has nothing to do with gravity) that are given by following formulae:

$$\vec{f} = U\nabla W, \quad \vec{g} = W\nabla U$$

where U and W are two different scalar fields (functions of three variables) that have derivatives at least up to 2nd order. Then

$$\vec{f}\vec{n} = U\frac{\partial W}{\partial n}, \quad \vec{g}\vec{n} = W\frac{\partial U}{\partial n} .$$

On the other hand

$$\mathrm{div}\ \vec{f} = \nabla\vec{f} = \nabla(U\nabla W) = \nabla U\nabla W + U\Delta W$$

$$\mathrm{div}\ \vec{g} = \nabla\vec{g} = \nabla(W\nabla U) = \nabla W\nabla U + W\Delta U \ .$$

Applying the Gauss formula successively to $\vec{f}$ and $\vec{g}$ we get

$$\oiint_S U \frac{\partial W}{\partial n}\, dS = \iiint_V (\nabla U\nabla W + U\Delta W)\, dV$$

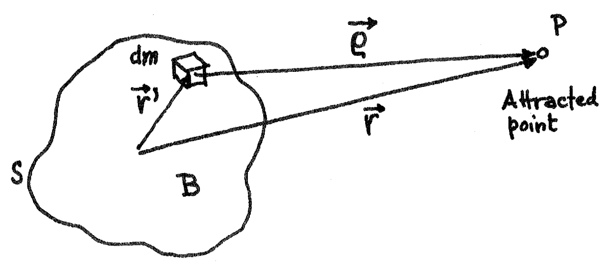
$$\oiint_S W \frac{\partial U}{\partial n}\, dS = \iiint_V (\nabla W\nabla U + W\Delta U)\, dV \ .$$

Substracting the second from the first formula yields:

$$\boxed{\oiint_S \left(U \frac{\partial W}{\partial n} - W \frac{\partial U}{\partial n}\right) dS = \iiint_V (U\Delta W - W\Delta U)\, dV}$$

that is known as <u>2nd Green's identity</u>.

Let us now have an attracting body B with volume and surface S. Let us take W in the 2nd Green's identity Equal to $1/\rho$. We get:



$$\oiint_S \left(U \frac{\partial}{\partial n}\, \frac{1}{\rho} - \frac{1}{\rho}\frac{\partial U}{\partial n}\right) dS = \iiint_V \left(U\Delta\frac{1}{\rho} - \frac{1}{\rho}\,\Delta U\right) dV.$$

It can be shown that

$$\iiint_V U\Delta\frac{1}{\rho}\, dV = \begin{cases} -4\pi U & \text{P outside V} \\ -2\pi U & \text{P on S} \\ 0 & \text{P inside V} \end{cases}$$

and we get

$$\boxed{\oiint_S \left(\frac{1}{\rho}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\frac{1}{\rho}\right) dS - \iiint_V \frac{1}{\rho}\,\Delta U\, dV = \begin{cases} +4\pi U & \text{P outside V} \\ +2\pi U & \text{P on S} \\ 0 & \text{P inside V.} \end{cases}}$$

This formula is usually known as the <u>third Green's identity</u> and can be regarded as the closed form solution to the third boundary-value problem for Poissons equation.

This would be the solution we would get when applying the Green's approach to the Poissons equation (with third boundary-values).

3) One application of 3rd Green's identity, Molodenskij equation

Let us consider the earth now, V being the volume of the earth, S the surface of the earth and W its potential. Then we get, for a point on the surface of the earth:

$$\iiint_V \frac{1}{\rho} \Delta W dV + 2\pi W + \oiint_S \left(W \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{1}{\rho}\frac{\partial W}{\partial n}\right) dS = 0 .$$

We know that for the potential of the earth the Poisson's equation

$$\Delta W = - 4\pi\kappa\sigma + 2\omega^2$$

holds inside the earth (V) and $\frac{\partial W}{\partial n} = g_n$ is the component of the actual gravity $\vec{g}$ in the normal to the earth surface. Hence we obtain:

$$\iiint_V \frac{1}{\rho} (-4\pi\kappa\sigma + 2\omega^2) dV + 2\pi W + \oiint_S \left(W \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{g_n}{\rho}\right) dS = 0 . \qquad (*)$$

On the other hand, we know that

$$W = \kappa \iiint_V \frac{\sigma}{\rho} dV + \frac{1}{2}\omega^2 r''^2$$

(see Physical Geodesy I, § 2.9) so that the first term in (*) can be written as

$$\iiint_V \frac{1}{\rho} (-4\pi\kappa\sigma) dV = - 4\pi\kappa \iiint_V \frac{\sigma}{\rho} dV = - 4\pi \left(W - \frac{1}{2}\omega^2 r''^2\right).$$

Hence the equation (*) can be rewritten as follows:

$$\boxed{-2\pi W + 2\pi\omega^2 r''^2 + 2\omega^2 \iiint_V \frac{dV}{\rho} + \oiint_S \left(W \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{g_n}{\rho}\right) dS = 0 .}$$

This equation bears the name of Molodenskij and may be considered as the origin of the whole theory. We may note that it relates the actual potential W to the surface S of the earth and contains (if we disregard the third, volume term) only the surface quantities. We shall see, how does Molodenskij get rid of the volume term in the forthcoming §.

4) <u>Molodenskij equation for disturbing potential</u>

In order to get rid of the volume term, let us apply the 3rd Greens identity to the normal potential U as we know it from classical geodesy. We end up with Molodenskij equation for U in following form:

$$-2\pi U + 2\pi\omega^2 r''^2 + 2\omega^2 \iiint_V \frac{dV}{\rho} + \oiint_S \left(W \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{\gamma_n}{\rho}\right) dS = 0 \ .$$

Subtracting this from the equation for the actual potential we get:

$$-2\pi (W-U) + \oiint_S \left(W \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{g_n}{\rho}\right) dS - \oiint_S \left(U \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{\gamma_n}{\rho}\right) dS = 0 \ .$$

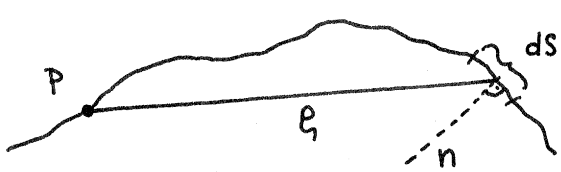
Here, according to definition, W-U=T, the disturbing potential, $g_n = \frac{\partial V}{\partial n}$, $\gamma_n = \frac{\partial U}{\partial n}$ and we get:

$$-\ 2\pi T + \oiint_S \left((W-U) \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{1}{\rho}\frac{\partial}{\partial n}(W-U)\right) dS = 0$$

or

$$\boxed{-2\pi T + \oiint_S \left(T \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{1}{\rho}\frac{\partial T}{\partial n}\right) dS = 0 \ .}$$

This is the Molodenskij equation for disturbing potential and is valid <u>on the surface</u> of the earth. Hence ρ is the distance between the point of interest P and the running point (dummy point in the integation) on the surface of the earth.

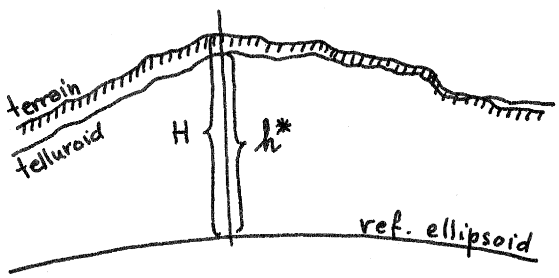


We can see that so far the development is precise -- there are no approximations and no hypotheses involved. The equation relates the disturbing potential T solely to surface elements $\frac{\partial}{\partial n}\frac{1}{\rho}$, $\frac{\partial T}{\partial n}$ and the surface S itself.

In order to be able to use the elements that are observable on the surface of the earth surface of the earth, we have to introduce an intermediate surface, telluroid.

5) Telluroid, quasigeoid

The telluroid (tellurian means earthly, terrestrial) is defined as a locus of normal heights h^* measured along the normal to the reference ellipsoid from the reference ellipsoid. For definition of normal heights see for instance Physical Geodesy II, § 6.5.



The difference between the ellipsoidal (geodetic) height H and the normal height h^* is called height anomaly and denoted by ζ. Is hence:

$$\zeta = H - h^* .$$

Recalling the definition of geoidal undulation N:

$$N = H - h,$$

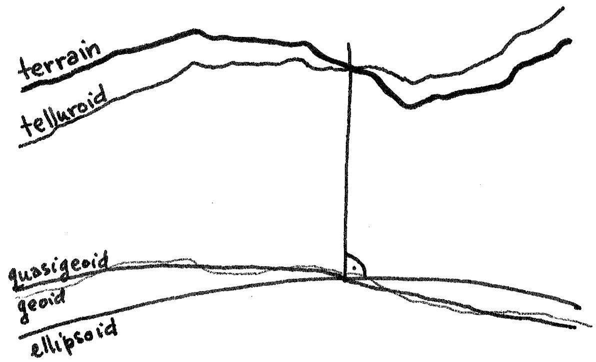
where h is the orthometric height, we can see the similarity between the two quantities. However, their deployment is very different and they serve two very different purposes.

Numerically, the two quantities ζ and N are very close. Combining the two above equations we see that

$$N - \zeta = h^* - h$$

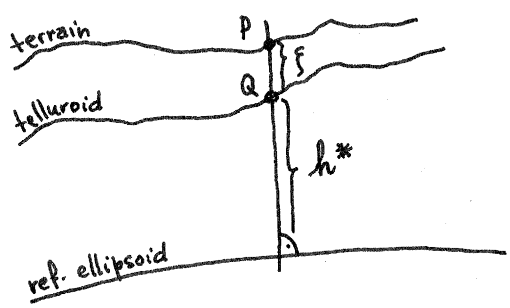
which is a fairly small quantity. It usually does not exceed 0.1 m and probably nowhere in the world surpasses a few meters. For instance, the value of $N - \zeta$ for Mt. Blanc is about -1.8 m. The telluroid does not depart from the surface of the earth by more than at most a hundred meters, i.e., roughly by as much as the geoid departs from the mean earth ellipsoid.

The height anomaly can be -- and very often is -- interpreted as height above the reference ellipsoid. The locus of such interpreted height anomalies is a surface known as quasigeoid. Since quasigeoid deviates from the geoid again by $N - \zeta$, the two surfaces are close together. The quasigeoid may represent the geoid with a precision better than is our today knowledge of the absolute geoid.



We may note that when defining the telluroid or quasigeoid we do not have to postulate any hypotheses. Both surfaces are purely conventional (mathematical as opposed to physical in case of the geoid) and are not even meant to represent any physical properties of the earth. Molodenskij was not, however, the first geodesist to advocate the use of a conventional surface. De Graaf-Hunter and Jeffreys had already been proposing the use of a quasigeoid before the Molodenskij theory was first published.

6) <u>Gravity anomaly and Bruns' formula in Molodenskij development</u>



Gravity anomaly is defined as

$$\Delta g = g_P - \gamma_Q$$

and refers to the terrain (or the telluroid). For comparison, let us recall the definition of gravity anomaly in classical geodesy: $\Delta g = g_P - \gamma_Q$ where P is a point on the geoid and Q on the ellipsoid. Hence in classical geodesy Δg refers to either the geoid or the ellipsoid, depending on the view we take.

The normal gravity in Molodenskij development refers to the telluroid and is obtained from the normal gravity γ_o on the ellipsoid by applying the negative free air correction computed for the normal height. For the free air reduction we may consider either linear or a more precise formula.

The Bruns' formula, as developed in Physical Geodesy I, § 3.11, can be derived in exactly the same way here (for ζ instead of N) and we get

$$\zeta = T/\gamma$$

where ζ is related either to P or Q, γ refers to the telluroid (Q) and the disturbing potential $T = W - U$ is taken for the point P on the terrain. Hence we can work with the disturbing potential on the surface of the earth.

7) Reformulation of the Molodenskij equation for disturbing potential

Getting back to the Molodenskij equation developed in 1.4, we can now introduce the first from the series of approximations. Because the telluroid is close to the actual surface of the earth we may evaluate the integral equation for T derived in § 4 on telluroid rather than on the earth surface. It is an approximation of the same order as the one used in classical geodesy, where we evaluate the Laplace's equation on the geoid instead of ellipsoid. As we have seen in Physical Geodesy II, § 4.4, in the classical approach we try to remove the effect of the masses above the geoid (instead of ellipsoid) which indicates that the boundary-value problem is solved for geoid. The order of approximation is roughly the same because the telluroid departs from the actual surface of the earth by approximately as much as the geoid departs from the ellipsoid.

Hence denoting the telluroid by Σ we can rewrite our integral equation as

$$T - \frac{1}{2\pi} \oint\!\!\!\oint_{\Sigma} \left(T \frac{\partial}{\partial n} \frac{1}{\rho} - \frac{1}{\rho} \frac{\partial T}{\partial n}\right) d\Sigma \simeq 0.$$

Providing we knew the telluroid and how to solve this integral equation we should be able to solve for the T if it was not for the second term under the integration sign. $\frac{\partial T}{\partial n}$ is the troublesome part of the equation. One thing we have to bear in mind is that n represents now the normal to the telluroid Σ . This normal has now replaced the normal to the topographic surface and departs from it by a quantity comparable to the classical deflection of vertical on the surface.

Molodenskij has derived the following approximate expression for $\frac{\partial T}{\partial n}$, the proof of which we are not going to give here:

$$\frac{\partial T}{\partial n} \simeq (-\Delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T + \gamma(\xi \text{ tg } \beta_1 + \eta \text{ tg } \beta_2)) \cos \beta$$

where Δg refers to the telluroid (or the terrain), γ and $\frac{\partial \gamma}{\partial n}$ refer to the telluroid, ξ, η are the components of the deflection of vertical on the telluroid (or, with a high degree of precision on the terrain as well), β_1, β_2 are the inclinations of the N-S and E-W telluroid profile (or, with very high precision these of the terrain) and finally β is the maximum inclination of the telluroid (terrain) at the point of interest. Substituting this result back into our integral equation, we obtain

$$T - \frac{1}{2\pi} \oiint_\Sigma (T \frac{\partial}{\partial n}\frac{1}{\rho} - \frac{\cos\beta}{\rho}(-\Delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T + \gamma(\xi \text{tg}\beta_1 + \eta \text{tg}\beta_2)))d\Sigma \simeq 0.$$

This can be reformulated as follows:

$$\boxed{T - \frac{1}{2\pi} \oiint_\Sigma (\frac{\partial}{\partial n}\frac{1}{\rho} - \frac{\cos\beta}{\rho\gamma}\frac{\partial \gamma}{\partial h}) T \, d\Sigma \simeq \frac{1}{2\pi} \oiint_\Sigma [\frac{\cos\beta}{\rho} \Delta g - \frac{\gamma \cos\beta}{\rho}(\xi \text{tg}\beta_1 + \eta \text{tg}\beta_2)] \, d\Sigma .}$$

In this equation everything is known apart from T. It can therefore be solved by the methods for solving the integral equations which are going to be shown in the appendix. We may notice, however, that the equation is rather cumbersome since it contains all the quantities ξ, η, β, β_1, β_2 that have to be known all over the earth globe. In the next paragraphs we shall see one of the tricks that can be used to avoid the evaluation of the right-hand side of our integral equation.

8) <u>Potential of a surface layer</u>

As we have said at the end of the last paragraph even when we know now how to solve the integral equation developed in 7) we want to get rid of the terms containing the quantities ξ, η, β_1, β_2. One way how to do it is through introducing a new physical quantity, <u>the potential of a surface layer</u>.

Suppose we have an enclosed shell ϕ of a finite thickness with density $\tilde{\sigma}$. The attracting potential of the shell will be given by (see Physical Geodesy I, § 2.8):

$$V = \kappa \iiint_{\phi} \frac{\tilde{\sigma}}{\rho} \, d\phi$$

where the meaning of κ and ρ remains the same as in §§ 2, 3). If we consider the shell to be infinitely thin the above equation will become

$$V = \kappa \oiint_{\phi} \frac{\sigma}{\rho} \, d\phi$$

where σ is a function of $\tilde{\sigma}$. Contrary to our intuitive physical expectation the "density" of the infinitely thin shell -- the surface layer -- does not grow beyond all limits. It can be understood as a consequence of the change of its physical meaning (if one can ever speak about the physical meaning of the density of a surface layer). The physical units of $\tilde{\sigma}$ are g cm^{-3} while σ is expressed in g cm^{-3} · cm = g cm^{-2}.

It can be shown that an attracting potential of a body B outside the body can always be expressed as a potential of a surface layer. In particular the surface layer may be placed on the surface S of the body B. On the other hand to each potential V of a surface layer S with density σ there exists at least one potential originated by the body B (enwrapped by the surface) with the distribution of density σ'. We can write

$$V = \kappa \oiint_S \frac{\sigma}{\rho} \, dS = \kappa \iiint_B \frac{\sigma'}{\rho} \, dB \ .$$

The problem of finding σ' in $\mathbb{B}$ that produces the same potential as σ on S is not unique, while given σ' in B, the σ on S can be determined and is unique.

9) Expression of disturbing potential by means of the potential of a surface layer, integral equation for surface layer density

Let us express the disturbing potential on the telluroid Σ in terms of the potential of a surface layer. We can write:

$$T = \kappa \oiint_\Sigma \frac{\sigma}{\rho}\, d\Sigma = \oiint_\Sigma \frac{\phi}{\rho}\, d\Sigma$$

where $\phi = \kappa\sigma$ is an as yet unknown function, determined uniquely by T.

On the other hand, by reasoning parallel to the one used in Physical Geodesy I, § 3.12, we can develop the "gravimetric equation", at the first view identical with the classical version:

$$\frac{\partial T}{\partial h} \simeq -\Delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T .$$

Here, however, the individual quantities are related to the telluroid the same way they are related to the ellipsoid in the classical geodesy. Note the difference between $\frac{\partial T}{\partial h}$ and $\frac{\partial T}{\partial n}$ in § 7.

Since we want to express the disturbing potential through the pseudo-density ϕ we have to derive the expression for $\partial T/\partial h$ first. We get

$$\frac{\partial T}{\partial h} = \frac{\partial}{\partial h} \oiint_\Sigma \frac{\phi}{\rho}\, d\Sigma = \oiint_\Sigma \frac{1}{\rho}\frac{\partial \phi}{\partial h}\, d\Sigma + \oiint_\Sigma \phi \frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) d\Sigma .$$

The first term on the right-hand side can be shown to be dependent on the position of our point of interest P (remember ρ is the distance between P and the dummy point in the integration) in the following way:

$$\oint\!\!\oint_\Sigma \frac{1}{\rho}\frac{\partial \phi}{\partial h}\, d\Sigma = \begin{cases} 0 & P \notin \Sigma \\ -2\pi\phi\cos\beta & P \in \Sigma \quad \text{outer side} \\ +2\pi\phi\cos\beta & P \in \Sigma \quad \text{inner side.} \end{cases}$$

Here β is the maximum inclination of the tellurooid at P or, which is the same, the angle between the normals to Σ and to the ellipsoid. Hence on the outer side of the telluroid we obtain

$$\frac{\partial T}{\partial h} = -\,2\pi\phi\cos\beta + \oint\!\!\oint_\Sigma \phi\,\frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) d\Sigma\ .$$

Substituting now for T and $\frac{\partial T}{\partial h}$ into the gravimetric equation we get:

$$-2\pi\phi\cos\beta + \oint\!\!\oint_\Sigma \phi\,\frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) d\Sigma \simeq -\Delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}\oint\!\!\oint_\Sigma \frac{\phi}{\rho}\, d\Sigma$$

This is usually written as

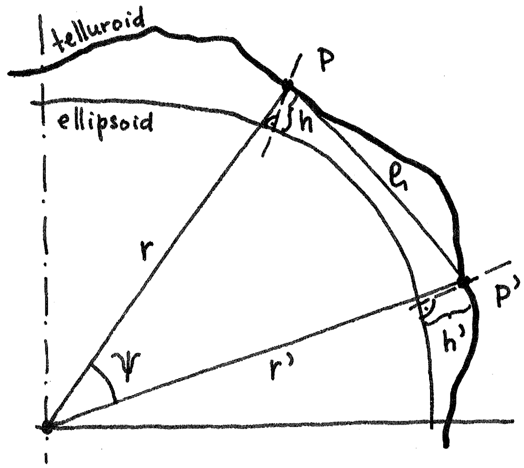
$$\boxed{\phi - \frac{1}{2\pi\cos\beta}\oint\!\!\oint_\Sigma \left(\frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) - \frac{1}{\gamma\rho}\frac{\partial\gamma}{\partial h}\right)\phi\, d\Sigma \simeq \frac{\Delta g}{2\pi\cos\beta}}$$

Note that h in the subintegral function is related to the point of interest. We classify the equation easily as Fredholm's. Hence, it can be solved by iterations. Note that ξ and η have been eliminated while β still persists.

Once ϕ is found from the above equation, T can be computed from the first equation in this paragraph and then ζ is given by Bruns' formula.

10) Simplification of the integral equation for the surface layer density

Even the integral equation for the surface layer density is still too complicated to be solved efficiently. We thus try to simplify it by introducing more approximations.



First of all we can write (see Physical Geodesy I, § 2.22):

$\rho = \sqrt{(r^2 + r'^2 - 2\,rr'\cos\psi)}$.

Since the direction of the ellipsoidal normal (or the plumb-line) departs from the direction of the radius vector by at most a few tens of minutes, we can approximate $\frac{\partial}{\partial h}$ by $\frac{\partial}{\partial r}$. We obtain:

$$\frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) \simeq \frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{\partial\rho}{\partial r}\,.$$

Since $\frac{\partial\rho^2}{\partial r} = 2\rho\,\frac{\partial\rho}{\partial r}$ we can write

$$\frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) \simeq -\frac{1}{\rho^2}\frac{1}{2\rho}\frac{\partial\rho^2}{\partial r} = -\frac{1}{2\rho^3}(2r - 2r'\cos\psi)$$

$$\simeq -\frac{r - r'\cos\psi}{\rho^3}\,.$$

For $\frac{\partial\gamma}{\partial h}$ on the ellipsoid we found in Physical Geodesy I, § 3.14:

$$\frac{\partial\gamma}{\partial h} \simeq -\frac{2\gamma}{a}(1 + m + f\cos 2\phi)$$

where a is the semimajor axis of the reference ellipsoid, f its flattening and $m = a^2 b_\omega{}^2/(\kappa M)$ is also the function of the ellipsoid. We can hence approximate

$$\frac{1}{\rho\gamma}\frac{\partial\gamma}{\partial h} \simeq \frac{-2}{\rho\gamma}\frac{\gamma}{a+h}(1 + m + f\cos 2\phi)\,.$$

This, second part of the kernel of our integral equation, is related (apart from ρ) to the point of interest. Hence for the integration purpose can be written as const./ ρ In any case it can be further approximated as

$$\frac{1}{\rho\gamma}\frac{\partial\gamma}{\partial h} \simeq -\frac{2}{\rho r}\,.$$

The kernel of the integral equation may then be expressed as

$$K(P,P') = \frac{\partial}{\partial h}\left(\frac{1}{\rho}\right) - \frac{1}{\rho\gamma}\frac{\partial\gamma}{\partial h} \simeq \frac{2}{\rho r} - \frac{r - r'\cos\psi}{\rho^3} .$$

Here we can get rid of ψ using following trick. We know that

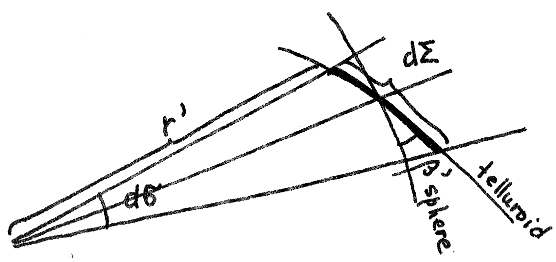
$$\rho^2 = r^2 + r'^2 - 2rr'\cos\psi .$$

Hence

$$r - r'\cos\psi = (\rho^2 - r'^2 + r^2)/(2r)$$

and the kernel becomes

$$\boxed{K(P,P') = \frac{2}{\rho\, r} - \frac{\rho^2 + r^2 - r'^2}{2r\rho^3} = \frac{3}{2r\rho} - \frac{r^2 - r'^2}{2r\rho^3} .}$$



Further, according to the diagram,

$$d\Sigma\, \cos\beta' = r'^2 d\sigma$$

where σ is the spatial angle (not to be confused with density!).

Substituting these two results back into our integral equation we get:

$$2\pi\phi\cos\beta - \oiint_\sigma \left(\frac{3}{2\rho} - \frac{r^2 - r'^2}{2\rho^3}\right)\frac{r'^2}{r}\,\frac{\phi}{\cos\beta'}\, d\sigma = \Delta g .$$

Since here r and r' are involved only in the "relative" sense, i.e., as a difference of squares and a ratio (multiplied by r'), we commit an error of the order 10^{-3} by approximating:

$$r \simeq R + h, \quad r' \simeq R + h', \quad R = \sqrt[3]{(a^2 b)} .$$

The approximation is as good as the approximation $\frac{\partial}{\partial h} \sim \frac{\partial}{\partial r}$ was.

Hence

$$\frac{r'^2}{r} \simeq \frac{(R+h')^2}{R+h} \simeq \frac{R^2+2Rh'}{R} \left(1 - \frac{h}{R}\right) \simeq R + (2h' - h)$$

and

$$r^2 - r'^2 \simeq (R^2 + 2Rh) - (R^2 - 2Rh') = 2R\,(h - h') \; .$$

Obviously even the term $2h'-h$ in the first equation can be neglected and we end up with

$$2\pi\phi \cos\beta - \oint\!\!\oint_\sigma \left(\frac{3R}{2\rho} + \frac{2R^2(h'-h)}{2\rho^3} \right) \tilde{\phi}\, d\sigma = \Delta g$$

after having denoted $\phi/\cos\beta'$ by $\tilde{\phi}$.

Now the spatial distance of P and P', ρ ,can be expressed in terms of R, h, h'. We can write

$$\rho^2 = r^2 + r'^2 - 2rr' \cos\psi$$

$$= R^2 + 2Rh + R^2 + 2Rh' + h^2 + h'^2 - 2(R+h)(R+h') \cos\psi$$

$$= 2R^2 + 2R\,(h+h') + h^2 + h'^2 - 2(R^2+R(h+h') + hh') \cos\psi$$

$$=(2R^2+2R(h+h'))(1-\cos\psi) + h^2 + h'^2 - 2hh' \cos\psi$$

$$= (2R^2+2R(h+h') + 2hh')(1-\cos\psi) + h^2 + h'^2 - 2hh'$$

$$= 2R^2 \left(1 + \frac{h+h'}{R} + \frac{hh'}{R^2} \right) (1-\cos\psi) + (h-h')^2$$

$$\simeq 4R^2 \sin^2\frac{\psi}{2} + (h-h')^2.$$

Denoting $4R^2 \sin^2\frac{\psi}{2} = \rho_o^2$ we can write finally:

$$\rho^2 = \rho_o^2 + (h-h')^2$$

and

$$\rho = \rho_o \sqrt{(1 + \frac{(h-h')^2}{\rho_o^2})} \ .$$

We obtain for the kernel of our integral equation:

$$K(P\ P') \simeq \frac{3R\rho^2 + 2R^2(h'-h)}{2\rho^3} \quad \frac{3R\rho_o^2 + 3R(h'-h)^2 + 2R^2(h'-h)}{2\rho_o^3 \ (\ 1 + \frac{(h'-h)^2}{\rho_o^2} \ )^{3/2}}$$

$$\simeq \frac{R}{2} \ [(\ \frac{3}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} + \frac{3(h'-h)^2}{\rho_o^3} \ ) \ (1 - \frac{3}{2} \ \frac{(h'-h)^2}{\rho_o^2} \ )]$$

$$\simeq \frac{R}{2} \ (\ \frac{3}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} - \frac{3(h'-h)^2}{2\rho_o^3} + \ldots \ ) \ .$$

We can thus write:

$$\boxed{\phi - \frac{R}{4\pi\cos\beta} \oiint_\sigma \ (\ \frac{3}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} - \frac{3(h'-h)^2}{2\rho_o^3} \ )\tilde{\phi} \ d\sigma \simeq \Delta g/(2\pi\cos\beta) \ .}$$

11) <u>Iterative solution of the Molodenskij equation for the surface layer density</u>

The equation developed in § 10 can now be finally solved. To do so let us rewrite it as

$$\tilde{\phi} - \frac{R}{4\pi\cos^2\beta} \oiint_\sigma \ (\frac{3}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} - \frac{3(h'-h)^2}{2\rho_o^3} \ ) \ \tilde{\phi} \ d\sigma \simeq \frac{\Delta g}{2\pi\cos^2\beta} \ .$$

Here $\frac{3}{\rho_o} >> \frac{2R(h'-h)}{\rho_o^3} >> \frac{3(h'-h)^2}{2\rho_o^3}$. Hence in the zero approximation the two last terms of the kernel may be neglected. Neglecting $\cos^2\beta$, which can be done for a flat terrain around the point of interest we can write the zero approximation as follows:

$$\tilde{\phi}_o = \frac{\Delta g}{2\pi} + \frac{3R}{4\pi} \oiint_\sigma \frac{\tilde{\phi}_o}{\rho_o} \, d\sigma \ .$$

Molodenskij has shown that the surface integral can be expressed in terms of our known Stokes' integral (see Physical Geodesy I, § 3.16)

$$\oiint_\sigma \frac{\tilde{\phi}_o}{\rho_o} \, d\sigma = \frac{1}{4\pi R} \oiint_\sigma \Delta g \; S(\psi) \, d\sigma$$

where Δg is the surface anomaly. Hence, the zero approximation is given by

$$\boxed{\tilde{\phi}_o = \frac{\Delta g}{2\pi} + \frac{3}{16\pi^2} \oiint_\sigma \Delta g \; S(\psi) \, d\sigma \ .}$$

For the first approximation, we take into consideration the linear term $2R(h'-h)/\rho_o^3$ and obtain

$$\tilde{\phi}_1 = \tilde{\phi}_2 + \delta\tilde{\phi}_1 = \frac{\Delta g}{2\pi} + \frac{R}{4\pi} \oiint_\sigma \left(\frac{3}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} \right) \tilde{\phi}_1 \, d\sigma.$$

The subintegral function can be rewritten as

$$\left(\frac{3}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} \right) (\tilde{\phi}_o + \delta\tilde{\phi}_1) \approx \frac{3\tilde{\phi}_o}{\rho_o} + \frac{2R(h'-h)}{\rho_o^3} \tilde{\phi}_o + \frac{3\sigma\tilde{\phi}_1}{\rho_o} \quad ,$$

neglecting the product of the two correcting terms. Thus the above integral equation becomes

$$\tilde{\phi}_o + \delta\tilde{\phi}_1 \simeq \frac{\Delta g}{2\pi} + \frac{R}{4\pi} \left(3 \oiint_\sigma \frac{\tilde{\phi}_o}{\rho_o} \, d\sigma + 2R \oiint_\sigma \frac{(h'-h)}{\rho_o} \tilde{\phi}_o \, d\sigma + 3 \oiint_\sigma \frac{\delta\tilde{\phi}_1}{\rho_o} \, d\sigma \right) .$$

Subtracting the zero approximation from this formula we finally end up with the expression for $\delta\tilde{\phi}_1$:

$$\delta\tilde{\phi}_1 \simeq \frac{R^2}{2\pi} \oint\!\!\oint_\sigma \frac{(h'-h)}{\rho o} \tilde{\phi}_o d\sigma + \frac{3R}{4\pi} \oint\!\!\oint_\sigma \frac{\delta\tilde{\phi}_1}{\rho_o} d\sigma$$

which is again an integral equation for $\delta\tilde{\phi}_1$. We may notice that the first term on the right-hand side is a constant, from the solution point of view. All the quantities there can be considered as known and we may denote

$$R^2 \oint\!\!\oint_\sigma \frac{h'-h}{\rho_o} \tilde{\phi}_o \, d\sigma = G_1 .$$

Hence the integral equation becomes

$$\delta\tilde{\phi}_1 \simeq \frac{G_1}{2\pi} + \frac{3R}{4\pi} \oint\!\!\oint_\sigma \frac{\delta\tilde{\phi}_1}{\rho_o} d\sigma .$$

It is easily seen that the final integral equation has exactly the same form as the integral equation for the zero approximation. The only difference is that here G_1 replaces Δg and $\delta\tilde{\phi}_1$ replaces $\tilde{\phi}_o$. Therefore the solution is again given by the Stokes integral and we can write:

$$\boxed{\delta\tilde{\phi}_1 \simeq \frac{G_1}{2\pi} + \frac{3}{16\pi^2} \oint\!\!\oint_\sigma G_1 \; S(\psi) \; d\sigma .}$$

The first approximation can then be written as

$$\boxed{\tilde{\phi}_1 \simeq \frac{\Delta g + G_1}{2\pi} + \frac{3}{16\pi^2} \oint\!\!\oint_\sigma (\Delta g + G_1) \; S(\psi) \; d\sigma}$$

where G_1 as well as Δg is a function of the point of interest, in the first term, or of the dummy point in the subintegral function.

This way, we could go on adding more terms if we wanted to make the solution more precise. The first approximation is, however, good enough for practical evaluation of the telluroid in flat areas. This iterative solution is less precise for mountaineous regions due to the negligence of the term $\cos^2\beta$. Treatment of the Molodenskij integral equation for the surface density layer in mountaineous areas remains still an open question.

12) Evaluation of the height anomalies

We have learnt in §§ 6 and 9 that the height anomaly ζ is related to the surface layer density ϕ by following formula:

$$\zeta = \frac{1}{\gamma} \oiint_\Sigma \frac{\phi}{\rho} \, d\Sigma$$

where the integration is carried out over the telluroid. We can again reformulate the integral, as we have done in § 10:

$$\oiint_\Sigma \frac{\phi}{\rho} \, d\Sigma = \oiint_\sigma \frac{\phi}{\rho} \frac{r'^2}{\cos \beta'} \, d\sigma \ .$$

Substituting $\tilde{\phi} \simeq (\tilde{\phi}_o + \delta\tilde{\phi}_1)$ for $\phi/\cos \beta$, R^2 for r'^2 and ρ_o for ρ we obtain:

$$\zeta \simeq \frac{R^2}{\gamma} \left[\oiint_\sigma \frac{\tilde{\phi}_o}{\rho_o} \, d\sigma + \oiint_\sigma \frac{\delta\tilde{\phi}_1}{\rho_o} \, d\sigma \right]$$

$$\boxed{= \frac{R}{4\pi\gamma} \left[\oiint_\sigma \Delta g \ S(\psi) \, d\sigma + \oiint_\sigma G_1 \ S(\psi) \, d\sigma \right] = \zeta_o + \delta\zeta_1 \ .}$$

The troublesome quantity here is G_1. It has been defined in § 11 as $R^2 \oiint_\sigma \frac{h'-h}{\rho_o^3} \tilde{\phi}_o \, d\sigma$. Let us see now whether it is possible to express it in terms of ζ_o, rather than $\tilde{\phi}_o$, to avoid the computation of the density of the surface layer altogether. We know that

$$\tilde{\phi}_o = \frac{\Delta g}{2\pi} + \frac{3}{16\pi^2} \oiint_\sigma \Delta g \ S(\psi) \, d\sigma \ .$$

On the other hand

$$\zeta_o = \frac{R}{4\pi\gamma} \oint\!\!\oint_\sigma \Delta g \; S(\psi) \; d\sigma \; .$$

(Note that ζ_o is given by the same formula -- the Stokes' formula -- as N in the classical geodesy (see Physical Geodesy I, § 3.17). The only difference is that here we are dealing with surface gravity anomalies as opposed to geoidal gravity anomalies in the classical approach). Hence $\tilde{\phi}_o$ and ζo are related by following equation

$$\tilde{\phi}_o = \frac{\Delta g}{2\pi} + \frac{3\gamma}{4\pi R} \zeta_o$$

and we can write for G_1:

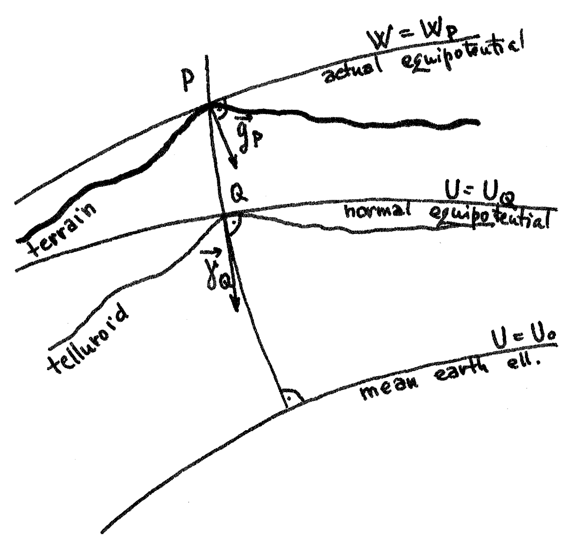
$$\boxed{G_1 = \frac{R^2}{2\pi} \oint\!\!\oint_\sigma \frac{h'-h}{\rho_o^3} \left(\Delta g + \frac{3\gamma}{2R} \zeta_o\right) d\sigma \; .}$$

$\delta\zeta_1$, as given by

$$\delta\zeta_1 = \frac{R}{4\pi\gamma} \oint\!\!\oint_\sigma G_1 \; S(\psi) \; d\sigma$$

is usually quite small as compared to ζ_o. It is therefore often neglected altogether leaving us again with the Stokes' formula alone.

13) Deflections of the vertical in Molodenskij theory

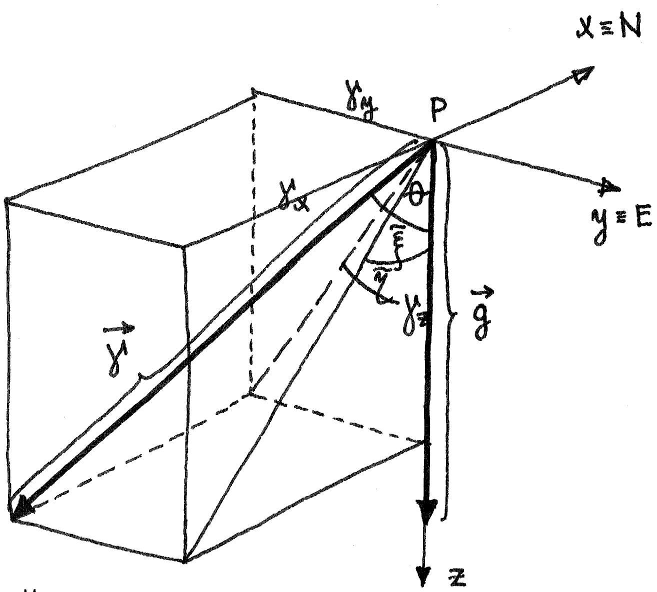


The deflection of the vertical in Molodenskij theory is defined (in much the same way as this of the classical geodesy) as the angle between the actual gravity vector g_P at the surface and the normal gravity vector γ_Q at the telluroid. For all practical purposes it could be expressed also as the angle between the actual gravity vector g_P and the normal gravity vector γ_P, both at the surface. These two definitions vary by at most a few hundreds of second of arc. They differ by the term due to the curvature of the normal plumb-line between the telluroid and the surface.

It can be shown that the components of the Molodenskij deflection, $\tilde{\xi},\tilde{\eta}$ are given by

$$\tilde{\xi} \doteq - \frac{1}{R}\frac{\partial\zeta}{\partial\phi}\Bigg|_{W=\text{const.}} \qquad \tilde{\eta} \doteq - \frac{1}{R\cos\phi}\frac{\partial\zeta}{\partial\lambda}\Bigg|_{W=\text{const.}} \quad .$$

To show this, let us take a look on the diagram. We can write:



$$\text{tg}\,\tilde{\xi} \simeq \gamma_x/\gamma_z, \quad \text{tg}\tilde{\eta} = \gamma_y/\gamma_z .$$

But $\gamma_x = \frac{\partial U}{\partial x}$, $\gamma_y = \frac{\partial U}{\partial y}$, $\gamma_z = \frac{\partial U}{\partial z}$.

Recalling that $U = W - T$ we get

$$\gamma_x = - \frac{\partial T}{\partial x} , \quad \gamma_y = - \frac{\partial T}{\partial y} , \quad \gamma_z = g - \frac{\partial T}{\partial z}$$

because in our coordinate system (xy plane tangent to the equipotential W = const.):

$$\frac{\partial W}{\partial x} = 0 , \quad \frac{\partial W}{\partial y} = 0 , \quad \frac{\partial W}{\partial z} = g .$$

Hence

$$\text{tg}\,\tilde{\xi} \simeq \tilde{\xi} \simeq - \frac{\partial T}{\partial x} / \left(g - \frac{\partial T}{\partial z}\right) , \quad \text{tg}\tilde{\eta} \simeq \tilde{\eta} \simeq - \frac{\partial T}{\partial y} / \left(g - \frac{\partial T}{\partial z}\right)$$

Here $\frac{\partial T}{\partial z} \simeq \frac{\partial T}{\partial h} \simeq -\Delta g + \zeta \frac{\partial \gamma}{\partial h} << g$ and can be neglected. Further, recalling the Bruns' formula: $T = \zeta\gamma$. Therefore

$$\frac{\partial T}{\partial x} = \frac{\partial \zeta}{\partial x}\gamma + \zeta\frac{\partial\gamma}{\partial x} , \quad \frac{\partial T}{\partial y} = \frac{\partial \zeta}{\partial y}\gamma + \zeta\frac{\partial\gamma}{\partial y} .$$

The second terms here are much smaller than the first and can again be neglected. Substituting $Rd\phi$ for dx and $R\cos\phi d\lambda$ for $d\lambda$ we finally obtain

$$\tilde{\xi} \simeq = \frac{\gamma}{g}\,\frac{1}{R}\,\frac{\partial\zeta}{\partial\phi}\bigg|_{W=\text{const.}} \simeq \frac{-1}{R}\,\frac{\partial\zeta}{\partial\phi}\bigg|_{W=\text{const.}} , \qquad \tilde{\eta} \simeq \frac{-1}{R\cos\phi}\,\frac{\partial\zeta}{\partial\lambda}\bigg|_{W=\text{const.}}$$

Formally, these formulae are completely analogous to the one used in classical geodesy, for the geoidal undulation:

$$\xi = -\frac{1}{R}\,\frac{\partial N}{\partial\phi}\bigg|_{W=\text{const.}} , \qquad \eta = -\frac{1}{R\cos\phi}\,\frac{\partial N}{\partial\lambda}\bigg|_{W\,=\,\text{const.}}$$

The only difference is obviously that here we deal with the height anomalies rather than geoidal undulations.

On the other hand there is a profound difference in evaluating the partial derivatives of ζ. In classical geodesy, N is a function of ϕ,λ defined along the geoid, i.e., the surface W = const. (see Physical Geodesy I, § 3.10). Here ζ is a function of ϕ,λ defined along the physical surface of the earth. Expressing this mathematically we can write N as a function of three arguments

$$N = N\ (\phi,\lambda,\ W=\text{const.})$$

while ζ would be given by

$$\zeta = \zeta(\phi,\lambda,\ W(\phi,\lambda)) .$$

Differentiation of ζ along the physical surface with respect to, say ϕ, then yields

$$\frac{\partial\zeta}{\partial\phi} = \frac{\partial\zeta}{\partial\phi}\bigg|_{W=\text{const.}} + \frac{\partial\zeta}{\partial W}\,\frac{\partial W}{\partial\phi} .$$

Hence we get:

$$\tilde{\xi} \doteq -\frac{1}{R}\,\frac{\partial\zeta}{\partial\phi}\bigg|_{W=\text{const.}} = -\frac{1}{R}\,\frac{\partial\zeta}{\partial\phi} + \frac{1}{R}\,\frac{\partial\zeta}{\partial W}\,\frac{\partial W}{\partial\phi} .$$

The second term on the right hand side can be evaluated, using dummy function h instead of W, which amounts to using another coordinates system, namely ϕ,λ,h for expressing ζ in:

$$\frac{1}{R}\frac{\partial \zeta}{\partial W}\frac{\partial W}{\partial \phi} = \frac{1}{R}\frac{\partial \zeta}{\partial h}\frac{\partial h}{\partial \phi} .$$

Moreover, $\frac{\partial h}{R\partial \phi} = \frac{\partial h}{\partial S_\phi} = \mathrm{tg}\ \beta_1$ where β_1 is the inclination of the N-S terrain profile.

For $\frac{\partial \zeta}{\partial h}$ we can write, using Bruns' theorem:

$$\frac{\partial \zeta}{\partial h} = \frac{\partial}{\partial h}\left(\frac{T}{\gamma}\right) = \frac{1}{\gamma}\frac{\partial T}{\partial h} - \frac{T}{\gamma^2}\frac{\partial \gamma}{\partial h} = \frac{1}{\gamma}\left(\frac{\partial T}{\partial h} - \frac{T}{\gamma}\frac{\partial \gamma}{\partial h}\right) .$$

The term in the brackets can be recognized, using the gravimetric equation (see § 9), as being equal to $-\Delta g$. Hence we finally get

$$\tilde{\xi} = -\frac{1}{R}\frac{\partial \zeta}{\partial \phi} - \frac{\Delta g}{\gamma}\ \mathrm{tg}\ \beta_1 .$$

Similarly, we derive

$$\tilde{\eta} = -\frac{1}{R\cos\phi}\frac{\partial \zeta}{\partial \lambda} - \frac{\Delta g}{\gamma}\ \mathrm{tg}\ \beta_2 ,$$

where β_2 is again the inclination of the E-W terrain profile, Δg is related to both the surface and the telluroid (see the definition in § 6) and γ is related to the telluroid.

Taking ζ in the first approximation given by the formula (as derived in 1.13)

$$\zeta = \frac{R}{4\pi\gamma} \oint\!\!\oint_\sigma (\Delta g + G_1)\ S(\psi)\ d\sigma$$

we end up with formulae analogous to the Vening-Meinesz's formulae in classical geodesy (see Physical Geodesy I, § 3.19).

$$\boxed{\begin{aligned} \tilde{\xi} &= \frac{1}{4\pi\gamma} \oint\!\!\oint_\sigma (\Delta g + G_1)\ \frac{dS}{d\psi}\ \cos\alpha\ d\sigma - \frac{\Delta g}{\gamma}\ \mathrm{tg}\ \beta_1 , \\ \tilde{\eta} &= \frac{1}{4\pi\gamma} \oint\!\!\oint_\sigma (\Delta g + G_1)\ \frac{dS}{d\psi}\ \sin\alpha\ d\sigma - \frac{\Delta g}{\gamma}\ \mathrm{tg}\ \beta_2 . \end{aligned}}$$

The derivation of these formulae is identical with that of Vening-Meinesz's.

14) Discussion of the Molodenskij deflections

The Molodenskij deflections are indeed different from both kinds of deflections used in classical geodesy. In classical geodesy (see Physical Geodesy II, § 6.3) we use: I) The gravimetric deflections (on the geoid) defined as angles between the actual gravity vector on the geoid (tangent to the local actual plumb-line) and normal gravity vector on the mean earth ellipsoid; ii) the surface or astronomic deflections defined as angles between the actual gravity vector on the surface and the normal to the reference ellipsoid (local or mean earth, depending on which one we use for the geodetic computations).

The Molodenskij deflections are obviously closer to the astro-deflections. They differ only by the term arising from the curvature of the normal plumb-line which can be computed. Due to the symmetry of the normal field with respect to λ the curvature of the normal plumb-line in E-W direction is zero. The curvature in the N-S direction that manifests itself as the difference of $\tilde{\xi}$ and ξ' can be obtained from the formula

$$\tilde{\delta\xi} = \int_{\text{ellipsoid}}^{\text{surface}} \frac{1}{\gamma} \frac{\partial \gamma}{\partial x} \, dh$$

derived in Physical Geodesy II, § 6.3, where x is the local Cartesian coordinate pointing to South. Since the normal gravity can be expressed approximately as

$$\gamma \doteq \gamma_a \left(1 + f^* \sin^2 \phi - \frac{2}{a} h \right)$$

(see Heiskanen-Moritz, Physical Geodesy, page 196)

we obtain

$$\frac{\partial \gamma}{\partial x} = - \frac{1}{R} \frac{\partial \gamma}{\partial \phi} \doteq - \frac{2\gamma_a}{R} f^* \sin\phi \cos\phi \doteq - \frac{\gamma}{R} f^* \sin 2\phi$$

where f* is the gravity flattening given by

$$f^* = \frac{\gamma_b - \gamma_a}{\gamma_a}$$

and γ_a, γ_b are the values of normal gravity on the equator and at the poles. Integrating $1/\gamma$-times the derivative above we get finally

$$\tilde{\delta}\xi \simeq - \frac{f^*}{R} \sin 2\phi \; H \simeq -0.17'' \sin 2\phi H$$

where H is the height above the ellipsoid.

Evidently, this correction has to be used if we are to work with Molodenskij deflections when dealing with the astronomic observations. The quantity

$$\tilde{\xi} - \tilde{\delta}\xi = \xi'$$

then provides the relation between the astronomic latitude Φ and the geodetic latitude ϕ related to the same ellipsoid as H. Note that $\tilde{\eta} = \eta'$ can then be used in relating the astronomic latitude Λ to the geodetic latitude λ.

Appendix

Brief note on integral equations

There are two classes of integral equations one can meet in practice:

1st class:

$$\int_a^b K(x,S)\, f(S)\, dS = g(x)$$

where the two functions $g(x)$, $K(x,S)$ are known and the function $f(x)$ is sought on $[a,b]$. The function $K(x,S)$ is called the kernel of the integral equation. The equations of this class are generally not solvable analytically.

There exist, however, numerical methods for solving them. One of such numerical methods is based on the idea of approximating the continuous functions f,g by numerical vectors and K by a numerical matrix. We write

$$x \in \{a, a+h, a+2h, \ldots, a+ih, \ldots, a+nh = b\} \equiv \{x_o, x_1, \ldots, x_i, \ldots, x_n\}$$

$$s \in \{a, a+h, a+2h, \ldots, a+ih, \ldots, a+nh = b\} \equiv \{s_o, s_1, \ldots, s_i, \ldots, s_n\}\ .$$

Then we may denote $hK(x_i, s_j) = K_{ij}$, $f(s_j) = f_j$, $g(x_i) = g_i$. The integral equation can hence be written approximately as

$$\sum_{j=0}^{n} K_{ij} f_j \simeq g_i$$

which represent a system of linear algebraic equations

$$A\vec{f} \simeq \vec{g}$$

that yields the solution:

$$\vec{f} \simeq A^{-1}\vec{g}$$

providing the matrix of the kernel is not singular.

2nd class contains two distinct subclasses

i) Fredholm's equations

$$f(x) - \lambda \int_a^b K(x,S)\ f(S)\ dS = g(x)$$

where f is the sought function on [a,b].

ii) Volterra's equations

$$f(x) - \lambda \int_a^x K(x,S)\ f(S)\ dS = g(x)$$

where f is sought on any subinterval of $[a,\infty]$.

We may note a certain resemblance of Fredholm's equations with the Fourier transform. There are actually some analytical methods for solving the Fredholm's type equations for special kernels. All the equations of the 2nd class may be solvable by the following iterative method, if the process converges:

$$f_0(x) = g(x).$$

$$f_1(x) = \lambda \int_a^{\bullet} K(x,S)\ f_0(s)\ dS + g(x)$$

$$\vdots$$

$$f_i(x) = \lambda \int_a^{\bullet} K(x,S)\ f_{i-1}(s)\ ds + g(x)$$

The upper limit of the integrals is either b, for Fredholm's equations, or x for Volterra's equations.

Fredholm's equations can again be approximated by a system of linear algebraic equations:

$$f_i - \lambda \sum_{j=0}^{n} K_{ij}\, f_j = g_i \qquad i = 0,1,2,\ldots, n$$

or

$$\sum_{j=0}^{n} (\lambda K_{ij} - \delta_{ij})\, f_j = -g_i \qquad i = 0,1,2,\ldots,n$$

where δ_{ij} is the Kronecker's δ.

The Volterra's equations are often transformed to differential equations and vice versa.

We have shown here only the integral equations formulated for one variable x. In geodesy, we deal with integral equations containing two variables. However, it is not difficult to see that the first two methods can be readily used even for two variables. The transformation of Volterra's equations leads in such a case to partial differential equations.