Interrelation between the geoid and orthometric heights

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Abstract. Height determination, like geoid modelling, is directly dependent on the gravity field. Hence it would be natural to link them for a comprehensive solution. It is known that the traditional methods for determining orthometric heights suffer from adopted approximations. In order to arrive at a more rigorous orthometric height, one must also account for the effects of the geoid-generated gravity disturbance, the shape of the topographical surface, and the density variations within topography. These effects are also considered for regional gravimetric geoid models. As a consequence, the implementation of the rigorous orthometric heights is simplified, further on, the heights become more compatible with regional geoid models. However, the two research areas have usually been discussed separately, and therefore the interrelation between them has only been vaguely considered. This contribution focuses on common features of geoid modelling and rigorous height determination. Relevant numerical results are presented and discussed.

Keywords: mean gravity, Stokes-Helmert, upward continuation, orthometric height

1 Introduction

The geoid plays an essential role in the national geodetic infrastructure, as the topographic heights and the depths of water bodies are reckoned from it. Over the past years, University of New Brunswick's geodesy group has been using Stokes-Helmert's method for regional geoid determination. Nowadays, many geodetic and engineering applications require that the two essential components of the vertical positioning – the height and the corresponding reference surface (geoid), are determined precisely. They both depend on the gravity field. It is thus appealing to examine the geoid and height computation issues together. With this in mind, we are herewith revisiting the principles of height determination.

In the past, three main approximations have been applied in practice to evaluate the orthometric heights. The Helmert method, as described in Heiskanen and Moritz (1967, Chap. 4), applies the Poincaré-Prey vertical gradient of gravity in conjunction with the measured gravity (at Earth's surface) for an estimation of the mean gravity. The Helmert height correction is simply proportional to the topographic heights. Niethammer (1932) and Mader (1954) refined the Helmert model by including the effect of the local shape of the topography. For a recent review of the three methods see Santos et al. (2006).

The simplest of the three – the Helmert orthometric height – has been adopted as the basis of the national vertical datum in many countries. Tenzer et al. (2005) showed that this commonly used method contains inadmissible approximations. Santos et al. (2006) presented the complete methodology with which to convert Helmert's heights to the rigorous orthometric heights.

It should be noted that the computation of the new rigorous heights is more involved, especially compared with the traditional Helmert approach. Indeed, as will be shown later, a number of components are needed to calculate the rigorous orthometric height. More specifically, one must account for the effect of the gravity disturbance generated by the geoid (Vaníček et al., 2004), the shape of the topographical surface, and the effects of density variations within topography.

The present contribution aims at demonstrating that the computations of the rigorous heights can be significantly simplified, if the Stokes-Helmert geoid modelling results are available. This is because several terms in this modelling are the same as those needed for computing the orthometric heights. Due to space limitations, however, the present contribution discusses only the most important causal relationships between the two research fields.

As such, this paper could also be considered as a complement to the earlier contributions by Tenzer et al. (2005), Kingdon et al. (2005) and Santos et al. (2006). Discussion on the relations between the normal heights (adopted in some countries) and the rigorous orthometric heights is considered to be outside the scope of the present contribution, but it can be found in Tenzer et al. (2005).

In Section 2 we continue with a brief recapitulation of the Stokes-Helmert geoid modelling principles. Section 3 is a review of the theory behind the rigorous orthometric heights. Section 4 deals with the assessment of components of the rigorous mean gravity. The relations between the constituents of Stokes-Helmert's geoid modelling and those needed for the rigorous heights are spelled out as well. Section 5 presents the results of the numerical investigations along a profile in the Canadian Rockies. A brief summary concludes the paper.

2 Helmert's condensation and solution to Stokes's boundary value problem

The solution of the boundary value problem by Stokes's (1849) method requires gravity to be known on the geoid, while in reality gravity measurements are taken at the topographic surface. Thus to satisfy the boundary condition gravity anomalies need to be downward continued to the geoid level. Harmonic quantities are needed for downward continuation; thus a number of different corrections related to the existence of topographic masses need be accounted for very carefully. [Strictly speaking, the effect of the atmospheric masses should also be considered in the geoid and rigorous orthometric height computations. Due to space limitations, however, these small effects are not discussed in this contribution]. One way of estimating the effect of topographical masses is to use Helmert's (1890) second condensation model. According to this model the Earth's topographical masses are replaced by an infinitesimally thin condensation layer on the geoid. So constructed gravity field becomes slightly different from the actual gravity field. The resulting Helmert anomalies, $\Delta g^{h}(r,\Omega)$, differ from the commonly used free-air anomalies, $\Delta g(r,\Omega)$. The relation between the two anomaly types can be expressed as (cf. Vaníček et al., 1999)

$$\Delta g^{h}(r_t,\Omega)=\Delta g(r_t,\Omega)+\frac{\partial\left[V^{T}(r_t,\Omega)-V^{CT}(r_t,\Omega)\right]}{\partial r}+\frac{2}{r_t(\Omega)}\left[V^{T}(r_t,\Omega)-V^{CT}(r_t,\Omega)\right]+e_{ellips}(r_t,\Omega) \tag{1}$$

where $V^{T}(r,\Omega)$ and $V^{CT}(r,\Omega)$ are the potentials of topographic masses and condensation layer, respectively. The geocentric position (r,Ω) is represented by the geocentric radius $r(\Omega)$ and a pair of geocentric spherical coordinates $\Omega=(f,l)$, where f and l are the latitude and longitude, respectively. All the quantities in Eq. (1) are referred to the surface of the Earth, $r_t(\Omega)=r_g(\Omega)+H^{o}(\Omega)$, where $r_g(\Omega)$ is the geocentric radius of the geoid surface and $H^{o}(\Omega)$ is the orthometric height. The term $e_{ellips}(r_t,\Omega)$ represents the ellipsoidal correction needed to account for the deviation of the actual shape of the Earth from the spherical approximation employed in fundamental gravimetric equation (Vaníček et al., 1999). The topographic terms in Eq. (1) can be evaluated by using the topographic elevation/density models in numerical quadrature methods (see e.g., Martinec, 1998). For more details on estimation of the components of Eq. (1), see e.g., Vaníček and Martinec (1994), Martinec (1998), Vaníček et al. (1999) and references therein. A recent review can also be found in Ellmann and Vaníček (2007).

Importantly, the product of corresponding Helmert anomaly and geocentric radius, $\Delta g^{h_\chi} r$, is harmonic above the geoid (Vaníček et al., 1996), and therefore such a field can be continued downward to the geoid level (note that this is not the case for the free-air anomalies!). For more details the reader is referred to (Vaníček et al., 1996). Thereafter the Helmert gravity anomalies serve as an input when solving the Stokes boundary value problem. The geoidal heights, $N(\Omega)$, after the application of the Helmert condensation are expressed as follows:

$$N(\Omega)=\frac{R}{4pg_0(f)}\iint_{\Omega'\in\Omega_0} S\left(y(\Omega,\Omega')\right)\Delta g^{h}(r_g,\Omega)\,d\Omega'+\frac{V^{T}(r_g,\Omega)-V^{CT}(r_g,\Omega)}{g_0(f)}\ , \tag{2}$$

where R is the mean radius of the Earth, $S\left(y(\Omega,\Omega')\right)$ is Stokes's function (Heiskanen and Moritz, 1967, Eq. 2-164), $y(\Omega,\Omega')$ is the geocentric angle between the computation Ω and integration points Ω'; $g_0(f)$ is the normal gravity (a function of latitude) at the reference ellipsoid, $d\Omega'$ is the area of the integration elements.

The geoid determination by the original Stokes formula requires the global coverage of gravity anomalies, $\Omega_0=\left[\,f\in\langle -p/2,p/2\rangle,\ l\in\langle 0,2p\rangle\right]$. Nowadays it is customary to use modifications of Stokes's formula (originally proposed by Molodensky et al., 1960) in conjunction with some global geopotential model. Here we skip the aspects of our usual modification scheme, since these are not relevant in the context of the present paper. For more details the interested reader is referred to Vaníček and Sjöberg (1991).

Recall that Stokes's integral employs Helmert's gravity anomalies. Note that Eq. (2) consists of two parts. The Stokesian integration (i.e. the first term on the right hand side of Eq. (2)) over these Helmert's anomalies results in the Helmert co-geoid. The Helmert condensation of the topographic masses yields the co-geoid which does not coincide with the actual geoid. The effect causing this change is called the primary indirect topographic effect (PITE). Accordingly, the last term in the right hand side of Eq. (2) is PITE, which transfers the Helmert co-geoid into the real geoid model.

3 Theoretical background of the rigorous orthometric heights

The orthometric height $H^o(\Omega)$ of a point on the Earth's surface is defined as the length of the somewhat curved plumb-line (reckoned from the geoid!) and is given by (e.g., Heiskanen and Moritz 1967, Eq. 4-21):

$$H^o(\Omega) = \frac{C(r_t,\Omega)}{\bar{g}(\Omega)} \;, \tag{3}$$

where $\bar{g}(\Omega)$ is the mean value of gravity between the geoid and the Earth's surface (along the plumb-line). $C(r_t,\Omega)$ is the geopotential number (see e.g. Heiskanen and Moritz, 1967, Chap. 4-2), which can be deduced from gravity measurements and spirit-levelling. Hence, the problem reduces to the determination of the mean gravity. The mean gravity is defined in an integral sense (e.g., Heiskanen and Moritz 1967, p. 166):

$$\bar{g}(\Omega) = \frac{1}{H^o(\Omega)} \int_{r_g(\Omega)}^{r_t(\Omega)} g(r,\Omega)dr \;, \tag{4}$$

where dr is an element of the plumb-line. Note that the integral is taken in radial direction, rather than along the (curved) plumb-line. This simplification is admissible, since this has a negligible influence (< 1 mm) on the orthometric height (Tenzer et al., 2005). Because the actual values of gravity $g(r,\Omega)$ cannot be measured inside the topographic masses, the integral-mean gravity $\bar{g}(\Omega)$ has to be computed from the observed surface gravity $g(r_t,\Omega)$, using a realistic and physically meaningful model of the vertical gravity gradient. For instance, in the computations of Helmert's mean gravity, $\bar{g}^H(\Omega)$, the approximate Poincaré-Prey vertical gradient is adopted as follows (Heiskanen and Moritz, 1967, Eq. 4-25):

$$\begin{aligned}\bar{g}^H(\Omega) &= g(r_t,\Omega) - \frac{1}{2}\left(\frac{\partial g}{\partial h} + 4p G r_0\right)H^o(\Omega) \\ &= g(r_t,\Omega) + 0.0424 H^o(\Omega)\end{aligned} \;, \tag{5}$$

where $\partial g/\partial h$ is the linear vertical gradient of normal gravity (evaluated at the surface) and r_0 is the mean topographic density (2670 kg/m^3). The Poincaré-Prey constant (= 0.0424 mGal/m) is thus obtained as a sum of the attraction of the Bouguer plate (+0.1119 mGal/m) and a half of the (negative) linear vertical gradient (-0.3086 mGal/m) of normal gravity. Consequently, the corresponding "Helmert correction to the measured surface gravity" is directly proportional to the topographic heights.

On the other hand, when computing the mean gravity rigorously one has to consider several terms. The gravity at a point $g(r,\Omega)$ can be decomposed into two terms; one comprising gravity generated by the masses inside geoid, $g^{NT}(r,\Omega)$ (in accordance with Vaníček et al. (2004) we call it *NoTopography* (NT) gravity, since the effect of the global topography has been subtracted from the "full" gravity), and another part, the gravity generated by the topography $g^T(r,\Omega)$. The geoid-generated gravity can be further decomposed into the contribution of the normal gravity and that of gravity disturbance caused by the masses inside the geoid (i.e. the *NoTopography* gravity disturbance, cf. Vaníček et al. 2004). This decomposition of the mean gravity can be expressed as follows (cf. Tenzer et al., 2005):

$$\bar{g}(\Omega) \approx \bar{g}^{NT}(\Omega) + \bar{g}^{T}(\Omega) \approx \bar{g}(\Omega) + \overline{dg}^{NT}(\Omega) + \bar{g}^{T}(\Omega) \tag{6}$$

where the approximate sign is due to neglecting the contribution of the atmosphere. Tenzer et al (2005, Appendix 1) have shown that this contribution is insignificant and can be neglected. To distinguish between $\bar{g}(\Omega)$ and the approximate Helmert mean gravity (Eq. (5)), the former will be referred to as 'rigorous mean gravity'.

4 Components of the rigorous mean gravity

The computation of the integral-mean (along the plumb-line) value of normal gravity, $\bar{g}(\Omega)$, in Eq. (6) is a rather trivial task. It can be evaluated accurately enough using a second-order Taylor expansion for the analytical downward continuation of normal gravity from the Earth's surface to the geoid. The final expression for computing the mean normal gravity can be found in Santos et al. (2006, Eq. 19).

It can be shown that the mean value of the topography-generated gravity can be evaluated (cf., Tenzer et al. 2005, Eqs. 16-18):

$$\bar{g}^T(\Omega) = \frac{1}{H^O(\Omega)} \int_{r_g(\Omega)}^{r_t(\Omega)} g^T(r,\Omega) dr =$$

$$\frac{-1}{H^O(\Omega)} \int_{r=r_g(\Omega)}^{r_t(\Omega)} \frac{\partial}{\partial r} V^T(r,\Omega) dr = \frac{\left[V^T\left(r_g,\Omega\right) - V^T\left(r_t,\Omega\right)\right]}{H^O(\Omega)} \quad . \quad (7)$$

In other words, the estimation of the $\bar{g}^T(\Omega)$ term reduces to the evaluation of the topographic potential at two points in the space: one at the surface of the earth and another one on the geoid level. Note that the terms in the numerator of Eq. (7) are already estimated during the Stokes-Helmert geoid determination, see Eqs. (1) and (2). If these terms (usually given on a grid) are made available, then evaluating Eq. (7) is quite straightforward.

Now we focus on the mean NT-gravity disturbance, the estimation of which is somewhat more involved. $\overline{dg}^{NT}(\Omega)$ is also evaluated as the integral mean in the radial direction, i.e. analogically to Eq. (4). Further on, since the geoid-generated gravity disturbance $dg^{NT}(r,\Omega)$ multiplied by r is harmonic above the geoid (because the NT-quantities by definition do not contain the contribution of the topographical masses!), then $\overline{dg}^{NT}(\Omega)$ can be evaluated by making use of Poisson's integral for upward continuation (Kellogg, 1929). Applying the integration limits the definite integral can be simplified (Santos et al., 2006, Eq. 37) as

$$\overline{dg}^{NT}(\Omega) = \frac{R}{4p H^O(\Omega)} \iint_{\Omega' \in \Omega_O} \bar{K}[r_t, y(\Omega,\Omega'), R] \, dg^{NT}\left(r_g,\Omega'\right) d\Omega' \ , \quad (8)$$

where $\bar{K}[r_t, y(\Omega,\Omega'), R]$ stands for the averaged Poisson's kernel. This new kernel is a function of two inverse distances relating: (i) the computation point (on the geoid level) and the integration element; and, (ii) the surface computation point and the integration element on the geoid (see also Santos et al., 2006, Eq. 38). Therefore, by no means this new kernel can be considered as an upward continuation of $dg^{NT}(r_g,\Omega)$ to some location in the space (e.g. geometrical mean between the geoid and the earth's surface), but just an integral average of $dg^{NT}(r,\Omega)$ in radial direction. The complete derivation of Eq. (8) can be found in Santos et al. (2006, Appendix A).

Equation (8) requires the NT gravity disturbance to be known on the geoid. To get it we make use of the Helmert gravity anomaly. The geoid-generated gravity disturbance $dg^{NT}(r_g,\Omega)$ for Eq. (8) is obtained (cf. Vaníček et al., 1999; Vaníček et al., 2004)

$$dg^{NT}\left(r_g,\Omega\right) = \Delta g^h\left(r_g,\Omega\right) + \frac{2}{R} T\left(r_g,\Omega\right) + \frac{\partial V^{CT}\left(r_g,\Omega\right)}{\partial r} + \frac{2}{R} V^{CT}\left(r_g,\Omega\right) - \frac{2}{R} V^T\left(r_g,\Omega\right). \quad (9)$$

As this expression shows, the NT-disturbance can be expressed as a collection of different terms, all related to the geoid level.

Considering the well-known Bruns (1878) formula, $T\left(r_g,\Omega\right) = N(\Omega) \cdot g_0(f)$, the disturbing potential $T(r_g,\Omega)$ can be taken from a regional geoid model. By denoting the PITE (the last term in Eq. (2)) as $dN_I(\Omega)$ Eq. (9) takes the following form:

$$dg^{NT}\left(r_g,\Omega\right) = \frac{2}{R}\left[N(\Omega) - dN_I(\Omega)\right] \cdot g_0(f) + \Delta g^h\left(r_g,\Omega\right) + \frac{\partial V^{CT}\left(r_g,\Omega\right)}{\partial r} \ . \quad (10)$$

Note that the three first terms on the right hand side are intermediate results of the Stokes-Helmert geoid determination. The remaining term in Eq. (10), the attraction of the condensation layer, need be evaluated on the geoid level (a suitable form of numerical expression can be deduced from Martinec, 1998). However, the latter term can also be computed as a geoid determination by-product. For this a relatively simple sub-routine can be added into the computer codes used in the Stokes-Helmert geoid modelling.

The resulting $dg^{NT}(r_g,\Omega)$ values are inserted into Eq. (8), providing the needed integral mean $\overline{dg}^{NT}(\Omega)$. This completes the methodology of linking the Stokes-Helmert geoid determination theory/results with the determination of rigorous orthometric heights. We see, that the availability of the Stokes-Helmert geoid results has important implications in the computational aspects of the rigorous orthometric heights.

5 Numerical Investigations

Using Canadian gravity and topographic data (provided by the Geodetic Survey Division of Natural Resources Canada) the rigorous mean gravity (cf. Eq. (6)) was computed along a profile across the Canadian Rocky Mountains. This profile coincides with the parallel 51°N and spans the longitudes between 235°E and 245°E. The mean topographic heights (with a spacing of 5 arc-minutes) range from 510 to 2384 m (with a mean of 1524 m). Due to high topographic elevations and very rugged landscape the mean gravity variations are expected to become significant.

Most of the terms in Eqs. (7) and (10) were 'borrowed' from the Stokes-Helmert geoid modelling results (Ellmann and Vaníček, 2007). We discuss only a few important aspects of the numerical estimation of Eq. (8), since it is the most laborious part of approach. Note that generally, the NT-gravity disturbances are negative over mountainous regions (as a rule of thumb, the larger the average of the local topography the larger the negative NT-disturbance). The $dg^{NT}(r_g, \Omega)$ is also a very smooth quantity (for an illustration see Kingdon et al., 2005, Fig. 4, erratum). Therefore, in spite to the high elevations, the mean $\overline{dg}^{NT}(\Omega)$ values remain very similar to the initial $dg^{NT}(r_g,\Omega)$ field. The differences $\overline{dg}^{NT}(\Omega) - dg^{NT}(r_g,\Omega)$ along the test profile do not exceed 10 mGal. At the same time the maximum upward continuation effect of $dg^{NT}(r_g,\Omega)$ from the geoid level to the earth's surface by using the Poisson integral formula remained smaller than 15 mGal.

The estimated Helmert mean gravity agrees generally well with $\bar{g}(\Omega)$. Nevertheless, in most cases $\bar{g}^H(\Omega)$ appears to be slightly weaker than the rigorous mean gravity (a few exceptions can be found inside of deep valleys). Along the selected profile the differences $\bar{g}(\Omega) - \bar{g}^H(\Omega)$ range between –21 and +30 mgal (with a mean of +6 mGal). These gravity differences can be then converted into the differences between the rigorous and Helmert orthometric heights (e.g. by using an approach in Heiskanen and Moritz, 1967, p.169). The height differences vary from –3.5 to +6.1 cm (with a mean of +1.1 cm). According to Kingdon et al. (2005) the differences at the higher elevations (> 3 km) may easily exceed a dm level. In a few extreme cases (at high elevations and very rugged areas), however, the relative height differences (due to $\bar{g}(\Omega) - \bar{g}^H(\Omega)$) disagree in about 7-8 cm for points located only some 10-20 km apart, see Fig. 1.

In most of the cases, especially over the mountain peaks, the Helmert orthometric heights appear to be higher than the corresponding rigorous mean heights, i.e. $H_H^o(\Omega) > H_{rig}^o(\Omega)$. This is due to the fact that the mean gravity has a 'reverse' effect on the height: the larger the mean gravity in the denominator the smaller the resulting height. As it was mentioned above, inside of some deep valleys $\bar{g}^H(\Omega) > \bar{g}(\Omega)$, yielding thus $H_H^o(\Omega) < H_{rig}^o(\Omega)$. Intuitively, this can be explained by the fact that in the Helmert approach the irregularity of the surrounding topography is entirely neglected (cf. Eq. (5)). In other words, the contributions due the mass deficiencies and excesses (with respect to the Bouguer plate, which is embedded in the Poincaré-Prey gradient) around the computation points are not accounted for. For instance, due to the mass deficiency around a computation point located on a mountain top the magnitude of $\bar{g}^H(\Omega)$ becomes underestimated (in Eq. (5) note the opposite signs for the constants in the brackets!). This gives an unreasonable rise to the resulting Helmert height. Conversely, the mass excess (with respect to the Bouguer plate) exists for the computation points inside of deep valleys. The magnitude of $\bar{g}^H(\Omega)$ is overestimated, the resulting Helmert height is thus lower than the rigorous orthometric height. We conclude that the quality of the rigorously computed

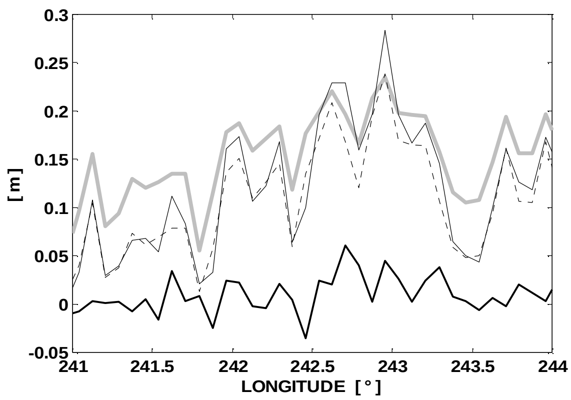

Fig. 1 Comparison among the rigorous and Helmert orthometric heights along a profile at 51°N across the Canadian Rocky Mountains. Thin black line represents $\left[\bar{g}(\Omega) - g(r_t,\Omega)\right]\cdot H(\Omega)/\bar{g}(\Omega)$ and the dotted line shows $\left[g^H(\Omega) - g(r_t,\Omega)\right]\cdot H(\Omega)/\bar{g}(\Omega)$. Note the two corrections to the measured heights are negative; the opposite sign is assigned here to illustrate their correlation with the topographic surface. The topographic heights, downscaled by the factor 0.0001, are denoted by the bold grey line. The bold black line represents the differences between the resulting orthometric heights, $H_H^o(\Omega) - H_{rig}^o(\Omega)$. Units in centimetres.

orthometric heights prevails over the traditional Helmert's heights. This effect becomes particularly transparent over mountainous regions.

Note that approaches by Mader (1954) and Niethammer (1932) attempt to improve the Helmert heights by accounting for the roughness of the topography. The resulting heights are more compatible with the rigorous orthometric height, see a numerical study by Santos et al. (2006).

6 Summary and conclusions

The aim of this paper is to demonstrate that the computations of the rigorous orthometric heights can be significantly simplified by making use of the typical by-products of the Stokes-Helmert geoid determination. Accordingly, the implementation of the rigorous height system becomes a relatively simple and straightforward task. An additional bonus is that the resulting orthometric heights are more compatible with regional gravimetric geoid models. Let us hope that these circumstances encourage those who currently use Helmert's approximate orthometric heights to upgrade them to a more rigorous height system.

The improved orthometric heights have a wide range of the practical and engineering applications. Therefore, the national agencies and organisations currently holding the Stokes-Helmert geoid determination results, should make them available to the users. For instance, both the Stokes-Helmert geoid methodology and the orthometric height system are adopted by the national agencies of the three North-American countries: Canada (Huang et al., in press), the U.S.A. (Roman et al., 2004) and Mexico (Hernandez, 2003). The existence of the needed components would allow the North-American users more easily to implement the rigorous orthometric heights, without having to recreate many of the results already calculated and held by their government agencies.

Note that the concept of the NT-gravity (as introduced in Vaníček et al., 2004) is exploited in this study. Hence, the usage of the NT-quantities, besides of its obvious value for different geophysical studies, has very promising geodetic applications as well.

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