CAN MEAN VALUES OF HELMERT'S GRAVITY ANOMALIES BE CONTINUED DOWNWARD DIRECTLY?

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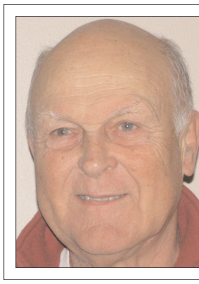
The computation of a precise gravimetric geoid based on the Stokes-Helmert approach requires the solution of the geodetic boundary value problem. For that, the mean Helmert's gravity anomaly on the earth's topographic surface must be reduced to the geoid, the surface that plays the role of the boundary. This reduction is a process known as downward continuation. This paper considers the downward continuation as a solution of the discrete inverse Poisson problem. It shows the derivation of a doubly-averaged upward continuation operator that relates mean Helmert's gravity anomaly from the boundary to the surface. Downward continuation is then carried out by the inversion of this operator. It is shown that this can be done rigorously if, and only if, the processes of averaging and downward continuation are commutative (mutually interchangeable).

Le calcul d'un géoïde gravimétrique précis fondé sur l'approche Stokes-Helmert requiert la solution au problème géodésique de la valeur à la limite. Pour ce faire, l'anomalie gravimétrique moyenne d'Helmert sur la surface topographique de la Terre doit être réduite au géoïde, la surface qui joue le rôle de limite. Cette réduction est un processus connu comme étant la « réduction vers le bas ». Le présent article considère la réduction vers le bas comme une solution au problème de la loi de Poisson inverse discrète. L'article montre la dérivation de la double moyenne d'un opérateur de réduction vers le haut qui relie l'anomalie gravimétrique moyenne d'Helmert de la limite à la surface. La réduction vers le bas est ensuite réalisée par l'inversion de cet opérateur. Il est démontré que ceci peut être effectué rigoureusement si, et uniquement si, les processus de calcul de la moyenne et de réduction vers le bas sont commutatifs (mutuellement interchangeables).



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Introduction

A quantity $u(x,y,z)$ that satisfies the Laplace partial differential equation:

$$\nabla^2 (u) = 0 , \qquad (1)$$

in a region of space is called *harmonic* in that region. Transforming the Cartesian coordinates x,y,z into the usual curvilinear coordinates $r,\varphi,\lambda \equiv r, \Omega$, we can define the direction "down" as going against the growth of r. In many geodetic applications, it is interesting to study the behaviour of a harmonic quantity (more generally the behaviour of a linear combination of a harmonic quantity and its vertical derivative) in the downward direction. For instance, the downward continuation of Helmert gravity anomalies from the Earth's topographic surface onto the boundary, the geoid, is a key process for the computation of a precise geoid following the Stokes-Helmert technique [*Vaníček and Martinec* 1994; Vaníček et al. 1999] formulated at and used by University of New Brunswick (UNB).

The downward continuation of a gravity anomaly to the geoid, a continuation process also known as the inverse Poisson problem, must precede the solution of the geodetic boundary value problem. By itself, it can be applied to several field quantities, for example, to observed gravity values (g), to gravity disturbances (δg), to disturbing potential (T), or any combination of these quantities. In the discussions that follow, the quantity of interest are Helmert's gravity anomalies measured on a mesh of cells at the Earth surface, and their corresponding mean cell values at the geoid. It is known [e.g., *Wong* 2002] that Helmert's gravity anomalies can be expressed as a linear combination of a harmonic quantity (T) and its vertical derivative $\left(\partial T/\partial r\right)$. Thus their downward continuation represents a proper inverse Poisson problem.

There are two different ways to compute downward continuation. One way is to formulate the downward continuation as an analytical problem, e.g., by Taylor series expansion of the quantity to be continued downward, or as a solution of Poisson integral equation [*Huang* 2002] shown here in spherical approximation as applied to gravity anomaly Δg:

$$\forall\Omega : \Delta g'(\Omega) \approx \frac{R}{4\pi\left[R + H(\Omega)\right]} \iint_s K\left[\Omega, H(\Omega), \Omega'\right] \Delta g(\Omega) d\Omega' \,, \tag{2}$$

where K is called the Poisson integration kernel and R is the mean radius of the Earth, S is the sphere of integration and $H(\Omega)$ is the terrain height at point Ω. This equation has to be discretised [*Sun and Vaníček* 1996] as:

$$\mathbf{\Delta g'} = \mathbf{B}\ \mathbf{\Delta g} \,, \tag{3}$$

where $\mathbf{\Delta g}$ is the gravity anomaly at the Earth surface, $\mathbf{\Delta g'}$ is the gravity anomaly on the geoid and $\mathbf{B}$ is a matrix of coefficients assembled from the values of Poisson's kernel. This discretisation results in the necessity of solving large systems of linear equations, complicated by situations when dealing with sparse and ill-conditioned coefficient matrices.

Discrete models can be based on *point-point, point-mean* and *mean-mean* relationships between gravity anomalies on the Earth surface and on the geoid [*Huang* 2002], each model is diagrammatically illustrated in Figure 1. In this paper, we focus on the *mean-mean* model to the exclusion of the other two models. We shall be trying to answer the following question: *Given mean Helmert's anomalies at the surface of the Earth, is it possible to directly derive mean Helmert's anomalies on the geoid?* The answer to this question is very important for those of us who are interested in computing the geoid by means of numerical integration of the product of Stokes's kernel with gravity anomalies. This numerical integration calls for the availability of mean anomalies on the geoid. On the other hand, the gravity anomalies available from the pre-processing of observed gravity data at the Earth surface are usually of the mean kind. Also, dealing with mean anomalies instead of point anomalies makes the computations much faster.

This paper presents the development of a rigorous formula for the downward continuation of mean Helmert's gravity anomalies using the discrete form of the Poisson integral equation Eq.(2). The downward continuation is accomplished numerically by inverting a doubly-averaged upward continuation linear operator $\overline{\overline{\mathbf{B}}}$ [*Sun and Vaníček* 1996].

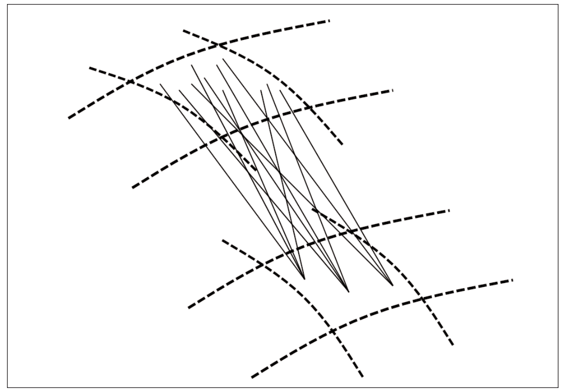

Figure 1a: Point-point model.

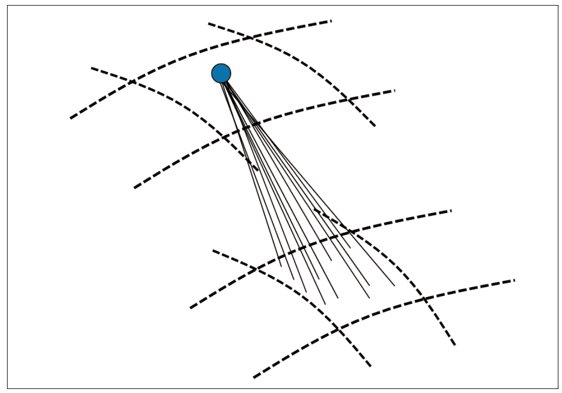

Figure 1b: Point-mean model.

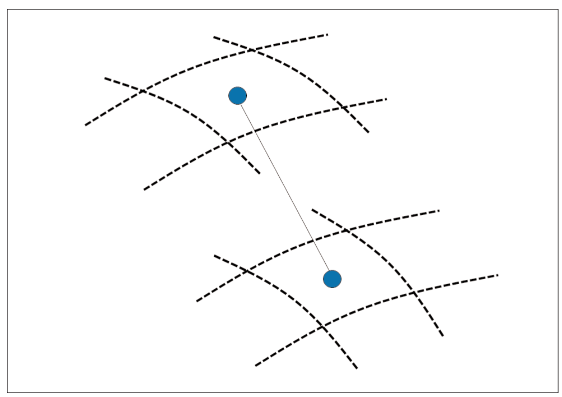

Figure 1c: Mean-mean model.

Cell-Averaging Operator

The development described in the next sections introduces a mathematical apparatus that we call *linear cell-averaging operator*, which transforms the set of gravity anomalies inside each cell into mean cell gravity anomaly, leading later to the development of the doubly-averaged upward continuation operator. Let's assume that we have Helmert's gravity anomalies on the geoid in a rectangle $\Re$ containing M times N, say 5′ by 5′ geographical cells [*Sun and Vaníček* 1998]. Such a rectangle is visually illustrated in Figure 2, which shows a representation on the geoid and its counterpart on the Earth's surface. Let us further assume that in each 5′ by 5′ cell C_{mn}, we have I_m times J_n values of regularly distributed Helmert's anomalies (Figure 3). Let us denote these values by:

$$\forall m = 1,2\ldots,M; n = 1,2,\ldots,N; i = 1,2,\ldots,I_m; j = 1,2,\ldots,J_n : \Delta g_g(\Omega_{ij}) = \Delta g_{ij}\,. \quad (4)$$

Let us define, as usual, the mean Helmert's gravity anomalies in the individual cells C_{mn} on the geoid as integral averages:

$$\forall m = 1,2,\ldots,M; n = 1,2,\ldots,N : \mu\left[\Delta g_g\right](\Omega_{mn}) = \mu[\Delta g]_{mn} = \frac{1}{A_{mn}} \int_{C_{mn}} \Delta g_g(\Omega)\, d\Omega\,, \quad (5)$$

where A_{mn} is the area of the C_{mn} cell. The mean gravity anomaly can be expressed approximately as:

$$\forall m = 1,2,\ldots,M; n = 1,2,\ldots,N : \mu\,[\Delta g]_{mn} \approx \frac{1}{I_{mn} J_n} \sum_{i=1}^{I_m} \sum_{j=1}^{J_n} \Delta g_{ij}\,. \quad (6)$$

Using matrix notation we can re-write Eq.(6) as follows:

$$\forall m = 1,2,\ldots,M; n = 1,2,\ldots,N : \mu\,[\Delta g]_{mn} \approx \mathbf{U}^T_m \Delta \mathbf{g} \mathbf{U}_n, \quad (7)$$

where $\mathbf{U}_n$ is the vector of J_n values of $(1/J_n)$, $\mathbf{U}_m$ is the vector of I_m values of $1/I_m$, and $\Delta\mathbf{g}$ is the I_m by J_n matrix of Helmert's anomaly values Δg_{ij} contained in the C_{mn} cell.

For simplicity, let us assume in the sequel that the number $I_m J_n$ of values of Δg_{ij} in each cell C_{mn} is "sufficiently high", and the same in both directions equal to $IJ = I^2 = \kappa$. We can thus visualize the distribution of these values as being on an identical regular square grid (say, 30″ by 30″) in each 5′ by 5′ cell. Further, let us introduce the vectorial form of Δg, $\mathbf{vec}_{mn}(\Delta g)$, that contains all κ values of Δg in the C_{mn} cell in the form of one numerical vector. Using the vectorial form, the mean value $\mu[\Delta g]_{mn}$ in each cell can be expressed as:

$$\mu[\Delta g]_{mn} = \kappa^{-1} \mathbf{V}^T\, \mathbf{vec}_{mn}(\Delta g), \quad (8)$$

where $\mathbf{V}$ is the vector of values of 1. The linear operation that transforms the complete vector of $M\,N\,\kappa$ gravity anomalies Δg in the whole rectangle $\Re$, let us denote it by $\mathbf{vec}(\Delta g)$, to a vector of $N\,M$ mean cell anomalies $\mu[\Delta g]_{mn}$, let us call it by just $\mathbf{vec}(\mu[\Delta g])$, reads:

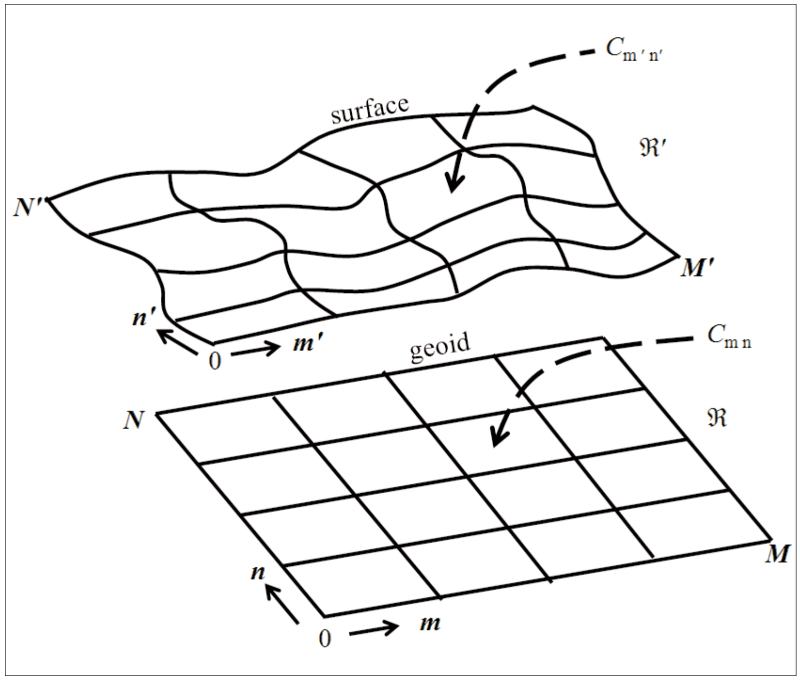

Figure 2: Rectangle $\Re$ on the geoid and its counterpart $\Re$' on the surface.

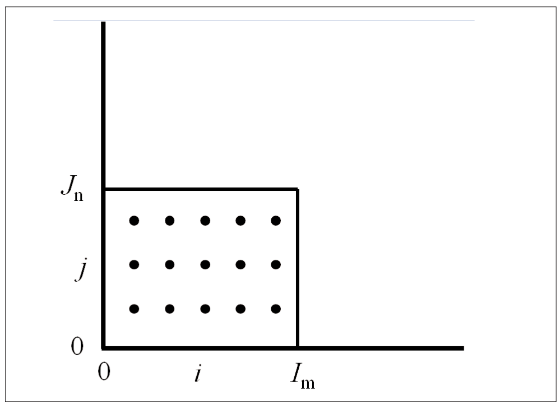

Figure 3: Regularly distributed gravity anomalies inside a cell.

$$\mathbf{vec}(\mu[\Delta g]) = \kappa^{-1} \mathbf{W}^T\, \mathbf{vec}(\Delta g), \quad (9)$$

where $\mathbf{W}^T$ is a $M\,N$ by $M\,N\,\kappa$ matrix composed of properly positioned $\mathbf{V}$ vectors surrounded by $\mathbf{0}$ vectors. This matrix can be called the *cell-averaging operator*. It is an orthogonal matrix with the following property:

$$\mathbf{W}^T \mathbf{W} = \mathbf{I}, \quad (10)$$

where $\mathbf{I}$ is a unit matrix of a dimension $M\,N$. We emphasize here that while the dimension of $\mathbf{vec}_{mn}(\Delta g)$ is κ, the dimension of $\mathbf{vec}(\Delta g)$ is $MN\,\boldsymbol{\kappa}$.

Discrete Poisson Upward Continuation

In the next step of the development, let us take the Poisson upward continuation of Helmert's gravity anomalies from the geoid to the earth surface as given by Eq. (2). In discrete form over the rectangle $\Re$ and using matrix notation, the Poisson upward continuation reads, cf., Eq. (3):

$$\mathbf{vec}(\Delta g') \cong \mathbf{B}\ \mathbf{vec}(\Delta g)\ , \tag{11}$$

where matrix **B** is composed of values $b_{kk'}$:

$$\forall k, k' = 1,2,\ldots, MN\kappa : b_{kk'} = \frac{K[\Omega_k, H(\Omega_k), \Omega k']}{1 + \frac{H(\Omega_k)}{R}}\ , \tag{12}$$

where subscripts k and k' denote the position of elements $(i,j)_{mn}$ and $(i,j)'_{mn}$ in the vectors **vec** (Δg) and $\mathbf{vec}(\Delta g')$ [*Vaníček et al*. 1996].

Doubly-Averaged Upward Continuation

In the next step, let's evaluate the mean Helmert gravity anomalies $\mu[\Delta g']$ on the Earth surface. The vector of these mean gravity anomalies can be obtained by applying the linear operation (9) as:

$$\mathbf{vec}\ (\mu[\Delta g']) = \kappa^{-1}\mathbf{W}^T\ \mathbf{vec}\ (\Delta g'). \tag{13}$$

Let us now have a look at the "other product" of matrices **W** and $\mathbf{W}^T$ which will come handy in the forthcoming development. The product $\mathbf{W}\ \mathbf{W}^T$ is a block-diagonal matrix, where each diagonal sub-matrix is composed of 1's. We can thus write:

$$\mathbf{W}\ \mathbf{W}^T = (\mathbf{I} + \mathbf{D}), \tag{14}$$

where **D** is a (square) block-diagonal matrix composed of off-diagonal elements that are all equal to 1 and diagonal elements equal to zero as well as all other off-diagonal block structures, and **I** is the identity matrix of a dimension $M\ N\ \kappa$, cf., Eq.(10).

Now, combining Eqs. (11) and (13), we obtain:

$$\mathbf{vec}\ (\mu[\Delta g']) = \kappa^{-1}\mathbf{W}^T\ \mathbf{B}\ \mathbf{vec}\ (\Delta g), \tag{15}$$

which gives us the vector of mean values on the Earth surface from the vector of point values on the geoid. To get the vector of mean values (on the geoid) on the right-hand side, we have to express **vec** (Δg) by means of the mean values **vec** $(\mu[\Delta g])$. It can be done by using Eq.(9). We rewrite Eq.(9) as:

$$\kappa\mathbf{W}\ \mathbf{vec}(\mu[\Delta g]) = \mathbf{W}\mathbf{W}^T\ \mathbf{vec}(\Delta g)\ . \tag{16}$$

Substituting for $\mathbf{W}\ \mathbf{W}^T$ from Eq.(14), we get:

$$\begin{aligned}\kappa\mathbf{W}\ \mathbf{vec}(\mu[\Delta g]) &= (\mathbf{I} + \mathbf{D})\ \mathbf{vec}(\Delta g)\\ &= \mathbf{vec}(\Delta g) + \mathbf{D}\mathbf{vec}(\Delta g)\\ &= \mathbf{vec}(\Delta g) + \kappa - 1)\ \mathbf{W}\ \mathbf{vec}(\mu[\Delta g]) - \\ &\quad \mathbf{vec}(r)\ ,\end{aligned} \tag{17}$$

where **vec** (r) is the vector of residuals $r = \Delta g - \mu[\Delta g]$ (subscripts are left out for simplicity) on the geoid. For each cell, the average of residuals, $\mu(r)$, is equal to zero, thus the expected value of the vector **vec** (r) is **0**.

Equation (17) can be rewritten as:

$$\mathbf{vec}(\Delta g) = \mathbf{W}\ \mathbf{vec}(\mu[\Delta g]) + \mathbf{vec}(r)\ , \tag{18}$$

and substituted back into Eq.(15) with the following result

$$\begin{aligned}\mathbf{vec}(\mu[\Delta g']) &= \kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \mathbf{vec}(\Delta g)\\ &= \kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\{\mathbf{W}\ \mathbf{vec}(\mu[\Delta g]) + \mathbf{vec}(r)\}.\end{aligned} \tag{19}$$

The quadratic form $\kappa^{-1}\mathbf{W}^T\ \mathbf{B}\ \mathbf{W}$ is the **doubly averaged upward continuation operator** $\overline{\overline{\mathbf{B}}}$ that can be also denoted by $\mu[\mu(\mathbf{B})]$. It is obtained from the values of **B** (cf. Eq. (12)) by taking areal averages over the locations contained in the individual cells, both on the geoid and on the surface of the Earth. For the simplified case described above, these would be the averages over the 100 ($\kappa = I^2$) 30″ values in each 5′ by 5′ cell. So finally we get:

$$\mathbf{vec}\ (\mu[\Delta g']) = \mu[\mu(\mathbf{B})]\ \mathbf{vec}\ (\mu[\Delta g]) - \kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \mathbf{vec}\ (r). \tag{20}$$

How about the second term in Eq. (20)? We have seen that the average of the residuals r in each cell on the geoid is equal to zero. Therefore, the mean of the residuals r on the geoid over the whole area $\Re$ is equal to 0:

$$\forall r \in \Re\colon E(r) \approx \mu(r) = \mathbf{W}^T\ \mathbf{vec}\ (r) = 0. \tag{21}$$

The same equation can be written about the residuals $r' = \Delta g' - \mu[\Delta g']$ on the surface of the Earth

$$\forall r' \in \Re\colon E(r') \approx \mu(r') = \mathbf{W}^T\ \mathbf{vec}\ (r') = 0. \tag{22}$$

On the other hand, the interpretation of the term $\mathbf{W}^T\ \mathbf{B}\ \mathbf{vec}\ (r)$ is that it represents the residuals on the geoid continued upward to the surface of the

Earth and is then averaged. This process is clearly different from the process needed to arrive at the residuals *r'* dealt with in Eq.(22). Thus only if the order of application of upward continuation and averaging are interchangeable, i.e., if these two operations are commutative, can the term in question be expected to go to 0. It seems to us that the two operations being both linear should be commutative, but we have not worked out a rigorous proof.

Solution

Assuming the commutativity of the two operations discussed in the previous section, the correct form for the upward continuation of mean Helmert's anomalies then has the following form:

$$\mathbf{vec}\ (\mu[\Delta g']) = \mu[\mu(\mathbf{B})]\ \mathbf{vec}\ (\mu[\Delta g]). \qquad (23)$$

showing that if the anomalies to be upward continued are averaged, and if we are also interested in producing mean anomalies on the surface of the Earth, the upward continuation operator has to be doubly averaged as well. The downward continuation is computed from the obvious formula:

$$\mathbf{vec}\ (\mu[\Delta g]) = \mu[\mu\ (\mathbf{B})]^{-1}\ \mathbf{vec}\ (\mu[\Delta g']) \qquad (24)$$

if the inverse of the doubly averaged operator exists. The regularity of $\mu[\mu\ (\mathbf{B})]$ for 5′ by 5′ cells was shown by *Vaníček et al*. [1996].

Still under the assumption of commutativity and the regularity of the doubly averaged upward continuation Poisson operator, our derivations show that it is possible to downward continue directly the mean Helmert's gravity anomalies from the earth surface to the geoid. The resulting mean Helmert's anomalies on the geoid can be then used directly in the numerical integration of the Stokes integral.

Concluding Remarks

Downward continuation of Helmert gravity anomalies, from the earth's topographic surface onto the geoid, is a very important part in the computation of a precise geoid following a procedure known as the Helmert-Stokes technique. A badly performed downward continuation will have pernicious effects on the quality of the computed geoid. This paper shows how the downward continuation of gravity anomalies can be accomplished by inverting a doubly-averaged upward continuation operator. A *linear cell-averaging* operator that transforms sets of gravity anomalies, inside rectangular cells, into mean gravity anomalies is then defined. The cell-averaging operator interacts with a matrix **B** composed of the product of the attenuation factor $[R\ /\ (R + H(\Omega))]$ with the Poisson Kernel values, which relates mean gravity anomalies on the geoid with corresponding mean gravity anomalies at height *H* on the Earth. Exploring further properties of the cell-averaging operator allows for the development of the doubly-averaged upward continuation operator.

The commutativity of upward continuation and averaging operators should be proved rigorously. So far we have done numerical evaluations, which ascertain the equivalence of operators derived using the commutative diagram presented in Appendix A. We have also evaluated the equivalence of Eq. (23) and (24) by a set of Δg′ mean values on the surface. These values were downward continued, upward continued, and compared to the original Δg′ mean values. The results were identical.

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Appendix A

A very interesting way to visualize the problem discussed in this paper is through the the use of a commutative diagram, such as the one shown in Figure A.1. In it, the two upper ellipses contain gravity anomalies, point and mean, referred to the Earth's surface. The two lower ellipses contain gravity anomalies, point and mean, on the surface of the geoid. The arrows represent the operations that connect those quantities.

To work with commutative diagrams, two basic rules should be followed: (i) start at the "point of arrival" and work backwards towards "point of departure"; and, (ii) must not commute operators. If used properly, the same results are obtained following different routes (arrows). For example, moving from the lower left (LL), which contains $\mathbf{vec}(\Delta g)$, to the upper left (UL), which contains $\mathbf{vec}(\Delta g')$, using operator **B**, we obtain Eq. (11) :

$$\mathbf{vec}(\Delta g') = \mathbf{B}\ \mathbf{vec}(\Delta g). \quad \text{(A.1)}$$

Moving now from the LL to the lower right (LR), which contains $\mathbf{vec}(\mu[\Delta g])$, and using operator $\mathbf{W}^T$, we arrive at:

$$\mathbf{vec}(\mu[\Delta g]) = \mathbf{W}^T\ \mathbf{vec}(\Delta g). \quad \text{(A.2)}$$

We note that if the operation is an "averaging operation" we have to divide the right-hand side by the number of elements, κ, in the cell changing the equation to (cf., Eq.(9)):

$$\mathbf{vec}(\mu[\Delta g]) = k^{-1}\mathbf{W}^T\ \mathbf{vec}(\Delta g). \quad \text{(A3)}$$

The same reasoning can be applied to moving from the UL to the upper right (UR), and using the same operator $\mathbf{W}^T$, we retrieve Eq. (13):

$$\mathbf{vec}(\mu[\Delta g']) = \kappa^{-1}\mathbf{W}^T\ \mathbf{vec}(\Delta g'). \quad \text{(A.4)}$$

Other relations can be established. For example, going from the LL to the UR, clockwise, using operators **B** and $\mathbf{W}^T$, we get back to Eq. (15):

$$\mathbf{vec}\ (\mu[\Delta g']) = \kappa^{-1} W^T\ \mathbf{B}\ \mathbf{vec}\ (\Delta g). \quad \text{(A.5)}$$

Going now from the LL to the UR, counter-clockwise, using operators $\mu[\mu(\mathbf{B})]$ and $\mathbf{W}^T$, we arrive at the equation:

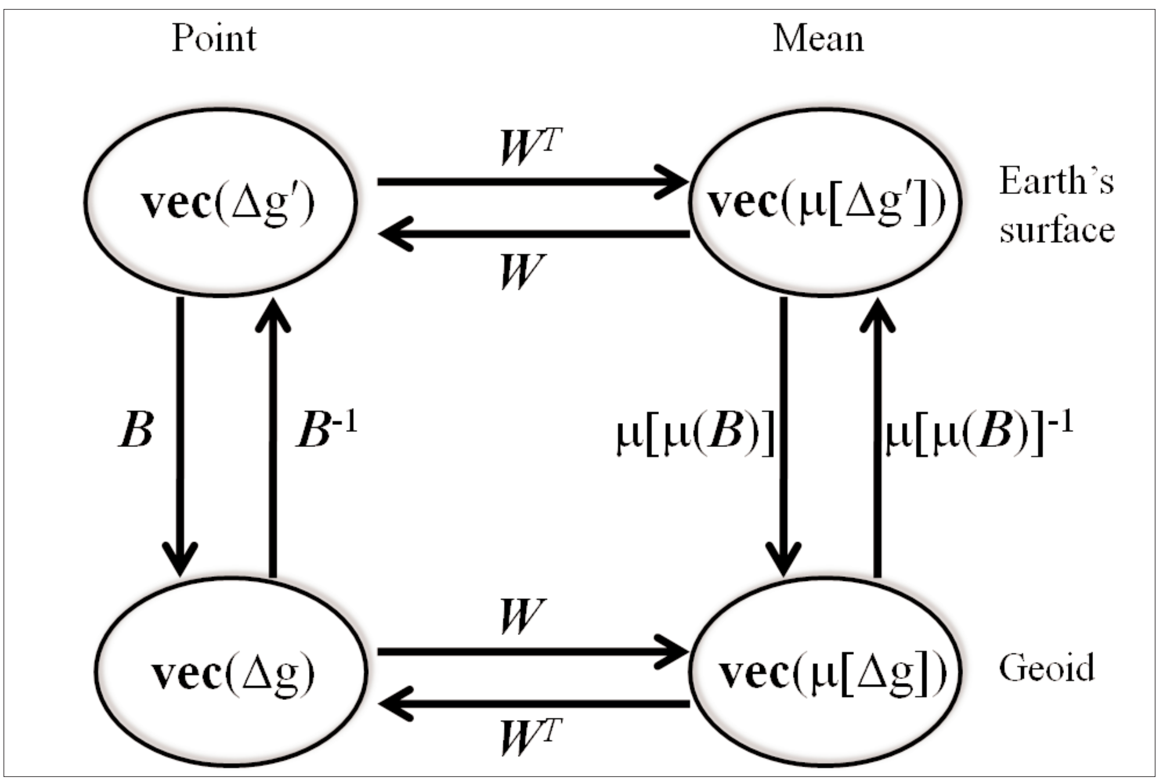

Figure A.1 – Commutative diagram.

$$\textbf{vec}\ (\mu[\Delta g']) = \kappa^{-1}\ \mu[\mu(\mathbf{B})]\ \mathbf{W}^T\ \textbf{vec}\ (\Delta g). \qquad (A.6)$$

In the absence of a rigorous proof on the equivalence between Eq. (A.5) and Eq. (A.6), we performed a numerical evaluation to ascertain their equivalence. We formed a set of regularly spaced simulated mean gravity values on the geoid in 4 cells, building the operator $\mathbf{W}^T$ accordingly. The result of the evaluation certified the equivalence of those equations.

Let us look back at Eq. (19):

$$\textbf{vec}\ (\mu[\Delta g']) = \kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \mathbf{W}\ \textbf{vec}\ (\mu[\Delta g]) - \kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \textbf{vec}\ (r), \qquad (A.7)$$

which should be equivalent to moving from the LR to the UR, using operator $\mu[\mu(\mathbf{B})]$:

$$\textbf{vec}\ (\mu[\Delta g']) = \mu[\mu(\mathbf{B})]\ \textbf{vec}\ (\mu[\Delta g]), \qquad (A.8)$$

where $\mu[\mu(\mathbf{B})]$ is equivalent to $\kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \mathbf{W}$. Eq. (A.8) is the same as the final solution presented in Eq. (23). The term $\kappa^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \textbf{vec}(r)$ is not part of the commutative diagram, meaning that the operations discussed are commutative only if the assumption that the expected value of the residuals r be equal to zero is true, resulting in the whole term $k^{-1}\ \mathbf{W}^T\ \mathbf{B}\ \textbf{vec}\ (r)$ tending to zero.

The downward continued solution coming from the commutative diagram is the same as Eq. (24):

$$\textbf{vec}\ (\mu[\Delta g]) = \mu[\mu(\mathbf{B})]^{-1}\ \textbf{vec}\ (\mu[\Delta g']). \qquad (A.9)$$

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