MATRICES

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LECTURE NOTES
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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

MATRICES

1. Introduction

Matrix notation is a powerful mathematical shorthand. Concepts and relationships which often otherwise become buried under a mass of symbols and equations, can in matrix notation be expressed with brevity and clarity, leading to greater understanding and less preoccupation with details of notation.

The next four sections of these notes review matrix notation and definitions; matrix addition, multiplication and transposition; determinants, inverse and orthogonal matrices; and partitioned matrices. This review is brief. More thorough treatments are available in any text on matrices. Thompson [1969] is recommended because of its applications - oriented approach. Ayres [1962] is recommended because of its widespread availability, many solved problems, and low cost.

The following two sections of these notes are lengthier and cover two areas in which matrices have important applications: the solution of linear equations and linear transformations.

The last section covers the differentiation of matrices and Taylor's series in matrix form.

Each point in these notes is illustrated by a solved example using matrices, usually of order 2 x 2 for simplicity.

2. Matrix Notation and Definitions

A matrix is a rectangular array of numbers which obeys certain rules of algebra to be introduced in this and the following three sections. The numbers making up a matrix are called the elements of the matrix. Only matrices with real elements will be discussed in these notes. Examples of matrices are:

$$ {}_2A_3 = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} , $$

$$ {}_3B_1 = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} , $$

$$ {}_2C_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . $$

In these notes a matrix will be denoted by a capital letter (e.g. A). The number of rows and the number of columns will be indicated by double subscripts (e.g. ${}_2A_3$, A_{23} or $A_{2,3}$ all indicate that A has two rows and three columns, in which case it is said to be of order 2 by 3 or 2 x 3). Whenever no confusion will result, the double subscripts will be dropped.

An element of a matrix will be denoted by a lower case letter with a double subscript which indicates at which row and column intersection it is located. For example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

where a_{13} is at the intersection of row 1 and column 3.

A matrix having the same number of rows as it has columns is called a <u>square</u> matrix (e.g. C, the third example above, is a square matrix). A matrix having only one row is called a <u>row</u> matrix or <u>row vector</u>. A matrix having only one column is called a <u>column</u> matrix or <u>column vector</u> (e.g. B, the second example above, is a column matrix).

A square matrix having all elements zero except along the diagonal running from top left to bottom right is called a <u>diagonal</u> matrix. A diagonal matrix which has all elements equal is called a <u>scalar</u> matrix. A scalar matrix which has elements equal to unity (1) is called a <u>unit</u> or <u>identity</u> matrix, and is denoted I or E. The unit matrix of order n x n is usually denoted I_n. The following are examples of diagonal, scalar and unit matrices, respectively.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

If all the elements of a matrix are zero, the matrix is called the <u>null</u> matrix and is equated:

$$A = 0 .$$

3. <u>Matrix Addition, Multiplication, and Transposition</u>

Matrices can be <u>added</u> only when they have the same number of rows and the same number of columns. They are then said to be <u>conformable for addition</u> (and subtraction). Matrices of different orders cannot be added. For example the matrices

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$$

are not conformable for addition, the first being of order 2 x 3 and the second being of order 3 x 1.

The sum of two matrices A and B which are conformable for addition, is a matrix C = A + B, each element of which is the sum of the corresponding elements of A and B. Expressed in terms of elements,

$$\boxed{c_{ij} = a_{ij} + b_{ij} .} \tag{1}$$

For example:

$$\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3+2 & 1+6 \\ 1+5 & 5+3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

Matrix addition has the following properties

$$\boxed{A + B = B + A \text{ (commutative)}} \tag{2}$$

$$\boxed{A + (B + C) = (A + B) + C \text{ (associative)}} \qquad (3)$$

If B is the sum of n matrices, all equal to A, then

$$B = n\,A\;.$$

Expressed in terms of elements $b_{ij} = na_{ij}$, that is each element of B is n times the corresponding element of A. More generally n can be <u>any</u> number, not just a positive integer. The above equation then defines <u>scalar multiplication</u>.

In the special case where n = -1,

$$B = -A$$

i.e. B is called the <u>negative</u> of A. Matrix <u>subtraction</u> is accomplished by adding the negative of the matrix to be subtracted.

The product of two matrices A B is defined only when the number of columns of A is equal to the number of rows of B. A and B are then said to be <u>conformable for multiplication</u> in the order A B, but not necessarily in the order B A. For example the matrices

$$_2A_3\;\,_3B_1 = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} ,$$

in that order, are conformable for multiplication, however they are <u>not</u> conformable in the reverse order

$$_3B_1\;\,_2A_3 = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} ,$$

Note that the two inner subscripts are equal in the first (conformable) case and are not equal in the second (nonconformable) case.

The product of two matrices A and B, which are conformable for multiplication, is a matrix C = A B whose (i, j)th element is the sum of the products of the elements in the ith row of A and the jth column of B, taken term by term. Expressed in terms of elements,

$$\boxed{c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}} \tag{4}$$

where m is the number of columns of A and the number of rows of B. For example:

$$C = AB = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3\times2 + 1\times6 + 4\times5 \\ 1\times2 + 5\times6 + 9\times5 \end{bmatrix} = \begin{bmatrix} 32 \\ 77 \end{bmatrix}.$$

Matrix multiplication has the following properties

$$A\,(B + C) = A\,B + A\,C \quad \text{(distributive)} \tag{5}$$

$$(A + B)\,C = A\,C + B\,C \quad \text{(distributive)} \tag{6}$$

$$A\,(B\,C) = (A\,B)\,C \quad \text{(associative)} \tag{7}$$

However, in general,

$$A\,B \neq B\,A \quad \text{(not commutative)} \tag{8}$$

$$A\,B = A\,C \quad \text{does } \underline{\text{not}} \text{ imply } B = C \tag{9}$$

$$A\,B = 0 \quad \text{does } \underline{\text{not}} \text{ imply } A = 0 \text{ or } B = 0 \tag{10}$$

If A and B are square matrices such that A B = B A, then A and B are called <u>commutative</u> matrices. Any square matrix will comute with itself and with the identity matrix of the same order. Another example of commutative matrices is

$$\begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ 24 & -2 \end{bmatrix}.$$

The matrix formed by interchanging the rows and columns of another matrix A is called the <u>transpose</u> of A, and is denoted A^T or A'. For example, if

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 5 \\ 4 & 9 \end{bmatrix}$$

then $B = A^T$ and $A = B^T$. Expressed in terms of their elements,

$$\boxed{b_{ij} = a_{ji}} \tag{11}$$

for $i = 1, 2, 3$ and $j = 1, 2$.

The transpose has the following properties

$$(A^T)^T = A \tag{12}$$

$$(A + B)^T = A^T + B^T \tag{13}$$

$$(nA)^T = n\, A^T \tag{14}$$

$$(A\ B)^T = B^T\ A^T \text{ (note \underline{reverse} order)}. \tag{15}$$

A square matrix which is equal to its transpose is called <u>symmetric</u>.

For example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$$

is symmetric ($A^T = A$). Expressed in terms of its elements

$$\boxed{a_{ij} = a_{ji}} \tag{16}$$

For any square matrix A, the matrices $(A + A^T)$ and (AA^T) will be symmetric.

4. Determinants, Inverses and Orthogonal Matrices

Associated with every square matrix A is a number called the determinant of A and denoted det A, or $|A|$. If A is of order n x n, its determinant is defined as

$$\boxed{|A| = \Sigma\ (\pm\ a_{1i}\ a_{2j}\ .\ .\ .\ a_{nk})} \tag{17}$$

where the summation is over all n! permutations of i, j, . . . k, where i, j, . . . k, are the integers 1 to n. A term in equation 17 is given a positive sign if the permutation involves an even number of exchanges (one exchange occurs whenever a larger integer precedes a smaller one) and a minus sign if the number of exchanges is odd.

Equation 17 can be expressed another way. If the elements in the ith row and jth column of A are removed, then the determinant of the remaining (n-1) x (n-1) matrix is called the minor of the element a_{ij}, and is denoted by $|M_{ij}|$. The signed minor of a_{ij} is called the cofactor of a_{ij} and is denoted by

$$\boxed{\alpha_{ij} = (-1)^{i+j}\ |M_{ij}|} \tag{18}$$

The value of the determinant of a matrix A can be expressed as the sum of the products of each element in a row or column of A times its cofactor.

$$\boxed{\begin{aligned} |A| &= \sum_{k=1}^{n} a_{ik}\ \alpha_{ik} \quad \text{(expansion along ith row)} \\ &= \sum_{k=1}^{n} a_{kj}\ \alpha_{kj} \quad \text{(expansion along jth column)} \end{aligned}} \tag{19}$$

For example, for a 2 x 2 matrix from equation 17

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{12}\ a_{21}$$

and for a 3 x 3 matrix from equation 19 expanding along the first row,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,\alpha_{11} + a_{12}\,\alpha_{12} + a_{13}\,\alpha_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32})$$

$$- a_{12}\,(a_{21}\,a_{33} - a_{23}\,a_{31})$$

$$+ a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31})$$

Determinants have the following properties

$$\boxed{|A^T| = |A|} \tag{20}$$

$$\boxed{|A\,B| = |A|\;|B|} \tag{21}$$

If the determinant of a matrix is zero, the matrix is called singular ($|A| = 0$). If the determinant is non-zero the matrix is called non-singular ($|A| \neq 0$).

If A and B are square matrices such that

$$\boxed{A\,B = B\,A = I} \tag{22}$$

then B is called the inverse of A and is denoted $B = A^{-1}$ (or equivalently A is called the inverse of B and is denoted $A = B^{-1}$). Only nonsingular square matrices have an inverse. Singular matrices do not have an inverse.

Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -6 \end{bmatrix}$$

the values of the determinants are $|A| = 1$ and $|B| = 0$, therefore A is nonsingular and has an inverse, but B is singular and does not have an inverse.

Systematic techniques for finding inverses of matrices are an important part of matrix mathematics. Details of several different methods are given in Ayers [1962] (particularly in Chapter 7). One method will be described here. Another is described in Appendix D.

If A is an n x n matrix, then the matrix obtained by replacing each element a_{ij} of A by the cofactor α_{ji} of the element a_{ji} (note the reversed order of the subscripts) is called the adjoint of A, and is denoted by adj A. For the matrix A given above, the matrix obtained by replacing each element by its own cofactor is

$$\begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 5 & 7 \\ -4 & -5 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ -2 & -5 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 2 & 5 \\ -2 & -4 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

and the adjoint of A is the transpose of this matrix

$$\text{adj } A = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} .$$

The inverse is related to the adjoint by

$$\boxed{A^{-1} = \frac{\text{adj } A}{|A|}} \tag{23}$$

In the above example $|A| = 1$, thus

$$A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

and

$$A\,A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which satisfies equation 22.

Inverses have the following properties:

$$\boxed{\begin{aligned} &(A^{-1})^{-1} = A && (24) \\ &(A\,B)^{-1} = B^{-1}\,A^{-1} \quad \text{(note \underline{reverse} order)} && (25) \\ &(k\,A)^{-1} = \frac{1}{k}\,A^{-1} && (26) \\ &|A^{-1}| = |A|^{-1} = \frac{1}{|A|} && (27) \end{aligned}}$$

If the inverse of a square matrix A is equal to the transpose of A (i.e. $A^{-1} = A^T$) then A is called an <u>orthogonal</u> matrix. If A is orthogonal, so are A^T and A^{-1}, and equation 22 becomes

$$\boxed{A\,A^T = A^T\,A = I} \qquad (28)$$

Examples of orthogonal matrices are

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

If an orthogonal matrix is considered to be composed of row (or column) vectors, then these <u>vectors</u> are orthogonal unit vectors. For example the columns of A are

$$X_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \qquad X_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

and

$$X_1^T\,X_2 = X_2^T\,X_1 = 0$$
$$X_1^T\,X_1 = X_2^T\,X_2 = 1$$

This property will be discussed in more detail in section 7f.

5. Partitioned Matrices

A matrix can be considered to be made up of smaller parts, or submatrices, which are themselves matrices. A matrix can be divided or partitioned into smaller submatrices in many ways. For example the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

could be partitioned into two row matrices

$$A = \left[\begin{array}{c} A_1 \\ \hdashline A_2 \end{array}\right] = \left[\begin{array}{ccc} 3 & 1 & 4 \\ \hdashline 1 & 5 & 9 \end{array}\right]$$

or into a square matrix and a column matrix

$$A = \left[\begin{array}{c:c} A_1 & A_2 \end{array}\right] = \left[\begin{array}{cc:c} 3 & 1 & 4 \\ 1 & 5 & 9 \end{array}\right]$$

In this section the rules for multiplying, transposing and inverting partitioned matrices will be discussed. If two matrices A and B are conformable for multiplication, then they can always be partitioned so that the corresponding submatrices are conformable for multiplication. It is only necessary that the columns of the left hand matrix (A) and the rows of the right hand matrix (B) be partitioned in exactly the same way. For the example A B, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 5 \end{bmatrix}$$

The most appropriate partitioning is

$$A\ B = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & 5 \end{array}\right] = \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right] \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right]$$

$$= A_1\ B_1 + A_2\ B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 15 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 1 & 6 \end{bmatrix}$$

However, these matrices could also be partitioned

$$A\ B = \left[\begin{array}{c|cc} 1 & 0 & 3 \\ \hline 0 & 1 & 1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ 1 & 5 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}\ B_{11} + A_{12}\ B_{21} & A_{11}\ B_{12} + A_{12}\ B_{22} \\ A_{21}\ B_{11} + A_{22}\ B_{21} & A_{21}\ B_{12} + A_{22}\ B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 3 & 0 + 15 \\ 0 + 1 & 0 + 6 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 1 & 6 \end{bmatrix}$$

Note that the submatrices follow the same rules in matrix multiplication as do <u>elements</u> of a matrix, subject only to the necessary condition that corresponding submatrices must be conformable for multiplication, and the order of the submatrices in a product must not be reversed.

In <u>transposing</u> a partitioned matrix, the submatrices again follow the rules for transposing elements of a matrix, with the important addition that the submatrix itself must be transposed. For example

$$\boxed{\left[\begin{array}{c|c} A_1 & A_2 \end{array}\right]^T = \left[\begin{array}{c} A_1^T \\ \hline A_2^T \end{array}\right]} \tag{29}$$

The inverse B of a square non-singular partitioned matrix A can be written as another partitioned matrix, with the submatrices of the inverse B functions of the submatrices of the original matrix A. There is a restriction on the partitioning of A; the submatrices along the diagonal must be square and non-singular. If A is partitioned into four submatrices, then B will also be partitioned into four submatrices, each of the same order as the corresponding A submatrix. Since B is the inverse of A

$$A \quad B = B \quad A = I$$

For example, if A and B are of order (m + n) x (m + n) A B = I becomes

$$\begin{bmatrix} \underset{mm}{A_{11}} & \underset{mn}{A_{12}} \\ \underset{nm}{A_{21}} & \underset{nn}{A_{22}} \end{bmatrix} \begin{bmatrix} \underset{mm}{B_{11}} & \underset{mn}{B_{12}} \\ \underset{nm}{B_{21}} & \underset{nn}{B_{22}} \end{bmatrix} = \begin{bmatrix} \underset{mm}{I_m} & \underset{mn}{0} \\ \underset{nm}{0} & \underset{nn}{I_n} \end{bmatrix}$$

where $|A_{11}| \neq 0$ and $|A_{22}| \neq 0$ (i.e. A_{11} and A_{22} have inverses).

Utilizing the rules of matrix multiplication

$$A_{11} \; B_{11} + A_{12} \; B_{21} = I_m \tag{30}$$

$$A_{11} \; B_{12} + A_{12} \; B_{22} = 0 \tag{31}$$

$$A_{21} \; B_{11} + A_{22} \; B_{21} = 0 \tag{32}$$

$$A_{21} \; B_{12} + A_{22} \; B_{22} = I_n \tag{33}$$

Similarly, B A = I can be expanded to give

$$B_{11} \; A_{11} + B_{12} \; A_{21} = I_m \tag{34}$$

$$B_{11} \; A_{12} + B_{12} \; A_{22} = 0 \tag{35}$$

$$\boxed{\begin{aligned} B_{21} A_{11} + B_{22} A_{21} &= 0 \qquad (36)\\ B_{21} A_{12} + B_{22} A_{22} &= I_n \,. \qquad (37)\end{aligned}}$$

From equations (34) and (35)

$$\boxed{\begin{aligned} B_{11} &= (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \qquad (38)\\ B_{12} &= - B_{11} A_{12} A_{22}^{-1} \,. \qquad (39)\end{aligned}}$$

From equations (32) and (33)

$$\boxed{\begin{aligned} B_{21} &= - A_{22}^{-1} A_{21} B_{11} \qquad (40)\\ B_{22} &= A_{22}^{-1} + A_{22}^{-1} A_{21} B_{11} A_{12} A_{22}^{-1} \qquad (41)\end{aligned}} \,.$$

Alternatively from equations (30), (31), (36) and (37)

$$\boxed{\begin{aligned} B_{11} &= A_{11}^{-1} + A_{11}^{-1} A_{12} B_{22} A_{21} A_{11}^{-1} \qquad (42)\\ B_{12} &= - A_{11}^{-1} A_{12} B_{22} \qquad (43)\\ B_{21} &= - B_{22} A_{21} A_{11}^{-1} \qquad (44)\\ B_{22} &= (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} \,. \qquad (45)\end{aligned}}$$

6. The Solution of Linear Equations

a) Rank of a matrix

A smaller submatrix can be obtained from a matrix by discarding some of the rows and columns of the original matrix. Each submatrix of a partitioned matrix is a special case of this, in which the discarded rows and columns are adjacent. More generally, the discarded rows and columns need not be adjacent. For example discarding the third and fifth rows, and the fourth column of the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 3 & 2 \\ 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & -3 \end{bmatrix}$$

gives the submatrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -1 & 2 \end{bmatrix} .$$

The rank of any matrix, which need not be square, is the order of its largest square non-singular submatrix.

In the above example the original matrix is of order 5 x 4. The largest square submatrix is 4 x 4. However, in this case, all possible 4 x 4 submatrices have zero determinants, and thus are singular. The determinant of the 3 x 3 submatrix found above is not zero (it is -10). Therefore, the order of the largest non-singular matrix in this case is 3 x 3, and the rank of the original matrix is 3. The systematic method of determining the rank of a matrix is known as "reducing the matrix to canonical form", and will not be covered in this review (see Ayres [1962], chapter 5 for details).

Rank has important applications in the solution of systems of linear equations, which will now be discussed.

b) Systems of linear equations in matrix notation

The set of m linear equations in n unknowns (x_i) can be written out explicitly as:

$$
\begin{array}{l}
a_{11}\,x_1 + a_{12}\,x_2 \;.\;.\;.\;.\;.\;.\;.\;.\; a_{1n}\,x_n = b_1 \\
a_{21}\,x_1 + a_{22}\,x_2 \;.\;.\;.\;.\;.\;.\;.\;.\; a_{2n}\,x_n = b_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
a_{m1}\,x_1 + a_{m2}\,x_2 \;.\;.\;.\;.\;.\;.\;.\;.\; a_{mn}\,x_n = b_m
\end{array}
$$

where the coefficients a_{ij} and constants b_j are known.

In matrix notation this can be written as

$$ {}_m A_n \; {}_n X_1 = {}_m B_1 $$

or simply

$$ \boxed{A\,X = B} \qquad (46) $$

where

$$ A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdot & & \\ \cdot & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix} $$

and A is called the <u>coefficient matrix</u>, X is called the <u>unknown vector</u>, and B is called the <u>constant vector</u>.

The <u>augmented matrix</u> of the system is formed by attaching the constant vector as an extra column to the right-hand side of the coefficient matrix, as:

$$ \left[A \mid B \right]. $$

The system is called <u>homogeneous</u> if the constant vector is zero (B = 0), and <u>non-homogeneous</u> if the constant vector is non-zero (B $\neq$ 0).

c) Systems which are inconsistent

If the rank of the coefficient matrix is equal to the rank of the augmented matrix, the system is said to be consistent. Homogeneous systems are always consistent. If the system is not consistent there is no solution for X.

The simplest example of an inconsistent system of equations is

$$x = 1$$
$$x = 2$$

There is obviously no solution for x which will satisfy both these equations. In this case the coefficient and augmented matrices are

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \left[A \,\vdots\, B\right] = \left[\begin{array}{c|c} 1 & 1 \\ 1 & 2 \end{array}\right]$$

having ranks 1 and 2 respectively.

Another example of a non-homogeneous system which is inconsistent is:

$$x_1 - 2x_2 + x_3 = 1$$
$$2x_1 - 3x_2 + 4x_3 = 5$$
$$2x_1 - 5x_2 = 0$$

In this case the coefficient and augmented matrices are

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ 2 & -5 & 0 \end{bmatrix} , \left[A \,\vdots\, B\right] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & -3 & 4 & 5 \\ 2 & -5 & 0 & 0 \end{array}\right]$$

having ranks 2 and 3 respectively. Because the ranks are different, this system is inconsistent. In fact the first two equations can be combined to eliminate x_3 to give

$$2x_1 - 5x_2 = -1$$

but the third equation is

$$2x_1 - 5x_2 = 0$$

which are obviously inconsistent.

d) Systems having a unique solution

If the rank of the coefficient matrix is equal to the number of unknowns (the number of rows in the unknown vector X), then there is one unique solution. For homogeneous systems this is the trivial solution X = 0. For non-homogeneous systems having square coefficient matrices (the number of equations equals the number of unknowns) this means the coefficient matrix is non-singular ($|A| \neq 0$) and therefore, has an inverse. In this special case the solution is given by:

$$\boxed{X = A^{-1} B} \quad . \tag{47}$$

For non-homogeneous systems having rectangular coefficient matrices (more equations than unknowns), this means that the matrix

$$A^T A$$

is non-singular ($|A^T A| \neq 0$), and therefore has an inverse. Thus the solution can be obtained by

$$A X = B$$

$$A^T A X = A^T B$$

$$\boxed{X = (A^T A)^{-1} A^T B} \tag{48}$$

(We will meet this solution again when we discuss the method of least squares, which is concerned with obtaining the best average solution from an inconsistent non-homogeneous system of equations.)

An example of a non-homogeneous system having a unique solution is

$$x_1 - 2x_2 + x_3 = 1$$

$$2x_1 - 3x_2 + 4x_3 = 5$$

$$2x_1 - 5x_2 + 2x_3 = 1$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ 2 & -5 & 2 \end{bmatrix}$$

which is non-singular ($|A| = 2$), and therefore has an inverse. Solving this system gives

$$X = A^{-1} B = \begin{bmatrix} 7 & -1/2 & -5/2 \\ 2 & 0 & -1 \\ -2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

or $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ is the unique solution.

e) Systems having an infinite number of solutions

If the rank of the coefficient matrix is less than the number of unknowns, then for both homogeneous and non-homogeneous systems there will be an infinite number of solutions. If there are n unknowns, and the rank is r, then (n - r) of the unknowns may be chosen so that the coefficient matrix of the remaining r unknowns is of rank r. When these (n - r) unknowns are assigned any whatever values, the remaining r unknowns will be uniquely determined.

An example of a non-homogeneous system having an infinite number of solutions is:

$$x_1 - 2x_2 + x_3 = 1$$

$$2x_1 - 3x_2 + 4x_3 = 5$$

$$2x_1 - 5x_2 = -1$$

The coefficient matrix is:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ 2 & -5 & 0 \end{bmatrix}$$

and has rank 2. By assigning one of the unknowns (in this case x_3) an arbitrary value, the other two unknowns are uniquely determined. In fact, the equations can be combined to give:

$$x_1 = 7 - 5x_3$$

$$x_2 = 3 - 2x_3$$

The table below summarizes this discussion of linear equations $(A\ X = B)$.

Value of Constant Vector B	homogeneous	non-homogeneous	Rank of Augmented Matrix [A B]	consistent	inconsistent	Rank of Coefficient Matrix A	no solution	unique solution $(X = 0)$	unique solution $(X \neq 0)$	infinite solutions
$B \neq 0$		•	$\neq$ rank A		•	---	•			
$B \neq 0$		•	= rank A	•		= dimension of X			•	
$B \neq 0$		•	= rank A	•		< dimension of X				•
$B = 0$	•		= rank A	•		= dimension of X		•		
$B = 0$	•		= rank A	•		< dimension of X				•

7. <u>Linear Transformations</u>

The matrix equation

$$\boxed{Y = A\ X} \tag{49}$$

where A is a matrix and X and Y are column vectors, can be regarded as a <u>linear transformation</u>, in which case the matrix A is called the <u>transformation matrix</u>. There are two related interpretations of such transformations. The first is that both X and Y are different vectors whose elements are referred to the same coordinate system, in which case the transformation matrix describes the coordinates of Y in terms of the coordinates of X, or the operations which must be performed on X to transform it into Y. The second interpretation is that both X and Y are the same vector, however their elements refer to different coordinate systems, in which case the transformation matrix describes the relationship between the two coordinate systems, or the operations which must be performed on the coordinate system to which X refers to transform it into the coordinate system to which Y refers.

Both of these interpretations of linear transformations will be of interest.

The discussion will be restricted to transformation matrices which are square and nonsingular ($|A| \neq 0$) in which case the inverse transformation exists, so that

$$X = A^{-1} Y$$

This restricted class of linear transformations are called <u>projective transformations</u>.

a) <u>Orthogonal transformations</u>

Within this class, transformations may be grouped according to the effect they have on the length of the vectors they are transforming.

There is a class of transformations which leave the lengths of vectors unchanged. The square of the length of a vector is given by $X^T X$. For example, if

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$$

For a transformation $Y = A X$ to leave the length of the vector unchanged, then it must also leave the square of the length of the vector unchanged, or

$$\boxed{Y^T Y = X^T X} \qquad (50)$$

but

$$Y = A X$$

Therefore

$$Y^T Y = (A X)^T A X = X^T (A^T A) X$$

Therefore, $A^T A = I$, that is the transformation matrix must be orthogonal. In this case the transformation is said to be an orthogonal transformation. Orthogonal transformations leave the lengths of vectors unchanged.

There are two kinds of orthogonal transformations, called reflections and rotations. Rotation matrices are proper orthogonal matrices (that is $|A| = +1$).

b) Reflections

Reflection matrices are improper orthogonal matrices (that is $|A| = -1$) which consist only of diagonal elements, an odd number of which are -1 and the rest +1. Any improper orthogonal matrix can be expressed as the product of a rotation and a reflection.

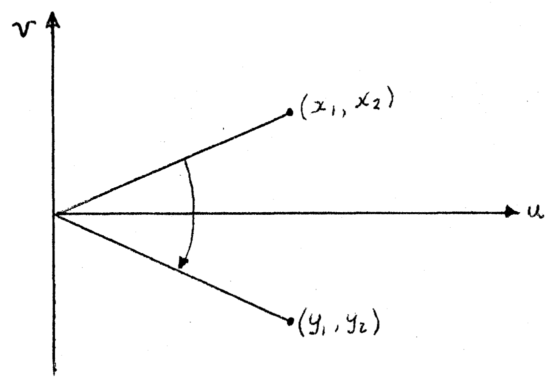
An example of a reflection in two dimensions is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

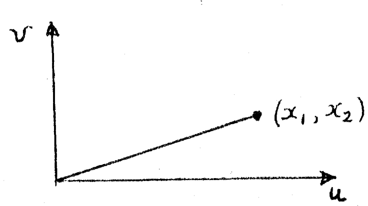
Expressing Y = A X explicitly and accepting the first interpretation of the transformation (that the coordinate system is the same, and the vector is changed);

$$y_1 = x_1$$
$$y_2 = -x_2$$

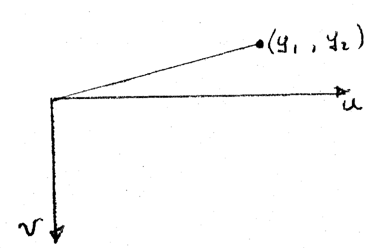
This concept is illustrated in the rectangular coordinate system (u,v)



The second interpretation (that the coordinate system changes and the vector remains the same) is depicted below.



original coordinate system



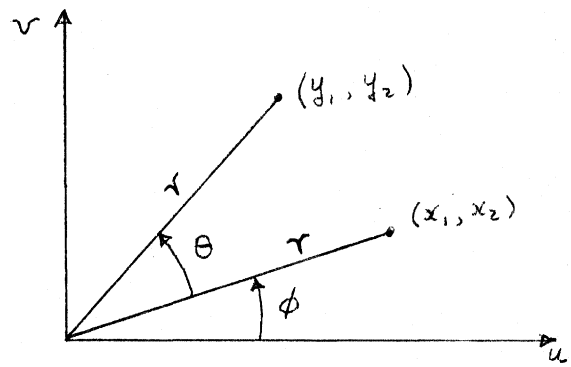
transformed coordinate system.

c) Rotations

An example of a rotation in two dimensions is:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{51}$$

Illustrating the first interpretation (the vector is transformed) in two dimensions:



From the diagram

$$x_1 = r \cos \phi$$

$$x_2 = r \sin \phi$$

$$y_1 = r \cos (\phi+\theta) = r \cos\phi \cos\theta - r \sin\phi \sin\theta$$

$$y_2 = r \sin (\phi+\theta) = r \cos\phi \sin\theta + r \sin\phi \cos\theta$$

or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

Note that R is orthogonal ($R R^T = I$), that is

$$RR^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta \cos\theta - \sin\theta \cos\theta \\ \sin\theta \cos\theta - \sin\theta \cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

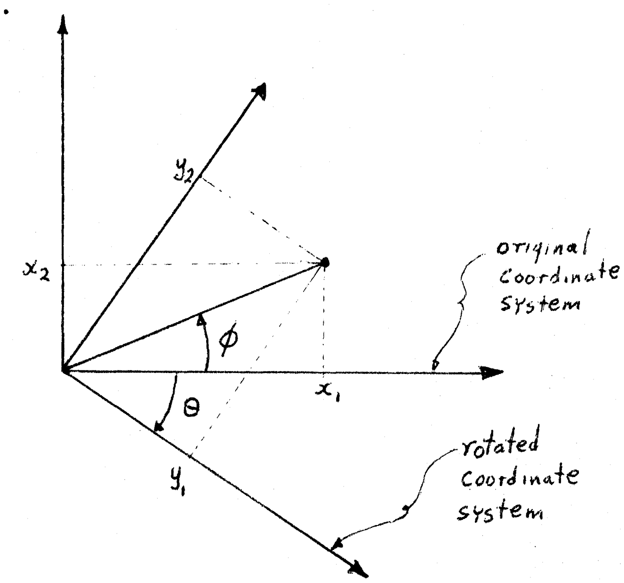
This means that the inverse transformation of R is R^T. We confirm this by noting that a negative rotation

$$R(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R^T(\theta)$$

results in a rotation matrix which is the transpose of a positive rotation. The product of a positive rotation followed by the same negative rotation is, of course, no change at all, or the identity transformation. It is a rule for rotation matrices that

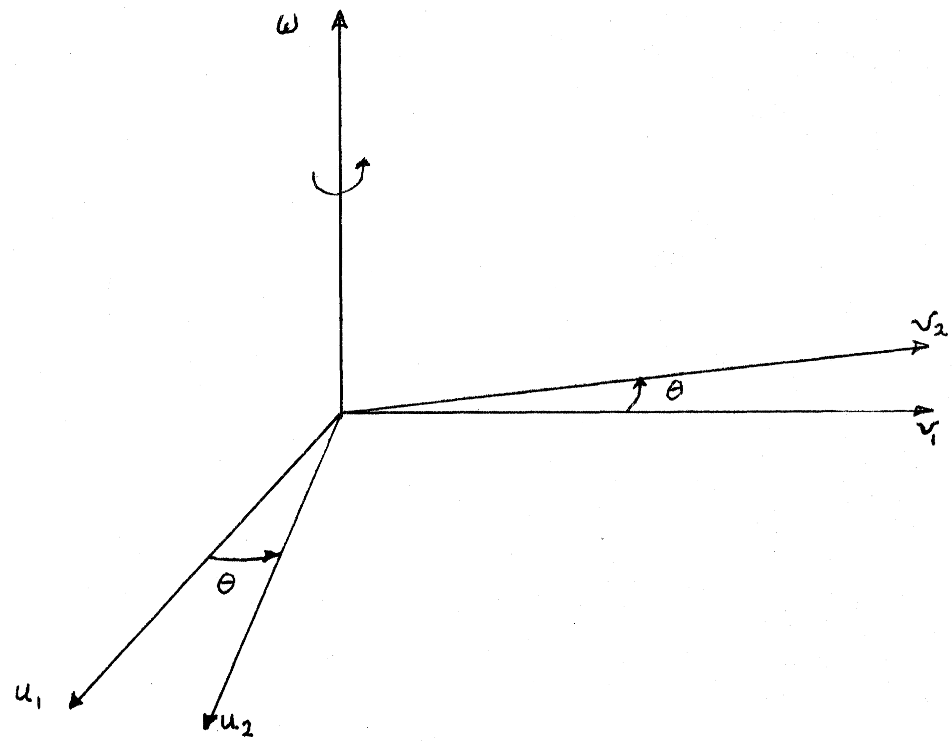
$$\boxed{R^{-1}(\theta) = R^T(\theta) = R(-\theta)} \tag{52}$$

See the figure below for the illustration of the second interpretation of a rotation (the coordinate system is transformed) in two dimensions.



Note that the two interpretations are related by the obvious fact that a rotation of the vector is equivalent to the same rotation (but in the opposite direction) of the coordinate system. We will now consider the <u>rotation of three dimensional</u> coordinate systems.

In two dimensions there is only one plane in which rotations can be made; in three dimensions there are three such planes, one perpendicular to each of the three axes of rectangular coordinate system (u, v, w). Consider a rotation in the uv plane, perpendicular to the w axis.



In this case the w axis is called the <u>rotation axis</u>. For a right handed coordinate system such as the one shown, a positive rotation is defined by the right hand rule as follows: when the rotation axis is grasped by the right hand such that the thumb points in the positive direction along that axis, then the fingers point in the direction of positive rotation. Positive rotations are counterclockwise when viewed from the positive end of the axis. The rotation shown is positive. The rotation matrix in this case is

$$R_3\ (\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{53}$$

where $R_3\ (\theta)$ denotes a positive rotation of angle θ about the "3 - axis" (or w axis in this case). The other two rotation matrices are

$$R_1\ (\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{54}$$

$$R_2\ (\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{55}$$

These rotation matrices define a counterclockwise rotation when applied to the rotation of right handed coordinate systems. They define a clockwise rotation when applied to the rotation of left handed coordinate systems.

Note that the two dimensional rotation matrix given in equation 51 can be replaced by the three dimensional rotation matrix $R_3\ (-\theta)$ given in equation 53, that is

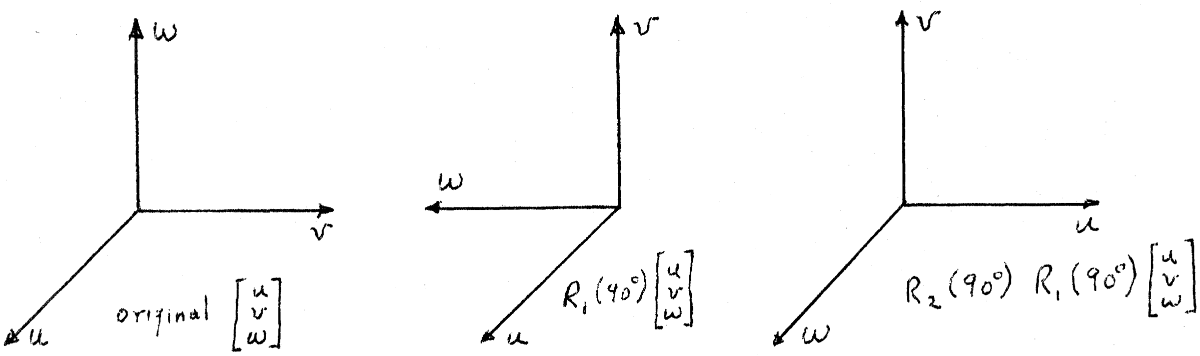
$$\begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = R_3\ (-\theta) \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

where

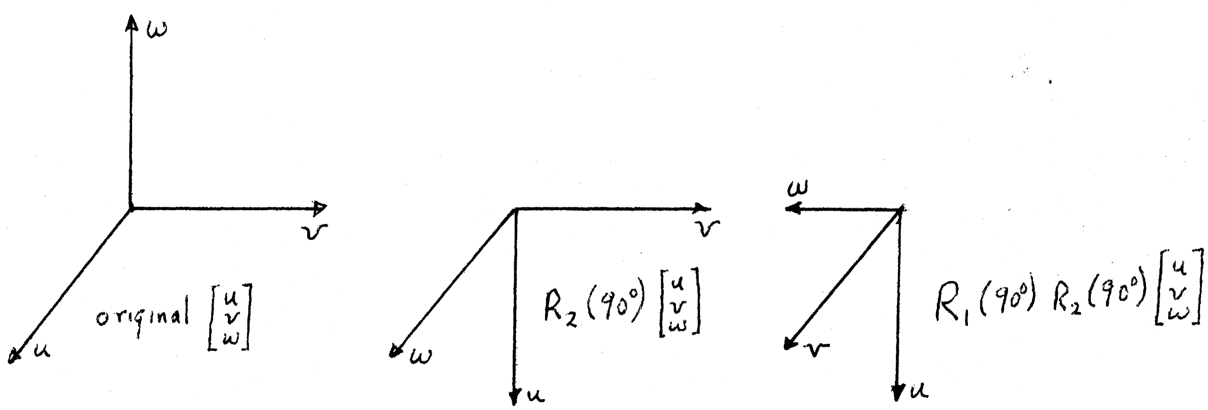
$$R_3\ (-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation which results from several rotations is represented by the product of the rotation matrices representing the individual rotations.* Successive rotation matrices are applied to the left of this product. For example let us consider what happens to a coordinate system subjected first to R_1 (90°) then to R_2 (90°).

* See Appendix C for an algorithm to compute the product of a sequence of rotations and reflections.



Applying the rotations in the reverse order gives a different result, which is a consequence of the fact that matrices do not commute.



d) <u>Scalar transformations</u>

So far, we have discussed only orthogonal transformations, which leave the length of the vector (or the scale of the coordinate system) unchanged. There is another special class of transformations which changes vector lengths (or coordinate scales), but produces the same change in length (or scale) whatever the vector. Such transformations are called <u>scalar</u> transformations and have matrices of the form

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k\ I$$

so that

$$Y = A\ X$$

can be written

$$Y = k\,X$$

e) Affine transformations

Projective transformations which are neither orthogonal nor scalar are called affine transformations. The effect of an affine transformation on a specific vector can be reproduced by a specific orthogonal transformation plus a specific scalar transformation. However, the effect of affine transformations on different vectors will, in general, be different. Therefore, different orthogonal and scalar transformations will be required for each vector, to reproduce the effect of the affine transformations. For example

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

is the matrix of an affine transformation which affects the vectors

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

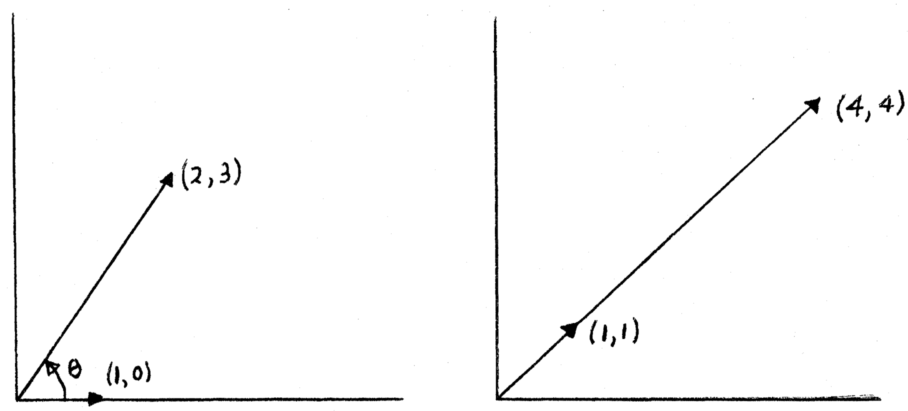
and

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in different ways.

$$Y_1 = A\,X_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$Y_2 = A\,X_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$



In the first case, the affine transformation can be reproduced by a rotation ($\theta = \tan^{-1}(\frac{3}{2})$) and a stretching ($k = \sqrt{13}$), having transformation matrices

$$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$$

$$S_1 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}$$

so that

$$S_1 R_1 = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

and

$$S_1 R_1 X_1 = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

which equals Y_1 above.

In the second case only a stretching ($k = 4$) is required, so that

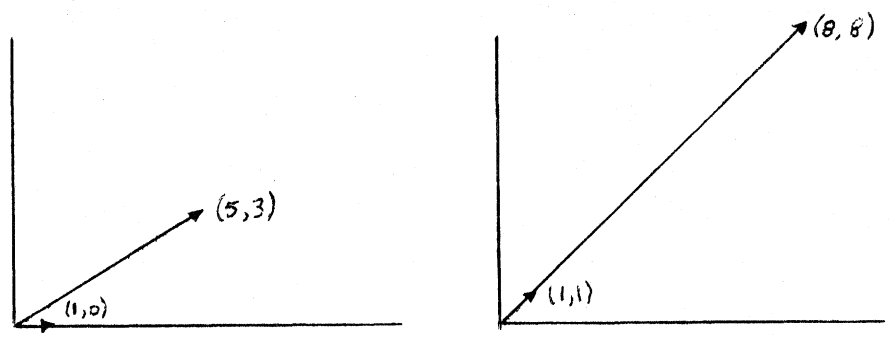
$$S_2 X_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

which equals Y_2 above.

Further discussion of affine transformations will be restricted to the special case in which the transformation matrix is symmetric. This class of transformations has useful properties leading to many important applications. An example of a symmetric affine transformation is

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

which changes the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in different ways as shown below. (Note, it is only a coincidence that both this and the previous example leave the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ vector changed only in length).



f) Eigenvalues and eigenvectors of symmetric matrices

A problem which often arises concerning a given linear transformation matrix A is to find the vectors X which will be changed in length but not in direction by A (for instance the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the examples above). Expressed in equation form, this problem is, given A find λ and X such that

$$\boxed{A\ X = \lambda\ X}. \tag{56}$$

Solutions will exist for any A, but this discussion will be restricted to non-singular symmetric matrices A.

The above matrix equation can be rewritten as

$$\boxed{(A - \lambda\ I)\ X = 0} \tag{57}$$

which is a system of homogeneous equations. As shown in the section on linear equations (section 6e), a non-trivial solution for X exists only when the rank of the coefficient matrix $(A - \lambda I)$ is less than the dimension of X, that is when $(A - \lambda I)$ is singular, or

$$\boxed{|A - \lambda I| = 0} \tag{58}$$

This equation is called the characteristic equation of the matrix A, and serves to determine n values of λ, where n is the order of the matrix A. These values of λ are called the eigenvalues (or characteristic roots or latent roots) of the matrix A. For example, the symmetric matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

has the characteristic equation

$$|A - \lambda I| = \begin{vmatrix} (5-\lambda) & 3 \\ 3 & (5-\lambda) \end{vmatrix} = (5-\lambda)^2 - 9 = 0$$

or

$$\lambda^2 - 10\lambda + 16 = 0$$

which has the solutions $\lambda_1 = 8$ and $\lambda_2 = 2$, that is, the eigenvalues of A are 8 and 2.

For each eigenvalue λ_i there will be a non-zero value of X that satisfies equation 56, and these are called the eigenvectors (or characteristic vectors or latent vectors) of A corresponding the eigenvalue λ_i. For example, for $\lambda_1 = 8$

$$(A - \lambda I)X = \left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The rank of the coefficient matrix is one less than the number of unknowns, therefore, as we found in the section on linear equations (section 6e), we must specify one of the unknowns (x_1, x_2) arbitrarily, and the remaining unknown will then be uniquely determined. In this case if $x_1 = c_1$ then $x_2 = c_1$ also, and equation 56 is satisfied by $X_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where c_1 is any constant. Similarly for $\lambda_2 = 2$, $X_2 = c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, that is the eigenvectors of A are $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These are the vectors which are changed by A only in length and not in direction.

Often the arbitrary constants c_1, c_2 are chosen so that the eigenvectors have unit length (or are normalized). This condition is expressed by

$$\boxed{X^T X = 1} \tag{59}$$

For X_1

$$X^T X = [c_1 \quad c_1] \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = c_1^2 + c_1^2 = 1 \text{ or } c_1 = \frac{1}{\sqrt{2}},$$

and

$$\hat{X}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Similarly for X_2

$$\hat{X}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

and

$$\boxed{\begin{aligned} A\,\hat{X}_1 &= \lambda_1\,\hat{X}_1 \\ A\,\hat{X}_2 &= \lambda_2\,\hat{X}_2 \end{aligned}} \tag{60}$$

Equations 60 can be combined

$$A\,[\hat{X}_1 \;\vdots\; \hat{X}_2] = [\hat{X}_1 \;\vdots\; \hat{X}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

or

$$\boxed{A\,P = P\,D} \tag{61}$$

where

$$P = [\hat{X}_1 \;\vdots\; \hat{X}_2] = \left[\begin{array}{c|c} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{array}\right]$$

and

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

Because the two eigenvectors are orthogonal unit vectors, that is

$$X_1^T\,X_2 = X_2^T\,X_1 = 0$$

$$X_1^T\,X_1 = X_2^T\,X_2 = 1$$

it follows that P is an orthogonal matrix, that is

$$\boxed{P^T\,P = I}$$

Explicitly

$$P^T\,P = [\hat{X}_1 \quad \hat{X}_2]^T\,[\hat{X}_1 \quad \hat{X}_2] = \begin{bmatrix} \hat{X}_1^T \\ \hat{X}_2^T \end{bmatrix} [\hat{X}_1 \quad \hat{X}_2]$$

$$= \begin{bmatrix} \hat{X}_1^T\,\hat{X}_1 & \hat{X}_1^T\,\hat{X}_2 \\ \hat{X}_2^T\,\hat{X}_1 & \hat{X}_2^T\,\hat{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Orthogonal matrices are nonsingular, so the inverse of P exists (in fact $P^{-1} = P^T$), therefore, equation 61 can be rewritten

$$\boxed{P^{-1}\,A\,P = D} \tag{62}$$

or

$$\boxed{P^T\,A\,P = D} \tag{63}$$

Now two square matrices are called <u>similar</u> if there exists a non-singular matrix R such that

$$R^{-1} A R = B$$

Two similar matrices have the same eigenvalues. If R is orthogonal, A and B are called <u>orthogonally similar</u>. Every <u>symmetric</u> matrix A is orthogonally similar to a diagonal matrix D. From the discussion leading to equation 62 it is evident that

a) the elements of the diagonal matrix D are the <u>eigenvalues</u> of A and

b) the columns of the similarity transformation matrix P are the <u>normalized eigenvectors</u> of A.

g) <u>Quadratic forms</u>

We have seen that the square of the length of a vector X is given by the form:

$$X^T X = x^2 + \ldots\ldots + x_n^2$$

More generally, any quadratic polynomial in $(x_1, x_2, \ldots x_n)$ can be represented by the <u>quadratic form</u>

$$\boxed{X^T A X} \qquad (64)$$

where A is a symmetric matrix called the matrix of the quadratic form whose elements are obtained from the polynomial coefficients. There are important applications of quadratic forms in statistics and the method of least squares.

Quadratic polynomial equations can be written in matrix notation as

$$\boxed{X^T A X = k} \qquad (65)$$

where k is the <u>value</u> of the quadratic form.

For example the quadratic polynomial equation

$$5x_1^2 + 6x_1 x_2 + 5x_2^2 = 8$$

can be written as

$$X^T A X = [x_1 \quad x_2] \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8$$

(Note that A could be written $\begin{bmatrix} 5 & 6 \\ 0 & 5 \end{bmatrix}$ but that the cross product coefficient is split in two halves to make A symmetric).

An important property of quadratic forms is that every quadratic form can be expressed as a sum of squares by a suitable change of variables (linear transformation).

If X has been obtained from some other vector Y by the orthogonal transformation

$$\boxed{X = B\ Y} \tag{66}$$

then the quadratic form is

$$X^T A X = (B\ Y)^T A(B\ Y) = Y^T(B^T A\ B)\ Y = k$$

where the value k of the quadratic form has not changed since B is an orthogonal transformation.

For $Y^T (B^T A\ B)\ Y$ to be a sum of squares (y_i^2) and have no cross product terms $(y_i \ y_j)$ then

$$B^T A\ B = D$$

where D is a diagonal matrix.

However, since A is symmetric, it will be orthogonally similar to the diagonal matrix D whose elements are the eigenvalues of A. In this case B must be the orthogonal matrix whose columns are the eigenvectors of A, and

$$\boxed{Y^T D\ Y = k} \tag{67}$$

For the above example it has already been shown that

$$B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \text{ and } D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

therefore

$$5x_1^2 + 6x_1\, x_2 + 5x_2^2 = 8$$

can be written

$$8y_1^2 + 2y_2^2 = 8$$

or

$$y_1^2 + \frac{y_2^2}{4} = 1$$

where

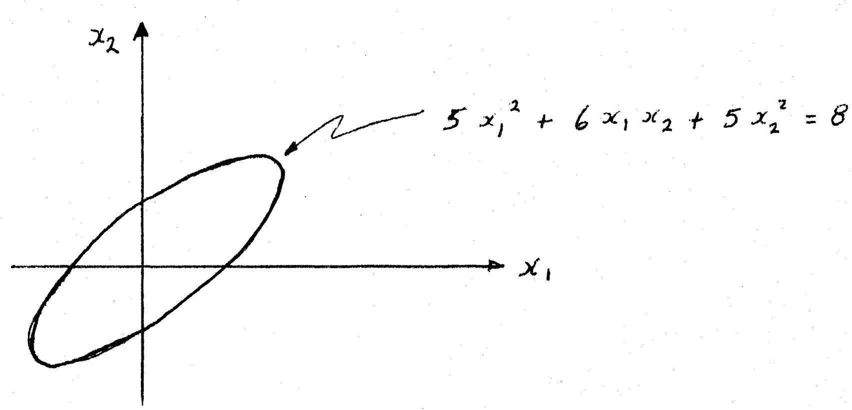
$$Y = B^{-1}\, X = B^T\, X$$

or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that $y_1^2 + \frac{y_2^2}{4} = 1$ is the equation of the ellipse having semi-axes of lengths 1 and 2.

What has been done geometrically? The equation for an ellipse was given and referred to a coordinate system whose axes were not coincident with the axes of the ellipse.



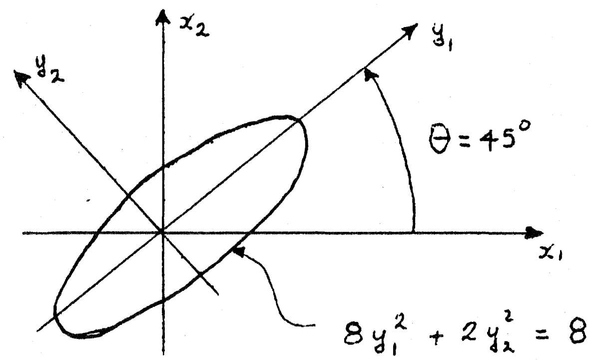
The coordinate system was then rotated counterclockwise by 45° (which is equivalent to rotating the ellipse clockwise by 45°) using the rotation matrix

$$B = R_3(-45°) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and the transformation

$$Y = B^{-1} X = B^T X = R_3(45°)\ X$$

to give a new coordinate system aligned to the axes of the ellipse



The axes of the new coordinate system are the eigenvectors of A (the matrix of the original quadratic form). The semi-axes a_i of the ellipse are related to the eigenvalues λ_i of A and the value of the quadratic form k by:

$$\boxed{a_i^2 = \frac{k}{\lambda_i}} \tag{68}$$

In our case k = 8, and $\lambda_i = 8, 2$ so $a_i = 1, 2$.

Quadratic forms are classified into five value classes which depend on the eigenvalues of the matrix of the quadratic form.

EIGENVALUES OF A	VALUE OF $X^T A X$	VALUE CLASS
all positive	positive for all X	positive definite
all negative	negative for all X	negative definite
all positive or zero	positive or zero for all X	positive semi definite
all negative or zero	negative or zero for all X	negative semi definite
some positive, some negative	$\left\{ \begin{array}{l} \text{positive for some } X, \\ \text{negative for some } X \end{array} \right\}$	indefinite

Positive definite quadratic forms have important properties of interest.

8. Differentiation of Matrices and Taylor's Series in Matrix Form

a) Derivative of a matrix

Assume the elements a_{ij} of a matrix A are differentiable functions of a variable x, rather than numbers as has been assumed so far; for example

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then the derivative of A is defined as the matrix whose elements are derivatives of the corresponding elements of A, for example

$$\boxed{\frac{d}{dx} A = \begin{bmatrix} \frac{da_{11}}{dx} & \frac{da_{12}}{dx} \\ \frac{da_{21}}{dx} & \frac{da_{22}}{dx} \end{bmatrix}} \qquad (69)$$

b) Derivative of a matrix product

If A is the product of two other matrices B and C, whose elements are also differentiable functions of x, then

$$A = B\,C$$

and the (i, j)th element of A is given by

$$a_{ij} = \sum_k b_{ik}\, c_{kj}$$

so that

$$\begin{aligned}
\frac{d}{dx} a_{ij} &= \frac{d}{dx} \sum_k b_{ik}\, c_{kj} \\
&= \sum_k \frac{d}{dx} (b_{ik}\, c_{kj}) \\
&= \sum_k \left(\frac{d}{dx}\left(b_{ik}\right) c_{kj} + b_{ik} \frac{d}{dx}\left(c_{kj}\right) \right) \\
&= \sum_k \left(\frac{d}{dx}\left(b_{ik}\right) c_{kj} \right) + \sum_k b_{ik} \frac{d}{dx}\left(c_{kj}\right) .
\end{aligned}$$

The first summation is $\frac{dB}{dx}$ C and the second B $\frac{dC}{dx}$. Therefore,

$$\boxed{\frac{dA}{dx} = \frac{d(BC)}{dx} = \frac{dB}{dx} C + B \frac{dC}{dx} .} \tag{70}$$

If C is non-singular, it has an inverse and

$$B = A\,C^{-1}.$$

From

$$\frac{dA}{dx} = \frac{dB}{dx}\, C + B \frac{dC}{dx}$$

$$\frac{dB}{dx} C = \frac{dA}{dx} - B \frac{dC}{dx} = \frac{dA}{dx} - AC^{-1} \frac{dC}{dx}$$

$$\boxed{\frac{dB}{dx} = \frac{dA}{dx}\, C^{-1} - AC^{-1} \frac{dC}{dx}\, C^{-1}} \tag{71}$$

(Note that both the above results are analogous to the scalar formulae for a = bc and b = a/c, except that in the matrix case the order of the terms in each product must not be altered).

c) Partial differentiation

Consider now a column vector Y, whose elements are functions of several variables $(x_1, x_2, \ldots x_n)$. Let X be the column vector whose elements are these variables for example:

$$Y = \begin{bmatrix} y_1 & (x_1, & x_2, & x_3) \\ \\ y_2 & (x_1, & x_2, & x_3) \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now the derivatives of y_1 are $\frac{\partial y_1}{\partial x_1}$, $\frac{\partial y_1}{\partial x_2}$, $\frac{\partial y_1}{\partial x_3}$. Adopt the convention that these derivatives form a row vector, denoted by

$$\frac{dy_1}{dX} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \end{bmatrix}$$

Then by this convention

$$\boxed{\frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{bmatrix}} \tag{72}$$

the (i, j)th element of which is

$$\frac{\partial y_i}{\partial x_j} \quad .$$

The total differential of Y is given by:

$$\boxed{d\,Y = \frac{\partial Y}{\partial X}\, d\,X} \tag{73}$$

where dY and dX are column vectors.

When X and Y have the same order, the matrix $\frac{\partial Y}{\partial X}$ is square and is called the Jacobian matrix of the transformation of X into Y, and its determinant is called the Jacobian of the transformation.

d) Derivative of the quadratic form

In the quadratic form

$$k = X^T A X$$

the elements of the matrix A are considered as constants, and the elements of the vector X as variables. The derivative of the quadratic form is

$$dk = dX^T A X + X^T A d X$$

but the value of each of these terms is unchanged after transposition, therefore

$$dX^T A X = (dX^T A X)^T = X^T A^T dX$$

and

$$dk = X^T A^T dX + X^T A dX$$

but A is symmetric ($A^T = A$) so

$$dk = 2X^T A dX$$

or

$$\boxed{\frac{d}{dX} (X^T AX) = 2X^T A} \tag{74}$$

e) Taylor's series in matrix form

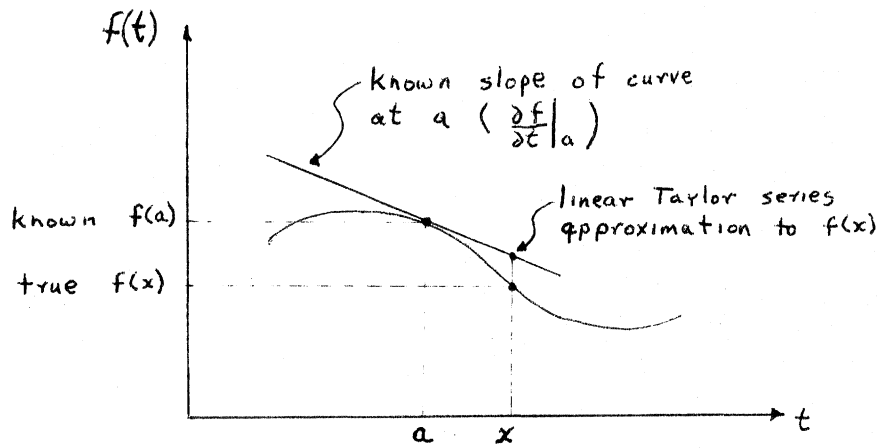
Given a single function f(t) of a single variable t, and a known value of this function f(a) at t = a, then values of the function at t = x are given by Taylor's series

$$f(x) = f(a) + \left.\frac{\partial f}{\partial t}\right|_a (x-a) + . . + \left.\frac{\partial^n f}{\partial t^n}\right|_a \frac{(x-a)^n}{n!} + \frac{1}{n!}\int_a^x (x-t)^n f(t)dt$$

For values of x close to a the <u>linear approximation</u> is used.

$$\boxed{f(x) = f(a) + \left.\frac{\partial f}{\partial x}\right|_a (x-a)} \tag{75}$$

The geometric meaning of this equation is that f(x) can be linearly approximated from a known value f(a) and the known slope of the f(t) curve at a, as shown below:



If f is a function of more than one variable, say $f(x_1, x_2)$ and its value is known at $x_1 = a_1$, $x_2 = a_2$, then for values of (x_1, x_2) close to (a_1, a_2) the linear approximation is

$$f(x_1, x_2) = f(a_1, a_2) + \left.\frac{\partial f}{\partial x_1}\right|_{a_1, a_2} (x_1-a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{a_1, a_2} (x_2-a_2)$$

Setting

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \Delta X = \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \end{bmatrix}, \quad X^\circ = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

$$\frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

Then

$$\boxed{f(X) = f(X^\circ) + \left.\frac{\partial f}{\partial X}\right|_{X^\circ} \Delta X} \tag{76}$$

If we now have more than one function of X we have a set of equations

$$f_1(X) = f_1(X^\circ) + \left.\frac{\partial f_1}{\partial X}\right|_{X^\circ} \Delta X$$

$$f_2(X) = f_2(X^\circ) + \left.\frac{\partial f_2}{\partial X}\right|_{X^\circ} \Delta X$$

Setting

$$F = \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix}, \qquad \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

Then

$$\boxed{F(X) = F(X^\circ) + \left.\frac{\partial F}{\partial X}\right|_{X^\circ} \Delta X}. \tag{77}$$

This is the Taylor's series linear approximation in matrix form.

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Ayres, F. (1962). Theory and Problems of Matrices. Schaum's Outline Series, McGraw-Hill, Toronto.

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APPENDIX A:

EXAMPLES IN MATRIX MANIPULATION

1) MATRIX NOTATION AND DEFINITIONS (section 2 in notes)

1) Which of the following matrices are square matrices? Which are row matrices? Which are column matrices? Which are none of these?

a) $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$

c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ d) $\begin{bmatrix} 7 & 8 \end{bmatrix}$

e) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ f) $\begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 2 & 0 \end{bmatrix}$ h) $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

2) Which of the following matrices are diagonal matrices? Which are scalar matrices? Which are identity matrices? Which are none of these?

a) $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$ b) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2) (cont'd)

g) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

h) $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

Answers: 1)

square	c, e
row	d, f
column	b, h
none	a, g

2)

diagonal	b, c, e, f, g, h
scalar	e, f, g, h
identity	f, g
none	a, d

II) ADDITION, MULTIPLICATION AND TRANSPOSITION (section 3 in notes)

3) From the following matrices, match those that are conformable for matrix addition, and add them.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad E = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \quad F = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$$

4) Multiply each of the following matrices by the scalar indicated.

a) $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$ by 3 b) $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ by k

5) From the following matrices, match the four pairs that are conformable for matrix multiplication, and multiply them.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 1 \\ 1 & 5 \\ 4 & 9 \\ 0 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

6) Which of the following pairs of matrices are communicative under matrix multiplication (i.e. A B = B A)?

a) $\begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 3 & 3 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$

e) $\begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$

7) Show that $(A\ B)^T = B^T\ A^T$ for the following pairs of matrices.

a) $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$ b) $\begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$

8) Which of the following matrices are symmetric?

a) $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

<u>Answers</u>: 3) $A + C = \begin{bmatrix} 4 & 3 & 7 \\ 5 & 10 & 15 \end{bmatrix}$ $B + D = \begin{bmatrix} 3 \\ 8 \\ 8 \end{bmatrix}$ $E + F = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$

4) a) $\begin{bmatrix} 9 & 3 & 12 \\ 3 & 15 & 27 \end{bmatrix}$ b) $\begin{bmatrix} 3k & k \\ k & 5k \end{bmatrix}$

5) $AB = \begin{bmatrix} 32 \\ 77 \end{bmatrix}$ $DC = \begin{bmatrix} 17 & 46 \end{bmatrix}$ $CA = \begin{bmatrix} 10 & 8 & 21 \\ 8 & 26 & 49 \\ 21 & 49 & 97 \\ 2 & 10 & 18 \end{bmatrix}$

$BD = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 6 & 12 & 18 & 24 \\ 5 & 10 & 15 & 20 \end{bmatrix}$

6) a, b, c

Answers (cont'd):

7) a) $\begin{bmatrix} 32 & 77 \end{bmatrix}$ b) $\begin{bmatrix} 10 & 30 \\ -8 & -2 \end{bmatrix}$

8) b, c

III) DETERMINANTS, INVERSES AND ORTHOGONAL MATRICES (section 4 of notes)

9) Find the determinants of the following matrices. Which are singular?

a) $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -6 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

10) Which pairs of the following matrices are inverses of each other? (Prove by showing $A A^{-1} = A^{-1} A = I$).

$A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

$D = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/5 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ $H = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

11) Show that $(A B)^{-1} = B^{-1} A^{-1}$ for the following pair of matrices.

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

12) Which of the following matrices are orthogonal? (Prove by showing $A\,A^T = A^T\,A = I$).

a) $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ b) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

<u>Answers</u>: 9) 14, -24, 9, 1, 0, 0 e, f are singular

10) AD = BE = CF = GH = I

11) $\begin{bmatrix} 5/3 & -2/5 \\ -2/3 & 1/5 \end{bmatrix}$

12) a, b, d, f

<u>IV) PARTITIONED MATRICES (section 5 in notes)</u>.

13) Partition the following pairs of matrices so that the corresponding submatrices are conformal for matrix multiplication (the partitioning will not be unique). Prove by multiplying the complete matrices,and the partitioned submatrices, and showing the results are the same.

a) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 1 & 4 & 0 \\ 1 & 5 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

<u>Answers:</u> 13) a) $\begin{bmatrix} 4 & 15 \\ 1 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

V) LINEAR EQUATIONS (Section 6 in notes.)

14) What is the rank of each of the following matrices? Write the largest nonsingular submatrix for each.

a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -6 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 3 & 2 \\ 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

15) Write the following systems of linear equations in matrix notation. Write the coefficient matrix, unknown vector, constant vector, and augmented matrix. Which of these systems are homogeneous? Which are non-homogeneous? Which are consistent? Which are inconsistent? Which have unique solutions? Find these unique solutions.

a) $\begin{aligned} x - 2y + z &= 1 \\ 2x - 3y + 4z &= 5 \\ 2x - 5y &= 0 \end{aligned}$ d) $\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 5y + 7z &= 0 \\ -2x - 4y - 6z &= 0 \end{aligned}$

b) $\begin{aligned} x - 2y + z &= 1 \\ 2x - 3y + 4z &= 5 \\ 2x - 5y + 2z &= 1 \end{aligned}$ e) $\begin{aligned} x + y &= 5 \\ x + 2y &= 7 \\ 2x + 3y &= 12 \end{aligned}$

c) $\begin{aligned} x - 2y + z &= 1 \\ 2x - 3y + 4z &= 5 \\ 2x - 5y &= -1 \end{aligned}$ f) $\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 5y + 7z &= 0 \\ -2x - 4y - 5z &= 0 \end{aligned}$

Answers: 14) Ranks = 1, 2, 2, 2, 3, 3

Largest nonsingular matrices for b, c, e $= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$,

$d = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ $f = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -1 & 2 \end{bmatrix}$

Answers (cont'd)

15) Homogeneous - d, f Non-homogeneous - a, b, c, e

Consistent - b, c, d, e, f Inconsistent - a

Unique solutions: b $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, e $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, f $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

VI) LINEAR TRANSFORMATIONS (Section 7 in notes).

16) Which of the following transformation matrices (matrix A in Y = A X) represent projective transformations (det A $\neq$ 0)? Which orthogonal transformations ($A A^T = I$)? Which reflections (orthogonal with det A = -1)? Which rotations (orthogonal with det A = +1)? Which scalar transformations? Which affine transformations? Draw a diagram for each transformation showing the original vector X = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the transformed vector Y = A X.

a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ g) $\begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$ i) $\begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}$

b) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ e) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ h) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ j) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ f) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

17) Find the eigenvalues, eigenvectors, normalized eigenvectors, and the diagonal matrix which is orthogonally similar to each of the following symmetric matrices.

a) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 7 & 4 \\ 4 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ g) $\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$

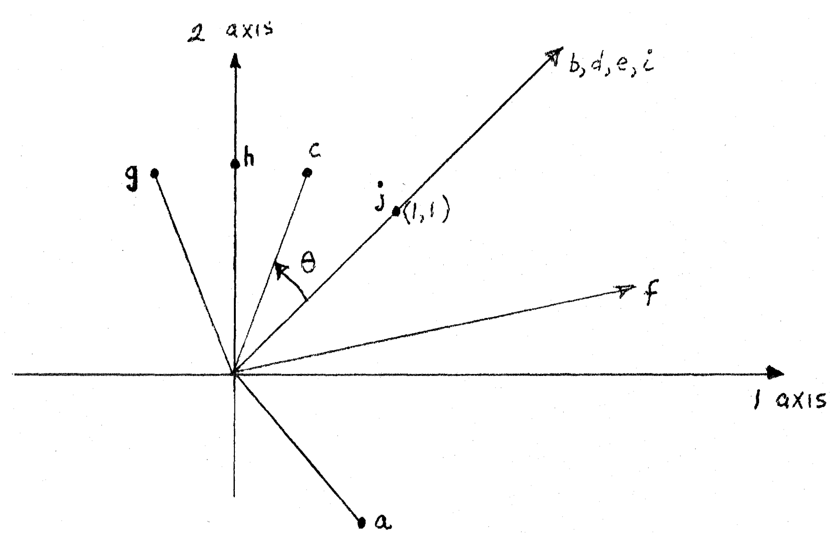
b) $\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$ f) $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ h) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$

18) Write the matrix of each of the following quadratic forms. What value class does each quadratic form belong to?

a)	$5x^2 + 6xy + 5y^2$	e)	$3x^2 + 8xy - 3y^2$
b)	$5x^2 + 4xy + 2y^2$	f)	$-5x^2 + 4xy - 2y^2$
c)	$7x^2 + 8xy + y^2$	g)	$9x^2 + 6xy + y^2$
d)	$5x^2 + 2xy + 3y^2$	h)	$-2x^2 + 4xy + y^2$

<u>Answers</u>: 16)

Projective	all but f.
Orthogonal	a, c, g, h, j.
Reflections	a.
Rotations	c, g, h, j.
Scalar	b, i.
Affine	d, e.



Answers (Cont'd)

17) a) 8, 2. e) ± 5

b) 6, 1. f) -6, -1.

c) 9, -1. g) 10, 0.

d) $4 \pm \sqrt{2}$ h) 3, -2.

18) Pos Def a, b, d.

Neg Def f.

Pos Semidef g.

Neg Semidef -

Indef c, e, h.

VII) DIFFERENTIATION OF MATRICES (Section 8 in notes).

19) Write the derivatives (with respect to x) of the following matrices.

a) $\begin{bmatrix} 1 & x \\ x^2 & x^3 \end{bmatrix}$ b) $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ c) $\begin{bmatrix} x & x^2 \\ x^2 & x \end{bmatrix}$

20) Write the derivatives of the quadratic forms listed in question 18) (with respect to x and y) both directly, and using the relation

$$\frac{d\ (X^T A X)}{d\ X} = 2X^T A.$$

21) Write the linear Taylor's series approximation for the following functions, using the relation

$$f(x) = f(a) + \left.\frac{\partial f}{\partial t}\right|_a (x - a).$$

a) $f(t) = 1 + t^2$ about $a = 0$

b) $f(t) = 1 + t^2$ about $a = 1$

c) $f(t) = (1 + t)^{1/2}$ about $a = 0$

d) $f(t) = \cos t$ about $a = \pi/2$

e) $f(t) = \tan^{-1} t$ about $a = 1$

Answers:

19) a) $\begin{bmatrix} 0 & 1 \\ 2x & 3x^2 \end{bmatrix}$ b) $\begin{bmatrix} -\sin x & -\cos x \\ \cos x & -\sin x \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2x \\ 2x & 1 \end{bmatrix}$

20) a) $\begin{bmatrix} 10x + 6y & 6x + 10y \end{bmatrix}$ e) $\begin{bmatrix} 6x + 8y & 8x - 6y \end{bmatrix}$

b) $\begin{bmatrix} 10x + 4y & 4x + 4y \end{bmatrix}$ f) $\begin{bmatrix} -10x + 4y & 4x - 4y \end{bmatrix}$

c) $\begin{bmatrix} 14x + 8y & 8x + 2y \end{bmatrix}$ g) $\begin{bmatrix} 18x + 6y & 6x + 2y \end{bmatrix}$

d) $\begin{bmatrix} 10x + 2y & 2x + 6y \end{bmatrix}$ h) $\begin{bmatrix} -4x + 4y & 4x + 2y \end{bmatrix}$

21) a) $f(x) = 1$ d) $f(x) = \pi/2 - x$

b) $f(x) = 2x$ e) $f(x) = \pi/4 + \frac{x - 1}{2}$

c) $f(x) = 1 + \frac{x}{2}$

APPENDIX B

PROPERTIES OF MATRIX TRACES*

Definitions:

Given a square matrix A, its trace is the sum of its diagonal elements

$$\text{Trace } A = Tr(A) = \sum_i a_{ii} \ .$$

Given a matrix A and a square matrix F which is a product of matrices including A, the partial derivative of the trace of F with respect to the matrix A is a matrix whose elements are the partial derivatives of the trace of F with respect to the corresponding elements of A, that is if

$$A = [a_{ij}]$$

then

$$\frac{\partial Tr(F)}{\partial A} = [\frac{\partial\ Tr(F)}{\partial a_{ij}}] \ .$$

Properties (Theorems):

$$Tr(A^T) = Tr(A)$$

Given a constant k

$$Tr(kA) = k\ Tr(A)$$

Given two matrices A and B conformable under addition

$$Tr(A + B) = Tr(A) + Tr(B)$$

Given two matrices A and B conformable under both multiplications AB and BA

$$Tr(\ A\ B) = Tr\ (B\ A)$$

---

* A complete discussion of these properties of traces is found in Blaha, G. (1971). "Inner Adjustment Constraints With Emphasis on Range Observations", Reports of the O.S.U. Department of Geodetic Science, Report No. 148.

Given two matrices A and B conformable under both multiplications $A^T B$ and AB^T

$$Tr\ (A^T B) = Tr\ (A B^T)$$

From the above properties it is evident that similar matrices have the same trace, that is for any nonsingular matrix R, and any matrix A of same order as R

$$Tr\ (R^{-1} A R) = Tr\ (A)$$

and in particular if R is the orthogonal matrix which diagonalizes A we have

$$Tr\ (A) = \sum_i \lambda_i$$

where λ_i are the eigenvalues of A.

A property of the derivative of Tr (F) is

$$\frac{\partial\ Tr(F)}{\partial\ A^T} = \left[\frac{\partial\ Tr(F)}{\partial A} \right]^T$$

For specific forms of F we have

$$F = AB \qquad \frac{\partial\ Tr(A\ B)}{\partial\ A} = \frac{\partial\ Tr\ (B\ A)}{\partial\ A} = B^T$$

$$F = ABA^T \qquad \frac{\partial Tr\ (A\ B\ A^T)}{\partial A} = A(B + B^T)$$

$$F = A^T BA \qquad \frac{\partial\ Tr(A^T BA)}{\partial A} = (B + B^T)A$$

$$F = ABA^T C \qquad \frac{\partial\ Tr\ (A\ B\ A^T C)}{\partial A} = C^T AB^T + CAB$$

APPENDIX C

ALGORITHM FOR THE PRODUCT MATRIX RESULTING FROM A SEQUENCE OF ROTATIONS AND REFLECTIONS

PROGRAM NO. ______________

EQUIPMENT IBM 370 / 155

UNIVERSITY OF NEW BRUNSWICK

SURVEYING ENGINEERING PROGRAM LIBRARY

PROGRAM DOCUMENTATION

SOURCE D. Wells

PROGRAM NAME ROTREF

PROGRAM TYPE Subroutine

DATE August 1973

PROGRAM LANGUAGE FORTRAN IV

PURPOSE To compute the product matrix rsulting from a sequence of rotations and reflections

METHOD OF USE

Double precision is used.

Calling statement is CALL ROTREF(NUM,NAXIS,ANGLE,ROT)

Inputs are

NUM = number of rotations and reflections in the input sequence (no limit)

NAXIS = vector of rotation and reflection axes (for rotations use 1,2,3 and for reflections use -1,-2,-3)

ANGLE = vector of rotation angles in radians (for reflections this angle ignored -set to 0)

Output is ROT = 3 x 3 product matrix

ATTACHMENTS

1) summary of rotations and reflections

2) flowchart

3) program listing

4) test results

SUMMARY OF REFLECTION AND ROTATION MATRICES

1 Orthogonal Transformations

The matrix equation

$$Y = A X$$

where A is a matrix and X and Y are column vectors, can be regarded as a linear transformation, in which case the matrix A is called the transformation matrix. If the two vectors X and Y have the same length, then both the transformation and the matrix are said to be orthogonal. Orthogonal matrices have the property that the product of the matrix and its transpose (or vice versa) is the identity matrix, that is

$$A^T A = A A^T = I.$$

From this property it follows that the determinant of an orthogonal matrix is either +1 or -1. There are two kinds of orthogonal transformations called reflections and rotations. The determinant of reflection matrices is -1, and the determinant of rotation matrices is +1.

There are two interpretations of the linear transformation above. The first is that the transformation describes the relationship between two coordinate systems, in which case X and Y are the same vector, but their elements refer to the two different systems. The second is that the transformation describes the relationship between different vectors X and Y in the same coordinate system. In these notes, we are interested only in the first interpretation.

2 Right and Left Handed Cartesian Coordinate Systems

A three dimensional Cartesian coordinate system can be orthogonally transformed in only six different ways. It can be rotated about each of its axes. Each of its axes can be reflected. In such a coordinate system, the vectors X and Y will have only three elements. Let us define the axis to which the first, second, and third elements of X and Y are referred as the 1-axis, 2-axis, and 3-axis respectively (we could equally well label them the x_1, x_2, x_3 axes or x, y, z axes).

These three axes may define either a right-handed or a left-handed coordinate system. Right handed systems follow the right hand rule: if the fingers of the right hand are curled around any axis so that the thumb points in the positive direction, then the fingers will point from a second axis to the third axis, numbered in cyclic fashion. Grasping the 1-axis, the fingers point from the 2-axis to the 3-axis. Grasping the 2-axis, the fingers point from the 3-axis to the 1-axis. Grasping the 3-axis, the fingers point from the 1-axis to the 2-axis. Left-handed coordinate systems follow the left hand rule, which differs from the above only in that the left hand is used.

3 Reflections

If we denote a reflection of the kth axis by P_k, then the following expressions define the three reflection matrices:

$$P_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that reflection matrices commute (e.g. $P_2P_3 = P_3P_2$), so that it makes no difference in what order a sequence of reflections are performed. Note also that an odd number of reflections changes the handedness of the coordinate system.

4 Rotations

If we denote a rotation of angle θ about the k^{th} axis by $R_k(\theta)$, then the following expressions define the three rotation matrices:

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that rotation matrices do <u>not</u> commute. The product of several rotations is performed from right to left, for example in

$$R_1(\alpha) \quad R_2(\beta) \quad R_3(\gamma)$$

the rotations are performed about the 3-axis of the original system, the 2-axis of the transformed system, and the 1-axis of the doubly transformed system, to yield the final triply transformed system.

If the rotation angles are all so small that their cosines can be assumed to be unity, then the rotation matrices become commutative. This is the case for <u>differential rotations</u>, for example.

The above expressions define <u>positive rotations</u>, which are right-hand rotations for right-handed coordinate systems and left-hand rotations for left-handed coordinate systems. A right-hand rotation is related to the right hand rule given above: if the fingers of the right hand are curled around the rotation axis so that the thumb points in the positive direction, then the fingers curl in the direction of a right hand rotation. A similar statement for left hand rotations is obvious.

5 Inverse Transformations

The inverse of a transformation A (denoted A^{-1}) is the transformation which returns conditions to their original state, that is

$$A^{-1}\ A = A\ A^{-1} = I.$$

Relfections are self-inverse, that is

$$P_k^{-1} = P_k$$

$$P_k\ P_k = I$$

Common sense tells us that the inverse of a positive rotation is a negative rotation, that is

$$R_k^{-1}(\theta) = R_k(-\theta)$$

and this conclusion is verified by taking the orthogonal property

$$A^T\ A = I$$

from which it is evident that for orthogonal matrices

$$A^{-1} = A^T$$

and for each of the above expressions for rotation matrices it can be shown that

$$R_k^T\ (\theta) = R_k(-\theta).$$

Applying the rule for the inverse of products

$$[A\ B]^{-1} = B^{-1}\ A^{-1}$$

we have

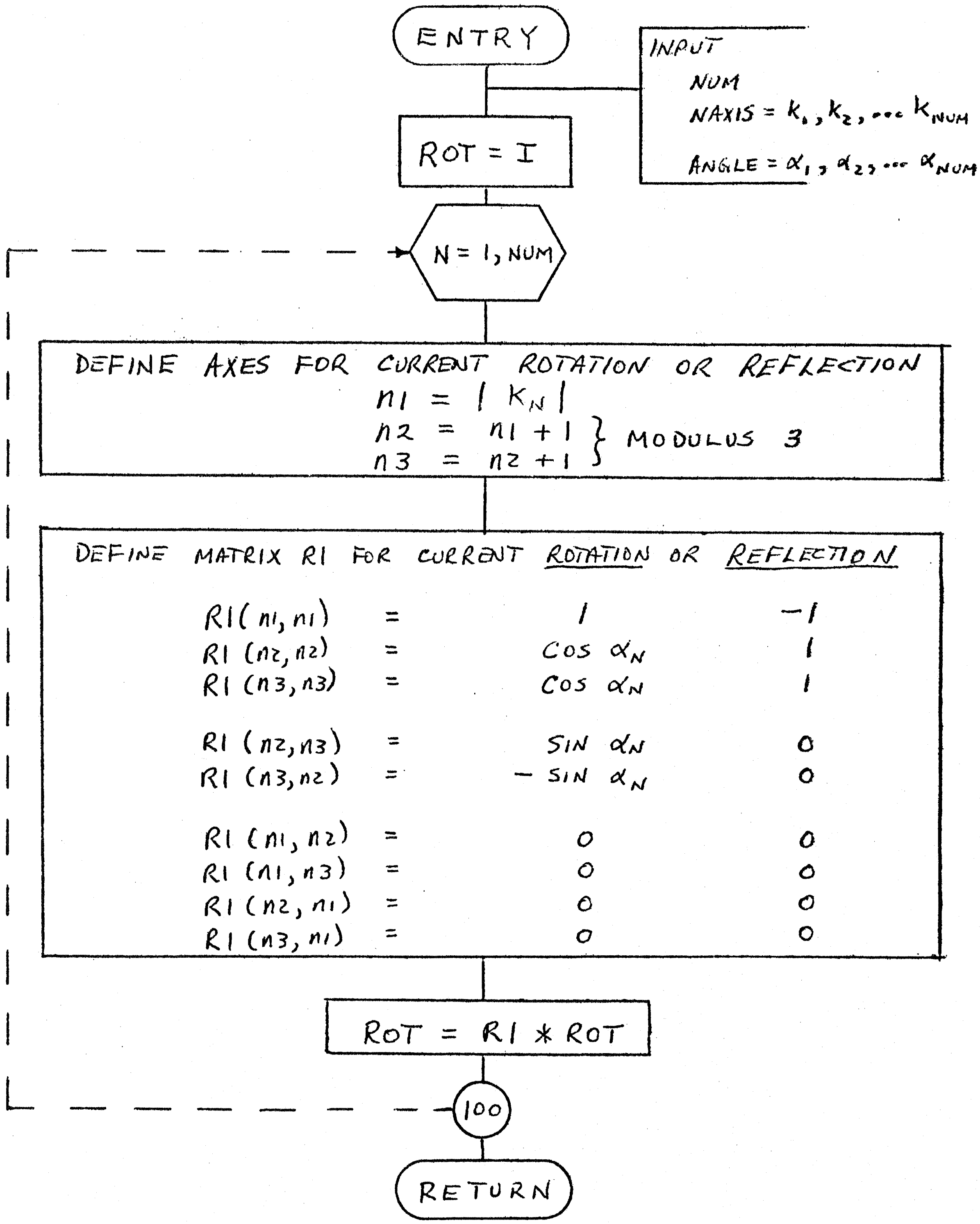
$$[R_j(\alpha)\ R_k(\beta)]^{-1} = R_k^T(\beta)\ R_j^T(\alpha) = R_k(-\beta)\ R_j(-\alpha)$$

A product transformation consisting of one rotation and one reflection commutes only if the rotation and reflection refer to the same axis, that is

$$P_j \, R_k = R_k \, P_j \quad \text{if } j = k$$

otherwise

$$P_j \, R_k = R_k^{-1} \, P_j \quad \text{if } j \neq k.$$


ENTRY
INPUT
NUM
NAXIS = $k_1, k_2, \ldots k_{NUM}$
ANGLE = $\alpha_1, \alpha_2, \ldots \alpha_{NUM}$
ROT = I
N = 1, NUM
DEFINE AXES FOR CURRENT ROTATION OR REFLECTION
$n1 = |K_N|$
$n2 = n1 + 1$
$n3 = n2 + 1$
MODULUS 3
DEFINE MATRIX R1 FOR CURRENT ROTATION OR REFLECTION
R1(n1, n1) = 1 -1
R1(n2, n2) = COS α_N 1
R1(n3, n3) = COS α_N 1
R1(n2, n3) = SIN α_N 0
R1(n3, n2) = - SIN α_N 0
R1(n1, n2) = 0 0
R1(n1, n3) = 0 0
R1(n2, n1) = 0 0
R1(n3, n1) = 0 0
ROT = R1 * ROT
100
RETURN

```
      SUBROUTINE ROTREF(NUM,NAXIS,ANGLE,ROT)
C
C COMPUTE PRODUCT MATRIX OF A SEQUENCE OF ROTATIONS AND REFLECTIONS
C
C INPUT ARGUMENTS
C NUM = NUMBER OF ROTATIONS AND REFLECTIONS IN SEQUENCE
C NAXIS(NUM) = SEQUENCE OF ROTATION AND REFLECTION AXES
C              FOR ROTATIONS USE 1,2, OR 3
C              FOR REFLECTIONS USE -1, -2, OR -3
C ANGLE(NUM) = SEQUENCE OF ROTATION ANGLES IN RADIANS
C              FOR REFLECTIONS THIS ANGLE IS IGNORED (SET TO ZERO)
C OUTPUT ARGUMENT
C ROT(3,3) = PRODUCT MATRIX
C
      DOUBLE PRECISION ROT,R1,R2,ANGLE,EPS,COS,DCOS,SIN,DSIN,ABS,DABS
      DIMENSION ROT(3,3),R1(3,3),R2(3),ANGLE(NUM),NAXIS(NUM)
      DATA EPS/1D-15/
      COS(EPS) = DCOS(EPS)
      SIN(EPS) = DSIN(EPS)
      ABS(EPS) = DABS(EPS)
C SET 'ROT' = IDENTITY MATRIX
      DO 1 I = 1,3
        DO 1 J = 1,3
          ROT(I,J) = 0.
          IF(I .EQ. J) ROT(I,J) = 1.
    1     CONTINUE
C CHECK ELEMENTS OF 'NAXIS' AND SET REFLECTION ELEMENTS OF 'ANGLE' = 0.
      DO 2 N = 1,NUM
        IF(NAXIS(N) .EQ. 0 .OR. NAXIS(N) .LT. -3 .OR. NAXIS(N) .GT. 3)
     *    GO TO 5
        IF(NAXIS(N) .LT. 0) ANGLE(N) = 0.
    2   CONTINUE
C PROCESS SEQUENCE OF ROTATIONS AND REFLECTIONS ONE AT A TIME
      DO 4 N = 1,NUM
C DEFINE THREE AXES FOR CURRENT ROTATION OR REFLECTION
        N1 = IABS(NAXIS(N))
        N2 = MOD(N1,3) + 1
        N3 = MOD(N2,3) + 1
C DEFINE DIAGONAL ELEMENTS
        R1(N1,N1) = 1.
        IF(NAXIS(N) .LT. 0.) R1(N1,N1) = -1.
        R1(N2,N2) = COS(ANGLE(N))
        R1(N3,N3) = R1(N2,N2)
C DEFINE NON-ZERO OFF-DIAGONAL ELEMENTS
        R1(N2,N3) = SIN(ANGLE(N))
        R1(N3,N2) = - R1(N2,N3)
C DEFINE ZERO OFF-DIAGONAL ELEMENTS
        R1(N1,N2) = 0.
        R1(N1,N3) = 0.
        R1(N2,N1) = 0.
        R1(N3,N1) = 0.
C FORM PRODUCT (SET 'ROT' = 'R1' * 'ROT')
        DO 4 J = 1,3
          DO 3 I = 1,3
            R2(I) = 0.
            DO 3 K = 1,3
    3         R2(I) = R2(I) + R1(I,K) * ROT(K,J)
          DO 4 I = 1,3
            ROT(I,J) = R2(I)
            IF(ABS(ROT(I,J)) .LT. EPS) ROT(I,J) = 0.
    4       CONTINUE
      RETURN
    5 WRITE(6,6) N,NAXIS(N)
    6 FORMAT(10X,'ILLEGAL VALUE NAXIS(',I3,') = ',I5,' IN ROTREF')
      RETURN
      END
```

TEST RESULTS

The program was tested by computing the product matrix for the following sequence of rotations and reflections:

$$R_3(-\tfrac{\pi}{2})\ R_2(-\tfrac{\pi}{2})\ R_1(\alpha)\ R_3(\beta-\tfrac{\pi}{2})\ P_2\ R_2(\tfrac{\pi}{2})\ R_3(\tfrac{\pi}{2})\ P_1\ R_2(\beta)\ R_3(\alpha)$$

for the case $\alpha = \beta = \frac{\pi}{4}$. For this example we have the following input to ROTREF:

NUM = 10

NAXIS = (3, 2, -1, 3, 2, -2, 3, 1, 2, 3)

ANGLE = $(\frac{\pi}{4}, \frac{\pi}{4}, 0, \frac{\pi}{2}, \frac{\pi}{2}, 0, -\frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{2})$

It can be shown (by drawing the new coordinate axes after each rotation and reflection, for example) that the above sequence results in a product matrix which is the identity matrix (i.e. the net effect of this sequence is to leave the coordinate system unchanged).

Attached is the test program listing and output. The product matrix was computed and printed as each of the above rotations and reflections were added to the sequence . In a production program, ROTREF would only be called once to compute the product matrix for the entire sequence.

```
C
C TEST SUBROUTINE ROTREF
C
      DOUBLE PRECISION ROT,ANGLE,PI
      DIMENSION ROT(3,3),ANGLE(20),NAXIS(20)
      PI = 3.14159265358979300
C READ INPUT SEQUENCE OF AXES AND ANGLES
      READ,NUMT
      READ,(NAXIS(K),K=1,NUMT)
      READ,(ANGLE(K),K=1,NUMT)
C PRINT INPUT SEQUENCE OF AXES AND ANGLES
      WRITE(6,10) (NAXIS(K),K=1,NUMT)
      DO 1 K = 1,NUMT
         IF(ANGLE(K) .NE. 0.) ANGLE(K) = 180. / ANGLE(K)
    1    CONTINUE
      WRITE(6,11) (ANGLE(K),K=1,NUMT)
      DO 2 K = 1,NUMT
    2    ANGLE(K) = ANGLE(K) * PI / 180.
C COMPUTE SEQUENCE OF PRODUCT MATRICES
      DO 3 NUM = 1,NUMT
         CALL ROTREF(NUM,NAXIS,ANGLE,ROT)
    3    WRITE(6,12) NUM,((ROT(I,J),J=1,3),I=1,3)
      RETURN
   10 FORMAT('1',5X,'INPUT AXES'/10X,20I5)
   11 FORMAT('0',5X,'INPUT ANGLES(IN DEGREES)'/11X,20F5.0)
   12 FORMAT('0',5X,'PRODUCT OF FIRST',I5,' ROTATIONS AND REFLECTIONS',
     *   3(/10X,3E20.10))
      END
```

```
INPUT AXES
          3     2    -1     3     2    -2     3     1     2     3

INPUT ANGLES(IN DEGREES)
         45.   45.    0.   90.   90.    0.  -45.   45.  -90.  -90.

PRODUCT OF FIRST     1 ROTATIONS AND REFLECTIONS
          0.7071067812D 00      0.7071067812D 00      0.0000000000D 00
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00
          0.0000000000D 00      0.0000000000D 00      0.1000000000D 01

PRODUCT OF FIRST     2 ROTATIONS AND REFLECTIONS
          0.5000000000D 00      0.5000000000D 00     -0.7071067812D 00
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00
          0.5000000000D 00      0.5000000000D 00      0.7071067812D 00

PRODUCT OF FIRST     3 ROTATIONS AND REFLECTIONS
         -0.5000000000D 00     -0.5000000000D 00      0.7071067812D 00
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00
          0.5000000000D 00      0.5000000000D 00      0.7071067812D 00

PRODUCT OF FIRST     4 ROTATIONS AND REFLECTIONS
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00
          0.5000000000D 00      0.5000000000D 00     -0.7071067812D 00
          0.5000000000D 00      0.5000000000D 00      0.7071067812D 00

PRODUCT OF FIRST     5 ROTATIONS AND REFLECTIONS
         -0.5000000000D 00     -0.5000000000D 00     -0.7071067812D 00
          0.5000000000D 00      0.5000000000D 00     -0.7071067812D 00
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00

PRODUCT OF FIRST     6 ROTATIONS AND REFLECTIONS
         -0.5000000000D 00     -0.5000000000D 00     -0.7071067812D 00
         -0.5000000000D 00     -0.5000000000D 00      0.7071067812D 00
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00

PRODUCT OF FIRST     7 ROTATIONS AND REFLECTIONS
          0.0000000000D 00      0.0000000000D 00     -0.1000000000D 01
         -0.7071067812D 00     -0.7071067812D 00      0.0000000000D 00
         -0.7071067812D 00      0.7071067812D 00      0.0000000000D 00

PRODUCT OF FIRST     8 ROTATIONS AND REFLECTIONS
          0.0000000000D 00      0.0000000000D 00     -0.1000000000D 01
         -0.1000000000D 01      0.0000000000D 00      0.0000000000D 00
          0.0000000000D 00      0.1000000000D 01      0.0000000000D 00

PRODUCT OF FIRST     9 ROTATIONS AND REFLECTIONS
          0.0000000000D 00      0.1000000000D 01      0.0000000000D 00
         -0.1000000000D 01      0.0000000000D 00      0.0000000000D 00
          0.0000000000D 00      0.0000000000D 00      0.1000000000D 01

PRODUCT OF FIRST    10 ROTATIONS AND REFLECTIONS
          0.1000000000D 01      0.0000000000D 00      0.0000000000D 00
          0.0000000000D 00      0.1000000000D 01      0.0000000000D 00
          0.0000000000D 00      0.0000000000D 00      0.1000000000D 01
```

APPENDIX D

CHOLESKI ALGORITHM FOR MATRIX INVERSION

PROGRAM NO. ______________

EQUIPMENT IBM 370 / 155

UNIVERSITY OF NEW BRUNSWICK

SURVEYING ENGINEERING PROGRAM LIBRARY

PROGRAM DOCUMENTATION

SOURCE D. Wells

PROGRAM NAME CHOLD

PROGRAM TYPE Subroutine

DATE August 1973

PROGRAM LANGUAGE FORTRAN IV

PURPOSE To compute the inverse and determinant of a given positive definite symmetric matrix, using the method of Choleski decomposition

METHOD OF USE

Double precision is used

Execution-time dimensioning of matrix is used

Matrix is inverted in place (input matrix is destroyed)

Calling statement is CALL CHOLD(A,IRDA,NA,DETA,&n)

Inputs are
- A = array containing input matrix to be inverted
- IRDA = row dimension of array A in calling program
- NA = size of input matrix contained in A

Outputs are
- A = now contains the inverse of the input matrix
- DETA = determinant of input matrix
- n = statement number to which control is transferred if NA is less than 1, or if DETA is less than 10^{-10}

ATTACHMENTS

1) discussion of algorithm

2) demonstration for 4 x 4 matrix

3) flowchart

4) program listing

5) test results for 4 x 4 Hilbert matrix $(a_{ij} = 1 / (i + j - 1))$

THE CHOLESKI MATRIX INVERSION ALGORITHM

The inversion of a triangular matrix is a much simpler process than the inversion of a full matrix. In the case of positive definite symmetric (PDS) full matrices, it is possible to take advantage of this fact.

Given a PDS matrix A, it is always possible to decompose A into a lower triangular matrix L such that

$$A = L\ L^T .$$

Then the inversion process can be performed on L, not on A, to obtain

$$\lambda = L^{-1}$$

after which a new full matrix B can be constructed from

$$B = \lambda^T\ \lambda \quad .$$

It is simple to see that B is the inverse of the original matrix A

$$B = \lambda^T\ \lambda = (L^{-1})^T\ L^{-1} = (L^T)^{-1}\ L^{-1} = (L\ L^T)^{-1} = A^{-1} .$$

The Choleski algorithm for matrix inversion incorporates these three steps, the latter two of which are relatively trivial. The important feature is the decomposition step which is performed using the Choleski decomposition (sometimes called the "square root method").

Given below are the algorithms for each of the three steps, assuming A to be positive definite symmetric and fully populated. It should be noted that when A is PDS and has a banded structure, these algorithms can be modified to be more efficient. The sequence of operations in each step has been arranged so that A, L, λ and B may all use the same storage array. Operations are omitted which would compute or use the zero elements of L and λ, and which would compute the redundant elements of B (due to its symmetry). Each algorithm is demonstrated for the case when A is a 4 x 4 matrix.

STEP 1 - Choleski decomposition of input matrix A

Given the n x n PDS matrix A, find the n x n lower triangular matrix L such that

$$A = L\, L^T .$$

First column of L

$$\ell_{11} = \sqrt{a_{11}}$$

$$\ell_{i1} = a_{i1} / \ell_{11} \qquad i = 2,3,\ldots,n$$

Subsequent columns of L (j = 2,3,...,n)

$$\ell_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2}$$

$$\ell_{ij} = \frac{\left(a_{ij} - \sum_{k=1}^{j-1} \ell_{ik}\,\ell_{jk}\right)}{\ell_{jj}} \qquad i = j+1, j+2, \ldots, n$$

$$\ell_{ij} = 0 \qquad i < j \quad \text{(omitted)}$$

STEP 2 - Inversion of lower triangular matrix L

Given the n x n lower triangular matrix L, find the n x n lower triangular matrix λ such that

$$\mathbf{L}\,\lambda = I .$$

Diagonal elements of λ

$$\lambda_{ii} = 1/\ell_{ii} \qquad i = 1,2,\ldots,n$$

Off-diagonal elements by columns (j = 1,2,...,n-1)

$$\lambda_{ij} = -\lambda_{ii} \sum_{k=j}^{i-1} \ell_{ik}\,\ell_{kj} \qquad i = j+1, j+2, \ldots, n$$

$$\lambda_{ij} = 0 \qquad i < j \quad \text{(omitted)}$$

STEP 3 - Construction of inverse matrix B

Given the n x n lower triangular matrix λ, find the n x n matrix B such that

$$B = \lambda^T \lambda .$$

First column of B

$$b_{i1} = \sum_{k=i}^{n} \lambda_{ki}\,\lambda_{k1} \qquad i = 1,2,\ldots,n$$

Subsequent columns of B (j = 2,3,...,n)

$$b_{ij} = \sum_{k=i}^{n} \lambda_{ki}\,\lambda_{kj} \qquad i = j, j+1, \ldots, n$$

$$b_{ij} = b_{ji} \qquad i < j$$

References:

Carnahan, Luther and Wilkes (1969). "Applied Numerical Methods" Wiley. (page 334)

Faddeev and Faddeeva (1963). "Computational Methods of Linear Algebra" Freeman. (page 144)

Thompson (1969). "Introduction to the Algebra of Matrices with some Applications" University of Toronto. (page 217)

STEP 1 - Demonstration for 4 x 4 matrix

The matrix equation is $A = L\ L^T$

where we know A and want to find L. For the 4 x 4 case the equation is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ & l_{22} & l_{32} & l_{42} \\ & & l_{33} & l_{43} \\ & & & l_{44} \end{bmatrix}$$

We now give the component equations for this matrix equation, and their solutions for the components of L, in the sequence in which they are determined in the above algorithm:

Component Equations	Solutions for l_{ij}
$a_{11} = l_{11}^2$	$l_{11} = \sqrt{a_{11}}$
$a_{21} = l_{21}\, l_{11}$	$l_{21} = a_{21} / l_{11}$
$a_{31} = l_{31}\, l_{11}$	$l_{31} = a_{31} / l_{11}$
$a_{41} = l_{41}\, l_{11}$	$l_{41} = a_{41} / l_{11}$
$a_{22} = l_{21}^2 + l_{22}^2$	$l_{22} = \sqrt{a_{22} - l_{21}^2}$
$a_{32} = l_{31}\, l_{21} + l_{32}\, l_{22}$	$l_{32} = (a_{32} - l_{31}\, l_{21}) / l_{22}$
$a_{42} = l_{41}\, l_{21} + l_{42}\, l_{22}$	$l_{42} = (a_{42} - l_{41}\, l_{21}) / l_{22}$
$a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2$	$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$
$a_{43} = l_{41} l_{31} + l_{42} l_{32} + l_{43} l_{33}$	$l_{43} = (a_{43} - l_{41} l_{31} - l_{42} l_{32}) / l_{33}$
$a_{44} = l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2$	$l_{44} = \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2}$

STEP 2 - Demonstration for 4 x 4 matrix

The matrix equation is $L \quad \lambda \quad = \quad I$

where we know L and want to find λ. For the 4 x 4 case the equation is

$$\begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} \lambda_{11} & & & \\ \lambda_{21} & \lambda_{22} & & \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

The components of this matrix equation and their solutions for the components of λ in the sequence of the above algorithm, are:

Component Equations	Solutions for λ_{ij}
$l_{11}\lambda_{11} = 1$	$\lambda_{11} = 1/l_{11}$
$l_{22}\lambda_{22} = 1$	$\lambda_{22} = 1/l_{22}$
$l_{33}\lambda_{33} = 1$	$\lambda_{33} = 1/l_{33}$
$l_{44}\lambda_{44} = 1$	$\lambda_{44} = 1/l_{44}$
$l_{21}\lambda_{11} + l_{22}\lambda_{21} = 0$	$\lambda_{21} = -\lambda_{22}(l_{21}\lambda_{11})$
$l_{31}\lambda_{11} + l_{32}\lambda_{21} + l_{33}\lambda_{31} = 0$	$\lambda_{31} = -\lambda_{33}(l_{31}\lambda_{11} + l_{32}\lambda_{21})$
$l_{41}\lambda_{11} + l_{42}\lambda_{21} + l_{43}\lambda_{31} + l_{44}\lambda_{41} = 0$	$\lambda_{41} = -\lambda_{44}(l_{41}\lambda_{11} + l_{42}\lambda_{21} + l_{43}\lambda_{31})$
$l_{32}\lambda_{22} + l_{33}\lambda_{32} = 0$	$\lambda_{32} = -\lambda_{33}(l_{32}\lambda_{22})$
$l_{42}\lambda_{22} + l_{43}\lambda_{32} + l_{44}\lambda_{42} = 0$	$\lambda_{42} = -\lambda_{44}(l_{42}\lambda_{22} + l_{43}\lambda_{32})$
$l_{43}\lambda_{33} + l_{44}\lambda_{43} = 0$	$\lambda_{43} = -\lambda_{44}(l_{43}\lambda_{33})$

STEP 3 - Demonstration for 4 x 4 matrix

The matrix equation is $B = \lambda^T \lambda$

where we know λ and want to find B. for the 4 x 4 case the equation is

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} & \lambda_{41} \\ & \lambda_{22} & \lambda_{32} & \lambda_{42} \\ & & \lambda_{33} & \lambda_{43} \\ & & & \lambda_{44} \end{bmatrix} \begin{bmatrix} \lambda_{11} & & & \\ \lambda_{21} & \lambda_{22} & & \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix}$$

The components of this equation (which are the solutions for b_{ij}) in the sequence of the above algorithm are:

$$b_{11} = \lambda_{11}^2 + \lambda_{21}^2 + \lambda_{31}^2 + \lambda_{41}^2$$
$$b_{21} = \lambda_{22}\lambda_{21} + \lambda_{32}\lambda_{31} + \lambda_{42}\lambda_{41}$$
$$b_{31} = \lambda_{33}\lambda_{31} + \lambda_{43}\lambda_{41}$$
$$b_{41} = \lambda_{44}\lambda_{41}$$

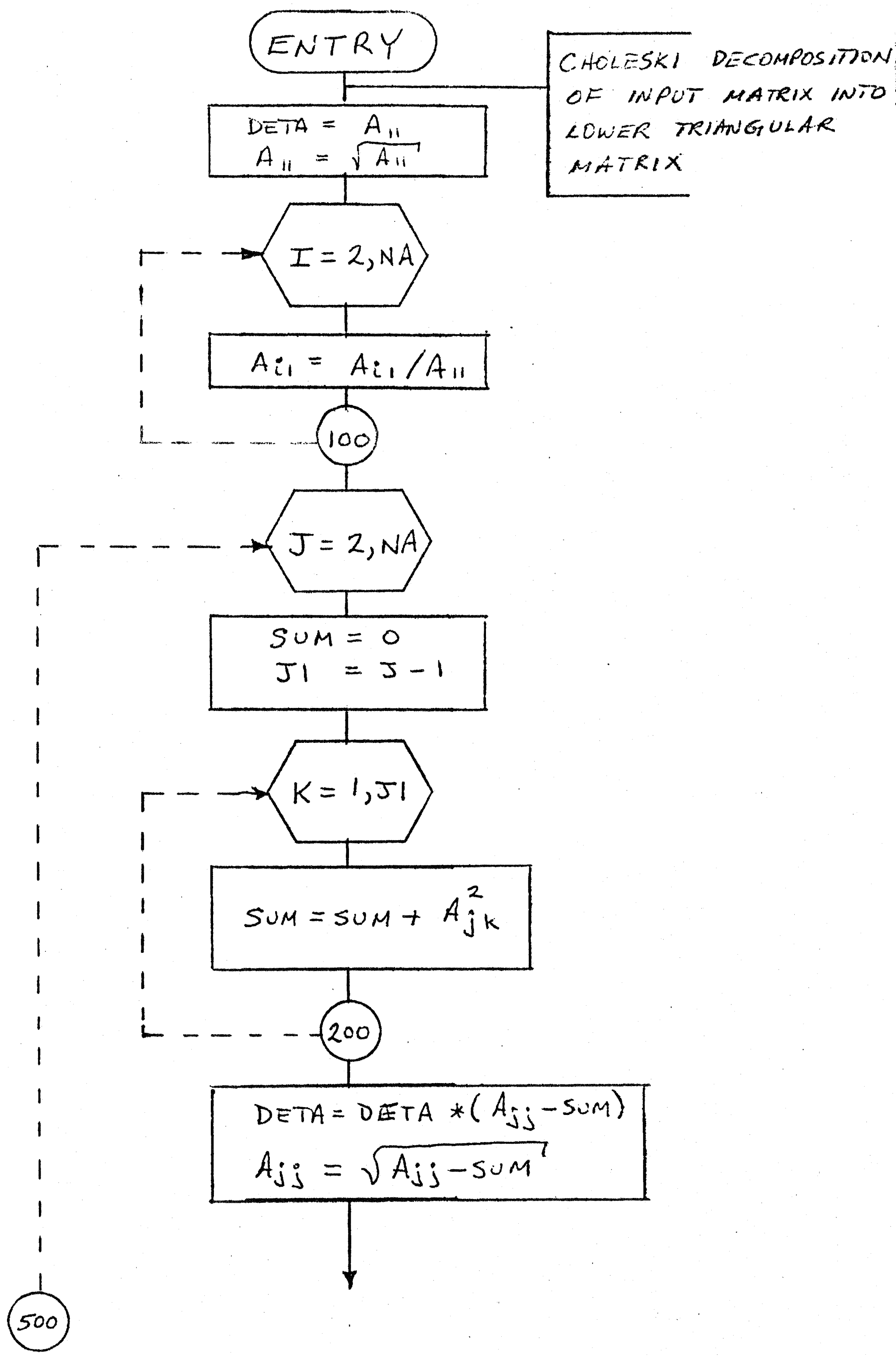
$$b_{12} = b_{21}$$
$$b_{22} = \lambda_{22}^2 + \lambda_{32}^2 + \lambda_{42}^2$$
$$b_{32} = \lambda_{33}\lambda_{32} + \lambda_{43}\lambda_{42}$$
$$b_{42} = \lambda_{44}\lambda_{42}$$

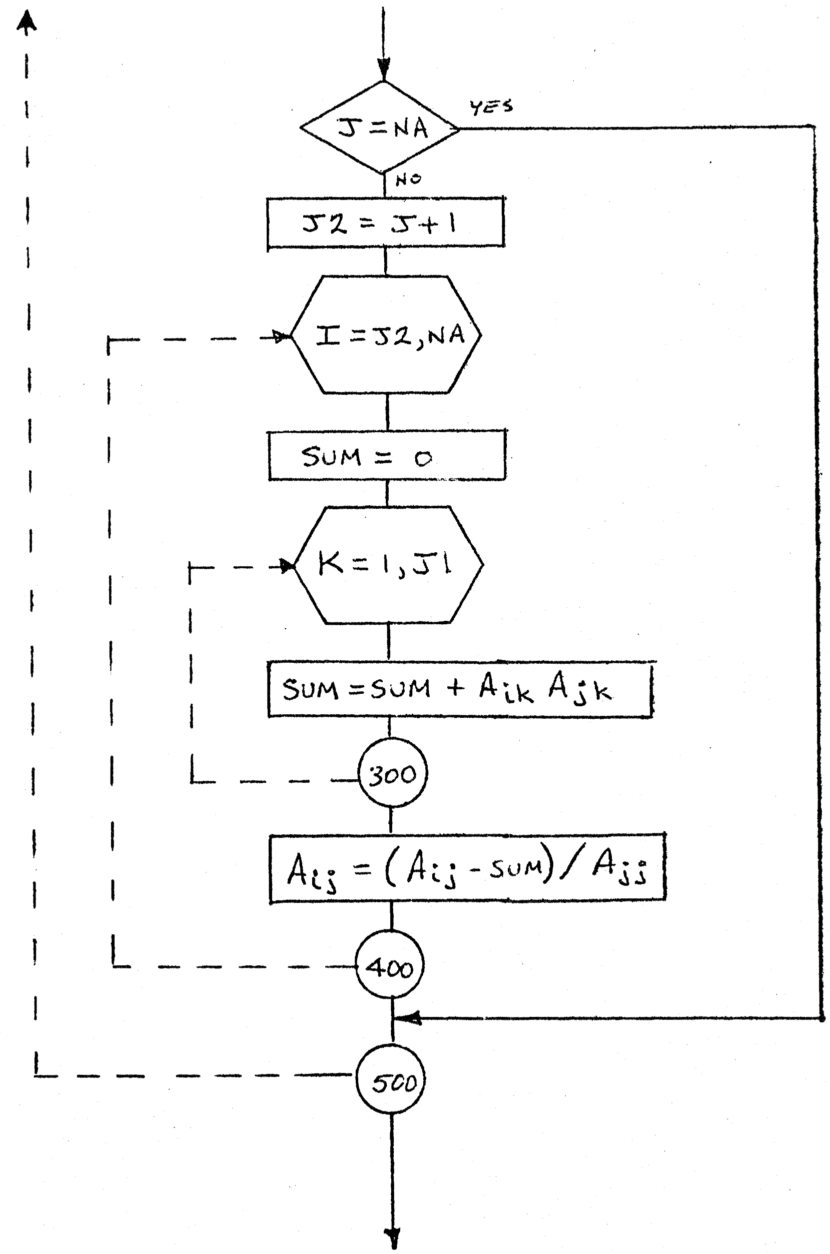
$$b_{13} = b_{31}$$
$$b_{23} = b_{32}$$
$$b_{33} = \lambda_{33}^2 + \lambda_{43}^2$$
$$b_{43} = \lambda_{44}\lambda_{43}$$

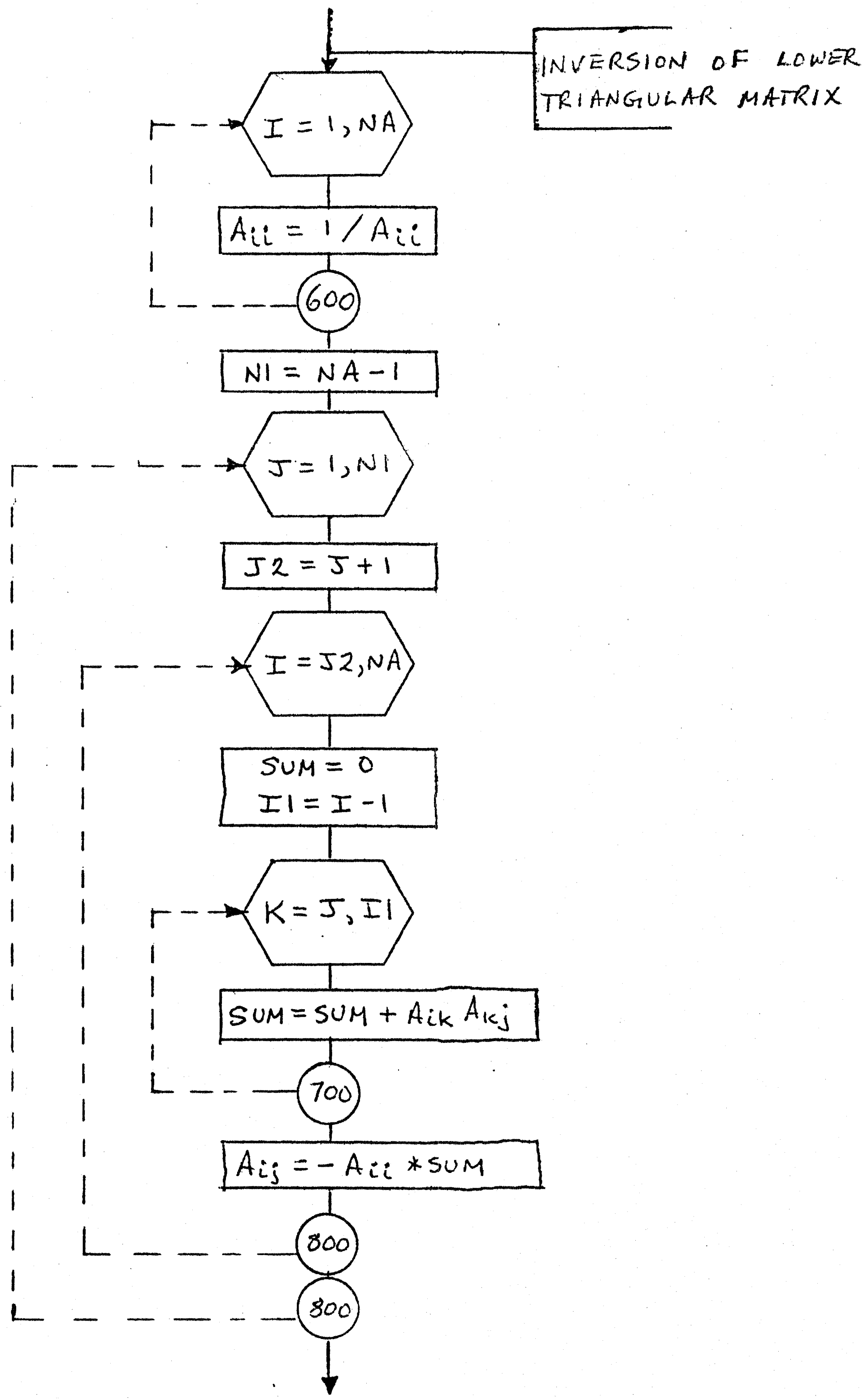
$$b_{14} = b_{41}$$
$$b_{24} = b_{42}$$
$$b_{34} = b_{43}$$
$$b_{44} = \lambda_{44}^2$$

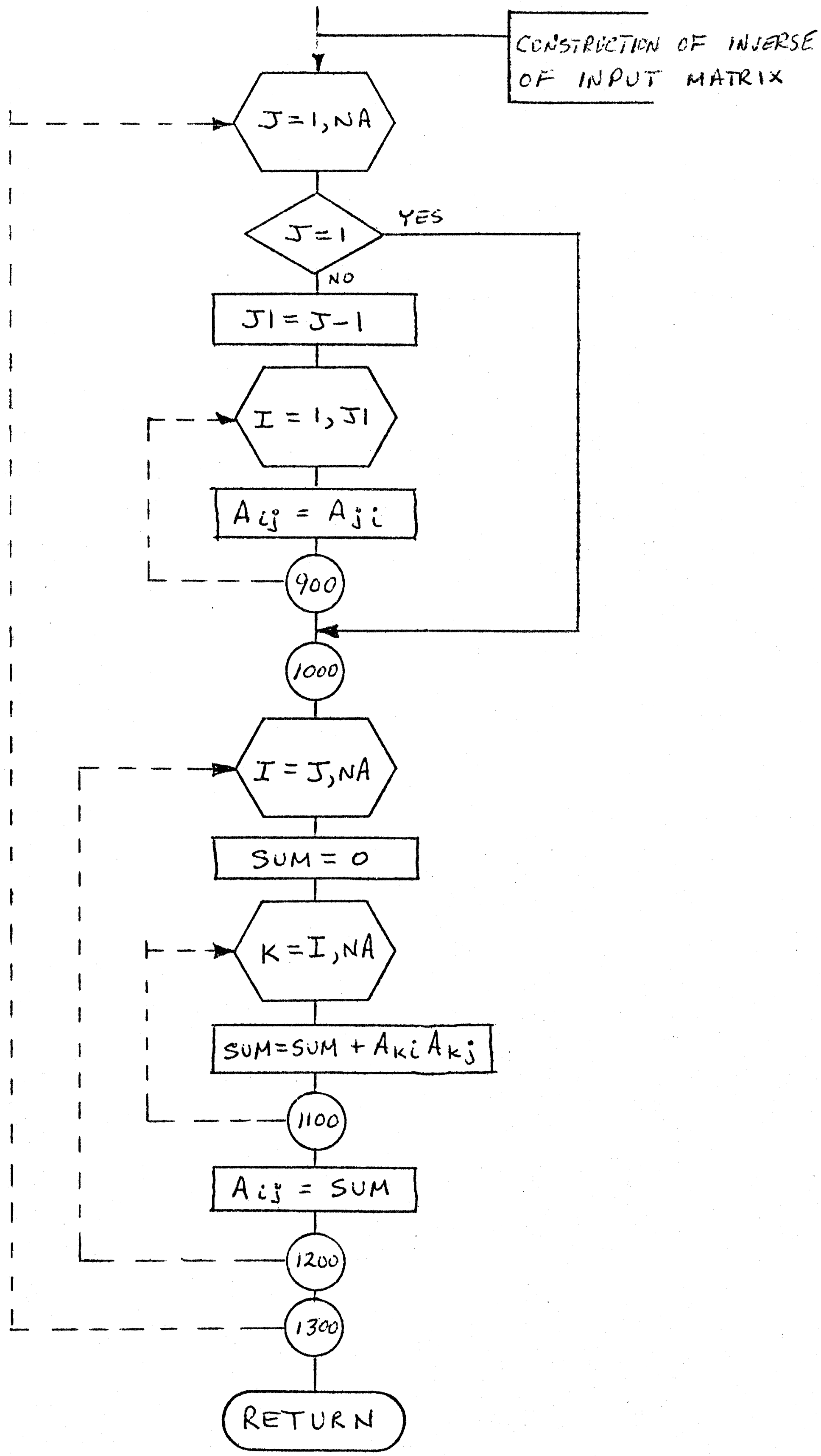
NUMBER OF OPERATIONS REQUIRED TO INVERT n x n

Step	Additions	Multiplications	Divisions	Roots
1	$\frac{n^3 - n}{6}$	$\frac{n^3 + 3n^2 - 4n}{6}$	0	n
2	$\frac{n^3 - 3n^2 + 2n}{6}$	$\frac{n^3 + 3n^2 - 4n}{6}$	n	0
3	$\frac{n^3 - n}{6}$	$\frac{n^3 + 3n^2 + 2n}{6}$	0	0
Total	$\frac{n^3 - n^2}{2}$	$\frac{n^3 + 3n^2 - 2n}{2}$	n	n


ENTRY
CHOLESKI DECOMPOSITION OF INPUT MATRIX INTO LOWER TRIANGULAR MATRIX
DETA = A_{11}
$A_{11} = \sqrt{A_{11}}$
I = 2, NA
$A_{i1} = A_{i1}/A_{11}$
100
J = 2, NA
SUM = 0
J1 = J − 1
K = 1, J1
SUM = SUM + A_{jk}^2
200
DETA = DETA * (A_{jj} − SUM)
$A_{jj} = \sqrt{A_{jj} - SUM}$
500


J = NA
YES
NO
J2 = J + 1
I = J2, NA
SUM = 0
K = 1, J1
SUM = SUM + $A_{ik} A_{jk}$
300
$A_{ij} = (A_{ij} - SUM) / A_{jj}$
400
500


INVERSION OF LOWER TRIANGULAR MATRIX
I = 1, NA
Aii = 1/Aii
600
N1 = NA - 1
J = 1, N1
J2 = J + 1
I = J2, NA
SUM = 0
I1 = I - 1
K = J, I1
SUM = SUM + Aik Akj
700
Aij = - Aii * SUM
800
800


CONSTRUCTION OF INVERSE OF INPUT MATRIX
J=1,NA
J=1
YES
NO
J1=J-1
I=1,J1
Aij = Aji
900
1000
I=J,NA
SUM=0
K=I,NA
SUM=SUM+Aki Akj
1100
Aij = SUM
1200
1300
RETURN

```
      SUBROUTINE CHOLD(A,IRDA,NA,DETA,*)
C
C MATRIX INVERSION USING CHOLESKI DECOMPOSITION
C
C INPUT ARGUMENTS
C A = ARRAY CONTAINING POSITIVE DEFINITE SYMMETRIC INPUT MATRIX
C IRDA = ROW DIMENSION OF ARRAY CONTAINING INPUT MATRIX
C NA = SIZE OF INPUT MATRIX
C OUTPUT ARGUMENTS
C A = CONTAINS INVERSE OF INPUT MATRIX (INPUT DESTROYED)
C DETA = DETERMINANT OF INPUT MATRIX
C * = ERROR RETURN (TAKEN IF NA .LT. 1 OR IF DETA .LT. SING)
C
      DOUBLE PRECISION A,DETA,SUM,SQRT,DSQRT,ABS,DABS,SING
      DIMENSION A(IRDA,NA)
      SQRT(SUM) = DSQRT(SUM)
      ABS(DETA) = DABS(DETA)
      DATA SING/1D-10/
C CHOLESKI DECOMPOSITION OF INPUT MATRIX INTO TRIANGULAR MATRIX
      IF(NA .LT. 1) GO TO 18
      DETA = A(1,1)
      A(1,1) = SQRT(A(1,1))
      IF(NA .EQ. 1) GO TO 6
      DO 1 I = 2,NA
    1   A(I,1) = A(I,1) / A(1,1)
      DO 5 J = 2,NA
        SUM = 0.
        J1 = J - 1
        DO 2 K = 1,J1
    2     SUM = SUM + A(J,K) ** 2
        DETA = DETA * (A(J,J) - SUM)
        A(J,J) = SQRT(A(J,J) - SUM)
        IF(J .EQ. NA) GO TO 5
        J2 = J + 1
        DO 4 I = J2,NA
          SUM = 0.
          DO 3 K = 1,J1
    3       SUM = SUM + A(I,K) * A(J,K)
    4     A(I,J) = (A(I,J) - SUM) / A(J,J)
    5   CONTINUE
    6 IF(ABS(DETA) .LT. SING) GO TO 16
C INVERSION OF LOWER TRIANGULAR MATRIX
      DO 7 I = 1,NA
    7   A(I,I) = 1. / A(I,I)
      IF(NA .EQ. 1) GO TO 10
      N1 = NA - 1
      DO 9 J = 1,N1
        J2 = J + 1
        DO 9 I = J2,NA
          SUM = 0.
          I1 = I - 1
          DO 8 K = J,I1
    8       SUM = SUM + A(I,K) * A(K,J)
    9     A(I,J) = - A(I,I) * SUM
C CONSTRUCTION OF INVERSE OF INPUT MATRIX
   10 DO 15 J = 1,NA
        IF(J .EQ. 1) GO TO 12
        J1 = J - 1
        DO 11 I = 1,J1
   11     A(I,J) = A(J,I)
   12   DO 14 I = J,NA
          SUM = 0.
          DO 13 K = I,NA
   13       SUM = SUM + A(K,I) * A(K,J)
   14     A(I,J) = SUM
   15   CONTINUE
      RETURN
   16 WRITE(6,17) DETA
   17 FORMAT(10X, 'SINGULAR MATRIX IN CHOLD. DET =',E20.5)
      RETURN 1
   18 WRITE(6,19)
   19 FORMAT(10X,'MATRIX OF DIMENSION ZERO IN CHOLD')
      RETURN 1
      END
```

```
C
C TEST SUBROUTINE CHOLD USING HILBERT MATRICES
C
      DOUBLE PRECISION A,DET,XI,XJ
      DIMENSION A(50,50)
      NDIM = 50
C OBTAIN TEST MATRIX
    1 READ     ,N
      IF(N .EQ. 0) GO TO 6
      DO 2 I = 1,N
        XI = I
        DO 2 J = 1,N
          XJ = J
    2     A(I,J) = 1. / (XI + XJ - 1.)
      WRITE(6,10)
      DO 3 I = 1,N
    3   WRITE(6,11) (A(I,J),J=1,N)
      WRITE(6,12) (A(I,I),I=1,N)
C INVERT TEST MATRIX, PRINT DETERMINANT, INVERSE, DIAGONAL ELEMENTS
      CALL CHOLD(A,NDIM,N,DET,&1)
      WRITE(6,13) DET
      WRITE(6,14)
      DO 4 I = 1,N
    4   WRITE(6,11) (A(I,J),J=1,N)
      WRITE(6,12) (A(J,J),J=1,N)
C REINVERT TO OBTAIN ORIGINAL TEST MATRIX, OUTPUT AS ABOVE
      CALL CHOLD(A,NDIM,N,DET,&1)
      WRITE(6,13) DET
      WRITE(6,15)
      DO 5 I = 1,N
    5   WRITE(6,11) (A(I,J),J=1,N)
      WRITE(6,12) (A(J,J),J=1,N)
      GO TO 1
    6 RETURN
   10 FORMAT('1',5X,'TEST MATRIX')
   11 FORMAT('0',5(10X,12E10.2,/))
   12 FORMAT('0',10X,'DIAGONAL ELEMENTS',50(/10X,E20.10))
   13 FORMAT('0',10X,'DETERMINANT'/10X,E20.10)
   14 FORMAT('0',5X,'INVERSE OF TEST MATRIX')
   15 FORMAT('0',5X,'REINVERTED MATRIX - SHOULD EQUAL TEST MATRIX')
      END
```

```
TEST MATRIX

        0.10D 01   0.50D 00   0.33D 00   0.25D 00

        0.50D 00   0.33D 00   0.25D 00   0.20D 00

        0.33D 00   0.25D 00   0.20D 00   0.17D 00

        0.25D 00   0.20D 00   0.17D 00   0.14D 00

      DIAGONAL ELEMENTS
         0.1000000000D 01
         0.3333333333D 00
         0.2000000000D 00
         0.1428571429D 00

      DETERMINANT
         0.1653439153D-06

INVERSE OF TEST MATRIX

        0.16D 02  -0.12D 03   0.24D 03  -0.14D 03

       -0.12D 03   0.12D 04  -0.27D 04   0.17D 04

        0.24D 03  -0.27D 04   0.65D 04  -0.42D 04

       -0.14D 03   0.17D 04  -0.42D 04   0.28D 04

      DIAGONAL ELEMENTS
         0.1600000000D 02
         0.1200000000D 04
         0.6480000000D 04
         0.2800000000D 04

      DETERMINANT
         0.6048000000D 07

REINVERTED MATRIX - SHOULD EQUAL TEST MATRIX

        0.10D 01   0.50D 00   0.33D 00   0.25D 00

        0.50D 00   0.33D 00   0.25D 00   0.20D 00

        0.33D 00   0.25D 00   0.20D 00   0.17D 00

        0.25D 00   0.20D 00   0.17D 00   0.14D 00

      DIAGONAL ELEMENTS
         0.1000000000D 01
         0.3333333333D 00
         0.2000000000D 00
         0.1428571429D 00
```